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EM scattering from perfectly conducting finite cones and cone-spheres

Choi, Jae Hoon, Ph.D.
The Ohio State University, 1989

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EM SCATTERING FROM PERFECTLY CONDUCTINGFINITE CONES
AND CONE-SPHERES

A Dissertation
Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the
Graduate School of the Ohio State University

by

Jae Hoon Choi, B.Sc., M.Sc.

* * * * *
The Ohio State University
1989

Dissertation Committee:
Roger C. Rudduck
Leon Peters, Jr.
Prabhatkar H. Pathak

Approved by:
Roger C. Rudduck
Advisor
Department of Electrical
Engineering
DEDICATION

To my wife Eunbong, my son Philip,
and my parents.
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VITA

May 28, 1957 ......................... Born - Seoul, Korea

1980 ........................................ B.Sc. Hanyang University, Seoul, Korea

1986 ........................................ M.Sc. The Ohio State University, Columbus, Ohio

1985-1989 ............................... Graduate Research Associate, ElectroScience Laboratory, Department of Electrical Engineering, The Ohio State University, Columbus, Ohio

PUBLICATIONS

I. Articles in journal

"Near Axial Backscattering from Finite Cones," Jaehoon Choi, Nan Wang, Leon Peters, Jr., and Paul Levy, submitted to be appeared in IEEE AP-S

II. Conference papers


"Off-Axis Backscattering from Cone-Sphere," Jaehoon Choi, Nan Wang, Leon Peters, Jr., and Paul Levy, presented at the IEEE AP-S Symposium and URSI Science Meeting at San Jose, June 26-30, 1989


III. Technical reports


IV. Dissertation and Thesis

"EM Scattering from Perfectly Conducting Finite Cones and Cone-Spheres," Ph.D. Dissertation, The Ohio State University, Department of Electrical Engineering, 1989

FIELDS OF STUDY

Major Field: Electrical Engineering

Studies in Antennas and Electromagnetics: Professors R. C. Rudduck,
L. Peters, Jr., P.H. Pathak and R. G. Kouyoumjian

Studies in Communications: Professor F. D. Garber

Studies in Mathematics: Professor J. T. Scheick
# TABLE OF CONTENTS

## LIST OF FIGURES

x

## LIST OF TABLES

xvii

## I. Introduction

1.1 The Problem and Technical Approaches ........................................... 1
1.2 Previous Research ................................................................................... 7
1.3 Format .......................................................................................................... 10

## II. THEORETICAL BACKGROUNDS

2.1 Introduction ......................................................................................... 12
2.2 UTD Field Expressions ............................................................................ 12
2.2.1 Curved Edge Diffraction .......................................................... 13
2.2.2 Surface Creeping Wave Mechanism ..................................... 16
2.2.3 Field radiated from sources on a convex surface ............. 22
2.3 Differential Geometry of a Cone .......................................................... 29
2.3.1 Differential Geometry of a Cone ............................................ 29
2.3.2 Geodesic on a Cone ................................................................. 34
2.4 Coordinate Transformation on the Spherical Base ......... 37
2.5 Equivalent Current Concept ............................................................... 40
2.6 Diffraction Coefficients for the Discontinuity in Curvature .... 42
III. RCS OF A PERFECTLY CONDUCTING FINITE CONE

3.1 Backscattering RCS of a Cone

3.1.1 Introduction

3.1.2 Equivalent Currents for the Singly Diffracted Field \( E_D \)

3.1.3 Equivalent Current for the Doubly Diffracted Field \( E_{DD} \)

3.1.4 Equivalent Current for the \( E_{DDG} \) Ray Mechanism

3.1.5 Equivalent Current for the \( E_{GDD} \) Ray Mechanism

3.1.6 Equivalent Current for the \( E_{GDDD} \) Ray Mechanism

3.1.7 Tip Diffraction

3.1.8 Numerical Results

3.2 Bistatic Scattering RCS of a Fully Illuminated Cone

3.2.1 Introduction

3.2.2 Equivalent Current for the Singly Diffracted Field \( E_D \)

3.2.3 Equivalent Current for the Doubly Diffracted Field \( E_{DD} \)

3.2.4 Equivalent Current for \( E_{DDG} \) Ray Mechanism

3.2.5 Numerical Results

3.3 Conclusion

IV. BACKSCATTERING RCS OF A CONE-SPHERE

4.1 Introduction

4.2 The Fully Illuminated Cone-Sphere

4.2.1 Diffracted Field from the Junction \( E_{JD} \)

4.2.2 Creeping Wave Around the Spherical Base \( E_C \)

4.3 Partially Illuminated Cone-Spheres

4.3.1 \( E_{CDG} \) Ray Mechanism

4.3.2 \( E_{GDC} \) Ray Mechanism
4.4 Numerical Results and Discussions .................................................... 130
4.5 Conclusion ............................................................................................. 146

V. CONCLUSION .......................................................................................... 147

A. PO SOLUTION FOR THE BACKSCATTERING RCS OF A
FINITE CONE AND A CONE-SPHERE .................................................... 150
   A.1 Backscattering from a Finite Cone .................................................... 150
   A.2 Backscattering from a Cone-Sphere .................................................... 156

B. EXPRESSION OF THE FIRST DIFFRACTION POINT IN
TERMS OF THE SECOND DIFFRACTION POINT FOR
THE DOUBLE DIFFRACTION ............................................................... 161

REFERENCES .............................................................................................. 165
LIST OF FIGURES

1.1 Diffracted field components of a cone ....................................................... 4
1.2 Diffracted field components of a cone-sphere ......................................... 5
2.1 Geometry of curved wedge with curved edge .......................................... 14
2.2 Transition function ........................................................................................ 17
2.3 Creeping wave geometry ............................................................................. 20
2.4 Plot of $e^{-j\frac{\pi}{4} p^*}(\zeta)$ vs $\zeta$ based on Logan's tabulated data [33] for $p(\zeta)$ 23
2.5 Plot of $e^{-j\frac{\pi}{4} q^*}(\zeta)$ vs $\zeta$ based on Logan's tabulated data [33] for $q(\zeta)$ 24
2.6 Surface diffracted ray tube and ray coordinates for the shadow region 26
2.7 Geometry of a cone ..................................................................................... 30
2.8 Ray path on a cone ..................................................................................... 31
2.9 Geometry for the coordinate transformation on the spherical base of a cone-sphere ............................................................................ 38
2.10 (a) A partial cone geometry and its associated equivalent currents on the rim, and (b) The equivalent currents of the partial cone radiating in free space ................................................................. 41
2.11 Geometry of the surface with curvature discontinuity ........................... 42
3.1 Geometry of a cone under consideration ................................................. 45
3.2 Diffracted field components included in the calculation of backscattering RCS of a cone ................................................................. 47
3.3 Geometry for the singly diffracted field at the rim ....................... 49

3.4 Comparison of the backscattering RCS of a cone with $\alpha = 15^\circ$ and 
$a = 1.435\lambda$: equivalent current method ($E_D$) (solid line), method of moments (dashed line). (a) vertical polarization (b) horizontal polarization ................................. 52

3.5 Geometry for the doubly diffracted field at the rim .................... 54

3.6 Relationship between $v'$ and $\phi'$ on the base ............................ 57

3.7 Comparison of the backscattering RCS of a cone with $\alpha = 15^\circ$ 
and $a = 1.435\lambda$: equivalent current method ($E_D + E_{DD}$) (solid line), method of moments (dashed line). (a) vertical polarization 
(b) horizontal polarization .............................................................. 60

3.8 Comparison of the E-plane backscattering RCS of $a = 2.2918\lambda$ hol-
low cylinder: equivalent current method (solid line), Wiener-Hopf[8] 
(black dots) ................................................................. 61

3.9 Comparison of the H-plane backscattering RCS of $a = 2.2918\lambda$ hol-
low cylinder: equivalent current method (dash-dot line), Wiener-
Hopf[8] (black dots) ................................................................. 62

3.10 Geometry for the creeping wave excited edge diffracted field ($E_{DGG}$) 64

3.11 Geometry for the edge diffracted field excited creeping wave ($E_{GDD}$) 69

3.12 Geometry for $E_{GDDG}$ ray ....................................................... 72

3.13 Comparison of the backscattering RCS of a cone with $\alpha = 15^\circ$ 
and $a = 1.435\lambda$: equivalent current technique (solid line), method of moments (dashed line). (a) vertical polarization (b) horizontal 
polarization ................................................................. 74

3.14 Backscattering RCS of a cone with $a = 1.435\lambda$ and $\alpha = 15^\circ$. (a) 
vertical polarization (b) horizontal polarization ................................. 76
3.15 Comparison of backscattering RCS of a cone with $ka = 9.642$ and $\alpha = 15^\circ$ between the present approach and [22]. (a) vertical polarization (b) horizontal polarization ................................. 80

3.16 Comparison of backscattering RCS of a cone with $ka = 11.527$ and $\alpha = 15^\circ$ between the present approach and [22]. (a) vertical polarization (b) horizontal polarization ................................. 81

3.17 Comparison of $\sigma_{\phi\phi}$ of a cone with $a = 1.435\lambda$ and $\alpha = 15^\circ$. (a) Wang and Mitschang[28] vs. method of moments (b) Present approach vs. method of moments .............................................................. 82

3.18 Comparison of backscattering RCS of a cone with $a = 3.88\lambda$ and $\alpha = 10.1586^\circ$ between Wang and Mitschang[28] (dashed line) and present approach (solid line). (a) vertical polarization (b) horizontal polarization .............................................................. 83

3.19 Axial backscattering RCS of a cone with $\alpha = 15^\circ$ ................ 84

3.20 Axial backscattering RCS of a cone with $\alpha = 30^\circ$ ................ 85

3.21 Backscattering RCS of a cone with $\alpha = 5^\circ$ and $a = 1.435\lambda$: horizontal polarization (solid line), vertical polarization (dash dot line) 86

3.22 Backscattering RCS of a cone with $\alpha = 10^\circ$ and $a = 1.435\lambda$: horizontal polarization (solid line), vertical polarization (dash dot line) 87

3.23 Backscattering RCS of a cone with $\alpha = 20^\circ$ and $a = 1.435\lambda$: horizontal polarization (solid line), vertical polarization (dash dot line) 88

3.24 Backscattering RCS of a cone with $\alpha = 15^\circ$ and $a = 1\lambda$: horizontal polarization (solid line), vertical polarization (dash dot line) .... 89

3.25 Backscattering RCS of a cone with $\alpha = 15^\circ$ and $a = 2\lambda$: horizontal polarization (solid line), vertical polarization (dash dot line) .... 90
3.26 Backscattering RCS of a cone with $\alpha = 15^\circ$ and $\alpha = 10\lambda$: horizontal polarization (solid line), vertical polarization (dash dot line) .... 91

3.27 Backscattering RCS of a cone with $\alpha = 15^\circ$ and $\alpha = 0.7175\lambda$: present approach (solid line), moment method (curve formed by dots). (a) vertical polarization, (b) horizontal polarization ............. 92

3.28 Backscattering RCS of a cone with $\alpha = 15^\circ$ and $\alpha = 2.87\lambda$: present approach (solid line), moment method (curve formed by dots). (a) vertical polarization, (b) horizontal polarization ............. 93

3.29 Geometry for the singly diffracted field for the bistatic scattering of a cone ................................. 95

3.30 Geometry for the doubly diffracted field for the bistatic scattering of a cone ................................. 99

3.31 $E_{DDG}$ ray mechanism for the bistatic scattering of a cone ................................. 102

3.32 Bistatic scattering cross section of a finite cone with a half cone angle $\alpha = 15^\circ$ and the angle of incidence $\theta_i = 0^\circ$ ......... 106

3.33 Bistatic scattering cross section of a finite cone with a half cone angle $\alpha = 15^\circ$ and the angle of incidence $\theta_i = 5^\circ$ ......... 107

3.34 Bistatic scattering cross section of a finite cone with a half cone angle $\alpha = 15^\circ$ and the angle of incidence $\theta_i = 10^\circ$ ......... 108

4.1 Cone-Sphere geometry under consideration ......................... 112

4.2 Diffracted field components in the calculation of the backscattering RCS of a cone-sphere ......................... 113

4.3 Geometry for the junction diffraction ............................. 115

4.4 Geometry for the creeping wave around the spherical base ...... 118

4.5 Geometry for $E_{CDG}$ ray mechanism ............................. 123

xiii
4.6 Geometry for $E_{GDC}$ ray mechanism ...................................................... 127

4.7 Backscattering cross section of a cone-sphere with $\alpha = 15^\circ$ and $\alpha = 1.435\lambda$ (solid line: present approach including only the junction diffraction, dashed line: method of moments). (a) horizontal polarization ($\sigma_{\theta\theta}$) (b) vertical polarization ($\sigma_{\phi\phi}$) .................................................... 132

4.8 Backscattering cross section of a cone-sphere with $\alpha = 15^\circ$ and $\alpha = 1.435\lambda$ (solid line: present approach including $E_{JD}$ and $E_C$, dashed line: method of moments). (a) horizontal polarization ($\sigma_{\theta\theta}$) (b) vertical polarization ($\sigma_{\phi\phi}$) .................................................... 133

4.9 Backscattering cross section of a cone-sphere with $\alpha = 15^\circ$ and $\alpha = 1.435\lambda$ (solid line: present approach including $E_{JD}$, $E_C$, and $E_{GDC}$, dashed line: method of moments). (a) horizontal polarization ($\sigma_{\theta\theta}$) (b) vertical polarization ($\sigma_{\phi\phi}$) .................................................... 134

4.10 Backscattering cross section of a cone-sphere with $\alpha = 15^\circ$ and $\alpha = 1.435\lambda$ (solid line: present approach including $E_{JD}$, $E_C$, $E_{GDC}$, and $E_{GDC}$, dashed line: method of moments). (a) horizontal polarization ($\sigma_{\theta\theta}$) (b) vertical polarization ($\sigma_{\phi\phi}$) .................................................... 135

4.11 Backscattering cross section of a cone-sphere with $\alpha = 15^\circ$ and $\alpha = 1.435\lambda$ (solid line: present approach including $E_{JD}$, $E_C$, $E_{GDC}$, $E_{GDC}$, and $E_{TD}$, dashed line: method of moments). (a) horizontal polarization ($\sigma_{\theta\theta}$) (b) vertical polarization ($\sigma_{\phi\phi}$) .................................................... 136

4.12 Backscattering RCS of a cone-sphere with $\alpha = 4.33^\circ$ and $\alpha = 1.0354\lambda$: measured data at frequency 15.2 GHz (solid line), calculated data (black dots). (a) horizontal polarization (b) vertical polarization ........................................... 137
4.13 Backscattering RCS of a cone-sphere with $\alpha = 4.33^\circ$ and $a = 0.962\lambda$: measured data at frequency 11.2 GHz (solid line), calculated data (black dots). (a) horizontal polarization (b) vertical polarization

4.14 Axial backscattering RCS of a cone-sphere with half cone angle $\alpha = 12.5^\circ$ ................................................................. 139

4.15 Backscattering RCS of a cone-sphere with $\alpha = 5^\circ$ and $a = 1.435\lambda$: vertical polarization (dash-dot line), horizontal polarization (solid line) ................................................................. 140

4.16 Backscattering RCS of a cone-sphere with $\alpha = 10^\circ$ and $a = 1.435\lambda$: vertical polarization (dash-dot line), horizontal polarization (solid line) ................................................................. 141

4.17 Backscattering RCS of a cone-sphere with $\alpha = 20^\circ$ and $a = 1.435\lambda$: vertical polarization (dash-dot line), horizontal polarization (solid line) ................................................................. 142

4.18 Backscattering RCS of a cone-sphere with $\alpha = 15^\circ$ and $a = 1\lambda$: vertical polarization (dash-dot line), horizontal polarization (solid line) ................................................................. 143

4.19 Backscattering RCS of a cone-sphere with $\alpha = 15^\circ$ and $a = 2\lambda$: vertical polarization (dash-dot line), horizontal polarization (solid line) ................................................................. 144

4.20 Backscattering RCS of a cone-sphere with $\alpha = 15^\circ$ and $a = 3\lambda$: vertical polarization (dash-dot line), horizontal polarization (solid line) ................................................................. 145

A.1 Geometry of a finite cone ................................................................. 151
A.2 Comparison of backscattering RCS of a finite cone with \( \alpha = 15^\circ \) and 
\( a = 1.435\lambda \) between the PO solution (solid line), and the equivalent 
current approach (dashed line: horizontal polarization, black dots: 
vertical polarization) .............................................................. 155

A.3 Geometry of a cone-sphere under consideration .................. 157

A.4 Comparison of backscattering RCS of a cone-sphere with \( \alpha = 15^\circ \)
and \( a = 1.435\lambda \) between the PO solution (solid line), and the equiv-
alent current approach (dashed line: horizontal polarization, black 
dots: vertical polarization) ...................................................... 160
LIST OF TABLES

2.1 Linear interpolation data for $F(x)$ .......................... 18
2.2 Torsion factors for a source on a convex surface .......... 28
3.1 Diffracted field components of a cone with half cone angle $\alpha$ in the different sections of scattering angles: Notes; 'O' means that the ray component exists and 'X' means that the ray component does not exist ......................................................... 58
CHAPTER I

Introduction

1.1 The Problem and Technical Approaches

The objective of this research is to analyze the electromagnetic (EM) scattering from a perfectly conducting finite cone and a cone-sphere for E- and H-plane incidence. Both monostatic and bistatic scattering for near axial plane wave incidence are of interest.

Much work had been done on the finite cone, and it is a geometry that is closely related to the cone-sphere. There were however a number of serious difficulties with the earlier work that needed to be eliminated. Since the finite cone represented an excellent starting point for the cone sphere analysis, these problem areas have been evaluated and then carried over into the cone-sphere analysis.

There are several methods to handle the electromagnetic scattering of complex surfaces. However, each method has advantages and disadvantages. Let’s consider the following two methods. One way of solving the problem is the method of moments (MM) which is often referred to as an exact solution. Even though the numerical results of this method may be good enough to be considered as the exact solution, the computational efficiency becomes poor as the size of the object under consideration becomes large in terms of the wave length. Another approach to handle the scattering problem is the uniform version of the geometrical theory of diffraction (UTD)[1,2,3] which has been widely applied to the various types of the
scattering and the radiation problems, such as the curved edge[1], the corner, the surface creeping wave[2], and the radiation of sources on a convex surface[3] at The Ohio State University (OSU). The UTD usually saves considerable computing time as compared to the MM and gives reasonably good results from the engineering point of view.

The UTD can be employed to calculate the field diffracted from the curved edge and the surface creeping wave diffracted field. However, the axial symmetry property of a cone and a cone-sphere creates a caustic in the axial direction. In order to overcome this caustic problem, the equivalent current concept[4,5] is employed to describe the different diffraction mechanisms. Also, Knott and Senior’s approach[6] can be used to treat these mechanisms. They evaluated the stationary phase contribution of the unknown current with the phase of the incident field at the rim (or discontinuity) and then compared the result to the field from the diffraction theory to obtain the magnitude of the unknown equivalent current.

In this research, the focus is on the calculation of the diffracted fields by using the equivalent current concept and the UTD. For a cone, the diffracted field components included in the calculation of the radar cross section (RCS) are the singly diffracted field, the doubly diffracted field, the creeping wave excited edge diffracted field, the edge diffracted field excited creeping wave, and the multiply diffracted fields which are a combination of the surface creeping wave diffracted field and the edge diffracted field. For a cone-sphere, the junction diffracted field, the creeping wave around the spherical base, and the two other multiply diffracted fields which are a combination of the creeping wave around the spherical base and the creeping wave associated with the conical surface. The tip diffracted field is included in both cone and cone-sphere cases. The tip diffracted field is calculated using the narrow cone angle approximation given by Felsen[7].
Before dealing with the approaches to solve the complicated ray mechanisms, it is convenient to introduce abbreviations and the pictorial illustrations of different ray mechanisms that are used in this dissertation.

Abbreviations:

\( E_D \): The singly diffracted field at the edge (figure 1.a),
\( E_{DD} \): the doubly diffracted field at the edge (figure 1.b),
\( E_{GDD} \): the creeping wave excited edge diffracted field (figure 1.c),
\( E_{DDC} \): the edge diffracted field excited creeping wave (figure 1.d),
\( E_{TD} \): the diffracted field at the tip of a cone (figure 1.e),
\( E_{GDDC} \): the surface creeping wave exciting a doubly edge diffracted field which is exciting a surface creeping wave toward the scattering direction (figure 1.f),
\( E_{JD} \): the diffraction at the junctio of a cone-sphere (figure 2.a),
\( E_C \): the creeping wave around the spherical base (figure 2.b),
\( E_{GDC} \): the incident ray creeps along the cone surface and then it diffracts through the junction and it continuously creeps on the spherical base until it diffracts toward the scattering direction (figure 2.c),
\( E_{CDG} \): the incident ray creeps on the spherical base and then it diffracts through the junction and continuously creeps along the cone surface until it diffracts toward the scattering direction (figure 2.d), and
\( E_{TD} \): the diffracted field at the tip of a cone-sphere (figure 2.e).
Figure 1.1: Diffracted field components of a cone
Figure 1.2: Diffracted field components of a cone-sphere
The diffracted ray mechanisms included in the RCS calculation of a cone are $E_D$, $E_{DD}$, $E_{GDD}$, $E_{DDG}$, and $E_{TD}$. The first five mechanisms are analyzed by the equivalent current concept[4,5] within the framework of the UTD. In order to verify the correctness of the equivalent current used for the singly and the doubly diffracted fields, the RCS of a hollow cylinder is calculated and compared to the result given in [8].

The diffracted fields in different regions are given as follows:

For a fully illuminated cone:
$$\bar{E}^d = \bar{E}_D + \bar{E}_{DD} + \bar{E}_{TD}$$

For a partially illuminated cone:
$$\bar{E}^d = \bar{E}_D + \bar{E}_{DD} + \bar{E}_{TD} + \bar{E}_{GDD} + \bar{E}_{DDG} + \bar{E}_{GDDG}$$

For a cone-sphere, the diffracted field components are $E_{JD}$, $E_C$, $E_{GDC}$, $E_{CDG}$, and $E_{TD}$. The equivalent current concept is applied to calculate the field components for the first four mechanisms. The diffraction coefficient presented by Senior[9] is utilized for the junction diffraction term. It is found in this research that the creeping wave diffracted field around the spherical base using the UTD diffraction coefficients yields about a 2 dB discrepancy compared to Senior’s result[10]. This is because the Pekeris function in the UTD diffraction coefficients for the surface creeping wave accounts for the leading term contribution only. On the other hand, Senior includes the higher order creeping wave contributions based on the perfectly conducting sphere. The diffraction coefficient including the higher order contribution can be obtained by comparing the resulting field expression of the equivalent current approach to Senior’s result [10]. Also, the correction term given by Voltmer [11] can be employed to account for the higher order contribution.

The diffracted field of a cone-sphere is given as follows:
$$\bar{E}^d = \bar{E}_{JD} + \bar{E}_C + \bar{E}_{TD}$$: for a fully illuminated cone-sphere,
\[ \vec{E}^d = \vec{E}_{JD} + \vec{E}_C + \vec{E}_{TD} + \vec{E}_{GDC} + \vec{E}_{CDG}; \] for a partially illuminated cone-sphere.

1.2 Previous Research

There has been substantial research on the cone and the cone-sphere. The following summary is not a complete overview. Its goal is to give the reader an indication of the amount of interest existing in the cone geometry.

Between 1952-1953, Siegel, et al.[12,13] published work which determined scattering from a cone and compared the theory to the experiment. These analyses were based on an eigenfunction expansion which is slowly convergent, and special summation techniques were used.

In 1955, L.B. Felsen[14] published an article on the backscattering from wide angle and narrow angle semi-infinite cones. In this paper, he used the method of characteristic Green's functions for the scalar problem and the spherical transmission theory for the vector problem.

In 1955, Siegel, et al.[15] published a paper on the electromagnetic and acoustical scattering from a semi-infinite cone. In this paper, they established the nose-on backscattering cross section of a semi-infinite cone by the exact electromagnetic and acoustical theory, and by physical optics.

In 1957, Felsen[16,7,17] published articles on alternative field representations for regions bounded by spheres, cones and planes. In these papers, he used the angular transmission eigenfunction expansion. This representation converges faster than the radial transmission version by Siegel, et al. In [17], Felsen extracted the asymptotic expansion of the tip diffracted field from the semi-infinite cone from his representation.

In 1960, J. B. Keller[18] published an article on backscatter from a finite cone (flat backed cone and cone with a spherical base). In his paper, he utilized the
GTD[19] to calculate the backscattering field.

In 1965, Bechtel[20] published work on a finite cone. In his paper, he applied the GTD to the scattering from a finite cone. However, he only considered the singly diffracted, doubly diffracted, and the specular reflection contributions.

In 1972, Burnside and Peters[21] published an article on the axial radar cross section of a finite cone. In this paper, they used the equivalent current method to calculate the singly and the doubly diffracted fields at the edge of the cone base. They indicated a 3 dB discrepancy between the measured and calculated data, and postulated this to be caused by a missing diffraction mechanism that is a tip-base interaction.

Senior and Uslenghi[22] published an article on the backscattering from a finite cone in 1973. They included the second order diffraction mechanisms and results were compared with experimental data.

Kouyoumjian[23] published work on the high frequency backscatter from a finite cone in 1977. This paper presented both acoustic and electromagnetic solutions. In this paper, an attempt was made to account for the tip base interaction.

Between 1977 and 1979, Felsen and Chan[24,25,26] published papers in which the integral form of the Green’s function for the cone was derived. The first of these papers extracted the creeping wave from two points on the surface which are sufficiently far from the tip. In the latter two papers, they investigated the transient behavior by transforming the Green’s function to the time domain.

Chu[27] calculated the backscattering RCS of a perfectly conducting finite cone including only the single diffraction from the edge. In that, he also used the equivalent current method to obtain the caustic solution. However, the omission of the doubly diffracted field and the creeping wave associated with the conical surface resulted in a serious error for the vertically polarized RCS.
In 1984, Wang and Mitschang[28] evaluated the backscattering RCS of a finite circular and an elliptic cone based on the physical theory of diffraction (PTD). They also investigated the effects of the creeping wave and the tip-base interaction based on the hybrid technique.

Trott[29] studied the uniform high frequency asymptotic solution for a fully illuminated perfectly conducting semi-infinite cone based upon PO as a part of his Ph.D. dissertation in 1986. In that, he obtained the solution in the format of UTD and extracted the tip base interaction term. However, his solution is limited to the fully illuminated case.


Senior[10] published an article on a backscattering RCS of a cone-sphere in 1965. He calculated the on-axis RCS of a cone-sphere based on the physical optics (PO) approximation and extracted the tip diffracted, the junction diffracted, and the creeping wave diffracted fields. Senior presented the diffraction coefficients for the surface with curvature discontinuity in the paper titled 'The Diffraction Matrix for a Discontinuity in Curvature'[9] in 1972.

Mitschang and Wang[32] obtained the solution for the bistatic scattering from a cone-sphere by using the hybrid method. In that, they incorporated the Fock solution for currents on the surface into the method of moments.

Even though much research on cone geometries has been conducted, the solution in the format of UTD is not firmly established yet. Also most of the solutions
have a poor numerical computing efficiency. This fact motivates the author to obtain the solution in the UTD form with a good computing efficiency. One advantage of the solution in the UTD form is that the different ray contributions can be identified. In turn, one can establish the RCS reduction by eliminating the most significant ray contribution. The reduction of RCS of a cone geometry might be achieved by either using the lossy dielectric material or reshaping the base structure of a cone geometry.

1.3 Format

In Chapter 2, the theoretical background involved in the calculation of the radar cross section of a perfectly conducting finite cone and a cone-sphere is discussed. The UTD field expressions for the curved edge diffraction, the surface creeping wave mechanism, and the radiation of sources on a convex surface are described. The differential geometry necessary to understand the generic characteristics of a cone is studied in detail. Also the equivalent current concept and the diffraction from a surface with curvature discontinuity is described briefly. In Chapter 3, the backscattering and bistatic scattering RCS of a perfectly conducting finite cone are studied in the framework of the equivalent current concepts based on the UTD. Some numerical results for the backscattering and the bistatic scattering radar cross sections (RCS) of finite cones are obtained and compared to the measured data and the method of moments. Backscattered field from a cone-sphere is calculated using the equivalent current approach in Chapter 4. Some data are obtained and compared to other methods, such as the method of moments and the measured data. In Chapter 5, some conclusions and the future research topics are discussed. The physical optics (PO) solution for the backscattered field of a perfectly conducting finite cone and a cone-sphere are calculated and compared to
results obtained by the present equivalent current approach in the Appendix A. In Appendix B, the solution for the first diffraction point in terms of the second diffraction point for the double diffraction from a finite cone is obtained by using the solution for a quartic equation.
CHAPTER II
THEORETICAL BACKGROUNDS

2.1 Introduction

This chapter describes the theoretical background necessary to solve the current problem. The UTD field expressions for the curved edge diffraction, the surface creeping wave mechanism, and the radiation of sources on a convex surface are reviewed. Also, some basic definitions required to understand the differential geometry of a cone are given and the transformation of the coordinate on the spherical base of a cone-sphere is described briefly. Finally, the equivalent current concept and the diffraction coefficients for the surface with curvature discontinuity are described.

An $e^{j\omega t}$ time dependence is assumed and will be suppressed in all field expressions.

2.2 UTD Field Expressions

In this section, the UTD field expressions for the curved edge diffraction, the surface creeping wave, and the radiation from a source on a convex surface are described briefly.
2.2.1 Curved Edge Diffraction

The field diffracted from a curved surface with a curved edge, as shown in Fig. 2.1, is given by [1]

\[
\vec{E}^d = \vec{E}^i(Q_e) \cdot (-\hat{\beta}' \hat{D}_s - \hat{\psi}' \hat{D}_h) A(s, s') e^{-jks}
\]

where \(\vec{E}^i(Q_e)\) is the incident field at the edge and the soft and the hard diffraction coefficients are given by

\[
D_{s,h} = \frac{-e^{-i\pi/4}}{2n\sqrt{2\pi\kappa}} \sin \beta_c \left\{ \cot\left(\frac{\pi + (\psi - \psi')}{2n}\right) F[kL^i a^+ (\psi - \psi')] \\
+ \cot\left(\frac{\pi - (\psi - \psi')}{2n}\right) F[kL^i a^- (\psi - \psi')] \\
+ \cot\left(\frac{\pi + (\psi + \psi')}{2n}\right) F[kL^r a^+ (\psi + \psi')] \\
+ \cot\left(\frac{\pi - (\psi + \psi')}{2n}\right) F[kL^r a^- (\psi + \psi')] \right\}
\]

The following parameters are used in this solution:

\[
A(s, s') = \frac{\rho_c}{\sqrt{s(s + \rho_c)}} \quad \text{2.3}
\]

\[
\frac{1}{\rho_c} = \frac{1}{\rho_e} - \frac{\hat{n}_e \cdot (\hat{s'} - \hat{s})}{\rho_g \sin^2 \beta_o} \quad \text{2.4}
\]

\(\rho_e\) = radius of curvature of incidence wavefront in the edge-fixed plane of incidence which contains unit vectors \(\hat{s}\) and \(\hat{e}\)

(infinite for plane, cylindrical, and conical waves: \(s'\) for spherical waves.)

\(\rho_g\) = radius of curvature of edge at diffraction point.

\(\hat{n}_e\) = unit vector normal to edge at \(Q\) and directed away from the center of curvature.

\(\hat{e}\) = unit vector tangent to edge at the point of diffraction.
Figure 2.1: Geometry of curved wedge with curved edge
\[ L^i = \frac{s(\rho_e^i + s)\rho_e^i}{\rho_e^i(\rho_e^i + s)(\rho_e^i + s)} \sin^2 \beta_0 \]  
\[ \rho_{1,2}^i = \text{principal radii of curvature of the incident wave front.} \]

\[ L^{r(n,0)} = \frac{s(\rho_e^r + s)\rho_e^r}{\rho_e^r(\rho_e^r + s)(\rho_e^r + s)} \sin^2 \beta_0 \]  
\[ (L^{r(n)} \text{ is } L^r \text{ for "N" face, } L^{r0} \text{ is } L^r \text{ for "0" face}) \]

\[ \rho_e^r = \text{reflected radius of curvature in the plane containing the reflected ray and edge} (\hat{e}): \]

\[ \frac{1}{\rho_e^r} = \frac{1}{\rho_e^0} - \frac{2(\hat{n} \cdot \hat{n}_e)(\hat{I} \cdot \hat{n})}{\rho_g \sin^2 \beta_0} \]  
\[ \rho_{1,2}^r = \text{principal radii of curvature of the reflected wave front[1].} \]

\[ a^\pm(\beta) = 2 \cos^2 \left( \frac{2n\pi N^\pm - \beta}{2} \right) \]

\[ N^\pm = \text{integers which mostly satisfy the equations: } 2n\pi N^\pm - \beta = \pm \pi. \]

The unit vectors in the edge-fixed coordinate system are defined by

\[ \hat{\psi}' = \frac{\hat{I} \times \hat{e}}{|\hat{I} \times \hat{e}|} \]  
\[ \hat{\beta}' = \frac{\hat{I} \times \hat{\psi}'}{|\hat{I} \times \hat{\psi}'|} \]  
\[ \hat{\psi} = \frac{\hat{e} \times \hat{s}}{|\hat{e} \times \hat{s}|}, \text{and} \]
\[ \hat{\beta} = \frac{\hat{s} \times \hat{\psi}}{|\hat{s} \times \hat{\psi}|} \]

where \( \hat{I} \) is the incident unit vector and \( \hat{s} \) is the unit vector in the scattering direction. The unit vectors \( \hat{\psi}' \) and \( \hat{\psi} \) are perpendicular to the edge-fixed plane of incidence which is formed by the edge unit vector \( \hat{e} \) and the incident unit vector \( \hat{I} \) and the plane of diffraction which is formed by \( \hat{e} \) and the diffraction direction.
unit vector $\hat{s}$, respectively. The unit vectors $\hat{\beta}'$ and $\hat{\beta}$ are parallel to the edge-fixed plane of incidence and the plane of diffraction, respectively.

The transition function $F(x)$ used in the diffraction coefficients is defined by

$$F(x) = 2j|\sqrt{x}|e^{jx} \int_{\sqrt{x}}^{\infty} e^{-jt^2} dt$$  \hspace{1cm} 2.13

and is illustrated in Fig. 2.2. In practical numerical calculations, this function is evaluated using the three schemes: the small argument form is given by

$$F(x) \simeq [\sqrt{\pi x} - 2xe^{\frac{jx}{4}}]e^{j\left(\frac{x}{4} + \pi\right)}$$  \hspace{1cm} 2.14

for $x < 0.3$; the large argument form is given by

$$F(x) \simeq 1 + \frac{j}{2x} - \frac{3}{4x^2}$$  \hspace{1cm} 2.15

for $x > 5.5$; and for $0.3 < x < 5.5$, $F(x)$ can be evaluated by linear interpolation using data given in Table 2.1 where

$$F(x) = F(x_n) + A_n(x - x_n);$$  \hspace{1cm} 2.16

when $x < 0$, $F(x) = F^*(|x|)$ where $*$ indicates the complex conjugate.

### 2.2.2 Surface Creeping Wave Mechanism

The creeping wave on a curved surface, as shown in Fig. 2.3, is given by \[2\]

$$\vec{E}_{\text{creep}} = \vec{E}^i(Q_1) \cdot \frac{\rho_2^d}{s(s + \rho_2^d)} e^{-jk\delta}$$  \hspace{1cm} 2.17

where $\vec{E}^i(Q_1)$ is the incident electric field at the attaching point $Q_1$ and the dyadic diffraction coefficients for the creeping wave are given by

$$\overline{T} = T_s \hat{b}_1 \hat{b}_2 + T_h \hat{n}_1 \hat{n}_2$$  \hspace{1cm} 2.18
Figure 2.2. Transition function

\[ F(K) = 2 \left( \frac{1}{K^{2}} \right) \left( \int_{0}^{1} e^{-i\theta} d\theta \right) \]
Table 2.1: Linear interpolation data for $F(x)$

<table>
<thead>
<tr>
<th>$x_n$</th>
<th>$F(x_n)$</th>
<th>$A_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3</td>
<td>0.5729 + j0.2677</td>
<td>0.0000 + j0.0000</td>
</tr>
<tr>
<td>0.5</td>
<td>0.6768 + j0.2682</td>
<td>0.5195 + j0.0025</td>
</tr>
<tr>
<td>0.7</td>
<td>0.7439 + j0.2549</td>
<td>0.3355 + j0.0665</td>
</tr>
<tr>
<td>1.0</td>
<td>0.8095 + j0.2322</td>
<td>0.2187 + j0.0757</td>
</tr>
<tr>
<td>1.5</td>
<td>0.8730 + j0.1982</td>
<td>0.1270 + j0.0680</td>
</tr>
<tr>
<td>2.3</td>
<td>0.9240 + j0.1577</td>
<td>0.0638 + j0.0506</td>
</tr>
<tr>
<td>4.0</td>
<td>0.9658 + j0.1073</td>
<td>0.0246 + j0.0296</td>
</tr>
<tr>
<td>5.5</td>
<td>0.9797 + j0.0828</td>
<td>0.0093 + j0.0163</td>
</tr>
</tbody>
</table>
\[ T_{s,h} = -\sqrt{m(Q_1)m(Q_2)} \sqrt{\frac{2}{k}} \left\{ \frac{e^{-j\xi}}{2\sqrt{\pi} \zeta^d} [1 - F(x^d)] \right\} + \tilde{P}_{s,h}(\zeta^d) \sqrt{\frac{\left[ m(Q_1) \right]}{\frac{\left[ m(Q_2) \right]}{\sqrt{\pi} \zeta^d}}} e^{-jkt} \]  

\[ \zeta^d = \int_Q^2 m(t') \frac{dt'}{\rho_g(t')} \]  

\[ m(Q) = \left[ \frac{k\rho_g(Q)}{2} \right]^{\frac{1}{3}} \]  

\[ \rho_g(Q) = \text{radius of curvature of the surface in the plane of incidence} \]  

\[ x^d = \frac{kL^d(\zeta^d)^2}{2m(Q_1)m(Q_2)} \]  

\[ L^d = s', \text{ in the far zone} \]  

\[ = \frac{ss'}{s + s'} \text{, in the near zone} \]  

where \( s' \) is the distance from the source to the attaching point \( Q_1 \) and \( s \) is the distance from the launching point \( Q_2 \) to the receiver. The unit vectors at \( Q_1 \) and \( Q_2 \) are defined as follows:

\( \hat{n}_{1,2} = \text{surface outward unit normal vector} \)

\( \hat{t}_{1,2} = \text{unit tangent vector to the surface ray at } Q_{1,2} \)

and the unit binormal vector is given by

\[ \hat{b}_{1,2} = \hat{t}_{1,2} \times \hat{n}_{1,2}. \]  

\( t = \text{arc length of geodesic ray path on the surface} \)

\[ \sqrt{\frac{d\varphi(Q_1)}{d\varphi(Q_2)}} = \text{energy spreading factor} \]

\( \rho^d_2 = \text{caustic distance at } Q_2. \)

The definition of the transition function \( F(x) \) is given in the previous section.

The Pekeris' Caret function \( \tilde{P}_{s,h} \) in the diffraction coefficients are defined in terms
Figure 2.3: Creeping wave geometry
of the related functions $p^*(\zeta)$ and $p^*(\zeta)$ are given by Logan [33]

$$\hat{P}_s(\zeta) = [p^*(\zeta) - \frac{1}{2\sqrt{\pi}\zeta}]e^{-j\frac{\pi}{\zeta}},$$ \hspace{1cm} 2.26

$$\hat{P}_h(\zeta) = [q^*(\zeta) - \frac{1}{2\sqrt{\pi}\zeta}]e^{-j\frac{\pi}{\zeta}}.$$ \hspace{1cm} 2.27

The function $p^*$ is shown in Fig. 2.4 and is given by

$$p^*(x) = \frac{1}{2x\sqrt{\pi}} + \hat{P}_s(x)e^{j\frac{\pi}{4}}$$ \hspace{1cm} 2.28

where

$$\hat{P}_s(x) = \frac{e^{-j\frac{\pi}{4}}}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{V(\tau)}{W_2(\tau)}e^{-j\omega \tau}d\tau$$ \hspace{1cm} 2.29

and $V(\tau)$ and $W_2(\tau)$ are Fock type Airy functions. The $p^*$ function is computed as follows:

For the deep lit region; $x \leq -3$

$$p^*(x) = \frac{1}{2x\sqrt{\pi}} + \frac{1}{2}\sqrt{|x|}(1 + jx^2)e^{j\frac{3\pi}{4}}e^{j\frac{\pi}{4}},$$ \hspace{1cm} 2.30

for the linear interpolation region; $-3 < x < 2$

$$p^*(x) = p^*(x_i) + \frac{x-x_i}{x_{i+1}-x_i}[p^*(x_{i+1}) - p^*(x_i)],$$ \hspace{1cm} 2.31

for the deep shadow region; $2 \leq x$

$$p^*(x) = \frac{1}{2x\sqrt{\pi}} - \frac{e^{j\frac{\pi}{8}}}{2\sqrt{\pi}} \sum_{n=1}^{5} \frac{e^{\pi n^2}}{[Ai'(-qn)]^2}$$ \hspace{1cm} 2.32

where $Ai'(t)$ is the derivative of the Miller type Airy function.

The function $q^*$ is shown in Fig. 2.5 and is given by

$$q^*(x) = \frac{1}{2x\sqrt{\pi}} + \hat{P}_h(x)e^{j\frac{\pi}{4}}$$ \hspace{1cm} 2.33
where

\[ \hat{P}_h(x) = \frac{e^{-j\frac{x^2}{4}}}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{QV(\tau)}{QW_2(\tau)} e^{-j\frac{\tau^2}{4}} d\tau \]

and \( V(\tau) \) and \( W_2(\tau) \) are Fock type Airy functions and \( Q = \frac{\partial^2}{\partial \tau^2} \). The \( q^* \) function is computed as follows:

For the deep lit region; \( x \leq -3 \)

\[ q^*(x) = \frac{1}{2x\sqrt{\pi}} - \frac{e^{j\frac{x^2}{6}}}{2\sqrt{\pi}} \sum_{n=1}^{\infty} \frac{e^{\frac{x^2}{4}}}{\sqrt{\pi}} e^{-j\frac{5n}{6}} \]

for the linear interpolation region; \(-3 < x < 2\)

\[ q^*(x) = q^*(x_i) + \frac{x - x_i}{x_{i+1} - x_i} [q^*(x_{i+1}) - q^*(x_i)] \]

for the deep shadow region; \( 2 < x \)

\[ q^*(x) = \frac{1}{2x\sqrt{\pi}} - \frac{e^{j\frac{x^2}{6}}}{2\sqrt{\pi}} \sum_{n=1}^{\infty} \frac{e^{xq_n e^{-j\frac{5n}{6}}}}{\sqrt{\pi} \cdot e^{-j\frac{x^2}{6}}} \]

where \( Ai(t) \) is the Miller type Airy function.

2.2.3 Field radiated from sources on a convex surface

The UTD electric fields in the shadow region of electric and magnetic dipoles located on a convex surface, as shown in Fig. 2.6, are given by [3]

\[ d\bar{E}_{m,e}(P_s) = d\bar{P}_{m,e}(Q') \cdot \bar{T}_{m,e}(Q', Q) \frac{\rho_c}{s + \rho_c} e^{-jks} \]

where \( d\bar{P}_e(Q') \) and \( d\bar{P}_m(Q') \) are the electric dipole and the magnetic dipole at \( Q' \), respectively. The dyadic diffraction coefficients \( \bar{T}_{m,e} \) are given by

\[ \bar{T}_{m,e}(Q', Q) = \frac{-j\kappa}{4\pi} \left[ \hat{\beta}^\dagger \hat{n} T_1(Q') H + \hat{\beta}^\dagger \hat{b} T_2(Q') S + \hat{\beta} \hat{b} T_3(Q') + \hat{\beta} \hat{n} T_4(Q') \right] e^{-jkt} \frac{d\psi_o}{d\eta(Q)} \left[ \frac{\rho_g(Q)}{\rho_g(Q')} \right]^\frac{1}{2} \]

\[ e^{-jkt} \frac{d\psi_o}{d\eta(Q)} \left[ \frac{\rho_g(Q)}{\rho_g(Q')} \right]^\frac{1}{2} \]
Figure 2.4: Plot of $e^{-j\frac{\pi}{4}}p'(\zeta)$ vs $\zeta$ based on Logan's tabulated data [33] for $p(\zeta)$
Figure 2.5: Plot of $e^{-j\frac{\pi}{4}} q^*(\zeta)$ vs $\zeta$ based on Logan's tabulated data [33] for $q(\zeta)$
\[ T_e(Q',Q) = \frac{-jkZ_o}{4\pi} [\hat{n}' \hat{n} T_b(Q') H + \hat{n}' \hat{b} T_b(Q') S] e^{-jkt} \]
\[ \sqrt{\frac{d\psi_o}{dn(Q)} \left[ \frac{\rho_g(Q)}{\rho_g(Q')} \right]^{\frac{1}{2}}} \]

where subscripts \( m \) and \( e \) are corresponding to the magnetic dipole case and the electric dipole case, respectively. Here \((\hat{i}', \hat{n}', \hat{b}')\) and \((\hat{i}, \hat{n}, \hat{b})\) are the tangent, normal, and binormal unit vectors to the surface at the source point \( Q' \) and diffraction point \( Q \), respectively. As seen from Fig. 2.6, \( \hat{i} \times \hat{n} = \hat{b} \) and \( \hat{i}' \times \hat{n}' = \hat{b}' \). The quantities \( T_1(Q'), ..., T_b(Q') \) are the torsion factors at \( Q' \) and are given by Table 2.2.

Also,

\[ H = g(\zeta) \]
\[ S = \frac{-j}{m(Q')} \tilde{g}(\zeta) \]

with

\[ g(\zeta) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-j\zeta t} \frac{e^{-jk/3}}{w_2(t)} dt, \text{ and} \]
\[ \tilde{g}(\zeta) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-j\zeta t} \frac{e^{-jk/3}}{w_2(t)} dt \]

which are known as the acoustic hard and soft Fock functions. The Fock type Airy function is given by

\[ w_2(\tau) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{\tau} e^{i\tau^2} \frac{e^{-j\zeta t}}{w_2(t)} dt \]

and \( w_2'(\tau) \) is the derivative of \( w_2(\tau) \) with respect to \( \tau \). The Fock parameter \( \zeta \) for the shadow region is given by

\[ \zeta = \int_{Q'}^{Q} \frac{m(t')}{\rho_g(t')} dt' \]

with

\[ m(t') = \left[ \frac{k \rho_g(t')}{2} \right]^{\frac{1}{3}}. \]
Figure 2.6: Surface diffracted ray tube and ray coordinates for the shadow region
Here $\rho_g(t')$ is the surface radius of curvature along the ray path at $t'$. The width of surface ray tube at $Q$, $d\eta(Q)$, is given by

$$d\eta(Q) = \rho_c d\psi$$  \hspace{1cm} 2.48

where the $\rho_c$ is the caustic distance at $Q$. The parameters $Z_0$ and $t$ are defined as the free space wave impedance and geodesic arc length from $Q'$ to $Q$, respectively.

Combining the above equations, the $\hat{n}$ and $\hat{b}$ directed components of $d\vec{E}_{m,e}(P_s)$ are given by [3]

(a) $d\vec{P}_m(Q')$ case:

$$d\vec{E}_m^n(P_s) = \frac{-jk}{4\pi} (d\vec{P}_m \cdot \hat{b}') H e^{-jkt} \left[ \frac{\rho_g(Q)}{\rho_g(Q')} \right]^{-\frac{1}{8}} \sqrt{\frac{d\psi_o}{d\psi}} \sqrt{\frac{1}{s(s + \rho_c)}} e^{-jks} + O[m^{-2}]$$  \hspace{1cm} 2.49

$$d\vec{E}_m^b(P_s) = \frac{-jk}{4\pi} [(d\vec{P}_m \cdot \hat{b}') T_0 S + (d\vec{P}_m \cdot \hat{t}') S] e^{-jkt} \left[ \frac{\rho_g(Q)}{\rho_g(Q')} \right]^{-\frac{1}{8}} \sqrt{\frac{d\psi_o}{d\psi}} \sqrt{\frac{1}{s(s + \rho_c)}} e^{-jks} + O[m^{-2}, m^{-3}]$$  \hspace{1cm} 2.50

where superscripts $n$ and $b$ are corresponding to the normal and binormal components, respectively.

(b) $d\vec{P}_e(Q')$ case:

$$d\vec{E}_e^n(P_s) = \frac{-jk Z_0}{4\pi} d\vec{P}_e(Q') H e^{-jkt} \left[ \frac{\rho_g(Q)}{\rho_g(Q')} \right]^{-\frac{1}{8}} \sqrt{\frac{d\psi_o}{d\psi}} \sqrt{\frac{1}{s(s + \rho_c)}} e^{-jks} + O[m^{-2}]$$  \hspace{1cm} 2.51

$$d\vec{E}_e^b(P_s) = \frac{-jk Z_0}{4\pi} d\vec{P}_e(Q') T_0 S e^{-jkt} \left[ \frac{\rho_g(Q)}{\rho_g(Q')} \right]^{-\frac{1}{8}} \sqrt{\frac{d\psi_o}{d\psi}} \sqrt{\frac{1}{s(s + \rho_c)}} e^{-jks} + O[m^{-2}]$$  \hspace{1cm} 2.52

where $T_0 = T(Q')\rho_g(Q')$ with $T(Q')$ being the surface torsion at the source location (refer to Table 2.2).
Table 2.2: Torsion factors for a source on a convex surface

<table>
<thead>
<tr>
<th>TYPES OF CONVEX SURFACE</th>
<th>SLOT OR $dF_m$ CASE</th>
<th>MONOPOLE OR $dF_e$ CASE</th>
<th>SURFACE RAY TORSION</th>
<th>SURFACE RADIUS OF CURVATURE IN $\gamma$ DIRECTION</th>
<th>DIFFRACTED RAY CAUSTIC DISTANCE</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$T_1(Q')$</td>
<td>$T_2(Q')$</td>
<td>$T_3(Q')$</td>
<td>$T_4(Q')$</td>
<td>$T_5(Q')$</td>
</tr>
<tr>
<td>SPHERE</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>CIRCULAR CYLINDER</td>
<td>1</td>
<td>1</td>
<td>$\sin \frac{E}{R} \cdot \cos \theta$</td>
<td>$\cos \frac{E}{R}$</td>
<td>0</td>
</tr>
<tr>
<td>ARBITRARY CONVEX SURFACE</td>
<td>1</td>
<td>1</td>
<td>$T_{10}' \mu(Q')$</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Note: (1) $\alpha'$ is defined by $\hat{\tau}_1 \cdot \hat{t}' = \cos \alpha'$ where $\hat{\tau}_1$ is the principal direction unit vector associated with $R_1(Q')$.

(2) The quantities $E$ and $G$ denote two of the three coefficients $E, F, G$ that appear in the "first fundamental form" of Differential Geometry.
2.3 Differential Geometry of a Cone

2.3.1 Differential Geometry of a Cone

To apply the UTD to electromagnetic scattering problems, the use of differential geometry[34] is necessary. The surface of a right circular cone, as shown in Fig. 2.7, is defined in vector notation by

$$\vec{R} = \hat{x}R_s \sin \alpha \cos \phi + \hat{y}R_s \sin \alpha \sin \phi + \hat{z}R_s \cos \alpha$$

where $\alpha$ is the half cone angle and $(R_s, \phi)$ are the two parameters defining the cone surface.

Consider a ray which propagates along a geodesic $Q_1Q_2$ on a cone, as shown in Fig. 2.8. At any point along the geodesic path, the three unit vectors $\hat{t}, \hat{n}$, and $\hat{b}$ are the unit geodesic tangent, the surface outward unit normal, and the unit binormal vectors. The unit surface outward normal vector is obtained from

$$\hat{n} = \frac{\vec{R}_\phi \times \vec{R}_R_s}{|\vec{R}_\phi \times \vec{R}_R_s|}$$

where

$$\vec{R}_\phi = \frac{\partial \vec{R}}{\partial \phi} = R_s \sin \alpha(-\hat{x} \sin \phi + \hat{y} \cos \phi),$$

$$\vec{R}_R_s = \frac{\partial \vec{R}}{\partial R_s} = \hat{x} \sin \alpha \cos \phi + \hat{y} \sin \alpha \sin \phi + \hat{z} \cos \alpha.$$ 

Therefore, the unit normal vector is given by

$$\hat{n} = \hat{x} \cos \alpha \cos \phi + \hat{y} \cos \alpha \sin \phi - \hat{z} \sin \alpha.$$ 

Let's introduce the mean and gaussian curvatures defined by:

mean curvature;
Figure 2.7: Geometry of a cone
Figure 2.8: Ray path on a cone
\[ K_M = \frac{K_1 + K_2}{2} = \frac{Eg - 2fF + eG}{2(EG - F^2)}, \]

Gaussian curvature:

\[ KG = K_1 K_2 = \frac{eg - f^2}{EG - F^2} \]

where \( K_1, K_2 \) are curvatures in the two principal directions. Unit vectors in the two principal directions are given by

\[ \hat{\mathbf{r}}_1 = \hat{x} \sin \alpha \cos \phi + \hat{y} \sin \alpha \sin \phi + \hat{z} \cos \alpha, \]

\[ \hat{\mathbf{r}}_2 = -\hat{x} \sin \phi + \hat{y} \cos \phi. \]

The first and the second fundamental forms used in the curvature notation are defined by

\[ E = \frac{\partial \hat{R}}{\partial R_s} \cdot \frac{\partial \hat{R}}{\partial R_s} = 1, \]

\[ F = \frac{\partial \hat{R}}{\partial R_s} \cdot \frac{\partial \hat{R}}{\partial \phi} = 0, \]

\[ G = \frac{\partial \hat{R}}{\partial \phi} \cdot \frac{\partial \hat{R}}{\partial \phi} = R_s^2 \sin^2 \alpha, \]

\[ e = \frac{\partial^2 \hat{R}}{\partial R_s^2} \cdot \hat{n} = 0, \]

\[ f = \frac{\partial^2 \hat{R}}{\partial R_s \partial \phi} \cdot \hat{n} = 0, \]

\[ g = \frac{\partial^2 \hat{R}}{\partial \phi^2} \cdot \hat{n} = -R_s \sin \alpha \cos \alpha. \]

Substituting the above quantities into the mean and Gaussian curvature notations, they are obtained by

\[ K_M = \frac{-\cos \alpha}{2R_s \sin \alpha}, \]

\[ K_G = 0. \]
Therefore, the principal curvatures $K_1$ and $K_2$ in $\hat{\tau}_1$ and $\hat{\tau}_2$ are given by

$$K_1 = K_M + \sqrt{K_M^2 - K_G} = 0,$$  \hspace{1cm} 2.70

$$K_2 = K_M - \sqrt{K_M^2 - K_G} = \frac{-1}{R_3 \tan \alpha}.$$  \hspace{1cm} 2.71

The incident direction unit vector $\hat{s}^i$ can be decomposed as

$$\hat{s}^i = \hat{n}_1(\hat{s}^i \cdot \hat{n}_1) + \hat{t}_1(\hat{s}^i \cdot \hat{t}_1)$$
$$= \cos \theta_1 \hat{n}_1 + \sin \theta_1 \hat{t}_1$$  \hspace{1cm} 2.72

where $\theta_1$ is the angle between the incident unit vector $\hat{s}^i$ and unit normal vector $\hat{n}_1$ at $Q_1$. Let the angle between the geodesic tangent unit vector at $Q_1$ and the principal direction unit vector $\tau_1$ be $\alpha'$. The scalar products of the incident unit vector $\hat{s}^i$ and the two principal direction unit vectors yield

$$\hat{s}^i \cdot \hat{\tau}_1 = \cos \theta_1 (\hat{n}_1 \cdot \hat{\tau}_1) + \sin \theta_1 (\hat{t}_1 \cdot \hat{\tau}_1),$$  \hspace{1cm} 2.73

$$\hat{s}^i \cdot \hat{\tau}_2 = \cos \theta_1 (\hat{n}_1 \cdot \hat{\tau}_2) + \sin \theta_1 (\hat{t}_1 \cdot \hat{\tau}_2).$$  \hspace{1cm} 2.74

Since the unit normal vector at $Q_1$ is perpendicular to the two principal direction unit vectors, the angle $\alpha'$ between $\hat{t}_1$ and $\hat{\tau}_1$ is obtained by

$$\alpha' = \tan^{-1} \left[ \frac{\hat{s}^i \cdot \hat{\tau}_2}{\hat{s}^i \cdot \hat{\tau}_1} \right].$$  \hspace{1cm} 2.75

Hence, the geodesic tangent unit vector at $Q_1$ is represented in terms of the two principal direction unit vectors by

$$\hat{t}_1 = \hat{\tau}_1 \cos \alpha' + \hat{\tau}_2 \sin \alpha'$$
$$= \hat{x}(\sin \alpha \cos \phi \cos \alpha' - \sin \alpha' \sin \phi) + \hat{y}(\sin \alpha \sin \phi \cos \alpha' + \sin \alpha' \cos \phi) + \hat{z} \cos \alpha \cos \alpha'.$$  \hspace{1cm} 2.76
The unit binormal vector at $Q_1$ is then obtained by

$$\hat{b}_1 = \hat{i}_1 \times \hat{n}_1$$

$$= -\hat{z}(\sin \phi \cos \alpha' + \sin \alpha \sin \alpha' \cos \phi) + \hat{y}(\cos \phi \cos \alpha' - \sin \alpha \sin \alpha' \sin \phi)$$

$$- \hat{z} \cos \alpha \sin \alpha'$$

From Euler's theorem, the radius of curvature along the geodesic path is specified by

$$\rho_g = \frac{1}{|K_1 \cos^2 \alpha' + K_2 \sin^2 \alpha'|} = \frac{R_s \tan \alpha}{\sin^2 \alpha'}$$

where $\alpha'$ is defined by equation 2.75. It is given by

$$\sin^2 \alpha' = \cos^2[\sin \alpha(\phi - \phi_o)]$$

where $\phi_o$ is a constant for the geodesic on a cone which is described in the following section.

### 2.3.2 Geodesic on a Cone

The geodesic path on a cone must satisfy the following differential equations [34];

$$\frac{d^2 \phi}{dt^2} + 2 \frac{d \phi}{dt} \frac{dz}{dt} = 0$$

$$\frac{d^2 z}{dt^2} - \frac{z \tan^2 \alpha}{1 + \tan^2 \alpha} \left(\frac{d \phi}{dt}\right)^2 = 0.$$ 

The differential arc length $dt$ satisfies

$$dt^2 = z^2 \tan^2 \alpha d\phi^2 + (1 + \tan^2 \alpha) dz^2.$$ 

Solving the above three differential equations, one obtains the equation for the geodesic path on a cone as

$$\sec|\sin \alpha(\phi - \phi_o)| = \frac{A_0 z}{\tan \alpha}$$

34
where $\phi$ and $z$ are the coordinate of the point on the conical surface and $\phi_0$ and $A_0$ are constants to be determined. They are uniquely determined by knowing the starting location on a cone surface and the initial geodesic tangent unit vector.

For the geometry given by Fig. 2.8,

\[
\phi_0 = \phi_1 - \frac{1}{\sin \alpha} \cos^{-1}\left[\frac{\tan \alpha}{A_0 z_1}\right], \quad 2.84
\]

\[
A_0 = \frac{\sin \alpha}{\sqrt{\cos^2 \alpha - \cos^2 \theta_i z_1}} \quad 2.85
\]

where parameters used are defined as follows:

$\alpha =$ half cone angle

$\theta_i =$ the angle of incidence

$\phi_1 =$ the azimuthal angle of $Q_1$

$z_1 =$ the $z$-coordinate of $Q_1$.

The geodesic arc length on a cone surface is obtained by solving the differential equation 2.82 as

\[
t = \frac{\tan^2 \alpha}{A_0 \sin \alpha} \left\{\tan[(\phi_2 - \phi_0) \sin \alpha] - \tan[(\phi_1 - \phi_0) \sin \alpha]\right\} \quad 2.86
\]

where $\phi_{1,2}$ represent the $\phi$ coordinates of $Q_1$ and $Q_2$, respectively. Also, the Fock parameter defined in the UTD field expression section is given by

\[
\zeta = \int_{Q_1}^{Q_2} \frac{m(t)}{\rho g(t)} dt = \left(\frac{k}{2}\right)^{\frac{1}{3}} \int_{Q_1}^{Q_2} [\rho g(t)]^{-\frac{2}{3}} dt. \quad 2.87
\]

Plugging equation 2.78 into the above equation yields

\[
\zeta = \left(\frac{k}{2}\right)^{\frac{1}{3}} (\tan \alpha)^{-\frac{2}{3}} \int_{Q_1}^{Q_2} \left[\frac{\sin^2 \alpha'}{R_s}\right]^{\frac{2}{3}} dt
\]

\[
= \frac{k}{2} R_s \sin \alpha \cos \omega_1^{\frac{1}{3}} [\phi_2 - \phi_1] \cos^{\frac{2}{3}} \alpha \quad 2.88
\]
where

$$w_1 = \sin^{-1}\left[-\frac{\cos \theta_i}{\cos \alpha}\right]$$

and $R_s$ is the distance from the origin to $Q_1$. 
2.4 Coordinate Transformation on the Spherical Base

When a cone-sphere is illuminated by a plane wave with the incident angle at $\theta_s$, the surface is shadowed as shown in Fig. 2.9. The original coordinate system $(x, y, z)$ is with respect to 0 and the primed coordinate system $(x', y', z')$ is with respect to $0'$. As one can see in the later chapter on the backscattering from a cone-sphere, it is desirable to make a coordinate transformation. The reason is that the equivalent current for the creeping wave is placed along the shadow boundary on the spherical base and the radiation integration is performed along the tilted ring. Therefore, the integration is very difficult if the original coordinate system is used. The $x'$ and $z'$ are chosen such that the $x'$ axis is perpendicular to the incident ray direction $s^\text{i}$ and $z'$ axis coincides with the incident ray direction $s^\text{i}$.

Let the rectangular coordinate of $Q$ on the spherical base be $(x, y, z)$. Then the coordinate of $Q$ with respect to the primed coordinate system $(x', y', z')$ can be obtained by

$$x' = (\hat{x} \cdot \hat{x}')x + (\hat{y} \cdot \hat{x}')y + (\hat{z} \cdot \hat{x}')(z - \frac{a}{\cos \alpha \sin \alpha})$$  \hspace{1cm} 2.90

$$y' = (\hat{x} \cdot \hat{y}')x + (\hat{y} \cdot \hat{y}')y + (\hat{z} \cdot \hat{y}')(z - \frac{a}{\cos \alpha \sin \alpha})$$  \hspace{1cm} 2.91

$$z' = (\hat{x} \cdot \hat{z}')x + (\hat{y} \cdot \hat{z}')y + (\hat{z} \cdot \hat{z}')(z - \frac{a}{\cos \alpha \sin \alpha}).$$  \hspace{1cm} 2.92

The dot products in equations 2.90 to 2.92 are given by

$$\hat{x} \cdot \hat{x}' = -\cos \theta_s, \hspace{1cm} 2.93$$

$$\hat{y} \cdot \hat{x}' = 0, \hspace{1cm} 2.94$$

$$\hat{z} \cdot \hat{x}' = \sin \theta_s, \hspace{1cm} 2.95$$

$$\hat{x} \cdot \hat{y}' = 0, \hspace{1cm} 2.96$$

$$\hat{y} \cdot \hat{y}' = 1, \hspace{1cm} 2.97$$

37
Figure 2.9: Geometry for the coordinate transformation on the spherical base of a cone-sphere
\[ \hat{z} \cdot \hat{y}' = 0, \]  
\[ \hat{x} \cdot \hat{z}' = -\sin \theta_s, \]  
\[ \hat{y} \cdot \hat{z}' = 0, \]  
\[ \hat{z} \cdot \hat{z}' = -\cos \theta_s. \]

Substituting the above equations into equations 2.90 to 2.92 yields

\[ x' = -x \cos \theta_s + z \sin \theta_s - \frac{a}{\cos \alpha \sin \alpha} \sin \theta_s, \]
\[ y' = y, \]
\[ z' = -x \sin \theta_s - z \cos \theta_s + \frac{a}{\cos \alpha \sin \alpha} \cos \theta_s \]

where \( \alpha \) is the half cone angle and \( \theta_s \) is the angle of incidence.
2.5 Equivalent Current Concept

The equivalent current concept was developed by Ryan and Peters [4,5] to obtain a valid solution for fields at or near caustics. For a cone, the singly and the doubly diffracted field from the edge form a caustic along the axial direction so that the finite number of rays does not predict the infinity of rays merging to one direction. The basic idea is that the equivalent currents can be obtained by matching the field radiated by the current filament along the edge with the field using the diffraction theory. After obtaining the equivalent currents, the finite edge scattered field is calculated by using a free space Green's function and integrating over the length of the edge.

From [4,5], the equivalent electric and magnetic currents are given by

\[
I_e(\ell) = -\frac{8\pi}{jk} \frac{\bar{E}^i(Q_E) \cdot \hat{e}}{Z_0 \sin \beta_o} D_s, \tag{2.105}
\]

\[
I_m(\ell) = -\frac{8\pi}{jk} \frac{\bar{H}^i(Q_E) \cdot \hat{e}}{Y_0 \sin \beta_o} D_h, \tag{2.106}
\]

where \(D_s,h\) are the diffraction coefficients for 2-D wedge, \(\hat{e}\) is the unit vector in the edge direction, \(\bar{E}^i(Q_E)\) and \(\bar{H}^i(Q_E)\) are the incident electric and magnetic field at the edge, respectively, \(Z_0, Y_0\) are the free space impedance and admittance, respectively, and \(k\) is the free space wave number. The angle \(\beta_o\) is determined by

\[
\beta_o = \cos^{-1}(\hat{s}^i \cdot \hat{e}) \tag{2.107}
\]

where \(\hat{s}^i\) is the incident unit vector. An example of the equivalent current concept is depicted in Fig. 2.10 for a finite cone structure.
Figure 2.10: (a) A partial cone geometry and its associated equivalent currents on the rim, and (b) The equivalent currents of the partial cone radiating in free space
2.6 Diffraction Coefficients for the Discontinuity in Curvature

The diffraction coefficients for the surface with discontinuity in curvature are given by Senior [9]

\[ D_{s,h} = \frac{e^{-j\frac{\pi}{4}}}{\sqrt{2\pi k}} (X \mp Y) \]  

2.108

where

\[ X = j \frac{a_1 - a_2}{2k} \frac{1 + \cos(\alpha + \theta)}{(\cos \alpha + \cos \theta)^3} \]  

2.109

\[ Y = j \frac{a_1 - a_2}{2k} \frac{1 + \cos(\alpha - \theta)}{(\cos \alpha + \cos \theta)^3} \]  

2.110

\( a_1 \) and \( a_2 \) are curvatures of the two surfaces at the discontinuity and the \( \alpha \) and \( \theta \) are angles defined in Fig. 2.11. The field diffracted from the curvature discontinuity can be obtained by replacing the above diffraction coefficients to the coefficients \( D_{s,h} \) of the edge diffracted field. The diffraction coefficients for the surface with curvature discontinuity given by equations 2.109 and 2.110 can be used to calculate
the diffrated field from the junction (i.e., curvature discontinuity) of a cone-sphere. This also could be done by using the GTD approach presented by Chu [27].
CHAPTER III

RCS OF A PERFECTLY CONDUCTING FINITE CONE

3.1 Backscattering RCS of a Cone

3.1.1 Introduction

In this section, the backscattering RCS of a perfectly conducting finite cone in the near axial region for the E- and H- plane are investigated. The conducting right circular cone of a half cone angle $\alpha$ and the base radius $a$ is illuminated by the plane wave incident at an angle $\theta_i$. The geometry of the problem under consideration is given in Fig. 3.1.

Much work has been done on the finite cone. Keller[18] used the GTD to a finite cone to calculate the backscattering field. He obtained the solution including the single and the double diffraction from the edge in the axial direction. However, no solution is available near the caustic region. Bechtel[20] applied the GTD to calculate the backscattering of a finite cone including the singly and the doubly diffracted field at the rim of the cone base. However, he did not account for the caustic which can be computed by using the equivalent current concept[4,5]. This caustic correction was first included by Ryan and Peters[4,5] and expanded to include double diffraction by Burnside and Peters[21]. In that, they also pointed out that the extra ray mechanisms (i.e., tip-rim interaction and rim-tip interaction) are required to improve the axial radar cross section. This has since been shown to be the case by Trott[29]. Recently, Wang and Mitschang[28] calculated the
\( \alpha; \) HALF CONE ANGLE
\( a; \) BASE RADIUS

Figure 3.1: Geometry of a cone under consideration
backscattering radar cross section of a finite circular and an elliptic cones based on the physical theory of diffraction (PTD).

All the earlier works ignored the creeping wave effects when the cone is partially illuminated by the incident plane wave. In this chapter, the creeping wave effect associated with the conical surface is incorporated into the equivalent current concept\cite{4,5} based on the uniform geometrical theory of diffraction (UTD\cite{1,3}). The ray mechanisms included are illustrated in Fig. 3.2. It is found that the creeping wave contributes significantly to the backscattering when the conical surface is partially illuminated by the incident field.

The present equivalent current approach is a straightforward procedure to implement for the study of backscattering from a cone. Furthermore, the contributions to the backscattering due to the each individual ray mechanism can be clearly identified. In the following sections, the equivalent current concept is presented systematically for the singly diffracted field, the doubly diffracted field, and the creeping waves. The present result shows good agreement when comparing to the measured data and the method of moments.

It has been demonstrated\cite{4,5} that the equivalent current technique is a more straightforward procedure to implement for the treatment of the scattered fields in the vicinity of the caustic. The basic assumption of this equivalent current method is that an equivalent current filament radiates in the scattering direction. In fact, only a finite number of points along the rim satisfy the generalized Fermat's principle. It can be shown that the stationary phase contribution of this radiation integral reduces to the GTD solution which can be represented as the sum of a finite number of individual rays. However, unlike the GTD solution, the equivalent current method yields a valid solution at or near the caustic.

The far field radiated by these equivalent electric and magnetic currents is
Figure 3.2: Diffracted field components included in the calculation of backscattering RCS of a cone
given by
\[
\bar{E} = 2 \frac{j k a Z_o e^{-j k R}}{4 \pi R} e^{-j 2 k a \frac{\cos \theta}{\tan \alpha}} \int_{\phi_1}^{\phi_2} [\hat{R} \times \hat{R} \times \hat{I}_e + Y_0 \hat{R} \times \hat{I}_m]_{\text{rim}} e^{j k \hat{R}' \cdot \hat{R} d \phi'} \tag{3.1}
\]
where \( R \) is the radial distance, \( \hat{R} \) is the unit vector in the radiation direction, \( \hat{R}' \) is the vector from the center of the cone base to the line element of equivalent current, \( a \) is the base radius, the \( k \) is the wave number in free space, \( Z_0 \) is the free space impedance, and \( Y_0 = \frac{1}{Z_0} \). The two vectors \( \hat{I}_e \) and \( \hat{I}_m \) are the equivalent electric and the equivalent magnetic currents, respectively. The integration is performed along the half rim of a cone base due to the symmetry property of a cone. The lower limit and the upper limit depend on the scattering angle.

3.1.2 Equivalent Currents for the Singly Diffracted Field \((E_D)\)

The equivalent electric and magnetic edge currents \( \bar{I}_e^1 \) and \( \bar{I}_m^1 \) for the singly diffracted field are positioned on the rim in the edge vector direction as illustrated in Fig. 3.3. They are given by [4,5]

\[
\bar{I}_e^1(\phi') = -Y_0 \sqrt{\frac{8 \pi}{j k}} \frac{D_s(\psi, \psi'; \beta_o^l)}{\sin \beta_o} [\bar{E}^i(Q_e) \cdot \hat{\phi'}]_{\text{rim}} \hat{\phi'}^l, \tag{3.2}
\]

and

\[
\bar{I}_m^1(\phi') = -Z_0 \sqrt{\frac{8 \pi}{j k}} \frac{D_h(\psi, \psi'; \beta_o^l)}{\sin \beta_o} [\bar{H}^i(Q_e) \cdot \hat{\phi'}]_{\text{rim}} \hat{\phi'}^l \tag{3.3}
\]

where \( \bar{E}^i(Q_e) \) and \( \bar{H}^i(Q_e) \) are the incident electric and magnetic fields at \( Q_e \) along the rim, respectively. The unit edge vector is given by

\[
\hat{\phi'} = -\hat{x} \sin \phi' + \hat{y} \cos \phi'. \tag{3.4}
\]

The quantities \( \beta_o \) and \( \beta_o^l \) are obtained from

\[
\beta_o = \cos^{-1}(\hat{s}^d \cdot \hat{\phi'}), \quad \text{and} \tag{3.5}
\]

\[
\beta_o^l = \cos^{-1}(\hat{s}^i \cdot \hat{\phi'}). \tag{3.6}
\]
Figure 3.3: Geometry for the singly diffracted field at the rim
Therefore,

\[
\sin \beta_0 = \sin \beta'_0 = \sqrt{1 - \sin^2 \theta_i \sin^2 \phi'}.
\]

3.7

The angles \( \psi \) and \( \psi' \) in the diffraction coefficients are given by

\[
\psi = \psi' = \begin{cases} 
\pi + \tan^{-1} \left| \frac{\cos \theta_i}{\sin \theta_i \cos \phi'} \right| & \text{if } 0 < \phi' < \frac{\pi}{2} \\
2\pi - \tan^{-1} \left| \frac{\cos \theta_i}{\sin \theta_i \cos \phi'} \right| & \text{otherwise}.
\end{cases}
\]

The diffraction coefficients \( D_{s,h} \) are given by

\[
D_{s,h}(\psi, \psi'; \beta''_0) = \frac{e^{-j\frac{\pi}{4}}}{{\sqrt{\pi}}} \frac{1}{\sin \beta'_0} \left\{ \frac{1}{\cos \frac{\pi}{n} - \cos \frac{\psi - \psi'}{n}} + \frac{1}{\cos \frac{\pi}{n} - \cos \frac{\psi - \psi'}{n}} \right\}
\]

where the negative sign is used for the soft case \( (D_s) \) and the positive sign is used for the hard case \( (D_h) \), and

\[
n = \frac{3}{2} + \frac{\alpha}{\pi}.
\]

3.9

Quantities \( \psi, \psi', \beta_0, \) and \( \beta'_0 \) in Equation 3.8 are well defined in [1]. Unit vectors \( \hat{\phi}', \hat{\psi}', \hat{s}', \) and \( \hat{\beta}_0 \) form the edge fixed coordinate system of diffracted ray and the edge fixed coordinate system of the incident ray consists of the unit vectors \( \hat{\phi}', \hat{\psi}', \hat{s}', \) and \( \hat{\beta}_0 \).

Once these equivalent currents are obtained, the singly diffracted field can be derived from the radiation integral given by Equation 3.1. The integration limits of the far field radiation integral are \( \phi_1 = 0 \) and \( \phi_2 = \pi \) for the fully illuminated case and \( \phi_1 = 0 \) for the partially illuminated case. The upper limit \( \phi_2 \) of the radiation integral for partially illuminated case is obtained from the fact that the scattering direction unit vector is orthogonal to the surface outward normal vector. From this fact, it is given by

\[
\phi_2 = \pi - \cos^{-1} \left| \frac{\tan \alpha}{\tan \theta_i} \right|.
\]

3.10

50
Vector cross products in Equation 3.1 are given by

\[ \mathbf{R} \times \mathbf{\hat{f}}' = -\hat{\theta} \cos \phi' - \hat{\phi} \cos \theta \sin \phi', \quad \text{and} \]

\[ \mathbf{R} \times \mathbf{R} \times \mathbf{\hat{f}}' = \hat{\theta} \cos \theta \sin \phi' - \hat{\phi} \cos \phi'. \]

Fig. 3.4 presents the backscattering RCS of a finite cone due to a plane wave incidence. The solid line is the singly diffracted fields using the equivalent currents and the dashed line is the backscattering RCS obtained by the method of moments developed by Chuang[35]. It can be seen that the singly diffracted field alone does not yield satisfactory results for \( \sigma_{\phi\phi} \) when the scattering angle is greater than 10°. However, the singly diffracted field for \( \sigma_{\theta\theta} \) shows good agreement comparing to that of the method of moments.

3.1.3 Equivalent Current for the Doubly Diffracted Field (\( E_{DD} \))

The doubly diffracted field, which is due to rays diffracted across the cone base to the opposite edge and then diffracted toward the source, can also be obtained by the equivalent magnetic current. The ray path for this mechanism is illustrated in Fig. 3.5. The equivalent magnetic current \( \mathbf{I}_m^2 \) for the doubly diffracted field is given by

\[ \mathbf{I}_m^2(v') = -Z_o \sqrt{\frac{8\pi}{jk}} \frac{1}{12} D_h(\psi_2, \psi'_2 = 0; \beta'_{02}) \left[ \mathbf{H}_{12}^i \cdot \mathbf{v}' \right]_{rim} \left[ \mathbf{v}' \frac{dv'}{d\phi'} \right] \]

and the second order equivalent electric current is insignificant and ignored here. The factor \( \left( \frac{dv'}{d\phi'} \right) \) is included in Equation 3.13 as the result of changing integration parameter from \( v' \) to \( \phi' \). Note that the factor \( \frac{1}{12} \) in front of \( D_h \) is introduced to account for the grazing incidence at \( Q_2 \) after diffracting at \( Q_1 \). The quantities \( \psi_2, \psi'_2, \beta_{02} \), and \( \beta'_{02} \) are evaluated at \( Q_2 \) and given by \( \psi'_2 = 0 \).
Figure 3.4: Comparison of the backscattering RCS of a cone with $\alpha = 15^\circ$ and $a = 1.435\lambda$: equivalent current method ($E_D$) (solid line), method of moments (dashed line). (a) vertical polarization (b) horizontal polarization.
\[ \psi_2 = \begin{cases} \pi + \tan^{-1} \left| \frac{\cos \theta_i}{\sin \theta_i \cos \varphi} \right| & \text{if } 0 < \varphi' < \frac{\pi}{2} \\ 2\pi - \tan^{-1} \left| \frac{\cos \theta_i}{\sin \theta_i \cos \varphi} \right| & \text{otherwise,} \end{cases} \]

\[ \sin \beta_{02} = \sqrt{1 - \sin^2 \theta_i \sin^2 \varphi'}, \quad \text{3.14} \]

\[ \sin \beta_{02}' = \sqrt{1 - \sin^2 \theta_i \sin^2 \phi'}. \quad \text{3.15} \]

The diffraction coefficients are the same as those in the singly diffracted field.

If we decompose \( \vec{H}_{12}^i \), which is the magnetic field at point \( Q_2 \) after diffracting from \( Q_1 \), into \( \hat{\beta}_{01} \) and \( \psi_1 \) components, \( \hat{\psi}_1 \cdot \hat{\varphi}' = 0 \). Hence, only the \( \hat{\beta}_{01} \) component of \( \vec{H}_{12}^i \) is necessary to calculate the equivalent magnetic current \( I_{m}^2 \). The unit vectors

\[ \hat{\varphi}' = -\hat{x} \sin \varphi' + \hat{y} \sin \varphi', \quad \text{3.16} \]

\[ \hat{\beta}_{01} = \hat{x} \sin \beta - \hat{y} \cos \beta \quad \text{3.17} \]

where

\[ \beta = \pi - \phi' - \sin^{-1}[\sin \theta_i \sin \phi']. \quad \text{3.18} \]

Therefore, the term in the bracket of Equation 3.13 becomes

\[ \vec{H}_{12}^i \cdot \hat{\varphi}' = H_{\beta_{01}}^i \hat{\beta}_{01} \cdot \hat{\varphi}' = -H_{\beta_{01}}^i \cos(\beta - \varphi') \quad \text{3.19} \]

where

\[ H_{\beta_{01}}^i = -H_{\beta_{01}}', D_h(\psi_1 = 0, \psi'_1; \beta_{01}') \sqrt{\frac{p}{s(s + \rho)}} e^{-jks}. \quad \text{3.20} \]

The quantities \( \psi_1, \psi'_1, \beta_{01}, \) and \( \beta_{01}' \) are evaluated at \( Q_1 \) and given by \( \psi_1 = 0, \)

\[ \psi'_1 = \begin{cases} \pi + \tan^{-1} \left| \frac{\cos \theta_i}{\sin \theta_i \cos \phi'} \right| & \text{if } 0 < \phi' < \frac{\pi}{2} \\ 2\pi - \tan^{-1} \left| \frac{\cos \theta_i}{\sin \theta_i \cos \phi'} \right| & \text{otherwise,} \end{cases} \]

53
Figure 3.5: Geometry for the doubly diffracted field at the rim
\[
\sin \beta_{01} = \sin \beta_{01}' = \sqrt{1 - \sin^2 \theta_i \sin^2 \phi'}.
\]  

3.21

\( \rho \) is the caustic distance at \( Q_1 \), \( s \) is the distance between \( Q_1 \) and \( Q_2 \). The caustic distance \( \rho \) at \( Q_1 \) is obtained from

\[
\frac{1}{\rho} = \frac{1}{\rho_e^i} - \frac{\hat{n}_e \cdot (\hat{s}^i - \hat{d})}{a_e \sin^2 \beta_0}.
\]

3.22

Since

\[
\rho_e^i = \infty,
\]

3.23

\[
a_e = a,
\]

3.24

\[
\hat{n}_e = \hat{x} \cos \phi' + \hat{y} \sin \phi',
\]

3.25

\[
\hat{d} = \hat{x} \cos \beta + \hat{y} \sin \beta,
\]

3.26

\[
\hat{s}^i = -\hat{x} \sin \theta_i - \hat{z} \cos \theta_i
\]

3.27

for a cone with the plane wave incidence case,

\[
\rho = \frac{a \sin^2 \beta_{01}}{\sin \theta_i \cos \phi' - \sin \beta_{01}}.
\]

3.28

The distance between the two diffraction points \( Q_1 \) and \( Q_2 \) is given by

\[
s = a \sqrt{2 - 2 \cos(\phi' - \nu')} = 2a \sin \beta_{01}.
\]

3.29

Since

\[
\beta_{01}' = \frac{1}{\sqrt{\cos^2 \theta_i + \sin^2 \theta_i \cos^2 \phi'}} [\hat{x} \cos^2 \theta_i \sin \phi' + \hat{y} \cos \phi' - \hat{z} \cos \theta_i \sin \theta_i \sin \phi'],
\]

3.30

\( H_{\beta_{01}} \) is obtained by \( \hat{H}^i \cdot \hat{\beta}_{01} \) as

\[
H_{\beta_{01}} = \frac{e^{jka \sin \theta_i \cos \phi'}}{Z_o} \left[ \frac{\cos \phi' E_{\theta}^i(Q_1)}{\sqrt{\cos^2 \theta_i + \sin^2 \theta_i \cos^2 \phi'}} + \frac{-\cos \theta_i \sin \phi' E_{\phi}^i(Q_1)}{\sqrt{\cos^2 \theta_i + \sin^2 \theta_i \cos^2 \phi'}} \right].
\]

3.31
$E_\theta^i$ and $E_\phi^i$ are the $\hat{\theta}$ and $\hat{\phi}$ components of the incident electric field at $Q_1$, respectively. The $\phi'$ and $v'$ are the azimuth angle in cylindrical coordinates of the first diffraction point ($Q_1$) and the second diffraction point ($Q_2$), respectively. Applying the law of diffraction at $Q_1$ (i.e., $\hat{s} \cdot \hat{\phi}' = \hat{d} \cdot \hat{\phi}'$), the relationship between the $v'$ and $\phi'$ is given by

$$v' = \pi + \phi' - 2 \sin^{-1}[\sin\theta \sin \phi']$$

and illustrated in Fig. 3.6. The solution of $\phi'$ in terms of $v'$ is described in Appendix B. Note that the original integration is along the parameter $v'$ for the diffraction rim. However, it is desirable to perform the radiation integration over the parameter $\phi'$ of the incident rim. Since the parameter $\phi'$ has double values for a given $v'$, the radiation integration sometimes yields the numerical difficulties. In order to avoid this numerical problem, the term $(\frac{dv'}{d\phi'})$ in Equation 3.13 is used. Evaluating $\frac{dv'}{d\phi'}$, one obtains

$$\frac{dv'}{d\phi'} = 1 - \frac{2 \sin \theta \cos \phi'}{\sqrt{1 - \sin^2 \theta \sin^2 \phi'}}.$$  \hspace{1cm} 3.33$$

Vector cross product in Equation 3.1 are given by

$$\hat{R} \times \hat{v}' = -\hat{\theta} \cos v' - \hat{\phi} \cos \theta \sin v'.$$  \hspace{1cm} 3.34$$

It should be noted that the lower and the upper limits of the integration given in Equation 3.1 depend upon the scattering angle and the included cone angle. When the cone is fully illuminated by the incident ray, the whole rim contributes. On the other hand, only part of the rim contributes when the cone is partially illuminated and the lower and the upper limits can be found from Equation 3.32. Also, the double diffraction exists only up to certain scattering
Figure 3.6: Relationship between $\psi'$ and $\phi'$ on the base
Table 3.1: Diffracted field components of a cone with half cone angle $\alpha$ in the different sections of scattering angles: Notes; 'O' means that the ray component exists and 'X' means that the ray component does not exist

<table>
<thead>
<tr>
<th>Scattering Angle</th>
<th>$0 - \alpha$</th>
<th>$\alpha - \theta_c$</th>
<th>$\theta_c - 90^\circ$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Field Components</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$E_D$</td>
<td>O</td>
<td>O</td>
<td>O</td>
</tr>
<tr>
<td>$E_{DD}$</td>
<td>O</td>
<td>O</td>
<td>X</td>
</tr>
<tr>
<td>$E_{DDG}$</td>
<td>X</td>
<td>O</td>
<td>O</td>
</tr>
<tr>
<td>$E_{GDD}$</td>
<td>X</td>
<td>O</td>
<td>O</td>
</tr>
<tr>
<td>$E_{GDDG}$</td>
<td>X</td>
<td>X</td>
<td>O</td>
</tr>
<tr>
<td>$E_{TD}$</td>
<td>O</td>
<td>O</td>
<td>O</td>
</tr>
</tbody>
</table>

angle which depends on the half cone angle and the incident direction. After the scattering angle exceeds this angle, the doubly diffracted field excites the creeping wave diffracted field which will be discussed in the next section. As an example, the diffracted field components in the different sections scattering angles are given in Table 3.1. For a cone with the half cone angle of 15°, $\theta_c = 30^\circ$.

Fig. 3.7 presents the backscattering RCS of a finite cone with the double diffraction included in the equivalent current technique. One observes that the agreement between the equivalent current solution and the moment method results is much better in the range between 0° and 25°. However, for scattering angle greater than 25°, improvement for $\sigma_{\phi\phi}$ is still needed. In the following section, we will show that the improvement can be achieved by including the creeping wave
contributions. In order to verify the correctness of the equivalent currents used for the singly and the doubly diffracted fields, the RCS of a hollow cylinder is calculated and compared to Wiener-Hopf result [8] in Figs. 3.8 and 3.9. There exists a good agreement between them except for null at the scattering angle around 48°.

3.1.4 Equivalent Current for the $E_{DDG}$ Ray Mechanism

When the cone is partially illuminated, the creeping wave contributions due to the shadowing of the conical surface are not negligible. In fact, these contributions turn out to be significant for a vertically polarized incident wave. As is shown in Figs. 3.2(b), (c), and (d), the doubly diffracted field from the rim ($E_{DD}$), the edge diffracted field excited creeping wave ($E_{DDG}$), and the creeping wave excited edge diffracted field ($E_{GDD}$) are closely related. The only difference is that rays associated with the $E_{DDG}$ and $E_{GDD}$ fields traverse additional geodesic path on the conical surface. Therefore, these two ray mechanisms can also be implemented into the equivalent current concept with some modification to include the creeping wave effects. The $E_{GDDG}$ ray mechanism is a combination of the above mentioned two ray mechanisms and the analysis of this ray follows exactly the same as those for $E_{DDG}$ and $E_{GDD}$.

Recently, Liang[36] and Chuang [37] studied the uniform GTD solution for the diffraction by an edge in a curved screen. A similar approach can be employed to study the backscattering of a finite cone when the cone is partially illuminated. In the present approach, the solution is obtained by using the equivalent current concept based on the UTD. As was mentioned in the previous section, we only need to include the magnetic equivalent currents for these three ray mechanisms.

Once the equivalent magnetic current $\bar{M} (\phi_2)$ for each ray mechanism is ob-
Figure 3.7: Comparison of the backscattering RCS of a cone with $\alpha = 15^\circ$ and $a = 1.435\lambda$: equivalent current method ($E_D + E_{DD}$) (solid line), method of moments (dashed line). (a) Vertical polarization (b) Horizontal polarization
Figure 3.8: Comparison of the E-plane backscattering RCS of $a = 2.2918\lambda$ hollow cylinder: equivalent current method (solid line), Wiener-Hopf[8] (black dots)
Figure 3.9: Comparison of the H-plane backscattering RCS of $a = 2.2918\lambda$ hollow cylinder: equivalent current method (dash-dot line), Wiener-Hopf[8] (black dots)
tained, the far field radiated by this current can be readily obtained by

\[ \tilde{E} = \frac{2jkae^{-jkR}}{4\pi R} e^{-j2ka \cos \theta_i \tan \alpha} \int_{\text{rim}} \hat{R} \times \overline{M(\phi_2)} e^{jk\hat{R} \cdot \hat{R}_s} d\phi_2 \]

where \( \hat{R} \) is the unit vector in the scattering direction, \( R \) is the radial distance of the source, \( \hat{R}_s \) is defined in each ray mechanism section, and the integration is performed along the partial rim associated with each ray mechanism. The partial rim for each ray mechanism is determined by tracing the individual ray incident upon the rim.

The incident field which hits the edge of the rim diffracts across the base and then excites the surface creeping wave diffracted field toward the source. This is the edge diffracted field excited creeping wave \( (E_{DDG}) \) and is illustrated in Fig. 3.10. Equivalent current for this ray mechanism can be positioned either along the shadow boundary on the conical surface or along the rim of a cone base. When the current is placed on the rim of the cone, then a “UTD Green’s function” for a current placed on the rim would be used; whereas, when equivalent current is placed on the shadow boundary, then a free space Green’s function is required. However, it is more convenient to place an equivalent current along the rim and the resulting field expression can be represented by the radiation integral as expressed in the previous section. Before discussing the details of the derivation of the equivalent magnetic current for \( E_{DDG} \), each diffraction point should be determined. The point \( Q_2 \) is determined by preserving the diffraction cone angle at \( Q_1 \) according to the law of diffraction. The diffraction angle at \( Q_2 \) (i.e., \( w \) in Fig. 3.10) which yields the unique geodesic (i.e., arc \( Q_2Q_3 \)) on the surface of a cone is adjusted such that the geodesic tangent vector at \( Q_2 \) is pointing toward the source.

The electric field radiated by the magnetic dipole \( \overline{dP_m} \) located on a cone in
Figure 3.10: Geometry for the creeping wave excited edge diffracted field ($E_{DDG}$)
the shadow region is given by [3] as

\[ dE_m = -\frac{jk e^{-jks}}{4\pi s} [dP_m(Q_2) \cdot \hat{b}_2] H(\zeta) e^{-jkt} \left[ \frac{\rho_3(Q_3)}{\rho_3(Q_2)} \right] \hat{n}_3 \] 3.36

where only terms proportional to the hard Fock function are retained and the function \( H(\zeta) \) is well discussed in [3]. Evaluating the limiting case of Equation 3.36 as \( Q_3 \) approaches to \( Q_2 \) by letting \( t \to 0, \zeta \to 0, \left[ \frac{\rho_3(Q_3)}{\rho_3(Q_2)} \right] \to 1, \) and \( \hat{n}_3 = \hat{n}_2 \)
yields

\[ dE_m = -\frac{jk e^{-jks}}{4\pi s} [dP_m(Q_2) \cdot \hat{b}_2] H(0) \hat{n}_2. \] 3.37

Note that the field given by the above Equation should reduce to the field radiated by the equivalent magnetic current \( M_2(\phi_2) \) associated with the double diffraction in the limiting case.

The field radiated by the equivalent magnetic current element \( \overline{M_2(\phi_2)} = M_2(\phi_2) \hat{\phi}_2 \) is given by Equation 3.1 as follows:

\[ dE_m = \frac{jk e^{-jks}}{4\pi s} [\hat{R} \times \overline{M_2(\phi_2)}] \Delta \ell \] 3.38

where \( \Delta \ell \) is the differential length along the edge of the rim. Since \( \hat{R} = \hat{n}_2 \times \hat{b}_2 \) and \( \hat{\phi} \cdot \hat{n}_2 = 0, \)

\[ \hat{R} \times \overline{M_2(\phi_2)} = -(\overline{M_2(\phi_2)} \cdot \hat{b}_2) \hat{n}_2. \] 3.39

Therefore, Equation 3.38 reduces to

\[ dE_m = -\frac{jk e^{-jks}}{4\pi s} [\overline{M_2(\phi_2)} \cdot \hat{b}_2] \hat{n}_2 \Delta \ell. \] 3.40

By equating Equation 3.37 and Equation 3.40, one obtains

\[ \frac{dP_m(Q_2) \cdot \hat{b}_2}{H(0)} \Delta \ell = \frac{M_2(\phi_2) \cdot \hat{b}_2}{H(0)} \Delta \ell \] 3.41
where

$$M_2(\phi_2) = -\frac{Z_0}{j k} \sqrt{\frac{8\pi}{\sin \beta_{02}}} \frac{D_h(\psi_2, \beta_{02})}{\sin \beta_{02}} [\tilde{H}_{12} \cdot \hat{b}_2] \sin \phi_2,$$

$$M_2(\phi_2) \cdot \hat{b}_2 = M_2(\phi_2) \sin \omega,$$

and

$$w = (\phi_2 - \phi_o) \sin \alpha.$$ 

Note that $\tilde{H}_{12}$ is the same as that for the double diffraction and parameters for the diffraction coefficient are given by

$$\psi'_2 = 0,$$

$$\psi_2 = \frac{3}{2} \pi + \alpha,$$

$$\sin \beta_{02} = \sqrt{1 - \sin^2 \theta_i \sin^2 \phi_2},$$

$$\sin \beta_{02} = \sin[\sin \alpha (\phi_2 - \phi_o)].$$

where

$$\phi_o = \pi + \cos^{-1} \left[ \frac{\tan \alpha}{\tan \theta_i} \right] - \frac{1}{\sin \alpha} \sec^{-1} \left[ \frac{\cos \alpha}{\sqrt{\cos^2 \alpha - \cos^2 \theta_i}} \right].$$

If one notices that $\hat{n}_3 = \hat{b}_2 \times \hat{R}$ at $Q_3$ and $\Delta \ell = a d \phi_2$, the equivalent magnetic current for the $E_{DDG}$ ray is obtained from Equations 3.36 and 3.41 as

$$M_{DDG}(\phi_2) = \frac{M_2(\phi_2) \cdot \hat{b}_2}{H(0)} H(\zeta) \left[ \frac{\rho_{g}(Q_3)}{\rho_{g}(Q_2)} \right]^{\frac{1}{6}} e^{-j k t \hat{b}_3},$$

where $t$ is the geodesic arc length from $Q_2$ to $Q_3$ on the conical surface. Since the surface outward normal vector at $Q_3$ is given by

$$\hat{n}_3 = \hat{x} \cos \alpha \cos \phi_A + \hat{y} \cos \alpha \sin \phi_A + \hat{z} \sin \alpha,$$

where

$$\phi_A = \pi + \cos^{-1} \left[ \frac{\tan \alpha}{\tan \theta_i} \right].$$
the unit binormal vector at $Q_3$ is given by

$$\hat{b}_3 = \hat{R} \times \hat{n}_3$$

$$= -\hat{z} \cos \theta_i \sin \phi_A + \hat{y} (\cos \theta_i \cos \alpha \cos \phi_A - \sin \theta_i \sin \alpha)$$

$$+ \hat{z} \sin \theta_i \cos \alpha \sin \phi_A.$$  \hspace{1cm} \text{(3.53)}

The argument of Fock function is given by

$$\zeta = \left[ \frac{k}{2} R_2 \cos w_1 \right]^{\frac{1}{3}} [\phi_3 - \phi_2] \cos^2 \alpha \cos \theta_i \sin \phi_A.$$  \hspace{1cm} \text{(3.54)}

where $R_2$ is the radial distance of $Q_2$ and $w_1 = \sin^{-1} \left[ \frac{\cos \theta_i}{\cos \alpha} \right]$. The surface radii of curvature at $Q_2$ and $Q_3$ are given by

$$\rho_g(Q_2) = \frac{a}{\cos \alpha \sin^2 u_2},$$  \hspace{1cm} \text{(3.55)}

$$\rho_g(Q_3) = \frac{R_3 \tan \alpha}{\sin^2 u_3}.$$  \hspace{1cm} \text{(3.56)}

where $R_3$ is the radial distance of $Q_3$ and

$$u_{2,3} = \cos^{-1}\{\sin \alpha (\phi_{2,3} - \phi_0)\}.$$  \hspace{1cm} \text{(3.57)}

The geodesic arc length $t$ is given by

$$t = \frac{\tan^2 \alpha}{A_0 \sin \alpha} \{ \tan[\sin \alpha (\phi_3 - \phi_0)] - \tan[\sin \alpha (\phi_2 - \phi_0)] \}.$$  \hspace{1cm} \text{(3.58)}

where

$$A_0 = \frac{\sin \alpha}{\sqrt{\cos^2 \alpha - \cos^2 \theta_i}} \frac{\tan \alpha}{a},$$  \hspace{1cm} \text{(3.59)}

$$\phi_0 = \phi_2 - \frac{1}{\sin \alpha} \cos^{-1} \left[ \frac{\sqrt{\cos^2 \alpha - \cos^2 \theta_i}}{\cos \alpha} \right].$$  \hspace{1cm} \text{(3.60)}

$\phi_2$ is obtained by applying the law of diffraction at $Q_1$ as

$$\phi_2 = \pi + \phi_1 - 2 \sin^{-1}[\sin \theta_i \sin \phi_1].$$  \hspace{1cm} \text{(3.61)}
Equation 3.50 is the equivalent magnetic current for the $E_{DDG}$ ray located on the rim and the field radiated by this current can be obtained from Equation 3.35. Note that $\vec{R}_s$ in Equation 3.35 is the vector from the center of the cone base to $Q_3$.

### 3.1.5 Equivalent Current for the $E_{GDD}$ Ray Mechanism

The incident ray which hits the shadow boundary on the conical surface at $Q'$ creeps along the surface until it diffracts at $Q_1$ on the rim. After diffracting at $Q_1$, a ray traverses across the base, and then diffracts at $Q_2$ toward the source. The ray that follows this path is denoted by $E_{GDD}$ and the ray path is shown in Fig. 3.11. Note that the equivalent magnetic current associated with this ray mechanism is essentially the same as that for the ray mechanism $E_{DD}$ except that the $H_{\beta_0}$ given by Equation 3.31 should be modified to take into account the creeping wave effect.

To account for the creeping wave effect on $H_{\beta_0'}$, the incident field at $Q_1$ is modified by the Fock function and the extra phase along the geodesic path $Q'Q_1$. The incident field at $Q_2$ after diffracting at $Q_1$ can be obtained as

$$E_{\psi_1}^{g12} = E_{\psi_1'}^g D_h(\psi_1', \psi_1; \beta_0') \sqrt{\frac{\rho}{s(s + \rho)}} e^{-jks}$$

where

$$E_{\psi_1}^g = [\hat{E}^i(Q') \cdot \hat{n}'] \frac{H(\zeta)}{H(0)} \left| \frac{\rho(Q_1)}{\rho(Q')} \right| e^{-jkt},$$

$\hat{n}'$ is the outward unit normal vector at $Q'$, and $\hat{\psi}$, $\hat{\psi}'$, $\hat{\beta}_{01}$, and $\hat{\beta}_{01}'$ are defined in the curved edge diffraction section in Chapter 2.
Figure 3.11: Geometry for the edge diffracted field excited creeping wave ($E_{GD}$)
Since
\[ \mathbf{H}_{\beta_01}^{g12} = \frac{1}{Z_0} \hat{\mathbf{d}} \times \mathbf{E}_{\psi_1}^{g12} \]
and
\[ \hat{\mathbf{d}} \times \hat{\mathbf{p}}_1 = \hat{\mathbf{p}}_{01} \]
\[ \mathbf{H}_{\beta_01}^{g12} = \mathbf{H}_{\beta_01}^{g} D_h(\psi_1, \psi_1'; \beta_{01}) \sqrt{\frac{\rho}{s(s + \rho)}} e^{-jks} \]
where
\[ \mathbf{H}_{\beta_01}^{g} = \frac{[\mathbf{E}^i(Q') \cdot \hat{n}']}{Z_0} \frac{H(Q)}{H(0)} \rho_g(Q_1) \rho_g(Q') \frac{1}{\delta} e^{-jkt}. \]

Note that the unit vector \( \hat{\mathbf{d}} \) is the unit vector from \( Q_1 \) to \( Q_2 \). The reason for using \( H(0) \) in the denominator is to ensure that the field at \( Q_2 \) (i.e., \( \mathbf{H}_{\beta_01}^{g12} \)) reduces to \( \mathbf{H}_{\beta_01}^{g12} \) of the doubly diffracted field section as \( Q_1 \) approaches to \( Q_A \) at the shadow boundary.

The equivalent magnetic current can be obtained by the same procedure used for the doubly diffracted field and is given by
\[ MGDD(\phi_2) = Z_0 \sqrt{\frac{8\pi}{jk}} \frac{D_h(\psi_2, \psi_2'; \beta_{02}')}{\sin \beta_{02}} \mathbf{H}_{\beta_01}^{g12} \cos(\beta - \phi_2) \hat{\phi}_2 \]
where \( \psi_2' = 0 \),
\[ \psi_2 = \begin{cases} \pi + \tan^{-1} \left| \frac{\cos \phi_1}{\sin \theta_1 \cos \phi_2} \right| & \text{if } 0 < \phi_2 < \frac{\pi}{2} \\ 2\pi - \tan^{-1} \left| \frac{\cos \phi_1}{\sin \theta_1 \cos \phi_2} \right| & \text{otherwise}, \end{cases} \]
\[ \sin \beta_{02} = \sqrt{1 - \sin^2 \phi_2 \sin^2 \theta_1}, \]
\[ \sin \beta_{02}' = \sin[\cos^{-1}(\sin(\beta - \phi_2))], \]
\[ \beta = \tan^{-1} \frac{\cos \phi_2 - \cos \phi_1}{\sin \phi_2 - \sin \phi_1}, \]
\[ \sin \beta_{01} = \sqrt{1 - \cos^2[\sin(\phi' - \phi_0)]}, \]
\[ \phi_\alpha = \pi - \cos^{-1} \left| \frac{\tan \alpha}{\tan \theta_1} \right| - \frac{1}{\sin \alpha} \sec^{-1} \left[ \frac{\cos \alpha}{\sqrt{\cos^2 \alpha - \cos^2 \theta_1}} \right], \]
\[ \hat{\phi}_2 = -\hat{\mathbf{e}} \sin \phi_2 + \hat{\mathbf{y}} \cos \phi_2, \]
and $\psi_1 = 0, \psi'_1 = \frac{3\pi}{2} + \alpha$. $\vec{E}^i(Q')$ is the incident electric field at point $Q'$, $\hat{n}'$ is the unit outward normal vector at $Q'$. Replacing $\overline{MGDD}(\phi_2)$ with $\overline{M}(\phi_2)$ in Equation 3.35 yields the electric field $E_{GDD}$. Vector $\vec{R}_s$ in Equation 3.35 is the vector from the center of the cone base to $Q_2$.

### 3.1.6 Equivalent Current for the $E_{GDDG}$ Ray Mechanism

This mechanism can be thought as a combination of $E_{GDD}$ and $E_{DDG}$. The ray hits the incident shadow boundary on the conical surface and creeps along the surface until it diffracts across the cone base. After diffracting across the base, this ray diffracts at the opposite edge of the rim and then excites the creeping wave which diffracts toward the source. Ray path is illustrated in Fig. 3.12 and this field is called $E_{GDDG}$ and can be analyzed similarly. The incident ray follows the $E_{GDD}$ ray mechanism until it hits the rim and then it follows the $E_{DDG}$ ray mechanism after diffracting at $Q_1$. One important thing is that the diffraction angle at $Q_2$ (i.e., $w$) is adjusted such that the creeping wave follows the geodesic path along the conical surface and diffracts at $Q_3$ toward the source.

From the previous analysis, the equivalent magnetic current for this ray mechanism can be written by

$$\overline{MGDDG}(\phi_2) = -[d\overline{P}_m(Q_2) \cdot \hat{b}_2] H_2(\zeta_2) e^{-j k t_2} \frac{\rho_g(Q_3)}{\rho_g(Q_2)} \hat{b}_3$$

where

$$d\overline{P}_m(Q_2) \cdot \hat{b}_2 = M_1(\phi_2) \frac{\sin w}{H(0)},$$

$$M_1(\phi_2) = \cos(\phi_2 - \beta) \sqrt{\frac{8\pi}{jk} \frac{1}{\sin \beta_{02}} D_h(\psi_2', \psi''_2; \beta_{02})} [\vec{E}^i(Q') \cdot \hat{n}'] \frac{H(\zeta_1)}{H(0)}$$

$$\frac{\rho_g(Q_1)}{\rho_g(Q')} \hat{b}_3 e^{-j k t_1} D_h(\psi_1, \psi'_1; \beta_{01}) \sqrt{\frac{\rho}{s(s + \rho)}} e^{-j ks},$$
Figure 3.12: Geometry for $E_{GDDG}$ ray
\[ \beta = \pi + \phi_1 - \sin^{-1}\left(\frac{\tan \alpha}{A_0 Z_2}\right), \] 3.79
\[ Z_2 = \frac{a}{\tan \alpha}, \] 3.80
\[ \hat{n}_3 = \hat{x} \cos \alpha \cos \phi_A + \hat{y} \cos \alpha \sin \phi_A + \hat{z} \sin \alpha, \] 3.81
\[ \hat{b}_3 = \hat{R} \times \hat{n}_3, \] 3.82
\[ \phi_A = \pi + \cos^{-1}\left|\frac{\tan \alpha}{\tan \theta_i}\right|, \] 3.83
\[ \vec{E}^i(Q') \cdot \hat{n}' = [E^i_\theta(-\cos \alpha \cos \phi_A \cos \theta_i + \sin \theta_i \sin \alpha) + E^i_\phi \cos \alpha \sin \phi_A]e^{jk\hat{R} \cdot \hat{r}'}, \] 3.84

where \( \hat{r}' \) is the vector from the coordinate origin to the attaching point \( Q' \), and \( \zeta_1, \zeta_2, t_2, \rho_g(Q_2,3) \) have the same definitions as those for \( E_{DDG} \) ray mechanism section. Note that the diffraction angle \( w \) at \( Q_2 \) in Fig. 3.12 is given by

\[ w = (\phi_2 - \phi_o) \sin \alpha \] 3.85

where

\[ \phi_o = \pi + \cos^{-1}\left|\frac{\tan \alpha}{\tan \theta_i}\right| - \frac{1}{\sin \alpha} \sec^{-1}\left[\frac{\cos \alpha}{\sqrt{\cos^2 \alpha - \cos^2 \theta_i}}\right]. \] 3.86

Definitions for all parameters other than those specified above are the same as those defined in the previous sections. Replacing \( M_{GDDG}(\phi_2) \) with \( M(\phi_2) \) in Equation 3.35 yields the electric field \( E_{GDDG} \). Note that \( R_s \) in Equation 3.35 is the vector from the center of the cone base to \( Q_2 \).

Fig. 3.13 presents the backscattering RCS of a finite cone with the creeping wave contributions included in the equivalent current technique. One can see that with the creeping wave effects included, the \( \sigma_{\phi \phi} \) obtained from the equivalent current technique (solid line) compare well with those obtained from the method of moments (dashed line). Bechtel[20] stated in his paper that the reason for poorer agreement in the vertically polarized \( \sigma_{\phi \phi} \) case is not clear. Referring to Fig. 3.13, there is evidence to suggest that the poorer agreement around \( \theta_i \approx 40^\circ \) is caused
Figure 3.13: Comparison of the backscattering RCS of a cone with $\alpha = 15^\circ$ and $a = 1.435\lambda$: equivalent current technique (solid line), method of moments (dashed line). (a) vertical polarization (b) horizontal polarization
by the omission of the creeping waves. In order to see the effect of the each ray contribution, the backscattering RCS including the single (black dots), the single and the double (dashed line), and the single plus the double plus the creeping wave (solid line) are plotted for a cone with $\alpha = 15^\circ$ and $a = 1.435\lambda$ in Fig. 3.14.

3.1.7 Tip Diffraction

In this research, we are interested in the scattering from the cone with a narrow included cone angle. For the narrow cone angle, the diffracted field from the tip is readily obtained by Felsen[7,38]. For a horizontally polarized plane wave incidence ($\hat{e}_{p}$ polarized incidence), the tip diffracted field is given by

$$\tilde{E}_{TD} \simeq \hat{\theta} \frac{-j}{\log(\sin^2 \frac{1}{2} \delta)} \tan \frac{\theta}{2} \tan \frac{\theta_i}{2} \frac{e^{-jkr}}{r}$$  \hspace{1cm} \text{3.87}

where

$\theta_i$ = incident angle

$\theta$ = scattering angle

$\delta$ = half cone angle.

For a vertically polarized plane wave incidence ($\hat{e}_{v}$ polarized incidence),

$$\tilde{E}_{TD} \simeq \frac{e^{-jkr}}{r} \left\{ \hat{\theta} jA \sin^2 \frac{\delta}{2} \sin(\phi - \phi_0) \right\}$$

$$+ \hat{\phi} \frac{j^2 \sin^2 \frac{\delta}{2}}{(\cos \theta + \cos \theta_0)^2} \left[ \sin \theta \sin \theta_0 + 2(1 + \cos \theta \cos \theta_0) \cos(\phi - \phi_0) \right]$$  \hspace{1cm} \text{3.88}

where $(r, \theta, \phi)$ is the spherical coordinate of a receiver location and $(\theta_0, \phi_0)$ is that of a source.

For the backscattering in the principal plane, $\theta = \theta_0 = \theta_i$ and $\phi = \phi_0 = 0$. Hence, the tip diffracted field for the backscattering, we have

$$\tilde{E}_{TD} \simeq \hat{\theta} \frac{-j}{\log(\sin^2 \frac{1}{2} \delta)} \tan^2 \frac{\theta_i}{2} \frac{e^{-jkr}}{2 \cos \theta_i}$$  \hspace{1cm} \text{3.89}
Figure 3.14: Backscattering RCS of a cone with $a = 1.435\lambda$ and $\alpha = 15^\circ$. (a) vertical polarization (b) horizontal polarization
for the horizontally polarized plane wave incidence, and
\[
\vec{E}_{TD} \simeq \frac{j \sin^2 \delta}{4 \cos \theta_i^3} \left[ \sin^2 \theta_i + 2(1 + \cos^2 \theta_i) \right] e^{-jkr} r
\]
for the vertically polarized plane wave incidence.

3.1.8 Numerical Results

Solutions developed in the previous sections are employed to calculate the backscattering RCS for a cone. The contributions of the single diffraction, the double diffraction, and the creeping wave effects are included in the calculation. In addition, the tip diffraction is also included. In order to see the effect of of each ray mechanism, the radar cross sections of a cone with \( \alpha = 15^\circ \) and \( a = 1.435\lambda \) with the single diffraction, the single and the double diffractions, and the total diffraction including the creeping wave effects are plotted separately in Fig. 3.14. For vertical polarization (\( \sigma_{\phi\phi} \)), the effect of each ray mechanism can be observed very clearly. The backscattering RCS for a cone with 15° half cone angle and \( ka = 9.642 \) are illustrated in Fig. 3.15 as a function of incident angle \( \theta_i \). These results are compared with those calculated and measured by Senior and Uslenghi [22]. It can be seen that the inclusion of the creeping wave effects improves the backscattering RCS for both polarizations. Similar comparisons are presented for a larger cone in Fig. 3.16.

Recently, Wang and Mitschang [28] investigated the electromagnetic scattering from circular and elliptic cones based on the physical theory of diffraction (PTD). The contributions associated with physical optics (PO), single edge diffraction, and double edge diffraction are included in their calculation for the backscattering RCS for a finite cone. Fig. 3.17(a) presents their calculation of \( \sigma_{\phi\phi} \) as a function of \( \theta_i \) for a cone with 15° half cone angle and \( a = 1.435\lambda \) [28, Fig. 7b]. Corresponding results obtained by present approach with the creeping wave effect
included are shown in Fig. 3.17(b). In Figs. 3.17(a) and 3.17(b), the results obtained from a method of moments code developed by Chuang[35] are also plotted for comparison. In Fig. 3.18, the backscattering RCS of a cone with $\alpha = 10.1586^\circ$ and $a = 3.88\lambda$ are presented. Comparison between the present approach and the result obtained by Wang and Mitschang [28] are made.

The axial backscattering RCS of a cone with a half cone angle $\alpha = 15^\circ$ is illustrated in Fig. 3.19 for $8 < ka < 12$. Also, measured data by Senior, computed data by Senior, and the first order result by Keller are plotted for comparison. Fig. 3.20 presents the axial backscattering RCS of a cone with $\alpha = 30^\circ$. Comparisons are made among the present approach, the first order result by Keller, and the measured data by Burnside and Peters.

Figs. 3.21, 3.22, and 3.23 present the backscattering RCS of a cone with a radius $a = 1.435\lambda$ and $\alpha = 5^\circ$, $\alpha = 10^\circ$, and $\alpha = 20^\circ$, respectively. In Figs. 3.24, 3.25, and 3.26, the backscattering RCS of a cone with $\alpha = 15^\circ$ and $a = 1\lambda$, $a = 2\lambda$, and $a = 10\lambda$, respectively. In each figure, the solid line represents the horizontal polarization and the dashed line represents the vertical polarization. As they are shown, there exists slight discontinuity in the horizontal polarization at the scattering angle $= \alpha$ (i.e., half cone angle). One possible explanation for this discontinuity is the lack of the contribution from the tip-base (or base-tip) interaction.

Figures 3.27 and 3.28 present the backscattering RCS of cones with a half cone angle $\alpha = 15^\circ$ and the base radii $a = 0.7175\lambda$ and $a = 2.87\lambda$, respectively. In each figure, the solid line represents the result from the present approach and dots represent the result from the method of moments. From Figs. 3.13, 3.27, and 3.28, one can observe that there exists better agreement between the two methods as the size of a cone becomes larger. This is an expected characteristic of a high
frequency method, namely, the GTD.
Figure 3.15: Comparison of backscattering RCS of a cone with $ka = 9.642$ and $\alpha = 15^\circ$ between the present approach and [22]. (a) vertical polarization (b) horizontal polarization
Figure 3.16: Comparison of backscattering RCS of a cone with $ka = 11.527$ and $\alpha = 15^\circ$ between the present approach and [22]. (a) Vertical polarization (b) Horizontal polarization
Figure 3.17: Comparison of $\sigma_{\phi\phi}$ of a cone with $a = 1.435\lambda$ and $\alpha = 15^\circ$. (a) Wang and Mitschang[28] vs. method of moments (b) Present approach vs. method of moments
Figure 3.18: Comparison of backscattering RCS of a cone with $\alpha = 3.88\lambda$ and $\alpha = 10.1586^\circ$ between Wang and Mitsch(28)(dashed line) and present approach (solid line). (a) vertical polarization (b) horizontal polarization
Figure 3.19: Axial backscattering RCS of a cone with $\alpha = 15^\circ$
Figure 3.20: Axial backscattering RCS of a cone with $\alpha = 30^\circ$
Figure 3.21: Backscattering RCS of a cone with $\alpha = 5^\circ$ and $a = 1.435\lambda$: horizontal polarization (solid line), vertical polarization (dash dot line)
Figure 3.22: Backscattering RCS of a cone with $\alpha = 10^\circ$ and $a = 1.435\lambda$: horizontal polarization (solid line), vertical polarization (dash dot line)
Figure 3.23: Backscattering RCS of a cone with $\alpha = 20^\circ$ and $\alpha = 1.435\lambda$: horizontal polarization (solid line), vertical polarization (dash dot line)
Figure 3.24: Backscattering RCS of a cone with $\alpha = 15^\circ$ and $\sigma = 1\lambda$: horizontal polarization (solid line), vertical polarization (dash dot line)
Figure 3.25: Backscattering RCS of a cone with $\alpha = 15^\circ$ and $\alpha = 2\lambda$: horizontal polarization (solid line), vertical polarization (dash dot line)
Figure 3.26: Backscattering RCS of a cone with $\alpha = 15^\circ$ and $a = 10\lambda$: horizontal polarization (solid line), vertical polarization (dash dot line)
Figure 3.27: Backscattering RCS of a cone with $\alpha = 15^\circ$ and $\alpha = 0.7175\lambda$: present approach (solid line), moment method (curve formed by dots). (a) vertical polarization, (b) horizontal polarization
Figure 3.28: Backscattering RCS of a cone with $\alpha = 15^\circ$ and $\lambda = 2.87\lambda$: present approach (solid line), moment method (curve formed by dots). (a) vertical polarization, (b) horizontal polarization
3.2 Bistatic Scattering RCS of a Fully Illuminated Cone

3.2.1 Introduction

Bistatic scattering RCS of a perfectly conducting finite cone for the E- and H- plane are studied when it is fully illuminated by the incident plane wave. The solution is obtained in the equivalent current technique based on the UTD. The derivation of each ray mechanism involved is very similar to that for the backscattering case. Only difference is that the phase term in the radiation integral for each ray mechanism becomes the function of the incident angle and the scattering angle instead of the function of incident angle only. Also, the limits of the radiation integral become a function of the scattering angle and dose not depend upon the incident angle.

3.2.2 Equivalent Current for the Singly Diffracted Field ($E_D$)

The procedure to derive the equivalent currents for the single diffraction for bistatic scattering from a finite cone is the same as that for the backscattering case. The geometry for the singly diffracted field is given in Fig. 3.29. From the backscattering case, we have the equivalent electric and magnetic currents as

$$I_e^i(\phi') = -Y_0 \sqrt{\frac{8\pi}{jk}} \frac{D_e(\psi, \psi'; \beta_C)}{\sin \beta_o} \left[ \vec{E}^i(Q_e) \cdot \hat{\phi}' \right]_{rim} \hat{\phi}'$$  \hspace{1cm} 3.91

and

$$I_m^i(\phi') = -Z_0 \sqrt{\frac{8\pi}{jk}} \frac{D_m(\psi, \psi'; \beta_C)}{\sin \beta_o} \left[ \vec{H}^i(Q_e) \cdot \hat{\phi}' \right]_{rim} \hat{\phi}'$$  \hspace{1cm} 3.92

where $\vec{E}^i(Q_e)$ and $\vec{H}^i(Q_e)$ are the incident electric and magnetic fields at $Q_e$ along the rim, respectively. The unit edge vector is given by

$$\hat{\phi}' = -\hat{x} \sin \phi' + \hat{y} \cos \phi'. \hspace{1cm} 3.93$$

The diffraction coefficients are the same as that for the backscattering case except that the parameters become the function of the incident angle and the scattering
Figure 3.27: Geometry for the singly diffracted field for the bistatic scattering of a cone
angle. The quantities $\beta_0$ and $\beta_0'$ are obtained from

$$\beta_0 = \cos^{-1}(\mathbf{s}^d \cdot \mathbf{\hat{e}}'),$$

$$\beta_0' = \cos^{-1}(\mathbf{s}^i \cdot \mathbf{\hat{e}}').$$

The scattering direction unit vector and the incident unit vector are given by

$$\mathbf{s}^d = \mathbf{x} \sin \theta_s + \mathbf{z} \cos \theta_s,$$

$$\mathbf{s}^i = -\mathbf{x} \sin \theta_i - \mathbf{z} \cos \theta_i,$$

where $\theta_s$ is the scattering angle and $\theta_i$ is the angle of incidence. Therefore,

$$\sin \beta_0 = \sqrt{1 - \sin^2 \theta_s \sin^2 \phi'},$$

$$\sin \beta_0' = \sqrt{1 - \sin^2 \theta_i \sin^2 \phi'},$$

where $\phi'$ is the angle measured along the rim of a cone base. The angles $\psi$ and $\psi'$ in the diffraction coefficients are given by

$$\psi = \tan^{-1}[\cos \theta_s/\sin \theta_s \cos \phi'],$$

$$\psi' = \tan^{-1}[\cos \theta_i/\sin \theta_i \cos \phi'].$$

Once the equivalent currents are obtained, the far field radiated from these currents is given by

$$\vec{E}_D \propto \frac{j2 \alpha \zeta_0 e^{-jk R - jka(\cos \theta_s + \cos \theta_i)}}{4\pi R} \int_{0}^{\phi'} \hat{R} \times \hat{R} \times \hat{I}_e d\phi'$$

$$+ Y_0 \hat{R} \times \hat{I}_m e^{jk \hat{R} \cdot \hat{R}'} d\phi'$$

where $\hat{R} = \mathbf{s}^d = \mathbf{x} \sin \theta_s + \mathbf{z} \cos \theta_s$ is the scattering direction unit vector and $\hat{R}'$ is the vector from the center of a cone base to $Q_e$ along the rim, and $R$ is the radial distance of the receiver. Vector cross products in Equation 3.102 are given by

$$\hat{R} \times \hat{I}_m = I_m(-\hat{\theta} \cos \phi' - \hat{\phi} \cos \theta_s \sin \phi'),$$

$$\hat{R} \times \hat{R} \times \hat{I}_e = I_e(\hat{\theta} \cos \theta_s \sin \phi' - \hat{\phi} \cos \phi').$$
The upper limit of the integration is obtained from the fact the scattering direction unit vector at the shadow boundary is orthogonal to the outward unit normal vector. Since

\[ \hat{s}^d = \hat{x} \sin \theta_s + \hat{y} \cos \theta_s \]  
\[ \hat{n} = \hat{x} \cos \alpha \cos \phi_A + \hat{y} \cos \alpha \sin \phi_A + \hat{z} \sin \alpha, \]  
\[ \hat{s}^d \cdot \hat{n} = \sin \theta_s \cos \alpha \cos \phi_A + \cos \theta_s \sin \alpha = 0. \]

By solving the above Equation for \( \phi_A \), one obtains

\[ \phi_A = \cos^{-1} \left(-\frac{\tan \alpha}{\tan \theta_s}\right) \] for \( \theta_s > \alpha. \]

When \( \theta_s < \alpha \), \( \phi_A = \pi. \)

### 3.2.3 Equivalent Current for the Doubly Diffracted Field \( (E_{DD}) \)

The doubly diffracted field, which due to rays diffracted across the cone base to the opposite edge and then diffracted toward the source, can also be obtained by the equivalent magnetic current. The ray path for this mechanism is illustrated in Fig. 3.30. The equivalent magnetic current \( I_m^2 \) for the doubly diffracted field is given by

\[ I_m^2(v') = -Z \sqrt{\frac{8\pi}{jk}} \frac{1}{2} D_{il}(\psi_2, \psi'_2, \beta_{02}) \frac{1}{\sin \beta_{02}} [\tilde{H}_{12} \cdot \hat{v}]_{rim} \hat{v}' \]

and the second order equivalent electric current is insignificant and ignored. Note that the diffraction coefficients are the same as those in the singly diffracted field.

The quantities \( \psi_2, \psi'_2, \beta_{02}, \) and \( \beta'_{02} \) are evaluated at \( Q_2 \) and given by \( \psi'_2 = 0, \)

\[ \psi_2 = \begin{cases} \pi + \tan^{-1} \frac{\cos \theta_s}{\sin \theta_s \cos v'} & \text{if } 0 < v' < \frac{\pi}{2} \\ 2\pi - \tan^{-1} \frac{\cos \theta_s}{\sin \theta_s \cos v'} & \text{otherwise}, \end{cases} \]

97
\[
\sin \beta_{02} = \sqrt{1 - \sin^2 \theta_s \sin^2 \nu'}, \quad 3.110 \\
\sin \beta_{02}' = \sqrt{1 - \sin^2 \theta_s \sin^2 \phi'}. \quad 3.111 
\]

If we decompose \( \vec{H}_{12} \), which is the magnetic field at point \( Q_2 \) after diffracting from \( Q_1 \), into \( \hat{\beta}_{01} \) and \( \hat{\psi}_1 \) components, \( \hat{\psi}_1 \cdot \hat{v}' = 0 \). Hence, only the \( \hat{\beta}_{01} \) component of \( \vec{H}_{12} \) is necessary to calculate the equivalent magnetic current \( \vec{j}_m^2 \). The unit vectors

\[
\hat{v}' = -\hat{x} \sin \nu' + \hat{y} \sin \nu', \quad 3.112 \\
\hat{\beta}_{01} = \hat{x} \sin \beta - \hat{y} \cos \beta \quad 3.113
\]

where

\[
\beta = \pi - \phi' - \sin^{-1}[\sin \theta_s \sin \phi']. \quad 3.114
\]

Therefore, the term in the bracket of the equivalent magnetic current expression becomes

\[
\vec{H}_{12} \cdot \hat{v}' = H_{\beta_{01}}^{i12} [\hat{\beta}_{01} \cdot \hat{v}'] = -H_{\beta_{01}}^{i12} \cos(\beta - v') \quad 3.115
\]

where

\[
H_{\beta_{01}}^{i12} = -H_{\beta_{01}} \frac{D_h(\psi_1, \psi_1'; \beta_{01})}{s(s + \rho)} \sqrt{\frac{\rho}{s(s + \rho)}} e^{-js} \quad 3.116
\]

and \( \rho \) is the caustic distance at \( Q_1 \), \( s \) is the distance between \( Q_1 \) and \( Q_2 \). The caustic distance \( \rho \) at \( Q_1 \) is obtained from

\[
\frac{1}{\rho} = \frac{1}{\rho_e^i} - \frac{n_e \cdot (\hat{s}^i - \hat{d})}{a_e \sin^2 \beta_o}. \quad 3.117
\]

Since

\[
\rho_e^i = \infty, \quad 3.118 \\
a_e = a, \quad 3.119
\]

98
Figure 3.28: Geometry for the doubly diffracted field for the bistatic scattering of a cone.
\[ \hat{n}_e = \hat{x} \cos \phi' + \hat{y} \sin \phi', \quad \text{3.120} \]
\[ \hat{d} = \hat{x} \cos \beta + \hat{y} \sin \beta, \quad \text{3.121} \]
\[ \hat{s} = -\hat{x} \sin \theta_i - \hat{z} \cos \theta_i \quad \text{3.122} \]

for a cone with the plane wave incidence case,
\[ \rho = \frac{a \sin^2 \beta_{01}}{\sin \theta_i \cos \phi' - \sin \beta_{01}}. \quad \text{3.123} \]

The distance between the two diffraction points \( Q_1 \) and \( Q_2 \) is given by
\[ s = a \sqrt{2 - 2 \cos(\phi' - \nu')} = 2a \sin \beta_{01}. \quad \text{3.124} \]

Since
\[ \beta'_{01} = \frac{1}{\sqrt{\cos^2 \theta_i + \sin^2 \theta_i \cos^2 \phi'}} \left| \hat{x} \cos \theta_i \sin \phi' + \hat{y} \cos \phi' - \hat{z} \cos \theta_i \sin \theta_i \sin \phi' \right|, \quad \text{3.125} \]

\( H_{\beta'_{01}} \) is obtained by \( \hat{H}' \cdot \hat{\beta}'_{01} \) as
\[ H_{\beta'_{01}} = \frac{e^{jka \sin \theta_i \cos \phi'}}{Z_0} \left[ \frac{\cos \phi' E^i_{\theta}(Q_1)}{\sqrt{\cos^2 \theta_i + \sin^2 \theta_i \cos^2 \phi'}} + \frac{\cos \theta_i \sin \phi' E^i_{\phi}(Q_1)}{\sqrt{\cos^2 \theta_i + \sin^2 \theta_i \cos^2 \phi'}} \right]. \quad \text{3.126} \]

The quantities \( \psi_1, \psi_1', \beta_{01}, \) and \( \beta'_{01} \) are evaluated at \( Q_1 \) and given by \( \psi_1 = 0, \)
\[ \psi_1' = \begin{cases} \pi + \tan^{-1} \left| \frac{\cos \theta_i}{\sin \theta_i \cos \phi'} \right| & \text{if } 0 < \phi' < \frac{\pi}{2} \\ 2\pi - \tan^{-1} \left| \frac{\cos \theta_i}{\sin \theta_i \cos \phi'} \right| & \text{otherwise}, \end{cases} \]
\[ \sin \beta_{01} = \sin \beta_{01}' = \sqrt{1 - \sin^2 \theta_i \sin^2 \phi'}, \quad \text{3.127} \]

and \( E^i_{\theta} \) and \( E^i_{\phi} \) are the \( \hat{\theta} \) and \( \hat{\phi} \) components of the incident electric field at \( Q_1 \), respectively. The \( \phi' \) and \( \nu' \) are the azimuth angle in cylindrical coordinates of
the first diffraction point \((Q_1)\) and the second diffraction point \((Q_2)\), respectively. Applying the law of diffraction at \(Q_1\), the relationship between the \(v'\) and \(\phi'\) is given by

\[
v' = \pi + \phi' - 2 \sin^{-1}[\sin \theta \sin \phi'].
\]

The far-zone electric field radiated by an equivalent magnetic current is given by

\[
\vec{E}_{DD} = \frac{j2\alpha e^{-jkR-jk\frac{(\cos \theta + \cos \phi)}{\tan \alpha}}}{4\pi R} \int_{\phi_B}^{2\pi} \vec{R} \times \vec{I}_m e^{jk\vec{R} \cdot \vec{R}' \cdot dv'}
\]

where

\[
\phi_B = \begin{cases} 
\pi & \text{if } 0 < \theta < \alpha \\
\pi + \cos^{-1} \frac{\tan \alpha}{\tan \theta} & \text{otherwise},
\end{cases}
\]

and

\[
\vec{R} \times \vec{I}_m = I_m(-\dot{\theta} \cos v' - \dot{\phi} \cos \theta \sin v').
\]

When the scattering angle exceeds the half cone angle \(\alpha\), only the partial rim contributes. The remaining part of the rim contributes in the form of the creeping wave which will be discussed in the following section.

3.2.4 Equivalent Current for \(E_{DDG}\) Ray Mechanism

When the scattering angle becomes greater than the half cone angle, there exists a creeping wave contribution. The ray incident upon the edge of a rim diffracts across the cone base to the opposite edge and then creeps along the conical surface until it diffracts toward the receiver. This ray is denoted by \(E_{DDG}\) and called the edge diffracted field excited surface creeping wave. The ray path is illustrated in Fig. 3.31. Since the derivation of the equivalent current for this ray mechanism is the same as that for the backscattering case, only the final expression of the equivalent current is given. Once the equivalent magnetic current
Figure 3.31: $E_{DDG}$ ray mechanism for the bistatic scattering of a cone
is obtained, the far field radiated by this current can be readily derived from the far field radiation integration as

\[
\vec{E}_{DDG} = \frac{2jkae^{-j\kappa R}}{4\pi R} e^{-j\kappa a_{\text{tan} \alpha}} \int_\pi \vec{R} \times M_{DDG}(\phi_2)e^{j\kappa \vec{R} \cdot \vec{R}_s} d\phi_2 \tag{3.131}
\]

where \( \vec{R} \) is the unit vector in the scattering direction, \( R \) is the radial distance of the receiver, \( \vec{R}_s \) is the vector from the center of a cone base to \( Q_2 \), and \( \phi_B \) is given in the previous section.

The equivalent magnetic current for \( E_{DDG} \) ray mechanism is given by

\[
M_{DDG}(\phi_2) = \left( \frac{M_2(\phi_2) \cdot \vec{b}_2}{H(0)} \right) H(\zeta) \left[ \frac{\rho_g(Q_3)}{\rho_g(Q_2)} \right] e^{-j\kappa t_3} \tag{3.132}
\]

where

\[
\frac{M_2(\phi_2)}{M_2(\phi_2) \cdot \vec{b}_2} = \frac{\pi}{\kappa} \frac{D_h(\psi_2, \psi_2'; \beta_2'; \beta_0)}{\sin \beta_0} [\vec{H}_1 \cdot \vec{b}_2] \text{rim} \phi_2, \tag{3.133}
\]

\[
w = (\phi_2 - \phi_0) \sin \alpha. \tag{3.134}
\]

Parameters for the diffraction coefficient \( D_h \) are given by

\[
\psi_2' = 0, \tag{3.135}
\]

\[
\psi_2 = \frac{3}{2} \pi + \alpha, \tag{3.136}
\]

\[
\sin \beta_0 = \sqrt{1 - \sin^2 \theta_i \sin^2 \phi_2}, \tag{3.137}
\]

\[
\sin \beta_0 = \sin[\sin \alpha(\phi_2 - \phi_0)]. \tag{3.138}
\]

The unit vectors at \( Q_3 \) are given by

\[
\vec{n}_3 = \hat{x} \cos \alpha \cos \phi_B + \hat{y} \cos \alpha \sin \phi_B + \hat{z} \sin \alpha, \tag{3.139}
\]

\[
\vec{b}_3 = \hat{R} \times \vec{n}_3. \tag{3.140}
\]
The geodesic arc length $t$ from $Q_2$ to $Q_3$ on the conical surface is given by

$$t = \frac{\tan^2 \alpha}{A_0 \sin \alpha}\left\{\tan[\sin \alpha(\phi_3 - \phi_0)] - \tan[\sin \alpha(\phi_2 - \phi_0)]\right\}$$  \hspace{1cm} 3.142$$

where

$$A_0 = \frac{\sin \alpha \tan \alpha}{\sqrt{\cos^2 \alpha - \cos^2 \theta_0}},$$  \hspace{1cm} 3.143$$

$$\phi_0 = \phi_2 - \frac{1}{\sin \alpha} \cos^{-1}\left[\frac{\sqrt{\cos^2 \alpha - \cos^2 \theta_0}}{\cos \alpha}\right],$$  \hspace{1cm} 3.144$$

$$\phi_2 = \pi + \phi_1 - 2 \sin^{-1}[\sin \theta_0 \sin \phi_1],$$  \hspace{1cm} 3.145$$

$$\phi_3 = \pi + \cos^{-1}\left[\frac{\tan \alpha}{\tan \theta_0}\right].$$  \hspace{1cm} 3.146$$

The argument of Fock function is given by

$$\zeta = \left[\frac{k}{2} R_2 \cos \omega_1\right]^{\frac{1}{2}} \phi_3 - \phi_2 \cos^2 \alpha \cos^2 \theta_0 \sin \theta_0 \sin \phi_1 \cos \phi_2 \cos \phi_3,$$  \hspace{1cm} 3.147$$

where $R_2$ is the radial distance of $Q_2$

$$w_1 = \sin^{-1}\left[\frac{\cos \theta_0}{\cos \alpha}\right].$$  \hspace{1cm} 3.148$$

The surface radii of curvature at $Q_2$ and $Q_3$ are given by

$$\rho_g(Q_2) = \frac{a}{\cos \alpha \sin^2 u_2},$$  \hspace{1cm} 3.149$$

$$\rho_g(Q_3) = \frac{R_3 \tan \alpha}{\sin^2 u_3}$$  \hspace{1cm} 3.150$$

where $R_3$ is the radial distance of $Q_3$ and

$$u_{2,3} = \cos^{-1}\{\sin[\sin \alpha(\phi_{2,3} - \phi_0)]\}.$$  \hspace{1cm} 3.151$$

### 3.2.5 Numerical Results

The bistatic scattering cross sections of a fully illuminated finite cone are calculated for different angles of incidence as a function of scattering angle. The
single diffraction, the double diffraction, the edge diffracted field excited creeping wave \((E_{DDG})\), and the tip diffracted field given in section 3.1.7 are included for the bistatic RCS patterns. The cone has half cone angle \(\alpha = 15^\circ\) and base radius \(a = 1.435\lambda\).

In Fig. 3.32, the bistatic scattering cross section of the cone with the incident angle of \(\theta_i = 0^\circ\) is calculated and compared to the result obtained by the method of moments code developed by Chuang[35]. Similar comparisons are made in Figs. 3.33 and 3.34 for the incident angle of \(\theta_i = 5^\circ\) and \(\theta_i = 10^\circ\), respectively. As is shown in the figures, there exists some discrepancies between two results. These discrepancies are believed to come from the lack of tip-base (or base-tip) interaction contribution. However, the nature of this tip-base interaction is still not clearly understood and needs further investigation.
Figure 3.32: Bistatic scattering cross section of a finite cone with a half cone angle $\alpha = 15^\circ$ and the angle of incidence $\theta_i = 0^\circ$.
Figure 3.33: Bistatic scattering cross section of a finite cone with a half cone angle $\alpha = 15^\circ$ and the angle of incidence $\theta_i = 5^\circ$
Figure 3.34: Bistatic scattering cross section of a finite cone with a half cone angle $\alpha = 15^\circ$ and the angle of incidence $\theta_i = 10^\circ$
3.3 Conclusion

The near axial backscattering and the bistatic scattering of a perfectly conducting finite cone is studied using the equivalent current concept based on the uniform geometrical theory of diffraction (UTD). The creeping wave associated with the conical surface are also incorporated into the equivalent current technique. It has been found that the contribution from the creeping waves are significant for the oblique incidence case. There is evidence to speculate that the poorer agreement between the previously calculated results and the measured data for the vertically polarized backscattering, $\sigma_{\phi\phi}$, is probably a result of the omission of the creeping wave contribution. Even with the creeping wave effects accounted for, some small discrepancies exist and the nature of the discrepancy needs further investigation. One possible ray mechanism missing is the tip-base interaction which is still not clearly understood.
CHAPTER IV
BACKSCATTERING RCS OF A CONE-SPHERE

4.1 Introduction

In this chapter, the backscattering radar cross section (RCS) of a cone-sphere in the near axial region for the E- and H-plane are investigated. The conducting cone-sphere of a half cone angle $\alpha$ and a radius of spherical base $b$ is illuminated by the plane wave incident at an angle $\theta_i$. The radius of a spherical base is $b$ and the radius of the rim at the junction is $a$. The geometry of the problem under consideration is illustrated in Fig. 4.1.

Few studies on the radar cross section of a cone-sphere have appeared. Ken-naugh and Moffatt [30] calculated the axial backscattering RCS of a cone-sphere by using the impulse response of the target. Blore [31] measured some numerical data for a axial RCS of a cone-sphere. Senior [10] used the physical optics (PO) approximation to calculate the axial backscattering from a cone-sphere. He obtained the tip diffracted field and the junction diffracted field based on the PO. To include the creeping wave contribution associated with the spherical base, the solution from a conducting sphere was added. Recently, Wang and Mitschuang [11] obtained the solution for the bistatic scattering from a cone-sphere based on the hybrid technique which incorporates the Fock solution for currents on the surface into the method of moments.

Earlier work done by Senior [10] treated only the axial backscattering. For
the axial backscattering, creeping wave contribution associated with the conical surface is less important. When the cone-sphere is partially illuminated, the inclusion of the junction diffraction and the creeping wave associated with the spherical base is not enough to predict the valid numerical value for the backscattering RCS. In this chapter, the creeping wave contribution associated with the conical surface is incorporated into the equivalent current concept [4,5] based on the uniform geometrical theory of diffraction (UTD) by Pathak et al [1,3]. The ray mechanisms included in this chapter are illustrated in Fig. 4.2. It is found that the creeping wave associated with the conical surface contributes significantly to the backscattering when the cone-sphere is partially illuminated by the incident field.

In this chapter the equivalent current approach is used to generate the backscattering from a conducting cone-sphere. Furthermore, the contributions to the backscattering due to the each individual ray mechanism can be clearly identified. In the following sections, the equivalent current concept is presented systematically for the junction diffraction, the creeping wave associated with the spherical base, and the creeping wave associated with the conical surface. The present results show good agreement when compared to either the method of moments solutions or to measured data.
Figure 4.1: Cone-Sphere geometry under consideration
Figure 4.2: Diffraction field components in the calculation of the backscattering
RCS of a cone-sphere
4.2 The Fully Illuminated Cone-Sphere

When a cone-sphere is fully illuminated, significant contributions come from the creeping wave around the spherical base and the diffraction from the junction associated with the curvature discontinuity. In this section, these two ray mechanisms are outlined in the framework of the equivalent current concept based on the high frequency diffraction theory.

4.2.1 Diffracted Field from the Junction \( (E_{JD}) \)

Even though a cone-sphere has smooth junction between a conical structure and a spherical base, there exists a higher order discontinuity, namely, a curvature discontinuity. This discontinuity causes diffraction and the diffracted field at the junction forms a caustic along the axial direction. This ray mechanism is illustrated in Fig. 4.3. Since the ray solution can not predict the valid result at or near the caustic, the equivalent current concept is employed to calculate the backscattered field due to the junction discontinuity. The derivation of the equivalent currents follows the same procedure described by Ryan and Peters [4,5].

The equivalent electric current \( I_e \) and the magnetic current \( I_m \) for the junction diffraction can readily be obtained as:

\[
I_e(\phi') = -Y_o \sqrt{\frac{8\pi}{jk \sin \beta_o}} D_s(\psi) [\vec{E}_i(Q_e) \cdot \hat{\phi}] \hat{\phi}',
\]

\[
I_m(\phi') = -Z_o \sqrt{\frac{8\pi}{jk \sin \beta_o}} D_h(\psi) [\vec{H}_i(Q_e) \cdot \hat{\phi}] \hat{\phi}'.
\]

where \( \vec{E}_i(Q_e) \) and \( \vec{H}_i(Q_e) \) are the incident electric and magnetic fields at the junction, respectively. The diffraction coefficients \( D_{s,h}(\psi) \) are given by Senior [9] and was described in the theoretical background chapter. For the backscattering case, \( \alpha = \theta = \psi \) in Equations 2.109 and 2.110. Therefore, the diffraction coefficients
Figure 4.3: Geometry for the junction diffraction
become

$$D_{s,h}(\psi) = \frac{e^{-j \frac{\pi}{4}}}{\sqrt{2\pi k \sin \beta_o}} [X \mp Y]$$

where

$$X = j \frac{1 + \cos 2\psi}{8kb \cos^3 \psi}$$

$$Y = j \frac{1}{4kb \cos^3 \psi}.$$  

The quantities $\beta_o, \beta_o'$, and $\psi$ are given by

$$\sin \beta_o = \sin \beta_o' = \sqrt{1 - \sin^2 \theta_i \sin^2 \phi'},$$

$$\psi = \begin{cases} \alpha + \frac{\pi}{2} - \tan^{-1} \left( \frac{\cos \theta_i}{\sin \theta_i \cos \phi'} \right) & \text{if } 0 < \phi' < \frac{\pi}{2} \\ \alpha - \frac{\pi}{2} + \tan^{-1} \left( \frac{\cos \theta_i}{\sin \theta_i \cos \phi'} \right) & \text{otherwise.} \end{cases}$$

The far field radiated by these equivalent currents is given by

$$\hat{E}_{JD} = \frac{j2kaZ_o e^{-jkr}}{4\pi R} e^{-j2kr \tan \phi_i} \int_0^{\phi_A} [\hat{R} \times \hat{R} \times \hat{l}_e + Y_c \hat{R} \times \hat{l}_m] e^{jkr \hat{R}_c} \hat{R} \cos \phi \, d\phi'$$

where $\hat{R}$ is the unit vector in the radiation direction and given by

$$\hat{R} = \hat{x} \sin \theta_i + \hat{z} \cos \theta_i.$$
\[ \phi_A = \pi - \cos^{-1} \left| \frac{\tan \alpha}{\tan \theta_1} \right| . \]

The rim in the shadow region gives a contribution to the backscattering as a creeping wave mechanism \( E_{\text{CDG}} \) associated with the conical surface which will be discussed in the following section.

### 4.2.2 Creeping Wave Around the Spherical Base \( (E_C) \)

The field incident upon the shadow boundary on the spherical base forms a caustic toward the backscattering direction. The geometry for the creeping wave around spherical base is given in Fig. 4.4. In order to account for the infinite rays, the equivalent current concept is used. The equivalent current for the creeping wave associated with the spherical base can be derived from the similar procedure for obtaining the equivalent magnetic current associated with the doubly diffracted field of a finite cone discussed by Burnside and Peters [21]. The only difference is that the diffraction coefficient for the edge diffraction has to be replaced by that for the surface creeping wave.

Following the equivalent current approach used by Ryan and Peters [4, 5], the equivalent magnetic current for the creeping wave around the spherical base can be represented as follows:

\[
\tilde{M}_c = M_c \hat{b}_2 = -Z_c \sqrt{\frac{8\pi}{jk}} T_h(t, \beta) (\tilde{H}^i(Q_1) \cdot \hat{b}_1) \hat{b}_2
\]

where \( \tilde{H}^i(Q_1) = \hat{\theta} H_\theta^i + \hat{\phi} H_\phi^i \) is the incident magnetic field at \( Q_1 \) and \( \hat{b}_1 \) and \( \hat{b}_2 \) are unit binormal vectors at \( Q_1 \) and \( Q_2 \) as shown in Figure 4.4. Note that the equivalent magnetic current is placed along the shadow boundary on the spherical base. The diffraction coefficient \( T_h(t, \beta) \) will be determined in the following analysis. Parameters \( t \) and \( \beta \) are defined as follows:

\[ t = \text{geodesic arc length between } Q_1 \text{ and } Q_2, \]

117
Figure 4.4: Geometry for the creeping wave around the spherical base
\[
\beta = \int_{Q_1}^Q \left( \frac{kb}{2} \right)^3 \sin \alpha \ \text{d} \phi.
\]

The far-zone electric field due to this equivalent magnetic current can be obtained from the far field radiation integral given by Equation 4.7 as:

\[
\bar{E}_C = \frac{j 2 k b e^{-jkR}}{4 \pi R} e^{-j k a \tan \alpha} \int_0^\pi [\hat{R} \times \hat{b}_2 M_c] e^{-jk \tan \alpha \cos \phi_1} d \phi_2. \quad 4.10
\]

where \( \phi_2 \) is the local polar angle in the \( O' \) coordinate system shown in Figure 4.4. Note that the contribution from the equivalent electric current is insignificant and ignored in this analysis. The vector cross product \( \hat{R} \times \hat{b}_2 \) at \( Q_2 \) is given by

\[
\hat{R} \times \hat{b}_2 = -\hat{n}_2. \quad 4.11
\]

Surface outward unit normal vector \( \hat{n}_2 \) at \( Q_2 \) is given by

\[
\hat{n}_2 = \hat{x} (\cos \theta_2 \cos \phi'_2 \sin \theta_2 - \sin \theta_2 \sin \phi'_2) - \hat{y} \sin \theta_2 \sin \phi'_2
- \hat{z} (\sin \theta_2 \cos \phi'_2 \sin \theta_2 + \cos \theta_2 \cos \phi'_2). \quad 4.12
\]

where \((\theta'_2, \phi'_2)\) are the primed spherical coordinate system of \( Q_2 \) along the shadow boundary. Since \( \theta'_2 = \frac{\pi}{2} \) in Fig. 4.4, we have

\[
\hat{n}_2 = \hat{x} \cos \theta'_2 \cos \phi'_2 - \hat{y} \sin \theta'_2 \sin \phi'_2
- \hat{z} \sin \phi'_2. \quad 4.13
\]

By noticing that the \( \phi'_1 = \phi'_2 - \pi \) and \( \theta'_1 = \frac{\pi}{2} \), the surface outward unit normal vector at \( Q_1 \) is given by

\[
\hat{n}_1 = -\hat{x} \cos \theta'_2 \cos \phi'_2 + \hat{y} \sin \phi'_2 + \hat{z} \sin \theta_1 \cos \phi'_2
= -\hat{x} \cos \phi'_2 \sin \theta_1 \sin \phi'_2. \quad 4.14
\]

The unit binormal vector at \( Q_1 \) is given by

\[
\hat{b}_1 = \hat{s}^i \times \hat{n}_1. \quad 4.15
\]
where \( \hat{s}^i \) is the incident unit vector.

Substituting the above vectors and Equation 4.9 into the radiation integration given by Equation 4.10, one obtains

\[
\vec{E}_C = \sqrt{\frac{jk}{2\pi}} \cdot 2bT_h(t, \beta)e^{-j2k\alpha \tan \alpha \cos \theta_i} - j2k\alpha \cos \theta_i \tan \alpha \frac{e^{-jkR}}{R} \times \\
[\hat{\theta}E^i_\theta \int_{\phi_s}^{\phi_e} \cos^2 \phi_2 d\phi_2 + \hat{\phi}E^i_\phi \int_0^\pi \sin^2 \phi_2 d\phi_2].
\]

For axial incidence, the upper and the lower limits in Equation 4.10 are \( \phi_s = 0 \) and \( \phi_e = \pi \), respectively. Therefore, the far-zone electric field due to the equivalent magnetic current can be obtained from Equation 4.10 as

\[
\vec{E}_C = \sqrt{\frac{jk}{2\pi}} b\pi T_h(t, \beta)e^{-j2k\alpha \tan \alpha \cos \theta_i} - j2k\alpha \cos \theta_i \tan \alpha \frac{e^{-jkR}}{R} \left[\hat{\theta}E^i_\theta + \hat{\phi}E^i_\phi\right]
\]

where \( E^i_\theta \) and \( E^i_\phi \) are \( \theta \) and \( \phi \) components of the incident electric field, respectively. By comparing the above equation to the Equation (5) of Senior [10], the diffraction coefficient \( T_h(t, \beta) \) can be expressed as follows:

\[
T_h(t, \beta) = -\left(\frac{kb}{2}\right)^{\frac{3}{2}} \sqrt{\frac{\pi}{2}} e^{-j\frac{k}{2}t} \hat{F}_h[\beta],
\]

\[
\hat{P}_h(\beta) = \frac{e^{-j\frac{k}{2}t}}{2\sqrt{\pi} \tilde{q}_1[A_i(-\tilde{q}_1)]^2} \left[1 + \frac{8\tilde{q}_1}{10} (1 + \frac{9}{32\tilde{q}_1^2})(\frac{2}{kb})^\frac{3}{2} e^{-j\frac{k}{2}t}\right]
\]

\[
\left[-\beta\tilde{q}_1 e^{j\frac{k}{2}t} - \beta(\frac{kb}{2})^\frac{3}{2} \frac{q_1^2}{60}(1 - \frac{q_1}{\tilde{q}_1})e^{-j\frac{k}{2}t}\right]
\]

where \( \tilde{q}_1 = 1.01879... \), \( A_i(-\tilde{q}_1) = 0.53565.... \) Note that the parameters for the diffraction coefficient \( T_h \) for the axial incidence are given by

\[
t = b\pi, \quad 4.20
\]

\[
\beta = \pi(\frac{kb}{2})^\frac{1}{3}. \quad 4.21
\]

When the cone-sphere is partially illuminated (i.e., when \( \theta_i > \alpha \)), due to the shadowing of the conical surface, only the part of the ring on the spherical base
contributes to the backscattering. The upper limit and the lower limits \( \phi_e \) and \( \phi_s \) can be readily determined from the geometry and they are given by

\[
\text{upper limit} = \phi_e = \cos^{-1}[x' \cdot \hat{O'Q}_A] = \cos^{-1}[-\cos \alpha \cos \phi_A \cos \theta_i + \sin \alpha \sin \theta_i], \quad 4.22
\]
\[
\text{lower limit} = \phi_s = \pi - \phi_e. \quad 4.23
\]

Evaluating the Equation 4.16 for the partially illuminated case, the creeping wave associated with the spherical base is obtained as follows:

\[
\mathbf{E}_C = -\sqrt{\frac{jk}{2\pi}} bT_h(t, \beta) e^{-j2k lessons=\frac{\alpha \cos \theta_i}{\alpha}} e^{-\frac{jkR}{R}} \{\tilde{\theta} E^i_{\phi}((\phi_e - \phi_s) + \frac{1}{2}(\sin 2\phi_e - \sin 2\phi_s)) + \hat{\phi} E^i_{\hat{\phi}}((\phi_e - \phi_s) - \frac{1}{2}(\sin 2\phi_e - \sin 2\phi_s))\} \quad 4.24
\]

where the diffraction coefficient \( T_h \) is given by Equation 4.19 and parameters \( t \) and \( \beta \) are defined by Equations 4.20 and 4.21, respectively. Note that the part of the ring corresponds to the creeping wave around the spherical base when the cone-sphere is partially illuminated. The remaining part will contribute to the backscattering in the form of a creeping wave associated with the conical surface. This is the \( E_{GDC} \) ray mechanism and is discussed in detail in the following section.
4.3 Partially Illuminated Cone-Spheres

When a cone-sphere is partially illuminated by the incident field, the diffraction mechanism becomes complicated. In addition to the junction diffraction and the creeping wave around the spherical base, there are two more creeping waves associated with the conical surface. These two ray mechanisms are resulted from the shadowing due to the conical surface. One of the ray mechanisms creeps around the spherical base and then continuously traverses along the conical surface and is denoted by $E_{CDG}$. Another ray mechanism is generated when the field incident upon the conical surface continuously traverses around the spherical base. This is denoted by $E_{GDC}$. As is shown in Figs. 4.2 (b), (c), and (d), all three creeping wave mechanisms are closely related. Therefore, the latter two ray mechanisms can also be implemented into the equivalent current concept with the creeping wave effect due to the additional geodesic path on the conical surface.

4.3.1 $E_{CDG}$ Ray Mechanism

The field incident upon the shadow boundary on the spherical base at $Q_1$ creeps along the great circle (geodesic path) until it diffracts at the opposite shadow boundary on the conical surface at $Q_3$. This ray is denoted by $E_{CDG}$ and illustrated in Fig. 4.5. The equivalent magnetic current can be placed at $Q_2$ along the rim where the curvature discontinuity exists. Since the equivalent magnetic current is shadowed from the observation point, it launches a creeping wave on the conical surface from $Q_2$ to $Q_3$.

Before discussing the derivation of the equivalent magnetic current in detail, it is necessary to determine the diffraction point $Q_3$ corresponding to $Q_1$. $Q_2$ is determined by following the geodesic on the spherical base after the incident field grazing at $Q_1$ until it intersects the junction rim at $Q_1$. $Q_3$ is determined by
Figure 4.5: Geometry for $E_{CDG}$ ray mechanism
adjusting the angle between \( \hat{e} \) and the geodesic tangent vector \( \hat{t}_2 \) at \( Q_2 \) such that the diffracted ray at \( Q_3 \) directs toward the source.

The electric field radiated by the magnetic dipole \( \overrightarrow{dP_m} \) located on an infinite cone in the shadow region is given by Pathak et. al [3] as

\[
\overrightarrow{dE_m} = \frac{-jke^{-jks}}{4\pi s}[\overrightarrow{dP_m(Q_2)} \cdot \hat{b}_2]H(\zeta)e^{-jkt}\left(\frac{\rho_2(Q_3)}{\rho_2(Q_2)}\right)^{\frac{1}{6}}\hat{n}_3
\]

where only the term proportional to the hard Fock function is retained and the function \( H(\zeta) \) has been discussed in the UTD radiation paper by Pathak et. al [3]. Evaluating the limiting case of the above equation as \( Q_3 \) approaches to \( Q_2 \), i.e. \( t \rightarrow 0, \zeta \rightarrow 0, \left(\frac{\rho_2(Q_3)}{\rho_2(Q_2)}\right) \rightarrow 1 \), and \( \hat{n}_3 = \hat{n}_2 \) yields

\[
\overrightarrow{dE_m} = \frac{-jke^{-jks}}{4\pi s}[\overrightarrow{dP_m(Q_2)} \cdot \hat{b}_2]H(0)\hat{n}_2.
\]

Note that the field given by the above equation reduces to the field radiated by the equivalent magnetic current \( \overrightarrow{M_c} \) at \( Q_2 \) given by Equation 4.9.

The field radiated by the equivalent magnetic current element \( \overrightarrow{M_c} = M_c\hat{b}_2 \) located at \( Q_2 \) is given by Equation 4.10 as follows:

\[
\overrightarrow{dE_m} = \frac{jke^{-jks}}{4\pi s}[\overrightarrow{M_c} \cdot \hat{b}_2]\hat{n}_2\Delta \ell
\]

where \( \Delta \ell \) is the differential length along the junction rim and given by \( \Delta \ell = ad\phi_2 \).

By equating Equation 4.26 and Equation 4.27, one obtains

\[
\overrightarrow{dP_m(Q_2)} \cdot \hat{b}_2 = \frac{-M_c \cdot \hat{b}_2}{H(0)}\Delta \ell.
\]

For the \( ECDG \) ray mechanism, parameters \( t \) and \( \beta \) in the diffraction coefficient \( T_h \) in Equation 4.9 are given by

\[
\beta = \left(\frac{kb}{2}\right)^\frac{1}{3}\gamma, \quad 4.29
\]

\[
t = b\gamma. \quad 4.30
\]

124
\( \gamma \) is the angular separation between \( Q_1 \) and \( Q_2 \) and is given by

\[
\gamma = \frac{\pi}{2} + \tan^{-1}\left[\frac{a}{b} \left( -\sin \theta_i \cos \phi_2 + \frac{\cos \theta_i}{\tan \alpha} - \frac{\cos \theta_i}{\cos \alpha \sin \alpha} \right) \right]
\]

where \( b \) is the radii of the spherical base and \( a \) is the radii of the junction rim. If one notices that \( \hat{n}_3 = \hat{b}_3 \times \hat{R} \) at \( Q_3 \), the equivalent magnetic current for the \( E_{CDG} \) mechanism is obtained from Equations 4.25 and 4.28 as

\[
\overline{M_{CDG}(\phi_2)} = -\frac{M_e \cdot \hat{b}_2}{H(0)} e^{-jkt_2} \left[ \rho_g(Q_2)^{\frac{1}{6}} \hat{b}_3 \right] \rho_g(Q_3) \]

where \( t_2 \) is the geodesic arc length from \( Q_2 \) to \( Q_3 \). The surface outward unit normal vector at \( Q_3 \) is given by

\[
\hat{n}_3 = \hat{x} \cos \alpha \cos \phi_A - \hat{y} \cos \alpha \sin \phi_A + \hat{z} \sin \alpha
\]

where

\[
\phi_A = \pi - \cos^{-1}\left[ \frac{\tan \alpha}{\tan \theta_i} \right]
\]

and binormal vector at \( Q_3 \) is given by

\[
\hat{b}_3 = \hat{R} \times \hat{n}_3.
\]

The argument \( \zeta \) of Fock function is given by

\[
\zeta = \frac{k \alpha}{2 \tan \alpha} \cos w_1 \left[ \phi_3 - \phi_2 \right] \cos \frac{1}{3} \phi_3 - \frac{1}{3} \cos^2 \alpha - \cos^2 \theta_i \right] \]

where \( w_1 = \sin^{-1} \left[ \frac{\cos \theta_i}{\cos \alpha} \right] \). The surface radii of curvature at \( Q_2 \) and \( Q_3 \) are given by

\[
\rho_g(Q_2) = \frac{a}{\cos \alpha \sin^2 u_2},
\]

\[
\rho_g(Q_3) = \frac{R_3 \tan \alpha}{\sin^2 u_3}
\]

where \( R_3 \) is the radial distance of \( Q_3 \) and

\[
u_{2,3} = \cos^{-1} \{ \sin[\sin \alpha (\phi_{2,3} - \phi_0)] \}.
\]
The geodesic arc length $t_2$ is given by

$$t_2 = \frac{\tan^2 \alpha}{A_0 \sin \alpha} \left\{ \tan[\sin \alpha(\phi_3 - \phi_0)] - \tan[\sin \alpha(\phi_2 - \phi_0)] \right\}$$

where

$$A_0 = \frac{\sin \alpha \tan \alpha}{\sqrt{\cos^2 \alpha - \cos^2 \theta_i}},$$

$$\phi_0 = \phi_2 - \frac{1}{\sin \alpha} \cos^{-1}\left[ \frac{\sqrt{\cos^2 \alpha - \cos^2 \theta_i}}{\cos \alpha} \right].$$

Substituting Equation 4.32 into the far field radiation integral, we have

$$\vec{E}_{CDG} = \frac{2^j k a e^{-j k R}}{4 \pi R} e^{-j k a \tan \theta_i} \int_{\phi_A}^\pi \vec{R} \times \vec{M}_{CDG}(\phi_2) e^{j k \vec{R} \cdot \vec{R}_3} d\phi_2.$$  

Note that the integration is performed from $\phi_A$ to $\pi$ and $\vec{R}_3$ is the vector from the center of the cone base to $Q_3$. The angular location of $Q_1$ (i.e., $\phi_1$) and $Q_3$ (i.e., $\phi_3$) are given in terms of $\phi_2$ by

$$\phi_1 = \tan^{-1}\left[ \frac{\sin \phi_1'}{\cos \theta_i \cos \phi_1'} \right],$$

$$\phi_1' = \phi_2 - \pi,$$

$$\phi_2' = \tan^{-1}\left[ \frac{\sin \phi_2}{\cos \theta_i \cos \phi_2 + \frac{\sin \theta_i}{\tan \alpha} - \frac{\sin \theta_i}{\cos \alpha \sin \alpha}} \right],$$

$$\phi_3 = \pi - \cos^{-1}\left[ \frac{\tan \alpha}{\tan \theta_i} \right].$$

### 4.3.2 $E_{GDC}$ Ray Mechanism

The field incident upon the shadow boundary at $Q_1$ on the conical surface creeps along the geodesic on a cone and then continuously traverses along the spherical base until it diffracts at $Q_3$ toward the source. This ray is denoted by $E_{GDC}$ and illustrated in Fig. 4.6. Note that the equivalent magnetic current associated with this ray mechanism is essentially the same as that for the ray.
Figure 4.6: Geometry for $E_{GDC}$ ray mechanism
mechanism $E_C$ except that the $H^i(Q_1)$ in Equation 4.9 should be modified to take into account the creeping wave effect associated with the conical surface.

The incident magnetic field at $Q_2$ can be obtained by modifying the incident field at $Q_1$ by the hard Fock function and the extra phase along the geodesic path $Q_1Q_2$. It is obtained as

$$\bar{H}^i(Q_2) = \frac{H^i(Q_1)}{H(0)} \left[ \frac{\rho_Q(Q_2)}{\rho_Q(Q_1)} \right]^{\frac{1}{2}} e^{-jkt_1} b_2$$  \hspace{1cm} 4.48

where $t_1$ is the geodesic arc length from $Q_1$ to $Q_2$ and $\rho_Q(Q_{1,2})$ are the surface radii of curvature at $Q_1$ and $Q_2$, respectively. The reason for using $H(0)$ in the denominator is to ensure that the field at $Q_2$ reduces to the incident field at $Q_1$ as $Q_2$ approaches to $Q_A$ at the shadow boundary. Following the same procedure as in $E_C$ ray mechanism, one obtains the equivalent magnetic current located along the shadow boundary on the spherical base as

$$\bar{M}_{GDC} = -Z \sqrt{\frac{8\pi}{jk}} T_h(t, \beta)[\bar{H}^i(Q_2) \cdot b_2] b_3.$$  \hspace{1cm} 4.49

The diffraction coefficient is given by

$$T_h(t, \beta) = -\left(\frac{kb}{2}\right)^{\frac{1}{2}} \sqrt{\frac{2}{k}} e^{\frac{jk}{2}} e^{-jkt} \hat{P}_h[\beta]$$  \hspace{1cm} 4.50

where

$$\beta = \left\{ \frac{\pi}{2} + \tan^{-1}\left(\frac{a}{b}\right)(-\sin \theta_i \cos \phi_2 + \frac{\cos \theta_i}{\tan \alpha} - \frac{\cos \theta_i}{\cos \alpha \sin \alpha})\right\} \left(\frac{kb}{2}\right)^{\frac{1}{3}}$$  \hspace{1cm} 4.51

$$t = \left\{ \frac{\pi}{2} + \tan^{-1}\left(\frac{a}{b}\right)(-\sin \theta_i \cos \phi_2 + \frac{\cos \theta_i}{\tan \alpha} - \frac{\cos \theta_i}{\cos \alpha \sin \alpha})\right\} b$$  \hspace{1cm} 4.52

and the function $\hat{P}_h$ is defined in $E_C$ ray mechanism section. Parameters appeared in the equivalent magnetic current are given by

$$\zeta_1 = (\pi Z_1 \sin \alpha)^{\frac{1}{3}} [\phi_2 - \phi_1][\cos^2 \alpha - \cos^2 \theta_i]^{\frac{1}{3}},$$  \hspace{1cm} 4.53
\[ Z_1 = \frac{\tan \alpha}{A_0 \cos[\sin \alpha(\phi_1 - \phi_0)]}, \quad 4.54 \]
\[ A_0 = \frac{\tan^2 \alpha}{a \cos[\sin \alpha(\phi_2 - \phi_0)]}, \quad 4.55 \]
\[ t_1 = \frac{\tan^2 \alpha}{A_0 \sin \alpha} \{ \tan[\sin \alpha(\phi_2 - \phi_0)] - \tan[\sin \alpha(\phi_1 - \phi_0)] \}, \quad 4.56 \]
\[ \phi_0 = \phi_A - \frac{1}{\sin \alpha} \cos^{-1} \left[ \frac{\sqrt{\cos^2 \alpha - \cos^2 \theta_i}}{\cos \alpha} \right], \quad 4.57 \]
\[ \phi_1 = \phi_A = \pi - \cos^{-1} \left[ \frac{\tan \alpha}{\tan \theta_i} \right], \quad 4.58 \]
\[ \phi_2 = \tan^{-1} \left[ \frac{\sin \theta_i}{\cos \theta_i \cos \phi_2} \right], \quad 4.59 \]
\[ \phi_2' = \phi_3' = \pi. \quad 4.60 \]

Finally, the far-zone electric field radiated by the equivalent magnetic current is readily obtained by using the far field radiation integral as:

\[ \mathbf{E}_{GDC} = \frac{j2ke^{-jkR}}{4\pi R} e^{-j2ka\tan \alpha} e^{-jk\cos \theta_i} \sin \alpha e^{j\hat{\mathbf{r}} \times \hat{\mathbf{M}}_{GDC}} e^{-jk\hat{R} \cdot \hat{R}_3} d\phi_3' \quad 4.61 \]

where \( \hat{R}_3 \) is the vector from the center of the spherical base to \( Q_3 \) and the upper limit for the above equation is given by

\[ \phi_e = \cos^{-1} [ - \cos \alpha \cos \phi_A \cos \theta_i + \sin \alpha \sin \theta_i ] . \quad 4.62 \]

Note that \( \phi_3' \) is the primed coordinate system illustrated in Fig. 4.6 and used as the parameter for the radiation integral.

The total scattered field for the partially illuminated cone-sphere is simply the superposition of each individual ray mechanism.
4.4 Numerical Results and Discussions

Solutions obtained in the previous sections are employed to calculate the backscattering RCS of a cone-sphere. The contributions due to the junction diffraction, the creeping wave around the spherical base, the creeping wave associated with the conical surface, and the tip diffracted field given in the cone chapter are included. The backscattering RCS of a cone-sphere with half cone angle $\alpha = 15^\circ$ and the junction radius (cone base radius) of $1.435\lambda$ is calculated as a function of incident angle $\theta_i$ in Figs. 4.7, 4.8, 4.9, 4.10, and 4.11. The results obtained from the method of moments code developed by Chuang [35] are also presented for comparison. In Figs. 4.7, 4.8, 4.9, 4.10, and 4.11, the solid line represents the present equivalent current approach and the dashed line represents the method of moments.

In Fig. 4.7, the result of the present approach including only the junction diffraction $E_{JD}$ is compared to that of the moments method. One can observe that there exist large disagreement between them. In Fig. 4.8, the creeping wave around the spherical base is added to the previous result given in Fig. 4.7 and compared to the moments method. It is seen that there exists a discontinuity at the scattering angle of $15^\circ$ for $\sigma_{\theta\theta}$ due to shadowing of the ray mechanism $E_C$. In Figure 4.9, $E_{CDG}$ ray contribution is added to the previous result given in Figure 4.8. The discontinuity for $\sigma_{\theta\theta}$ at $15^\circ$ becomes smaller. $E_{GDC}$ ray mechanism is included in Fig. 4.10 and the result is compared to that of the moments method. Finally, the tip diffracted field is added in Fig. 4.11 and the final result is compared to that of the moments method. It is observed that the inclusion of the two creeping waves associated with the conical surface contribute significantly in the off axial direction. However, there still exists a small discontinuity at the scattering angle.
of 15° and the lobe structures do not match well in the scattering angles between 30° and 60° for σθθ. It is clear that there are still some other mechanisms that need to be investigated. The previous study on a semi-infinite cone by Trott [29] proved that the discrepancy in the near axial region is caused by the tip diffracted field along the cone surface.

In Fig. 4.12, the backscattering RCS of a cone-sphere with half cone angle of α = 4.33° and base radius of 1.0354λ is presented and compared to the measured data obtained by Dominek[39] at the OSU compact range. The measurement was performed at the frequency of 15.2 GHz. Similar comparison is made in Fig. 4.13 and the frequency is 11.2 GHz. In both figures, the solid line is measured data and the black dots are the calculated data from the present approach.

The axial backscattering RCS of a cone-sphere with half cone angle α = 12.5° is calculated and compared to results obtained by Blore [31] and Senior [10] in Fig. 4.14. As it is shown, the present approach yields the same results as those obtained by Senior [10] without considering the creeping wave enhancement.

In Figs. 4.15, 4.16, and 4.17, the backscattering RCS of cone-spheres with the base radius a = 1.435λ and the half cone angle of 5°, 10° and 20° are given. Also the backscattering cross sections of cone-spheres with the half cone angle α = 15° and the base radii of 1λ, 2λ, and 3λ are given in Figs. 4.18, 4.19, and 4.20, respectively. In Figs. 4.15 through 4.20, the solid line represents the horizontal polarization and the dash-dot line represents the vertical polarization.
Figure 4.7: Backscattering cross section of a cone-sphere with $\alpha = 15^\circ$ and $a = 1.435\lambda$ (solid line: present approach including only the junction diffraction, dashed line: method of moments). (a) horizontal polarization ($\sigma_{\theta\theta}$) (b) vertical polarization ($\sigma_{\phi\phi}$).
Figure 4.8: Backscattering cross section of a cone-sphere with $\alpha = 15^\circ$ and $a = 1.435 \lambda$ (solid line: present approach including $E_JD$ and $E_C$, dashed line: method of moments). (a) horizontal polarization ($\sigma_{\theta\theta}$) (b) vertical polarization ($\sigma_{\phi\phi}$)
Figure 4.9: Backscattering cross section of a cone-sphere with $\alpha = 15^\circ$ and $a = 1.435\lambda$ (solid line: present approach including $E_J$, $E_C$, and $E_{CDG}$, dashed line: method of moments). (a) horizontal polarization ($\sigma_{\theta\theta}$) (b) vertical polarization ($\sigma_{\phi\phi}$).
Fig. 4.10: Backscattering cross section of a cone-sphere with $\alpha = 15^\circ$ and $a = 1.435\lambda$ (solid line: present approach including $E_{JD}$, $E_{JC}$, $E_{CDG}$, and $E_{GDC}$, dashed line: method of moments). (a) horizontal polarization ($\sigma_{\theta\theta}$), (b) vertical polarization ($\sigma_{\phi\phi}$).
Figure 4.11: Backscattering cross section of a cone-sphere with $\alpha = 15^\circ$ and $a = 1.435\lambda$ (solid line: present approach including $E_{JD}$, $E_{C}$, $E_{CDG}$, $E_{GDC}$, and $E_{TD}$, dashed line: method of moments). (a) horizontal polarization ($\sigma_{\theta\theta}$) (b) vertical polarization ($\sigma_{\phi\phi}$).
Figure 4.12: Backscattering RCS of a cone-sphere with $\alpha = 4.33^\circ$ and $\alpha = 1.0354\lambda$: measured data at frequency 15.2 GHz (solid line), calculated data (black dots). (a) horizontal polarization (b) vertical polarization
Figure 4.13: Backscattering RCS of a cone-sphere with $\alpha = 4.33^\circ$ and $a = 0.962\lambda$: measured data at frequency 11.2 GHz (solid line), calculated data (black dots). (a) horizontal polarization (b) vertical polarization.
Present Approach: ————

Measured by Senior: ↑

Calculated by Senior: ————

(enhancement factor included)

Calculated by Blore: ————

Figure 4.14: Axial backscattering RCS of a cone-sphere with half cone angle $\alpha = 12.5^\circ$
Figure 4.15: Backscattering RCS of a cone-sphere with $\alpha = 5^\circ$ and $a = 1.435\lambda$: vertical polarization (dash-dot line), horizontal polarization (solid line)
Figure 4.16: Backscattering RCS of a cone-sphere with $\alpha = 10^\circ$ and $a = 1.435\lambda$: vertical polarization (dash-dot line), horizontal polarization (solid line)
Figure 4.17: Backscattering RCS of a cone-sphere with $\alpha = 20^\circ$ and $a = 1.435\lambda$: vertical polarization (dash-dot line), horizontal polarization (solid line)
Figure 4.18: Backscattering RCS of a cone-sphere with $\alpha = 15^\circ$ and $a = 1\lambda$: vertical polarization (dash-dot line), horizontal polarization (solid line)
Figure 4.19: Backscattering RCS of a cone-sphere with $\alpha = 15^\circ$ and $\alpha = 2\lambda$: vertical polarization (dash-dot line), horizontal polarization (solid line)
Figure 4.20: Backscattering RCS of a cone-sphere with \( \alpha = 15^\circ \) and \( a = 3\lambda \): vertical polarization (dash-dot line), horizontal polarization (solid line)
4.5 Conclusion

The near axial backscattering cross section from a perfectly conducting cone-sphere is studied using the equivalent current concept based on the high frequency diffraction theory. The creeping waves associated with the spherical base and the conical surface are incorporated into the equivalent current concept. It has been found that the contributions from the creeping waves associated with the conical surface are significant for the oblique incidence case. Even with the creeping wave accounted for, there still exists some small discrepancies and the nature of them need further investigation. One possible explanation is the omission of the creeping wave enhancement as it was stated by Senior [10] and this takes the form of the tip diffraction along the conical surface.
CHAPTER V

CONCLUSION

Solutions for the EM backscattering of a perfectly conducting finite cone and a cone-sphere in the near axial region are calculated by the equivalent current concept based on the uniform geometrical theory of diffraction (UTD). Some numerical results are obtained and compared to the method of moments (MM) and measured data. The bistatic cross section of a finite cone is calculated by the same approach.

The main contribution of this research to the analyses of a cone and a cone-sphere is that the creeping wave effects associated with the shadowing of the conical surface and the spherical base are formulated. One advantage of the present equivalent current approach is that each individual ray mechanism can be identified so that the RCS reduction can be achieved by eliminating the most significant ray contribution. The reduction of the radar cross section can be achieved by either using the lossy dielectric material coated on the surface of interest or modifying the scattering structure which does not have the edge or the curvature discontinuity. Also, the computational efficiency of the present approach is very good compared to that of the method of moments (MM). For example, the CPU time required to calculated the backscatter RCS of a cone with $\alpha = 15^\circ$ and $a = 1.435\lambda$ is approximately 1 minute. On the other hand, the MM requires about 50 minutes CPU time to obtained the backscattering pattern.

The ray mechanisms included in the analysis of a cone are the single diffrac-
tion, the double diffraction, the tip diffraction, and the creeping wave contributions. The tip diffracted field is calculated by using the narrow cone angle approximation given by Felsen [7]. As is shown in the previous chapters, the inclusion of the creeping wave contributions yields good agreement between the calculated data and those obtained by the MM. It has been shown that the contribution from the creeping waves are significant for the oblique incidence case. There exists evidence to speculate that the poorer agreement between the previously calculated results and the measured data for the vertically polarized backscatter, $\sigma_{\phi\phi}$, is probably resulted from the omission of the creeping wave contribution. Even with the creeping wave effects accounted for, some small discrepancies exist and the nature of the discrepancy needs further investigation. One possible ray mechanism missing is the tip-base interaction which can readily be computed for non-axial incidence.

For a cone-sphere, the junction diffraction, the creeping wave around the spherical base, and the creeping wave associated with the conical surface are analyzed by the equivalent current concept based on the high frequency diffraction theory. The tip diffracted field is the same as that of a cone. As numerical results show, there is good agreement between the calculated data and results obtained by MM. However, there still exists some discrepancies and the discontinuity at the scattering angle of $\alpha$ for $\sigma_{\theta\theta}$ (horizontal polarization). The nature of this discrepancy and the discontinuity needs further investigation. One possible explanation for this is similar to the cone case, i.e. a tip-base interaction.

In the future, the tip-base interaction of a finite cone and the creeping wave enhancement of a cone-sphere need thorough study to improve the calculated results to compare well with the measured data. Since the military application of the similar object requires the low radar cross section, the scattering from the conical structure without the edge or the curvature discontinuity is necessary to
construct the low radar cross section target. In addition to the modification of the base structure, the lossy dielectric material coated on the surface of a scatterer will produce even lower radar cross section. The effect of the lossy dielectric material will also be studied in the near future.
APPENDIX A

PO SOLUTION FOR THE BACKSCATTERING RCS OF A FINITE CONE AND A CONE SPHERE

The physical optics (PO) solution for the backscattering RCS of a perfectly conducting finite cone and a cone-sphere are briefly described in this appendix.

A.1 Backscattering from a Finite Cone

Let the incident electric field of a plane wave be

\[ \hat{E}^i = (\hat{\theta}E_\theta^i + \hat{\phi}E_\phi^i)e^{jk\hat{r}\cdot\hat{r}'} \]  

where

\[ \hat{r} = \hat{x}\sin \theta_s + \hat{z}\cos \theta_s, \]  

\[ \hat{r}' = \hat{x}r' \sin \alpha \cos \phi' + \hat{y}r' \sin \alpha \sin \phi' + \hat{z}r' \cos \alpha, \]  

\( \theta_s \) is the angle of incidence, \( \alpha \) is the half cone angle, and \( \hat{r}' \) is the vector from the cone apex to the cone surface. Since the magnetic field is given by

\[ \hat{H}^i = \frac{1}{Z_0} \hat{E}^i \times \hat{r}, \]  

\[ \hat{H}_i = \frac{1}{Z_0} (\hat{\theta}E_\theta^i - \hat{\phi}E_\phi^i)e^{jk\hat{r}\cdot\hat{r}'} \]  

The geometry of a cone under study is illustrated in Fig. A.1.

The geometrical optics (GO) current on the cone surface can be obtained from

\[ \tilde{J}_s = 2\hat{n} \times \hat{H}^i \]  

150
Figure A.1: Geometry of a finite cone
where \( \hat{n} \) is the surface outward normal vector. Since

\[
\hat{n} = \hat{x} \cos \alpha \cos \phi' + \hat{y} \cos \alpha \sin \phi' - \hat{z} \sin \alpha ,
\]

the GO current on the illuminated part of a conical surface is given by

\[
\bar{J}_o = \frac{2}{Z_0} e^{jk\hat{r} \cdot \hat{r}'} \{ \hat{x}( - \cos \alpha \sin \phi' \sin \theta_s E_{\phi}' - \sin \alpha E_{\theta}' ) \\
+ \hat{y}( - \sin \alpha \cos \theta_s E_{\phi}' + \cos \alpha \sin \theta_s \cos \phi' E_{\theta}' ) \\
+ \hat{z}( - \cos \alpha \cos \phi' E_{\theta}' - \cos \alpha \cos \theta_s \sin \phi' E_{\phi}' ) \}
\]

where \( \phi' \) is the azimuthal angle of the point \( P \) on the cone surface and \( \theta_s \) is the angle of incidence.

The electric vector potential due to this current is given by

\[
\bar{A} = \int_{-\phi_A}^{\phi_A} \int_{0}^{\alpha} \bar{J}_o \sin \alpha \frac{e^{-jkR}}{4\pi R^2} r' dr' d\phi'
\]

where

\[
\phi_A = \begin{cases} 
\pi & \text{if } \pi - \alpha < \theta_s < \pi \\
\pi - \cos^{-1} \left( \frac{\tan \alpha}{\tan \theta_s} \right) & \text{otherwise}.
\end{cases}
\]

In the far zone, the radial distance of the field point is given by

\[
R = \begin{cases} 
r & \text{for } \frac{1}{R} \text{ term} \\
 r - \hat{r} \cdot \hat{r}' & \text{for phase term}.
\end{cases}
\]

Evaluating the electric vector potential for \( \hat{\theta} \) polarized plane wave incidence, one obtains
The far-zone electric field in terms of the electric vector potential is given by

\[ \vec{E} = -j\omega\mu\vec{A} - (\hat{r} \cdot \vec{A})\hat{r}. \]  

Therefore, the electric field radiated by the GO current with the \( \hat{\theta} \) polarized incident plane wave is obtained by

\[ \vec{E} = \hat{\theta}jE_\theta^{i} e^{-jkr} \int_{-\phi_{A}}^{\phi_{A}} \int_{0}^{\frac{a}{\sin \alpha}} \left( \sin \alpha \cos \theta_{s} - \cos \alpha \sin \theta_{s} \cos \phi' \right) \sin \alpha \cos \theta_{s} \left( \sin \alpha \sin \theta_{s} \cos \phi' + \sin \alpha \cos \theta_{s} \right) r' dr' d\phi'. \]  

Similarly, the electric field radiated by the GO current with the \( \hat{\phi} \) polarized incident plane wave is obtained by

\[ \vec{E} = \hat{\phi}jE_\phi^{i} e^{-jkr} \int_{-\phi_{A}}^{\phi_{A}} \int_{0}^{\frac{a}{\sin \alpha}} \left( \sin \alpha \cos \theta_{s} - \cos \alpha \sin \theta_{s} \cos \phi' \right) \sin \alpha \cos \theta_{s} \left( \sin \alpha \sin \theta_{s} \cos \phi' + \sin \alpha \cos \theta_{s} \right) r' dr' d\phi'. \]  

Note that the same field expressions are obtained for both polarizations.

The backscattering RCS of a cone using the PO approximation is presented and compared to the results obtained by the equivalent current approach in Fig. A.2. The radar cross section is defined by

\[ \sigma = \lim_{r \to \infty} \frac{4\pi r^2}{|E_t|^2} |E_s|^2 \]  

and the backscatter angle in Fig. A.2 is defined by

\[ \text{'BACKSCATTER ANGLE'} = \pi - \theta_s. \]  

The cone has a radius \( a = 1.435\lambda \) and the half cone angle \( \alpha = 15^{\circ} \). As it is shown
in Fig. A.2, the PO solution does not yield good result in the near axial region. The reason for that is the end point contribution from the PO radiation integral does not predict the correct edge diffracted field. Also, the creeping wave effects can not be accounted for when a cone is partially illuminated by a plane wave.
Figure A.2: Comparison of backscattering RCS of a finite cone with $\alpha = 15^\circ$ and $\alpha = 1.435\lambda$ between the PO solution (solid line), and the equivalent current approach (dashed line: horizontal polarization, black dots: vertical polarization).
A.2 Backscattering from a Cone-Sphere

Before going into the derivation of the PO solution for the backscattering of a cone-sphere, let's consider the geometry given in Fig. A.3. Since the PO solution requires the surface integral over the illuminated area (not shaded) depicted in Fig. A.3, the procedure can be broken into two parts. The first part is the surface integration over the conical surface and the second part is the integration over the partial spherical base. Since the contribution from the cone surface is readily obtained in the previous section, only the contribution from the spherical section is considered here. The backscattered field of a cone-sphere is simply the superposition of contributions from each surface integration.

Now introduce the primed coordinate system referred to the center of the spherical base as shown in Fig. A.3. Then the vector from the origin of the original coordinate system (i.e., x, y, and z) to the spherical base in terms of the new primed system (i.e., x', y', and z') is given by

\[ \vec{r}_s = \hat{x}(-b \cos \theta_s \sin \theta' \cos \phi' - b \cos \theta' \sin \theta_s) + \hat{y}b \sin \theta' \sin \phi' \]
\[ + \hat{z}(b \sin \theta_s \sin \theta' \cos \phi' - b \cos \theta_s \cos \theta' + \frac{a}{\tan \alpha} + a \tan \alpha) \]

where \( \theta_s \) is the angle of incidence, \((b, \theta', \phi')\) is the spherical coordinate of the spherical base with respect to the new primed coordinate system, \( b \) is the radius of the spherical base, and \( a \) is the radius of curvature at the junction. The surface outward normal vector at \( P \) on the spherical base is given by

\[ \hat{n}' = \hat{x}(-\cos \theta_s \cos \phi' \sin \theta' - \sin \theta_s \cos \theta') + \hat{y} \sin \theta' \sin \phi' \]
\[ + \hat{z}(\sin \theta_s \sin \theta' \cos \phi' - \cos \theta_s \cos \theta'). \]

For the incident field given by the equation A.1, the GO current on the spher-
Figure A.3: Geometry of a cone-sphere under consideration
The base is given by
\[
\mathcal{J}_s = \frac{2}{Z_0} e^{jk\hat{r}\cdot\hat{r}_s} \{ E^i_\theta[\hat{x}(\sin \theta_s \cos \phi' \sin \theta' - \cos \theta_s \cos \theta')] \\
+ \hat{z}(\cos \theta_s \sin \theta' \cos \phi' + \sin \theta_s \cos \theta') \}
\]
\[
+ E^i_\phi[-\hat{x} \sin \theta_s \sin \phi' \sin \theta' - \hat{y} \cos \theta' - \hat{z} \cos \theta_s \sin \phi' \sin \theta']\}. \quad \text{A.17}
\]
Calculating the electric vector potential by the same procedure as that for a cone and evaluating the electric far-field radiated by this current, one obtains
\[
\bar{E} = \hat{\theta} b^2 E^i_\theta \frac{e^{-jk\hat{r}}}{r} \int_{\theta_1}^{\theta_2} \int_{\phi_1}^{\phi_2} e^{jk[(\frac{a}{\tan \alpha} + a \tan \alpha) \cos \theta_s - b \cos \theta']} \sin \theta' \cos \theta' d\phi' d\theta' \\
+ \hat{\phi} b^2 E^i_\phi \frac{e^{-jk\hat{r}}}{r} \int_{\theta_1}^{\theta_2} \int_{\phi_1}^{\phi_2} e^{jk[(\frac{a}{\tan \alpha} + a \tan \alpha) \cos \theta_s - b \cos \theta']} \sin \theta' \cos \theta' d\phi' d\theta'. \quad \text{A.18}
\]
For a cone-sphere, we also have the same backscattered field expressions for both polarized plane wave incidences. The integration limits are given by
\[
\theta_1 = \begin{cases} 
\frac{\pi}{2} & \text{if } \frac{\pi}{2} < \theta_s < \pi \\
0 & \text{otherwise},
\end{cases}
\]
\[
\theta_2 = \begin{cases} 
\frac{3\pi}{2} - \theta_s + \alpha & \text{if } \frac{3\pi}{2} < \theta_s < \pi \\
\pi & \text{otherwise}.
\end{cases}
\]
Note that the upper limit \(\phi_2\) of the \(\phi'\) integration depends upon the \(\theta'\) through the following equations:
\[
\cos \theta' = \frac{-a \sin \theta_s \cos \phi_2 - a \cos \theta_s + a \cos \theta_s}{b}, \quad \text{A.19}
\]
\[
\cot \phi_2 = \frac{a \sin \phi_2}{-a \cos \theta_s \cos \phi_2 + a \cos \theta_s + a \sin \theta_s}. \quad \text{A.20}
\]
The backscattering RCS of a cone-sphere using the PO approximation is presented and compared to the results obtained by the equivalent current approach in Fig. A.4. The backscatter angle in Fig A.4 is defined by

'BACKSCATTER ANGLE' = \pi - \theta_s.

The cone-sphere has radius \( a = 1.435\lambda \) and the half cone angle \( \alpha = 15^\circ \). As it is shown, the PO solution does not yield good result in the near axial region. The reason for that is the end point contribution from the PO radiation integral does not predict the correct creeping wave contributions.
Figure A.4: Comparison of backscattering RCS of a cone-sphere with $\alpha = 15^\circ$ and $a = 1.435\lambda$ between the PO solution (solid line), and the equivalent current approach (dashed line: horizontal polarization, black dots: vertical polarization)
APPENDIX B

EXPRESSION OF THE FIRST DIFFRACTION POINT IN TERMS OF THE SECOND DIFFRACTION POINT FOR THE DOUBLE DIFFRACTION

As was described in the double diffraction section for a finite cone, the relationship between the two diffraction points on the rim of the cone is given by

\[ v' = \pi + \phi' - \sin^{-1}(\sin \theta_i \sin \phi') \]  \hspace{1cm} B.1

where \( \theta_i \) is the angle of incidence and \( \phi' \) and \( v' \) represent azimuth angles of the first and the second diffraction points along the rim, respectively. As one notices, it is a simple matter to find \( v' \) for given \( \phi' \). However, it is not simple to obtain \( \phi' \) in terms of given \( v' \). We are now to solve equation (B.1) for \( \phi' \).

Equation (B.1) can be rewritten as

\[ \cos\left(\frac{\phi' - v'}{2}\right) = \sin \theta_i \sin \phi'. \] \hspace{1cm} B.2

Expanding the left hand side of equation (B.2), one obtains

\[ \cos \frac{\phi'}{2} \cos \frac{v'}{2} + \sin \frac{\phi'}{2} \sin \frac{v'}{2} = \sin \theta_i \sin \phi'. \] \hspace{1cm} B.3

Squaring both sides of the above equation yields

\[ \cos^2 \frac{\phi'}{2} \cos^2 \frac{v'}{2} + \sin^2 \frac{\phi'}{2} \sin^2 \frac{v'}{2} + \frac{1}{2} \sin v' \sin \phi' = \sin^2 \theta_i \sin^2 \phi'. \] \hspace{1cm} B.4
Since
\[ \cos^2 \frac{\phi'}{2} = \frac{1 + \cos \phi'}{2}, \]  
\[ \sin^2 \frac{\phi'}{2} = \frac{1 - \cos \phi'}{2}, \]
equation (B.4) becomes
\[ 2 \sin^2 \theta_i \sin^2 \phi' - \sin v' \sin \phi' - 1 = \cos v' \cos \phi'. \]  
Squaring both sides and letting \( x = \sin \phi' \), we have
\[ x^4 - \frac{\sin^2 v'}{\sin^4 \theta_i} x^3 + \frac{1 - 4 \sin^2 \theta_i}{4 \sin^4 \theta_i} x^2 + \frac{\sin v'}{2 \sin^4 \theta_i} x + \frac{\sin^2 v'}{4 \sin^4 \theta_i} = 0. \]  
Let
\[ p_3 = \frac{\sin^2 v'}{\sin^4 \theta_i}, \]  
\[ p_2 = \frac{1 - 4 \sin^2 \theta_i}{4 \sin^4 \theta_i}, \]  
\[ p_1 = \frac{\sin v'}{2 \sin^4 \theta_i}, \]  
\[ p_0 = \frac{\sin^2 v'}{4 \sin^4 \theta_i}. \]  
Then we have the quartic equation of \( x \) as
\[ x^4 + p_3 x^3 + p_2 x^2 + p_1 x + p_0 = 0. \]  
The quartic equation can be solved by using the solution of the cubic equation.
The quartic equation given by (B.13) can be rewritten as
\[ (x^2 + \frac{p_3}{2} x + y)^2 = [\frac{p_3}{4} - P_2 + 2y] x^2 + (P_3 y - P_1) x + y^2 - P_0 \]  
where \( y \) is such that the right hand side of the above equation becomes the perfect square; i.e., set the discriminant equal to 0. The discriminant of right hand side becomes
\[ y^3 - \frac{P_2 y^2}{2} - \frac{8P_0 - 2P_1 P_3}{8} y - \frac{P_1^2 + P_3^2 P_0 - 4P_2 P_0}{8} = 0. \]
which is called resolvent cubic. By letting

\[ p = \frac{-P_2}{2}, \]  
\[ q = \frac{-8P_0 - 2P_1P_3}{8}, \]  
\[ r = \frac{P_1^2 + P_3^2 - 4P_2P_0}{8}, \]  
\[ a = \frac{1}{3}(q - \frac{p^2}{3}), \text{ and} \]  
\[ b = \frac{1}{54}(2p^3 - 9qr + 27r) \]

\( y \) is obtained from equation B.15 as

\[ y = -\frac{p}{3} - \frac{3}{2}\sqrt[3]{b - \sqrt{b^2 + a^2}} - \frac{3}{2}\sqrt[3]{b + \sqrt{b^2 + a^2}}, \text{ or} \]  
\[ y = -w\frac{p}{3} - 3\sqrt[3]{b - \sqrt{b^2 + a^2}} - w\sqrt[3]{b + \sqrt{b^2 + a^2}}, \text{ or} \]  
\[ y = -w^2\frac{p}{3} - 3\sqrt[3]{b - \sqrt{b^2 + a^2}} - w^2\sqrt[3]{b + \sqrt{b^2 + a^2}} \]

where

\[ w = \frac{-1 + j\sqrt{3}}{2}, \]  
\[ w^2 = \frac{-1 - j\sqrt{3}}{2}, \]  
\[ w^3 = 1. \]

Plugging any one of the values of \( y \)'s into equation (B.14), we have

\[ (x^2 + \frac{P_3}{2}x + y)^2 = (\alpha x + \beta)^2 \]

where

\[ \alpha = \pm \sqrt{\frac{P_3^2}{4} - P_2 + 2y}, \]  
\[ \beta = \pm \sqrt{y^2 - P_0}. \]
Note that we have to choose $\alpha$ and $\beta$ such that $2\alpha\beta = P_2y - P_1$. By solving the equation (B.27), we obtain

$$x = \frac{1}{2}[\alpha - \frac{P_3}{2} \pm \sqrt{(\frac{P_3}{2} - \alpha)^2 + 4(\beta - y)}, \text{or} \quad \text{B.30}$$

$$x = \frac{1}{2}[-\alpha - \frac{P_3}{2} \pm \sqrt{(\frac{P_3}{2} + \alpha)^2 - 4(\beta + y)}. \quad \text{B.31}$$

Since $0 < \phi' < \pi$ and $x = \sin \phi'$, one has to choose the proper root among four possible roots such that $0 < x < 1$. Also, $\sin \phi'$ must satisfy equation B.1.
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