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Birth and death processes with disasters

Peng, Nan-Fu, Ph.D.
The Ohio State University, 1989
BIRTH AND DEATH PROCESSES WITH DISASTERS

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By

Nan-Fu Peng, Ph.D.

*****

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1989

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Chapter I

Introduction

From the beginning of the century there has been an increasing interest in the study of systems which vary in time in a random manner. Mathematical models of such systems are known as stochastic processes. In a great many population processes of practical importance, the rate of appearance of individuals, i.e. birth, or the rate of disappearance of existing individuals, i.e. death, is proportional to the present population size, which is referred as a linear birth and death process and establishes the basis of the topic that we are going to research.

Consider a collection of particles which act independently in giving rise to succeeding generations of particles. Suppose that in an infinitesimal time of length $\Delta t$, only two types of transitions are possible, namely that the chance of any individual giving birth is $\lambda \Delta t$, and the chance of dying is $\mu \Delta t$. Hence the chance of remaining unchanged is $1-\lambda \Delta t-\mu \Delta t$, and we neglect the chance of giving birth to more than one descendant. The population process obtained in this manner is called a linear birth and death process with birth rate $\lambda$ and death rate $\mu$. This process has been widely used in physics and biology, and most of the stochastic behaviors of the
population such as mean, variance and probability of a certain population size at a given time are well known, e.g. [3].

In 1975, a new model of population processes influenced by a certain factor from the environment, namely disasters or catastrophes, was proposed by Kaplan et al. [18]. We shall restrict our attention to a supercritical linear birth and death process, $\lambda > \mu$, subject to disasters. Supercriticality is required in order to avoid almost sure extinction of the process even without disasters, see [3], p. 95. Those disasters occur at random times that are assumed to be the epochs of a renewal process, or specifically a Poisson process with parameter $\beta$, which proceeds independently of the population size. Each disaster is an instantaneous event consisting of a binominal killing, with constant probability $1-\delta$, of the members of the population. Hence the survival of any particle is independent of the survival of all other particles. This process can be applied to many physical phenomena provided appropriate interpretation is given for the disaster. For example, disasters could be epidemics, famine, wars or earthquakes.

Notice that this process is not equivalent to adding the killing rate $\beta(1-\delta)$ (the product of the occurrence rate and the killing probability of disasters) to the death rate of a linear birth and death process. In fact, the members of the population subject to disasters are not independent since they suffer disasters at the same time. Hence this fact complicates the problem. A few papers have studied some stochastic processes of a similar kind. For
example, Brockwell et al. [7] considered the discrete-state models where immigration is possible, [8] emphasized a continuous-state space, Brockwell [5] and [6] studied the distribution of extinction time, Bartoszynski et al. [4] and Buhler and Puri [9] found several properties of the model mentioned in the last paragraph. Some of those papers did not use the binomial killing (or included the binomial killing as a special case), for instance, [5], [6], [7], [8]. But they have a feature in common that the killing probabilities depend at most on the present population size. Being interested in this problem, I continued this work with great help from Dr. Robert Bartoszynski, Dr. Dennis K. Pearl and Dr. Wen Yaw Chan.

We generalize the above model by assuming that the killing probability 1-δ (or the surviving probability δ) in a disaster is, instead of a constant, a function of the time that has elapsed since the last disaster. The motivation for studying this type of processes may be as follows. Some disasters become more serious as the time length since the last disaster increases, such as earthquakes (this will be explained in Chapter 5) or some types of epidemics. For instance, an epidemic leaves survivors with some immunity. As time passes, this immunity may be lost, so that the next epidemic is more serious when it comes long after the preceding one. On the other hand, in regarding disasters as applications of treatments for some cancers, we find that each disaster renders some members of the (cell) population immune to
treatment, so that subsequent treatments coming close together kill smaller and smaller fractions of cancer cells.

Many stochastic properties of this generalized process will be stated in Chapter 2 after the crucial derivation of the probability generating function conditioned on a finite sequence of disaster times, has been obtained. Letting this finite sequence pass to an infinite one, we obtain, with the help of the Law of Large Numbers, an important discovery of a necessary and sufficient condition for sure extinction (this is a condition causing the probability of eventual extinction of the process to be equal to one). The theorem implies that the "average" of the time-dependent killing probabilities in the disaster cannot be too small. Of course, if the above condition is not satisfied, then the probability of eventual extinction is always less than one. Hence, instead of being a real number, it is a variable with randomness because of its conditioning on an infinite sequence of disaster times, which are random. In order to find the distribution of the extinction probability in the special case of surviving probability being constant, we face a differential-difference equation with both constant scale and location delay parameters. A transformation will be used in Chapter 2 in order to obtain another equation with constant location delay parameter only which can be solved by a computer with the method proposed in [21]. This saves considerable computer time and is more accurate than simulation method that usually takes a much longer computer time. As for
the general case of time-dependent surviving probabilities, a
differential-difference equation with both variable scale and
location delay parameters is involved. Since no general technique
can be applied to search for the solutions, a statistical
approximation method named Edgeworth expansion is used to
acquire the distribution of the probability of eventual extinction.
One point we need to mention here is that the conventional
probability of eventual extinction is the mean of above
distribution. Hence our result contains more information than the
traditional one about the probability of eventual extinction. The
Laplace transforms of mean and variance will also be derived in
Chapter 2.

A generalization of time-dependent surviving probability to
multiple type time-dependent surviving probability is obtained in
Chapter 3. This extension is quite natural if different types of
disasters apply together to a population process. The relationship
among different types of disasters are divided by two categories,
independent and dependent. The degree of dependence will be
defined in Chapter 3. Almost all results obtained in Chapter 2 will
reappear in Chapter 3 with some modification except the Laplace
transforms of mean and variance of time-dependent disasters
which are independent of other types. Time-independent random
disasters are also discussed in Chapter 3.

In Chapter 4, we extend the models of Chapter 2 and Chapter
3 to nonhomogeneous birth and death processes with the formulas
in [3]. An amazing consequence is that we only have to modify the "process part" and leave the "disaster part" unaltered.

In Chapter 5, the inverse Laplace transforms of mean and variance which can be theoretically obtained is found and some examples are demonstrated. For those Laplace transforms which are not listed in standard tables of Laplace transforms, e.g. [23], numerical method is obtained and also some numerical results are shown.
§2.1 Conditioned Probability Generating Function

Consider a supercritical linear birth and death process with disasters which arrive in a Poisson stream. Assume that the disasters have binomial killing of the population with killing probability \(1-\delta(t)\) which only depends on the time that has elapsed since last disaster. From [3], p. 93, the probability generating function (pgf) of a linear birth and death process without disasters is, if the initial size is \(a\),

\[
F(s, t) = \left( \frac{\mu - \mu e^{\rho t} - (\lambda - \mu e^{\rho t})s}{\mu - \lambda e^{\rho t} - (\lambda - \lambda e^{\rho t})s} \right)^a
\]

\[
= \left(1 + \frac{(\lambda - \mu) e^{\rho t}(s - 1)}{\lambda e^{\rho t} - \mu - \lambda (e^{\rho t} - 1)s} \right)^a
= \left(1 + \frac{(\lambda - \mu) e^{\rho t}(s - 1)}{(\lambda - \mu) - \lambda (e^{\rho t} - 1)(s - 1)} \right)^a
\]

\[
= \left(1 + \frac{1}{e^{-\rho t} s - 1 - \frac{\lambda}{\rho} (1 - e^{-\rho t})} \right)^a
\]
where \( \lambda \) and \( \mu \) are birth and death rates, respectively, and \( \rho = \lambda - \mu > 0 \) as assumed. Let \( \{ \tau_1, \tau_2, \ldots \} \) be i.i.d. intercatastrophe times which are exponentially distributed with mean \( \frac{1}{\beta} \), and let

\[
\begin{aligned}
\delta_i &= \delta(\tau_i) \\
\phi_i &= e^{\rho \tau_i} \\
\Phi_i &= \frac{1}{\delta_i} e^{-\rho \tau_i} = \frac{1}{\delta_i \phi_i} \\
A_i &= \frac{\lambda}{\beta}(1 - e^{-\rho \tau_i})
\end{aligned}
\]

(2.2)

\( i = 1, 2, 3, \ldots \)

Define \( g(t^+) = \lim_{\epsilon \to 0} g(t + \epsilon) \) for any function \( g \), then the compound p.g.f., [13], p. 287, conditioned on \( \tau_1 \), is

\[
F(s, \tau_1 \mid \tau_1) = \left( 1 + \frac{1}{\delta_i s + 1 - \delta_i - 1 - \frac{\lambda}{\beta}(1 - e^{-\rho \tau_i})} \right)^a
\]

(2.3)

\[
= \left( 1 + \frac{1}{\Phi_i \frac{s}{s - 1} - A_i} \right)^a
\]

and that of conditioned on \( \tau_1 \) and \( \tau_2 \) is

\[
F(s, (\tau_1 + \tau_2)^+ \mid \tau_1, \tau_2) = \left( 1 + \frac{1}{\Phi_i \frac{s^2}{s^2 - 1} - A_i} \right)^a
\]

(2.4)
\[ 1 + \frac{1}{\Phi_1 \Phi_2} = \left( \frac{\Phi_2}{s-1 - \Phi_1 A_2 - A_1} \right)^a. \]

Hence, by induction, it is easy to see that by defining \( \Phi_1^* = \prod_{k=1}^{n-1} \Phi_k \)
and \( A_1^* = \sum_{j=1}^{n-1} A_j \prod_{k=1}^{j-1} \Phi_k \) and by using the case \( n=2 \) above, we have

\[ F(s, (\tau_1 + \ldots + \tau_n) \mid \tau_i, \ldots, \tau_n) = \left( 1 + \frac{1}{\prod_{i=1}^n \Phi_i \left( \frac{\prod_{i=1}^n \Phi_i}{s-1} - \sum_{j=1}^n A_j \prod_{k=1}^{j-1} \Phi_k \right)} \right)^a \]

if we define \( \prod_{k=1}^0 \Phi_k = 1 \).

\( \S 2.2 \) Probability of Extinction

Obviously, with probability 1, there will be infinitely many disasters. One way to understand the event 'eventual extinction', denoted by \( E \), is to condition on an infinite sequence of \( \tau_i \)'s. The conditional probability of extinction at \( (\tau_1 + \ldots + \tau_n)^+ \) is

\[ P_n(E) = F(0, (\tau_1 + \tau_2 + \ldots + \tau_n) \mid \tau_1, \tau_2, \ldots, \tau_n) = \left( 1 - \frac{1}{\sum_{j=1}^n A_j \prod_{k=1}^{j-1} \Phi_k + \prod_{i=1}^n \Phi_i} \right)^a. \]
Let

$$Y = \lim_{n \to \infty} \sum_{i=1}^{n} A_{i} \prod_{k=1}^{i-1} \Phi_{k}$$

If $Y=\infty$, then $P_{n}(E) \to P(E) = 1$, a degenerate random variable, since

$$\prod_{i=1}^{n} \Phi_{i} \geq 0 \quad \forall \ n.$$ 

If $Y<\infty$, then

$$P_{n}(E) \to P(E) = (1 - \frac{1}{Y})^{a},$$

a random variable depending on $Y$, because $\prod_{i=1}^{n} \Phi_{i} \to 0$ by the independence of $A_{j}$ and $\prod_{i=1}^{j} \Phi_{i}$, and by the fact that $P(A_{j}>0)=1$.

Hence we have the following fact,

$$P(E) = 1 \text{ if and only if } Y=\infty.$$ 

Using (2.9), we can show another useful necessary and sufficient condition of almost sure extinction which is a generalization of the theorem in [9], which deals with constant probability of surviving. Before writing theorem 2.1, let's restate our crucial assumptions.
$A_1$: The disasters occur according to a Poisson stream with parameter $\beta$.

$A_2$: The killing rate of the $i$-th disaster depends only on $\tau_i$.

**Theorem 2.1**

Under the assumptions of $A_1$ and $A_2$, a necessary and sufficient condition for $P(\xi)$ being degenerated to 1, i.e. almost sure extinction, is

\[(2.10) \quad E \log \delta(\tau_1) \leq -\rho \ E \tau_1 = -\frac{\rho}{\beta}.\]

**Proof:**

First we show that $Y=\infty$ implies $E \log \delta(\tau) \leq -\rho \ E \tau$.

If $Y=\infty$, then $\exists \{m_n\} \to \infty$, such that

\[A_{m_n} \prod_{j=1}^{m_n-1} \Phi_j > \frac{1}{m_n^2} \Rightarrow e^{-\sum_{j=1}^{m_n-1} (\rho \tau_j + \log \delta_j)} > \frac{1}{m_n^2 A_{m_n}}\]

\[\Rightarrow \frac{1}{m_n} \sum_{j=1}^{m_n-1} (\rho \tau_j + \log \delta_j) < \frac{1}{m_n} \log( m_n^2 A_{m_n}) \Rightarrow 0\]

Hence $E(\rho \tau + \log \delta(\tau)) \leq 0$ by the strong Law of Large Numbers.
Secondly, we show $Y < \infty$ implies $E \log \delta(\tau) \geq -\rho Et$.

If $Y < \infty$, then $\forall \epsilon \in \mathbb{R}$ such that $\forall n \geq N$ we have

$$A_n \prod_{j=1}^{n-1} \Phi_j < \epsilon \Rightarrow \epsilon - \sum_{j=1}^{n-1} (\rho \tau_j + \log \delta_j) < \frac{\epsilon}{A_n}$$

$$\therefore \frac{1}{n} \sum_{j=1}^{n-1} (\rho \tau_j + \log \delta_j) > -\frac{1}{n} \log \frac{\epsilon}{A_n} \rightarrow 0 \text{ a. s.}$$

$$\therefore E(\rho \tau + \log \delta) \geq 0.$$ 

Last, we demonstrate that $E \log \delta(\tau) = -\rho Et$ implies $Y = \infty$.

If $E \log \delta(\tau) = -\rho Et$, then

$$Y = \lim_{n \rightarrow \infty} \sum_{i=1}^{n} A_i \prod_{j=1}^{i-1} \Phi_j = \lim_{n \rightarrow \infty} \sum_{i=1}^{n} A_i e^{-\sum_{j=1}^{i-1} (\rho \tau_j + \log \delta_j)} = \infty$$

since $P \left[ \sum_{j=1}^{i-1} (\rho \tau_j + \log \delta_j) \leq 0 \text{ i.o.} \right] = 1$ by [10], p. 264.

§ 2.3 Density of Probability of Extinction

If the condition in theorem 2.1 is not satisfied, that is, extinction is not certain, then $P(E)$ depends on $Y$, hence depends on intercatastrophe times, and is less than 1 with randomness. To
get the density of \( P(E) \), we have to obtain that of \( Y \) first because of (2.8).

Since

\[
Y = \lim_{n \to \infty} \sum_{j=1}^{n} A_j \prod_{k=1}^{j-1} \Phi_k,
\]

in addition with (2.2), we have the following equation

\[
(2.11) \quad Y = \Phi(\tau)Y + A(\tau) = \frac{Y}{\delta(\tau)\phi(\tau)} + \frac{\lambda}{\rho} (1 - \frac{1}{\phi(\tau)}) ,
\]

where \( \Phi \) and \( A \) are independent of \( Y \) and distributed as \( \Phi_1 \) and \( A_1 \) respectively, and \( \overset{d}{=} \) stands for equality in distribution. After some simple computations, we have

\[
\frac{\rho}{\lambda} \frac{Y-1}{Y} = \frac{\frac{\rho}{\lambda} (Y-1) + 1 - \delta(\tau)}{\phi(\tau)\delta(\tau)}
\]

hence

\[
(2.12) \quad X = \frac{X + 1 - \delta(\tau)}{\phi(\tau)\delta(\tau)}
\]

where we define

\[
(2.13) \quad X = \frac{\rho}{\lambda} Y^{-1},
\]

which is a one to one function of \( Y \). Working on \( Y \) is equivalent to working on \( X \), and
(2.14) \[ P(E) = \left(1 - \frac{1}{Y}\right)^a = \left(\frac{\lambda + \mu X}{\lambda + \mu X}\right)^a. \]

We also claim that if \( Y < \infty \), then

(2.15) \[ X = \sum_{j=1}^l (1 - \delta_j) \prod_{k=1}^{j-1} \frac{1}{\delta_k} e^{-\rho_k} > 0. \]

From (2.7), \( Y = \lim_{n \to \infty} \sum_{j=1}^n A_j \prod \Phi_k = \sum_{j=1}^n \lambda \prod \left(1 - e^{-\rho_k}\right) \prod_{k=1}^{j-1} \frac{1}{\delta_k} \prod_{k=1}^{j-1} e^{-\rho_k} \)

and \( \frac{\rho}{\lambda} Y = \sum_{j=1}^n \left(1 - e^{-\rho_k}\right) \prod_{k=1}^{j-1} \frac{1}{\delta_k} e^{-\rho_k} \), so that

\[
\frac{\rho}{\lambda} Y - \sum_{j=1}^n (1 - \delta_j) \prod_{k=1}^{j-1} \frac{1}{\delta_k} e^{-\rho_k} = \sum_{j=1}^n (1 - e^{-\rho_k}) \prod_{k=1}^{j-1} \frac{1}{\delta_k} e^{-\rho_k} = \sum_{j=1}^n \left(1 - \prod_{k=1}^{j-1} \frac{1}{\delta_k} e^{-\rho_k}\right) \prod_{k=1}^{j-1} \frac{1}{\delta_k} e^{-\rho_k}
\]

\[
= \sum_{j=1}^n \left[ \prod_{k=1}^{j-1} \frac{1}{\delta_k} e^{-\rho_k} - \prod_{k=1}^{j-1} \frac{1}{\delta_k} e^{-\rho_k}\right]
\]

\[
= 1 - \prod_{k=1}^{j-1} \frac{1}{\delta_k} e^{-\rho_k} = 1 - e^{-\lambda \sum_{k=1}^{j-1} \rho_k + \log \delta_k} = 1 - e^{-\lambda \sum_{k=1}^{j-1} \rho_k + \log \delta_k} = 1,
\]

since \( \sum_{k=1}^{j-1} (\rho_k + \log \delta_k) = \infty \) w.p.1 by theorem 2.1. This implies (2.15) because of (2.13).

To find the density of \( X \), we consider two cases.
(i) \( \delta \) is a constant.

Let \( V = \frac{1}{\delta \phi(\tau)} = \frac{1}{\delta} e^{-\rho \tau} \). Since \( \tau \sim \exp(\beta) \),

\[
P(V \leq \nu) = P\left(\frac{1}{\delta} e^{-\rho \tau} \leq \nu\right) = P\left(\tau \geq \frac{-\log \nu}{\beta} \right) = e^{\frac{\beta}{\rho} \log(\nu \delta)} = (\nu \delta)^{\frac{\beta}{\rho}}.
\]

Hence,

\[
P(V \leq \nu) = \begin{cases} 
0 & \text{if } \nu < 0 \\
(\nu \delta)^{\alpha} & \text{if } 0 \leq \nu \leq \frac{1}{\delta} \\
1 & \text{if } \nu > \frac{1}{\delta},
\end{cases}
\]

where \( \alpha = \frac{\beta}{\rho} \), and so the density of \( V \) is

\[
f(\nu) = \begin{cases} 
0 & \text{if } \nu < 0 \\
\alpha \delta^{\alpha} \nu^{\alpha-1} & \text{if } 0 \leq \nu \leq \frac{1}{\delta} \\
0 & \text{if } \nu > \frac{1}{\delta}.
\end{cases}
\]

By (2.12), \( X = \frac{X \delta + 1 - \delta}{\phi(\tau) \delta} = (X \delta + 1 - \delta)V \), so that the c.d.f. \( G(x) \) of \( X \) satisfies

\[
G(x) = \min \left( \frac{1}{\delta \cdot \frac{x}{1-\delta}} \right) \int_0^x f(\nu)G\left(\frac{X}{\nu} - 1 + \delta\right) d\nu,
\]

hence by (2.16) the density function \( g(x) \) of \( X \) satisfies
(2.17) \[ g(x) = \begin{cases} \int_0^{1-\delta} \alpha v^{-2\alpha} g(\frac{x}{v} - 1 + \delta) dv & \text{if } 0 < x \leq \frac{1-\delta}{\delta} \\ \int_{1-\delta}^{x} \alpha v^{-2\alpha} g(\frac{x}{v} - 1 + \delta) dv & \text{if } x > \frac{1-\delta}{\delta} \end{cases} \]

Substituting \( \frac{x}{v} - 1 + \delta \) by \( z \) in (2.17), we obtain

\begin{equation}
(2.18) \quad g(x) = \begin{cases} \int_0^{1-\delta} \alpha \delta^\alpha \frac{x^{\alpha-2}}{(z+1-\delta)^\alpha} g(z) dz & \text{if } 0 < x \leq \frac{1-\delta}{\delta} \\ \int_{1-\delta}^{x} \alpha \delta^\alpha \frac{x^{\alpha-2}}{(z+1-\delta)^\alpha} g(z) dz & \text{if } x > \frac{1-\delta}{\delta} \end{cases}
\end{equation}

Dividing both sides of (2.18) by \( c = \alpha \delta^\alpha \int_0^{1-\delta} \frac{g(z)}{(z+1-\delta)^\alpha} dz \) and differentiating it with respect to \( x \) for \( x > \frac{1-\delta}{\delta} \), we have the following delayed differential equation, with both scale and location delay arguments,

\begin{equation}
(2.19) \quad \begin{cases} h(x) = x^{\alpha-1} & \text{if } 0 < x \leq \frac{1-\delta}{\delta} = \frac{1}{\delta} - 1 \\ h'(x) = \frac{\alpha - 1}{x} h(x) - \frac{\delta \alpha}{x} h(\delta x - 1 + \delta) & \text{if } x > \frac{1-\delta}{\delta} = \frac{1}{\delta} - 1 \end{cases}
\end{equation}

where \( h(x) = g(x)/c \). In order to remove the scale change from (2.19), we define:
\begin{align}
\tag{2.20}
\zeta(t) &= \delta^{-1} - 1 \Rightarrow \delta \zeta(t) - 1 + \delta = \zeta(t) - 1 \\
\psi(t) &= h(\zeta(t))
\end{align}

We have, replacing $x$ by $\xi(t)$ in (2.19),

\begin{align}
\tag{2.21}
\begin{cases}
\psi(t) = (\delta^{-1} - 1)^{\alpha-1} & \text{if } 0 < t \leq 1 \\
\psi'(t) = [\alpha - 1] \psi(t) - \alpha \delta \psi(t - \eta) \left( \log \frac{\delta}{\delta' - 1} \right) & \text{if } t > 1
\end{cases}
\end{align}

Now there is only a location delay argument in the above equation. If we use the method proposed by [21], an approximate solution of $\psi(t)$, and hence $h(x)$, can be found. Finding the density function $g(x)$ of $X$ is then easy because $h(x) = g(x) / c$ and $\int_{0}^{1} g(x) \, dx = 1$. Two examples of densities of $P(E)$'s when $\delta = 0.6$ and $\delta = \frac{1}{\sqrt{2}}$ with $\beta = 2$, $\lambda = 4$ and $\mu = 2$ are illustrated in Figure 1 and Figure 2.

(ii) $\delta$ is a function of time.

From (2.12) we get $X^n = \frac{X + 1 \cdot \delta(\tau)}{\phi(\tau) \delta(\tau)}^n$, $n \geq 1$. If $M_n = EX^n < \infty$, then integrate both sides of the above equation to obtain

\[ M_n = \beta \sum_{j=0}^{n} \binom{n}{j} M_j \int_{0}^{1} e^{-\beta \tau} \frac{1}{(\phi(\tau) \delta(\tau))^n (1 - \delta(\tau))^{n-j}} \, d\tau. \]

So that
Figure 1: Density Function of Probability of Extinction for \( \delta = 0.6 \)
Figure 2: Density Function of Probability of Extinction for $\delta = \frac{1}{\sqrt{2}}$
Using an Edgeworth expansion [12], [11] and [19], we can get an approximate density of $X$. Some results are shown in Figure 3 and Figure 4.

There is an advantage of using Edgeworth expansion if all moments of $X$ exist. That is, we can show that the Carleman's condition, [10], p. 98, holds so that the sequence of finite moments of $X$ uniquely determines the distribution of $X$. To see this, let's look at the following result if $M_n$ exists for all $n$.

Since $X > 0$, by (2.15), and the numerator of (2.22) is positive, it is obvious that if $M_n$ is finite, then the denominator of (2.22) satisfies

\[(2.23) \quad 1 - \beta \int_0^\infty e^{-\beta \tau} \left( \frac{1}{\delta(\tau) \phi(\tau)} \right)^n d\tau > 0.\]

Consequently, we have

\[(2.24) \quad \frac{1}{\delta(\tau) \phi(\tau)} \leq 1 \quad \text{a.e. and} \quad P\left( \frac{1}{\delta(\tau) \phi(\tau)} = 1 \right) < 1\]

if (2.23) holds for all $n$.\[M_n = \frac{\beta \sum_{j=0}^{n-1} \binom{n}{j} M_j \int_0^\infty e^{-\beta \tau} \frac{1}{(\phi(\tau) \delta(\tau))^n (1 - \delta(\tau))^{n-j}} d\tau}{1 - \beta \int_0^\infty e^{-\beta \tau} \frac{1}{(\phi(\tau) \delta(\tau))^n} d\tau}\]
Figure 3: Density Function of Probability of Extinction for $\delta = (0.5198)/(0.5198 + t)$
Figure 4: Density Function of Probability of Extinction for $\delta = \exp(-t)$
Now we show that $M_n$'s satisfy Carleman's condition

\begin{equation}
2.25 \quad \sum_{r=1}^{\infty} \frac{1}{M_{2r}^{2r}} = \infty
\end{equation}

From (2.24), it is obvious that we can assume

$$c = \sup_{m \to 0} \frac{\beta \int_0^\infty e^{-\beta \tau} \left( \frac{1}{\delta(\tau) \phi(\tau)} \right)^m d\tau}{1 - \beta \int_0^\infty e^{-\beta \tau} \left( \frac{1}{\delta(\tau) \phi(\tau)} \right)^m d\tau} < \infty$$

From (2.22), we have

$$M_n = \frac{\beta \sum_{j=0}^{n-1} j^n M_j \int_0^\infty e^{-\beta \tau} \left( \frac{1}{\delta(\tau) \phi(\tau)} \right)^n (1 - \delta(\tau))^{n-j} d\tau}{1 - \beta \int_0^\infty e^{-\beta \tau} \left( \frac{1}{\delta(\tau) \phi(\tau)} \right)^n d\tau} \leq c \sum_{j=0}^{n-1} j^n M_j$$

Fix an $n$ and let $b$ be such that $M_j \leq (b^j)^j \leq (a^j)^j \quad \forall j = 1,2,\ldots,n-1$

where $a = \max (1,b,c)$. Then
Hence the constant $a$ does not depend on $n$. Thus, Carleman's condition (2.25) is satisfied.

In the next section we shall discuss the mean and variance of the population size $Z(t)$. We start from the conditional p.g.f. of $Z(t)$, and then derive their Laplace Transforms. Inverse Laplace Transforms and some examples will be discussed in Chapter 5.

§2.4 Laplace Transform of Mean and Variance of $Z(t)$

We know from (2.5) that

$$
F(s,(x_1 + \ldots + x_n)^+|x_1, \ldots, x_n) = \left(1 + \frac{1}{\prod_{i=1}^{n} \Phi_i} \right)^a \frac{1}{s - 1 - \sum_{i=1}^{n} \sum_{k=1}^{j-1} A_k \prod_{i=1}^{j-1} \Phi_i}
$$

and so

$$
E(Z(x_1 + \ldots + x_n)^+|x_1, \ldots, x_n) = \left. \frac{dF(s,(x_1 + \ldots + x_n)^+|x_1, \ldots, x_n)}{ds} \right|_{s=1}
$$
Let $N(t)$ be the number of disasters having occurred before $t$.

Then by [3], p. 94,

\[(2.27) \quad \mathbb{E}(Z(t)|\tau_1, ..., \tau_n, N(t) = n) = \mathbb{E}(Z(\tau_1 + ... + \tau_n)|\tau_1, ..., \tau_n) \cdot e^{p(t-(\tau_1 + ... + \tau_n))} \]

\[= a \left[ \prod_{i=1}^{n} \delta_i \cdot e^{p \tau_i} \right] \cdot e^{p(t-(\tau_1 + ... + \tau_n))} = ae^{p^t} \prod_{i=1}^{n} \delta_i . \]

It is well known that conditioned on $N(t) = n$, the joint distribution of $\tau_i, (\tau_1 + \tau_2), ..., (\tau_1 + ... + \tau_n)$ is the same as that of ordered statistics from a Uniform distribution on $(0, t)$. Since the Jacobian of transforming $\{ (\tau_1 + \tau_2), ..., (\tau_1 + ... + \tau_n) \}$ to $\{ \tau_1, \tau_2, ..., \tau_n \}$ is 1, if we define $\tilde{\tau}$ to be the vector $(\tau_1, \tau_2, ..., \tau_n)$, then

\[(2.28) \quad \mathbb{E}(Z(t)) = \mathbb{E}_N \mathbb{E}_N^{\tilde{\tau}} \mathbb{E}(Z(\tilde{\tau}, \tau_2, ..., \tau_n, N(t) = n) \]

\[= \sum_{n=0}^{\infty} \left( \int \cdot e^{p^t} \prod_{i=1}^{n} \delta_i n! \frac{1}{t^n} d\tilde{\tau} \right) \cdot \frac{(\beta t)^n}{n!} e^{-\beta t} = a \sum_{n=0}^{\infty} \left( \frac{e^{-(\rho - \beta)}t^n}{ \prod_{i=1}^{n} \delta_i } \right) \int \prod_{i=1}^{n} \delta_i d\tilde{\tau} \]
\[ a \sum_{n=0}^{\infty} e^{(0-\beta)n} \left( (\delta \ast \delta \ast \ldots \ast \delta) \ast \delta(t) \right) \]

where \( \delta(s) = \int_0^s \delta(y) dy \) and '\( \ast \)' stands for convolution of functions.

Now, let us define \( L_1(s) \) to be the Laplace Transform of a function \( f(t) \). Since \( L_1(s) = \frac{L_1(s)}{s} \), by (2.28) the Laplace Transform of \( EZ(t) \) is

\[ L_{E}(s) = \sum_{n=0}^{\infty} \beta_n L_{\delta}(s + \beta - \rho) \frac{a}{s + \beta - \rho} \frac{a}{(s + \beta - \rho) \Gamma(\beta) L_{\delta}(s + \beta - \rho) } \]

More complicated work is needed to find the variance. We start with the similar procedures as the case of the mean above.

From (2.5), we have

\[ \frac{d^2 F(s, (\tau_1 + \ldots + \tau_n)^+ | \tau_1, \ldots, \tau_n)}{ds^2} \bigg|_{s=1} \]

\[ = \frac{2a}{n} \sum_{j=1}^{n} \left( \prod_{k=1}^{j-1} \Phi_k + a(a-\gamma) \right)^{-1} \frac{1}{(\prod_{j=1}^{n} \Phi_j)^2} \]

And from (2.26) and (2.30), we also have

\[ \text{Var}(Z((\tau_1 + \ldots + \tau_n)^+ | \tau_1, \ldots, \tau_n)) \]

\[ = \frac{d^2 F(s, (\tau_1 + \ldots + \tau_n)^+ | \tau_1, \ldots, \tau_n)}{ds^2} \bigg|_{s=1} \]
\[ + E(Z((\tau_1 + \ldots + \tau_n)^+) | \tau_1, \ldots, \tau_n) - [E(Z((\tau_1 + \ldots + \tau_n)^+) | \tau_1, \ldots, \tau_n)]^2 \]

\[ = \frac{2a}{n} \sum_{i=1}^{n} \Phi_i \prod_{k=1}^{l-1} \phi_k + a(a - \gamma) \frac{1}{n} \sum_{i=1}^{n} \prod_{k=1}^{l-1} \phi_k \]

\[ = a \left[ \frac{2}{(\prod_{i=1}^{n} \phi_i)^2} \sum_{i=1}^{n} \lambda_i (1 - e^{-\rho \tau}) \prod_{k=1}^{l-1} (\delta_k e^{-\rho \tau}) + \prod_{i=1}^{n} (\delta_i e^{\rho \tau}) - \prod_{i=1}^{n} (\delta_i e^{\rho \tau})^2 \right]. \]

Again, let \( N(t) \) be the number of disasters having occurred before time \( t \). Then by [3], p. 94,

\[ \text{Var}[Z(t) | \tau_1, \ldots, \tau_n, N(t) = n] \]

\[ = Z((\tau_1 + \ldots + \tau_n)^+) \frac{\lambda + \mu}{\rho} e^{\rho(t-(\tau_1 + \ldots + \tau_n))} (e^{\rho(t-(\tau_1 + \ldots + \tau_n))} - 1) \]

So far we have obtained the variance of \( Z(t) \) conditioned on \( \tau \), \( N(t) \) and \( Z((\tau_1 + \ldots + \tau_n)^+) \). In order to get the unconditioned variance, we have to do the following steps:

First, we eliminate the condition of \( Z((\tau_1 + \ldots + \tau_n)^+) \).

(2.32) \[ \text{Var}(Z(t) | \tau_1, \ldots, \tau_n, N(t) = n) \]

\[ = E[Z((\tau_1 + \ldots + \tau_n)^+) \frac{\lambda + \mu}{\rho} e^{\rho(t-(\tau_1 + \ldots + \tau_n))} (e^{\rho(t-(\tau_1 + \ldots + \tau_n))} - 1) | \tau_1, \ldots, \tau_n, N(t) = n] \]
Thus, we can proceed the second step to eliminate the conditions of \( \tau \) and \( N(t) \). From (2.32), we have

\[
(2.33) \quad \text{Var}Z(t) = \lambda \mu \rho \left[ \sum_{j=1}^{n} \lambda t \left( 1 - e^{-\rho t} \right)^{j-1} \prod_{k=1}^{j} \left( \frac{1}{\delta_k} e^{-\rho t} \right) \right] \\
+ a e^{2 \rho t} \left[ \sum_{i=1}^{n} \delta_i \rho t \left( 1 - e^{-\rho t} \right)^{j-1} \prod_{k=1}^{j} \left( \frac{1}{\delta_k} e^{-\rho t} \right) \right] \\
+ a^2 e^{2 \rho t} \left( \prod_{i=1}^{n} \delta_i \right)^2 - \left[ \lambda \mu \rho e^{\rho t} \right]^2 \\
= W - \frac{\lambda + \mu}{\rho} \text{EZ}(t) - (\text{EZ}(t))^2
\]
where

\[ W = aE^{\xi,N} \left[ \frac{2\lambda}{\rho} \prod_{i=1}^{n} \delta_i e^{\rho_1 e^{(t-(\tau_1+\ldots+\tau_n))}} \right. \]
\[ + 2e^{2\rho_1} \prod_{i=1}^{n} \delta_i^2 \sum_{j=1}^{n} \frac{\lambda}{\rho} (1 - e^{-\rho_1}) \prod_{k=1}^{j-1} \left( \frac{1}{\delta_k} e^{-\rho_1} \right) \]
\[ \left. + (a^2 - a)E^{\xi,N} \left( e^{2\rho_1 \prod_{i=1}^{n} \delta_i^2} \right) \right] \]

To write \( W \) explicitly, we find that

\[ W = a\frac{2\lambda}{\rho} \sum_{n=0}^{\infty} e^{(2p-\beta)l} \beta_n \int_{\tau_1+\ldots+\tau_n \leq t} \rho^{(1-(\tau_1+\ldots+\tau_n))} \prod_{i=1}^{n} \delta_i(\tau_i) d\tilde{\tau} \]
\[ + 2a \sum_{n=0}^{\infty} e^{(2p-\beta)l} \beta_n \int_{\tau_1+\ldots+\tau_n \leq t} \sum_{l=1}^{n} \frac{\lambda}{\rho} (1 - e^{-\rho_1}) \prod_{k=1}^{n} \delta_i^2(\tau_i) \prod_{k=1}^{n} \frac{1}{\delta_i(\tau_k)} e^{-\rho_1} d\tilde{\tau} \]
\[ + (a^2 - a)\sum_{n=0}^{\infty} e^{(2p-\beta)l} \beta_n \int_{\tau_1+\ldots+\tau_n \leq t} \prod_{i=1}^{n} \delta_i^2(\tau_i) d\tilde{\tau} \]

\[ = a\frac{2\lambda}{\rho} \sum_{n=0}^{\infty} e^{(2p-\beta)l} \beta_n \int_{\tau_1+\ldots+\tau_n \leq t} \rho^{(1-(\tau_1+\ldots+\tau_n))} \prod_{i=1}^{n} \delta_i(\tau_i) d\tilde{\tau} \]
\[ + \frac{2\lambda}{\rho} \sum_{n=0}^{\infty} e^{(2p-\beta)l} \beta_n \sum_{l=1}^{n} \int_{\tau_1+\ldots+\tau_n \leq t} \prod_{k=1}^{l-1} \delta_i(\tau_k) e^{-\rho_1} \prod_{k=1}^{n} \delta_i^2(\tau_i) d\tilde{\tau} \]
\[ - \frac{2\lambda}{\rho} \sum_{n=0}^{\infty} e^{(2p-\beta)l} \beta_n \sum_{l=1}^{n} \int_{\tau_1+\ldots+\tau_n \leq t} \left( \prod_{k=1}^{l-1} \delta_i(\tau_k) e^{-\rho_1} \right) \delta_i^2(\tau_i) e^{-\rho_1} \left( \prod_{l=1}^{n} \delta_i^2(\tau_i) \right) d\tilde{\tau} \]
\[ + (a^2 - a)\sum_{n=0}^{\infty} e^{(2p-\beta)l} \beta_n \int_{\tau_1+\ldots+\tau_n \leq t} \prod_{i=1}^{n} \delta_i^2(\tau_i) d\tilde{\tau} \]
\[
= a^{2}\frac{2\lambda}{\rho} \sum_{n=0}^{\infty} e^{(\nu-\beta)t} \beta^n \left( e^{\nu t} \sum_{j=1}^{n} \left( \delta e^{-\nu t} \delta e^{-\nu t} \cdots \delta e^{-\nu t} \right) \right)
\]

\[
+ a^{2}\frac{2\lambda}{\rho} \sum_{n=0}^{\infty} e^{(2\nu-\beta)t} \beta^n \sum_{j=1}^{n} \left( \delta e^{-\nu t} \delta e^{-\nu t} \cdots \delta e^{-\nu t} \right)
\]

\[
\left( n-j \right) \delta e^{-\nu t} \cdots \delta e^{-\nu t} \left( s^2(t) \right)
\]

\[
\left( n-j \right) \delta^2 e^{-\nu t} \cdots \delta^2 e^{-\nu t} \left( s^2(t) \right)
\]

\[
\left( n-j \right) \delta^2 e^{-\nu t} \cdots \delta^2 e^{-\nu t} \left( s^2(t) \right)
\]

\[
+ (a^2 - a) \sum_{n=0}^{\infty} e^{(2\nu-\beta)t} \beta^n \left( \delta^2 \cdots \delta^2 \right) \left( s^2(t) \right)
\]

where \( s^2(s) = \int_{0}^{s} \delta^2(y)dy \).

By (2.33), \( \text{Var}Z(t) \) can be written in terms of \( \text{EZ}(t) \) and \( W \). The Laplace transform of \( \text{EZ}(t) \) was obtained from (2.29), and the Laplace transform of \( W \) can be obtained as follows:
\[(2.35) \quad L_w(s) = a \frac{2\lambda}{p} \sum_{n=0}^{\infty} \beta^n \frac{1}{L_\delta(s + \beta - \rho)} \frac{1}{s + \beta - 2\rho} \]

\[+ a \frac{2\lambda}{p} \sum_{n=0}^{\infty} \beta^n L_{s+1}(s + \beta - \rho) L_{\delta+1}(s + \beta - 2\rho) \frac{1}{s + \beta - 2\rho} \]

\[- a \frac{2\lambda}{p} \sum_{n=0}^{\infty} \beta^n L_{\delta+1}(s + \beta - \rho) L_{\delta+1}(s + \beta - 2\rho) \frac{1}{s + \beta - 2\rho} \]

\[+ (a^2 - a) \sum_{n=0}^{\infty} \beta^n L_{\delta+1}(s + \beta - 2\rho) \frac{1}{s + \beta - 2\rho} \]

\[= \frac{2\lambda a}{p(s + \beta - 2\rho)} \left[ \frac{1}{1 - \beta L_\delta(s + \beta - \rho)} \right] \]

\[+ \sum_{n=0}^{\infty} \beta^n \frac{L_{\delta+1}(s + \beta - 2\rho) L_\delta(s + \beta - \rho)}{L_\delta(s + \beta - \rho)} \frac{L_\delta(s + \beta - \rho)}{L_\delta(s + \beta - 2\rho)} - \frac{L_{\delta+1}(s + \beta - \rho)}{L_{\delta+1}(s + \beta - 2\rho)} \]

\[+ \sum_{n=0}^{\infty} \beta^n \frac{L_{\delta+1}(s + \beta - 2\rho) L_\delta(s + \beta - \rho)}{L_\delta(s + \beta - \rho)} \frac{L_\delta(s + \beta - \rho)}{L_\delta(s + \beta - 2\rho)} - \frac{L_{\delta+1}(s + \beta - \rho)}{L_{\delta+1}(s + \beta - 2\rho)} \]

\[+ \frac{a^2 - a}{(s + \beta - 2\rho)\left[1 - \beta L_\delta(s + \beta - 2\rho)\right]} \]
Hence, the Laplace transform of $\text{Var}Z(t)$ is found completely. A few examples of inverse Laplace Transform will be shown in Chapter 5.
Chapter III

Multiple-Type and Time-Independent Random Disasters

In this chapter we shall investigate some results about time-dependent multiple-type disasters, which is a generalization of Chapter 2, and time-independent random disasters. In the former case we classify those into two categories. The first category are those multiple disasters which are independent of other types. This is a straight extension of Chapter 2 if a few different types of disasters are applied to the same population. The second category are those disasters depending on other types through the assumption that once a disaster occurs, whichever type it is, the probability of surviving of the population for every type of disasters will start all over again, that is, the probability of surviving depends only on the time that has elapsed since the last disaster, without regarding to the type it is. This model can be applied to the case of disasters with not only different types, but also having cross effect on one another. For example, members of population could produce resistance against all types of treatments if any one of these treatments is applied to the population.
§3.1 Time-dependent Multiple Disasters of Category One

Consider a linear birth and death process with disasters of \( M \) different types, where each type is independent of one another and arrives in a Poisson stream with rate \( \beta_i \), occurring times \( \tau^{(i)} \), and consisting of binomial killing of the population with probability of surviving \( \delta_i(t) \), which is a function of the time elapsed since the last \( i \)-th type disaster, \( i=1,...,M \). Let \( \tau_k^{(i)} \) be the time length between the \((k-1)\)st occurrence and the \( k \)-th occurrence of the \( i \)-th type disaster, and let \( \tau_j \) be the time length between the \((j-1)\)st occurrence and the \( j \)-th occurrence of any disasters. By the renewal property of exponential distribution, these \( \tau_k^{(i)} \)'s are independent and identically distributed for fixed type \( i \), and so are \( \tau_j \)'s since

\[
(3.1) \quad \tau_j \overset{d}{=} \min\{\tau^{(0)},...,\tau^{(M)}\} \sim \exp(\sum_{i=1}^{M} \beta_i).
\]

Also, let \( I_k \) be the random variable indicating the type of the \( k \)-th disaster such that
(3.2) \[ P(I_k = i) = \frac{\beta_i}{\sum_{j=i}^{M} \beta_j} \]

Conditioning on \( \tau_j \)'s and \( I_j \)'s is equivalent to conditioning on \( \tau_j \)'s, \( I_j \)'s and \( \tau_{k_j} \)'s, where \( k_j \) denotes the number of occurrences of type \( I_j \) disaster up to the \( j \)-th disaster and depends only on \( I_1, ..., I_j \), for \( k_j \leq j \). From (2.1), we have the p.g.f. of a linear birth and death process

\[ F(s, t) = \left( 1 + \frac{1}{\frac{\phi}{s - 1} - \frac{\lambda}{\rho} \left( 1 - e^{-\rho t} \right)} \right)^s. \]

Now define

(3.3) \[ \delta_j = \delta_{I_j}^{(k_j)}. \]

We then have from equation (2.3)

(3.4) \[ F(s, \tau_i | I_1, \tau_i) = F(s, \tau_i | I_1 \tau_i \tau_k) \]

\[ = \left( 1 + \frac{1}{\frac{e^{-\rho t_i}}{\delta_i, s + 1 - \delta_i - 1} - \frac{\lambda}{\rho} \left( 1 - e^{-\rho t_i} \right)} \right)^s \]

\[ = \left( 1 + \frac{1}{\Phi_i, s - 1 - A_i} \right)^s, \]

where \( A_1 \) and \( \Phi_1 \) are defined as in (2.2). It is obvious that the
p.g.f. conditioned on $I_1, \ldots, I_n$ and $\tau_1, \ldots, \tau_n$ are the same as (2.5).

Thus,

\begin{align*}
(3.5) \quad & \mathbb{P}(s, (\tau_1 + \ldots + \tau_n)^+ \mid I_1, \ldots, I_n, \tau_1, \ldots, \tau_n) \\
& = \mathbb{P}(s, (\tau_1 + \ldots + \tau_n)^+ \mid I_1, \ldots, I_n, \tau_1, \ldots, \tau_n, \tau_k^{(1)}, \ldots, \tau_k^{(l)}) \\
& = \left(1 + \frac{1}{\prod_{i=1}^{n} \Phi_i} \right) \left(\frac{s - 1}{s - 1} - \sum_{j=1}^{n} \frac{A_j \prod_{k=1}^{j} \Phi_k}{\prod_{i=1}^{n} \Phi_i} \right)^a.
\end{align*}

The conditional probability of extinction at $(\tau_1 + \ldots + \tau_n)^+$ is

\begin{align*}
(3.6) \quad & \mathbb{P}_n(\mathbb{E}) := \mathbb{P}(0, (\tau_1 + \tau_2 + \ldots + \tau_n)^+ \mid I_1, \ldots, I_n, \tau_1, \ldots, \tau_n) \\
& = \left(1 - \frac{1}{\sum_{j=1}^{n} \frac{A_j \prod_{k=1}^{j-1} \Phi_k + \prod_{i=1}^{n} \Phi_i}{\prod_{i=1}^{n} \Phi_i}} \right)^a.
\end{align*}

Let

\begin{align*}
(3.7) \quad & Y = \lim_{n \to \infty} \sum_{i=1}^{n} \frac{A_j \prod_{k=1}^{j-1} \Phi_k}{\prod_{i=1}^{n} \Phi_i}.
\end{align*}

If $Y = \infty$, then $\mathbb{P}_n(\mathbb{E}) \to \mathbb{P}(\mathbb{E}) = 1$, a degenerated random variable,

since $\prod_{i=1}^{n} \Phi_i \geq 0 \quad \forall n.$
If \( Y < \infty \), then \( P_n(E^\infty) \to P(E^\infty) = (1 - \frac{1}{Y})^a \), a random variable depending on \( Y \), because \( \prod_{i=1}^{n} \Phi_i \to 0 \) by the independence of \( A_j \) and \( \prod_{i=1}^{j-1} \Phi_i \).

Indeed, \( A_j = \frac{\lambda_j}{\rho} (1 - e^{-\rho \tau_j}) \) and \( \Phi_i = \frac{1}{\delta_i} e^{-\rho \tau_i} = \frac{1}{\delta_i (\tau^{(i)}_{k_i})} e^{-\rho \tau_i} \), which is independent of \( \tau_j \) hence independent of \( A_j \); observe that \( P(A_j > 0) = 1 \).

We have therefore the following fact

\[(3.8) \ P(E^\infty) = 1 \ if \ and \ only \ if \ Y = \infty.\]

Now we restate the assumption mentioned in the beginning of this chapter.

\( A_3 \) : There are disasters of \( M \) different types which are independent of one another.

\( A_4 \) : The killing rate of the \( i \)-th type disaster depends only on \( \tau^{(i)} \).

After \( \delta_j \) was defined in (3.3), the proof of the following theorem 3.1 is a straightforward extension of the proof of the theorem 2.1.
**Theorem 3.1**

Under the assumptions of $A_3$ and $A_4$, a necessary and sufficient condition of $P(E)$ being degenerated to 1, i.e. almost sure extinction, is

$$E^i_{\tau_i} \log \delta_1(\tau_i) \leq -\rho E \tau_1 = -\frac{\rho}{\sum_{n=1}^M \beta_n}.$$

**Proof:**

Exactly the same as the proof of Theorem 2.1 except

$$\frac{1}{N} \sum_{j=1}^{N} (\rho \tau_j + \log \delta_j) \to \rho E \tau_1 + E^i_{\tau_i} \log \delta_1(\tau_i) \text{ by SLLN.}$$

With similar procedures as those of Chapter 2, we obtain for the probability of extinction the formula

$$P(E) = (1 - \frac{1}{Y})^a = (\frac{\mu + \lambda X}{\lambda + \lambda X})^a$$

analogous to (2.14), where $Y$ is given by (3.7) and $X = \frac{\rho}{\lambda} Y - 1$.

Moreover, $X$ satisfies the following equation
where $\phi(\tau_i)$ is given by (2.2).

In order to obtain the density of $P(E)$, we have to find that of $X$ first. Similar to the procedure mentioned in Chapter 2, finding the moments of $X$ and using the Edgeworth expansion are essential. From (3.11), the $n$-th moment of $X$, denoted by $M_n$, satisfies the following equation

$$M_n = \frac{\sum_{j=0}^{n-1} \binom{n}{j} \frac{1}{M_j} E^{I_i}_{\tau_i^{(i)}} \left(1 - \delta_i(\tau_i^{(i)})\right)^{n-j}}{1 - E^{I_i}_{\tau_i^{(i)}} \left(\frac{1}{\phi(\tau_i)\delta_i(\tau_i^{(i)})}\right)^n}$$

To solve (3.12) numerically, we first have to find the joint density of $\tau_i$ and $\tau_i^{(i)}$. For $y \leq x$

$$P(\tau_i = y, \tau_i^{(i)} = x)$$

$$= P(\min\{\tau_i^{(i)}, ..., \tau_i^{(M)}\} = y, \tau_i^{(i)} = x)$$

$$= P(\min\{\tau_i^{(i)}, ..., \tau_i^{(M)}\} = y = x) + P(\min\{\tau_i^{(i)}, ..., \tau_i^{(M)}\} = y < x, \tau_i^{(i)} = x)$$

$$= P(\min\{\tau_i^{(i)}, ..., \tau_i^{(i-1)}, \tau_i^{(i+1)}, ..., \tau_i^{(M)}\} > x, \tau_i^{(i)} = x)$$
\[ + P(\min\{ \tau_1^{(n)}, \ldots, \tau_i^{(l-n)}, \tau_i^{(l+n)}, \ldots, \tau_i^{(M)} \} = y < x, \quad \tau_i^{(l)} = x) \]

\[ = \beta_i e^{-\theta_1^i} x e^{-\sum_{j=1}^{M} \beta_j} d \chi \left( \sum_{j=1}^{M} \beta_j \right) e^{-\sum_{j=1}^{M} \beta_j} d \chi \left( \sum_{j=1}^{M} \beta_j \right) \]

Thus, the joint density function of \( \tau_1 \) and \( \tau_i^{(l)} \) is

\[ (3.13) \quad \beta_i e^{-\theta_1^i} \tau_1^{(l)} e^{-\sum_{j=1}^{M} \beta_j} \left( \sum_{j=1}^{M} \beta_j \right) e^{-\sum_{j=1}^{M} \beta_j} \]

Hence (3.12) can be computed recursively, where \( E^l \) represents the weighted average with weights \( \frac{\beta_i}{\sum_{n=1}^{M} \beta_n} \)'s. The density of \( X \), after using the Edgeworth expansion, can be obtained and can be transformed to the density of \( P(E) \) because of (3.10). The Laplace transform of the mean and variance of \( Z(t) \) in this case appears to be difficult to obtain.

§3.2 Time-Dependent Multiple Disasters of Category Two

As mentioned at the beginning of this chapter, we assume that the probabilities of surviving depend only on the time elapsed since the last disaster, no matter what type it is. This assumption makes the problem simpler than that of section 3.1 for first category disasters. Because the dependence of \( \delta 's \) on the time is
"uniform", the notations of \( \tau^{(i)}_k \)'s are no longer needed. Let \( \tau_j \)'s be the same as that of section 3.1 and define \( \delta_j = \delta_j(\tau_j) \). Then

\[
(3.14) \quad F(s,(\tau_1 + \ldots + \tau_n)^+|I_1, \ldots, I_n, \tau_1, \ldots, \tau_n)
\]

\[
= \left(1 + \frac{1}{\prod_{i=1}^{n} \Phi_i \left(\frac{1}{s - 1} - \sum_{j=1}^{n} A_j \prod_{k=1}^{j-1} \Phi_k\right)}\right)^a
\]

and the conditional probability of extinction at \((\tau_1 + \ldots + \tau_n)^+\) is

\[
(3.15) \quad P_n(E) = F(0,(\tau_1 + \tau_2 + \ldots + \tau_n)^+|I_1, \ldots, I_n, \tau_1, \ldots, \tau_n)
\]

\[
= \left(1 - \frac{1}{\sum_{j=1}^{n} A_j \prod_{k=1}^{j-1} \Phi_k + \prod_{i=1}^{n} \Phi_i}\right)^a.
\]

Let

\[
(3.16) \quad Y = \lim_{n \to \infty} \sum_{j=1}^{n} A_j \prod_{k=1}^{j-1} \Phi_k
\]

We still have

\[
(3.17) \quad \hat{P}(E) = 1 \quad \text{if and only if} \quad Y = \infty.
\]

Summarize the above assumptions, we have:
\( A_5 \) : There are disasters of \( M \) different types which depend on one another.

\( A_6 \) : The probability of surviving every type of disasters depend only on the time elapsed since the last disaster, no matter what type it is.

Similar to the proof of theorem 3.1, the proof of the following theorem 3.2 is a straightforward extension of the proof of the theorem 2.1.

**Theorem 3.2**

Under the assumptions of \( A_5 \) and \( A_6 \), a necessary and sufficient condition of \( P(E) \) being degenerated to 1, i.e. almost sure extinction, is

\[
E \left[ E^T \log \delta \left( \tau_1 \right) \right] \leq - \rho E \tau_s = - \frac{\rho}{\sum_{n=1}^{M} \beta_n}.
\]

To find the nondegenerated probability of extinction, we have, as (3.10) of section 3.1,

\[
P(E) = \left(1 - \frac{1}{\gamma} \right)^a = \left( \frac{\mu + \lambda X}{\lambda + \lambda X} \right)^a,
\]
and \( X_1 = \frac{\rho}{\lambda} \) \( Y - 1 \) satisfies the following equation

\[
X = \frac{X + 1 - \delta_{i_1}(\tau)}{\phi(\tau)\delta_{i_1}(\tau)}.
\]

where \( \phi(\tau) \) is given by (2.2). The \( n \)-th moment of \( X \) satisfies the following equation,

\[
M_n = \sum_{j=0}^{n-1} \binom{n}{j} M_j E^{i_1} E^{i_2} \left( \frac{1 - \delta_{i_1}(\tau)}{\phi(\tau)\delta_{i_1}(\tau)} \right)^{n-j-1}.
\]

Here \( \tau_j \sim \exp(\sum_{i=1}^{M} \beta_i) \), and \( E^I \) represents the weighted average with weights \( \frac{\beta_i}{\sum_{n=1}^{M} \beta_n} \), and by (3.19) we can get the density of probability of extinction. An example of \( \lambda = 4, \mu = 2, \beta_1 = 2, \delta_1(t) = \frac{0.5198}{0.5198 + t} \), and \( \beta_2 = 2, \delta_2 = \exp(-t) \) is illustrated in Figure 5.

Similarly, if we differentiate the conditioned p.g.f. we have

\[
E(Z(\tau_1 + \ldots + \tau_n)\mid \tau_1, \ldots, \tau_n, I_1, \ldots, I_n) = a \prod_{i=1}^{n} \delta_i e^{\rho x_i}
\]

\[
= a e^{\sum_{\tau_i}^{n} \delta_i(\tau)}.
\]
Figure 5: Density Function of Probability of Extinction for Category Two Delta Being the Combination of 
\( (0.5198)/(0.5198+t) \) and \( \exp(-t) \)
Hence

\[(3.23) \quad E(Z(t) \mid \tau_1, \tau_2, N(t) = n) = e^{\rho t} \prod_{i=1}^{n} \delta_1(\tau_i) \]

and

\[(3.24) \quad E(Z(t)) = E^{\tau,N} \int e^{\rho t} \prod_{i=1}^{n} \delta_1(\tau_i) \]

\[= E^{\tau,N} \int e^{\rho t} \prod_{i=1}^{n} I_i(\tau_i) \]

\[= E^{\tau,N} \int e^{\rho t} \prod_{i=1}^{n} \delta(\tau_i) \]

where \( \delta(t) = E^{\tau} I_1(t) = \sum_{j=1}^{M} \frac{\beta_j}{\sum_{n=1}^{M} \beta_n} \delta_j(t) \).

Hence we have the weighted average \( \delta(t) \), which is of the same form as the \( \delta(t) \) in Chapter 2. Thus, the formula (2.29) of the mean can be used here. Similarly, by the mutual independence of \( I_i \)'s, we have, as in (2.33),

\[ \text{Var}Z(t) = W - \frac{\lambda + \mu}{\rho} EZ(t) - (EZ(t))^2 \]

where \( W \) is the same as (2.34) except
Hence the Laplace Transform of the mean and variance for this case may be easily obtained following the procedures of Chapter 2.

\[\begin{align*}
\delta(t) &= E\left[I\left(\delta(t)\right)\right] = \sum_{j=1}^{M} \frac{\beta_j}{\sum_{n=1}^{M} \beta_n} \delta_j(t) \\
\delta^2(t) &= E\left[I\left(\delta^2(t)\right)\right] = \sum_{j=1}^{M} \frac{\beta_j}{\sum_{n=1}^{M} \beta_n} \delta_j^2(t)
\end{align*}\]

(3.25)

\[\left\{\right.\]

§3.3 Time-Independent Random Disasters

If the probabilities of surviving, \(\delta\), of the binomial killing during disasters are independent of time but, instead, are random with some distribution, then we also have similar results except a few changes on functions of \(\delta\). Let \(\delta_j's\) be i.i.d. random variables representing the probabilities of surviving at the \(j\)-th disasters, \(j=1,2,...\) The p.g.f. now should be conditioned on \(\tau_i's\) and \(\delta_i's\). Now we make the assumption

\[A_7: \text{The killing probabilities are independent of time but are}\]

i.i.d. random variables.

Follow the same ways as theorem 2.1, we have
Theorem 3.3

Under the assumption $A_\gamma$, a necessary and sufficient condition of $P(E)$ being degenerated to 1, i.e. almost sure extinction, is

$$(3.26) \quad E^E \log \delta \leq -\rho \tau = -\frac{\rho}{\beta}. \quad \text{One point needs to be mentioned:}$$

If $\delta$ has only $M$ possible values, then this is a special case of multiple type disasters with $M$ different types of constant disasters, whichever category one or category two since the disasters now are independent of time. Suppose $P(\delta = a_i) = \theta_i$, $i = 1, \ldots, M$, and $\sum_{i=1}^{M} \theta_i = 1$. If we define $\beta_i = \beta \theta_i$, $\delta_i = a_i$, $i = 1, \ldots, M$, where $\beta$ is the occurring rate of disasters, then this is equivalent to the multiple type disaster case because

$$P(I = i) = \frac{\beta_i}{\sum_{n=1}^{M} \beta_n} = \frac{\beta \theta_i}{\beta} = \theta_i = P(\delta = a_i),$$

where $I$ is defined in (3.2) as an indicator random variable.
We now use a different way to find the density of $P(E)$. We start from the following equation:

\[(3.27) \quad X = \frac{1 - \delta + X}{\phi(\tau)\delta} = V(1 - \delta + X)\]

\[(3.28) \quad \text{where} \quad V(\tau) = \frac{1}{\phi(\tau)\delta}.\]

Conditioning on $\delta$, $V$ and $(1-\delta+X)$ are independent of each other because $\phi(\tau)$ is independent of $X$. Let $g(\delta)$ and $h(x)$ be the density function of $\delta \in (0,1)$ and $X$, respectively. Then from (3.27) we have the following integral equation, with initial condition $h(0)=0$,

\[
h(x) = \int_0^{\min(\frac{1}{\delta}, \frac{x}{1-\delta})} g(\delta) \int_0^{\frac{x}{1-\delta}} \alpha v^{-2} \delta^a h\left(\frac{x}{v} - 1 + \delta\right) dv d\delta
\]

\[
= \int_0^{\frac{1}{1+X}} g(\delta) \int_0^{\frac{x}{1-\delta}} \alpha v^{-2} \delta^a h\left(\frac{x}{v} - 1 + \delta\right) dv d\delta
\]

\[
+ \int_{\frac{1}{1+X}}^{\frac{1}{\delta}} g(\delta) \int_0^{\frac{x}{1-\delta}} \alpha v^{-2} \delta^a h\left(\frac{x}{v} - 1 + \delta\right) dv d\delta
\]

, because $\delta < \frac{1}{1+X}$ if $\min\left(\frac{X}{1-\delta}, \frac{1}{\delta}\right) = \frac{X}{1-\delta}$. If we change the variable $z = \frac{x}{v} - 1 + \delta$, then
\[ h(x) = \int_0^{1+X} \alpha \delta g(\delta) \left( \frac{x}{z+1-\delta} \right)^{\alpha-2} h(z) \frac{x}{(z+1-\delta)^2} \, dz \, d\delta \]

\[ + \int_{\frac{1}{1+X}}^1 \alpha \delta g(\delta) \int_{\frac{1}{\delta-1+X}}^1 \left( \frac{x}{z+1-\delta} \right)^{\alpha-2} h(z) \frac{x}{(z+1-\delta)^2} \, dz \, d\delta \]

\[ = \alpha x^{\alpha-1} \int_0^{1+X} \delta g(\delta) \left( \frac{1}{z+1-\delta} \right)^{\alpha} h(z) \, dz \, d\delta \]

\[ + \alpha x^{\alpha-1} \int_{\frac{1}{1+X}}^1 \delta g(\delta) \int_{\frac{1}{\delta-1+X}}^1 \left( \frac{1}{z+1-\delta} \right)^{\alpha} h(z) \, dz \, d\delta \]

Multiplying by \( x^{1-\alpha} \) and differentiating both sides of (3.29), we have

\[ \frac{d(x^{1-\alpha}h(x))}{dx} \]

\[ = -\alpha \left( \frac{1}{1+x} \right)^2 \left( \frac{1}{1+x} \right)^{\alpha} g\left( \frac{1}{1+x} \right) \int_0^1 \frac{1}{(z+1-\frac{1}{1+x})^{\alpha}} h(z) \, dz \]

\[ + \alpha \left( \frac{1}{1+x} \right)^2 \left( \frac{1}{1+x} \right)^{\alpha} g\left( \frac{1}{1+x} \right) \int_0^1 \frac{1}{(z+1-\frac{1}{1+x})^{\alpha}} h(z) \, dz \]

\[ - \alpha \int_{\frac{1}{1+X}}^1 \delta^{\alpha+1} g(\delta) \left( \frac{1}{\delta x} \right)^{\alpha} h(\delta x - 1+\delta) \, d\delta \]
We have therefore the Volterra Integral-Differential Equation

\[ (3.31) \quad \zeta(x) = -\alpha \int_0^x (1 + x)^{-2} \left( \frac{1 + w}{1 + x} \right)^{\alpha-1} (1 + w) \zeta(w) dw \]

with initial condition \( \zeta(0) = c > 0 \), where \( \zeta(x) = x^{1-\alpha} h(x), \int_0^x h(x)dx = 1 \).

This can be solved for \( \zeta(x) \) numerically by the method proposed by Linz [20]. An example of the random variable \( \delta \) having Beta (2,2) distribution is illustrated in Figure 6. It is interesting to observe that the density of \( P(E) \) tends to infinity as \( P(E) \) tends to 1. It is plausible that although the possibility of severe disasters is small, \( P(1 - \delta_i > 0.99) = 0.0003 \), we still have \( P(1 - \delta_i > 0.99 \ i.o.) = 1 \), i.e. with probability 1 that with infinitely many times there will occur extremely serious disasters because the probability of surviving is time independent. Comparing this with the accelerating killing cases ( \( \delta(t)'s \) are decreasing functions of time \( t \) ) of Figure 3 and
Figure 6: Density Function of Probability of Extinction for the Distribution of Random Delta Being Beta(2,2)
Figure 4, we understand that although severe disasters will also occur infinitely many times for the latter two cases (Fig. 3 and 4), but those disasters will happen long time apart, hence the population size will have already become large.

Since the $\delta_i$'s are i.i.d., by the same reasons as above, the Laplace transform of mean and $W$, hence the variance, are (2.29) and (2.35), respectively, except that $\delta$ and $\delta^2$ have to be replaced by $E\delta$ and $E\delta^2$. 
Chapter IV

Simple Nonhomogeneous Birth and Death Processes with Disasters

So far we have been investigating the influence of disasters on a linear birth and death process, in which population members contain constant birth and death rates. If the environment is bad enough to affect the birth and death rates of population members immediately after disasters and recover back to normal gradually, then the birth and death rates should not be constant. Hence we now extend that process to be a nonhomogeneous one and will observe that many results in Chapter 2 and Chapter 3 are going to reappear in this chapter, except that we have to change the "process part", and leave the "disaster part" unaltered. The previous four theorems will also reappear in this chapter and only the proof of Theorem 4.1 will be given since the rest of three are straightforward from the proof of Theorem 2.1.

Consider a nonhomogeneous birth and death process with birth rate $\lambda=\lambda(t)$ and death rate $\mu=\mu(t)$, both of which are functions of the time elapsed since the last disaster, i.e. the birth
rate and death rate functions will start all over again immediately after each disaster. The condition on disasters are the same as those discussed in the previous chapters. By the formula in Bailey's book [3], p. 112, the p.g.f. of a simple nonhomogeneous birth and death process is

\begin{equation}
F(s, t) = \left(1 + \frac{1}{s - 1} \int_{0}^{t} e^{-\rho(\tau)} \lambda(\tau) e^{-\rho(\tau)} d\tau \right)^{s},
\end{equation}

where

\begin{equation}
\rho(t) = \int_{0}^{t} (\lambda(x) - \mu(x)) \, dx.
\end{equation}

In order to avoid sure extinction, it is necessary to have

\begin{equation}
\lim_{t \to \infty} \int_{0}^{t} \mu(x) e^{-\rho(x)} \, dx < \infty,
\end{equation}

which is the supercritical condition; see Bailey [3], p. 113.

§ 4.1 Single Type Time-Dependent Disasters

In this section we generalize the results of Chapter 2. The assumptions $A_1$ and $A_2$ are assumed here.

From (4.1), we have
\[ F(s, \tau_1^+ | \tau_1) = \left(1 + \frac{1}{e^{-\rho(\tau_1)} - \int_0^{\tau_1} e^{-\rho(x)} \, dx} \right)^s \]

where

\[
\begin{align*}
\delta_i &= \delta(\tau_i) \\
A_i &= \int_0^{\tau_i} \lambda(x) e^{-\rho(x)} \, dx \\
\rho_i &= \rho(\tau_i)
\end{align*}
\]

\(i=1,2,3,...\)

If we define

\[
\begin{align*}
\phi_i &= e^{\rho_i} \\
\Phi_i &= \frac{1}{\delta_i} e^{-\rho_i} = \frac{1}{\delta_i \phi_i}
\end{align*}
\]

then the p.g.f. conditioned on \(\tau_1, \tau_2, \ldots, \tau_n\) is exactly the same as (2.5),

\[ F(s, (\tau_1 + \ldots + \tau_n)^+ | \tau_1^+ \ldots \tau_n) = \left(1 + \frac{1}{\prod_{i=1}^n \Phi_i - \sum_{j=1}^n A_i \prod_{k=j}^n \Phi_k} \right)^s \]

The conditional probability of extinction at \((\tau_1 + \ldots + \tau_n)^+\) is

\[ P_n(E) = F(0, (\tau_1 + \tau_2 + \ldots + \tau_n)^+ | I_1, \ldots, I_n, \tau_v^+, \ldots, \tau_n) \]
Let

\[(4.9) \quad Y = \lim_{n \to \infty} \sum_{j=1}^{n} A_{i} \prod_{k=1}^{j-1} \Phi_{k} \]

Then as usual

\[(4.10) \quad P_{n}(E) \to P(E) = (1 - \frac{1}{Y})^{a} \]

and we still have

\[(4.11) \quad P(E) = 1 \text{ if and only if } Y = \infty. \]

**Theorem 4.1**

Under the assumptions $A_{1}$ and $A_{2}$, a necessary and sufficient condition of $P(E)$ being degenerated to 1, i.e. almost sure extinction, in the nonhomogeneous case is

\[(4.12) \quad E \log \delta(\tau) \leq -E \rho(\tau). \]

**Proof:**

Exactly the same as the proof of Theorem 2.1 except

\[\frac{1}{N} \sum_{j=1}^{N} (\rho_{j} + \log \delta_{j}) \to E \rho(\tau) + E \log \delta(\tau) \text{ by SLLN.} \]
Unlike the results of previous chapters, we are not able to use the random variable \( X \) of (2.13) to find the density of \( P(E) \) because \( A(\tau) \), given by (4.5), does not have an exact form since it depends on \( \lambda(t) \) and \( \mu(t) \). Instead, we should use \( Y \) to obtain the density of \( P(E) \) because of (4.10). There is a disadvantage of using \( Y \) since \( Y > 1 \) in view of (4.10), and so the moments of \( Y \) must tend to infinity. From (4.9), it is obvious that \( Y \) satisfies the following equation:

\[
Y = Y \Phi(\tau) + A(\tau) = \frac{Y}{\delta(\tau)e^{p(\tau)}} + A(\tau),
\]

where \( \Phi(\tau) \) and \( A(\tau) \) are independent of \( Y \).

Consequently,

\[
Y^n = \left( \frac{Y}{\delta(\tau)e^{p(\tau)}} + A(\tau) \right)^n.
\]

Integrating both sides of (4.14), if both are finite, we see that the \( n \)-th moment of \( Y \) satisfies the following equation:

\[
M_n = \sum_{j=0}^{n} \binom{n}{j} M_j E\frac{A^{n-j}(\tau)}{\left(\delta(\tau)e^{p(\tau)}\right)^j},
\]

so that
Applying these moments to the Edgeworth expansion method proposed in Chapter 2, we obtain an approximate density of $Y$, hence that of $P(E)$ by (4.10).

The Laplace transform of the mean and $W$, hence variance, of $Z(t)$ can be obtained in a similar way as used in Chapter 2. From (4.7), we have

$$
E(Z(t) | \tau_1, ..., \tau_n) = \frac{dF(s, (\tau_1, ..., \tau_n)^+)}{s} \bigg|_{s=t}
$$

$$
= a \prod_{i=1}^{n} \delta_i e^{\rho_i} = a \prod_{i=1}^{n} \frac{1}{\Phi_i}
$$

Let $N(t)$ be the number of disasters having occurred before $t$.

Then by Bailey [3], p. 114,

$$
E(Z(t) | \tau_1, ..., \tau_n, N(t) = n) = E(Z(t_1 + ... + t_n) | \tau_1, ..., \tau_n) e^{\rho(t-(\tau_1 + ... + \tau_n))}
$$

$$
= a \left( \prod_{i=1}^{n} \delta_i e^{\rho_i} \right) e^{\rho(t-(\tau_1 + ... + \tau_n))}
$$

As discussed in section 2.4, we have

$$
E(Z(t)) = E^N E^N E(Z(t) | \tau_1, \tau_2, ..., \tau_n, N(t) = n)
$$
The Laplace Transform of \( EZ(t) \) is

\[
\mathcal{L}_X(s + \beta) = a \sum_{n=0}^{\infty} \left( \int_{\tau_1 + \cdots + \tau_n \leq t} \frac{\prod_{i=1}^{n} \delta(\tau_i) e^{\rho(\tau_i)}}{n!} \frac{1}{t^n} d\tau \right) \left( \frac{\beta t}{s} \right)^n e^{-\beta t}.
\]

\[
= a \sum_{n=0}^{\infty} e^{-\beta t} \beta^n \int_{\tau_1 + \cdots + \tau_n \leq t} e^{\rho(t-(\tau_1 + \cdots + \tau_n))} \prod_{i=1}^{n} \delta(\tau_i) e^{\rho(\tau_i)} d\tau
\]

\[
= a \sum_{n=0}^{\infty} e^{-\beta t} \beta^n \left( e^{\rho(t)} \frac{\delta e^{\rho(t)} \delta e^{\rho(t)} \cdots \delta e^{\rho(t)}}{n \delta e^{\rho(t)}} \right).
\]

The Laplace Transform of \( EZ(t) \) is

\[
(4.19) \quad \mathcal{L}_{EZ}(s) = a \sum_{n=0}^{\infty} \beta^n L(e^{s}(s + \beta))L(e^{s}(s + \beta)) = \frac{a \mathcal{L}_e(s + \beta)}{1 - \beta L(e^{s}(s + \beta))},
\]

while (4.19) reduces to (2.29) when \( \lambda \) and \( \mu \) are constants.

To find the variance, we follow the same steps proceeded in Chapter 2, obtain the variance conditioned on \( \tau, N(t) \) and \( Z(\tau_1 + \cdots + \tau_n)^+ \) first and remove those conditions one by one. First, we observe that

\[
\frac{d^2 F(s, (\tau_1 + \cdots + \tau_n)^+) | \tau_1, \ldots, \tau_n)}{ds^2} \bigg|_{s=1} = \frac{2a}{\prod_{i=1}^{n} \Phi_i} \sum_{j=1}^{n} A_j \prod_{k=1}^{j-1} \Phi_k + a(a - 1) - \frac{1}{\prod_{i=1}^{n} \Phi_i^2},
\]

hence

\[
(4.20) \quad \text{Var}(Z((\tau_1 + \cdots + \tau_n)^+) | \tau_1, \ldots, \tau_n)
\]
\[
\begin{align*}
&= \frac{d^2 F(s, (\tau_1 + \ldots + \tau_n)^+, \tau_1, \ldots, \tau_n)}{ds^2} \bigg|_{s=1} \\
&\quad + E(Z((\tau_1 + \ldots + \tau_n)^+)|\tau_1, \ldots, \tau_n) - \left[ E(Z((\tau_1 + \ldots + \tau_n)^+)|\tau_1, \ldots, \tau_n) \right]^2 \\
&= \frac{2a}{(\prod \Phi_i)^2} \sum_{j=1}^{n} A_j \prod_{k=1}^{j-1} \Phi_k + a(a - 1) \frac{1}{(\prod \Phi_i)^2} + \frac{a}{\prod \Phi_i} - \frac{a^2}{(\prod \Phi_i)^2} \\
&\quad = a \left[ 2 \left( \sum_{j=1}^{n} A_j \prod_{k=1}^{j-1} \Phi_k \right) + \frac{1}{\delta} \prod_{k=1}^{j-1} \Phi_k \right] + \prod_{i=1}^{n} (\delta_i e^\rho) - \prod_{i=1}^{n} (\delta_i e^\rho)^2 \right].
\end{align*}
\]

Again, let \( N(t) \) be the number of disasters having occurred before \( t \). Then by Bailey [3], p.114,

\[(4.21) \quad \text{Var} \left[ \left( Z(t) | \tau_1, \ldots, \tau_n, N(t) = n \right) | Z((\tau_1 + \ldots + \tau_n)^+) \right] \]

\[= Z((\tau_1 + \ldots + \tau_n)^+) e^{2p(t-(\tau_1 + \ldots + \tau_n))} \int \left( (\lambda(x) + \mu(x)) e^{-p(x)} \right) dx. \]

Now we remove the condition of \( Z((\tau_1 + \ldots + \tau_n)^+) \). From (4.20) and (4.21), we have

\[(4.22) \quad \text{Var}(Z(t) | \tau_1, \ldots, \tau_n, N(t) = n) \]

\[= E^Z \left( \text{Var} \left[ \left( Z(t) | \tau_1, \ldots, \tau_n, N(t) = n \right) | Z((\tau_1 + \ldots + \tau_n)^+) \right] \right) + \text{Var} \left( E \left[ (Z(t) | \tau_1, \ldots, \tau_n, N(t) = n) | Z((\tau_1 + \ldots + \tau_n)^+) \right] \right) \]

\[= E^Z \left( Z((\tau_1 + \ldots + \tau_n)^+) e^{2p(t-(\tau_1 + \ldots + \tau_n))} \right). \]
\[
\int_0^{t-(\tau_1+\ldots+\tau_n)} \left((\lambda(x) + \mu(x))e^{-\rho(x)}dx\right)_{\tau_\nu, \ldots, \tau_n, N(t) = n} \]

\[+ \text{Var}_Z(Z((\tau_1+\ldots+\tau_n)^+)e^{\rho(t-(\tau_1+\ldots+\tau_n))} | \tau_\nu, \ldots, \tau_n, N(t) = n) \]

\[= e^{2\rho(t-(\tau_1+\ldots+\tau_n))} \left(\int_0^n (\lambda(x) + \mu(x))e^{-\rho(x)}dx \prod_{i=1}^n \delta_i e^{\rho_i}\right) \]

\[+ e^{2\rho(t-(\tau_1+\ldots+\tau_n))} \left[2\left(\prod_{i=1}^n \delta_i e^{\rho_i}\right)^2 \sum_{j=1}^n A_j \prod_{k=1}^{j-1} \left(\frac{1}{\delta_k} e^{-\rho_k}\right) + \prod_{i=1}^n (\delta_i e^{\rho_i}) - \prod_{i=1}^n (\delta_i e^{\rho_i})^2 \right] \]

where

\[(4.23) \quad B(t-(\tau_1+\ldots+\tau_n)) = \int_0^{t-(\tau_1+\ldots+\tau_n)} (\lambda(x) + \mu(x))e^{-\rho(x)}dx.\]

We are now ready to remove the last two conditions, \(\tilde{\tau}\) and \(N(t)\).

From (4.22)

\[(4.24) \quad \text{Var}_Z(t)\]

\[= E^{\tau,N} \left(\text{Var}_Z[(Z(t) | \tau_\nu, \ldots, \tau_n, N(t) = n)]\right) \]

\[+ \text{Var}^{\tau,N} \left(E[(Z(t) | \tau_\nu, \ldots, \tau_n, N(t) = n)]\right) \]

\[= aE^{\tau,N} \left\{ e^{2\rho(t-(\tau_1+\ldots+\tau_n))} \left(\prod_{i=1}^n \delta_i e^{\rho_i}\right) B(t-(\tau_1+\ldots+\tau_n)) \right\} \]
\[ + e^{2p(t-\tau_{1}+\ldots+\tau_{n})} \left\{ 2 \prod_{i=1}^{n} \delta_{i} e^{\rho_{i}} \right\} ^{2} \sum_{j=1}^{n} A_{j} \prod_{k=1}^{j-1} \left( \frac{1}{\delta_{k}} e^{-\rho_{k}} \right) + \prod_{i=1}^{n} \delta_{i} e^{\rho_{i}} - \prod_{i=1}^{n} \delta_{i} e^{\rho_{i}} \right\} \right\}^{2} \]

\[ + a^{2} E^{N} \left( e^{2p(t-\tau_{1}+\ldots+\tau_{n})} \prod_{i=1}^{n} \delta_{i} e^{\rho_{i}} \right) - \left[ e^{2p(t-\tau_{1}+\ldots+\tau_{n})} \prod_{i=1}^{n} \delta_{i} e^{\rho_{i}} \right]^{2} \]

\[ = W - (E^{2}(t)) \]

where

\[ W = aE^{N} \left\{ e^{2p(t-\tau_{1}+\ldots+\tau_{n})} \prod_{i=1}^{n} \delta_{i} e^{\rho_{i}} \right\} B(t - (\tau_{1} + \ldots + \tau_{n})) \]

\[ + e^{2p(t-\tau_{1}+\ldots+\tau_{n})} \left\{ 2 \prod_{i=1}^{n} \delta_{i} e^{\rho_{i}} \right\} ^{2} \sum_{j=1}^{n} A_{j} \prod_{k=1}^{j-1} \left( \frac{1}{\delta_{k}} e^{-\rho_{k}} \right) + \prod_{i=1}^{n} \delta_{i} e^{\rho_{i}} - \prod_{i=1}^{n} \delta_{i} e^{\rho_{i}} \right\} \right\}^{2} \]

\[ + a^{2} E^{N} \left( e^{2p(t-\tau_{1}+\ldots+\tau_{n})} \prod_{i=1}^{n} \delta_{i} e^{\rho_{i}} \right) \]

\[ W \] can also be written as the following form:

(4.25) \[ W = aE^{N} \left\{ e^{2p(t-\tau_{1}+\ldots+\tau_{n})} \prod_{i=1}^{n} \delta_{i} e^{\rho_{i}} \right\} B(t - (\tau_{1} + \ldots + \tau_{n})) \]

\[ + e^{2p(t-\tau_{1}+\ldots+\tau_{n})} \left\{ 2 \prod_{i=1}^{n} \delta_{i} e^{\rho_{i}} \right\} ^{2} \sum_{j=1}^{n} A_{j} \prod_{k=1}^{j-1} \left( \frac{1}{\delta_{k}} e^{-\rho_{k}} \right) + \prod_{i=1}^{n} \delta_{i} e^{\rho_{i}} - \prod_{i=1}^{n} \delta_{i} e^{\rho_{i}} \right\} \right\}^{2} \]

\[ + (a^{2} - a) E^{N} \left( e^{2p(t-\tau_{1}+\ldots+\tau_{n})} \prod_{i=1}^{n} \delta_{i} e^{\rho_{i}} \right) \]

\[ = a \sum_{n=0}^{\infty} e^{-\beta_{t}} \beta^{n} \int_{\tau_{1}+\ldots+\tau_{n}}^{t} e^{2p(t-\tau_{1}+\ldots+\tau_{n})} B(t - (\tau_{1} + \ldots + \tau_{n})) \prod_{i=1}^{n} \delta(\tau_{i}) e^{\rho(\tau_{i})} d\tau \]

\[ + a \sum_{n=0}^{\infty} e^{-\beta_{t}} \beta^{n} \int_{\tau_{1}+\ldots+\tau_{n}}^{t} e^{2p(t-\tau_{1}+\ldots+\tau_{n})} \left\{ 2 \prod_{i=1}^{n} \delta(\tau_{i}) e^{\rho(\tau_{i})} \right\} ^{2} \sum_{j=1}^{n} A(\tau_{j}) \prod_{k=j}^{n} \delta(\tau_{k}) e^{\rho(\tau_{k})} \]
\[ + (a^2 - a) \sum_{n=0}^{\infty} e^{-\beta t} p^n \left( e^{2p(t)} \right)^* \frac{\delta e^{2p(t)} \delta e^{2p(t)} \cdots}{\delta e^{2p(t)}_{s}} \]

Like (2.35), the formula of \( W \) in the linear case, the Laplace transform of \( W \) in the nonhomogeneous case can be obtained easily.

\[
(4.26) \quad L_w(s) = a \sum_{n=0}^{\infty} \beta^n \left[ L_{e_r^2 B}(s + \beta)L_{e^r}(s + \beta) + L_{e_r^2}(s + \beta)L_{e^r}(s + \beta) \right] \\
+ 2 \sum_{j=1}^{n} L_{e_r^2}(s + \beta)L_{e^r}(s + \beta)L_{\lambda e^2 r^2}(s + \beta)L_{\lambda e^2 r^2}(s + \beta) \\
+ (a^2 - a) \sum_{n=0}^{\infty} \beta^n L_{e_r^2}(s + \beta)L_{e^r}(s + \beta) \\

= L_{e_r^2 B}(s + \beta) \frac{a}{1 - \beta L_{e^r}(s + \beta)} + L_{e_r^2}(s + \beta) \frac{a}{1 - \beta L_{e^r}(s + \beta)} \\
+ 2a \sum_{n=0}^{\infty} \beta^n \frac{L_{e_r^2}(s + \beta)L_{\lambda e^2 r^2}(s + \beta)L_{\lambda e^2 r^2}(s + \beta)}{L_{e^r}(s + \beta)} \sum_{j=1}^{n} \left( \frac{L_{e^r}(s + \beta)}{L_{\lambda e^2 r^2}(s + \beta)} \right)^j \\
+ (a^2 - a) \frac{L_{e_r^2}(s + \beta)}{1 - \beta L_{\lambda e^2 r^2}(s + \beta)} \\

= \frac{aL_{e_r^2 B}(s + \beta) + aL_{e_r^2}(s + \beta)}{1 - \beta L_{e^r}(s + \beta)} + \frac{2a \beta L_{e_r^2}(s + \beta)L_{\lambda e^2 r^2}(s + \beta)}{\left(1 - \beta L_{\lambda e^2 r^2}(s + \beta)\right)\left(1 - \beta L_{e^r}(s + \beta)\right)}
Again, some inverse Laplace transforms will be demonstrated in Chapter 5.

§4.2 Time-dependent Multiple Type Disasters of Category One

The definition of category one, i.e. independent multiple type disasters, can be found in section 3.1, and the assumptions $A_3$ and $A_4$ are still assumed here. The extension from single type to multiple type in the nonhomogeneous case is similar to the extension in homogeneous case.

Let $\tau_k^{(i)}$ be the time length between $(k-1)\text{st}$ occurrence and $k\text{-th}$ occurrence of $i\text{-th}$ type disaster; let $\tau_j$ be the time length between $(j-1)\text{st}$ occurrence and $j\text{-th}$ occurrence of any disaster, and let $I_k$ be the random variable indicating the type of the $k\text{-th}$ disaster in the definition of Chapter 3. As before, we have to condition on $\tau_j's$ and $I_j's$, $j=1,2,\ldots,n$, in order to obtain the conditioned p.g.f. In the same way, we have, leaving the disaster
part unaltered, the following theorem:

**Theorem 4.2**

Under the assumptions of $A_3$ and $A_4$, a necessary and sufficient condition of $P(E)$ being degenerated to 1, i.e. almost sure extinction, in the nonhomogeneous case is

\[(4.27) \quad E^I \left[ E_{1_1}^{I_1} \log \delta_i (\tau_i) \right] \leq -Ep(\tau)\]

To find the density of $Y$, we have

\[(4.28) \quad Y = \frac{Y}{\delta_i (\tau_i) e^{\rho(\tau)}} + A(\tau)\]

hence

\[(4.29) \quad Y^n = \left( \frac{Y}{\delta_i (\tau_i) e^{\rho(\tau)}} + A(\tau) \right)^n\]

Integrating both sides of (4.29), we have

\[M_n = \sum_{j=0}^{n} \left( \binom{n}{j} M_j \right) E^I \left[ \varepsilon^{\tau_i(\tau)} \right] \frac{A^{n-j}(\tau)}{\left( \delta_i (\tau) e^{\rho(\tau)} \right)^j}\]

so that
The joint density of \( \tau_1 \) and \( \tau_1^{(i)} \) can be found in (4.13) of Chapter 3. We can use these moments to find the approximate density of \( Y \), and transform it to the density of \( P(E) \).

The Laplace transform of mean and variance are difficult to obtain.

§ 4.3 Time-Dependent Multiple Type Disasters of Category Two

The assumptions \( A_5 \) and \( A_6 \) remain unchanged as in section 3.2. As before, we have to condition on \( \tau_j \)'s and \( I_j \)'s, \( j=1,2,\ldots,n \), in order to obtain the conditioned p.g.f. In the same way, we have the following theorem:
Theorem 4.3

Under the assumptions of \( A_5 \) and \( A_6 \), a necessary and sufficient condition of \( P(E) \) being degenerated to 1, i.e. almost sure extinction, in the nonhomogeneous case is

\[
E^1 E^T \log \delta_i(\tau_i) \leq -E\rho(\tau_i)
\]

To find the density of \( Y \), we have

\[
Y = \frac{Y}{\delta_i(\tau_i)e^{\rho(\tau_i)} + A(\tau_i)}
\]

hence

\[
Y^n = \left(\frac{Y}{\delta_i(\tau_i)e^{\rho(\tau_i)} + A(\tau_i)}\right)^n
\]

Integrating both sides of (4.33), if both are finite, we have

\[
M_n = \sum_{j=0}^{n} \binom{n}{j} M_j E^1 E^T \frac{A^{n-j}(\tau)}{\delta_i(\tau_i)e^{\rho(\tau)}}
\]

so that
We can use these moments to find the approximate density of \( Y \), and transform it to the density of \( P(E) \).

The Laplace transforms of mean and \( W \), hence the variance, are the same as (2.29) and (2.35), respectively, except

\[
\delta = E^t_1(\delta_1(\tau)) \quad \delta^2 = E^t_1(\delta_1^2(\tau)).
\]

§4.4 Time-Independent Random Disasters

We need the assumption \( A_7 \) of section 3.3 here. Conditioned on \( \tau_j \)'s, \( j=1,2,\ldots,n \), we still have the following theorem:

**Theorem 4.4**

Under the assumption of \( A_7 \), a necessary and sufficient condition of \( P(E) \) being degenerated to 1, i.e. almost sure extinction, in the nonhomogeneous case, is

\[
\sum_{j=0}^{n-j}^n \binom{n}{j} M_j E^t_1 e^{\tau_i} \frac{A^{n-j}(\tau_i)}{\left(\delta_1(\tau_i) e^{\rho(\tau_i)}\right)^j}.
\]

\[(4.34) \quad M_n = \frac{1 - E^t_1 e^{\tau_i}}{1 - \left(\delta_1(\tau_i) e^{\rho(\tau_i)}\right)^n}.
\]
To find the density of $Y$, we have

$$Y^d = \frac{Y}{\delta e^{\rho(\tau)}} + A(\tau),$$

hence

$$Y^n = \left( \frac{Y}{\delta e^{\rho(\tau)}} + A(\tau) \right)^n.$$

Integrating both sides of (4.38), if both are finite, we have

$$M_n = \sum_{j=0}^{n} \binom{n}{j} M_j \delta^n \tau^j \frac{A^{n-j}(\tau)}{(\delta e^{\rho(\tau)})^j}.$$ 

, so that

$$M_n = \frac{\sum_{j=0}^{n} \binom{n}{j} M_j \delta^n \tau^j \frac{A^{n-j}(\tau)}{(\delta e^{\rho(\tau)})^j}}{1 - \delta^n \tau^n \frac{1}{(\delta e^{\rho(\tau)})^n}}.$$

We can also use these moments to find the approximate density of $Y$, and transform it to the density of $P(E)$.

The Laplace transform of mean and variance is the same as (2.29) and (2.35), respectively, except

$$\begin{cases} \delta = \delta^\delta \\ \delta^2 = \delta^2 \end{cases}.$$
In this chapter, we shall discuss the inverse Laplace Transform of the mean and variance of the population, whose Laplace Transforms were obtained in the previous chapters. We begin with some examples.

§5.1 Some Examples

The following are some examples whose exact means and variances can be found by using the inverse Laplace Transform. Those probabilities δ(t)'s of surviving not mentioned in this section will be discussed in the next section, which mostly deals with numerical computing.

(i) When δ is a constant, then

\[ L_\delta(s) = \frac{\delta}{s} \text{ and } L_\delta^2(s) = \frac{\delta^2}{s}. \]

Then by (2.29)

\[ L_{\alpha\delta}(s) = \frac{a}{(s + \beta - \rho) \left(1 - \frac{\beta \delta}{s + \beta - \rho}\right)} = \frac{a}{s + \beta - \rho - \beta \delta}. \]
so that

\[(5.3) \quad E(Z(t)) = e^{(\rho - \beta(1 - \delta))t} .\]

By \((2.35)\)

\[
L_w(s) = \frac{2\lambda a}{s + \beta - 2\rho - \beta \delta} \frac{1 - \frac{\beta \delta^2}{s + \beta - \rho}}{1 - \frac{\beta \delta^2}{s + \beta - 2\rho}} \left(1 - \frac{\beta \delta}{s + \beta - \rho}\right) = \frac{a^2 - a}{(s + \beta - 2\rho)} \left[1 - \frac{\beta \delta}{s + \beta - 2\rho}\right]
\]

and therefore

\[(5.4) \quad W(t) = -\frac{2\lambda a}{\rho} \frac{\beta \delta - \beta \delta^2}{\rho - \beta \delta + \beta \delta^2} e^{-(\rho - \beta \delta) t} + \left(\frac{2\lambda a}{\rho - \beta \delta + \beta \delta^2} + a^2 - a\right) e^{-(2\rho - \beta \delta^2) t} .\]

Thus by \((2.33)\)

\[
\text{Var}Z(t) = W(t) - \frac{\lambda + \mu}{\rho} E(Z(t)) - (E(Z(t)))^2
\]
is obtained completely because of (5.3) and (5.4).

(ii) When \( \delta(t) = e^{-kt} \) (accelerating killing power), then

\[
L_\delta(s) = \frac{1}{s+k} \quad \text{and} \quad L_\delta^2(s) = \frac{1}{s+2k},
\]

where \( k > 0 \). Therefore

\[
L_{\eta z}(s) = \frac{a}{(s + \beta - \rho) \left( 1 - \frac{\beta}{s + k + \beta - \rho} \right)} = a \left( \frac{k}{k - \beta} - \frac{\beta}{s + k - \rho} \right)
\]

so that

\[
H_{\eta z}(t) = a e^{\rho t} \left( \frac{k}{k - \beta} e^{-\beta t} - \frac{\beta}{k - \beta} e^{-kt} \right),
\]

and

\[
L_w(s) = \frac{2\lambda a}{\rho(s + \beta - 2\rho)} \frac{1}{1 - \frac{\beta}{s + \beta - 2\rho + 2k}} \left( 1 - \frac{\beta}{s + \beta - 2\rho + 2k} \right)
\]

\[
= \frac{2\lambda a}{\rho} \frac{(s + \beta - 2\rho + 2k)(s - \rho + 2k)(s + \beta - \rho + k)}{(s + \beta - 2\rho)(s - 2\rho + 2k)(s + \beta - \rho + 2k)(s - \rho + k)}
\]

To find the inverse Laplace Transform of \( L_w(s) \), we can use the
method proposed in Thomson [23]. If there are only poles for $f(t)$, then

$$f(t) = \sum_i \text{Res}_i(e^{sL_f(s)})$$

where $\text{Res}_i(e^{sL_f(s)})$ stands for residues of $e^{sL_f(s)}$ at $i$-th pole of $f(t)$ when defined in the complex plane. Thus,

$$W(t)$$

$$= \frac{2A_p}{\beta} \left\{ \frac{2k(\rho - \beta + 2k)(\rho + k)}{(2k - \beta)(\rho + 2k)(\rho - \beta + k)} e^{(2\rho - \beta)t} + \frac{\beta(\beta + \rho - k)}{(\beta - 2k)(\beta + \rho)(\rho - k)} e^{(2\rho - 2k)t} + \frac{\beta \rho k}{(\rho + 2k)(\beta + \rho)(\beta + k)} e^{(\rho - 2k)t} \right\}$$

$$+ (a^2 - a) \left\{ \frac{2k}{-\beta + 2k} e^{(2\rho - \beta)t} + \frac{\beta}{\beta - 2k} e^{(2\rho - 2k)t} \right\}$$

and $\text{Var}Z(t) = W(t) - \frac{\lambda + \mu}{\beta} EZ(t) - (EZ(t))^2$.

(iii) If $\delta(t) = 1-e^{-kt}$ (decelerating killing power), then

$$L_\delta(s) = \frac{1}{s} - \frac{1}{s + k} = \frac{k}{s(s + k)}$$

and

$$L_{\delta'}(s) = \frac{1}{s} - \frac{2}{s + k} + \frac{1}{s + 2k}$$

where $k > 0$. Then, after some algebra,
(5.12) \[ L_{bw}(s) = \frac{a}{(s + \beta - \rho)} \left( 1 - \frac{\beta k}{(s + k + \beta - \rho)(s + \beta - \rho)} \right) \]

\[
= a \left[ \frac{1}{s + \beta - \rho + \frac{k - \sqrt{\Delta}}{2}} - \frac{1}{\sqrt{\Delta}} \left( \frac{k}{2} - \frac{\sqrt{\Delta}}{2} \right) \right] + \frac{1}{\sqrt{\Delta}} \left( \frac{k}{2} - \frac{\sqrt{\Delta}}{2} \right)
\]

where

(5.13) \[ \Delta = k^2 + 4\beta k. \]

Thus,

(5.14) \[ E_Z(t) = a \left[ \left( \frac{k}{2\sqrt{\Delta}} + \frac{1}{2} \right) e^{-\left(\beta - \rho + \frac{k - \sqrt{\Delta}}{2}\right)t} - \left( \frac{k}{2\sqrt{\Delta}} - \frac{1}{2} \right) e^{-\left(\beta - \rho + \frac{k + \sqrt{\Delta}}{2}\right)t} \right]. \]

By (5.11), we have

(5.15) \[ L_w(s) \]

\[
= \frac{2\lambda a}{\rho(s + \beta - 2\rho)} \left( \frac{1 - \frac{\beta}{s + \beta - 2\rho} + \frac{2\beta}{s + \beta - 2\rho + k} - \frac{\beta}{s + \beta - 2\rho + 2k}}{1 - \frac{\beta}{s + \beta - \rho} + \frac{\beta}{s + \beta - \rho + k}} \right)
\]

\[
+ \frac{a^2 - a}{s + \beta - 2\rho} \left( \frac{1 - \frac{\beta}{s + \beta - 2\rho} + \frac{2\beta}{s + \beta - 2\rho + k} - \frac{\beta}{s + \beta - 2\rho + 2k}}{1 - \frac{\beta}{s + \beta - 2\rho} + \frac{2\beta}{s + \beta - 2\rho + k} - \frac{\beta}{s + \beta - 2\rho + 2k}} \right)
\]
\[
\frac{(\xi + k)(\xi + 2k)}{\xi^3 + 3k\xi^2 + 2k^2\xi - 2\beta k^2} \left[ \frac{2\lambda}{\rho} + a^2 - a \right.
\left. + \frac{2\lambda\beta k(s + \beta - \rho)}{\rho(s + \beta - \rho + 2k)(s + \beta - \rho + \frac{k + \sqrt{\Delta}}{2})(s + \beta - \rho + \frac{k - \sqrt{\Delta}}{2})} \right],
\]

where

\[(5.16) \quad \xi = s + \beta - 2\rho\]

and \(\Delta\) is given by (5.13).

If we can factorize the denominator of the term involving \(\xi\) in (5.15), then the inverse Laplace transform could be obtained easily. By Cardan's formula, [17], p. 251, the roots of

\[
\xi^3 + 3k\xi^2 + 2k^2\xi - 2\beta k^2 = 0
\]

(assuming that \(k^2 > 27\beta^2\)), are

\[
\frac{2k}{\sqrt{3}}\cos\left(\frac{\theta}{3}\right) - k, \quad -\frac{k}{\sqrt{3}}\cos\left(\frac{\theta}{3}\right) - k\sin\left(\frac{\theta}{3}\right) - k \quad \text{and}
\]

\[
-\frac{k}{\sqrt{3}}\cos\left(\frac{\theta}{3}\right) + k\sin\left(\frac{\theta}{3}\right) - k
\]

where

\[
sin\theta = \sqrt{\frac{k^6}{27} - \beta^2k^4}, \quad \cos\theta = \frac{\beta k^2}{\frac{k^3}{3\sqrt{3}}} = \frac{k^3}{3\sqrt{3}}.
\]

After some algebra, we have, by (5.8)

\[
f(t) = \sum_i \Re s_i(e^{s_i}L_i(s))
\]
(5.17) \[ W(t) = \sum_{i=1}^{6} Q_i e^{-V_i t} \]

where

\[ V_1 = \beta + 2\rho + 2\alpha_1 - k, \quad V_2 = \beta + 2\rho - \alpha_1 - \alpha_2 - k, \]
\[ V_3 = \beta + 2\rho - \alpha_1 + \alpha_2 - k, \]
\[ V_4 = \beta - \rho + 2k, \quad V_5 = \beta - \rho + \alpha_3, \quad V_6 = \beta - \rho + \alpha_4, \]
\[ \alpha_1 = \frac{k}{\sqrt{3}} \cos \frac{\theta}{3}, \quad \alpha_2 = k \sin \frac{\theta}{3}, \quad \alpha_3 = \frac{k + \sqrt{\Delta}}{2}, \quad \alpha_4 = \frac{k - \sqrt{\Delta}}{2}, \]

and \( Q_i = \text{Res}_{V_i} (L_W(s)) \), the residues of \( L_W(s) \) at \( V_i \), \( i=1,...,6 \).

(iv) Earthquake model

The probability of surviving \( \delta(\tau) = k(\frac{1}{\tau})^c \) may be deduced from Gutenberg-Richter equation [16]. This relationship says that

\[ \log N = a - b M, \]

where \( N \) is the number of earthquakes per unit time, \( M \) is the number of shocks of a certain magnitude and \( a \) and \( b \) are positive constants. Let \( r \) be the risk per event or shock.

Then the probability of surviving equals \( (1-r)^M \).

Hence

\[ \delta(\tau) = (1 - r)^M = \left(1 - r\right)^{\frac{1}{b} \log \frac{1}{N(\tau)}} \]

\[ = \left(1 - r\right)^{\frac{1}{b} \log \left(\frac{1}{\tau}\right)} \]
for some constants $k$ and $c$ and only for large $x$ since $\delta(t) \leq 1$. So we can modify it to be

$$\delta(t) = k \left( \frac{1}{\tau} \right)^c,$$

then $\delta(t) = k \left( \frac{1}{\tau} \right) \sim \frac{k}{t}$.

when $t$ is large.

Then $\delta(t) \leq 1$ and $\delta(t) \to 1$ as $t \to 0$ with $\delta$ being a decreasing function of $t$.

Choosing $k=1$ and $c=1$, we set $\delta(t) = \frac{1 - e^{-t}}{t}$. After some computation, we have

$$L_s(s) = \log \left( 1 + \frac{1}{s} \right),$$

so that

$$L_{\text{Ir}}(s) = \frac{a}{(s + \beta - \rho) \left( 1 - \beta \log \left( 1 + \frac{1}{s + \beta - \rho} \right) \right)}.$$
If we can find \( f(t) \) whose the Laplace Transform is

\[
(5.21) \quad L_f(s) = \frac{a}{s(1 - \beta \log(1 + \frac{1}{s}))},
\]

then, by (5.20),

\[
(5.22) \quad E(t) = ae^{(\rho - \beta)t}f(t).
\]

In order to find the exact form of \( f(t) \), the following inverse Laplace Transform of \( L_f(s) \), [23], is required:

\[
(5.23) \quad f(t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} L_f(s)e^{st}ds
\]

for any real \( \gamma \) that is greater than the real parts of any singularities, if they are not poles, of \( L_f(s) \) in the complex plane.

The singularities of \( L_f(s) \), (5.21), are

\[
(5.24) \quad s=0 \quad \text{and} \quad s = \frac{1}{1 - \beta} = Q, \text{ say.}
\]
In view of Figure 7 we have, by the property of analytic function,

\[(5.25) \quad \int_{AB} + \int_{BC+DA} + \int_{CO} + \int_{QQ} + \int_{QQ} + \int_{OD} + \int_{O} + \int_{O} = 0.\]

Our purpose is to find, by (5.23),

\[
f(t) = \frac{1}{2\pi i} \int_{r-i\infty}^{\gamma+i\infty} L_t(s)e^{st} ds = \lim_{AB \to \infty} \frac{1}{2\pi i} \int_{AB} L_t(s)e^{st} ds.
\]

By Jordan's Lemma (Thomson [23]), \(\int_{BC+DA} \to 0\).

Hence from (5.25)

\[(5.26) \quad f(t) = \lim_{AB \to \infty} \frac{1}{2\pi i} \int_{AB} L_t(s)e^{st} ds\]
Thus, we are required to search for the right hand side of (5.26).

We proceed this with the following four steps (a) - (d):

(a)

\[
\int_{0}^{Q} + \int_{Q}^{0} = 0.
\]

On \( Q \), \( s = x \), \( \frac{1}{s} = \frac{1}{x} \), \( \log(t + \frac{1}{s}) = \log(t + \frac{1}{x}) \), \( ds = dx \), hence

\[
\int_{0}^{Q} = \int_{0}^{Q} \frac{e^{xt}}{x \left(1 - \beta \log(t + \frac{1}{x})\right)} dx.
\]

On \( OQ \), \( s = x \), \( \frac{1}{s} = \frac{1}{x} \), \( \log(t + \frac{1}{s}) = \log(t + \frac{1}{x}) \), \( ds = dx \), hence

\[
\int_{0}^{Q} = \int_{0}^{Q} \frac{e^{xt}}{x \left(1 - \beta \log(t + \frac{1}{x})\right)} dx.
\]

Thus, we have \( \int_{0}^{Q} + \int_{Q}^{0} = 0 \).

(b)

\[
\lim_{A \to -\infty} \lim_{B \to +\infty} \frac{1}{2\pi i} \left( \int_{OC} + \int_{OC} \right) = -\frac{1}{\beta} \int_{-\infty}^{\frac{1}{\beta} + y} e^{\frac{1}{\beta} + y} \frac{1}{\frac{1}{\beta} + y + 1} \frac{1}{y^2 + \pi^2} dy.
\]

On \( OC \), \( s = -x \), \( \frac{1}{s} = -\frac{1}{x} \), \( ds = -dx \), and
\[
\log(1 + \frac{1}{s}) = \begin{cases} 
\log(1 - \frac{1}{x}) & \text{if } x > 1 \\
\log\left(\frac{1}{x} - 1\right) - i\pi & \text{if } x < 1 \\
-\infty & \text{if } x = 1,
\end{cases}
\]

so that

\[
\int_{OC}^1 \frac{e^{-xt}}{(-x)[1 - \beta(\log(\frac{1}{x} - 1) - i\pi)]} (-dx)
\]

\[
+ \int_{1}^{R} \frac{e^{-xt}}{(-x)[1 - \beta \log(1 - \frac{1}{x})]} (-dx)
\]

where \(R\) is the length of \(OC\).

On DO, \(s = -x, \frac{1}{s} = -\frac{1}{x}\), \(ds = -dx\), and

\[
\log(1 + \frac{1}{s}) = \begin{cases} 
\log(1 - \frac{1}{x}) & \text{if } x > 1 \\
\log\left(\frac{1}{x} - 1\right) + i\pi & \text{if } x < 1 \\
-\infty & \text{if } x = 1,
\end{cases}
\]

so that

\[
\int_{DO}^{0} \frac{e^{-xt}}{(-x)[1 - \beta(\log(\frac{1}{x} - 1) + i\pi)]} (-dx)
\]

\[
+ \int_{1}^{R} \frac{e^{-xt}}{(-x)[1 - \beta \log(1 - \frac{1}{x})]} (-dx)
\]

Therefore

\[
\int_{OC} + \int_{DO}
\]
\[
\begin{align*}
\int_0^1 e^{-xt} & \left( \frac{1}{1 - \beta \left( \log \left( \frac{1}{x} - \gamma \right) - i \pi \right)} - \frac{1}{1 - \beta \left( \log \left( \frac{1}{x} - \gamma \right) + i \pi \right)} \right) \, dx \\
= \int_0^1 e^{-xt} & \left( \frac{-2 i \beta \pi}{\left( 1 - \beta \log \left( \frac{1}{x} - \gamma \right)^2 + (\beta \pi)^2 \right)} \right) \, dx \\
= -2 \pi i & \frac{1}{\beta} \int_0^1 \frac{e^{-xt}}{x} \left( \frac{1}{\left( \frac{1}{\beta} - \log \left( \frac{1}{x} - \gamma \right)^2 + \pi^2 \right)} \right) \, dx \\
= -2 \pi i & \frac{1}{\beta} \int_{-\infty}^{\frac{1}{\beta} + y} \frac{1}{e^{\frac{1}{\beta} + y} + 1} \, dy \\
\end{align*}
\]

if we let \( y = \log \left( \frac{1}{x} - \gamma \right) - \frac{1}{\beta} \).

(c)

\[(5.29) \quad \lim_{r \to 0} \frac{1}{2 \pi i} \int_0^r = 0.\]

On \( O \), \( s = re^{i \theta}, \quad ds = ire^{i \theta} \, d\theta, \quad \frac{1}{s} = \frac{1}{r} e^{-i \theta} \) and

\[
\begin{align*}
\int_0^r & = \int_0^r \frac{e^{r (\cos \theta + i \sin \theta) \cdot ire^{i \theta} \, d\theta}}{1 - \beta \left( \log \sqrt{1 + \frac{1}{r^2} + \frac{2}{r} \cos \theta + i \psi} \right)} \\
= \int_0^r & \frac{e^{r (\cos \theta + i \sin \theta) \cdot i \, d\theta}}{1 - \beta \left( \log \sqrt{1 + \frac{1}{r^2} + \frac{2}{r} \cos \theta + i \psi} \right)} \\
\end{align*}
\]

where \( \psi = \text{Arg}(1 - \beta \log(i + \frac{1}{r})) \). Hence \( \int_0^r \to 0 \) as \( r \to 0 \).
(d)

\[(5.30) \quad \lim_{r \to 0} \frac{1}{2\pi i} \int_a^b \frac{Q+1}{\beta} e^{\alpha t} \, dt = \frac{Q+1}{\beta} e^{\alpha t} \]

On \( Q, s = r e^{i\theta} + Q = r e^{i\theta}, \) \( \frac{1}{s} = \frac{1}{r} e^{-i\theta}, \) \( ds = ire^{i\theta} d\theta \) and

\[\log(1 + \frac{1}{s}) = \log \sqrt{1 + \left(\frac{1}{r}\right)^2 + 2 \frac{r \cos \theta}{r^2 + Q^2 + 2rQ \cos \theta}} + i \psi, \]

where

\[r = \sqrt{r^2 + Q^2 + 2rQ \cos \theta}, \quad w = \sin^{-1} \left(\frac{r \sin \theta}{\sqrt{r^2 + Q^2 + 2rQ \cos \theta}}\right),\]

\[\sin \psi = \frac{-\frac{1}{r} \sin w}{\sqrt{1 + \left(\frac{1}{r}\right)^2 + 2 \frac{r \cos \theta}{r^2 + Q^2 + 2rQ \cos \theta}}}, \quad \cos \psi = \frac{1 + \frac{1}{r} \cos w}{\sqrt{1 + \left(\frac{1}{r}\right)^2 + 2 \frac{r \cos \theta}{r^2 + Q^2 + 2rQ \cos \theta}}}.\]

After some computation we have, as \( r \) tends to 0,

\[\int_a^b = \int_0^{2\pi} \frac{e^{\alpha t} e^{i\theta} \, d\theta}{Q - \beta \left(\frac{Q^2}{Q} - \frac{2 \cos \theta}{Q^2} - \frac{\sin \theta}{1 + \frac{1}{Q}}\right)} = 2\pi i \frac{Q+1}{\beta} e^{\alpha t} \]

With the help of (5.27), (5.28), (5.29) and (5.30), we obtain

\[(5.31) \quad f(t) = \frac{1}{\beta} \left(\frac{1}{e^\frac{1}{\beta} - 1} + 1\right) e^{\frac{1}{\beta} t - 1} - \int_0^{\frac{1}{\beta+y}} e^{\frac{1}{\beta+y} - 1} e^{-\frac{1}{\beta+y} + 1} \frac{1}{y^2 + \pi^2} \, dy\]

and from (5.22)
(5.32) \[ \mathbb{E}Z(T) = a e^{(\rho - \beta)T} f(t). \]

It is easy to show that \( \left( \frac{1}{\beta} + 1 \right) e^{\frac{1}{\beta} - 1} = 1 + o(\beta) \) and

\[
\int - \frac{1}{\beta + y} e^{-\frac{1}{\beta} + y} \frac{1}{y^2 + \pi^2} dy = 1 - \beta + o(\beta).
\]

Hence \( \mathbb{E}Z(t) \to e^{\rho t} \) as \( \beta \to 0 \).

To obtain the variance of \( Z(t) \), we find first

(5.33) \[ L_\delta(s) = (2 + s)\log(1 + \frac{2}{\delta}) - 2(s + \gamma)\log(1 + \frac{1}{\delta}). \]

By the same way as above and after some more complicated computation, we have

\[
W(t) = e^{(\rho - \beta)t} f(t), \text{ where }
\]

(5.34) \[ f(t) = \frac{2\lambda \alpha}{\rho} \left\{ 1 - \beta \left[ (2 + \rho) \log(1 + \frac{2}{\rho}) - 2(\rho + \gamma) \log(1 + \frac{1}{\rho}) \right] \right. \\
\left. \left( 1 - \beta \log(1 + \frac{1}{\rho}) \right) \right\} \\
\left. + \frac{\left\{ 1 - \beta \left[ (2 + \rho + \beta) \log(1 + \frac{2}{\rho}) - 2(\rho + \gamma + \beta) \log(1 + \frac{1}{\rho}) \right] \} \beta }{\left( 1 - \beta \log(1 + \frac{2}{\rho}) - 2(\rho + \gamma) \log(1 + \frac{1}{\rho}) \right)} \right\} \beta \\
\left[ (U + \rho)(U + \rho + \gamma)e^{ut} \right] \\
\]

\[
[(U + \rho)(U + \rho + \gamma)e^{ut}].
\]
\[
\begin{align*}
&+ \left\{ 1 - \beta \left[ (2 + V + \rho) \log(1 + \frac{2}{V + \rho}) - 2(V + \rho + 1) \log(1 + \frac{1}{V + \rho}) \right] \right\} \mathbb{e}^{Vt} \\
&\quad \cdot \left[ \frac{1 - \beta \log(1 + \frac{1}{V + \rho}) \left[ 2 \log(1 + \frac{1}{V}) - \log(1 + \frac{2}{V}) \right]}{\beta} \right] V \\
&\quad + \frac{1}{0} \left[ H_1(x) e^{-\lambda t} dx + \int_{1}^{2} H_2(x) e^{-\lambda t} dx + \int_{p}^{p+1} H_3(x) e^{-\lambda t} dx + \int_{p+1}^{p+2} H_4(x) e^{-\lambda t} dx \right] \\
&\quad + (a^2 - a) \left\{ \frac{1}{1 - \beta \log 4} + \frac{\mathbb{e}^{Vt}}{\beta \left[ 2 \log(1 + \frac{1}{V}) - \log(1 + \frac{2}{V}) \right]} \right\} V \\
&\quad + \int_{0}^{1} H_5(x) e^{-\lambda t} dx + \int_{1}^{2} H_6(x) e^{-\lambda t} dx \\
\end{align*}
\]

where

\[ U = \frac{1}{e^\beta - 1} - \rho, \quad (2 + V) \log(1 + \frac{2}{V}) - 2(V + 1) \log(1 + \frac{1}{V}) = \frac{1}{\beta} \]

\[ H_1(x) = \frac{G_1(x)}{G_2(x)} H_5(x), \quad H_2(x) = \frac{G_4(x)}{G_2(x)} H_6(x), \]

\[ H_3(x) = \frac{\beta}{G_3(x) \left\{ \left[ 1 - \beta \log(\frac{1}{x - \rho} - 1) \right]^2 + \beta^2 \pi^2 \right\}} \]

\[ \left\{ (x - \rho) \left[ 1 - \beta \log(\frac{1}{x - \rho} - 1) \right] \right\} \]

\[- \left\{ 1 - \beta \left[ (2 - x + \rho) \log(\frac{2}{x - \rho} - 1) - 2(-x + \rho + 1) \log(\frac{1}{x - \rho} - 1) \right] \right\}, \]
\[ H_4(x) = \frac{1}{G_3(x)} \frac{\beta(2 - x + \rho)}{1 - \beta \log(1 + \frac{1}{\rho - x})}, \]

\[ H_5(x) = \frac{-\beta x}{\left(1 - \beta \left(2 - x) \log(\frac{2}{x} - 1) - 2(1 - x) \log(\frac{1}{x} - 1)\right)\right)^2 + \beta^2 x^2 \pi^2}, \]

\[ H_6(x) = \frac{-\beta(2 - x)}{\left(1 - \beta \left(2 - x) \log(\frac{2}{x} - 1) - 2(1 - x) \log(\frac{1}{x} - 1)\right)\right)^2 + \beta^2 (2 - x)^2}, \]

\[ G_1(x) = 1 - \beta \left(2 - x + \rho\right) \log\left(1 + \frac{2}{\rho - x}\right) - 2 \left(-x + \rho + 1\right) \log\left(1 + \frac{1}{-x + \rho}\right). \]

\[ G_2(x) = -x \left[1 - \beta \log\left(1 + \frac{1}{\rho - x}\right)\right]. \]

\[ G_3(x) = -x \left[1 - \beta \left(2 - x) \log\left(1 - \frac{2}{x}\right) - 2 \left(-x + 1\right) \log\left(1 - \frac{1}{x}\right)\right]\]

and \[ \text{VarZ}(t) = W(t) - \frac{\lambda + \mu}{\rho} EZ(t) - (EZ(t))^2. \]

§5.2 Numerical Computation

In I.M.S.L. (Intenational Mathematical and Statistical Library), there is a subroutine called INLAP which can be used to find an approximate solution of some Inverse Laplace Transforms. So if we can find the exact forms of the Laplace Transform of \( \delta \) and \( \delta^2 \), hence the exact forms of the Laplace Transform of the mean and variance, then the INLAP subroutine can be used. But if we have strange \( \delta(t) \), then we have to use other numerical computation techniques. One method is shown below next.

Suppose we cannot obtain the exact forms of the Laplace
transform of $\delta$ and $\delta^2$. By (5.23), the inverse Laplace Transform of a function $f(t)$ is

$$f(t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} L_f(s) e^{st} ds.$$  

Let $s = \gamma + ix$, $x \in (-\infty, \infty)$ and let $\gamma$ be a constant greater than any real part of the singularities of the Laplace Transform of $f(t)$. Then by (2.29),

$$L_{\mathcal{L}}(s) = \frac{a}{(s + \beta - \rho)[1 - \beta L_{\delta}(s + \beta - \rho)]}$$

, which is to be replaced in the above inverse Laplace transform formula and obtain its real part only. First, find the real part and imaginary part of $L_{\delta}(s+\beta-\rho)$.

$$L_{\delta}(s + \beta - \rho) = L_{\delta}(\gamma + ix + \beta - \rho) = \int_0^{\infty} e^{-(\gamma + ix + \beta - \rho)t} \delta(t) dt$$

$$= \int_0^{\infty} e^{-(\gamma + \beta - \rho)t} \delta(t) (\cos(x t) - i \sin(x t)) dt$$

$$= M_1(\gamma, x) + i N_1(\gamma, x)$$

where

$$\begin{align*}
M_1(\gamma, x) &= \int_0^{\infty} e^{-(\gamma + \beta - \rho)t} \delta(t) \cos(x t) dt \\
N_1(\gamma, x) &= -\int_0^{\infty} e^{-(\gamma + \beta - \rho)t} \delta(t) \sin(x t) dt
\end{align*}$$

(5.35)
It is easy to show that

\[(5.36) \quad \begin{cases} M_1(\gamma, x) = M_1(\gamma, -x) \\ -N_1(\gamma, x) = N_1(\gamma, -x) \end{cases}\]

Note that Riemann-Lebesque Lemma, [1], p. 313, implies

\[(5.37) \quad M_i(\gamma, x) \to 0 \text{ and } N_i(\gamma, x) \to 0 \text{ as } x \to \infty.\]

Second, replace \(L_g(s+\beta\cdot\rho)\) with \(M_1\) and \(N_1\) of first step into the formula of \(L_{EZ}(s)\), and obtain its real and imaginary parts also.

\[(5.38) \quad L_{EZ}(\gamma + i x) = \frac{a}{(\gamma + i x + \beta - \rho)[1 - \beta M_1(\gamma, x) - i\beta N_1(\gamma, x)]}\]

\[= a / \{[(\gamma + \beta - \rho)(1 - \beta M_1(\gamma, x)) + \beta xN_1(\gamma, x)]\}
\[\quad + i [x(1 - \beta M_1(\gamma, x)) - (\gamma + \beta - \rho)\beta N_1(\gamma, x)]\}\]

\[= a \{[(\gamma + \beta - \rho)(1 - \beta M_1(\gamma, x)) + \beta xN_1(\gamma, x)]\}
\[- i [x(1 - \beta M_1(\gamma, x)) - (\gamma + \beta - \rho)\beta N_1(\gamma, x)]\}/\]

\[\{[(\gamma + \beta - \rho)(1 - \beta M_1(\gamma, x)) + \beta xN_1(\gamma, x)]^2\]
\[+ [x(1 - \beta M_1(\gamma, x)) - (\gamma + \beta - \rho)\beta N_1(\gamma, x)]^2\}\]

\[= a (f_1(\gamma, x) + ig_1(\gamma, x))\]

where

\[(5.39) \quad f_1(\gamma, x)\]
Using (5.36), (5.39) and (5.40), it is also easy to show that
(5.41) \( f_1(\gamma, x) = f_1(\gamma, -x) \), \(- g_1(\gamma, x) = g_1(\gamma, -x) \)

and, by (5.37),

(5.42) \( f_3(\gamma, x) \propto o\left(\frac{1}{x}\right) \), \( g_3(\gamma, x) \propto \frac{1}{x} \).

Next, we search for the real part of the inverse Laplace transform.

(5.43) \( E(t) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} L_{E_{xy}}(s) e^{st} ds = \frac{1}{2\pi i} \int_{-\infty}^{\infty} L_{E_{xy}}(\gamma + i x) e^{(\gamma + i x)t} i \ dx \)

\[ = \frac{ae^{\gamma t}}{2\pi} \text{Real}\left\{ \int_{-\infty}^{\infty} \left[ f_1(\gamma, x) + i g_1(\gamma, x) \right] \cos(x t) + i \sin(x t) dx \right\} \]

\[ = \frac{ae^{\gamma t}}{2\pi} \int_{-\infty}^{\infty} f_1(\gamma, x) \cos(x t) - g_1(\gamma, x) \sin(x t) dx \]

\[ = \frac{ae^{\gamma t}}{2\pi} \left\{ \int_{-\infty}^{0} f_1(\gamma, x) \cos(x t) - g_1(\gamma, x) \sin(x t) dx \right\} \]

\[ + \int_{0}^{\infty} f_1(\gamma, x) \cos(x t) - g_1(\gamma, x) \sin(x t) dx \]

\[ = \frac{ae^{\gamma t}}{2\pi} \left[ \int_{-\infty}^{0} f_1(\gamma, -x) \cos(-x t) - g_1(\gamma, -x) \sin(-x t) d(-x) \right] \]

\[ + \int_{0}^{\infty} f_1(\gamma, x) \cos(x t) - g_1(\gamma, x) \sin(x t) dx \]

\[ = \frac{ae^{\gamma t}}{\pi} \int_{0}^{\infty} f_1(\gamma, x) \cos(x t) - g_1(\gamma, x) \sin(x t) dx \].

We can use Romberg integration to find approximate \( M_1(\gamma, x) \) and \( N_1(\gamma, x) \), hence \( f_1(\gamma, x) \) and \( g_1(\gamma, x) \), then use the same integration
method to get, \( \text{EZ}(t) \), (5.43). One disadvantage of (5.43) is that \( g_1(\gamma,x) \) is proportional to \( \frac{1}{x} \) by (5.42), so the convergence of (5.43) is very slow. We can improve this by integrating by parts in (5.43), and reducing the order of \( g_1(\gamma,x) \). A result using this method by letting \( \delta(t)=0.6 \) is illustrated in Table 1.

To find the variance after obtaining the mean first, we have to find \( W \) given by (2.35) first.

Let \( W=W_1+W_2 \), where

\[
L_{w_1}(s) = \frac{2\lambda a [1-\beta L_\delta^2(s+\beta-\rho)]}{\rho(s+\beta-2\rho)[1-\beta L_\delta^2(s+\beta-2\rho)][1-\beta L_\delta^2(s+\beta-\rho)]}
\]

and

\[
L_{w_2}(s) = \frac{\lambda^2 - a}{(s+\beta-2\rho)[1-\beta L_\delta^2(s+\beta-2\rho)]}
\]

Then, by the same method, we have

\[
W_1(t) = \frac{2\lambda a e^{\gamma t}}{\pi \rho} \int_0^\infty \gamma_2(\gamma,x) \cos(x t) - g_2(\gamma,x) \sin(x t) \, dx
\]

and

\[
W_2(t) = \frac{(\lambda^2 - a) e^{\gamma t}}{\pi} \int_0^\infty \gamma_3(\gamma,x) \cos(x t) - g_3(\gamma,x) \sin(x t) \, dx
\]
Table 1: NCEZ stands for numerically computed expected $Z(t)$ and TRUE stands for the true value of $EZ(t)$.

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<th>TRUE</th>
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</tr>
</tbody>
</table>
where

(5.48) \[ f_2(\gamma, x) \]

\[
= \frac{(1 - \beta M_2)[Z_i(1 - \beta M_1) + Y_i\beta N_1] - \beta N_2[- Z_i\beta N_1 + Y_i(1 - \beta M_1)]}{[Z_i(1 - \beta M_1) + Y_i\beta N_1]^2 + [- Z_i\beta N_1 + Y_i(1 - \beta M_1)]^2}
\]

(5.49) \[ g_2(\gamma, x) \]

\[
= \frac{- (1 - \beta M_2)[- Z_i\beta N_1 + Y_i(1 - \beta M_1)] - \beta N_2[Z_i(1 - \beta M_1) + Y_i\beta N_1]}{[Z_i(1 - \beta M_1) + Y_i\beta N_1]^2 + [- Z_i\beta N_1 + Y_i(1 - \beta M_1)]^2}
\]

(5.50) \[ f_3(\gamma, x) \]

\[
= \{(\gamma + \beta - \rho)(1 - \beta M_1 \rho^\prime) + \beta x N_1 \rho^\prime\} / \\
[\{(\gamma + \beta - \rho)(1 - \beta M_1 \rho^\prime)^2 + [\beta x N_1 \rho^\prime]^2\} - [x(1 - \beta M_1 \rho^\prime)]^2 + [(\gamma + \beta - \rho)\beta N_1 \rho^\prime]^2\}
\]

(5.51) \[ g_3(\gamma, x) \]

\[
= \{- [x(1 - \beta M_1 \rho^\prime) - (\gamma + \beta - \rho)\beta N_1 \rho^\prime]\} / \\
[\{(\gamma + \beta - \rho)(1 - \beta M_1 \rho^\prime)^2 + [\beta x N_1 \rho^\prime]^2\} - [x(1 - \beta M_1 \rho^\prime)]^2 + [(\gamma + \beta - \rho)\beta N_1 \rho^\prime]^2\}
\]

(5.52) \[ Z_i = (\gamma + \beta - 2\rho)(1 - \beta \rho^\prime M_2) + x \beta \rho^\prime N_2 \]

and
(5.53) \[ Y_1 = x\left(1 - \beta e^{\rho t} M_2\right) - (\gamma + \beta - 2\rho)\beta e^{\rho t} N_2. \]

\( M_1 \) and \( N_1 \) are the same as (5.35), and

\[
\begin{cases}
M_2(\gamma, x) = \int_0^\infty e^{-(\gamma + \beta - \rho)t} \delta^2(t) \cos(x t) \, dt \\
N_2(\gamma, x) = -\int_0^\infty e^{-(\gamma + \beta - \rho)t} \delta^2(t) \sin(x t) \, dt
\end{cases}
\]

(5.54)

Again, using Romberg integration to obtain approximate \( M_2 \) and \( N_2 \), then \( f_2, g_2, f_3 \) and \( g_3 \) are found. Therefore, by (5.46) and (5.47), \( W \) can be found completely. Then the variance is at hand.
REFERENCES


