INFORMATION TO USERS

The most advanced technology has been used to photograph and reproduce this manuscript from the microfilm master. UMI films the text directly from the original or copy submitted. Thus, some thesis and dissertation copies are in typewriter face, while others may be from any type of computer printer.

The quality of this reproduction is dependent upon the quality of the copy submitted. Broken or indistinct print, colored or poor quality illustrations and photographs, print bleedthrough, substandard margins, and improper alignment can adversely affect reproduction.

In the unlikely event that the author did not send UMI a complete manuscript and there are missing pages, these will be noted. Also, if unauthorized copyright material had to be removed, a note will indicate the deletion.

Oversize materials (e.g., maps, drawings, charts) are reproduced by sectioning the original, beginning at the upper left-hand corner and continuing from left to right in equal sections with small overlaps. Each original is also photographed in one exposure and is included in reduced form at the back of the book. These are also available as one exposure on a standard 35mm slide or as a 17” x 23” black and white photographic print for an additional charge.

Photographs included in the original manuscript have been reproduced xerographically in this copy. Higher quality 6” x 9” black and white photographic prints are available for any photographs or illustrations appearing in this copy for an additional charge. Contact UMI directly to order.

UMI

University Microfilms International
A Bell & Howell Information Company
300 North Zeeb Road, Ann Arbor, MI 48106-1346 USA
313/761-4700 800/521-0500
Impedance properties of periodic linear arrays conformal to a
dielectric-clad infinite PEC cylinder

Simon, Jay Irwin, Ph.D.
The Ohio State University, 1989
IMPEDEANCE PROPERTIES OF PERIODIC LINEAR ARRAYS
CONFORMAL TO A DIELECTRIC-CLAD INFINITE PEC CYLINDER

A Dissertation
Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the
Graduate School of the Ohio State University

by
Jay Irwin Simon, B.S., M.S.

* * * * *
The Ohio State University
1989

Dissertation Committee:
Professor Benedikt A. Munk
Professor Leon Peters, Jr.
Professor Prabakar H. Pathak

Approved by:
Advisor
Department of Electrical Engineering
DEDICATION

To My Parents
ACKNOWLEDGEMENTS

I would like to thank my advisor and friend, Dr. Benedikt A. Munk, who encouraged me academically and personally throughout my doctoral program at Ohio State. I wish to extend thanks to my colleagues Paul A. Alexander and Bill Lange at Hughes Aircraft, Fullerton, CA. who were most helpful and supportive during my years as a Hughes Master's Fellow and later as a Hughes Doctoral Fellow. I also wish to thank my friend Jeff Hughes for many interesting and enjoyable late night discussions during the early stages of my dissertation research. I am also pleased to thank Leon Peters, Jr. and Prabhakar H. Pathak for being on my reading committee and for their helpful suggestions and insights.

Jay I. Simon 6/3/89
VITA

October 6, 1950 ......................... Born in Cincinnati, Ohio.


June, 1989 ............................... Ph.D. Electrical Engineering, The Ohio State University, Columbus, Ohio

FIELDS OF STUDY

Electromagnetic Theory .............................. Profs. B.A. Munk, R.G. Kouyoumjian

Communication Theory ............................. R. T. Compton, Jr.

Mathematics ........................................... Prof. J. T. Scheik

Statistics ............................................. Prof. Mark Berliner
# TABLE OF CONTENTS

**DEDICATION**

**ACKNOWLEDGEMENTS**

**VITA**

**LIST OF FIGURES**

**LIST OF SYMBOLS**

<table>
<thead>
<tr>
<th>CHAPTER</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>I. INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>II. OVERVIEW OF THEORY</td>
<td>5</td>
</tr>
<tr>
<td>2.1 The Electric Field Integral Equation</td>
<td>5</td>
</tr>
<tr>
<td>2.2 Helical Array Elements</td>
<td>13</td>
</tr>
<tr>
<td>2.3 Subsectional Array Field</td>
<td>15</td>
</tr>
<tr>
<td>2.4 Array Mutual Impedance</td>
<td>19</td>
</tr>
<tr>
<td>III. ARRAY MUTUAL IMPEDANCE</td>
<td>22</td>
</tr>
<tr>
<td>3.1 The Array Fields</td>
<td>22</td>
</tr>
</tbody>
</table>
<pre><code>| 3.1.1 Shorthand Notation for Bessel Functions | 23 |
| 3.1.2 Vertical Hertzian Elements | 23 |
| 3.1.3 Transverse Hertzian Elements | 28 |
</code></pre>
3.2 Discussion ......................................................... 32
3.3 Explicit Series Representation for Mutual Impedance .... 34
  3.3.1 Special Case: Dielectric Layer Absent ............... 35
3.4 Series Convergence ............................................. 38
  3.4.1 Vertical Directed Dipoles, No Dielectric Layer .... 39
  3.4.2 Transverse Dipoles, No Dielectric Layer .......... 43
  3.4.3 Arbitrary Element Pitch, No Dielectric Layer .... 46
  3.4.4 Convergence Tests ........................................ 52
3.5 Numerical Results .............................................. 52

IV. INDUCED VOLTAGE ............................................. 76
  4.1 Incident TM plane wave ....................................... 78
    4.1.1 Axial fields in region 1 ............................. 79
    4.1.2 Axial fields in region 2 ............................. 79
    4.1.3 Induced Voltage ........................................ 82
  4.2 Incident TE plane wave ....................................... 85
    4.2.1 Axial fields in region 1 ............................. 85
    4.2.2 Axial fields in region 2 ............................. 86
    4.2.3 Induced voltage ........................................ 88
  4.3 Summary of Results ......................................... 90

V. RADIATION FIELDS ............................................... 91
  5.1 Preliminary Remarks ......................................... 91
  5.2 Scattered Field due to Arrays ............................. 92
    5.2.1 Special Case: No Dielectric, Vertical Dipoles .... 98
    5.2.2 Special Case: No Dielectric, Transverse Dipoles .. 100
  5.3 Field Scattered from the Dielectric Coated Cylinder ...... 101
5.3.1 TM Incident Plane Wave ......................................................... 101
5.3.2 TE Incident Plane Wave ............................................................ 102

VI. A HYBRID GTD APPROACH .......................................................... 105
6.1 Introduction .................................................................................................. 105
6.2 Overview of Method ................................................................................. 106
6.3 Mathematical Description ........................................................................ 110
6.4 The Ray-Optic Approximation ................................................................. 111
6.5 Numerical Results .................................................................................... 117

APPENDICES ........................................................................................................... 120
A. FIELDS RADIATED BY PERIODIC LINEAR LINEAR ARRAYS OF HERTZIAN ELEMENTS IN THE PRESENCE OF AN INFINITE PEC CYLINDER ................................................................. 129
A.1 Preliminary Remarks ................................................................................ 129
A.2 Stratton-Chu Formulation ..................................................................... 131
A.3 The Scalar Boundary-Value Problems ............................................. 135
A.4 Solutions for the Scalar Green’s Functions ...................................... 135
  A.4.1 Single Point Source ....................................................................... 136
  A.4.2 Periodic Array of Point Sources .................................................. 137
A.5 Solution for the Radial Green’s Function ......................................... 138
A.6 Array Field Solutions ............................................................................... 139
  A.6.1 (z) Directed Hertzian Current Elements .................................... 141
  A.6.2 (ϕ) Directed Hertzian Current Elements .................................... 143
  A.6.3 (ϕ) Directed Hertzian Current Elements .................................... 146
A.7 Free Space Fields .................................................................................... 148

vii
LIST OF FIGURES

1 Cutaway view of a periodic linear array embedded in an annular dielectric layer and radiating in the presence of an infinite PEC cylinder................................................. 6

2 Decomposition of a linear periodic array into 3 periodic subsectional arrays when 3 PWS expansion functions are used. .......... 10

3 Subsectional element geometry in primed and unprimed coordinate system................................................................. 13

4 Truth table for array field matrix and scattering coefficients for various combinations of scan and dielectric layer ............... 33

5 Convergence of array self-reactance as a function of the angular modal index \( m \) for the \( n = 0 \) Floquet wave, for various dipole pitch angles; uncoated cylinder case ......................... 49

6 Convergence of array self-reactance as a function of the angular modal index \( m \) for the \( n = 1 \) Floquet wave, for various dipole pitch angles; uncoated cylinder case ......................... 50

7 Convergence of array self-reactance as a function of the number Floquet wave modes included; for various dipole pitch angles; uncoated cylinder case ........................................... 51
Array self-impedance as a function of frequency with the number of PWS modes as a parameter; vertical element case; no dielectric;
\[ r_w = 0.001 \lambda_0, D_z = 0.6 \lambda_0, 2L = 0.5 \lambda_0, a = 0.227 \lambda_0, \rho_o = 0.475 \lambda_0 \]...

Array self-impedance as a function of frequency with the number of PWS modes as a parameter; transverse element case; no dielectric;
\[ r_w = 0.001 \lambda_0, D_z = 0.6 \lambda_0, 2L = 0.5 \lambda_0, a = 0.227 \lambda_0, \rho_o = 0.475 \lambda_0 \]...

Array self-impedance as a function of frequency with the number of PWS modes as a parameter; vertical element case; dielectric present;
\[ r_w = 0.001 \lambda_0, D_z = 0.6 \lambda_0, 2L = 0.5 \lambda_0, a = 0.227 \lambda_0, \rho_o = 0.475 \lambda_0, \epsilon_r = 2.0, w = 0.248 \lambda_0 \]...

Array self-impedance as a function of frequency with the number of PWS modes as a parameter; transverse element case; dielectric present;
\[ r_w = 0.001 \lambda_0, D_z = 0.6 \lambda_0, 2L = 0.5 \lambda_0, a = 0.227 \lambda_0, \rho_o = 0.475 \lambda_0, S_z = 0, \epsilon_r = 2.0, w = 0.248 \lambda_0 \]...

Array self-impedance as a function of frequency for various element pitch angles; without cylinder; no dielectric; \[ r_w = 0.001 \lambda_0, D_z = 0.6 \lambda_0, 2L = 0.5 \lambda_0, \rho_o = 0.475 \lambda_0, S_z = 0 \]...

Array self-impedance as a function of frequency for various element pitch angles; with cylinder; no dielectric; \[ r_w = 0.001, D_z = 0.6 \lambda_0, 2L = 0.5 \lambda_0, a = 0.227 \lambda_0, \rho_o = 0.475 \lambda_0, S_z = 0 \]...

Array self-impedance as a function of frequency for various element pitch angles; with cylinder; scan case: \[ S_z = 0.5; no \ dielectric; r_w = 0.001 \lambda_0, D_z = 0.6 \lambda_0, 2L = 0.5 \lambda_0, a = 0.227 \lambda_0, \rho_o = 0.475 \lambda_0 \]...
15 Array mutual impedance $Z_{2,1}^P$ in the complex plane as a function of the array angular separation; vertical elements; with and without dielectric; $r_w = 0.001\lambda_0$, $D_z = 0.6\lambda_0$, $2L = 0.5\lambda_0$, $a = 0.227\lambda_0$, $\rho_0 = 0.475\lambda_0$, $\varepsilon_r = 2.0$, $w = 0.248\lambda_0$................................. 64

16 Direct component of array mutual impedance $Z_{2,1}^{P,\text{dir}}$ as a function of the array angular separation; vertical elements; no dielectric; $r_w = 0.001\lambda_0$, $D_z = 0.6\lambda_0$, $2L = 0.5\lambda_0$, $a = 0.227\lambda_0$, $\rho_0 = 0.475\lambda_0$ ....... 65

17 Scatter component of array mutual impedance $Z_{2,1}^{P,\text{scat}}$ as a function of the array angular separation; no dielectric; vertical elements; $r_w = 0.001\lambda_0$, $D_z = 0.6\lambda_0$, $2L = 0.5\lambda_0$, $a = 0.227\lambda_0$, $\rho_0 = 0.475\lambda_0$, $\varepsilon_r = 2.0$, $w = 0.248\lambda_0$................................. 66

18 Direct component of array mutual impedance $Z_{2,1}^{P,\text{dir}}$ as a function of the array angular separation; with dielectric; vertical elements; $r_w = 0.001\lambda_0$, $D_z = 0.6\lambda_0$, $2L = 0.5\lambda_0$, $a = 0.227\lambda_0$, $\rho_0 = 0.475\lambda_0$, $\varepsilon_r = 2.0$, $w = 0.248\lambda_0$................................. 67

19 Scatter component of array mutual impedance $Z_{2,1}^{P,\text{scat}}$ as a function of the array angular separation; with dielectric; vertical elements; $r_w = 0.001\lambda_0$, $D_z = 0.6\lambda_0$, $2L = 0.5\lambda_0$, $a = 0.227\lambda_0$, $\rho_0 = 0.475\lambda_0$, $\varepsilon_r = 2.0$, $w = 0.248\lambda_0$................................. 68

20 Array mutual impedance $Z_{2,1}^P$ in the complex plane as a function of the array angular separation; transverse elements; with and without dielectric; $r_w = 0.001\lambda_0$, $D_z = 0.6\lambda_0$, $2L = 0.5\lambda_0$, $a = 0.227\lambda_0$, $\rho_0 = 0.475\lambda_0$, $\varepsilon_r = 2.0$, $w = 0.248\lambda_0$................................. 69

21 Direct component of array mutual impedance $Z_{2,1}^{P,\text{dir}}$ as a function of the array angular separation; transverse elements; no dielectric; $r_w = 0.001\lambda_0$, $D_z = 0.6\lambda_0$, $2L = 0.5\lambda_0$, $a = 0.227\lambda_0$, $\rho_0 = 0.475\lambda_0$ ....... 70
22 Scatter component of array mutual impedance $Z_{2,1}^{P,scat}$ as a function of the array angular separation; no dielectric; transverse elements; 
    \( r_w = .001 \lambda_o, \quad D_z = 0.6 \lambda_o, \quad 2L = 0.5 \lambda_o, \quad a = .227 \lambda_o, \quad \rho_o = .475 \lambda_o \) .... 71

23 Direct component of array mutual impedance $Z_{2,1}^{P,dir}$ as a function of the array angular separation; with dielectric; transverse elements; 
    \( r_w = .001 \lambda_o, \quad D_z = 0.6 \lambda_o, \quad 2L = 0.5 \lambda_o, \quad a = .227 \lambda_o, \quad \rho_o = .475 \lambda_o, \)
    \( \epsilon_r = 2.0, w = .248 \lambda_o \) ............................... 72

24 Scatter component of array mutual impedance $Z_{2,1}^{P,scat}$ as a function of the array angular separation; with dielectric; transverse elements; 
    \( r_w = .001 \lambda_o, \quad D_z = 0.6 \lambda_o, \quad 2L = 0.5 \lambda_o, \quad a = .227 \lambda_o, \quad \rho_o = .475 \lambda_o, \)
    \( \epsilon_r = 2.0, w = .248 \lambda_o \) ............................... 73

25 Array mutual impedance $Z_{2,1}^{P}$ as a function of the relative axial offset \( \Delta z \); no dielectric; vertical elements; angular separation \( \Delta \phi = 12.06 \text{deg.} \); 
    \( r_w = .001 \lambda_o, \quad D_z = 0.6 \lambda_o, \quad 2L = 0.5 \lambda_o, \quad a = .227 \lambda_o, \quad \rho_o = .475 \lambda_o \) .... 74

26 Array mutual impedance $Z_{2,1}^{P}$ as a function of the relative axial offset \( \Delta z \); no dielectric; transverse elements; angular separation \( \Delta \phi = 0 \text{deg.} \); 
    \( r_w = .001 \lambda_o, \quad D_z = 0.6 \lambda_o, \quad 2L = 0.5 \lambda_o, \quad a = .227 \lambda_o, \quad \rho_o = .475 \lambda_o \) 75

27 Incident plane wave impinging on array structure (dielectric layer is not shown). ................................. 77

28 Truth table for matrix and scattering coefficients; TM or TE plane wave incidence on dielectric coated cylinder; for various combinations of scan and dielectric layer. ................................. 83
29 Truth table for array field scattering coefficients; field components for various combinations of scan and dielectric layer; results for axial and transverse current components. .............................................................. 96

30 Truth table for cylinder field scattering coefficients; field components for various combinations of scan and dielectric layer; TM and TE incident plane wave cases. ................................................................. 97

31 Polarization of field scattered from antenna structure due to an incident plane wave of arbitrary polarization; for various combinations of scan and dielectric layer. ................................................................. 104

32 The rays and regions associated with scattering by a smooth convex cylinder. .................................................................................................................... 107

33 Mutual coupling between two arrays of vertical dipoles radiating in the presence of an uncoated PEC cylinder as a function angular displacement; comparison of eigenfunction-MM and hybrid GTD solutions; cyl radius $a = .1\lambda$, array displacement $\rho_o = .75\lambda$ ........ 119

34 Mutual coupling between two arrays of vertical dipoles radiating in the presence of an uncoated PEC cylinder as a function angular displacement; comparison of eigenfunction-MM and hybrid GTD solutions; cyl radius $a = .25\lambda$, array displacement $\rho_o = .75\lambda$ ....... 120

35 Mutual coupling between two arrays of vertical dipoles radiating in the presence of an uncoated PEC cylinder as a function angular displacement; comparison of eigenfunction-MM and hybrid GTD solutions; cyl radius $a = .5\lambda$, array displacement $\rho_o = .75\lambda$ ....... 121
Mutual coupling between two arrays of vertical dipoles radiating in the presence of an uncoated PEC cylinder as a function angular displacement; comparison of eigenfunction-MM and hybrid GTD solutions; cyl radius $a = .65\lambda$, array displacement $\rho_o = .75\lambda$ . . . 122

Mutual coupling between two arrays of vertical dipoles radiating in the presence of an uncoated PEC cylinder as a function angular displacement; comparison of eigenfunction-MM and hybrid GTD solutions; cyl radius $a = .1\lambda$, array displacement $\rho_o = 1.25\lambda$ . . . 123

Mutual coupling between two arrays of vertical dipoles radiating in the presence of an uncoated PEC cylinder as a function angular displacement; comparison of eigenfunction-MM and hybrid GTD solutions; cyl radius $a = .25\lambda$, array displacement $\rho_o = 1.25\lambda$ . . . 124

Mutual coupling between two arrays of vertical dipoles radiating in the presence of an uncoated PEC cylinder as a function angular displacement; comparison of eigenfunction-MM and hybrid GTD solutions; cyl radius $a = .5\lambda$, array displacement $\rho_o = 1.25\lambda$ . . . 125

Mutual coupling between two arrays of vertical dipoles radiating in the presence of an uncoated PEC cylinder as a function angular displacement; comparison of eigenfunction-MM and hybrid GTD solutions; cyl radius $a = .75\lambda$, array displacement $\rho_o = 1.25\lambda$ . . . 126

Mutual coupling between two arrays of vertical dipoles radiating in the presence of an uncoated PEC cylinder as a function angular displacement; comparison of eigenfunction-MM and hybrid GTD solutions; cyl radius $a = 1.0\lambda$, array displacement $\rho_o = 1.25\lambda$ . . . 127
Mutual coupling between two arrays of vertical dipoles radiating in the presence of an uncoated PEC cylinder as a function angular displacement; comparison of eigenfunction-MM and hybrid GTD solutions; cyl radius $a = 1.15\lambda$, array displacement $\rho_o = 1.25\lambda$.

Pattern function for transverse (zero pitch) elements as a function of the angle mode index $m$, for the first few Floquet modes.

Pattern function for elements with 30 deg. pitch angle as a function of the angle mode index $m$, for the first few Floquet modes.

Pattern function for elements with 45 deg. pitch angle as a function of the angle mode index $m$, for the first few Floquet modes.

Pattern function for elements with (steep) 89.9 deg. pitch angle as a function of the angle mode index $m$, for the first few Floquet modes.

Pattern function for vertical elements as a function of the angle mode index $m$, for the first few Floquet modes.

Pattern function as a function of the angle mode index $m$, with frequency as a parameter; element pitch $\alpha = 55$ deg.; first few Floquet modes are shown.

Pattern function as a function of the angle mode index $m$, with the dipole length as a parameter; element pitch $\alpha = 55$ deg.; first few Floquet modes are shown.

Pattern function as a function of the angle mode index $m$, with the obliquity factor $S_2$ as a parameter; element pitch $\alpha = 55$ deg.; first few Floquet modes are shown.

Equivalent wire radius as a function of dipole length for linear array of vertical dipoles radiating in free space.
52 Comparison of self-reactance using the axial filament approximation versus the exact integration method .................................................. 176
53 Self impedance of periodic linear array of vertical dipoles as a function of frequency; comparison of exact integration and axial filament methods; number of PWS current modes as a parameter; wire radius
\[= .001\lambda \text{ at } f_o\] .................................................. 177
54 Self impedance of periodic linear array of vertical dipoles as a function of frequency; comparison of exact integration and axial filament methods; number of PWS current modes as a parameter; wire radius
\[= .015\lambda \text{ at } f_o\] .................................................. 178
55 Magnitude and phase of current distribution on reference element of a periodic linear array of axial dipoles; 1, 3 and 5 PWS current modes; freq=0.8\(f_o\);wire radius = .001\(\lambda\) ........................................... 180
56 Magnitude and phase of current distribution on reference element of a periodic linear array of axial dipoles; 1, 3 and 5 PWS current modes; freq=\(f_o\);wire radius = .001\(\lambda\) ........................................... 181
57 Magnitude and phase of current distribution on reference element of a periodic linear array of axial dipoles; 1, 3 and 5 PWS current modes; freq=1.2\(f_o\);wire radius = .001\(\lambda\) ........................................... 182
58 Magnitude and phase of current distribution on reference element of a periodic linear array of axial dipoles; 1, 3 and 5 PWS current modes; freq=0.8\(f_o\);wire radius = .015\(\lambda\) ........................................... 183
59 Magnitude and phase of current distribution on reference element of a periodic linear array of axial dipoles; 1, 3 and 5 PWS current modes; freq=\(f_o\);wire radius = .015\(\lambda\) ........................................... 184
60 Magnitude and phase of current distribution on reference element of a periodic linear array of axial dipoles; 1, 3 and 5 PWS current modes; freq=1.2/\omega_0;wire radius = 0.015\lambda

61 Current tube geometry with image current tube; equivalent axial filament model with image

62 Self-impedance as a function of position of test array on the averaging circle; vertical element case (\alpha = 90 deg.)

63 Self-impedance as a function of position of test array on the averaging circle; element pitch \alpha = 75 deg.

64 Self-impedance as a function of position of test on the averaging circle; element pitch \alpha = 30 deg.

65 Self-impedance as a function of position of test on the averaging circle; transverse elements (\alpha = 0 deg.)

66 Coupling diagram for slant set-up. In this example, there are 3 array groups, 2 tiers, and 3 PWS current modes.

67 Structure of impedance matrix; 3 array groups, 2 tiers, 3 PWS modes

68 Non-redundant impedance matrix entries for vertical elements, unscanned case.

69 Non-redundant impedance matrix entries for vertical elements, scanned case.

70 Non-redundant impedance matrix entries for transverse elements, unscanned case.

71 Non-redundant impedance matrix entries for transverse elements, scanned case.
Non-redundant impedance matrix entries for slanted elements, un-scanned case. .............................................. 218

Non-redundant impedance matrix entries for slanted elements, scanned case. ................................................. 219
LIST OF SYMBOLS

α .................. Array element pitch angle.

a .................. Cylinder radius.

\( a_m \) .............. Soft scattering coefficient = \( \frac{J_m(\beta u_p a)}{H_m^{(2)}(\beta u_p a)} \).

l ..................... Radial distance to outermost boundary of dielectric layer.

\( b_m \) .............. Hard scattering coefficient = \( \frac{J'_m(\beta u_p a)}{H_m^{(2)}(\beta u_p a)} \).

\( \beta_i \) ............. Wave number \( 2\pi/\lambda \) in region \( i \); without subscript, free-space wave number understood.

\( D_z \) .............. Array vertical interelement spacing.

\( \Delta L \) ............. Subsectional element arm length.

\( \epsilon_m \) ............. equals 1 when \( m=0 \), 2 otherwise.

\( \epsilon_r \) ............. Relative permittivity (real) in dielectric layer.

d\( \tilde{E}_{x_{i,j}} \) ............... \( x_i \) component of the electric field in region \( j \) due to a periodic Hertzian array of short current elements oriented in the \( \hat{x}_k \) direction, where \( (x_1, x_2, x_3) = (\rho, \phi, z) \).

\( \tilde{E}^{\text{inc}} \) ............. Incident electric field impinging on array structure.

\( E_t^{\text{inc}} \) ............. Tangential component of incident electric field evaluated on a particular array reference element.

\( E_t \) .............. Tangential component of total electric field due to all \( N \) arrays evaluated on a particular array reference element.

\( F_0 \) ............. Reference frequency; design distance parameters are given in wavelengths at this frequency.
\( f_j^q \) \( j^{th} \) PWS current mode on \( q^{th} \) array.

\( f(q) \) Normalized PWS current mode on \( q^{th} \) array; depends on subsectional element length.

\( \overline{\Gamma}_\pi \) Periodic Green’s dyadic; satisfies boundary conditions on cylinder surface.

\( G_{\rho m} \) Radial Green’s function.

\( G_{T_m} \) Transverse Green’s function.

\( G^A \) Composite Green’s function for single point source.

\( G^B \) Composite Green’s function for periodic point source.

\( dH_{i,j}^{x_k} \) \( x_i \) component of the electric field in region \( j \) due to a periodic Hertzian array of short current elements oriented in the \( \hat{x}_k \) direction, where \((x_1, x_2, x_3) = (\rho, \phi, z)\).

\( H_{ix} \) Hankel function of order \( m \) with argument equal to a distance variable \( x \) times \( \beta_i u_{ip} \) referred to the \( i^{th} \) region, where \( i=1,2 \).

\( H_x \) same as \( H_{2x} \), used in the case that dielectric layer is absent.

\( I_j^q \) Amplitude of \( j^{th} \) PWS current mode of the \( q^{th} \) array; array index \( q \) may be suppressed.

\( J_{ix} \) Bessel function of order \( m \) with argument equal to a distance variable \( x \) times \( \beta_i u_{ip} \) referred to the \( i^{th} \) region, where \( i=1,2 \).

\( J_x \) same as \( J_{2x} \), used in the case that dielectric layer is absent.

\( \kappa_z \) Axial wavenumber \( = \beta_i u_{zi} \) in region \( i \); has same value in each layer.

\( I_e^P \) Equivalent electric line current amplitude which radiates same propagating mode emitted by array of vertical elements.

\( I_i^q \) Element of array current vector; terminal current at reference terminals of subarray \( q \) of array \( i \).

\( \Delta L \) Subsectional element arm length.

\( L \) Dipole arm length.

\( \lambda_o \) Wavelength at reference frequency \( F_o \).
\( m \) ................. Angular mode index.

\( M \) ................. Number of PWS current modes.

\( n \) ................. Floquet cylinder wave index.

\( N \) ................. Number of arrays placed about cylinder.

\( \hat{p} \) ................. Unit tangent vector at point on dipole wire axis; in reference to Hertzian element arrays, \( \hat{p} \) is the vector direction of the current moment.

\( \phi_{\Delta L} \) ................. Azimuthal angle span of subsectional element arm.

\( \phi_L \) ................. Azimuthal angle span of physical element arm.

\( \phi_p \) ................. Azimuthal angle of incident plane wave.

\( P_{n}^{\pm}(m) \) .......... Plus/minus element pattern function corresponding to \( n^{th} \) Floquet mode and \( m^{th} \) angular mode.

\( P_n \) ................. Element pattern function: vertical elements; function of \( n \) but not \( m \).

\( P(m) \) ................. Element pattern function: transverse elements; function of \( m \) but not \( n \).

\( PWS \) ................. Piece-Wise Sinusoidal; refers to subsectional current basis function used in this dissertation.

\( \rho_o \) ................. Radial distance to array terminals from \( z \)-axis; used when all arrays are equidistant from \( z \)-axis.

\( R_{nm}^{\hat{\phi} \rightarrow \hat{\phi}} \) .......... Fourier coefficient representing interactions between \( \hat{\phi} \) current vector components on source and test arrays.

\( R_{nm}^{\hat{z} \rightarrow \hat{z}} \) .......... Fourier coefficient representing interactions between axial current vector components on source and test arrays.

\( R_{nm}^{\hat{z} \rightarrow \hat{\phi}} \) .......... Fourier coefficient representing interactions between \( \hat{\phi} \) current vector component on source array and \( \hat{z} \) current vector components on test array.

\( R_{nm}^{\hat{\phi} \rightarrow \hat{z}} \) .......... Fourier coefficient representing interactions between \( \hat{z} \) current vector component on source array and \( \hat{\phi} \) current vector components on test array.

xxi
\( \vec{R}_o \) ............... Position vector of terminals of reference element of a given array with respect to (absolute) \((x,y,z)\) coordinate system.

\( \vec{R}_q \) ............... Position vector of point on reference element of array \(q\) with respect to (absolute) \((x,y,z)\) coordinate system.

\( \vec{R}_s \) ............... Local position vector of point on reference element on a given array (say the \(q^{th}\) array) with respect to coordinate system whose origin is located at \(R_o\).

\( r_{eq} \) ............... Equivalent wire radius.

\( r_w \) ............... Physical wire radius.

\( S_z \) ............... Obliquity factor; in passive mode: \( S_z = \cos \theta_p \).

\( \theta_p \) ............... Elevation angle of incident plane wave with respect to positive \(z\)-axis (for source at \(z = -\infty\) with \(\theta_p = 0\)).

\( u_{iz} \) ............... Normalized axial wavenumber in region \(i\), \(i = 1,2\), \( u_{iz} = \frac{S_z}{\sqrt{\epsilon_r i}} + \frac{n \lambda_i}{D_z} \); when dielectric layer is absent the \(i\) subscript is suppressed.

\( u_z \) ............... Normalized axial wavenumber in free-space, see above.

\( u_{ip} \) ............... Normalized radial wavenumber = \(\sqrt{1 - u_{iz}^2}\); when dielectric layer is absent the \(i\) subscript is suppressed.

\( u_{\rho} \) ............... Normalized radial wavenumber in free-space, see above.

\( V_i^q \) ............... Element of voltage vector; open-circuit voltage induced at reference terminals of subarray \(q\) of array \(i\).

\( w \) ............... Width of annular dielectric layer.

\( \zeta \) ............... Tangent of \(\alpha\), the element pitch angle.

\( Z_A^q \) ............... Active array scan impedance of array \(q\).

\( z_o \) ............... Relative axial displacement of array reference element terminals with respect to fixed coordinate origin.

\( Z_i \) ............... Characteristic impedance of medium in region \(i\), \(i = 1,2\).

\( Z_d \) ............... Characteristic impedance in dielectric layer; same as \(Z_1\).
$Z_o$  ............... Characteristic impedance of free-space.

$Z_L$  ............... Dipole load impedance.

$Z_{i,j}$  ............. Port mutual impedance between arrays $i$ and $j$.

$Z_{q,p}^{ij}$  ........ Element of mutual impedance matrix; subsectional array mutual between subarrays $i$ and $j$ on arrays $q$ and $p$, respectively.

$Z_{i,j}^{dir}$  ........ Direct component (cylinder absent) of array mutual between subarrays $i$ and $j$; physical array superscripts $q, p$ are usually suppressed.

$Z_{i,j}^{scat}$  ......... Scatter component of array mutual between subarrays $i$ and $j$; physical array superscripts are $q, p$ usually suppressed.

$Z_{i,j}^{tot}$  ........... Total array mutual coupling ($= Z_{i,j}^{dir} + Z_{i,j}^{scat}$); between subarrays $i$ and $j$; physical array superscripts $q, p$ are usually suppressed; "tot" superscript often suppressed but understood.

$Z_{ij}^{\hat{\phi}-\hat{\phi}}$  Contribution to mutual coupling between subsectional arrays $i$ and $j$ due to interactions between $\hat{\phi}$ current vector components on source and test arrays.

$Z_{ij}^{\hat{z}-\hat{z}}$  Contribution to mutual coupling between subsectional arrays $i$ and $j$ due to interactions between $\hat{z}$ current vector components on source and test arrays.

$Z_{ij}^{\hat{\phi}-\hat{z}}$  Contribution to mutual coupling between subsectional arrays $i$ and $j$ due to interactions between $\hat{\phi}$ current vector component on source array and $\hat{z}$ current component on test array.

$Z_{ij}^{\hat{\phi}-\hat{z}}$  Contribution to mutual coupling between subsectional arrays $i$ and $j$ due to interactions between $\hat{\phi}$ current vector component on source array and $\hat{z}$ current component on test array.
CHAPTER I
INTRODUCTION

In this dissertation the mutual impedance between periodic linear arrays of curved wire dipoles conformal to an infinite perfectly conducting cylinder is calculated. Mutual coupling must be properly compensated for in connection with the pattern synthesis problem. Similarly, the bistatic scattering from the array structure operating passively is strongly influenced by array coupling interactions.

A periodic version of Galerkin's method (MM) [1][2] is employed to solve the associated electric field integral equation (EFIE) for the array dipole currents. We emphasize that the periodicity of the antenna geometry is incorporated into the solution form, resulting in a significant reduction in the impedance matrix fill time when compared to that required for an large but finite (truncated) ensemble of arrays. In applying the moment method, piecewise sinusoid (PWS) basis and testing functions are used.

For a class of array elements considered in this paper, viz, elements lying on helical arcs, the moment method integrals may be evaluated in closed form. This element class includes vertical and transverse elements as special cases and so is more general than it might first appear. To be sure, the purely vertical or transverse element orientations alone or in combination seem more likely in application than arbitrarily pitched elements. Nevertheless, the helical dipole model is appealing because it provides a mathematical link between the vertical and transverse...
element cases and allows a deeper understanding of the convergence properties of the field and impedance series representations.

The electromagnetic field radiated by the array structure is expanded in a sum of inhomogeneous cylinder waves [3]; the mutual coupling terms are expressed in the form of double Fourier series in the axial and azimuthal coordinates. Essentially, the approach described in this paper is the cylindrical wave analog to the plane-wave expansion technique developed by Munk [4] for the analysis of periodic planar arrays. The present paper is a generalization of work recently completed by Hughes [5], in which the impedance properties of periodic linear arrays of vertical collinear dipoles was considered. Since the rate of series convergence is proportional to the cylinder diameter, the eigenfunction-MM approach described here is most suitable for small cylinders with diameters on the order of a few wavelengths or less.

The organization of this dissertation is as follows:

In chapter II, a general outline of the theoretical development is presented beginning with the EFIE for the unknown array dipole currents and culminating in the general series formulation for the array reaction impedances, i.e., the elements of the generalized impedance matrix resulting from an application of the moment method. The series representations for the array reaction impedances developed in chapter II are general in the sense that they involve expansion coefficients which are not explicitly derived until chapter III.

The solution for the field expansion coefficients is the subject of chapter III and entails the solution of a two region electromagnetic boundary value problem. The periodicity of the array fields in the azimuth and axial coordinates is exploited. A double Fourier series representation is introduced for the axial field components in the dielectric layer and in the free-space region. The other field components are
found from Maxwell's equations. The fields everywhere are determined (except for a proportionality constant) by boundary conditions enforced at the air-dielectric interface and at the cylinder surface. In order to determine the proportionality constant it is necessary to determine the fields due to a periodic linear array of arbitrarily oriented Hertzian current elements radiating in an infinite homogeneous dielectric space. This calculation is presented in detail in Appendix A and is based on an application of the Stratton-Chu vector wave formula, superposition, and the Poisson sum formula. Chapter III concludes with a section on numerical results.

In chapter IV, we consider the voltage induced at the reference terminals of an array due to a plane-wave excitation, when the antenna is operated in the passive mode. In order to solve the matrix equation given in chapter V for the unknown modal current amplitudes it is necessary to calculate the voltage vector or forcing function. When the array is operating in the active mode, the voltage vector consists of non-zero entries at each position corresponding to a subsectional element containing the physical reference element terminals (excited arrays only) and zero everywhere else. This corresponds to a delta-gap source model.

When the antenna operates as a scatterer, the voltage vector calculation entails solving a two-region boundary-value problem (BVP), similar to the one analysed in chapter III which pertained to the mutual coupling problem. The two BVPs differ only in their vector source functions — the source for the induced voltage problem considered in chapter IV is an oblique plane wave of arbitrary polarization.

Formulas for the "far-zone" fields radiating from the arrays-plus-cylinder structure are included in chapter V. The radiation field is decomposed into two components: the first component is the field due to the arrays radiating in the presence of the cylinder. The second component is the scatter contribution from
the cylinder. The total field radiated from the antenna structure is the sum of these two components. The far-zone fields are found without difficulty by using large argument approximations (in the Hankel functions containing the field point) in the field expressions derived previously in chapters III and IV.

In chapter VI a hybrid GTD solution is developed for the array mutual impedance in the special case for which vertical element arrays radiate in the presence of an uncoated cylinder. This result provides a check on the eigenfunction expansion method and also provides a means of calculating array impedances and antenna patterns for the large cylinder case.
CHAPTER II
OVERVIEW OF THEORY

2.1 The Electric Field Integral Equation

The antenna geometry is illustrated in figure 1. \( N \) linear periodic arrays are placed about the PEC cylinder, one of which is displayed in figure 1. By linear we mean that the terminals of all elements in an array are collinear; in general, the array elements are neither straight nor collinear. All dipoles in a given array are assumed to be identical with uniform vertical element spacing \( D_z \). Each element lies on a helical arc whose pitch may vary from array to array. The element pitch angle \( \alpha \) may range from zero (transverse elements) to 90 degrees (vertical elements). The dipole length and radial distance from the cylinder axis to the dipole terminals may also vary from array to array. Elements on the \( q^{th} \) array may be terminated with a load \( Z_{Lq} \) (not necessarily center loaded) which may vary from array to array but may not differ for elements of the same array. By assumption, the magnitude of the currents on all elements in an array are identical while the phase varies linearly from element to element. This situation corresponds to an incident plane wave (passive mode) or a linearly phased, uniform amplitude phased array (active mode). Thus, one need consider only the current distribution on a representative element of each array which is designated as the array reference element. The current distribution on the \( m^{th} \) element of the \( q^{th} \) array is given by

\[
I^{(q)}(\ell) = I^{(q)}_0(\ell)e^{-j\beta S_x m D_z}
\]  

(2.1)
Figure 1: Cutaway view of a periodic linear array embedded in an annular dielectric layer and radiating in the presence of an infinite PEC cylinder
where \( \ell \) is a length parameter and where \( I_0^{(q)}(\ell)e^{-j\beta S_z\ell_q} \) is the current distribution on the reference element centered at \( z = z_q \). The array phase slope is \( \beta S_z \) where \( \beta = \omega \sqrt{\mu_0 \varepsilon_0} \). An \( e^{j\omega t} \) time dependence is understood.

The electric field due to the array ensemble radiating in the presence of the cylinder and evaluated at a point \( \mathbf{R}_q \) on reference element \( q \) may be expressed as

\[
\mathbf{E}(\mathbf{R}_q) = j\beta Z_c \sum_{p=1}^{N} \int \int \bar{\mathbf{\Gamma}}_{\pi}(\mathbf{R}_q, \mathbf{R}_p) \cdot \mathbf{I}(\mathbf{R}_p) \, dS_p \tag{2.2}
\]

where the integrations are over the surface of the reference elements of each array. The endcap contributions are neglected. \( \bar{\mathbf{\Gamma}}_{\pi} \) is a periodic electric Green's dyadic and \( Z_c = \sqrt{\mu_0 / \varepsilon_0} \). \( \mathbf{I}(\mathbf{R}_p) \) is the surface current density on the reference element of the \( p \)th array (of \( N \) arrays total).

It is emphasized that we regard (2.2) as a formal integral equation; we do not propose to derive \( \bar{\mathbf{\Gamma}}_{\pi} \) for the general case in which the dielectric index of refraction differs from unity. (The free-space electric Green's dyadic \( \bar{\mathbf{\Gamma}}_{\pi_0} \), a periodic version of the Green’s dyadic formulated by Tai [6] for the case of a single Hertzian current element, is derived in Appendix A).

For thin wires we may replace the current tubes by current filaments and evaluate the electric field at several angular positions about the perimeter of the wire surface [7]. This averaging process is only necessary when the source and test arrays coincide or overlap. We denote the average field value at the position \( \ell_q \) on the reference element of array \( q \) as \( \bar{\mathbf{E}}(\ell_q) \). In this case, equation (2.2) is approximated by

\[
\bar{\mathbf{E}}(\ell_q) = j\beta Z_c \sum_{p=1}^{N} \int \bar{\mathbf{\Gamma}}_{\pi}(\ell_q, \ell_p) \cdot \mathbf{I}(\ell_p) \, d\ell_p \tag{2.3}
\]

where \( d\ell_p \) is tangent to the wire axis at the point \( \ell_p \).
The mathematical justification for impedance averaging in the context of thin-wire theory is discussed in detail in Appendix C. In Appendix D we describe the mechanics of the averaging process as well as the effects of the dielectric and array scan on mutual impedance, as a function of the test array position on the averaging circle. Appendix C also includes a detailed development of the equivalent radius concept as applied specifically to linear periodic arrays.

The component of $\mathbf{E}$ tangent to the wire axis (denoted by $E_t$) is

$$E_t(\ell_q) = j \mu u \sum_{p=1}^{N} \hat{\mathbf{e}}_q \cdot \mathbf{\tilde{I}}(\ell_q, \ell_p) \cdot I(\ell_p) \, d\hat{\mathbf{e}}_p$$

$$= -E_{t}^{inc}(\ell_q) + Z_{L_q} I(\ell_q) \delta(\ell_q - \ell_{L_q}) \quad (2.4)$$

where $\hat{\mathbf{e}}_q$ is the unit tangent vector. $E_{t}^{inc}(\ell)$ is the tangential component of the incident field excitation. $Z_{L_q}$ is the load on array $q$ at position $\ell_{L_q}$.

Applying Galerkin’s method using PWS subsectional basis and testing functions [8] we approximate the current distribution on the reference element of the $q^{th}$ array by the series

$$I^{(q)}(\ell) = \sum_{j=-M_o}^{M_o} I_j^q f_j^q(\ell), \quad (2.5)$$

where

$$f_j^q(\ell) = f^{(q)}(\ell - j \Delta L_q)$$

$$f^{(q)}(\ell) = \left\{ \begin{array}{ll}
\frac{\sin \beta(\Delta L_q - |\ell|)}{\sin \beta \Delta L_q}, & |\ell| < \Delta L_q \\
0, & |\ell| > \Delta L_q
\end{array} \right. \quad (2.6)$$

With each PWS current mode, $\Delta L_q = L_q/(M+1)$ is the subsectional element arm length, $M$ is the number of PWS current modes per array (for simplicity assumed the same, an odd integer, for all arrays) and $M_o = (M-1)/2$.
or subarray each element of which is equipped with a a pair of terminals (located in the center of each segment). These terminals are normally short-circuited with the possible exception of the physical port terminals in the center of segment \( i_0 \) (the loaded port). The situation is illustrated in figure 2, which shows the decomposition of an array into three subarrays, corresponding to three PWS modes.

To avoid confusion we must distinguish between physical (non-subdivided) arrays and subsectional arrays (also called subarrays) and their respective attributes. The term "array terminals" is understood to mean the terminals of the array reference element. Further, "the array element" or simply "the element" will always refer to the array reference element. For example "...integrate over element \( q \)" refers to integration over the length of the reference element of the \( q^{th} \) array. An "array" may refer to either a physical array or a subsectional array. In cases where the distinction is not clear from the context, we will distinguish between physical and subsectional arrays. Note that if only one PWS basis function is employed per array (a good approximation for very thin wire dipoles operating near resonance) there is no difference between subsectional and physical arrays. Array attributes (e.g., terminal current, element length, etc.) will generally refer to subsectional arrays; otherwise we will add the descriptors "port" or "physical". An exception is the active scan impedance (see below) which is by definition a port quantity. For example, the "port terminal current" refers to the current into the physical dipole terminals of the array reference element, whereas the "terminal current" refers to the current into the reference terminals of a subsectional array (the same as the amplitude coefficient of the corresponding PWS current mode). Thus, in equation (2.5), \( I^{(q)}(j \Delta L_q) \) is the port terminal current for the \( q^{th} \) array (assuming the dipole terminals are in the center of segment \( j \)) whereas \( I^{q}_{j} j^{q}_{j}(0) \) is the terminal
Figure 2: Decomposition of a linear periodic array into 3 periodic subsectional arrays when 3 PWS expansion functions are used.
current of the \( j^{th} \) subsectional array corresponding to PWS mode \( j \) on (physical) array \( q \).

Returning again to the EFIE, we obtain upon substituting equations (2.5),(2.6) and (2.7) into equation (2.3), multiplying both sides of equation (2.3) by \( f_i \) and integrating over the support of subsegment \( i \) on array \( q \) the matrix equation

\[
V_i^q = \sum_{p=1}^{N} \sum_{j=1}^{M} Z_{i,j}^q T_j^q + Z_{L_q} T_i^q \delta(i,i_o) \quad (2.8)
\]

where

\[
\begin{align*}
q &= 1, \ldots, N \\
i &= 1, \ldots, M \\
\delta(i,k) &= \begin{cases} 1, & i = k \\ 0, & i \neq k \end{cases}
\end{align*}
\]

assuming there are \( N \) arrays, each comprised of \( M \) modal subarrays. The index \( i_o \) corresponds to the port PWS segment.

Suppressing the array index (superscript), we have

\[
V_i = -\int E_i f_i d\ell_i = \int E^\text{inc}_i f_i d\ell_i \quad (2.9)
\]

and

\[
Z_{ij} = -\int E_{(ij)} f_i d\ell_i. \quad (2.10)
\]

where \( \ell_i \) is a length parameter measured from the terminal position of segment \( i \) and \( E_{(ij)}(\ell_i) \), the tangential electric field due to the \( j^{th} \) basis function \( f_j \) evaluated at element \( i \), is given by

\[
E_{(ij)}(\ell_i) = j\beta Z_c \int \hat{\ell}_i \cdot \overline{\Gamma_\pi}(\ell_i,\ell_j) \cdot \hat{\ell}_j f_j(\ell_j) \, d\ell_j \quad (2.11)
\]

We will refer the elements \( Z_{ij}^q \) (equation (2.10)) of the impedance matrix \([Z]\) as array reaction impedances; they are equal to the array mutual impedance between two subsectional arrays with PWS current distributions. The \( Z_{ij} \) are called reaction impedances because they are equivalent to the mutual impedance obtained
via the induced e.m.f. method [9]. It is desirable to introduce an array element indexing scheme, such that any symmetries inherent in the cylinder-plus-arrays geometry are reflected to advantage in the impedance matrix structure. In this way, one may systematically extract all redundancies arising from geometrical symmetries. One possible indexing system which we have found to be effective is introduced and discussed in Appendix E.

When the antenna operates in the scattering mode, $V_i^q$ is found from equation (2.9) where $E^{inc}$ is the incident electric field impinging on the antenna structure (see chapter IV for details). In the antenna mode, we may set $V_i^q = A^q \delta (i, i_o)$, corresponding to a delta-gap source model, where $A^q$ is the amplitude of an ideal voltage generator placed across the reference terminals (located on segment $i_o$) of the array $q$. Inverting the matrix equation for the modal current amplitudes, we obtain the active scan impedance of array $q$ defined as the ratio

$$Z_A^q = \frac{V_i^q}{I_i^q} = \frac{A^q}{I_i^q}$$

(2.12)

The array port mutual impedance $Z_{p,q}^P$ is defined as the open-circuit voltage at the terminals of the reference element of array $p$ when a unit current is impressed at the terminals of the reference element of array $q$, with all other array ports open and with all other subsectional terminals short-circuited. (Note that a unit magnitude current is also impressed on all elements of array $q$ in accordance with (2.1)). The superscript $P$ reminds us that $[Z^P]$ is a port impedance matrix to be distinguished from $[Z]$, the moment method impedance matrix. $[Z^P] = [Z]$ only if one PWS mode per array is employed. $[Z^P]$ is obtained from $[Z]$ by deleting the rows and columns of the short circuit admittance matrix $[Y] = [Z]^{-1}$ involving reactions between non-port subsegments. The resultant admittance matrix is inverted to obtain $[Z^P]$. 

12
2.2 Helical Array Elements

For arbitrary array elements, the integrals involved in the impedance expression (2.10) must be evaluated numerically. For the class of array elements lying on helical arcs one can evaluate the integrals in closed form and consequently obtain exact series representations for the array fields and array mutuals, as will now be demonstrated.

Consider a subsectional array with reference element terminals located in the center of the segment at $R_o = (\rho_o, \phi_o, z_o)$. The position vector for a point lying
on the reference element relative to the unprimed coordinate origin is
\[ \vec{R}' = \vec{R}_0 + \vec{R}_s \]
as illustrated in figure 3.

The local position vector \( \vec{R}_s \) may be parameterized alternately with respect to arc length \( \ell \), azimuthal angle displacement \( \phi' \), or axial displacement \( z' \). The parameterization \( \vec{R}_s = z' \hat{z} \) is obviously appropriate for the special vertical element case. The angular parametrization is useful for the more general pitched element case for which
\[ \vec{R}_s(\phi') = \rho_o \left[ \cos \phi' \hat{x}' + \sin \phi' \hat{y}' + \zeta \phi' \hat{z}' \right], -\phi_{\Delta L} \leq \phi' \leq \phi_{\Delta L} \]  
(2.13)
where
\[ \phi_{\Delta L} = \frac{\Delta L}{\rho_o \sqrt{1 + \zeta^2}} \]
\( \zeta \) is the element pitch parameter, and where the subsectional element length is 2\( \Delta L \).

The differential arc length along an element is related to the projected azimuthal angle displacement \( d\phi' \) by
\[ dl = \frac{dl}{d\phi'} d\phi' \]
where
\[ \frac{dl}{d\phi'} = \left| \frac{d\vec{R}_s}{d\phi'} \right| = \rho_o \sqrt{1 + \zeta^2} \]
Hence,
\[ dl = \rho_o \sqrt{1 + \zeta^2} d\phi'. \]  
(2.14)
Since the derivative of the axial position of a point on the element with respect to \( \phi' \) is \( \rho_o \zeta \) it follows from the chain rule that
\[ \frac{dl}{dz} = \frac{\sqrt{1 + \zeta^2}}{\zeta} \]
We define the element pitch angle \( \alpha \) as

\[
\alpha = \arcsin \frac{dz}{d\ell} = \arcsin \frac{\zeta}{\sqrt{1 + \zeta^2}}
\]

Thus, the parameter \( \zeta \) equals the tangent of \( \alpha \). It is the increase in vertical displacement per unit of horizontal displacement along the element arc. Note that \( \zeta = 0 \) and \( \zeta = \infty \) represent the vertical and transverse element cases respectively.

The dipole moment of a short element with orientation \( \hat{p} \) is \( I(\ell)d\ell \hat{p} \), where \( \hat{p} \), the unit tangent vector at the element axis is given by

\[
\hat{p} = \frac{d\hat{R}_s}{d\phi^i} \left| \frac{dR_s}{d\phi^j} \right| = \frac{1}{\sqrt{1 + \zeta^2}} \left( \hat{\phi} + \zeta \hat{z} \right)
\]  

(2.15)

2.3 Subsectional Array Field

In this section, an eigenfunction series representation for the electric field radiated by a subsectional array (embedded in a annular dielectric layer surrounding the PEC cylinder). It is possible to obtain a general series formulation for the electric field without actually solving the associated electromagnetic boundary value problem. That is, we may infer the general form of the fields from a knowledge of the fields due to a Hertzian array of current elements radiating in free space, as derived in Appendix A. Specific expressions for the series expansion coefficients are derived in chapter III.

Throughout this section the subarray and array indices will be suppressed. The electric field \( \overline{E} \) is understood to be the electric field radiated by a given subsectional array, say the \( i^{th} \) subarray of array \( q \). The currents on a given subsectional array will be further decomposed as an infinite ensemble of Hertzian arrays, as described below. The electric field radiated by a Hertzian array is denoted by \( d\overline{E} \). The total subsectional array field \( \overline{E} \) is found by integrating the field contributions from all the Hertzian arrays comprising that subarray.
The electric field due to an array of arbitrarily oriented Hertzian elements with reference element located at \((\rho_o, \phi_o, z_o)\) may be expressed as:

\[
d\vec{E} = j\omega\mu Ids \vec{1}_\pi \cdot \hat{p} = j\omega\mu Ids \vec{1}_\pi \cdot \left[(\hat{p} \cdot \hat{z}) \hat{z} + (\hat{p} \cdot \hat{\phi}) \hat{\phi} + (\hat{p} \cdot \hat{\rho}) \hat{\rho}\right] \tag{2.16}
\]

For the class of elements considered in this paper, the array currents have no radial component so the last term will not be present. The second line of equation (2.16) emphasizes the main point: each vector component of the electric field may be decomposed into contributions due to the \(\hat{z}\), \(\hat{\phi}\) and \(\hat{\rho}\) components of the current moment \(Ids \hat{p}\). Knowing the fields generated by a Hertzian array radiating in the presence of an uncoated cylinder (see Appendix A) together with equation (2.16), one infers that a general representation for the electric field in the dielectric layer may be expressed in the form of a double Fourier series:

\[
dE_z = \frac{Ids Z_c \beta}{\sqrt{1 + \zeta^2 4D_z}} \sum_{n=-\infty}^{\infty} e^{-j\beta u_z \Delta z} \sum_{m=0}^{\infty} \left[\zeta R_{nm}^{\hat{z}\rightarrow \hat{z}} \cos m \Delta \phi + R_{nm}^{\hat{z}\rightarrow \hat{\rho}} \sin m \Delta \phi\right] \tag{2.17}
\]

\[
dE_\phi = \frac{Ids Z_c \beta}{\sqrt{1 + \zeta^2 4D_z}} \sum_{n=-\infty}^{\infty} e^{-j\beta u_z \Delta z} \sum_{m=0}^{\infty} \left[R_{nm}^{\hat{\phi}\rightarrow \hat{\phi}} \cos m \Delta \phi + \zeta R_{nm}^{\hat{z}\rightarrow \hat{\phi}} \sin m \Delta \phi\right] \tag{2.18}
\]

\[
dE_\rho = \frac{Ids Z_c \beta}{\sqrt{1 + \zeta^2 4D_z}} \sum_{n=-\infty}^{\infty} e^{-j\beta u_z \Delta z} \sum_{m=0}^{\infty} \left[R_{nm}^{\hat{\rho}\rightarrow \hat{\rho}} \cos m \Delta \phi + R_{nm}^{\hat{z}\rightarrow \hat{\rho}} \sin m \Delta \phi\right] \tag{2.19}
\]

where \(\Delta \phi = \phi - \phi_o\), \(\Delta z = z - z_o\), and \(Z_c = \sqrt{\mu_o/\epsilon_o}\).

The superscript \(\hat{z} \leftarrow \hat{\phi}\) refers to the contribution to the axial field component due to the \(\hat{\phi}\) component of the current moment, etc. The Fourier coefficients
\( R_{nm} = R_{nm}(\rho, \rho_o) \) depend on the radial displacements to the field and source points and are also functions of frequency, the Floquet mode index \( n \), the angular mode index \( m \) and the obliquity factor \( S_z \).

In order to obtain the electric field due a subsectional array of finite elements we integrate the contributions from an infinite ensemble of Herztian element arrays juxtaposed along the entire length of the element. The contribution of each Hertzian array in the ensemble is weighted by the amplitude of the current density at the position of that subarray. Note that by assumption, the subsectional current distribution is PWS. The composite field is the vector sum of contributions from the infinite ensemble of Hertzian element arrays, viz,

\[
\bar{E} = j\beta Z_c \int \mathbf{\overrightarrow{I}} \cdot \mathbf{\hat{p}} I(\ell) \, d\ell
\]  

(2.20)

The integration takes place over the reference element of the subsectional array. For the class of elements considered in this paper, the integral in (2.20) may be evaluated in closed form. To this end we define the element pattern function for a subsectional array with element length \( 2\Delta L \):

\[
P_n^z(m) = \frac{1}{\rho_o I(0)} \int_{-\Delta L}^{\Delta L} I(\ell) e^{-j\beta \mu z'z(\ell)} e^{\pm j m \phi'(\ell)} \, d\ell
\]  

(2.21)

where \( I(0) \) is the current at the terminals of the reference element terminals located at \( \bar{R}_o \). The element pattern function incorporates information about the element slant angle, length and curvature. In general it is a function of both the Floquet mode index \( n \) and the azimuthal eigenvalue \( m \). With the exception of the vertical element case, the function \( P_n^z(m) \) is distinct from and should not be confused with the far-field pattern function of a single array element. The pattern function for a particular element configuration strongly influences the rate of convergence of the field series representations, as discussed in some detail in Appendix B.
In view of (2.14), the PWS current distribution may be parameterized as a function of the angular displacement of a point on the element relative to the angular position of the reference terminals as

\[ I(\phi') = I(R_o) \frac{\sin \beta \rho \sqrt{1 + \zeta^2 (\phi \Delta L - |\phi'|)}}{\sin \beta \rho \sqrt{1 + \zeta^2 \phi \Delta L}} \tag{2.22} \]

Noting that \( \phi'(\phi') = \zeta \rho \phi' \) on the element arc, we may re-express the pattern function in terms of the azimuth angle coordinate \( \phi' \) as

\[ P_n^{\pm}(m) = \frac{\sqrt{1 + \zeta^2}}{I(R_o)} \int_{-\phi \Delta L}^{\phi \Delta L} I(\phi') e^{-j(\beta \rho \zeta \mp m) \phi'} d\phi' \tag{2.23} \]

On integrating (2.23) using the current distribution (2.22), we find after simplification that

\[ P_n^{\pm}(m) = \frac{2}{\beta \rho} \left[ \frac{\zeta u^2 + \frac{m}{\beta \rho^2}}{\sqrt{1 + \zeta^2}} \right] - 1 \frac{\cos \beta \Delta L - \cos \frac{\beta \Delta L}{\sqrt{1 + \zeta^2}} (u \zeta + \frac{m}{\beta \rho})}{\sin \beta \Delta L} \tag{2.24} \]

Two special cases, namely, \( \zeta = \infty \) (corresponds to vertical dipoles) and \( \zeta = 0 \) (corresponds to transverse dipoles) are of particular interest: For vertical dipoles, the element pattern function reduces to

\[ P_n \equiv P_n^{\pm}(m) \big|_{\zeta = \infty} = \frac{2}{\beta \rho (1 - u^2)} \frac{[\cos \beta u \Delta L - \cos \beta \Delta L]}{\sin \beta \Delta L} \tag{2.25} \]

Thus, for the vertical element case, the pattern function is not a function of the azimuthal mode index \( m \) and so can be extracted from the summation over the angular modal index \( m \). Also, the plus and minus functions are the same. The vertical element pattern function will be subsequently denoted by \( P_n \).

In the second special case,

\[ P(m) = P_n^{\pm}(m) \big|_{\zeta = 0} = \frac{2 \beta \rho}{(\beta \rho)^2 - m^2} \frac{[\cos \frac{\Delta L}{\beta \rho} m - \cos \beta \Delta L]}{\sin \beta \Delta L} \tag{2.26} \]
so that for the \( \phi \) oriented element, the pattern function is a function of \( m \) but not \( n \). For this case also the plus and minus functions are identical.

Integrating the differential field representations given by (2.17), (2.18), and (2.19) over the source coordinate \( \phi' \), we obtain the following expressions for the field due to a subsectional array:

\[
E_z = \frac{I(\bar{R}_o)Z_c \beta \rho_o}{\sqrt{1 + \zeta^2 Dz}} \sum_{n=-\infty}^{\infty} e^{-j\beta u_z \Delta z} \sum_{m=0}^{\infty} \frac{1}{2} \left\{ \zeta R_{nm} [P_n^+(m)e^{jm\Delta\phi} + P_n^-(m)e^{-jm\Delta\phi}] \right\} 
\]

\[
E_{\phi} = \frac{I(\bar{R}_o)Z_c \beta \rho_o}{\sqrt{1 + \zeta^2 Dz}} \sum_{n=-\infty}^{\infty} e^{-j\beta u_z \Delta z} \sum_{m=0}^{\infty} \frac{1}{2} \left\{ \xi R_{nm}^\phi [P_n^+(m)e^{jm\Delta\phi} + P_n^-(m)e^{-jm\Delta\phi}] \right\} 
\]

\[
E_{\rho} = \frac{I(\bar{R}_o)Z_c \beta \rho_o}{\sqrt{1 + \zeta^2 Dz}} \sum_{n=-\infty}^{\infty} e^{-j\beta u_z \Delta z} \sum_{m=0}^{\infty} \frac{1}{2} \left\{ \xi R_{nm}^\rho [P_n^+(m)e^{jm\Delta\phi} + P_n^-(m)e^{-jm\Delta\phi}] \right\} 
\]

where \( \Delta z = z - z_o \) and \( \Delta \phi = \phi - \phi_o \).

### 2.4 Array Mutual Impedance

The reaction impedances \( Z_{ij}^{dp} \) may now be obtained from (2.10). Dropping the array superscripts again for brevity, we consider the reaction \( Z_{21} \) between two subsectional arrays with reference element terminals at \( \bar{R}_1 = (\rho_1, \phi_1, z_1) \) and \( \bar{R}_2 = (\rho_2, \phi_2, z_2) \). All pertinent element parameters will be tagged "1" or "2" to distinguish between the two subarrays. We emphasize that the element pitch
\( \alpha_2 \), the length \( L_2 \) and the displacement \( \rho_2 \) from the z-axis of the dipole terminals of subarray 2 may in general be different than that of subarray 1 (unless both subarrays belong to the same array).

Substituting (2.27) and (2.28) into (2.10) and using the definition of the pattern factor as in (2.21), one finds that

\[
Z_{2,1} = CZ_c \sum_{n=-\infty}^{\infty} e^{-j\beta n_z \Delta z} \sum_{m=0}^{\infty} \frac{1}{2} \left\{ \zeta_1 \zeta_2 R_{nm}^{\hat{z}^- \hat{z}} + R_{nm}^{\hat{\phi} - \hat{\phi}} \right\}
\]

\[
= \zeta_1 R_{nm}^{\hat{\phi} - \hat{\phi}} \left[ P_n^{(1)+}(m)P_n^{(2)+}(m)e^{jm\Delta \phi} + P_n^{(1)-}(m)P_n^{(2)-}(m)e^{-jm\Delta \phi} \right] - j \left[ \zeta_2 R_{nm}^{\hat{z}^- \hat{z}} \right]
\]

where

\[
C = \frac{\beta \rho_1 \rho_2}{4D_2} \frac{1}{\sqrt{1 + \zeta_1^2}} \frac{1}{\sqrt{1 + \zeta_2^2}}
\]

Also, the axial and angular displacements between the source and test array reference element terminals is

\[
\Delta z = z_2 - z_1 \quad \text{and} \quad \Delta \phi = \phi_2 - \phi_1
\]

where array 1 is understood to be the driven array. The coefficients \( R_{nm} = R_{nm}(\rho_2, \rho_1) \) have been introduced previously.

Computationally, it is useful to decompose (2.30) into "direct" and "scatter" components. The direct component \( Z_{2,1}^{\text{dir}} \) is the mutual impedance between two subsectional arrays radiating in free-space (i.e., in the absence of the cylinder). The scatter component \( Z_{2,1}^{\text{cat}} \) is the difference \( Z_{2,1} - Z_{2,1}^{\text{dir}} \) and obviously accounts for the coupling due to the currents induced on the cylinder. Since the asymptotic behavior of the direct and scatter coupling components is markedly different, it is
advantageous to compute each separately using independent convergence criteria. It is noted in passing that the physical array port mutual impedance matrix \[ Z^P \]
also may be decomposed in a similar fashion. While this is helpful conceptionally, there is no compelling numerical advantage in doing so.
CHAPTER III
ARRAY MUTUAL IMPEDANCE

3.1 The Array Fields

The field radiated by a periodic subsectional array with reference elements located at \((z_o, \rho_o, \phi_o)\) can be represented as an infinite series of cylinder waves in both the dielectric layer and free-space region, as discussed in chapter II.

As indicated in section 2.3, the fields due to a periodic array of arbitrarily pitched elements can be expressed as the sum of fields radiated by two superimposed ensembles of \(\hat{z}\)- and \(\hat{\phi}\)-directed Herztian element arrays; accordingly, we treat the two element orientations separately. In either case, the electric field in region 1 is decomposed into direct and scattered field components. The direct field component is the field radiated by the array in an infinite dielectric space in the absence of the cylinder. The direct field components for the \(\hat{z}\) and \(\hat{\phi}\) directed element cases are derived in Appendix A.

The scattered field is the difference between the total field and the direct field within the dielectric layer. This component obviously accounts for the presence of the cylinder and the air-dielectric interface. Since the source condition is already accounted for by the direct field contribution in region 1 (the dielectric region, \(a < \rho < b\)), a single series representation for the total field in region 2 (the free-space region, \(\rho > b\)) suffices.
3.1.1 Shorthand Notation for Bessel Functions

In this chapter and elsewhere, the shorthand notation $J_{iz}$ means the Bessel function of order $m$ with argument equal to a distance variable $x$ times $\beta_i u_{i\rho}$ referred to the $i^{th}$ region ($i = 1$ or 2), where $u_{i\rho}$ is the normalized radial wavenumber in region $i$ corresponding to the $n^{th}$ Floquet mode. Viz,

$$u_{i\rho} = \sqrt{1 - u_{i\rho}^2}, \quad i = 1, 2 \quad \text{where}$$

$$u_{1z} = \frac{S_z}{\sqrt{\epsilon_r}} + \frac{n\lambda_1}{D_z} \quad \text{whereas}$$

$$u_{2z} = S_z + \frac{n\lambda_2}{D_z}$$

Hankel functions of the second kind are denoted similarly (replace $J$ by $H$). For example, $H_m^{(2)}(\beta_1 u_{\rho, z})$ would be denoted simply by $H_{1z}$. In the case that the dielectric layer is absent (so that the entire region exterior to the cylinder is free-space) we will omit the first subscript denoting the region number. Thus, $J_z$ means $J_m(\beta u_{\rho, z})$ where all propagation parameters are understood to be that of free-space.

3.1.2 Vertical Hertzian Elements

The general form of the fields in both regions may be deduced from the form of the direct field component in region 1 (see section A.6.1, Appendix A). Let the terminals of the reference element be located at $(\rho_o, \phi_o, z_o)$. Knowing the form of the direct field in region 1, one finds with a bit of hindsight that the axial fields in regions 1 and 2 at a field point $(\rho, \phi, z)$ may be written as follows.

axial fields in region 1

$$dE_{z, \text{dir}}^{z, \text{dir}} = \frac{I d I}{4 D_z} \sum_{n=-\infty}^{\infty} e^{-j\kappa_z \Delta z} u_{\rho, 1}^2 \sum_{m=0}^{\infty} \epsilon_m J_{1\rho} H_{1\rho} > \cos m \Delta \phi$$
\[ dh_{z,1}^{\hat{z}, \text{dir}} = 0 \]
\[ dE_{z,1}^{\hat{z}, \text{scat}} = \frac{\text{Id} \beta_1 Z_1}{4D_z} \sum_{n=-\infty}^{\infty} e^{-j\kappa_2 \Delta z} u_{p_1}^2 \sum_{m=0}^{\infty} \epsilon_m [\epsilon_m J_1 \rho + B_m^z H_1 \rho] \cos m \Delta \phi \]
\[ dH_{z,1}^{\hat{z}, \text{scat}} = \frac{\text{Id} \beta_1}{4D_z} \sum_{n=-\infty}^{\infty} e^{-j\kappa_2 \Delta z} \sum_{m=0}^{\infty} \epsilon_m m [C_m^z J_1 \rho + D_m^z H_1 \rho] \sin m \Delta \phi \]

axial fields in region 2

\[ dE_{z,2}^{\hat{z}, \text{tot}} = \frac{\text{Id} \beta_1 Z_2}{4D_z} \sum_{n=-\infty}^{\infty} e^{-j\kappa_2 \Delta z} u_{p_2}^2 \sum_{m=0}^{\infty} \epsilon_m E_m^z H_2 \rho \cos m \Delta \phi \]
\[ dH_{z,2}^{\hat{z}, \text{tot}} = \frac{\text{Id} \beta_1}{4D_z} \sum_{n=-\infty}^{\infty} e^{-j\kappa_2 \Delta z} \sum_{m=0}^{\infty} \epsilon_m m F_m^z H_2 \rho \sin m \Delta \phi \]

where the "dir" and "scat' superscripts denote the direct and scattered contributions, respectively and where \( \rho \geq \frac{max}{\min} (\rho, \rho_o) \). Also, \( \Delta z = z - z_o \) and \( \Delta \phi = \phi - \phi_o \). \( \kappa_z = \beta_1 u_{z_1} = \beta_2 u_{z_2} \) is the axial component of the propagation vector (the same for regions 1 and 2).

For TM (TE) waves, \( E_z (H_z) \) plays the role of a potential function from which the remaining field components may be obtained. If a \( z \)-dependence \( e^{-j\beta u_z z} \) is assumed, Maxwell’s equations reduce as follows:

\[ E_T \rho = \frac{1}{(\beta u_z)^2 \rho} \frac{\partial^2 E_T}{\partial z \partial \rho} \]
\[ E_T \phi = \frac{1}{(\beta u_z)^2 \phi} \frac{\partial^2 E_T}{\partial z \partial \phi} \]
\[ E_T z = 0 \]
\[ E_T \rho = \frac{1}{j \omega \mu u_{z}^2 \rho} \frac{\partial E_T}{\partial \rho} \]
\[ E_T \phi = \frac{1}{j \omega \mu u_{z}^2 \phi} \frac{\partial E_T}{\partial \phi} \]

\[ H_T \rho = \frac{1}{j \omega \mu u_z^2 \rho} \frac{\partial H_T z}{\partial \rho} \]
\[ H_T \phi = \frac{1}{j \omega \mu u_z^2 \phi} \frac{\partial H_T z}{\partial \phi} \]

\[ H_T z = 0 \]

(3.1)
where \( u_{\rho} = \sqrt{1 - u_{z}^2} \) is the normalized radial wavenumber. These equations may be further simplified as shown in Appendix A.

The \( \rho \) and \( \phi \) components of the electric and magnetic fields may be found from relations (3.1) and (3.2). The results are tabulated in Appendix A, section A.6.1.

We need six independent boundary conditions to uniquely determine the six unknown expansion coefficients introduced in the series expansions for the axial fields. They are:

\[
\begin{align*}
E_z \mid_{\rho=a} &= 0 \quad (3.3) \\
E_{\phi} \mid_{\rho=a} &= 0 \quad (3.4) \\
E_z \mid_{\rho=b-} &= E_z \mid_{\rho=b+} \quad (3.5) \\
E_{\phi} \mid_{\rho=b-} &= E_{\phi} \mid_{\rho=b+} \quad (3.6) \\
H_z \mid_{\rho=b-} &= H_z \mid_{\rho=b+} \quad (3.7) \\
H_{\phi} \mid_{\rho=b-} &= H_{\phi} \mid_{\rho=b+} \quad (3.8)
\end{align*}
\]

where the cylinder radius is \( a \) and the dielectric-air interface is at \( \rho = b \).

Applying the boundary conditions, one obtains a 6 by 6 matrix equation. The equation set can be reduced to two independent equations in say \( A_m \) and \( C_m \); the remaining coefficients can be expressed in terms of these two. The matrix equation for \( A_m^z \) and \( C_m^z \) is of the form

\[
\begin{bmatrix}
T_m^z & U_m^z \\
V_m^z & W_m^z
\end{bmatrix}
\begin{bmatrix}
A_m^z \\
C_m^z
\end{bmatrix}
= 
\begin{bmatrix}
S_{1m}^z \\
S_{2m}^z
\end{bmatrix}
\quad (3.9)
\]

After some manipulation, we find that \( T, U, V, \) and \( W \) reduce to

\[
T_m^z = Z_1 \left[ \frac{u_{z_1}}{\beta_1 b} - \frac{u_{z_2}}{\beta_2 b} \left( \frac{u_{\rho_1}}{u_{\rho_2}} \right)^2 \right] \left[ J_{1b} - \frac{J_{1a}}{H_{1a}} H_{1b} \right] \quad (3.10)
\]
\[
U_m^z = \frac{Z_1}{u_{\rho_1}} \left[ J'_1 - \frac{J'_{1a}}{H'_{1a}} H'_1b \right] - \frac{Z_2}{u_{\rho_2}} \frac{H'_{2b}}{H_{2b}} \left[ J'_1 - \frac{J'_{1a}}{H'_{1a}} H'_1b \right] \quad (3.11)
\]

\[
V_m^z = u_{\rho_1} \left[ J'_1b - \frac{J'_{1a}}{H'_{1a}} H'_1b \right] - \frac{Z_2 u_{\rho_1}^2}{Z_2 u_{\rho_2}} \frac{H'_{2b}}{H_{2b}} \left[ J'_1b - \frac{J'_{1a}}{H'_{1a}} H'_1b \right] \quad (3.12)
\]

\[
W_m^z = \left[ \frac{m^2 u_{\rho_1}}{u_{\rho_1}^2 \beta_1b} - \frac{m^2 u_{\rho_2}}{u_{\rho_2}^2 \beta_2b} \right] \left[ J'_1b - \frac{J'_{1a}}{H'_{1a}} H'_1b \right] \quad (3.13)
\]

The excitation functions \( S_{2m}^z \) and \( S_{2m}^z \) are found to be

\[
S_{1m}^z = Z_1 \left[ \frac{u_{\rho_1}}{\beta_1b} - \frac{u_{\rho_2}}{\beta_2b} \left( \frac{u_{\rho_1}}{u_{\rho_2}} \right)^2 \right] \left[ J_{1\rho_0} - \frac{J'_{1a}}{H'_{1a}} H_{1\rho_0} \right] H_{1b} \quad (3.14)
\]

\[
S_{2m}^z = \left[ u_{\rho_1} H'_{1b} - \frac{Z_1}{Z_2} \frac{H'_{2b}}{H_{2b}} u_{\rho_2} \right] \left[ J_{1\rho_0} - \frac{J'_{1a}}{H'_{1a}} H_{1\rho_0} \right] \quad (3.15)
\]

Inverting equation (3.9), we obtain

\[
A_m^z = \frac{W_m^z S_{1m}^z - U_m^z S_{2m}^z}{T_m^z W_m^z - V_m^z U_m^z} \quad (3.16)
\]

\[
C_m^z = \frac{T_m^z S_{2m}^z - V_m^z S_{1m}^z}{T_m^z W_m^z - V_m^z U_m^z} \quad (3.17)
\]

It is necessary to pair Bessel and Hankel functions or their derivatives multiplicatively to avoid arithmetic overflow errors on the computer owing to the exponential behavior of the individual functions for large order. For example, for \( m \gg 1 \),

\[
J_m(z_1) H_m^{(2)}(z_2) \rightarrow \frac{j}{m \pi} \left| \frac{z_1}{z_2} \right|^m
\]

If \( |z_1| \ll |z_2| \) (always the case in the present context) then the \( m^{-1} \) dependence for large order presents no numerical difficulty. As seen from equations (3.10)—(3.13), the matrix coefficients \( T_m, U_m, V_m \) and \( W_m \) each exhibit a "dangling" Bessel

26
function, so that these coefficients behave asymptotically like $m^{-m^{3/2}}$ for large $m$.
Since the excitation functions $S_{1m}$ and $S_{2m}$ are $O(1/m)$ for large $m$, the coefficients $A_m^z$ and $C_m^z$ grow exponentially for $m \gg 1$. In order to avoid numerical overflow and underflow errors it is useful to introduce a normalized coefficient set as follows. The matrix elements in equation (3.9) are divided by $J_1\rho$. This results in the normalized matrix equation

$$
\begin{bmatrix}
\tilde{T}_m^z & \tilde{U}_m^z \\
\tilde{V}_m^z & \tilde{W}_m^z
\end{bmatrix}
\begin{bmatrix}
\tilde{A}_m^z \\
\tilde{C}_m^z
\end{bmatrix} = 
\begin{bmatrix}
S_{1m}^z \\
S_{2m}^z
\end{bmatrix}
$$

(3.18)

where the normalized scattering coefficients are given by

$$
\tilde{A}_m^z = A_m^z J_1\rho \\
\tilde{B}_m^z = B_m^z H_1\rho \\
\tilde{C}_m^z = C_m^z J_1\rho \\
\tilde{D}_m^z = D_m^z H_1\rho \\
\tilde{E}_m^z = E_m^z H_2\rho \\
\tilde{F}_m^z = F_m^z H_2\rho
$$

(3.19) - (3.24)

These normalized coefficients are the actual quantities of interest in the field expansions as may be seen from the expressions for the axial fields. Whereas the $A_m^z$ and $C_m^z$ grow exponentially with increasing $m$ for large order, the magnitude of the corresponding normalized coefficients $\tilde{A}_m^z$ and $\tilde{C}_m^z$ are at worst $O(1)$ (with convergence guaranteed by presence of the element pattern factors).

The remaining coefficients $\tilde{B}_m^z, \tilde{D}_m^z, \tilde{E}_m^z$ and $\tilde{F}_m^z$ are determined by applying the boundary conditions (3.3), (3.4), (3.5) and (3.7) respectively with the result
that

\[ \vec{B}_m^z = \frac{J_{1a}}{H_{1a}} \left[ H_{1\rho o} - \frac{\tilde{A}_m^z}{J_{1\rho}} \right] H_{1\rho} \]  

(3.25)

\[ \vec{D}_m^z = -\frac{H_{1\rho}}{J_{1\rho}} \frac{J_{1a}}{H_{1a}} \vec{C}_m^z \]  

(3.26)

\[ \vec{E}_m^z = \frac{Z_1}{Z_2} \left( u_{\rho_1} \right)^2 \frac{H_{2\rho}}{H_{2b}} \left[ \left( \frac{J_{1b}}{J_{1\rho}} - \frac{J_{1a}}{H_{1a}} \frac{H_{1b}}{J_{1\rho}} \right) \tilde{A}_m^z \right. \\
\left. - \frac{J_{1\rho}}{H_{1a}} \frac{H_{1b}}{J_{1\rho}} \right] \]  

(3.27)

\[ \vec{F}_m^z = \frac{1}{J_{1\rho} H_{2b}} \left[ J_{1b} - \frac{J_{1a}}{H_{1a}} H_{1b} \right] H_{2\rho} \vec{C}_m^z \]  

(3.28)

### 3.1.3 Transverse Hertzian Elements

In this case, each array element has the vector orientation \( \hat{\phi} \). As in the vertical element case, the fields in regions 1 and 2 are expressed as double Fourier series with unknown coefficients which are determined by applying boundary conditions. The direct field generated by an array with \( \hat{\phi} \) directed elements is comprised of both a TE and a TM component, as derived in Appendix A (section A.6.2). One infers that the axial field components in regions 1 and 2 may be expressed in the form:

**axial fields in region 1**

\[ dE^\phi_{z,1, dir} = -\frac{j \bar{d}l \beta_1 Z_1}{4 D_z} \sum_{n=-\infty}^{\infty} e^{-j \kappa_z \Delta z} u_{\rho_1} \sum_{m=1}^{\infty} 2m J_{1\rho < H_{1\rho >} \sin \Delta \phi} \]
\[ dH_{z,1}^{\phi,dir} = \frac{jIdl\beta_1}{4Dz} \sum_{n=-\infty}^{\infty} e^{-j\kappa_z \Delta z} u_{\rho_1} \sum_{m=0}^{\infty} \epsilon_m \left\{ \frac{J_{1\rho_1} H_{1\rho}}{J_{1\rho} H_{1\rho_1}} \right\} \cos m \Delta \phi, \text{for } \{ \rho > \rho_0 \} \]

\[ dE_{z,1}^{\phi,scat} = -\frac{jIdl\beta_1 Z_1}{4Dz} \sum_{n=-\infty}^{\infty} e^{-j\kappa_z \Delta z} \frac{u_{z_1}}{\beta_1 \rho_0} \sum_{m=0}^{\infty} \epsilon_m m \left[C_m \phi J_{1\rho} + D_m \phi H_{1\rho} \right] \sin m \Delta \phi \]

\[ dH_{z,1}^{\phi,scat} = \frac{jIdl\beta_1}{4Dz} \sum_{n=-\infty}^{\infty} e^{-j\kappa_z \Delta z} u_{\rho_1} \sum_{m=0}^{\infty} \epsilon_m \left[A_m \phi J_{1\rho} + B_m \phi H_{1\rho} \right] \cos m \Delta \phi \]

axial fields in region 2

\[ dE_{z,2}^{\phi,tot} = -\frac{jIdl\beta_2 Z_2}{4Dz} \sum_{n=-\infty}^{\infty} e^{-j\kappa_z \Delta z} \frac{u_{z_2}}{\beta_1 \rho_0} \sum_{m=1}^{\infty} 2m F_m \phi H_{2\rho} \sin m \Delta \phi \]

\[ dH_{z,2}^{\phi,tot} = \frac{jIdl\beta_1}{4Dz} \sum_{n=-\infty}^{\infty} e^{-j\kappa_z \Delta z} u_{\rho_2} \sum_{m=0}^{\infty} \epsilon_m E_m \phi H_{2\rho} \cos m \Delta \phi, \]

where the reader will recall that \( \kappa_z = \beta_1 u_{z_1} = \beta_2 u_{z_2} \) is the axial component of the propagation vector (the same for regions 1 and 2). The \( \rho \) and \( \phi \) components of the electric and magnetic field components may be found from relations (3.1) and (3.2). The results are tabulated in Appendix A, section A.6.2.

Again, we have six unknown coefficients which are determined by applying the boundary conditions (3.3) through (3.8) at the cylinder and air-dielectric interface. As in the axial element case, we obtain, upon applying the boundary equations, a 6 by 6 inhomogeneous matrix equation which is reducible to two linearly independent equations in the TM and TE waveguide coefficients \( A_m^\phi \) and \( C_m^\phi \), respectively:

\[
\begin{bmatrix}
T_m^\phi & U_m^\phi \\
V_m^\phi & W_m^\phi
\end{bmatrix}
\begin{bmatrix}
C_m^\phi \\
A_m^\phi
\end{bmatrix}
= \begin{bmatrix}
S_{1m}^\phi \\
S_{2m}^\phi
\end{bmatrix}
\tag{3.29}
\]
After manipulation $T, U, V,$ and $W$ simplify to

$$T_m^\phi = \frac{m^2 Z_1}{(\beta_1 \rho_0)(\beta_1 b)} \left[ \left(\frac{u_{z_1}}{u_{\rho_1}}\right)^2 - \left(\frac{u_{z_2}}{u_{\rho_2}}\right)^2 \right] \left[ J_{1b} - \frac{J_{1a}}{H_{1a}} H_{1b} \right]$$

(3.30)

$$U_m^\phi = Z_1 \left[ J'_{1b} - \frac{J'_{1a}}{H'_{1a}} \right] - Z_2 \frac{u_{\rho_1}}{u_{\rho_2}} \frac{H'_{2b}}{H_{2b}} \left[ J_{1b} - \frac{J'_{1a}}{H_{1a}} H_{1b} \right]$$

(3.31)

$$V_m^\phi = \frac{u_{z_1}}{u_{\rho_1}} \frac{J'_{1b} - \frac{J_{1a}}{H_{1a}} H_{1b}}{J_{1b} - \frac{J_{1a}}{H_{1a}} H_{1b}} - \frac{1}{\beta_1 \rho_0} Z_2 \frac{u_{z_1}}{u_{\rho_2}} \frac{H'_{2b}}{H_{2b}} \left[ J_{1b} - \frac{J_{1a}}{H_{1a}} H_{1b} \right]$$

(3.32)

$$W_m^\phi = \frac{1}{\beta_1 b} \left[ \frac{u_{z_1}}{u_{\rho_1}} - \frac{Z_2 \frac{u_{z_2}}{u_{\rho_2}}}{Z_1} \right] \left[ J_{1b} - \frac{J'_{1a}}{H'_{1a}} H_{1b} \right]$$

(3.33)

The excitation functions $S_{1m}^\phi$ and $S_{2m}^\phi$ are found to be

$$S_{1m}^\phi = \frac{m^2 Z_1}{(\beta_1 \rho_0)(\beta_1 b)} \left[ \left(\frac{u_{z_2}}{u_{\rho_2}}\right)^2 - \left(\frac{u_{z_1}}{u_{\rho_1}}\right)^2 \right] \left[ J_{1\rho_0} - \frac{J_{1a}}{H_{1a}} H_{1\rho_0} \right] H_{1b}$$

$$- \left[ Z_1 H'_{1b} - Z_2 \frac{u_{\rho_1}}{u_{\rho_2}} \frac{H_{1b}}{H_{2b}} H'_{2b} \right] \left[ J_{1\rho_0} - \frac{J_{1a}}{H_{1a}} H'_{1a} \right]$$

(3.34)

$$S_{2m}^\phi = \left[ \frac{1}{\beta_1 b} \frac{Z_2 \frac{u_{z_2}}{u_{\rho_1}}}{Z_1} - \frac{1}{\beta_1 b} \frac{u_{z_1}}{u_{\rho_1}} \right] \left[ J_{1\rho_0} - \frac{J_{1a}}{H_{1a}} H_{1\rho_0} \right] H_{1b}$$

$$- \frac{1}{\beta_1 \rho_0} \frac{u_{z_1}}{u_{\rho_1}} \left[ J_{1\rho_0} - \frac{J_{1a}}{H_{1a}} H_{1\rho_0} \right] H_{1b} + \frac{1}{\beta_1 \rho_0} \frac{Z_1 \frac{u_{z_1}}{u_{\rho_2}}}{Z_2} \frac{H'_{2b}}{H_{2b}}$$

$$\left[ J_{1\rho_0} - \frac{J_{1a}}{H_{1a}} H_{1\rho_0} \right] H_{1b}$$

(3.35)

Inverting equation (3.29), we find that

$$C_m^\phi = \frac{W_m^\phi S_{1m}^\phi - U_m^\phi S_{2m}^\phi}{T_m^\phi W_m^\phi - V_m^\phi U_m^\phi}$$

(3.36)
\[ A_m^\phi = \frac{T_m^\phi S_{2m}^\phi - V_m^\phi S_{1m}^\phi}{T_m^\phi W_m^\phi - V_m^\phi U_m^\phi} \]  

(3.37)

To facilitate numerical computation (see previous section), let us introduce a normalized matrix equation

\[
\begin{bmatrix}
\tilde{T}_m^\phi & \tilde{U}_m^\phi \\
\tilde{V}_m^\phi & \tilde{W}_m^\phi \\
\end{bmatrix}
\begin{bmatrix}
\tilde{C}_m^\phi \\
\tilde{A}_m^\phi \\
\end{bmatrix}
= 
\begin{bmatrix}
S_{1m}^\phi \\
S_{2m}^\phi \\
\end{bmatrix}
\]  

(3.38)

where the normalized matrix elements are equal to the unnormalized matrix elements divided by \( J_{1\rho} \) and the normalized scattering coefficients \( \tilde{A}_m^\phi, \tilde{B}_m^\phi, \) etc. are defined as in equations (3.19)-(3.24), where the superscript \( z \) is replaced by the superscript \( \phi \).

Applying boundary conditions (3.3) through (3.8), one finds that the remaining coefficients \( \tilde{B}_m^\phi, \tilde{D}_m^\phi, \tilde{E}_m^\phi \) and \( \tilde{F}_m^\phi \) are related to \( \tilde{A}_m^\phi \) and \( \tilde{C}_m^\phi \) according to

\[
\tilde{B}_m^\phi = -\frac{J_{1a}}{H_{1a}} \left[ H_{1\rho_0} - \frac{\tilde{C}_m^\phi}{J_{1\rho}} \right] H_{1\rho} \]  

(3.39)

\[
\tilde{B}_m^\phi = -\frac{J_{1a}'}{H_{1a}} \left[ \frac{1}{J_{1\rho}} \frac{\tilde{A}_m^\phi}{J_{1\rho}} - H_{1\rho_0}' \right] \]  

(3.40)

\[
\tilde{E}_m^\phi = \left( \frac{Z_1}{Z_2} \right)^2 \frac{H_{1\rho}}{H_{2b}} \left[ \left( J_{1\rho_0} - \frac{J_{1a}}{H_{1a}} H_{1\rho_0} \right) H_{1b} \right. \\
+ \left( J_{1b} - \frac{J_{1a}}{H_{1a}} H_{1b} \right) \frac{\tilde{C}_m^\phi}{J_{1\rho}} \]  

(3.41)

\[
\tilde{F}_m^\phi = \frac{u_{\rho_1}}{u_{\rho_2}} \frac{H_{1\rho}}{H_{2b}} \left[ \left( J_{1\rho_0}' - \frac{J_{1a}'}{H_{1a}'} H_{1\rho_0}' \right) H_{1b} \right. \\
+ \left( J_{1b}' - \frac{J_{1a}'}{H_{1a}'} H_{1b} \right) \frac{\tilde{A}_m^\phi}{J_{1\rho}} \]  

(3.42)
3.2 Discussion

The presence of a field component in the proximity of the arrays does not imply that that component will survive in the far-zone. It is of interest to know under what circumstances a particular vector component of the array field is excited.

We consider four possibilities: the arrays are scanned or unscanned and the dielectric is present or absent.

From the definitions of the matrix coefficients $T_m^z, T_m^\phi, U_m^z, U_m^\phi$ and so on introduced in the previous two sections one may ascertain when the expansion coefficients $A_{m}, A_m^\phi, \ldots F_m^z, F_m^\phi$ are non-zero. With this information one may determine which field components will be present for a particular set-up. Obviously none of the "z" coefficients will come into play if the array elements are transverse; likewise, the "\phi" coefficients are zero for vertical elements. The results for all possibilities are presented in the form of a truth table as shown in figure 4. The coefficients of immediate interest are the $A_m^z, A_m^\phi, \ldots F_m^z, F_m^\phi$, since the electromagnetic field is expressed directly in terms of them. The intermediary matrix coefficients $T_m^z, T_m^\phi, U_m^z, U_m^\phi$, etc. are included for completeness.

A discussion of the implications for the scattered field (region 2 solution) will be deferred to chapter V, following a discussion of the induced array voltage due to an incident plane wave (the subject of chapter IV). We may determine from figure 4 which components of the scattered field will propagate assuming current is in fact induced on the arrays; the material in chapter III is needed to determine under what conditions array currents are excited. That is, the presence of a scattered field component is not only determined by the presence or absence of the dielectric layer and the array element orientation but also on the polarization and direction of the incident wave.
Figure 4: Truth table for array field matrix and scattering coefficients for various combinations of scan and dielectric layer

<table>
<thead>
<tr>
<th>NO LAYER</th>
<th>NO SCAN</th>
<th>( T_{m}^{z} U_{m}^{z} V_{m}^{z} W_{m}^{z} S_{m}^{z} S_{m}^{z} A_{m}^{z} B_{m}^{z} C_{m}^{z} D_{m}^{z} E_{m}^{z} F_{m}^{z} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>NO LAYER</td>
<td>SCAN</td>
<td>( X \times X \times X \times X \times X \times X \times X \times X \times X \times X )</td>
</tr>
<tr>
<td>LAYER</td>
<td>NO SCAN</td>
<td>( X \times X \times X \times X \times X \times X \times X \times X \times X \times X )</td>
</tr>
<tr>
<td>LAYER</td>
<td>SCAN</td>
<td>( X \times X \times X \times X \times X \times X \times X \times X \times X \times X )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>NO LAYER</th>
<th>NO SCAN</th>
<th>( T_{m}^{\phi} U_{m}^{\phi} V_{m}^{\phi} W_{m}^{\phi} S_{m}^{\phi} S_{m}^{\phi} A_{m}^{\phi} B_{m}^{\phi} C_{m}^{\phi} D_{m}^{\phi} E_{m}^{\phi} F_{m}^{\phi} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>NO LAYER</td>
<td>SCAN</td>
<td>( X \times X \times X \times X \times X \times X \times X \times X \times X \times X )</td>
</tr>
<tr>
<td>LAYER</td>
<td>SCAN</td>
<td>( X \times X \times X \times X \times X \times X \times X \times X \times X \times X )</td>
</tr>
</tbody>
</table>

○ COEFFICIENT IS ZERO  \( \times \) NON-ZERO ONLY FOR EVANESCENT MODES (n>0) \( \otimes \) NON-ZERO FOR ALL CYLINDER WAVE MODES
3.3 Explicit Series Representation for Mutual Impedance

In the previous section, explicit representations were presented for the electric field due to an array of Hertzian dipoles for two "canonical" element orientations, axial and transverse. As discussed in section 2.3, the slant array field is a superposition of these two contributions, each weighted according to the element pitch.

We are now in a position to give explicit formulas for the general Fourier coefficients $R_{nm}$ introduced in the field equations (2.17)—(2.19) and reappearing in the series expansion for the array mutual impedance, as in equation (2.30). On comparing equation (2.17) with the total electric field in the dielectric region (see section 3.1.2) we find that

$$R_{nm}^{\hat{z}-\hat{z}} = u_{\rho_1}^2 \left[ A_m^z J_{1\rho} + B_m^z H_{1\rho} - J_{1\rho < H_{1\rho}>} \right]$$  \hspace{1cm} (3.43)

If the arrays are scanned and the dielectric layer is present, the $\phi$ component of the electric field radiated by vertical elements exhibits both TE and TM wave components (see figure 4) and is found according to equations (A.14) and (A.20), with the result that

$$R_{nm}^{\hat{\phi}-\hat{z}} = jm \left\{ \frac{u_{z_1}}{\beta_{1\rho}} \left[ A_m^z J_{1\rho} + B_m^z H_{1\rho} - J_{1\rho < H_{1\rho}>} \right] \right. + \left[ \frac{1}{u_{\rho_1}} \left[ C_m^{\phi} J_{1\rho} + D_m^{\phi} H_{1\rho} \right] \right\}$$  \hspace{1cm} (3.44)

Similarly, the $\hat{\phi}$ component of the electric field radiated by transverse elements exhibits both TM and TE wave components, as shown in section A.6.2 and is computed using equations (A.14) and (A.20) with the result that

$$R_{nm}^{\hat{\phi}-\hat{\phi}} = -\frac{m^2}{(\beta_{1\rho})(\beta_{1\rho})} \left( \frac{u_{z_1}}{u_{\rho_1}} \right)^2 \left[ J_{1\rho < H_{1\rho}>} + C_m^{\phi} J_{1\rho} + D_m^{\phi} H_{1\rho} \right]$$

$$- \left[ J_{1\rho < H_{1\rho}>} + A_m^{\phi} J_{1\rho} + B_m^{\phi} H_{1\rho} \right]$$  \hspace{1cm} (3.45)
Lastly, comparison of the expression for the axial electric field component radiated by a $\hat{\phi}$ directed element array with equation (2.17) yields

$$R_{nm}^{\hat{z} \cdot \hat{\phi}} = -j m \frac{u_{z1}}{\beta_{1\rho_o}} \left[ C_m^\phi J_{1\rho} + D_m^\phi H_{1\rho} + J_{1\rho}H_{1\rho} \right]$$  \hspace{1cm} (3.46)

The series expansion for the subsectional array mutual impedance is given by equation (2.30), rewritten below for convenience; for the remainder of this chapter we will assume for simplicity that (sub)arrays 1 and 2 are identical in all element attributes and that the radial displacement from the z-axis to the terminals of each array is the same (denoted by $\rho_o$).

$$Z_{2,1} = - \frac{Z_1}{1 + \zeta^2} (\beta\rho_o)^2 \sum_{n=-\infty}^{\infty} e^{-j\kappa z\Delta z} \sum_{m=0}^{\infty} \frac{1}{2} \left\{ \zeta^2 R_{nm}^{\hat{z} \cdot \hat{\phi}} + R_{nm}^{\hat{\phi} \cdot \hat{z}} \right\}$$  \hspace{1cm} (3.47)

\[ \cdot \left[ (P_n^+(m))^2 e^{j m \Delta \phi} + (P_n^-(m))^2 e^{-j m \Delta \phi} \right] - j \left[ R_{nm}^{\hat{z} \cdot \hat{\phi}} + \zeta R_{nm}^{\hat{\phi} \cdot \hat{z}} \right] \]

\[ \cdot \left[ (P_n^+(m))^2 e^{j m \Delta \phi} - (P_n^-(m))^2 e^{-j m \Delta \phi} \right] \]

where the pattern factor is given by equation (2.24).

3.3.1 Special Case: Dielectric Layer Absent

For the no dielectric case, we suppress the subscripts 1 and 2 in all coefficient expressions (including the argument of the Bessel functions) and set $\varepsilon_r = 1$. First, consider the case of $\hat{z}$ oriented elements. With the dielectric absent, $T_n^\pm, W_n^\pm$ and the excitation functions $S_{1m}^\pm$ and $S_{2m}^\pm$ all vanish. From the matrix equation (3.9), one finds that this implies $A_n^\pm, C_n^\pm$ and therefore $D_n^\pm$ and $F_n^\pm$ are identically zero. $B_n^\pm$ however does not vanish. From equation (3.25) one finds

$$B_n^\pm = \frac{J_n}{H_a} H_{\rho_o}$$
The coefficients corresponding to \( \hat{\phi} \) directed elements reduce in a similar fashion. \( W_m^\phi \), \( T_m^\phi \) and \( S_m^\phi \) all vanish. This implies, by equations (3.36) and (3.37), that \( A_m^\phi \) and \( C_m^\phi \) equal zero, whereas
\[
D_m^\phi = -\frac{J_a}{H_a} H_{\rho_0} \\
B_m^\phi = -\frac{J'_a}{H'_a} H'_{\rho_0}
\]
In general the exterior coefficients \( E_m^\phi \) and \( F_m^\phi \) will both be non-zero when the dielectric layer is absent, even if the array is unscanned. Substituting the previous results in the expressions for the \( R_{nm} \) coefficients above, we have for the no-dielectric case
\[
R_{nm}^{\hat{z} \rightarrow \hat{z}} = -u_\rho^2 \left[ J_{\rho < H_{\rho_0}} - \frac{J_a}{H_a} H_{\rho} H_{\rho_0} \right] \quad (3.48)
\]
\[
R_{nm}^{\hat{\phi} \rightarrow \hat{z}} = -j m \frac{u_z}{\beta_\rho} \left[ J_{\rho < H_{\rho_0}} - \frac{J_a}{H_a} H_{\rho} H_{\rho_0} \right] \quad (3.49)
\]
\[
R_{nm}^{\hat{\phi} \rightarrow \hat{\phi}} = -\frac{m^2}{(\beta_{\rho_0})(\beta_\rho)} \left( \frac{u_z}{u_\rho} \right)^2 \left[ J_{\rho < H_{\rho_0}} - \frac{J_a}{H_a} H_{\rho} H_{\rho_0} \right] \quad (3.50)
\]
\[
- \left[ J_{\rho < H_{\rho_0}} - \frac{J'_a}{H'_a} H'_{\rho} H'_{\rho_0} \right]
\]
\[
R_{nm}^{\hat{z} \rightarrow \hat{\phi}} = -j m \frac{u_z}{\beta_{\rho_0}} \left[ J_{\rho < H_{\rho_0}} - \frac{J_a}{H_a} H_{\rho} H_{\rho_0} \right] \quad (3.51)
\]
where \( \beta \) and the various normalized wavenumbers are understood to be the free-space values.
Substitution of equation (3.48) into (3.47) with $\zeta = \infty$ so that $P_n^\pm(m) \doteq P_n$ (see pattern function, equation (2.25)) we obtain

$$Z_{2,1} = \frac{Z(\beta_0)^2}{4\beta D_z} \sum_{n=-\infty}^{\infty} e^{-j\beta u_z(s_2-z_1)} P_n^2 \sum_{m=0}^{\infty} \epsilon_m H_m^{(2)}(\beta u_\rho \rho) \epsilon_m\left[J_m(\beta u_\rho \rho) + a_m H_m^{(2)}(\beta u_\rho \rho)\right] \cos m(\phi_2 - \phi_1)$$

(3.52)

where the subscript "o" denotes "free space".

In a similar fashion, we substitute equation (3.50) into (3.47) with $\zeta = 0$ so that $P_n^\pm(m) \doteq P(m)$ (as defined by equation (2.26)) to obtain

$$Z_{2,1} = \frac{Z_c}{4\beta D_z} \sum_{n=-\infty}^{\infty} e^{-j\beta u_z(s_2-z_1)} P_n^2 \sum_{m=0}^{\infty} \epsilon_m J_{\rho}^{(2)}(\beta u_\rho \rho) \left[J_m(\beta u_\rho \rho) + b_m H_m^{(2)}(\beta u_\rho \rho)\right] \cos m(\phi_2 - \phi_1)$$

(3.53)

where

$$a_m = -\frac{J_m(\beta u_\rho \rho)}{H_m^{(2)}(\beta u_\rho \rho)} \quad \text{and} \quad b_m = -\frac{J_m(\beta u_\rho \rho)}{H_m^{(2)}(\beta u_\rho \rho)}$$

Using results (3.48) through (3.51), it is simple to verify our previous claim (see discussion on pattern function, section E) that if $S_z = 0$ then $\hat{R}_{nm}^{\rho-i\tilde{z}}$ and $\hat{R}_{nm}^{\rho-i\tilde{z}}$ are even functions of $n$ whereas $\hat{R}_{nm}^{\rho-i\tilde{z}}$ and $\hat{R}_{nm}^{\rho-i\tilde{z}}$ are odd functions of $n$, in the case that the dielectric is absent. This follows from the facts that for $S_z = 0$

(i) $u_z = \frac{n\lambda}{D_z}$ and (ii) $u_\rho = \sqrt{1 - u_z^2}$ (an even function of $n$) and (iii) the arguments of the Bessel functions are of the form $\beta u_\rho z$ and are therefore even functions of $n$. 37
It may also be shown through an inspection of the matrix coefficients $T_m^z$, $U_m^z$, $V_m^z$, and $V_m^z$, as well as $T_m^\phi$, $U_m^\phi$, $V_m^\phi$ and $V_m^\phi$, that these even/odd relations hold even when the dielectric layer is present, assuming $S_z = 0$.

3.4 Series Convergence

It is sufficient to consider the convergence properties of the general series representation for the mutual array impedance given on page 20. That is, the array self-impedance is (computationally) equivalent to the mutual impedance between two filament arrays separated by a wire radius, as shown in Appendix C. Our primary interest in this section is in the convergence properties of the "self" or diagonal terms of the generalized impedance matrix. The off-diagonal terms, while by no means insignificant, are comparatively small and rapidly convergent if the subsectional array separation is an appreciable fraction of a wavelength.

It is useful to decompose the mutual impedance function into a "direct" coupling component $Z_{2,1}^{\text{dir}}$ and a "scatter" coupling component $Z_{2,1}^{\text{scat}}$. The scatter component represents coupling due to energy scattered from the PEC cylinder as well as trapped wave contributions propagating in the dielectric channel. We emphasize that as far as the direct coupling contribution is concerned, the reference array is radiating in an unbounded homogeneous space whose permittivity is that of the dielectric layer.

These two impedance components are considered separately because the asymptotic behaviors of their eigenfunction representations are markedly different. The direct coupling component depends only on the position of reference element 2 relative to reference element 1 (assuming that both arrays are embedded in the dielectric layer) and the index of refraction in the dielectric medium. The functional form and therefore the series convergence of the direct coupling component
is independent of the PEC cylinder and the thickness of the dielectric layer.

While the expressions for the scatter coupling components are more complicated than that of the direct component, the direct component converges far more slowly at least when the array separation is less than a quarter wavelength. For larger array separations, the direct and scatter coupling components tend to be comparable in magnitude and converge rapidly. When the test array is shadowed by the cylinder, the direct and scatter impedance contributions tend to cancel each other.

3.4.1 Vertical Directed Dipoles, No Dielectric Layer

We first consider the simple case of two periodic arrays of vertical dipoles radiating in the presence of a PEC cylinder with no dielectric layer present.

In the vertical dipole case, the pattern function is a constant in $m$-space for a given $n$ and so plays no role in the convergence of the $m$ sum. The series on $n$, the Floquet wave index, converges very rapidly for axial array elements because the pattern function is proportional to $u_z^{-2}$ for large $n$, where $u_z = S_z + \frac{n\lambda}{D_z}$. Therefore, the $n^{th}$ term in the Floquet series decreases as $n^{-4}$ for large $n$ since the coupling associated with the $n^{th}$ cylinder wave is proportional to the product of two pattern functions.

The angular modal sum is more difficult to compute for near-vertical dipoles because the series behaves asymptotically like an oscillating harmonic series. For strictly vertical elements, one may invoke the addition theorem for the Hankel function of order zero to obtain a closed-form result. For near-vertical dipoles, the pattern function oscillates with a large period in $m$-space. Recall from section B that the location of the element pattern beam peak in $m$-space is to first
approximation
\[ m_n = \beta \rho_0 \zeta \left( S_z + \frac{n \lambda}{D_z} \right), \quad n = 0, 1, 2, \ldots \]

For a given Floquet mode \( n \), the beamwidth of the corresponding element pattern function in \( m \)-space is
\[ \Omega = 10 \frac{\rho_0}{L} \sqrt{1 + \zeta^2} \]

Thus, for steeply pitched elements both \( m_n \) and \( \Omega \) are large numbers, implying that a large number of azimuthal modal terms must be summed (and an enormous number of Bessel functions must be computed).

However, in this case one cannot apply the addition theorem because the pattern factor is a function of \( m \) and therefore cannot be taken out of the \( m \)-sum. In consequence, for antenna diameters on the order of a wavelength, our computer program becomes unstable and fails if the element pitch exceeds 87 degrees or so.

For vertical elements, the expression for the array mutual impedance simplifies to
\[
Z_{2,1} = -\frac{Z_c(\beta \rho_1)(\beta \rho_2)}{4\beta D_z} \sum_{n=-\infty}^{\infty} e^{-j\kappa_z \Delta z} P_n^{(1)} P_n^{(2)} \sum_{m=0}^{\infty} \epsilon_m [R_{nm}^{z-z_{\text{dir}}} + R_{nm}^{z-z_{\text{scat}}}] \cos m \Delta \phi \tag{3.54}
\]

where we have decomposed the Fourier coefficient \( R_{nm}^{z-z} \) into direct and scatter components. For simplicity, we will assume that \( \rho_1 \approx \rho_2 \) (subsequently denoted by \( \rho_0 \)), that is the radial distance to arrays 1 and 2 are equal. We will also assume the two pattern factors are the same, that is, \( P_n^{(1)} \approx P_n^{(2)} \) (denoted simply by \( P_n \)), which means that the elements in arrays 1 and 2 are identical in every way. (If the pattern factors are not the same, then substitute \( P_n^{(1)} P_n^{(2)} \) for \( P_n^2 \) in the remainder of this section).
Under these assumptions, we have

$$Z_{2,1} = -\frac{Z_c(\beta \rho_o)^2}{4\beta D_z} \sum_{n=-\infty}^{\infty} e^{-j\kappa_z \Delta z} P_n^2 \sum_{m=0}^{\infty} \epsilon_m [R_{nm}^{\hat{z},dir} + R_{nm}^{\hat{z},scat}] \cos m\Delta \phi$$

$$= Z_{2,1}^{dir} + Z_{2,1}^{scat}$$  \hspace{1cm} (3.55)

That is, we may view the mutual impedance as a sum of direct and scatter coupling contributions.

From equation (3.48) one finds that with no dielectric present

$$R_{nm}^{\hat{z},dir} = -u_p^2 J_{\rho_o} H_{\rho_o}$$  \hspace{1cm} (3.56)

$$R_{nm}^{\hat{z},scat} = u_p^2 J_a (H_{\rho_o})^2$$  \hspace{1cm} (3.57)

$R_{nm}^{\hat{z},dir}$ incorporates the coupling between arrays 1 and 2 due to the direct field radiated by array 1. $R_{nm}^{\hat{z},scat}$ describes the reaction at array 2 due to currents induced on the PEC cylinder. The direct component of the array impedance is

$$Z_{2,1}^{dir} = \frac{Z_c(\beta \rho_o)^2}{4\beta D_z} \sum_{n=-\infty}^{\infty} e^{-j\kappa_z \Delta z} u_p^2 P_n^2 \sum_{m=0}^{\infty} \epsilon_m J_{\rho_o} H_{\rho_o} \cos m\Delta \phi$$  \hspace{1cm} (3.58)

$$= \frac{Z_c(\beta \rho_o)^2}{4\beta D_z} \sum_{n=-\infty}^{\infty} e^{-j\kappa_z \Delta z} u_p^2 P_n^2 H_0^{(2)} (\beta u_p |\vec{p}_2 - \vec{p}_1|)$$

where the second equality follows from the addition theorem for the Hankel function of order zero. The coupling due to the scattered field is

$$Z_{2,1}^{scat} = \frac{Z_c(\beta \rho_o)^2}{4\beta D_z} \sum_{n=-\infty}^{\infty} e^{-j\kappa_z \Delta z} u_p^2 P_n^2 \sum_{m=0}^{\infty} \epsilon_m J_a (H_{\rho_o})^2 \cos m\Delta \phi$$  \hspace{1cm} (3.59)

Let us further simplify matters by assuming that the axial displacement $\Delta z$ of array 2 relative to array 1 is zero. $\Delta z$ only appears as an argument in the exponential factor. This exponential contributes an effective phase delay for each cylinder wave indexed by $n$ impinging on the test array element terminals.
As will be shown presently, the mutual resistance converges very rapidly, whereas the mutual reactance (particularly the direct component) exhibits much slower convergence in comparison. For element spacings sufficiently small (so that no grating lobes propagate), the arguments of the Bessel functions are purely negative imaginary for $|n| > 0$, assuming the dielectric loss is negligible. Since the product of $J_m(-jx)$ and $H_m^{(2)}(-jy)$ (or its reciprocal) is purely imaginary for all $x, y > 0$, it follows that only the $n = 0$ term contributes to $R_{2,1}$, assuming no grating lobes propagate.

Consider now the array self-reactance. The direct coupling contribution (see equation (3.58)) behaves asymptotically like

$$J_m(\kappa \rho <) H_m^{(2)}(\kappa \rho >) \cos m \Delta \phi \sim \frac{j}{m \pi} \left( \frac{\rho <}{\rho >} \right)^m \cos m \Delta \phi \quad (3.60)$$

For a small wire radius, the ratio $\rho < / \rho >$ is close to unity and the angular displacement between reference and test arrays is small. Therefore, for large $m$, the series behaves like an alternating harmonic series whose oscillation period in $m$ is $\frac{2\pi}{\Delta \phi}$, a large number for thin wires. For very thin wires, the tail of the angular modal series resembles a simple harmonic series and therefore diverges in the limit as the wire radius tends to zero. Observe also from equation (3.60) that the $m^{th}$ term tends asymptotically to a purely imaginary value. Therefore, not only is the self-resistance comprised of only the $n = 0$ cylinder wave but the real part of the angular modal series associated with the $n = 0$ term evidently converges before $m$ enters the asymptotic regime. This is why the self-resistance converges so rapidly.

For the reactive contribution from the scatter coupling component, we use the asymptotic result

$$\frac{J_m(\kappa \rho a)}{H_m^{(2)}(\kappa \rho a)} \frac{J_m(\kappa \rho <) H_m^{(2)}(\kappa \rho >) \cos m \Delta \phi}{J_m(\kappa \rho a)} \sim \frac{j}{m \pi} \left( \frac{a}{\rho <} \right)^m \left( \frac{a}{\rho >} \right)^m \cos m \Delta \phi$$

42
Since $a$, the cylinder radius, is less than $\rho_o$, the reactive coupling due to the scattered field contribution converges rapidly if the ratio $\frac{a}{\rho_o}$ is appreciably less than unity. Comparing the last result with equation (3.60), we see that for the $n^{th}$ Floquet mode, the magnitude of the $m^{th}$ scatter coupling term is $(\frac{a}{\rho_o})^{2m}$ times smaller than the corresponding direct term for the vertical dipole case.

### 3.4.2 Transverse Dipoles, No Dielectric Layer

The vertical dipole case represents one extreme in that the convergence of the sum over the Floquet modes is rapid "at the expense of" the angular modal sum, which converges slowly. The $\phi$ directed dipole case represents the opposite extreme in that the Floquet (index $n$) series converges relatively slowly while the angular modal series (index $m$) for a given $n$ converges rapidly. This should come as no surprise. The far-field pattern for a vertical dipole is uniform azimuthally, while the field due $\phi$ directed dipoles exhibits little variation in the elevation plane. Since each index $n$ is associated with the (possibly complex) elevation angle of an (inhomogeneous) cylinder wave, one expects that a relatively large number of evanescent cylinder waves will contribute to the reactive coupling between transverse elements.

The series representation for the array mutual impedance for two arrays with transverse elements reduces to

$$Z_{2,1} = -\frac{Z_c(\beta \rho_1)(\beta \rho_2)}{4\beta D_z} \sum_{n=-\infty}^{\infty} e^{-j\kappa_z \Delta z} \sum_{m=0}^{\infty} \epsilon_m \left[ R_{nm, \phi, \phi, \text{dir}} + R_{nm, \phi, \phi, \text{scat}} \right] \cdot P^{(1)}(m) P^{(2)}(m) \cos m\Delta \phi$$

(3.62)

In section 2.3 it was shown that for transverse dipoles the pattern function

$$P(m) \dagger P_n^H(m)|_{\zeta=0} = \frac{2\beta \rho_o}{(\beta \rho_o)^2 - m^2} \left[ \frac{\cos \frac{L}{\rho_o} m - \cos \beta L}{\sin \beta L} \right].$$

43
where \( \rho_o \) is the radial distance to the arrays from the z-axis.

As in the vertical element case, we will assume for simplicity that \( \rho_1 \simeq \rho_2 \) (which we denote by \( \rho_o \)) and that the pattern functions for the two arrays are identical. Then equation (3.62) simplifies to

\[
 Z_{2,1} = - \frac{Z_\infty (\beta \rho_o)^2}{\delta D_z} \sum_{n=-\infty}^{\infty} e^{-j \kappa_z \Delta z} \sum_{m=0}^{\infty} \epsilon_m \left[ R_{nm}^\phi \hat{\phi}_{dir} + R_{nm}^\phi \hat{\phi}_{scat} \right] P^2(m) \cos m \Delta \phi 
\]  

(3.63)

The direct and scatter Fourier coefficients may be further decomposed into TM and TE wave contributions:

\[
 R_{nm}^\phi \hat{\phi}_{dir} = R_{nm}^\phi \hat{\phi}_{dir,TM} + R_{nm}^\phi \hat{\phi}_{dir,TE} 
\]

(3.64)

\[
 R_{nm}^\phi \hat{\phi}_{scat} = R_{nm}^\phi \hat{\phi}_{scat,TM} + R_{nm}^\phi \hat{\phi}_{scat,TE} 
\]

(3.65)

With the dielectric layer absent, the \( R_{nm} \) are given by (see section 3.3.1)

\[
 R_{nm}^\phi \hat{\phi}_{TM,dir} = - \frac{m^2}{(\beta \rho_o)^2} \left( \frac{u_z}{u_\rho} \right)^2 \hat{J}_{\rho_o} H_{\rho_o} 
\]

(3.66)

\[
 R_{nm}^\phi \hat{\phi}_{TE,dir} = J_{\rho_o}^I H_{\rho_o}^I 
\]

(3.67)

\[
 R_{nm}^\phi \hat{\phi}_{TM,scat} = \frac{m^2}{(\beta \rho_o)^2} \left( \frac{u_z}{u_\rho} \right)^2 \hat{J}_a \left( \frac{H_{\rho_o}}{H_a} \right)^2 
\]

(3.68)

\[
 R_{nm}^\phi \hat{\phi}_{TE,scat} = J_a^I \left( \frac{H_{\rho_o}^I}{H_a} \right)^2 
\]

(3.69)

so that one may decompose the impedance function as

\[
 Z_{2,1} = Z_{2,1}^{TM,dir} + Z_{2,1}^{TM,scat} + Z_{2,1}^{TE,dir} + Z_{2,1}^{TE,scat} 
\]

(3.70)

As is the previous section, we set the relative axial displacement \( \Delta z \) to zero for simplicity. The mutual resistance is captured in the \( n = 0 \) Floquet mode alone for reasonably small element spacings. The reason is that every product of Bessel functions (or their derivatives) in equation (3.62) is purely imaginary for \(|n| > 0\)
provided that the element spacing is sufficiently small such that $|S_x + \frac{n}{2D_x}| > 1$ for $|n| > 0$.

To evaluate the behavior of the impedance series for large $m$ we need some asymptotic results:

$$m^2 J_m(\kappa \rho <) H_m^{(2)}(\kappa \rho >) P_m^2 \sim \frac{jC}{\pi m^3} \left( \frac{\rho <}{\rho >} \right)^m$$

$$J'_m(\kappa \rho <) H_m^{(2)}(\kappa \rho >) P_m^2 \sim -\frac{jC}{\pi m^3 (\kappa \rho <)(\kappa \rho >)} \left( \frac{\rho <}{\rho >} \right)^m$$

$$\sim -\frac{jC}{\pi m^3 (\kappa \rho_o)^2}$$

$$m^2 \frac{J_m(\kappa \rho a)}{H_m^{(2)}(\kappa \rho a)} J_m(\kappa \rho <) H_m^{(2)}(\kappa \rho >) P_m^2 \sim$$

$$\frac{jC}{\pi m^3} \left( \frac{a}{\rho <} \right)^m \left( \frac{a}{\rho >} \right)^m \sim \frac{jC}{\pi m^3} \left( \frac{a}{\rho_o} \right)^{2m}$$

$$m^2 \frac{J'_m(\kappa \rho a)}{H'_m^{(2)}(\kappa \rho a)} J'_m(\kappa \rho <) H'_m^{(2)}(\kappa \rho >) P_m^2 \sim$$

$$-\frac{jC}{\pi m^3 (\kappa \rho <)(\kappa \rho >)} \left( \frac{a}{\rho <} \right)^m \left( \frac{a}{\rho >} \right)^m \sim -\frac{jC}{\pi m^3} \left( \frac{1}{\kappa \rho_o} \right)^2 \left( \frac{a}{\rho_o} \right)^{2m}$$

where we have used the fact that

$$P_n^\pm(m)|_{\zeta=0} \equiv P(m) \sim \frac{C}{m^2}$$

where

$$C = \frac{2 \beta \rho_o}{\sin \beta L} \left( \cos \frac{L}{\rho_o} m - \cos \beta L \right)$$

Whereas for vertical dipoles the azimuthal modal series for the direct impedance component tends to a slowly converging oscillating harmonic series for large $m$,
the corresponding series for transverse elements goes as $m^{-3}$. In both cases, the corresponding scatter impedance component is roughly $(\frac{a}{\rho_o})^{2m}$ times smaller than its direct coupling counterpart, where $\rho_o$ is the average radial distance to the reference and test arrays.

### 3.4.3 Arbitrary Element Pitch, No Dielectric Layer

In this case, the element pitch $\alpha$ is in the range $0 < \alpha < 90$ degrees so that the "cross-coupling" coefficients $R_{nm}^{\hat{z} \rightarrow \hat{z}}$ and $R_{nm}^{\hat{\phi} \rightarrow \hat{\phi}}$ now come into play. Recall from chapter II that an array of slant elements is envisioned as an infinite ensemble of Herztian elements juxtaposed along the length of the dipole. We may further imagine that each Hertzian element is decomposed into its $\hat{z}$ and $\hat{\phi}$ vector components, resulting in two superimposed sub-ensembles of Hertzian arrays. The non-zero coefficient $R_{nm}^{\hat{z} \rightarrow \hat{z}}$ reflects the fact that there is a net coupling between the $\hat{z}$- and $\hat{\phi}$- components of current on the reference and test arrays, respectively. The coefficient $R_{nm}^{\hat{\phi} \rightarrow \hat{\phi}}$ is similarly interpreted. In general of course, the "$\hat{z} \rightarrow \hat{z}$" and "$\hat{\phi} \rightarrow \hat{\phi}$" type interactions will also occur; the associated coefficients $R_{nm}^{\hat{z} \rightarrow \hat{z}}$ and $R_{nm}^{\hat{\phi} \rightarrow \hat{\phi}}$ were discussed in the previous two sections.

If we make the previous simplifying assumptions that $\rho_o \simeq \rho_1 \simeq \rho_2$, $\Delta z = 0$ and that the reference and test arrays are similar (i.e., same pitch, length and element spacing), the array mutual impedance as given by equation (2.30) reduces to

$$Z_{2,1} = -\frac{Z_c}{1 + \zeta^2} \frac{(\beta \rho_o)^2}{4 \beta D_z} \sum_{n=-\infty}^{\infty} \sum_{m=0}^{\infty} \frac{1}{2} \left\{ \frac{\epsilon^2 R_{nm}^{\hat{z} \rightarrow \hat{z}} + R_{nm}^{\hat{\phi} \rightarrow \hat{\phi}}}{4} \right\} \left[ (P_n^+ (m))^2 e^{j m \Delta \phi} + (P_n^- (m))^2 e^{-j m \Delta \phi} \right] - \zeta R_{nm}^{\hat{z} \rightarrow \hat{z}} \left[ (P_n^+ (m))^2 e^{j m \Delta \phi} - (P_n^- (m))^2 e^{-j m \Delta \phi} \right]$$

(3.71)
Notice that although the expressions for the coefficients \( R_{nm} \) and \( \hat{R}_{nm} \) are identical to those given in the previous two sections, the impedance components \( Z_{2,1}^{\hat{z} \rightarrow \hat{z}} \) and \( Z_{2,1}^{\hat{z} \rightarrow \hat{z}} \) in general differ from the corresponding expressions (3.55) and (3.63); only in the special cases \( \zeta = \infty \) (vertical elements) and \( \zeta = 0 \) (transverse elements), respectively, are the two corresponding pairs of impedance expressions equal. For example if \( \zeta = \infty \), then equation (3.71) reduces to equation (3.55). Whereas in general \( Z_{2,1}^{\hat{z} \rightarrow \hat{z}} \) is a measure of the coupling between the \( \hat{z} \) vector components of the current distributions on the reference and test arrays, the expression given by equation (3.55) is the mutual impedance between two arrays with vertical elements.

This distinction is due to the fact that the pattern factor is a function of \( \zeta \) and so varies with element pitch. Also the relative magnitude of the contribution from each type of interaction (\( \hat{z} \leftarrow \hat{z}, \hat{\phi} \leftarrow \hat{\phi}, \text{etc.} \)) is "scaled" by the \( \zeta \) factors multiplying each contribution in (3.71). The \( \zeta \) factors effectively suppress the insignificant interactions while bringing into prominence the most characteristic type of interaction for a given element pitch. As an example, for near vertical elements, \( \zeta \) will be a large number. Because of the \( (1 + \zeta^2)^{-1} \) factor out in front, all interactions except the \( \hat{z} \leftarrow \hat{z} \) type will be very small, though not necessarily negligible. If \( \zeta \) is near unity, corresponding to an element pitch of 45 degrees, all four types of interactions will be comparable in magnitude.

The coefficients \( R_{nm}^{\hat{z} \rightarrow \hat{z}} \) and \( \hat{R}_{nm}^{\hat{z} \rightarrow \hat{z}} \) are approximately equal if \( \rho_1 \approx \rho_2 \approx \rho_o \) (see section 3.3.1) with

\[
R_{nm}^{\hat{z} \rightarrow \hat{z}} \approx \hat{R}_{nm}^{\hat{z} \rightarrow \hat{z}} \approx jm \frac{u_z}{\beta \rho_o} \left[ J_{\rho_o} H_{\rho_o} - \frac{J_a}{H_a} H_{\rho_o}^2 \right] \tag{3.73}
\]

\[
= jm \frac{u_z}{\beta \rho_o} R_{nm}^{\hat{z} \rightarrow \hat{z}} \tag{3.74}
\]

47
In view of the equality of $R_{nm}^\hat{z} \to \hat{\phi}$ and $R_{nm}^{\hat{\phi} \to \hat{z}}$ under the forementioned assumptions (i.e., $\rho_1 \simeq \rho_2$ and $\Delta z = 0$), we will subsequently refer only to $\hat{\phi} \to \hat{z}$ type interactions in the following observations.

First, $R_{nm}^{\hat{z} \to \hat{\phi}}$ is of the same form (but in phase quadrature) as $R_{nm}^\hat{z} \to \hat{\phi}$ except for a multiplicative constant $m$. Since $R_{nm}^\hat{z} \to \hat{\phi}$ decreases as $m^{-1}$ for large $m$, $R_{nm}^{\hat{z} \to \hat{\phi}}$ is at worst $O(1)$ for large $m$, for each $n$. (However, the series involving $R_{nm}^\hat{z} \to \hat{\phi}$ still converges with help from the pattern function).

Secondly, for small scan angles ($|S_z| < 1$) the corresponding cross-coupling $Z_{2,1}^{\hat{z} \to \hat{\phi}}$ is small and in fact vanishes if $S_z$ is identically zero and $\Delta z = 0$. This is due to the presence of the multiplicative factor $u_z = S_z + \frac{n \lambda}{D_z}$ and the fact that one sums over all negative and positive values of $n$. That is, the plus and minus terms combine to produce a factor $\sin \frac{\beta \lambda \Delta z}{D_z}$ which vanishes if $\Delta z$ is zero. Note that if $\Delta z \neq 0$, then the exponential phase factor comes into play, and so there will be a non-vanishing cross-coupling contribution even for zero scan. Figures 5 and 6 show the convergence of the mutual impedance (see equation (3.47)) function of the angular mode index $m$ for the $n = 0$ (propagating) and $n = 1$ (lowest order evanescent) Floquet wave modes with the dipole pitch angle as a parameter. The arrays radiate in the presence of an uncoated PEC cylinder. The reduction in the rate of convergence with increasing pitch angle (as discussed in section B) is illustrated in Figure 5. Figure 7 shows that the convergence of the sum on $n$, is rapid over a wide range of pitch angles. More terms are required for near horizontal elements ($\alpha \simeq 0$) as expected since the near-transverse dipoles radiate very broad beams in the vertical plane.
Figure 5: Convergence of array self-reactance as a function of the angular modal index $m$ for the $n = 0$ Floquet wave, for various dipole pitch angles; uncoated cylinder case.
Figure 6: Convergence of array self-reactance as a function of the angular modal index \( m \) for the \( n = 1 \) Floquet wave, for various dipole pitch angles; uncoated cylinder case.
Figure 7: Convergence of array self-reactance as a function of the number Floquet wave modes included; for various dipole pitch angles; uncoated cylinder case
3.4.4 Convergence Tests

The rules for terminating each interaction series are summarized as follows: For "direct" self-impedance components, the convergence criterion need not be applied for a given Floquet mode \( n \) until \( m > m_n + \frac{\Omega}{2} \) where \( m_n \) is the location of the \( n^{th} \) beam peak in \( m \)-space and \( \Omega \) is the corresponding beamwidth. For mutual impedances, we test for convergence early on in the series (applies to both direct and scatter components).

Two types of convergence criterion are implemented in our numerical code:

1. percentage test: the absolute value of the sum of the last \( k_p \) terms in the series divided by \( k_p \) is less than \( p \) per cent of the absolute value of the cumulative sum.

2. tail magnitude test: the absolute value of the sum of the last \( k_m \) terms is less than \( Z_{\text{thresh}} \) ohms.

These tests are applied independently to the \( n \) and \( m \) sums. To ensure that the series are not terminated prematurely, we require that each series test positively on three consecutive testings. The values of \( k_p \) and \( k_m \) should be large enough such that false terminations do not occur due to local extrema in the impedance sums. We have found that \( k_p = k_m = 10 \) work well for typical cylinder-plus-layer radial dimensions on the order of a half wavelength. To insure adequate precision, \( p \) should be on the order of 0.5 per cent. We have set the threshold impedance \( Z_{\text{thresh}} \) in the 0.1 to 0.25 ohm range with satisfactory results.

3.5 Numerical Results

In this section numerical results for the array self-impedance and the mutual impedance between two arrays are presented. The coupling between coinciding or
overlapping subarrays belonging to the same array is computed as the averaged mutual impedance between two thin filament subarrays, where the averaging takes place on a circle of radius \( r_{eq} \) lying in a plane normal to a line tangent to the wire axis at the subarray terminals. \( r_{eq} \), the equivalent wire radius, is not precisely equal to the physical wire radius but is smaller by an amount proportional to the ratio of the subarray element diameter to length, viz,

\[
 r_{eq} = r_w \left[ 1 - 0.40976 \frac{r_w}{\Delta L} \right]
\]

where \( r_w \) is the physical wire radius, \( \Delta L \) is the subsectional array element arm length, with \( \Delta L = \frac{L}{M+1} \), where \( M \) is the number of PWS modes used and \( 2L \) is the physical dipole length. This first-order correction was originally derived by Imbriale [7] for the case of a single linear dipole radiating in free-space. It is shown in Appendix C that this result is also valid for periodic linear arrays of vertical arrays radiating in the presence of a 2-d conducting body, providing the separation between the array and the conducting body is much greater than a wire radius. We also apply this approximation to periodic arrays of curved elements with less theoretical justification as the true equivalent radius is not known for this more general case.

In all numerical cases considered below, the array element terminals are located in the center of each dipole. All length parameters are given as fraction of a wavelength \( \lambda_o \) in the dielectric medium at \( F_o \), an arbitrary reference frequency. The cylinder radius \( a \) is fixed at .227\( \lambda_o \) and the thickness \( w \) of the dielectric layer if present is .248\( \lambda_o \) while the the interelement spacing \( D_z \) is 0.6\( \lambda_o \).

The wire radius is .001\( \lambda_o \) in all test cases. The radial distances \( \rho_1 \) and \( \rho_2 \) to the source and test array terminals are chosen to be equal with \( \rho_1 = \rho_2 = .475\lambda_o \) and denoted by \( \rho_o \).
Figures 8 and 9 show the array self-impedance as a function of frequency with the number of PWS current modes as a parameter, for the vertical and transverse element cases, respectively. Figures 10 and 11 are similar but include a dielectric layer of relative permittivity $\varepsilon_r = 2.0$ and thickness $0.248 \lambda_o$. It is observed in figures 8 through 11 that the difference between the 3 and 5 PWS mode MM solutions for the case of transverse elements is significantly larger than for the corresponding vertical element case. It is likely that the error incurred in using the correction (3.75) for tightly curved elements is at least partly responsible.

In the remaining cases, 3 PWS modes per array were employed. Figures 12 shows the direct component (i.e., cylinder is absent) of the array self-impedance $Z_{2,1}^{P,\text{dir}}$ as a function of frequency for various element pitch angles (with no dielectric present). The self-resistance is greatest for vertical elements while the self-reactance is relatively insensitive to the element pitch angle. Figure 13 shows the same case now with a small (0.227$\lambda_o$ radius) cylinder present. The presence of the cylinder tends to increase the radiation resistance in the higher frequency range. With a 30 degree scan and an interelement spacing $D_2 = 0.6\lambda_o$, one observes (14) a discontinuous increase in the radiation resistance as the first grating lobe emerges at 1.11 $F_o$. Below this frequency, the family of self-impedance trajectories (as a function of element pitch) tend to coalesce in comparison to the unscanned case.

Figure 15 shows the array mutual impedance between two arrays of vertical dipoles as function of the angular displacement as the test array is moved about a circle about the cylinder axis, with and without a dielectric layer present. Note that the position $\Delta\phi = 0$ corresponds to the array self-impedance.

In figures 16 and 17, the array port mutual impedance for vertical elements (no dielectric case) is decomposed into its direct and scatter components, where
the real and imaginary parts are plotted separately. Figures 18 19 are similar except a dielectric layer is included. Observe that the non-propagating Floquet modes \(|n| > 0\) contribute only to the mutual reactance and then only when the test array is in the immediate vicinity of the source array. In the present case, the radial distance to the arrays from the cylinder surface is \(0.248\lambda_o\), which is sufficiently large that the evanescent mode contribution to \(X_{2,1}^{P,\text{scat}}\) is very small although not identically zero.

In group of figures 20 through 24 the previous study is repeated for transverse array elements. The relative axial offset between the test and source arrays is \(0.5D_z\). For \(\rho_o = 0.475\lambda_o\), a half-wavelength transverse dipole spans a 60.3 degree arc; thus, for the small cylinder in these examples there is significant mutual coupling even when the source and test arrays are diametrically opposed about the cylinder, as seen in figure 12. One observes on comparing figures 16 and 18 with figures 21 and 23 that the non-propagating Floquet modes strongly influence the direct component of the mutual reactance over a greater angular separation range (compared with the vertical element case) because the source and test arrays overlap azimuthally until \(\Delta \phi > 60.3\) deg.

In figures 25 and 26 the variation in the mutual coupling as a function of the relative axial offset position \(\Delta z\) is considered for the vertical and transverse element cases, respectively. In each case, the mutual resistance and reactance is decomposed into direct, scatter, and total (direct + scatter) components. For un-scanned arrays, only the propagating \((n = 0)\) Floquet mode ordinarily contributes to the mutual resistance (for a sufficiently small interelement spacing). But with \(S_z = 0\), the \(n = 0\) mode is independent of the axial offset. Therefore, the mutual resistance (which depends only on the \(n=0\) Floquet mode) remains constant as \(\Delta z\) is varied. The non-propagating modes do contribute to the mutual reactance and
these modal contributions are a periodic function of $\Delta z$, where the period is $D_z$. 
Figure 8: Array self-impedance as a function of frequency with the number of PWS modes as a parameter; vertical element case; no dielectric; $r_w = .001 \lambda_o$, $D_z = 0.6 \lambda_o$, $2L = 0.5 \lambda_o$, $a = .227 \lambda_o$, $\rho_o = .475 \lambda_o$
Figure 9: Array self-impedance as a function of frequency with the number of PWS modes as a parameter; transverse element case; no dielectric; \( r_w = 0.001 \lambda_o \), \( D_z = 0.6 \lambda_o \), \( 2L = 0.5 \lambda_o \), \( a = 0.227 \lambda_o \), \( \rho_o = 0.475 \lambda_o \).
Figure 10: Array self-impedance as a function of frequency with the number of PWS modes as a parameter; vertical element case; dielectric present; $r_w = 0.001 \lambda_0$, $D_z = 0.6 \lambda_0$, $2L = 0.5 \lambda_0$, $a = 0.227 \lambda_0$, $\rho_o = 0.475 \lambda_0$, $\epsilon_r = 2.0$, $w = 0.248 \lambda_0$.
Figure 11: Array self-impedance as a function of frequency with the number of PWS modes as a parameter; transverse element case; dielectric present; $r_w = 0.001\lambda_0$, $D_z = 0.6\lambda_0$, $2L = 0.5\lambda_0$, $a = 0.227\lambda_0$, $\rho_0 = 0.475\lambda_0$, $S_z = 0$, $\varepsilon_r = 2.0$, $w = 0.248\lambda_0$.
Figure 12: Array self-impedance as a function of frequency for various element pitch angles; without cylinder; no dielectric; \( r_w = .001 \lambda_o, D_z = 0.6 \lambda_o, 2L = 0.5 \lambda_o, \rho_o = .475 \lambda_o, S_z = 0 \)
Figure 13: Array self-impedance as a function of frequency for various element pitch angles; with cylinder; no dielectric; $r_w = .001$, $D_z = 0.6\lambda_o$, $2L = 0.5\lambda_o$, $a = .227\lambda_o$, $\rho_o = .475\lambda_o$, $S_z = 0$
Figure 14: Array self-impedance as a function of frequency for various element pitch angles; with cylinder; scan case: $S_z = .5$; no dielectric; $\tau_w = .001\lambda_o$, $D_z = 6.6\lambda_o$, $2L = 0.5\lambda_o$, $a = .227\lambda_o$, $\rho_o = .475\lambda_o$
Figure 15: Array mutual impedance $Z_{2,1}^P$ in the complex plane as a function of the array angular separation; vertical elements; with and without dielectric; $r_w = .001 \lambda_o$, $D_z = 0.6 \lambda_o$, $2L = 0.5 \lambda_o$, $a = .227 \lambda_o$, $\rho_o = .475 \lambda_o$, $\epsilon_r = 2.0$, $w = .248 \lambda_o$
Figure 16: Direct component of array mutual impedance $Z_{2,1}^{P,\text{dir}}$ as a function of the array angular separation; vertical elements; no dielectric; $r_w = 0.001\lambda_0$, $D_z = 0.6\lambda_0$, $2L = 0.5\lambda_0$, $a = 0.227\lambda_0$, $\rho_0 = 0.475\lambda_0$. 

65
Figure 17: Scatter component of array mutual impedance $Z_{2,1}^{p,scat}$ as a function of the array angular separation; no dielectric; vertical elements; $r_w = .001\lambda_0$, $D_z = 0.6\lambda_0$, $2L = 0.5\lambda_0$, $a = .227\lambda_0$, $\rho_o = .475\lambda_0$.
Figure 18: Direct component of array mutual impedance $Z_{2,1}^{P,\text{dir}}$ as a function of the array angular separation; with dielectric; vertical elements; $r_w = 0.001\lambda_o$, $D_z = 0.6\lambda_o$, $2L = 0.5\lambda_o$, $a = 0.227\lambda_o$, $\rho_o = 0.475\lambda_o$, $\epsilon_r = 2.0$, $w = 0.248\lambda_o$.
Figure 19: Scatter component of array mutual impedance $Z_{2,1}^{p,scat}$ as a function of the array angular separation; with dielectric; vertical elements; $r_w = 0.001\lambda_o$, $D_z = 0.6\lambda_o$, $2L = 0.5\lambda_o$, $a = 0.227\lambda_o$, $\rho_o = 0.475\lambda_o$, $\epsilon_r = 2.0$, $w = 0.248\lambda_o$. 
Figure 20: Array mutual impedance $Z_{2,1}^P$ in the complex plane as a function of the array angular separation; transverse elements; with and without dielectric; $r_w = 0.001\lambda_o$, $D_z = 0.6\lambda_o$, $2L = 0.5\lambda_o$, $a = 0.227\lambda_o$, $\rho_o = 0.475\lambda_o$, $\epsilon_r = 2.0$, $w = 0.248\lambda_o$. 

69
Figure 21: Direct component of array mutual impedance $Z_{2,1}^{P,dir}$ as a function of the array angular separation; transverse elements; no dielectric; $r_w = 0.001 \lambda_o$, $D_z = 0.6 \lambda_o$, $2L = 0.5 \lambda_o$, $a = 0.227 \lambda_o$, $\rho_o = 0.475 \lambda_o$
Figure 22: Scatter component of array mutual impedance $Z_{2,1}^{P,\text{scat}}$ as a function of the array angular separation; no dielectric; transverse elements; $r_w = .001 \lambda_o$, $D_z = 0.6 \lambda_o$, $2L = 0.5 \lambda_o$, $a = .227 \lambda_o$, $\rho_o = .475 \lambda_o$. 

71
Figure 23: Direct component of array mutual impedance $Z_{2,1}^{P,dir}$ as a function of the array angular separation; with dielectric; transverse elements; $r_w = 0.001 \lambda_0$, $D_z = 0.6 \lambda_0$, $2L = 0.5 \lambda_0$, $a = 0.227 \lambda_0$, $\rho_o = 0.475 \lambda_0$, $\epsilon_r = 2.0$, $w = 0.248 \lambda_0$.
Figure 24: Scatter component of array mutual impedance $Z_{2,1}^{p,scat}$ as a function of the array angular separation; with dielectric; transverse elements; $r_w = .001\lambda_0$, $D_z = 0.6\lambda_0$, $2L = 0.5\lambda_0$, $a = .227\lambda_0$, $\rho_0 = .475\lambda_0$, $\epsilon_r = 2.0$, $w = .248\lambda_0$
Figure 25: Array mutual impedance $Z_{2,1}$ as a function of the relative axial offset $\Delta z$; no dielectric; vertical elements; angular separation $\Delta \phi = 12.06\,\text{deg.}$; $r_w = .001\lambda_0$, $D_z = 0.6\lambda_0$, $2L = 0.5\lambda_0$, $a = .227\lambda_0$, $\rho_o = .475\lambda_0$
ARRAY MUTUAL COUPLING: TRANSVERSE ELEMENTS

Figure 26: Array mutual impedance $Z_{2,1}^P$ as a function of the relative axial offset $\Delta z$; no dielectric; transverse elements; angular separation $\Delta \phi = 0$deg.; $r_w = .001\lambda_0$, $D_z = 0.6\lambda_0$, $2L = 0.5\lambda_0$, $a = .227\lambda_0$, $\rho_o = .475\lambda_0$
CHAPTER IV

INDUCED VOLTAGE

In this chapter we consider the open-circuit voltage induced at the reference terminals of a periodic linear array due to an oblique plane wave of arbitrary polarization impinging on the antenna structure. This entails solving a two-region boundary value problem for the fields in the dielectric layer and in the free space region. An eigenfunction solution is developed. The resultant field in the dielectric layer is used to calculate the voltage induced at the dipole terminals of each array. The solution form valid in free-space region gives the scattered field contribution radiated by the layered cylinder structure in the absence of all arrays. We focus on the induced voltage calculation in this chapter. The scattered field results are given in the following chapter.

The antenna excitation consists of a plane wave of arbitrary polarization obliquely incident on the array-plus-cylinder configuration. The terminals of the reference element are located at \((\rho_q, \phi_q, z_q)\). The plane wave propagates in the direction

\[
\hat{s}_p = \cos \theta_p \hat{z} - \sin \theta_p \hat{\rho}_p
\]  

(4.1)

where

\[
\hat{\rho}_p = \cos \phi_p \hat{x} + \sin \phi_p \hat{y}
\]  

(4.2)
Figure 27: Incident plane wave impinging on array structure (dielectric layer is not shown).
The induced voltage across the reference terminals of the $q^{th}$ array is given by

$$V_{\text{o.c.}} = -\frac{1}{I_{q}(R_{q})} \int_{\text{element}} I_{q}(s) \overline{E} \cdot \hat{p} \, ds \quad (4.3)$$

where $I(R_{q})$ is the terminal current and $I_{q}(s)$ is understood to be the current distribution under transmitting conditions and $\hat{p}$ is the unit element orientation vector as defined in equation (2.15). $\overline{E}$ is the total scattered electric field in the dielectric layer (containing the arrays).

We decompose the plane wave into $TM$ and $TE$ components and treat each polarization separately. We consider the $TM$ plane wave case first. The fields corresponding to the $TE$ case follow through a duality argument.

The approach employed here is the same as that used in the previous chapter to obtain the array fields. In the present case, the fields in the free space region $\rho > b$ are decomposed into incident and scattered field components, whereas a single series represents the total field in the dielectric layer (where there are no active sources).

4.1 Incident TM plane wave

First, a remark on notation: In the remainder of this chapter, as elsewhere, the shorthand notation $J_{\iota \rho}$ means the Bessel function of order $m$ with argument equal to a distance variable $\rho$ times $\beta_{i}u_{i \rho}$ referred to the $i^{th}$ region ($i = 1$ or 2) and where $u_{i \rho}$ is the normalized radial wavenumber in region $i$; in the present situation $u_{i \rho} = \sqrt{1 - u_{i z}^{2}}$, where $u_{i z} = \frac{S_{z}}{\sqrt{\varepsilon_{i}}}$.

In the case of $TM$ plane wave incidence, $\overline{E}^{\text{inc}}$ is in the plane of incidence, defined by the $z$-axis and $\hat{s}_{p}$, the plane wave propagation vector. The cylindrical
wave expansion of an oblique $TM$ plane wave is given by Harrington [10]. In the free-space region $2$,

\[
E_{x,2}^{inc} = E^{TM} u_{\rho_2} e^{-j\kappa_2 x} \sum_{m=0}^{\infty} \epsilon_m j^m J_{2\rho} \cos m(\phi - \phi_p)
\]

\[
H_{x,2}^{inc} = 0
\]

where $u_{\rho_2} = \sin \theta_p$ for the incident plane wave.

In view of (4.4), we infer that the axial fields in region 1 and 2 may be expressed in the form

4.1.1 Axial fields in region 1

\[
E_{x,1}^{tot} = E^{TM} u_{\rho_1} e^{-j\kappa_1 x} \sum_{m=0}^{\infty} \epsilon_m j^m [A_m J_{1\rho} + B_m H_{1\rho}] \cos m(\phi - \phi_p)
\]

\[
H_{x,1}^{tot} = \frac{E^{TM}}{Z_0} e^{-j\kappa_1 x} \sum_{m=0}^{\infty} \epsilon_m m j^m [C_m J_{1\rho} + D_m H_{1\rho}] \sin m(\phi - \phi_p)
\]

4.1.2 Axial fields in region 2

\[
E_{x,2}^{inc} = E^{TM} u_{\rho_2} e^{-j\kappa_2 x} \sum_{m=0}^{\infty} \epsilon_m j^m J_{2\rho} \cos m(\phi - \phi_p)
\]

\[
H_{x,2}^{inc} = 0
\]

\[
E_{x,2}^{scat} = E^{TM} u_{\rho_2} e^{-j\kappa_2 x} \sum_{m=0}^{\infty} \epsilon_m j^m E_m^{TM} H_{2\rho} \cos m(\phi - \phi_p)
\]

\[
H_{x,2}^{scat} = \frac{E^{TM}}{Z_2} e^{-j\kappa_2 x} \sum_{m=0}^{\infty} \epsilon_m m j^m F_m^{TM} H_{2\rho} \sin m(\phi - \phi_p)
\]

The $\rho$ and $\phi$ components of the electric and magnetic field components may be found by applying equations (3.1) and (3.2). The unknown Fourier coefficients $A_m^{TM}$ through $F_m^{TM}$ are determined by applying boundary conditions. The boundary conditions and their mathematical expression are given by
\* \textit{E}_z \text{ vanishes at cylinder surface } (\rho = a): \\\\
\[ \frac{J_{1a}}{H_{1a}} A^T_{m} + B^T_{m} = 0 \] (4.4) \\
\* \textit{E}_\phi \text{ vanishes at cylinder surface } (\rho = a): \\
\[ \frac{J'_{1a}}{H_{1a}} C^T_{m} + D^T_{m} = 0 \] (4.5) \\
\* \text{continuity of } \textit{E}_z \text{ at air-dielectric interface:} \\
\[ \frac{u_{\rho 1}}{u_{\rho 2}} \left[ A^T_{m} J_{1b} + B^T_{m} H_{1b} \right] - E^T_{m} H_{2b} = J_{2b} \] (4.6) \\
\* \text{continuity of } \textit{E}_\phi \text{ at air-dielectric interface:} \\
\[ \frac{u_{z 1}}{u_{\rho 1} \beta_{1b}} \left[ J_{1b} - \frac{J_{1a}}{H_{1a}} H_{1b} \right] A^T_{m} + \frac{1}{u_{\rho 1}} \left[ J'_{1b} - \frac{J'_{1a}}{H'_{1a}} H_{1b} \right] C^T_{m} \] - \[ \frac{u_{z 2}}{u_{\rho 2} \beta_{2b}} E^T_{m} H_{2b} - \frac{1}{u_{\rho 2}} F^T_{m} H_{2b} = \frac{u_{z 2}}{u_{\rho 2} \beta_{2b}} J_{2b} \] (4.7) \\
\* \text{continuity of } H_z \text{ at air-dielectric interface:} \\
\[ \frac{Z_2}{Z_1} \left[ J_{1b} - \frac{J'_{1a}}{H'_{1a}} H_{1b} \right] C^T_{m} - F^T_{m} H_{2b} = 0 \] (4.8) \\
\* \text{continuity of } H_\phi \text{ at air-dielectric interface:} \\
\[ \left[ J'_{1b} - \frac{J_{1a}}{H_{1a}} H_{1b} \right] A^T_{m} + \frac{m^2 u_{z 1}}{u_{\rho 1} \beta_{1b}} \left[ J_{1b} - \frac{J'_{1a}}{H'_{1a}} H_{1b} \right] C^T_{m} \] - \[ \frac{Z_1}{Z_2} \frac{E^T_{m} H_{2b}}{Z_2} - \frac{Z_1}{Z_2} m^2 u_{z 2}^2 F^T_{m} H_{2b} = \frac{Z_1}{Z_2} J'_{2b} \] (4.9) \\

Although we have tagged all of the unknown coefficients with the superscript \textit{TM}, corresponding to the case of \textit{TM} plane wave incidence, it should be noted that the
coefficients \( C^{TM}, D^{TM} \) and \( F^{TM} \) are actually associated with \( TE \) waves. That is, the scattered field will have a non-vanishing axial magnetic field component for oblique incidence on a coated cylinder. Similarly, in the case of \( TE \) wave incidence, we will label all the unknown coefficients with the superscript \( TE \) even though the scattered fields are a combination of \( TM \) and \( TE \) components.

The preceding six equations may be reduced to two independent equations in the coefficients \( A_m^{TM} \) and \( C_m^{TM} \). In matrix form:

\[
\begin{bmatrix}
T_m^{TM} & U_m^{TM} \\
V_m^{TM} & W_m^{TM}
\end{bmatrix}
\begin{bmatrix}
A_m^{TM} \\
C_m^{TM}
\end{bmatrix}
= \begin{bmatrix}
S_1^{TM} \\
S_2^{TM}
\end{bmatrix}
\]

(4.10)

After some manipulation, we find that the \( T, U, V, \) and \( W \) coefficients reduce to

\[
T_m^{TM} = \left[ \frac{u_{z2} u_{\rho 1}}{u_{\rho 2} \beta_2 b} - \frac{u_{z1}}{u_{\rho 1} \beta_1 b} \right]\left[ J_{1b} - \frac{J_{1a}}{H_{1a}} H_{1b} \right]
\]

(4.11)

\[
U_m^{TM} = \frac{1}{u_{\rho 2} Z_1 H_{2b}} \left[ J_{1b} - \frac{J_{1a}}{H_{1a}} H_{1b} \right] - \frac{1}{u_{\rho 1}} \left[ J_{1b} - \frac{J_{1a}}{H_{1a}} H_{1b} \right]
\]

(4.12)

\[
V_m^{TM} = \left[ J_{1b} - \frac{J_{1a}}{H_{1a}} H_{1b} \right] - \frac{u_{\rho 1} Z_1 H_{1b}}{u_{\rho 2} Z_2 H_{2b}} \left[ J_{1b} - \frac{J_{1a}}{H_{1a}} H_{1b} \right]
\]

(4.13)

\[
W_m^{TM} = \left[ \frac{m^2 u_{z1}}{u_{\rho 1} \beta_1 b} - \frac{m^2 u_{z2}}{u_{\rho 2} \beta_2 b} \right]\left[ J_{1b} - \frac{J_{1a}}{H_{1a}} H_{1b} \right]
\]

(4.14)

The excitation functions are

\[
S_{1m}^{TM} = 0
\]

(4.15)

\[
S_{2m}^{TM} = \frac{Z_1}{Z_2} \left[ J_{2b} - \frac{H_{1b}'}{H_{2b}} J_{2b} \right]
\]

(4.16)

Inverting the matrix equation, one obtains

\[
A_m^{TM} = \frac{W_m^{TM} S_{1m}^{TM} - U_m^{TM} S_{2m}^{TM}}{T_m^{TM} W_m^{TM} - V_m^{TM} U_m^{TM}}
\]

(4.17)

81
For antenna structures on the order of a wavelength in diameter, the $A_{m}^{TM}$ and $C_{m}^{TM}$ diminish very rapidly as a function of $m$; for this reason, it is not necessary to introduce a normalized coefficient set as was the case at this point in our analysis in chapter III. The remaining coefficients $B_{m}^{TM}$, $D_{m}^{TM}$, $E_{m}^{TM}$ and $F_{m}^{TM}$ are related to $A_{m}^{TM}$ and $C_{m}^{TM}$ through the boundary equations (4.4) through (4.9). In particular, the free space scattering coefficients are given by

\[ E_{m}^{TM} H_{2b} = \frac{u_{\rho_{1}}}{u_{\rho_{2}}} \left[ J_{1b} - \frac{J_{1a}}{H_{1a}} H_{1b} \right] A_{m}^{TM} - J_{2b} \]  \hspace{1cm} (4.19)

\[ F_{m}^{TM} H_{2b} = \frac{Z_{2}}{Z_{1}} \left[ J_{1b} - \frac{J'_{1a}}{H_{1a}} H_{1b} \right] C_{m}^{TM} \]  \hspace{1cm} (4.20)

**No Dielectric Layer Case**

With the aid of figure 28, equations (4.17) and (4.18), together with equations (4.11)—(4.16) one finds that in the absence of the dielectric

\[ A_{m}^{TM} = \frac{S_{2m}^{TM}}{V_{m}^{TM}} = 1 \]

\[ B_{m}^{TM} = E_{m}^{TM} = - \frac{J_{a}}{H_{a}} \]

whereas

\[ C_{m}^{TM} = D_{m}^{TM} = F_{m}^{TM} = 0 \]

**4.1.3 Induced Voltage**

Upon substitution of the eigenfunction series representations for the electric field in region 1 into equation (4.3) and integrating over the reference element one obtains the open-circuit voltage induced across the reference terminals of the $q^{th}$
Figure 28: Truth table for matrix and scattering coefficients; TM or TE plane wave incidence on dielectric coated cylinder; for various combinations of scan and dielectric layer.
array:

\[ V_{oc}^{TM,q} = -\frac{E^{TM} \rho_q u_{\rho_2} e^{-j_\varphi z q} P_0^2}{\sqrt{1 + \zeta^2}} \sum_{m=0}^{\infty} \frac{\epsilon_m}{2} \left\{ \zeta R_{nm}^{TM,z}(\rho_q) \left[ P_0^+(m) e^{j m (\phi_q - \phi_p)} + P_0^-(m) e^{-j m (\phi_q - \phi_p)} \right] - j R_{nm}^{TM,\phi}(\rho_q) \left[ P_0^+(m) e^{j m (\phi_q - \phi_p)} - P_0^-(m) e^{-j m (\phi_q - \phi_p)} \right] \right\} \] (4.21)

where the pattern functions \( P_0^\pm \) are given by equation (2.24) evaluated for \( n = 0 \) and where

\[ R_{m}^{TM,z} = u_{\rho_1} \rho_1 \left[ A_m^{TM} J_{1 \rho_q} + B_m^{TM} H_{1 \rho_q} \right] \] (4.22)

\[ R_{m}^{TM,\phi} = \frac{j m u_{\rho_1} \rho_1}{\beta_{1 \rho_q \rho_1} Z_1} R_{m}^{TM,z} + j (m+1) \rho_1 Z_1 \left[ C_m^{TM} J_{1 \rho_q} + D_m^{TM} H_{1 \rho_q} \right] \] (4.23)

**Special Case: Vertical Elements, No Dielectric**

For vertical elements, \( \zeta = \infty \). Recall also that for this case, the plus and minus element pattern functions are equal and are independent of the angular index \( m \) (see equation (2.25)).

In the absence of the dielectric, \( A_m^{TM} = 1 \) and \( B_m^{TM} = -\frac{J_{a}}{H_{a}} \) while \( C_m^{TM} \) and \( D_m^{TM} \) vanish. Thus, the \( R_{m}^{TM} \) Fourier coefficients reduce to

\[ R_{m}^{TM,z} = u_{\rho_2} \rho_2 \left[ J_{\rho_q} - \frac{J_{a}}{H_{a}} H_{\rho_q} \right] \] (4.24)

\[ R_{m}^{TM,\phi} = 0 \] (4.25)

The resultant voltage induced at the terminals the terminals of the \( q \)th array is

\[ V_{oc}^{TM,q} = -E^{TM} \rho_q u_{\rho_2} e^{-j_\varphi z q} P_0^2 \sum_{m=0}^{\infty} \frac{\epsilon_m}{2} \left\{ J_{\rho_q} - \frac{J_{a}}{H_{a}} H_{\rho_q} \right\} \cdot \cos m (\phi_q - \phi_p) \] (4.26)

where \( P_0 \) denotes the element pattern function for a vertical element given by equation (2.25).
Special Case: Transverse Elements, No Dielectric

For transverse elements, $\zeta = 0$. As in the previous case, the plus and minus element patterns are identical though they are functions of the index $m$ in present case. Substitution of equations (4.24) and (4.23) into equation (4.21) yields

$$V_{oc}^{TM,q} = -jE^{TM}\frac{u_{p}}{\beta u_{z}} e^{-j\kappa zq} \sum_{m=1}^{\infty} j^m P(m) \left[ J_{\rho q} - \frac{J_a}{H_a} H_{\rho q} \right] \sin m(\phi_q - \phi_p)$$

where $P(m)$ is the pattern function for transverse dipoles.

4.2 Incident TE plane wave

In this case $H^{inc}$ lies in the plane of incidence. The associated boundary value problem is not quite the dual of the TM case, as one would have to replace the cylinder with a perfect magnetic conducting cylinder for complete duality. Nevertheless, one can exploit this partial duality to obtain a solution for the TE plane wave case with little additional effort.

The cylinder wave expansion for an obliquely incident $TE$ plane wave propagating in the direction $\delta_p$ in region 2 is given by

$$H_{z,inc}^{2} = H_{o}^{TE} u_{\rho z} e^{-j\kappa z} \sum_{m=0}^{\infty} \epsilon_m j^m J_{2\rho} \cos m(\phi - \phi_p)$$

$$E_{z,inc}^{2} = 0$$

One may express the axial fields in regions 1 and 2 in the form

4.2.1 Axial fields in region 1

$$E_{z,1}^{tot} = -H_{o}^{TE} Z_{1} e^{-j\kappa z} \sum_{m=0}^{\infty} \epsilon_m j^m \left[ C_{m}^{TE} J_{1\rho} + D_{m}^{TE} H_{1\rho} \right] \sin m(\phi - \phi_p)$$

$$H_{z,1}^{tot} = H_{o}^{TE} u_{\rho z} e^{-j\kappa z} \sum_{m=0}^{\infty} \epsilon_m j^m \left[ A_{m}^{TE} J_{1\rho} + B_{m}^{TE} H_{1\rho} \right] \cos m(\phi - \phi_p)$$
4.2.2 Axial fields in region 2

\[ E_{z,2}^{\text{inc}} = 0 \]
\[ H_{z,2}^{\text{inc}} = H_0^{TE} u_{\rho_2} e^{-j\kappa z} \sum_{m=0}^{\infty} \epsilon_m j^m J_{1\rho} \cos m(\phi - \phi_p) \]
\[ E_{z,2}^{\text{scat}} = -H_0^{TE} Z_2 e^{-j\kappa z} \sum_{m=0}^{\infty} \epsilon_m m J_m^{TE} H_{1\rho} \sin m(\phi - \phi_p) \]
\[ H_{z,2}^{\text{scat}} = H_0^{TE} u_{\rho_2} e^{-j\kappa z} \sum_{m=0}^{\infty} \epsilon_m j^m E_m^{TE} H_{1\rho} \cos m(\phi - \phi_p) \]

The \( \rho \) and \( \phi \) components of the electric and magnetic field components may be found by applying equations (3.1) and (3.2). The boundary equations to be satisfied are

- \( E_z \) vanishes at cylinder surface \( (\rho = a) \):
  \[ \frac{J_{1a}}{H_{1a}} C_m^{TE} + D_m^{TE} = 0 \]  \hspace{1cm} (4.28)

- \( E_\phi \) vanishes at cylinder surface \( (\rho = a) \):
  \[ \frac{J_{1a}}{H_{1a}} A_m^{TE} + B_m^{TE} = 0 \]  \hspace{1cm} (4.29)

- Continuity of \( E_z \) at air-dielectric interface:
  \[ \frac{Z_1}{Z_2} \left[ J_{1b} - \frac{J_{1a}}{H_{1a}} H_{1b} \right] C_m^{TE} - F_m^{TE} H_{2b} = 0 \]  \hspace{1cm} (4.30)

- Continuity of \( E_\phi \) at air-dielectric interface:
  \[ \frac{m^2 u_{\rho_1}}{u_{\rho_1}^2} \left[ J_{1b} - \frac{J_{1a}}{H_{1a}} H_{1b} \right] C_m^{TE} + \left[ J_{1b}' - \frac{J_{1a}'}{H_{1a}'} H_{1b}' \right] A_m^{TE} \]
  \[ -\frac{Z_2}{Z_1} \frac{m^2 u_{\rho_2}}{u_{\rho_2}^2} F_m^{TE} H_{2b} - \frac{Z_2}{Z_1} E_m^{TE} H_{2b} = \frac{Z_2}{Z_1} J_{1b}' \]  \hspace{1cm} (4.31)
• continuity of $H_z$ at air-dielectric interface:

$$\frac{u\rho_1}{u\rho_2} [A_m^{TE} J_{1b} + B_m^{TE} H_{1b}] - E_m^{TE} H_{2b} = J_{2b} \quad (4.32)$$

• continuity of $H_\phi$ at air-dielectric interface:

$$\frac{u\rho_1}{u\rho_1\beta_2} \left[ J_{1b} - \frac{J'_{1a}}{H_{1a}} H_{1b} \right] A_m^{TE} + \frac{1}{u\rho_1} \left[ J'_{1b} - \frac{J_{1a}}{H_{1a}} H'_{1b} \right] C_m^{TE} \quad (4.33)$$

$$- \frac{u\rho_2}{u\rho_2\beta_2} E_m^{TE} H_{2b} - \frac{1}{u\rho_2} F_m^{TE} H'_{2b} = \frac{u\rho_2}{u\rho_2\beta_2} J_{2b}$$

Comparison of these boundary equations with those corresponding to the incident $TM$ plane wave case (equations (4.4) - (4.9)) reveals the expected duality relationships between the two equation sets. That is, one obtains equations (4.28) through (4.33) (with the roles of the electric and magnetic fields interchanged) by making the following symbol replacements in the "TM" equations (4.4) through (4.9):

$$A_m^{TM} \rightarrow A_m^{TE} \quad B_m^{TM} \rightarrow B_m^{TE}$$

$$C_m^{TM} \rightarrow C_m^{TE} \quad D_m^{TM} \rightarrow D_m^{TE}$$

$$E_m^{TM} \rightarrow E_m^{TE} \quad F_m^{TM} \rightarrow F_m^{TE}$$

$$\epsilon_r \rightarrow \frac{1}{\epsilon_r} \quad \frac{1}{\epsilon_r} \rightarrow \epsilon_r$$

$$\frac{J_{1a}}{H_{1a}} \rightarrow \frac{J'_{1a}}{H_{1a}} \quad \frac{J'_{1a}}{H_{1a}} \rightarrow \frac{J_{1a}}{H_{1a}}$$

The presence of the last replacement pair is a manifestation of the incomplete duality between the $TM$ and $TE$ plane wave cases, in that we must interchange "hard" and "soft" boundary conditions at the cylinder surface.

In summary, the solutions for the $TE$ coefficients are obtained by applying the duality mapping (4.34) to equations (4.4)—(4.9) in the $TM$ set-up. Thus,
equations (4.28) through (4.33) reduce to two independent equations in $A_{m}^{TE}$ and $C_{m}^{TE}$ (compare with (4.10)):

\[
\begin{bmatrix}
T_{m}^{TE} & U_{m}^{TE} \\
V_{m}^{TE} & W_{m}^{TE}
\end{bmatrix}
\begin{bmatrix}
A_{m}^{TE} \\
C_{m}^{TE}
\end{bmatrix}
= \begin{bmatrix}
S_{1m}^{TE} \\
S_{2m}^{TE}
\end{bmatrix}
\]  

(4.35)

where $U_{m}^{TE}$, $T_{m}^{TE}$, $V_{m}^{TE}$, $W_{m}^{TE}$, $S_{1m}^{TE}$, and $S_{2m}^{TE}$ are obtained from $U_{m}^{TM}$, $T_{m}^{TM}$, $V_{m}^{TM}$, $W_{m}^{TM}$, $S_{1m}^{TM}$, and $S_{2m}^{TM}$, respectively, by making the symbol interchanges $\epsilon_r \leftrightarrow \epsilon_r^{-1}$ and $j\frac{1}{H_1} \leftrightarrow j\frac{1}{H_1}$. The remaining coefficients $B_{m}^{TE}$, $D_{m}^{TE}$, $E_{m}^{TE}$ and $F_{m}^{TE}$ are related to $A_{m}^{TE}$ and $C_{m}^{TE}$ through the boundary equations (4.28)—(4.33).

4.2.3 Induced voltage

Proceeding on as in the TM incident plane wave case, we obtain the open-circuit voltage induced across the reference terminals of the $q^{th}$ array by substituting the series expansion of the electric field into equation (4.3). We find that

\[
V_{oc}^{TE,q} = -\frac{H_0^{TE} z_1 \rho_q e^{-j \kappa z_q}}{\sqrt{1 + \zeta^2}} \sum_{m=0}^{\infty} \frac{\epsilon_m}{2} \left\{ R_{m}^{TE,\phi}(\rho_q) \left[ P_{0}^{+}(m)e^{jm(\phi_q - \phi_p)} \right. \\
+ P_{0}^{-}(m)e^{-jm(\phi_q - \phi_p)} \right. \\
- \left. \left. j \zeta R_{m}^{TE,z}(\rho_q) \left[ P_{0}^{+}(m)e^{jm(\phi_q - \phi_p)} \right. \\
- \left. P_{0}^{-}(m)e^{-jm(\phi_q - \phi_p)} \right] \right\}
\]  

(4.36)

where the pattern functions $P_{0}^{\pm}$ are given by equation (2.24) with $n = 0$, and where

\[
R_{m}^{TE,z} = -mj^{m} \left[ C_{m}^{TE} J_{1}\rho_q + D_{m}^{TE} H_{1}\rho_q \right] 
\]  

(4.37)

\[
R_{m}^{TE,\phi} = -j^{m} \frac{\mu_{z_q}}{\beta_1 \rho_q u_{\phi_1}} R_{m}^{TE,z} + j^{(m+1)} \left[ A_{m}^{TE} J_{1}\rho_q + B_{m}^{TE} H_{1}\rho_q \right]
\]  

(4.38)

**Special Case: Vertical Elements, No Dielectric Layer**

We infer from the results obtained for the TM incidence case together with the duality relations (4.34) that $C_{m}^{TE} = D_{m}^{TE} = 0$ in the absence of the dielectric
layer. This means $R_{m}^{TE,z}$ is zero and so the induced voltage is zero.

**Special Case: Transverse Elements, No Dielectric Layer**

In this case, $A_{m}^{TE} = 1$ and $B_{m}^{TE} = -\frac{J_{m}^{a}}{H_{1a}},$ so that

$$R_{m}^{TE,\phi} = j^{m+1} \left[ J_{\rho q}^{l} - J_{1a}^{l} H_{\rho q}^{l} \right] \quad (4.39)$$

It follows from equation (4.36) that

$$V_{oc}^{TE,q} = -E_{z}^{TE} Z_{o} \rho_{q} e^{-j\kappa z_{q}} \rho_{q} \sum_{m=0}^{\infty} \epsilon_{m} j^{m+1} \left[ J_{\rho q}^{l} - J_{1a}^{l} H_{\rho q}^{l} \right] \cdot \cos m(\phi_{q} - \phi_{p}) \quad (4.40)$$

For the general slant element case, we combining equations (4.21) and (4.36), to obtain the total voltage induced at the reference terminals of the $q^{th}$ array due to an incident wave of arbitrary polarization:

$$V_{oc}^{q} = V_{oc}^{TM,q} + V_{oc}^{TE,q} \quad (4.41)$$

$$= -\frac{1}{\sqrt{1 + \zeta^{2}}} e^{-j\kappa z_{q}} \rho_{q} \sum_{m=0}^{\infty} \frac{\epsilon_{m}}{2} \left[ \zeta E_{o}^{TM} R_{m}^{TM,z}(\rho_{q}) + \frac{E_{z}^{TE}}{\sqrt{\epsilon_{r}}} R_{m}^{TE,\phi}(\rho_{q}) \right] \cdot \left[ P_{0}^{+}(m)e^{jm(\phi_{q} - \phi_{p})} + P_{0}^{-}(m)e^{-jm(\phi_{q} - \phi_{p})} \right] - j \left[ E_{o}^{TM} R_{m}^{TM,\phi}(\rho_{q}) \right]$$

$$+ \zeta \frac{E_{o}^{TE}}{\sqrt{\epsilon_{r}}} R_{m}^{TE,z}(\rho_{q}) \left[ P_{0}^{+}(m)e^{jm(\phi_{q} - \phi_{p})} - P_{0}^{-}(m)e^{-jm(\phi_{q} - \phi_{p})} \right] \right\}$$

where $R_{m}^{TM,z}$, $R_{m}^{TM,\phi}$, $R_{m}^{TE,z}$ and $R_{m}^{TE,\phi}$ are given by (4.22), (4.23), (4.37) and (4.38), respectively, and where

$$E_{o}^{TE} = Z_{2} H_{o}^{TE} \quad (4.42)$$

is the amplitude of the incident electric field.
4.3 Summary of Results

From the definitions of the matrix elements $U_m^{TE}$, $U_m^{TM}$, $T_m^{TE}$, $T_m^{TM}$, etc., introduced in previous sections one may determine the conditions under which the field expansion coefficients $A_m^{TM}$, $A_m^{TE}$, $F_m^{TM}$, $F_m^{TE}$ are non-zero for an incident plane wave of arbitrary polarization. A complete summary of results for all four combinations of scan and dielectric layer is tabulated in figure 28. These results will prove useful in chapter V where the scattered field contribution due to the cylinder will be classified according to these four cases. Note that the truth table entries for the TE incident plane wave case are identical to the corresponding entries for the TM case, a manifestation of the duality between the TM and TE cases. It is not true that the corresponding entries have the same numerical value however. In order to obtain corresponding numerical equalities we would have to replace the PEC cylinder with a PMC cylinder in the TE case or vice versa.
5.1 Preliminary Remarks

In chapter III and IV, two independent but related boundary value problems (B.V.P.) were considered. In chapter III we considered the fields generated due to currents distributed on the array elements radiating in the presence of the layered cylinder. The field solution valid in the dielectric layer was subsequently used to calculate the mutual coupling between arrays. This information is incorporated in the impedance matrix.

In chapter III we encountered a similar boundary-value problem but with a different source located in the free-space region rather than in the dielectric layer. That is, an obliquely incident TM (TE) plane wave may be generated by a linearly phased electric (magnetic) axial line source removed to infinity. Its angular position with respect to the cylinder axis corresponds to the azimuthal direction of the incoming plane wave.

Thus, while the previous two chapters addressed two distinct boundary-value problems, the focus in each case was on the respective solutions in the dielectric layer. In the present chapter, we consider the field solutions in the exterior region. While the interior fields are of interest primarily as intermediary quantities leading to expressions for the array mutuals and the induced voltage, the exterior fields are obviously of interest in their own right. The formulas and results in this chapter are
included for completeness. Calculated and measured field pattern data is included in a separate report [11].

A note on terminology: Since the antenna structure as modelled is infinite and periodic in the z-direction it would be incorrect to refer to “the far-field”, in that the antenna can never appear to be a point source. We will nevertheless employ the term “far zone” in referring to an observation point located a large radial distance from the cylinder axis; specifically, this distance must be sufficiently large such that $\beta u_\rho u \gg 1$, so that the large argument approximation may be used in the Hankel functions containing the field point.

As in previous chapters, we decompose the far zone fields (region 2) into scatter contributions from the arrays and from the layered cylinder. The corresponding boundary-value problems have already be solved previously in chapters III and IV, respectively.

5.2 Scattered Field due to Arrays

Upon inverting equation (2.8) for the unknown current amplitude coefficients, one may calculate the scattered field radiated by the array ensemble by using the large argument approximation in the Hankel functions containing the field point. The eigenfunction expansion for the electric field radiated by the the $q^{th}$ subarray is given by equations (2.27) and (2.29), which will be rewritten here in slightly modified form:

$$E_z^{(q)} = \frac{I(R_q)Z_0}{\sqrt{1 + \zeta^2}} \frac{\beta_1 \rho_0}{4Dz} \sum_{n=-\infty}^{\infty} e^{-j\kappa_z \Delta z} \sum_{m=0}^{\infty} \frac{1}{2} \left\{ \xi R_{nm} \left[ P_{nm}^+(m)e^{jm\Delta \phi} + P_{nm}^-(m)e^{-jm\Delta \phi} \right] - jR_{nm}^+ \left[ P_{nm}^+(m)e^{jm\Delta \phi} - P_{nm}^-(m)e^{-jm\Delta \phi} \right] \right\}$$

92
\[ E_{(q)}^{(q)} = \frac{I(R_{q})Z_0 \beta_1 \rho_o}{\sqrt{1 + \zeta^2}} \sum_{n=-\infty}^{\infty} \sum_{m=0}^{\infty} e^{-j\kappa_z \Delta z} \left\{ \frac{1}{2} R_{nm}^{\phi \phi} \left[ P_n^+(m)e^{jm\Delta \phi} + \right] ight\} + \left(5.2\right) \]

\[ P_n^-(m)e^{-jm\Delta \phi} - j\zeta R_{nm}^{\phi \phi} \left[ P_n^+(m)e^{jm\Delta \phi} - P_n^-(m)e^{-jm\Delta \phi} \right] \]

\[ E_{(q)}^{(q)} = \frac{I(R_{q})Z_0 \beta_1 \rho_o}{\sqrt{1 + \zeta^2}} \sum_{n=-\infty}^{\infty} \sum_{m=0}^{\infty} e^{-j\kappa_z \Delta z} \left\{ \zeta R_{nm}^{\phi \phi} \left[ P_n^+(m)e^{jm\Delta \phi} + \right] \right\} \]

\[ P_n^-(m)e^{-jm\Delta \phi} - jR_{nm}^{\phi \phi} \left[ P_n^+(m)e^{jm\Delta \phi} - P_n^-(m)e^{-jm\Delta \phi} \right] \]

where \( \Delta z = z - z_q \) and \( \Delta \phi = \phi - \phi_q \), and where the reference element terminals of array \( q \) are located at \( (z_q, \phi_q) \). \( \rho_o \) is the radial displacement to the array from the \( z \)-axis (assume to be the same for all arrays). Because of the assumed form of the fields in region 2 (see section 2.1), the free-space (rather than the dielectric) characteristic impedance \( Z_o \) appears in these expressions.

Note that we have included the radial scattered field component above. In general, the radial component of the scattered electric field \textit{due to the arrays} will be present in the far zone for oblique plane wave incidence (receive mode) or for scanned arrays (transmit mode). While the radial component may also be present in the dielectric layer, it plays no explicit role in the mutual impedance and induced voltage calculation for the dipole orientations considered in this paper and so was ignored in chapters III and IV.

Note also that the \( R_{nm} \) coefficients given explicitly in section 3.3 are valid only in the dielectric region. In the free-space region, one simply replaces the region 1 scattering coefficients with the region 2 scattering coefficients in an obvious way. The Fourier coefficients for the radial field component are determined from the axial component by invoking the reduced form of Maxwell's equations (equations (3.1) and (3.2). In the free-space region 2, the \( R_{nm} \) Fourier coefficients
For $\beta \rho \gg 1$, the TM components of $R_{\text{nm}}^{\hat{z}\rightarrow\hat{z}}$ and $R_{\text{nm}}^{\hat{\rho}\rightarrow\hat{\rho}}$ as well as the TE components of $R_{\text{nm}}^{\hat{\rho}\rightarrow\hat{z}}$ and $R_{\text{nm}}^{\hat{\rho}\rightarrow\hat{\rho}}$ tend to zero. Furthermore, only the $n = 0$ Floquet mode in the field expressions given above survives at large distances from the array structure (assuming no grating lobes are propagating). This means that the cross-polarization coefficients $R_{\text{nm}}^{\hat{z}\rightarrow\hat{z}}$ and $R_{\text{nm}}^{\hat{z}\rightarrow\hat{\rho}}$ are identically zero for normal incidence. This follows because

$$u_{z_2} = S_z + \frac{n \lambda_0}{D_z} \bigg|_{n=0}$$

which vanishes if the obliquity factor $S_z$ is zero.

Using the large argument approximations

$$H^{(2)}_m(x) \sim \sqrt{\frac{2j}{\pi x}} j^m e^{-jx}$$

$$H'^{(2)}_m(x) \sim -j \sqrt{\frac{2j}{\pi x}} j^m e^{-jx}$$

we obtain the radiation field due to the arrays alone:

$$E^{(q)}_z = \frac{C I(\overline{R}_q)}{\sqrt{1 + \zeta^2}} \frac{Z_0\beta_1 \rho_0}{4D_z} e^{-j\beta \cos \theta_d \Delta z} \sum_{m=0}^{\infty} \frac{j^m}{2} \left\{ \zeta \sin^2 \theta_d E_m^z \left( P_n^+(m) e^{jm\Delta \phi} \right) \right\}$$
\[ + P_n^-(m)e^{-jm\phi} \] 

\[ - m \cdot \frac{\cos \theta_p}{\beta_1 \rho_0} \cdot F_m^\phi \left[ P_n^+(m)e^{jm\phi} - P_n^-(m)e^{-jm\phi} \right] \] 

\[ E_{\phi}^{(q)} = j \frac{CI(\bar{R}_q)}{\sqrt{1 + \zeta^2}} \cdot \frac{Z_0 \beta_1 \rho_0}{4D_z} \cdot e^{-jm\phi} \cos \theta_p \Delta z \sum_{m=0}^{\infty} \frac{j^m}{2} \left\{ E_m^\phi \left[ P_n^+(m)e^{jm\phi} - P_n^-(m)e^{-jm\phi} \right] \right\} \] 

\[ E_{\rho}^{(q)} = - \frac{CI(\bar{R}_q)}{\sqrt{1 + \zeta^2}} \cdot \frac{Z_0 \beta_1 \rho_0}{4D_z} \cdot e^{-jm\phi} \cos \theta_p \Delta z \sum_{m=0}^{\infty} \frac{j^m}{2} \left\{ \zeta \sin \theta_p \cos \theta_p \cdot E_m^\rho \right\} \] 

\[ \cdot \left[ P_n^+(m)e^{jm\phi} + P_n^-(m)e^{-jm\phi} \right] - \frac{m}{\beta_1 \rho_0} \cot \theta_p \cos \theta_p \cdot E_m^\phi \] 

\[ \cdot \left[ P_n^+(m)e^{jm\phi} - P_n^-(m)e^{-jm\phi} \right] \] 

where

\[ C = \left[ \frac{2j}{\pi \beta \rho \sin \theta_p} \right]^{\frac{1}{2}} e^{-j\beta \rho \sin \theta_p} \]

and \((\cos \theta_p, \cos \phi_p)\) are the direction cosines of the normal \(\hat{n}_i\) to the incident wave phase front, i.e., \(\cos \theta_p = S_z = \hat{n}_i \cdot \hat{z}\) and \(\cos \phi_p = \hat{n}_i \cdot \hat{\phi}\).

Each electric field vector component is comprised of two terms, contributions from the \(\hat{\phi}\) and \(\hat{\rho}\) vector components of the dipole current, respectively. Whether or not a field component is present depends on the direction and polarization of the incident plane wave as well as the presence or absence of the dielectric layer. That is, we must first determine if the terminal current \(I(\bar{R}_q)\) is non-zero.

Obviously \(I(\bar{R}_q)\) will be non-zero if a voltage is induced at the terminals of the reference element (assuming the dipoles are not open-circuited). This information is obtained directly from figure 30. For example, suppose the dipoles are transverse and the dielectric embedded antenna is illuminated by a obliquely incident TM
Figure 29: Truth table for array field scattering coefficients; field components for various combinations of scan and dielectric layer; results for axial and transverse current components.
Figure 30: Truth table for cylinder field scattering coefficients; field components for various combinations of scan and dielectric layer; TM and TE incident plane wave cases.
plane wave. Obviously only the phi-component of the electric field in the dielectric layer can manifest an e.m.f. at the terminals of a transverse element. Looking up the entry corresponding to $E^{(1)}_\phi$ for the layered scanned case, we find that $E^{(1)}_\phi$ is indeed non-zero. We conclude that array currents are excited for this case.

In order to determine if this current actually radiates a particular electric field component, say the radial component, in the far zone, we could (if in doubt) look up the appropriate entry for $E^{(2)}_\rho$ in figure 29. The table shows the asymptotic dependence to be $\rho^{-1/2}$, so that this component propagates in the far zone. Note that a few of the entries in figure 29 show a radial dependence of $\rho^{-3/2}$. See for example the phi-component radiated by scanned arrays of vertical dipoles with the dielectric layer absent. In this case, the propagating mode has an azimuthal electric field component which, while significant in the vicinity of the arrays, falls off faster than $\sqrt{1/\rho}$ and may therefore be neglected in the far zone. This situation should be compared with that of the evanescent or non-propagating Floquet modes for which $|n| > 0$. The electric field contribution due to the non-propagating Floquet modes is also significant in the vicinity of the antenna and strongly influences the reactive coupling between arrays. Unlike the propagating mode, these modes, which exhibit an exponential fall-off, never survive in the far zone and convey no real power from the arrays.

Before turning to the field scattered by the coated cylinder, the array fields for two special cases will be briefly discussed.

5.2.1 Special Case: No Dielectric, Vertical Dipoles

As was shown in section 3.3.1, the TE scattering coefficients $F^z_m$ vanish for the no-layer case. Obviously the $E^\phi_m$ and $F^\phi_m$ are zero for vertical elements. Only
the \( E_m^z \) survive with (see section 3.1.2)

\[
E_m^z = - \left[ J_m(\beta \rho_o \sin \theta_p) - \frac{J_m(\beta a \sin \theta_p)}{H_m^{(2)}(\beta \rho_o \sin \theta_p)} H_m^{(2)}(\beta a \sin \theta_p) \right] \tag{5.15}
\]

where we have used the result that the region 1 scattering coefficient \( A_m^z \) vanishes when the dielectric layer is absent. Also the plus and minus pattern functions are equal and independent of the index \( m \) for vertical elements (i.e., \( \zeta = \infty \)) with

\[
P_0 \equiv P_n^\pm(m)|_{\zeta=\infty} = \frac{2}{\beta \rho_o \sqrt{1 - S_2^2}} \frac{[\cos \beta S_2 L - \cos \beta L]}{\sin \beta L} \tag{5.16}
\]

where \( L \) is the element half-length. Using these results in equations (5.12) through (5.14), one finds that the electric field reduces to

\[
E_z^{(q)} = -\frac{C I (R_q)}{4 D_z} Z_0(\beta \rho_o) \sin^2 \theta_p P_0 e^{-j\beta(z - z_q) \cos \theta_p}
\]

\[
\cdot \sum_{m=0}^\infty \epsilon_m j^m \left[ J_m(\beta \rho_o \sin \theta_p) - \frac{J_m(\beta a \sin \theta_p)}{H_m^{(2)}(\beta \rho_o \sin \theta_p)} H_m^{(2)}(\beta a \sin \theta_p) \right]
\]

\[
\cdot \cos m(\phi - \phi_q) \tag{5.17}
\]

\[
E_\phi^{(q)} = 0 \tag{5.18}
\]

\[
E_\rho^{(q)} = \frac{C I (R_q)}{4 D_z} Z_0(\beta \rho_o) \sin \theta_p \cos \theta_p P_0 e^{-j\beta(z - z_q) \cos \theta_p}
\]

\[
\cdot \sum_{m=0}^\infty \epsilon_m j^m \left[ J_m(\beta \rho_o \sin \theta_p) - \frac{J_m(\beta a \sin \theta_p)}{H_m^{(2)}(\beta \rho_o \sin \theta_p)} H_m^{(2)}(\beta a \sin \theta_p) \right]
\]

\[
\cdot \cos m(\phi - \phi_q) \tag{5.19}
\]

where

\[
C = \left[ \frac{2j}{\pi \beta \rho \sin \theta_p} \right]^{1/2} e^{-j \beta \rho \cos \theta_p}
\]

99
5.2.2 Special Case: No Dielectric, Transverse Dipoles

For transverse elements, the scattering coefficients $E_m^\phi$ and $F_m^\phi$ are identically zero. Although the $E_m^\phi$ and $F_m^\phi$ coefficients are in general not zero even if $S_z = 0$, it is seen from equations (5.12) that the axial and radial electric field components vanish for normal incidence (due to the multiplicative $\cos \theta_p$ factors). For oblique incidence, all three vector components will be non-zero (unless $\phi = \phi'$, in which case the axial and radial components vanish).

With the dielectric layer absent then, the exterior scattering coefficients are

$$F_m^\phi = J_m(\beta \rho_0 \sin \theta_p) - \frac{J_m(\beta a \sin \theta_p)}{H_m^{(2)}(\beta a \sin \theta_p)} H_m^{(2)}(\beta \rho_0 \sin \theta_p)$$

$$E_m^\phi = J'_m(\beta \rho_0 \sin \theta_p) - \frac{J'_m(\beta a \sin \theta_p)}{H_m^{(2)}(\beta a \sin \theta_p)} H_m^{(2)}(\beta \rho_0 \sin \theta_p)$$

as seen from equations (3.41) and (3.42), where $\hat{A}_m^\phi = \hat{C}_m^\phi = 0$, and where the tilde denotes the normalization defined on page 31.

Using these results in equations (5.12)—(5.14) we obtain for the electric field:

$$E_z^{(q)} = -\frac{jCI(\overline{R}_q)}{4D_z} Z_0 \cos \theta_p P_0(m) e^{-j \beta (z - z_q) \cos \theta_p} \sum_{m=0}^{\infty} \epsilon_m m j^m$$

$$E_\phi^{(q)} = \frac{jCI(\overline{R}_q)}{4D_z} Z_0(\beta \rho_0) P_0(m) e^{-j \beta (z - z_q) \cos \theta_p} \sum_{m=0}^{\infty} \epsilon_m m j^m$$

$$E_\rho^{(q)} = \frac{jCI(\overline{R}_q)}{4D_z} \cot \theta_p \cos \theta_p P_0(m) e^{-j \beta (z - z_q) \cos \theta_p} \sum_{m=0}^{\infty} \epsilon_m m j^m$$

100
For transverse elements, the plus and minus pattern functions are equal and functions of the angular index \( m \) with

\[
P_0(m) = P_n^\pm(m)|_{\zeta=0} = \frac{2\beta\rho_0}{(\beta\rho_0)^2 - m^2} \left[ \frac{\cos \frac{L}{\rho_0} m - \cos \beta L}{\sin \beta L} \right]
\]  

(5.25)

5.3 Field Scattered from the Dielectric Coated Cylinder

The relevant boundary value problem was solved in chapter IV. In that chapter, we split the problem into TM and TE plane wave incidence. The response due to an incoming wave of arbitrary polarization may then be found by superposition. We consider the TM case first.

5.3.1 TM Incident Plane Wave

From equations (4.4) and (4.4) and the reduced set of Maxwell's equations (A.12) through (A.23), together with the large argument approximations (5.10) and (5.11), we obtain the electric field scattered by a dielectric clad cylinder illuminated by a TM plane wave:

\[
E_{cyl,TM}^z \sim C E^{TM} \sin \theta_p e^{-j\beta z} \cos \theta_p \sum_{m=0}^{\infty} \epsilon_m (-1)^m E_m^{TM} \cos m(\phi - \phi_p)
\]  

(5.26)

\[
E_{cyl,TM}^\phi \sim \frac{C E^{TM}}{\sin \theta_p} e^{-j\beta z} \cos \theta_p \sum_{m=1}^{\infty} 2m (-1)^m F_m^{TM} \sin m(\phi - \phi_p)
\]  

(5.27)

\[
E_{cyl,TM}^\rho \sim -\cot \theta_p E_{cyl,TM}^z
\]  

(5.28)

where

\[
C = \sqrt{\frac{2j}{\pi \beta \rho \sin \theta_p}} e^{-j\beta \rho \sin \theta_p}
\]

and where \( E_{TM} \) is the amplitude of the incident wave. The scattering coefficients \( E_m^{TM} \) and \( F_m^{TM} \) are given in terms of the dielectric region scattering coefficients \( A_m^{TM} \) and \( C_m^{TM} \) in equations (4.19) and (4.20).
No Dielectric Layer:

In the absence of the dielectric layer (see paragraph following equations (4.19) and (4.20)), the scattered fields reduce to the more familiar expressions:

\[ E_{cyl,TM} = -CE^{TM} \sin \theta_p e^{-j\beta_z \cos \theta_p} \sum_{m=0}^{\infty} (-1)^m \frac{\mathcal{J}_m(\beta a \sin \theta_p)}{H_m^{(2)}(\beta a \sin \theta_p)} \cos m(\phi - \phi_p) \]  
(5.29)

\[ E_{\phi,TM} = 0 \]  
(5.30)

\[ E_{\rho,TM} = -\cot \theta_p E_{cyl,TM} \]  
(5.31)

5.3.2 TE Incident Plane Wave

Starting with equations (4.28) and proceeding as in the TM case, we find

\[ E_{cyl,TE} = -CE^{TE} e^{-j\beta_z \cos \theta_p} \sum_{m=1}^{\infty} 2m(-1)^m E_{m}^{TE} \sin m(\phi - \phi_p) \]  
(5.32)

\[ E_{\phi,TE} = CE^{TE} e^{-j\beta_z \cos \theta_p} \sum_{m=0}^{\infty} \epsilon_m(-1)^m E_{m}^{TE} \cos m(\phi - \phi_p) \]  
(5.33)

\[ E_{\rho,TE} \sim -\cot \theta_p E_{cyl,TE} \]  
(5.34)

No Dielectric Layer:

In the absence of the dielectric layer

\[ E_{m}^{TE} = 0 \quad \text{and} \quad E_{m}^{TE} = -\frac{\mathcal{J}_m(\beta a \sin \theta_p)}{H_m^{(2)}(\beta a \sin \theta_p)} \]

so that

\[ E_{cyl,TE} = 0 \]  
(5.35)

\[ E_{\phi,TE} \sim -CE^{TE} e^{-j\beta_z \cos \theta_p} \sum_{\epsilon_m=0}^{\infty} (-1)^m \frac{\mathcal{J}_m(\beta a \sin \theta_p)}{H_m^{(2)}(\beta a \sin \theta_p)} \]  
(5.36)
Figure 30 summarizes the conditions under which the exterior scattering coefficients $E_{m}^{TM}$, $E_{m}^{TE}$ and $F_{m}^{TM}$ are non-zero for various combinations of scan and dielectric layer. This information is tabulated in the form of a truth table. Figure 30 also gives the radial dependence of each electric field component in the free space region. For the antenna scattering problem, the information in figures 29 and 30 is combined in figure 31; for a TM or TE incident plane wave, this table gives the polarization of the scattered field for each possible combination of scan and dielectric layer. The scattered field is broken down into contributions due to the induced cylinder currents, vertical dipoles, transverse dipoles and slanted dipoles, respectively.

This concludes the presentation of results for the scattered field from the antenna structure due to an incident plane wave of arbitrary polarization. We have obtained the total field scattered from the antenna structure, which is composed of two parts: the field re-radiated by the arrays in the presence of the cylinder as derived in section 5.2 plus the field scattered by the coated cylinder as derived in section 5.3.

\[ E_{cyl,TE}^{\phi} \sim 0 \]
### SCATTERED FIELD POLARIZATION

**TM INCIDENT WAVE**

<table>
<thead>
<tr>
<th></th>
<th>No Layer No Scan</th>
<th>No Layer Scan</th>
<th>Layer No Scan</th>
<th>Layer Scan</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Cylinder</strong></td>
<td>TM</td>
<td>TM</td>
<td>TM</td>
<td>TM + TE</td>
</tr>
<tr>
<td><strong>Array Contribution:</strong> Vertical Elements</td>
<td>TM</td>
<td>TM</td>
<td>TM</td>
<td>TM + TE</td>
</tr>
<tr>
<td><strong>Array Contribution:</strong> Transverse Elements</td>
<td>0</td>
<td>TM + TE</td>
<td>0</td>
<td>TM + TE</td>
</tr>
<tr>
<td><strong>Array Contribution:</strong> Slant Elements</td>
<td>TM + TE</td>
<td>TM + TE</td>
<td>TM + TE</td>
<td>TM + TE</td>
</tr>
</tbody>
</table>

**TE INCIDENT WAVE**

<table>
<thead>
<tr>
<th></th>
<th>No Layer No Scan</th>
<th>No Layer Scan</th>
<th>Layer No Scan</th>
<th>Layer Scan</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Cylinder</strong></td>
<td>TE</td>
<td>TE</td>
<td>TE</td>
<td>TM + TE</td>
</tr>
<tr>
<td><strong>Array Contribution:</strong> Vertical Elements</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>TM + TE</td>
</tr>
<tr>
<td><strong>Array Contribution:</strong> Transverse Elements</td>
<td>TE</td>
<td>TM + TE</td>
<td>TE</td>
<td>TM + TE</td>
</tr>
<tr>
<td><strong>Array Contribution:</strong> Slant Elements</td>
<td>TM + TE</td>
<td>TM + TE</td>
<td>TM + TE</td>
<td>TM + TE</td>
</tr>
</tbody>
</table>

Figure 31: Polarization of field scattered from antenna structure due to an incident plane wave of arbitrary polarization; for various combinations of scan and dielectric layer.
CHAPTER VI
A HYBRID GTD APPROACH

6.1 Introduction

The mathematical analysis of the cylindrical array configuration described in the previous chapters involved an eigenfunction expansion of the electromagnetic fields in the dielectric and free-space regions. The method of moments was employed to obtain the unknown array current distributions. The eigenfunction field representation is well suited for applications in which the antenna diameter is small. Since the number of significant series terms is roughly proportional to the antenna diameter (in wavelengths), the eigenfunction expansion converges slowly for diameters much larger than a few wavelengths. This is a significant practical limitation. Further, the analysis for the circular cylinder can not be easily modified to accommodate non-circular geometries. An example of interest is the elliptical cylinder. The eigenfunction solution for the elliptical cylinder problem is difficult – especially if dielectric layers are added – due to the complicated nature of the elliptic cylinder functions.

In this chapter we introduce an alternative hybrid GTD approach. The moment method construction developed in chapter II is employed; the GTD is used to calculate the (subsectional) array fields formally defined in equation (2.11), which in turn are needed to compute the reaction mutuals defined in equation (2.10). In this chapter, we limit consideration to the case of unscanned periodic linear arrays
of vertical dipoles radiating in the presence of an infinite uncoated PEC convex cylinder.

Some advantages of the hybrid GTD approach are:

- Provide corroborative data for the eigenfunction-MM analysis.
- Applicable to convex cylinders of non-circular cross-section.
- Large cylinders can be treated.
- Our computer simulations indicate a significant impedance matrix fill time reduction over the eigenfunction field expansion approach.

Preliminary computer simulations involving PEC circular cylinders indicate that the hybrid GTD approach is quite accurate, provided that the cylinder and the array-to-cylinder spacing is not too small electrically. On the debit side, we recognize that for array-to-cylinder spacings less than \( \lambda/4 \) and cylinder diameters equally small, some loss of accuracy is expected; of course this is precisely the regime for which the eigenfunction field expansion is well-suited.

6.2 Overview of Method

The purpose of this section is to provide a heuristic description of the hybrid GTD approach. A more rigorous discussion follows in the next section. As we are concerned with the calculation of the mutuals between subsectional arrays (i.e., the elements of the moment method impedance matrix), it will be understood that the term "arrays" refers to linear periodic subsectional arrays and "array mutuals" or "mutual impedance" refers to reaction mutuals.

First we note that the \( z \)-component of the electric field radiated by a periodic array of vertical dipoles plays the role of a potential function, which we will denote
Figure 32: The rays and regions associated with scattering by a smooth convex cylinder.
by \( u \), from which the remaining field components (both electric and magnetic) may be derived. For unscanned arrays, \( u \) is essentially a 2-D scalar field. Suppose the source array is located at \( Q' \) as illustrated in figure 32. Then we may express \( u \) as:

\[
u \sim \begin{cases} 
  u^i + u^r + u^d, & \text{field point in lit region} \\
  u^d, & \text{field point in shadow region}
\end{cases}
\]  

(6.1)

where \( u^i, u^r \) and \( u^d \) are the incident, reflected and diffracted fields, respectively, observed at the field point. Ordinarily the diffracted field component in the lit sector is negligible. We further decompose \( u^i \) into propagating and non-propagating wave contributions:

\[
u^i = u^i_p + u^i_N
\]  

(6.2)

where the subscripts \( p \) and \( N \) denote propagating and non-propagating wave components, respectively. (Normally \( u^i_p \) will consist of a single propagating wave if \( D_z \), the vertical element spacing, is reasonably small). Note that \( u^i_N \), which includes the evanescent wave modes, cannot be interpreted as a ray-optical field.

If the source is not appreciably closer than a quarter-wavelength to the cylinder, then \( u^r \) is approximately proportional to \( u^i_p(Q_R) \) but is independent of the rapidly decaying evanescent component \( u^i_N \). Similarly the diffracted ray field \( u^d \) is proportional to \( u^i_p(Q_a) \) and independent of \( u^i_N \). (Here \( Q_R \) and \( Q_a \) are the cylinder reflection and surface ray attachment points, respectively—see figure 32). Summarizing, the scattered field is independent of the non-propagating waves generated by the source array, provided that the array-to-cylinder spacing is not too small. Therefore at high frequencies the vehicle for mutual coupling between arrays via interactions with the cylinder consists of ordinary space and surface ray fields. This implies that for the purpose of calculating the array mutual coupling contributions due to reflection and diffraction from the cylinder, the (discrete) source array can
be replaced by an equivalent line source which radiates the same propagating wave excitation, as shall be shown presently.

The incident field contribution to the array mutual impedance comes into play only when the test array is in the lit sector. Included in the incident contribution are the non-propagating modal contributions (which are significant when computing the array self-impedance) as well as the propagating mode contribution. The incident coupling component is the same as the direct coupling component (as defined earlier in chapter III) in the lit sector and is equal to zero in the shadow region. The incident contribution is independent of the cylinder and can be computed exactly (and easily) as a Floquet sum of cylinder waves. The evanescent wavemodes strongly affect the array self-reactance. While the scatter coupling (includes coupling due to reflection and diffraction) is generally weaker than the "incident" coupling interaction in the lit sector, it is still significant and the eigenfunction series for it is considerably more difficult to compute, especially for large cylinders.

We can also evaluate the direct coupling (evanescent as well as propagating wave contributions) asymptotically if we wish, provided the test array is sufficiently distant from the source array (but still in the lit region) so that the large argument form of \( H_0^{(2)}(\beta |\vec{r}_1 - \vec{r}_2|) \) can be used, where \( |\vec{r}_1 - \vec{r}_2| \) is the separation between the source and test arrays. This minimum distance ranges from roughly \( \lambda/4 \) to \( 1\lambda \), depending on the accuracy desired.

In summary, our approach is to evaluate the array coupling due to scattering from the cylinder using diffraction theory. In contrast, the direct line-of-sight array interactions may be evaluated easily with or without asymptotic approximation.
6.3 Mathematical Description

We consider first the coupling between two arrays of vertical elements in the presence of an infinite uncoated PEC circular cylinder of radius \(a\). Expressed as a spectral sum, the mutual impedance between array 1 (the "source" array) and array 2 (the "test" array) is given by (see chapter III):

\[
Z_{2,1} = \frac{(\beta \rho_o)^2 Z_0}{4 \beta D_z} \sum_{n=-\infty}^{\infty} e^{-j \beta u_z \Delta z} P_n^2 \rho_o^2 \sum_{m=0}^{\infty} \epsilon_m \left\{ H_m^{(2)}(\beta u \rho_o) \right\} \cos m \Delta \phi \tag{6.3}
\]

where \(u_z = S_z - \frac{\Delta z}{D_z}\) and \(u_\rho = \sqrt{1 - u_z^2}\) are the normalized axial and radial wavenumbers respectively. \(\Delta z = z_2 - z_1, \Delta \phi = \phi_2 - \phi_1,\) the axial and angular separation between the reference terminals of arrays 1 and 2, where array 1 is understood to be the driven array. \(\rho_o\) is the displacement from the z-axis to the dipole terminals (which for simplicity we will assume is the same for both arrays). \(P_n\) is the vertical element pattern factor (see chapter II) given by

\[
P_n = \frac{2}{\beta \rho_o (1 - u_z^2)} \frac{[\cos \beta u_z \Delta L - \cos \beta \Delta L]}{\sin \beta \Delta L} \tag{6.4}
\]

where \(2 \Delta L\) is the subsectional element length and where by assumption the element current distribution is piecewise sinusoid. Using the addition theorem, we may write equation (6.3) as

\[
Z_{2,1} = Z_{2,1}^{dir} + Z_{2,1}^{scat} \tag{6.5}
\]

where

\[
Z_{2,1}^{dir} = \frac{(\beta \rho_o)^2 Z_0}{4 \beta D_z} \sum_{n=-\infty}^{\infty} e^{-j \beta u_z \Delta z} P_n^2 \rho_o^2 H_0^{(2)}(\beta u \rho_o |\vec{\rho}_1 - \vec{\rho}_2|) \tag{6.6}
\]

where

\[
\vec{\rho}_1 = x \cos \phi_1 + y \sin \phi_1
\]

110
and

\[
\bar{p}_2 = \hat{x} \cos \phi_2 + \hat{y} \sin \phi_2
\]

and

\[
Z_{2,1}^{\text{scatter}} = - \frac{(\beta \rho_o)^2 Z_0}{4 \beta D_z} \sum_{n=-\infty}^{\infty} e^{-j \beta u_z \Delta z} p_n^2 u^2 \rho \cdot \sum_{m=0}^{\infty} \epsilon_m \frac{J_m(\beta u \rho a)}{H_m^{(2)}(\beta u \rho a)} \left( H_m^{(2)}(\beta u \rho o) \right)^2 \cos m \Delta \phi.
\]  

(6.7)

\(Z_{2,1}^{\text{dir}}\) represents the direct coupling contribution, i.e., the coupling between two arrays in free space. \(Z_{2,1}^{\text{scatter}}\) is the the scatter coupling contribution, that is, the array coupling due to induced currents on the cylinder.

As seen in figures 33 through 42, \(Z_{2,1}^{\text{dir}}\) dominates in an angular sector corresponding to about a half-wavelength neighborhood of the source array. The lower two plots in figures 33–42, show the direct and scatter coupling components as a function of \(\Delta \phi\), using the exact eigenfunction method. Outside this sector \(Z_{2,1}^{\text{dir}}\) and \(Z_{2,1}^{\text{scatter}}\) are comparable, but the overall coupling in this region is relatively small (though not negligible). If the array-to-cylinder spacing is \(\lambda/4\) or more, the non-propagating (\(|n| > 0\)) wave contributions to \(Z_{2,1}^{\text{scatter}}\) are negligibly small for all angular locations of the test array. Subsequent developments hinge on this important result.

6.4 The Ray-Optic Approximation

Let us summarize the key results of the previous section:

1. For arrays no closer than about a quarter-wavelength to the cylinder, the non-propagating Floquet wave modes have a negligible impact on the scatter coupling component of the array mutual impedance. That is, only the \(n = 0\) propagating mode contributes to \(Z_{2,1}^{\text{scatter}}\).
2. The non-propagating Floquet wave modes strongly influence the direct component of the mutual reactance. However, the direct coupling component is independent of the cylinder and may be evaluated easily without approximation.

Result 1 implies that for the purpose of calculating $Z_{2,1}^{\text{cat}}$ we may replace the source array with an equivalent electric line source of amplitude $I_p^e$. The superscript "p" reminds us that the equivalence holds only with respect to the coupling contributions due to the $n = 0$ propagating mode. That is, $I_p^e$ radiates the field $u_p^e$, as defined on page 108.

Consequently, the array mutual impedance $Z_{2,1}$ as defined in equation (6.5) may be re-expressed according to the following decomposition:

$$Z_{2,1} = \begin{cases} Z_{2,1}^i + Z_{2,1}^r + Z_{2,1}^d, & \text{test array in lit region} \\ Z_{2,1}^d, & \text{test array in shadow region} \end{cases}$$

(6.8)

where we have adapted the superscript notation of the previous section: $Z_{2,1}^i$ is the total direct coupling component, the mutual impedance between two arrays in free-space. $Z_{2,1}^i$ equals $Z_{2,1}^{\text{dir}}$ (see equation (6.6)) when the test array is in the lit sector and zero elsewhere. $Z_{2,1}^r$ is the scatter coupling component resulting from interactions through the reflected ray path. In computing $Z_{2,1}^r$ only the propagating wave mode emitted by the source array is retained, that is, the field radiated by the equivalent line source $I_p^e$. Similarly, $Z_{2,1}^d$ is the scatter coupling component due to surface ray(s) excited by $I_p^e$. Explicit asymptotic formulas for the various components of $Z_{2,1}$ will be given in the next section.

In general, the source and test (sub)array reference elements will be axially offset with respect to each other. A question arises as to how one should account for the effect of this relative offset. Upon replacing the source array by an equivalent line source, one is at a loss to assign a reference phase origin to the equivalent
source. This is not a real problem since the array axial offset does not affect $Z_{2,1}^\text{cat}$ if the arrays are unscanned. To see this we observe that the axial offset only comes into play in the exponential factor $e^{-j\beta u_z (z_2-z_1)}$, where $u_z$, the normalized axial wavenumber, is given by $u_z = S_z - \frac{n\lambda}{D_z}$. If, as assumed, the array is unscanned, then $S_z = 0$ and so $u_z = 0$ for the $n = 0$ propagating mode. Therefore, the propagating wave mode is indistinguishable from the propagating wave generated by an ordinary unphased electric line source. The amplitude of the equivalent line source is determined as follows: The $z$-component of the $n = 0$ propagating mode of the free-space electric field radiated by the source array (in the absence of the cylinder) and evaluated at a point $P$ is

$$u_p^i = -\frac{\beta P_0}{4D_z} I(0) Z_0 P_0 H_0^2(\beta|\bar{\rho}|)$$  \hspace{1cm} (6.9)

where $I(0)$ is the source array reference terminal current. Observe that since the source array is unscanned by assumption, the radial wavenumber is simply $\beta$.

On the other hand, the field radiated by an electric line source of amplitude $I_P^0$ in free-space is given by [10]

$$u_p^i = -\frac{\beta Z_0}{4} I_P^0 R(2) (\beta|\bar{\rho}|)$$  \hspace{1cm} (6.10)

Relations (6.9) and (6.10) imply that

$$I_P^0 = \frac{\rho_0 P_0 I(0)}{D_z}$$  \hspace{1cm} (6.11)

where $P_0$ is the zeroth order vertical element pattern function given by

$$P_0|_{S_z=0} = \frac{2}{\beta\rho_0 \sqrt{1-S_z^2}} \left[ \frac{\cos \beta S_z \Delta L - \cos \beta \Delta L}{\sin \beta \Delta L} \right]_{S_z=0} = -\frac{2}{\beta\rho_0} \cot \beta \Delta L$$  \hspace{1cm} (6.12)

(Note that $I_P^0$ is independent of radial displacement $\rho_0$ since the element pattern function $P_0$, as we have defined it, is inversely proportional to $\rho_0$). Note also that since the source array is unphased by assumption, the surface ray geodesic is simply a circular arc about the cylinder.
GTD Analysis

The equivalent line source concept is useful because it provides an entry point into established GTD analyses [12][13] for convex surface diffraction, the key results of which are summarized as follows:

We consider an electric line source located at $Q'$ radiate in the presence of an infinite PEC cylinder. Let $u$, a scalar field, represent $E_z$, the total electric field excited by the equivalent line source radiating in the presence of the cylinder. $u$ may be decomposed in the lit and shadow sectors as shown in equation (6.1) which we rewrite here for convenience

$$u \sim \begin{cases} u^i + u^r + u^d, & \text{field point in lit region} \\ u^d, & \text{field point in shadow region} \end{cases} \quad (6.13)$$

where $u^i$ is the incident ray field which arrives directly from the source along the incident GO path (see figure 32, $u^r$ is the field which arrives along the reflected ray path, and $u^d$ is the field which arrives via the surface diffracted ray path.

As explained on page 108, the reflected and diffracted ray fields are proportional to the propagating component $u_p^i$ of the total incident field $u^i$ (and unaffected by the non-propagating components) provided that the source and test arrays are not too close to the cylinder. For a convex PEC cylinder, it may be shown [12] that the reflected field at the point $P_L$ (in the lit region) is given by

$$u^r(P_L) = u_p^i(Q_R) \left[ -\sqrt{-\frac{1}{\gamma^2}} e^{-j(\xi^L)} \frac{e^{-j \pi/4}}{2\xi^L \sqrt{\pi}} [1 - F(x^L)] \right. \\
\left. + \hat{P}_s(\xi^L) \right] \cdot \sqrt{\rho^r + s^r} e^{-j \beta r^r}. \quad (6.14)$$

The diffracted field at the point $P_s$ (in the shadow region) is

$$u^d(P_s) = u_p^i(Q_1) \left[ -\sqrt{m(Q_1)m(Q_2)} e^{-j \beta t} \sqrt{\frac{2}{\beta}} \right. \\
\left. \cdot \left\{ \frac{e^{-j \pi/4}}{2\xi^d \sqrt{\pi}} [1 - F(x^d)] + \hat{P}_s(\xi^d) \right\} \right] e^{-j \beta s^d} \sqrt{s^d}. \quad (6.15)$$
The various transition functions in equations (6.14) and (6.15) are defined as follows:

\[ F(x) = 2j \sqrt{\pi} e^{jxz} \int_{\sqrt{x}}^{\infty} dr e^{-jr^2} \]

\[ \hat{P}_s(y) = p^*(y)e^{-j\pi/4} - \frac{e^{-j\pi/4}}{2y\sqrt{\pi}} \]

where

\[ p^*(y) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} d\tau \frac{V(\tau)}{W_2(\tau)} e^{-j\sqrt{\tau}}. \]

\[ V(\tau) \] and \[ W_2(\tau) \] are the Fock type Airy functions given by

\[ 2jV(\tau) = W_1(\tau) - W_2(\tau) \quad \text{and} \quad W_{1/2}(\tau) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dt e^{j2\pi/3} e^{rt-t^3/3} \]

where \( \epsilon \) is a positive number.

We will also need the following parameters, which simplify for the circular cylinder case as follows:

\[ m(Q) = \left[ \frac{\beta a \sqrt{2}}{2} \right]^{1/3} \]

\[ \xi_L = -2m \cos \theta^i, \quad \xi^d = m(\phi_b - \phi_a) \]

\[ x_L = 2\beta L^L \cos^2 \theta^i, \quad x^d = \frac{\beta L^d(\xi^d)^2}{2m^2} \]

\[ L^L = \frac{s^r s'}{s^r + s'}, \quad L^d = \frac{s^d s_o}{s^d + s_o} \]

where \( \cos \theta^i \) is the cosine of the angle between the cylinder normal and reflected ray and \( \phi_a \) and \( \phi_b \) are the angular positions of the attachment and launching points for the surface diffracted rays as shown in figure 32. are tangent to the cylinder.

The path lengths \( s', s^r, s_o \) and \( s_d \) are also shown figure 32.

The scatter component of the array mutual impedance, \( Z_{2,1}^{\text{cat}} \) is obtained by integrating the scattered (reflected-diffracted) components of \( u \) over the reference
element of the test array with the PWS current distribution as a weighting function, and then multiplying by -1. Since \( u^d \) and \( u^r \) do not vary with \( z \) in the unscanned case, the integral is simply the Fourier transform of the PWS current distribution evaluated at the origin in the transform domain. The result is that

\[
Z^{\text{scat}}_{2,1} \sim \begin{cases} 
-u_p(Q_R)\rho_0P_0\mathcal{R}_s\sqrt{\frac{\beta}{\beta + s}}e^{-j\beta s} & \text{array 2 in lit region} \\
-u_p(Q_a)\rho_0P_0\mathcal{T}_s\sqrt{\frac{\beta}{s}}e^{-j\beta s} & \text{array 2 in shadow region}
\end{cases}
\]  

(6.16)

where \( Q_R \) and \( Q_a \) are reflection and surface ray attachment points, respectively.

The integration over the test reference element is manifested in the \( \rho_0P_0 \) factor, where \( P_0 \) is the zero\(^{th} \) order vertical element pattern function given by equation (6.12)

Note that \( Z^{\text{scat}}_{2,1} \) is actually proportional to \( P^2_0 \) since the equivalent line source \( I_p^s \) and therefore \( u_p^i \) includes a \( P_0 \) factor (a consequence of the integration over the source array reference element).

The surface reflection and diffraction functions \( \mathcal{R}_s \) and \( \mathcal{T}_s \) are given by

\[
\mathcal{R}_s = -\left[ -\sqrt{\frac{4}{\xi L}}e^{-j(\xi L)^3/12}\left\{ \frac{e^{-j\pi/4}}{2\xi L\sqrt{\pi}}[1 - F(X^L)] + \hat{P}_s(\xi^L) \right\} \right] 
\]  

(6.17)

and

\[
\mathcal{T}_s = -\left[ \sqrt{m(Q_1)m(Q_2)}e^{j\beta t}\sqrt{\frac{2}{\beta}} \cdot \left\{ \frac{e^{-j\pi/4}}{2\xi \sqrt{\pi}}[1 - F(x^d)] + \hat{P}_s(\xi) \right\} \right] 
\]  

(6.18)

The total array mutual impedance is then

\[
Z_{2,1} = Z^i_{2,1} + Z^{\text{scat}}_{2,1}
\]  

(6.19)

where

\[
Z^i_{2,1} = \begin{cases} 
Z^{\text{dir}}_{2,1} & \text{test array in lit region} \\
0 & \text{test array in shadow region}
\end{cases}
\]  

(6.20)

116
and where $Z_{2,1}^{dir}$ is given in equation (6.6). The array self-impedance is computed as the average array mutual impedance as the test array is moved around a circular arc about the source array at a wire radius (see Appendices C and D).

It is clear that this approach may be extended to non-circular cylinder geometries; the analysis remains the same except that the local curvature of the surface varies and so one must use the more general parameter forms which depend on the local radius of curvature on the cylinder. See reference [12] for details.

### 6.5 Numerical Results

Figures 33 through 42 show the complex mutual array impedance as a function of angular separation from 0 degrees (self-term) to 180 degrees (test array in deep shadow zone) in 10 degree steps for various combinations of circular cylinder radius and array radial displacements. The real and imaginary parts of $Z_{2,1}^{dir}$ and $Z_{2,1}^{scat}$ are plotted as a function of angular separation using the exact eigenfunction field representation method (see lower two plots of each figure). The propagating ($n = 0$ term only) contribution is compared with the complete series solution (all terms included) in each case study.

In all cases, the element spacing $D_z = 0.55\lambda$, element length is $0.5\lambda$ and the wire radius is $0.01\lambda$. The element current was approximated by one PWS mode. The cylinder radius and the radial distance to the arrays is varied. A study of these results reveals that the accuracy of the GTD method is more sensitive to the spacing between the arrays and the cylinder than to the cylinder radius. For example, in figure 35 the cylinder radius is only $0.5\lambda$ while the test array is only a quarter-wavelength from the cylinder surface, yet the GTD results are still accurate.

For very small cylinders (radius is less than a quarter wavelength), the hybrid
GTD solution is self-correcting in the sense that while the scatter coupling error may be large on a percentage basis, this coupling contribution is small in magnitude in comparison with the strong direct coupling component in the deep lit region.
Figure 33: Mutual coupling between two arrays of vertical dipoles radiating in the presence of an uncoated PEC cylinder as a function angular displacement; comparison of eigenfunction-MM and hybrid GTD solutions; cyl radius $a = .1\lambda$, array displacement $\rho_o = .75\lambda$.
Figure 34: Mutual coupling between two arrays of vertical dipoles radiating in the presence of an uncoated PEC cylinder as a function angular displacement; comparison of eigenfunction-MM and hybrid GTD solutions; cyl radius $a = .25\lambda$, array displacement $\rho_a = .75\lambda$
Figure 35: Mutual coupling between two arrays of vertical dipoles radiating in the presence of an uncoated PEC cylinder as a function angular displacement; comparison of eigenfunction-MM and hybrid GTD solutions; cyl radius $a = 0.5\lambda$, array displacement $\rho_o = 0.75\lambda$. 

<table>
<thead>
<tr>
<th>FREQ</th>
<th>SCAN</th>
<th>ELEMENT PITCH</th>
<th>CURRENT MODES</th>
<th>ARRAYS</th>
<th>Dz</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>0.01</td>
<td>90 deg.</td>
<td>3</td>
<td>2</td>
<td>.55</td>
</tr>
<tr>
<td>$\varepsilon_r$</td>
<td>WIRE RADIUS</td>
<td>CYL RADIUS</td>
<td>RADIUS TO ARRAYS</td>
<td>LAYER THICKNESS</td>
<td>DIPOLE LENGTH</td>
</tr>
<tr>
<td>1.0</td>
<td>.01</td>
<td>.50</td>
<td>.75</td>
<td>--</td>
<td>0.5</td>
</tr>
</tbody>
</table>
Figure 36: Mutual coupling between two arrays of vertical dipoles radiating in the presence of an uncoated PEC cylinder as a function angular displacement; comparison of eigenfunction-MM and hybrid GTD solutions; cyl radius $a = 0.65\lambda$, array displacement $\rho_x = 0.75\lambda$. 

<table>
<thead>
<tr>
<th>FREQ</th>
<th>SCAN</th>
<th>ELEMENT PITCH</th>
<th>CURRENT MODES</th>
<th>ARRAYS</th>
<th>Dz</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>0.01</td>
<td>90 deg.</td>
<td>3</td>
<td>2</td>
<td>0.55</td>
</tr>
<tr>
<td>$\varepsilon_r$</td>
<td>WIRE RADIUS</td>
<td>CYL RADIUS</td>
<td>RADIUS TO ARRAYS</td>
<td>LAYER THICKNESS</td>
<td>DIPOLE LENGTH</td>
</tr>
<tr>
<td>1.0</td>
<td>.01</td>
<td>.65</td>
<td>.75</td>
<td>---</td>
<td>0.5</td>
</tr>
</tbody>
</table>
Figure 37: Mutual coupling between two arrays of vertical dipoles radiating in the presence of an uncoated PEC cylinder as a function angular displacement; comparison of eigenfunction-MM and hybrid GTD solutions; cyl radius \( a = 0.1 \lambda \), array displacement \( \rho_0 = 1.25 \lambda \).
Figure 38: Mutual coupling between two arrays of vertical dipoles radiating in the presence of an uncoated PEC cylinder as a function angular displacement; comparison of eigenfunction-MM and hybrid GTD solutions; cyl radius $a = .25\lambda$, array displacement $\rho_o = 1.25\lambda$
Figure 39: Mutual coupling between two arrays of vertical dipoles radiating in the presence of an uncoated PEC cylinder as a function angular displacement; comparison of eigenfunction-MM and hybrid GTD solutions; cyl radius $a = .5\lambda$, array displacement $\rho_o = 1.25\lambda$.
Figure 40: Mutual coupling between two arrays of vertical dipoles radiating in the presence of an uncoated PEC cylinder as a function angular displacement; comparison of eigenfunction-MM and hybrid GTD solutions; cyl radius $a = .75 \lambda$, array displacement $\rho_o = 1.25 \lambda$.
Figure 41: Mutual coupling between two arrays of vertical dipoles radiating in the presence of an uncoated PEC cylinder as a function angular displacement; comparison of eigenfunction-MM and hybrid GTD solutions; cyl radius $a = 1.0\lambda$, array displacement $\rho_o = 1.25\lambda$
Figure 42: Mutual coupling between two arrays of vertical dipoles radiating in the presence of an uncoated PEC cylinder as a function angular displacement; comparison of eigenfunction-MM and hybrid GTD solutions; cyl radius $a = 1.15\lambda$, array displacement $\rho_0 = 1.25\lambda$.
APPENDIX A

FIELDS RADIATED BY PERIODIC LINEAR LINEAR ARRAYS
OF HERTZIAN ELEMENTS IN THE PRESENCE OF AN
INFINITE PEC CYLINDER

A.1 Preliminary Remarks

In this appendix, we derive the fields due to an infinite periodic linear array
of short arbitrarily directed dipoles radiating in the presence of an infinite PEC
cylinder in an otherwise homogeneous lossless non-magnetic dielectric medium.
As a special subcase, we may let a, the cylinder radius go to zero, and so obtain
the the “direct” field solutions referred to in chapter II. The results for the fields
with the cylinder present (but no layer) derived in this appendix can be used to
check the results obtained in chapter II (using the mode-matching method) for the
special case in which the dielectric relative permittivity is unity (layer absent).

The direction \( \hat{p} \) of the array elements is completely arbitrary but we require
that all elements in the array are identically oriented. Since we can decompose
an arbitrary current moment \( I_0 d\hat{p} \) into radial, azimuthal and axial vector com­
nents, we treat these as three separate elementary element orientations. The fields
radiated by an linear array of radially directed Hertzian elements is included in
this analysis for completeness.

The radiated fields may be obtained through the use of a free-space vector
potential. While well-suited for the special case of axial dipoles, this method is
cumbersome to apply in the case of arbitrarily oriented dipoles. For example, having obtained the periodic magnetic vector potential \( \vec{A} \), one must compute the electric field according to

\[
\vec{E} = \frac{1}{j\omega \epsilon} \nabla \times \nabla \times \vec{A},
\]

a very tedious calculation if \( \vec{A} \) is not an axial vector.

Instead, we make use of an alternative Green's function formulation using an approach introduced by Wait [14] in his study of cylindrical wedge scattering. According to this approach, the axial electric and magnetic field components are obtained through an application of the Stratton-Chu vector wave formulation [3]. The problem of determining the radiated vector fields (both electric and magnetic) is essentially reduced to two associated scalar boundary value problems. This method has the virtue that we focus our attention entirely on the axial field components. The remaining field components are related in terms of a TM/TE field decomposition, and may be obtained from the axial components by inspection as demonstrated in sections A.6.1, A.6.2 and A.6.3. Thus, having invested an initial effort in the derivation of the associated scalar periodic Green's functions, we find that the fields corresponding to any dipole orientation are obtained with a minimal amount of differentiation and algebraic manipulation.

It is also possible to obtain the fields through a modification of the electric dyadic Green's function [6] associated with a single current element radiating in the presence of a PEC cylinder. However, in order to obtain the H-field we must either compute the curl

\[
\vec{H} = -\frac{1}{j\omega \mu} \nabla \times \vec{E}
\]

or develop a dyadic for the H-field. The periodic electric Green's dyadic for the PEC cylinder configuration is presented in section A.8.
A.2 Stratton-Chu Formulation

It will be useful at the outset to classify two related boundary value problems, designated as configurations A and B. In configuration A, a single dipole radiates in the presence of a perfectly conducting infinite cylinder. In configuration B, the dipole of set-up A is replaced by a periodic linear array, each element of which has the same dipole orientation as A.

According to the Stratton-Chu vector wave formulation, \( \bar{E}(\bar{R}) \) and \( \bar{H}(\bar{R}) \) within a homogeneous isotropic region V bounded by a surface S with outward pointing normal \( \hat{n} \) are given by

\[
\bar{E}(\bar{R}) = \int_V [\mathbf{J}(\bar{R}') \cdot \mathbf{G}_1(\bar{R}', \bar{R}) + \frac{1}{\varepsilon} \rho(\bar{R}') \nabla' \mathbf{G}_1(\bar{R}', \bar{R}) + \nabla' \mathbf{G}_1(\bar{R}', \bar{R})] dV' - \int_S [\mathbf{n} \times \mathbf{E}(\bar{R}')] \cdot \mathbf{G}_1(\bar{R}', \bar{R}) dS' + \int_S [\mathbf{n} \times \mathbf{E}(\bar{R}')] \cdot \nabla' \mathbf{G}_1(\bar{R}', \bar{R}) dS' \tag{A.1}
\]

\[
\bar{H}(\bar{R}) = \int_V [\mathbf{J}_m(\bar{R}') \times \nabla' \mathbf{G}_2(\bar{R}', \bar{R}) + \frac{1}{\mu} \rho_M(\bar{R}') \nabla' \mathbf{G}_2(\bar{R}', \bar{R}) + \nabla' \mathbf{G}_2(\bar{R}', \bar{R})] dV' - \int_S [\mathbf{n} \times \mathbf{H}(\bar{R}')] \times \nabla' \mathbf{G}_2(\bar{R}', \bar{R}) dS' + \int_S [\mathbf{n} \times \mathbf{H}(\bar{R}')] \times \nabla' \mathbf{G}_2(\bar{R}', \bar{R}) dS' \tag{A.2}
\]

where the subscript \( M \) denotes magnetic source density and \( \varepsilon = \varepsilon_r \varepsilon_0 \) is the permittivity of the dielectric medium.

The scalar Green's functions \( \mathbf{G}_1 \) and \( \mathbf{G}_2 \) each satisfy the inhomogeneous Helmholtz equation

\[
(\nabla'^2 + \beta^2) \mathbf{G}(\bar{R}', \bar{R}) = -\delta(\bar{R}' - \bar{R}) \tag{A.3}
\]

We assume that the functional dependence of \( \mathbf{G}_1 \) and \( \mathbf{G}_2 \) in the axial coordinate is of the form \( e^{-j\beta z} \). We further require that the \( \mathbf{G}_1 \) and \( \mathbf{G}_2 \) satisfy boundary conditions such that the \( \hat{z} \) components of the surface integrals in equations (A.1) and
(A.2) vanish. In addition, $G_1$ and $G_2$ must satisfy the radiation condition. Under these conditions, for electric filament sources only, the volume integrals degenerate to line integrals and the equations for the axial field components $(E_z, H_z)$ reduce to

\begin{align}
E_z(R) &= \int_{wire} \left[ -j\omega \mu I_z(R')G_1(R', \bar{R}) + \frac{1}{\epsilon} \rho(\bar{R}') \frac{\partial}{\partial z} G_1(R', \bar{R}) \right] dl'
\tag{A.4}
\end{align}

\begin{align}
H_z(\bar{R}) &= \int_{wire} \hat{z} \cdot [\hat{I}(\bar{R}') \times \nabla'] G_2(\bar{R}', \bar{R})] dl'
\tag{A.5}
\end{align}

For TM (TE) waves, $E_z$ ($H_z$) plays the role of a potential function from which the remaining field components may be obtained. Assuming an axial $e^{-j\beta_z z}$ dependence, it may be shown that Maxwell's equations reduce as follows:

\begin{align}
E_{\rho}^{TM} &= \frac{1}{(\beta \eta)^2} \frac{\partial^2 E_{\rho}^{TM}}{\partial \rho^2} \quad & E_{\phi}^{TM} = 0 \\
E_{\phi}^{TM} &= \frac{1}{(\beta \eta)^2} \frac{\partial^2 E_{\phi}^{TM}}{\partial \phi^2} \quad & E_{\phi}^{TE} = -\frac{1}{j\omega \epsilon \eta^2} \frac{\partial H_{\phi}^{TE}}{\partial \phi}
\tag{A.6}
\end{align}

\begin{align}
H_{\rho}^{TM} &= 0 \\
H_{\rho}^{TE} &= -\frac{1}{j\omega \mu \eta^2} \frac{\partial E_{\rho}^{TM}}{\partial \phi} \\
H_{\phi}^{TM} &= \frac{1}{j\omega \mu \eta^2} \frac{\partial E_{\phi}^{TM}}{\partial \phi} \\
H_{\phi}^{TE} &= \frac{1}{(\beta \eta)^2} \frac{\partial^2 H_{\phi}^{TE}}{\partial \phi^2}
\tag{A.7}
\end{align}

where $\eta = \sqrt{1 - \beta^2}$ is the normalized radial wavenumber and where $E_{z}^{TM}$ and $H_{z}^{TE}$ are given by equations (A.4) and (A.5). The TM and TE components are added to give the composite field.

We can express equations (A.6) and (A.7) in a simpler more explicit form as follows: Let $E_{z}^{TM}(\bar{r}; h, m)$ and $H_{z}^{TE}(\bar{r}; h, m)$ be defined as the $m^{th}$ term (in the Fourier series expansion in the azimuthal coordinate) of the axial electric field and magnetic intensity, respectively, evaluated at the spectral frequency $h$. The axial component of an arbitrary field with $z$-dependence $e^{-jhz}$ may be expanded in a
spectral summation:

\[
E_z^{TM}(\bar{r}) = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} E_z^{TM}(\bar{r}; h_i, m_j) \quad (A.8)
\]

\[
H_z^{TM}(\bar{r}) = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} H_z^{TE}(\bar{r}; h_i, m_j) \quad (A.9)
\]

If the field is periodic in the azimuthal coordinate, as is the case for the set-up considered in this dissertation, then \(\{m_j\}\) is the set of integers, i.e., \(m_j = m\). Note that for a finite set of radiating elements, the sum over the \(h_i\) is replaced by an integral (over a continuous spectrum). For a line source the sum on \(i\) degenerates to a single term. Otherwise, for a periodic linear array, the radiated field exhibits an infinite discrete spatial spectrum as indicated in equations (A.8) and (A.9).

Assuming that \(E_z^{TM}(\bar{r}; h, m)\) and \(H_z^{TE}(\bar{r}; h, m)\) are of the general form

\[
E_z^{TM}(\bar{r}; h, m) = E_0 e^{-jhz} R^{TM}(\beta\eta\rho) \left\{ \begin{array}{c} \cos m\Delta\phi \\ \sin m\Delta\phi \end{array} \right\} 
\]

\[
H_z^{TE}(\bar{r}; h, m) = H_0 e^{-jhz} R^{TM}(\beta\eta\rho) \left\{ \begin{array}{c} \cos m\Delta\phi \\ \sin m\Delta\phi \end{array} \right\} 
\]

then the remaining vector field components are related to the axial components as shown on the following page.

Therefore, having obtained the axial field components for a given element orientation, we obtain the remaining components with little additional effort by making the substitutions indicated in equations (A.12) through (A.23) where the prime denotes differentiation with respect to the argument.
TM Field Components

\[ E^TME_r(h,m) = E_oe^{-jhz R^TM} (\beta \eta \rho) \left\{ \begin{array}{c} \cos m \Delta \phi \\ \sin m \Delta \phi \end{array} \right\} \] (A.12)

\[ E^TME_\phi(h,m) = -\frac{j \eta}{\beta \rho} E_oe^{-jhz R^TM} (\beta \eta \rho) \left\{ \begin{array}{c} \cos m \Delta \phi \\ \sin m \Delta \phi \end{array} \right\} \] (A.13)

\[ H^TMz(h,m) = 0 \] (A.15)

\[ H^TME_\rho(h,m) = \frac{j m \eta}{Z_c \eta \beta \rho} E_oe^{-jhz R^TM} (\beta \eta \rho) \left\{ \begin{array}{c} \sin m \Delta \phi \\ \cos m \Delta \phi \end{array} \right\} \] (A.16)

\[ H^TM_\phi(h,m) = -\frac{j}{Z_c \eta} E_oe^{-jhz R^TM} (\beta \eta \rho) \left\{ \begin{array}{c} \cos m \Delta \phi \\ \sin m \Delta \phi \end{array} \right\} \] (A.17)

TE Field Components

\[ E^TETE_r(h,m) = 0 \] (A.18)

\[ E^TETE_\rho(h,m) = \frac{j m \eta}{Z_c \eta \beta \rho} H_oe^{-jhz R^TE} (\beta \eta \rho) \left\{ \begin{array}{c} \sin m \Delta \phi \\ \cos m \Delta \phi \end{array} \right\} \] (A.19)

\[ E^TETE_\phi(h,m) = \frac{j Z_c}{\eta} H_oe^{-jhz R^TE} (\beta \eta \rho) \left\{ \begin{array}{c} \cos m \Delta \phi \\ \sin m \Delta \phi \end{array} \right\} \] (A.20)

\[ H^TETEz(h,m) = H_oe^{-jhz R^TE} (\beta \eta \rho) \left\{ \begin{array}{c} \cos m \Delta \phi \\ \sin m \Delta \phi \end{array} \right\} \] (A.21)

\[ H^TETE_\rho(h,m) = -\frac{j \eta}{H_oe^{-jhz R^TE} (\beta \eta \rho) \left\{ \begin{array}{c} \cos m \Delta \phi \\ \sin m \Delta \phi \end{array} \right\} \] (A.22)

\[ H^TETE_\phi(h,m) = \frac{j m \eta}{\beta \rho \eta^2} H_oe^{-jhz R^TE} (\beta \eta \rho) \left\{ \begin{array}{c} \sin m \Delta \phi \\ \cos m \Delta \phi \end{array} \right\} \] (A.23)
A.3 The Scalar Boundary-Value Problems

The boundary value problems for $G_1$ and $G_2$ are as follows: $G_1$ and $G_2$ each satisfy
\[
(\nabla^2 + \beta^2)G(R, R') = -\delta(R - R') \quad (A.24)
\]
We subject $G_1$ to the Dirichlet ("soft") boundary condition
\[
G_1|_{\rho=a} = 0 \quad (A.25)
\]
where $a$ is the cylinder radius and where the outward unit normal on the cylinder surface is $\hat{n} = -\hat{\rho}$. It follows that $\nabla G_1 \cdot \hat{z} |_{\rho=a} = 0$ as well. Thus, the $\hat{z}$ component of the surface integral in equation (A.1) vanishes.

On the other hand, we impose on $G_2$ the Neumann ("hard") boundary condition
\[
\frac{\partial G_2}{\partial \rho} |_{\rho=a} = 0 \quad (A.26)
\]
By a vector identity,
\[
\hat{z} \cdot (\hat{\rho} \times \overline{H}(R)) \times \nabla G_2(R, R') = -\hat{\phi} \cdot (\hat{\rho} \times \overline{H}(R)) \frac{\partial}{\partial \rho} G_2(R', R) = 0
\]
Since the normal component of $\overline{H}$ and the tangential component of $\overline{E}$ are zero on $S$, we conclude that the $\hat{z}$ component of $\int_S$ in equation (A.2) vanishes as desired, so that the axial field components are given by the reduced equations (A.4) and (A.5).

Note that we have implicitly invoked the symmetry relation $G(R, R') = G(R', R)$ in that the roles of the primed and unprimed coordinates is reversed in equations (A.4) and (A.5).

A.4 Solutions for the Scalar Green's Functions

In this section we examine the general form of the scalar Green's functions corresponding to set-ups A and B. In subsequent sections, we will often suppress
the type 1 or 2 subscripts on the various Green’s functions; in those instances, the
reference to the “soft” or “hard” boundary condition cases will be clear from the
context. We will add the superscript A or B to the Green’s functions wherever the
context requires such a distinction for clarity.

A.4.1 Single Point Source

We first consider the case of a single source element (set-up A). We define \( G_{\rho m} \)
to be the radial Green’s function satisfying the inhomogeneous Bessel equation
\[
\left[ \frac{\partial}{\partial \rho} \left( \rho \frac{\partial}{\partial \rho} \right) + (\beta \eta)^2 \rho - \frac{m^2}{\rho} \right] G_{\rho m} = -\delta(\rho - \rho_0)
\]
(A.27)
together with as yet unspecified boundary conditions. Recalling that \( \eta = \sqrt{1 - \vec{h}^2} \)
is the normalized radial wavenumber, we choose the negative square root for \( |\vec{h}| > 1 \)
(radiation condition).

Clearly, \( G \), the composite Green’s function, is even-symmetric and periodic in \( \phi \) so that the transverse component of \( G \), denoted by \( G_{Tm} \) has the form
\[
G_{Tm} = \sum_{m=0}^{\infty} \frac{\epsilon_m}{2\pi} G_{\rho m} \cos m\Delta\phi
\]
(A.28)
where
\[
\epsilon_m = \begin{cases} 1 & m = 0 \\ 2 & m = 1,2,\ldots \end{cases}
\]
Then, the composite Green’s function for a single point source located at
\( \vec{R}' = (\rho_o, \phi_o, z') \) is related to \( G_{Tm} \) through summation over the axial wave
spectrum:
\[
G^A = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp \left[ -j \beta h(z - z') \right] G_{Tm} \ d(\beta h)
\]
(A.29)
\[
= \left( \frac{1}{2\pi} \right)^2 \int_{-\infty}^{\infty} \exp \left[ -j \beta h(z - z') \right] \sum_{m=0}^{\infty} \epsilon_m G_{\rho m} \cos m\Delta\phi \ d(\beta h)
\]
The additional \( (2\pi)^{-1} \) factor is a normalization factor for the axial exponential
eigenfunctions.
A.4.2 Periodic Array of Point Sources

We now consider an array of point sources parallel to the z axis and periodic in z (configuration B) with uniform element spacing \( D_z \) and having linear phase slope \( \beta S_z \). We may view \( S_z \) as the cosine of the elevation angle (as measured from the cylinder axis) of an incident plane wave illuminating the array. The Green’s function for this set-up is given by the superposition of point source responses:

\[
G^B = \int_{-\infty}^{\infty} G^A \sum_{n=-\infty}^{\infty} \delta(z' - z_0 - n D_z) e^{-j \beta S_z z'} dz'
\]  

We may regard \( z_0 \) as the location of an arbitrarily chosen reference element, while the absolute phase origin is located at \( z = 0 \). Substitution of equation (A.29) into equation (A.30) gives

\[
G^B = \left( \frac{1}{2\pi} \right)^2 \int_{-\infty}^{\infty} e^{-j \beta h} \sum_{m=0}^{\infty} c_m G_{\rho m} \cos m \Delta \phi
\]

\[
\cdot \left[ \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(z' - z_0 - n D_z) e^{-j \beta z'(S_z-h)} dz' \right] d(\beta h)
\]

The square-bracketed term equals

\[
e^{-j \beta z_0 (S_z-h)} \sum_{n=-\infty}^{\infty} e^{-j n \beta D_z (S_z-h)}
\]

According to the Poisson sum formula [15]

\[
\sum_{n=-\infty}^{\infty} \delta(t + nT) = \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{in2\pi t/T}
\]

Identifying \( t = \beta (S_z - h) \) and \( T = \frac{2\pi}{D_z} \) leads to

\[
\sum_{n=-\infty}^{\infty} e^{-j n \beta D_z (S_z-h)} = \frac{2\pi}{D_z} \sum_{n=-\infty}^{\infty} \delta \left( \beta h - \beta (S_z - \frac{n\lambda}{D_z}) \right)
\]

Substituting equations (A.34) and (A.32) into (A.31) and integrating on \( \beta h \), we obtain for the composite (periodic) Green’s function:
where \( u_{zn} = S_z + \frac{n\lambda}{D_z} \) and \( u_{\rho n} = \sqrt{1 - u_{zn}^2} \) are the normalized axial and radial wavenumbers respectively. Without loss of generality, the constant phase bias \( e^{-jz_0\beta S_z} \) will be suppressed in subsequent calculations.

**A.5 Solution for the Radial Green’s Function**

The radial Green’s function is a solution to equation (A.27) where for a periodic array the normalized radial wavenumber \( \eta \) takes on the discrete eigenvalues \( u_{\rho n} \) as shown in section A.4. It may be shown [16] that \( G_{\rho m} \) may be expressed in the form

\[
G_{\rho m} = \frac{U(\rho_\leq)T(\rho_\geq)}{W} \tag{A.36}
\]

where the conjunct \( W = \rho[U_{\rho}^T \frac{dU}{d\rho} - U_{\rho}^T \frac{dU}{d\rho}]_{\rho=\rho_0} \) and \( \rho_\geq = \max \min [\rho, \rho_0] \). The functions \( T \) and \( U \) are found to be

\[
T(\rho) = H_m^{(2)}(\kappa_\rho \rho) \tag{A.37}
\]

\[
U(\rho) = J_m(\kappa_\rho \rho) + D_1 \frac{H_m^{(2)}(\kappa_\rho \rho)}{H_m^{(2)}(\kappa_\rho \rho)} \tag{A.38}
\]

with \( D_1 = \begin{cases} b_m, & \text{for Dirichlet b.c.} \\ a_m, & \text{for Neumann b.c.} \end{cases} \)

where

\[
a_m = -\frac{J'_m(\kappa_\rho a)}{H_m^{(2)}(\kappa_\rho a)} \quad \text{and} \quad b_m = -\frac{J_m(\kappa_\rho a)}{H_m^{(2)}(\kappa_\rho a)}
\]

and where \( \kappa_\rho = \beta u_{\rho n} \) is the radial wavenumber and where \( \beta S_z \) and \( \lambda \) are the linear phase slope and wavelength, respectively. We find that the conjunct \( W = \frac{2i}{\pi} \), where
we have used the Wronskian formula

\[ J_m(x)H'_m(x) - J'_m(x)H_m(x) = \frac{-2j}{\pi x} \]  

(A.39)

Thus, the radial Green's function reduces to

\[ G^B_{\rho m}(\rho, \rho_o) = \frac{\pi}{2j} H_m^2(\kappa_{\rho o} x) \left[ J_m(\kappa_{\rho o} x) + \left\{ \begin{array}{c} \frac{b_m}{a_m} \\ \end{array} \right\} H_m^2(\kappa_{\rho o} x) \right] \]  

(A.40)

The \( b_m \) coefficients are selected if a Dirichlet boundary condition prevails at the
cylinder (applicable to \( E_z \) computation, as in equation (A.4)) and \( a_m \) is selected
for a Neumann boundary condition (applicable to \( H_z \) computation as in equa
tion (A.5)). Substituting result (A.40) into equation (A.35) we find the composite
scalar Green's function:

\[ G^B(R, R') = \frac{1}{4j D_z} \sum_{n=-\infty}^{\infty} e^{-j\kappa_z \Delta z} \sum_{m=-\infty}^{\infty} J_m(\kappa_{\rho o} x) + \left\{ \begin{array}{c} \frac{b_m}{a_m} \\ \end{array} \right\} H_m^2(\kappa_{\rho o} x) \]  

(A.41)

where \( \Delta z = z - z_o \) and \( \Delta \phi = \phi - \phi_o \) and where \( a_m \) and \( b_m \) are given by equa
tions (A.39).

A.6 Array Field Solutions

The axial field components for the \( \tilde{z}, \tilde{\phi} \) and \( \tilde{\rho} \) directed element cases are obtained using equations (A.4) and (A.5). The remaining field components may then be obtained according to the TM/TE decomposition given by equations (A.6) and (A.7).
For Hertzian array elements, the integrals in (A.4) and (A.5) involving the line current \( I(s) \hat{p} \) are of the form

\[
A = \int_{-\Delta s/2}^{\Delta s/2} I(s') f(s') ds'
\]

where \( s' \) is the arc length parameter and \( f \) is a continuous function proportional to \( G \) (or its derivative). Therefore, for \( \Delta s/\lambda \ll 1 \)

\[
A \simeq I(0)f(0)\Delta s = \int_0^\infty I(0)\Delta s \delta(s') f(s') ds'
\]

(A.42)

Let us define

\[
I_\delta(s) = I(0)\Delta s \delta(s)
\]

(A.43)

to be the equivalent line current distribution for an infinitesimal current element. Its dimensional units are amperes.

The term in equation (A.4) involving the charge density is

\[
B = \frac{1}{\epsilon} \int_{-\Delta s/2}^{\Delta s/2} \rho_e(s') \frac{\partial G(s')}{\partial s'} ds'
\]

(A.44)

where \( \frac{\partial G(s')}{\partial s'} \) is a continuous function of \( s' \). This term is problematical in that the charge density is singular at both ends of the current element, owing to the assumption that the current element \( I(s) \) is non-zero at the endpoints. We may define the equivalent line charge distribution for an infinitesimal current element to be

\[
\rho_e(s') = \frac{-1}{j\omega} \frac{dI_\delta(s')}{ds'}
\]

(A.45)

\[
= \frac{-I(0)}{j\omega} \Delta s \delta(s')
\]

(4.45)

\( \rho_e(s') \) has units of coulombs per meter.

140
\( I_\delta(s') \) and \( \rho_{e\delta}(s') \) are equivalent current and charge distributions in the sense that the same integrated result is obtained when they are substituted for \( I(s') \) and \( \rho_0(s') \).

Substituting \( \rho_{e\delta}(s') \) for \( \rho_e(s') \) in equation (A.44) we obtain

\[
B = \frac{I(0)\Delta s}{jw} \left. \frac{\partial^2 G}{\partial s' \partial z'} \right|_{\vec{H}=(\rho_0, \phi_0, z_0)} \tag{A.46}
\]

We have used the following operational properties of the delta function:

\[
\int_{-\infty}^{\infty} \delta(x - z)f(x) \, dx = f(z) \tag{A.47}
\]
\[
\int_{-\infty}^{\infty} \delta'(x - z)f(x) \, dx = -f'(z) \tag{A.48}
\]

The substitution of \( \rho_{e\delta} \) as given by (A.46) into equation (A.44) to obtain (A.46) may be justified through an argument similar to that leading to equation (A.42) where we use the facts that (1) both the charge density and \( \frac{\partial G(s')}{\partial z'} \) are continuous functions in the open interval \(-\Delta s' < s' < \Delta s'\) (2) the charge density is singular at the endpoints (3) \( \Delta s' \) is arbitrarily small.

\textbf{A.6.1} \ \hat{z} \textbf{ Directed Hertzian Current Elements}

An array with \( \hat{z} \) directed Hertzian current elements radiates in the presence of the cylinder. Consider the array reference element located at \( (\rho_o, \phi_o, z_o) \) with equivalent current distribution given by

\[
I_{\hat{z}\delta} = I_0 \Delta s \delta(z' - z_o) \tag{A.49}
\]

and corresponding equivalent charge distribution

\[
\rho_{e\hat{z}\delta} = -\frac{I_0 \Delta s}{j\omega} \delta'(z' - z_o). \tag{A.50}
\]

The differential axial field components are obtained from equations (A.4) and (A.5) together with results (A.42) and (A.46). We find that
\[
\begin{align*}
\text{d}E_{z,\text{TM}}^z &= -jI_0 \Delta s Z_c \left[ \beta + \frac{1}{\beta} \frac{\partial^2}{\partial z^2} \right] G_B^I |_{R'=(\rho_0, \phi_0, z_0)} \quad (A.51) \\
\text{d}H_{z,\text{TE}}^z &= 0 \quad (A.52)
\end{align*}
\]

where \( Z_c = \sqrt{\frac{\mu}{\varepsilon}} \), the characteristic impedance of the medium. Substituting equation (A.41) into (A.51), we obtain

\[
\text{d}E_{z,\text{TM}}^z = -\frac{I_0 \Delta s \beta Z_c}{4D_z} \sum_{n=-\infty}^{\infty} e^{-j \kappa_z \Delta z_n} u_{\rho_n} \sum_{m=0}^{\infty} \epsilon_m H_m^{(2)}(\beta u_{\rho_n} \rho) \left[ J_m(\beta u_{\rho_n} \rho) + b_m H_m^{(2)}(\beta u_{\rho_n} \rho) \right] \cos m \Delta \phi \quad (A.53)
\]

where \( \rho \geq \frac{\max}{\min} [\rho, \rho_0] \)

The remaining field components are obtained by inspection from equations (A.12) through (A.23), where the continuous spectra \( \eta \) and \( h \) are replaced by the discrete values \( u_{\rho_n} \) and \( u_{z_n} \). The results for the \( \rho \) and \( \phi \) electric and magnetic field components are as follows:

\[
\begin{align*}
\text{d}E_{\rho,\text{TM}}^z &= j \frac{I_0 \Delta s \beta Z_c}{4D_z} \sum_{n=-\infty}^{\infty} e^{-j \kappa_z \Delta z_n} u_{\rho_n} \sum_{m=0}^{\infty} \epsilon_m \left\{ \begin{array}{c}
H_m^{(2)}(\beta u_{\rho_n} \rho) \\
H_m^{(2)}(\beta u_{\rho_n} \rho)
\end{array} \right\} \cos m \Delta \phi \\
\text{d}E_{\phi,\text{TM}}^z &= -j \frac{I_0 \Delta s \beta Z_c}{4D_z} \sum_{n=-\infty}^{\infty} e^{-j \kappa_z \Delta z_n} \frac{1}{\beta \rho} \sum_{m=1}^{\infty} 2m H_m^{(2)}(\beta u_{\rho_n} \rho) \left[ J_m(\beta u_{\rho_n} \rho) + b_m H_m^{(2)}(\beta u_{\rho_n} \rho) \right] \sin m \Delta \phi \\
\text{d}H_{\rho,\text{TM}}^z &= \frac{I_0 \Delta s \beta}{4D_z} \sum_{n=-\infty}^{\infty} e^{-j \kappa_z \Delta z_n} \frac{1}{\beta \rho} \sum_{m=1}^{\infty} 2m H_m^{(2)}(\beta u_{\rho_n} \rho) \left[ J_m(\beta u_{\rho_n} \rho) + b_m H_m^{(2)}(\beta u_{\rho_n} \rho) \right] \sin m \Delta \phi
\end{align*}
\]

142
\[ dH_{\phi}^{\tilde{z},TM} = j \frac{I_0 \Delta s \beta}{4 D_z} \sum_{n=-\infty}^{\infty} e^{-j \kappa_z \Delta z} u_{\rho_n} \sum_{m=0}^{\infty} \epsilon_m \left\{ \begin{array}{c} H_m^{(2)}(\beta u_{\rho_n} \rho) \\ H_m^{(1)}(\beta u_{\rho_n} \rho_0) \end{array} \right\} \]

\[ \{ [J_m(\beta u_{\rho_n} \rho_0) + b_m H_m^{(2)}(\beta u_{\rho_n} \rho_0) ] + b_m H_m^{(2)}(\beta u_{\rho_n} \rho) \} \cos m \Delta \phi \]  

(A.57)

\[ dH_{\phi}^{\tilde{z}} = 0 \]  

(A.58)

for \( \{ \rho > \rho_0 \} \).

Here and elsewhere it is understood that both \( \rho \) and \( \rho_0 \) are greater than \( a \), the cylinder radius. From the foregoing, we observe that the field radiated by an array of \( \tilde{z} \) directed elements radiating in the presence of an uncoated cylinder is purely transverse magnetic.

A.6.2 \( \phi \) Directed Hertzian Current Elements

For an array with \( \phi \) directed elements, we postulate an equivalent line current distribution

\[ I_{\phi_\phi} = I_0 \Delta s \frac{\delta(\phi' - \phi_0)}{\rho_0} \]

together with the corresponding equivalent charge distribution

\[ \rho e_\phi = -\frac{I_0 \Delta s}{j \omega \rho_0^2} \delta'(\phi' - \phi_0) \]

Substituting these sources into equation (A.4) and noting that the axial current component is zero, we have

\[ dE_{\phi}^{\phi,TM} = \frac{1}{\epsilon} \int_{-\infty}^{\infty} \rho e_\phi(\phi') \frac{\partial}{\partial z'} G_1^B \rho_0 d\phi' \]

\[ = \frac{I_0 \Delta s}{j \omega \rho_0^2} \int_{-\infty}^{\infty} \delta'(\phi' - \phi_0) \frac{\partial}{\partial z'} G_1^B \rho_0 d\phi' \]

\[ = \frac{I_0 \Delta s Z_c}{j \beta \rho_0} \frac{\partial^2}{\partial z' \partial \phi'} G_1^B \bigg|_{R'=(\rho_0,\phi_0,z_0)} \]

143
Inserting the composite Green’s function given by (A.41) into this result we find the \( z \) component of the electric field:

\[
\begin{aligned}
dE_{z\,TM} &= -j \frac{I_0 \Delta s \beta Z_C}{4D_z} \sum_{n=-\infty}^{\infty} e^{-j\kappa_z \Delta z} \frac{u_{zn}}{\beta \rho_0} \sum_{m=1}^{\infty} 2m H_m^{(2)}(\beta u_{\rho_n} \rho) \\
&= \left[J_m(\beta u_{\rho_n} \rho) + b_m H_m^{(2)}(\beta u_{\rho_n} \rho)\right] \sin m\Delta\phi
\end{aligned}
\]

Next, we turn to the calculation of \( H_z \) as given by equation (A.5). By a vector identity, \( \hat{z} \cdot I_{\phi} \hat{\phi} \times \nabla'G = -\frac{\partial G}{\partial \rho'} I_{\phi}|_{\rho'=\rho_0} \). Then, substituting \( I_{\phi} \) for the current distribution and performing the integration indicated in equation (A.5), we obtain

\[
\begin{aligned}
dH_{z\,TM} &= j \frac{I_0 \Delta s \beta}{4D_z} \sum_{n=-\infty}^{\infty} e^{-j\kappa_z \Delta z} u_{\rho_n} \sum_{m=0}^{\infty} \epsilon_m \left\{ \begin{array}{l}
H_m^{(2)}(\beta u_{\rho_n} \rho) \\
H_m^{(2)}(\beta u_{\rho_n} \rho) \end{array} \right\} \\
&= \left\{ \begin{array}{l}
J_m(\beta u_{\rho_n} \rho) + a_m H_m^{(2)}(\beta u_{\rho_n} \rho) \\
J_m(\beta u_{\rho_n} \rho) + a_m H_m^{(2)}(\beta u_{\rho_n} \rho) \end{array} \right\} \cos m\Delta\phi, \text{ for } \rho > \rho_0
\end{aligned}
\]

The \( \rho \) and \( \phi \) electric and magnetic field components are readily obtained from equations (A.12) through (A.23), where the continuous spectra \( \eta \) and \( h \) are replaced by the discrete values \( u_{\rho_n} \) and \( u_{zn} \). The results are as follows:

\[
\begin{aligned}
dE_{\phi\,TM} &= -I_0 \frac{\Delta s \beta Z_C}{4D_z} \sum_{n=-\infty}^{\infty} e^{-j\kappa_z \Delta z} \frac{1}{(\beta \rho)(\beta \rho_0)} \left(\frac{u_{zn}}{u_{\rho_n}}\right)^2 \sum_{m=1}^{\infty} 2m^2 \cos m\Delta\phi
\end{aligned}
\]

\[
\begin{aligned}
dE_{z\,TE} &= -I_0 \frac{\Delta s \beta Z_C}{4D_z} \sum_{n=-\infty}^{\infty} e^{-j\kappa_z \Delta z} \sum_{m=0}^{\infty} \epsilon_m \\
&= \left\{ \begin{array}{l}
J_m(\beta u_{\rho_n} \rho) + b_m H_m^{(2)}(\beta u_{\rho_n} \rho) \\
J_m(\beta u_{\rho_n} \rho) + b_m H_m^{(2)}(\beta u_{\rho_n} \rho) \end{array} \right\} \cos m\Delta\phi
\end{aligned}
\]

144
\[
dE_{\rho,TM} = \frac{I_0}{4D_z} \sum_{n=-\infty}^{\infty} e^{-j\kappa_z \Delta z} \frac{u_{2n}}{u_{\rho_0}} \sum_{m=1}^{\infty} 2m \left\{ H_m^{(2)}(\beta u_{\rho_0}) [J_m(\beta u_{\rho_0}) + \beta u_{\rho_0} H_m^{(2)}(\beta u_{\rho_0})] \right\} \sin m \Delta \phi (A.62)
\]

\[
dE_{\rho,TE} = \frac{I_0}{4D_z} \sum_{n=-\infty}^{\infty} e^{-j\kappa_z \Delta z} \frac{1}{u_{\rho_0}} \sum_{m=1}^{\infty} 2m \left\{ H_m^{(2)}(\beta u_{\rho_0}) [J_m'(\beta u_{\rho_0}) + \beta u_{\rho_0} H_m^{(2)}(\beta u_{\rho_0})] \right\} \sin m \Delta \phi (A.63)
\]

\[
dH_{\phi,TM} = \frac{I_0}{4D_z} \sum_{n=-\infty}^{\infty} e^{-j\kappa_z \Delta z} \frac{u_{2n}}{u_{\rho_0}} \sum_{m=1}^{\infty} 2m \left\{ H_m^{(2)}(\beta u_{\rho_0}) \right\} \sin m \Delta \phi (A.64)
\]

\[
dH_{\phi,TE} = \frac{I_0}{4D_z} \sum_{n=-\infty}^{\infty} e^{-j\kappa_z \Delta z} \frac{u_{2n}}{u_{\rho_0}} \sum_{m=1}^{\infty} 2m \left\{ J_m'(\beta u_{\rho_0}) + \beta u_{\rho_0} H_m^{(2)}(\beta u_{\rho_0}) \right\} \sin m \Delta \phi (A.65)
\]

\[
dH_{\rho,TM} = \frac{I_0}{4D_z} \sum_{n=-\infty}^{\infty} e^{-j\kappa_z \Delta z} \frac{1}{u_{\rho_0}} \sum_{m=1}^{\infty} 2m^2 H_m^{(2)}(\beta u_{\rho_0}) \left[ J_m(\beta u_{\rho_0}) + \beta u_{\rho_0} H_m^{(2)}(\beta u_{\rho_0}) \right] \cos m \Delta \phi (A.66)
\]

\[
dH_{\rho,TE} = \frac{I_0}{4D_z} \sum_{n=-\infty}^{\infty} e^{-j\kappa_z \Delta z} \sum_{m=0}^{\infty} \epsilon_m \left[ J_m'(\beta u_{\rho_0}) + \beta u_{\rho_0} H_m^{(2)}(\beta u_{\rho_0}) \right] \cos m \Delta \phi (A.67)
\]

for \( \rho > \rho_0 \).

145
A.6.3  Directed Hertzian Current Elements

For an array of radially directed Hertzian elements, the equivalent current distribution is

$$I_{\rho_{\delta}} = I_0 \Delta s \delta(\rho' - \rho_o)$$

together with the equivalent charge distribution

$$\rho_{\epsilon\delta} = -\frac{I_0 \Delta s}{j\omega} \delta'(\rho' - \rho_o)$$

The differential axial field components are obtained from equations (A.4) and (A.5). With

$$dE_{z,TM}^\delta = \frac{1}{\epsilon} \int_{-\infty}^{\infty} \rho_{\epsilon\delta} (\rho') \frac{\partial}{\partial z'} G_1^B d\rho'$$  \hspace{1cm} (A.67)

We find after simplification that

$$dE_{z,TM}^\delta = \frac{I_0 \Delta s Z_e}{j\beta} \frac{\partial^2}{\partial z' \partial \rho^2} G_1^B \bigg|_{R = (\rho_o, \phi_o, z_o)}$$  \hspace{1cm} (A.68)

Substituting (A.41) into (A.68) we obtain

$$dE_{z,TM}^\delta = -j \frac{I_0 \Delta s \beta}{4D_e Z_c} \sum_{n=-\infty}^{\infty} e^{-j\kappa_z \Delta z u z_n u \rho_n} \sum_{m=0}^{\infty} \epsilon_m \left\{ \begin{array}{l} H_m^{(2)}(\beta u \rho_n \rho) \\ H_{m'}^{(2)}(\beta u \rho_n \rho) \end{array} \right\} \cdot \left\{ \begin{array}{l} J_{m'}(\beta u \rho_n \rho_o) + b_m H_{m'}^{(2)}(\beta u \rho_n \rho_o) \\ J_m(\beta u \rho_n \rho) + b_m H_m^{(2)}(\beta u \rho_n \rho) \end{array} \right\} \cos m \Delta \phi$$  \hspace{1cm} (A.69)

for \( \rho > \rho_o \)

According to (A.5) the axial component of \( \bar{dH}^\rho \) is

$$dH_{z,TE}^\rho = \int_{-\infty}^{\infty} \dot{z} \cdot [I_{\rho_{\delta}}(\rho') \hat{\rho} \times \nabla' G_2^B(\rho, \rho')] d\rho'$$

$$= \int_{-\infty}^{\infty} I_{\rho_{\delta}}(\rho') \nabla' G_2^B(\rho, \rho') \cdot (\dot{z} \times \hat{\rho}) d\rho'$$

$$= \int_{-\infty}^{\infty} I_{\rho_{\delta}}(\rho') \frac{1}{\rho_o} \frac{dG_2^B(\rho, \rho')}{d\phi'} d\rho'$$

$$= \frac{I_0 \Delta s \beta dG_2^B}{\beta \rho_o} \bigg|_{R = (\rho_o, \phi_o, z_o)}$$  \hspace{1cm} (A.70)
Substitution of equation (A.41) into (A.70) gives

\[ dH_z^{\phi,TE} = -j \frac{I_0 \Delta s \beta}{4D_z} \sum_{n=-\infty}^{\infty} e^{-j \kappa_z \Delta z} \frac{1}{\beta \rho_o} \sum_{m=1}^{\infty} 2m H_m^{(2)}(\kappa \rho_o) \left[ J_m(\kappa \rho_o) + a_m H_m^{(2)}(\kappa \rho_o) \right] \sin m \Delta \phi \]  

(A.71)

The \( \rho \) and \( \phi \) electric and magnetic field components are obtained from equations (A.12) and (A.23), where the continuous spectra \( \eta \) and \( \kappa \) are replaced by the discrete values \( u_{\rho_n} \) and \( u_{\kappa_n} \). The results are as follows:

\[ dE_{\phi}^{\rho, TM} = \frac{I_0 \Delta s \beta Z_c}{4D_z} \sum_{n=-\infty}^{\infty} e^{-j \kappa_z \Delta z} \frac{u_{\rho_n}^2}{u_{\rho_n} \beta \rho_o} \sum_{m=1}^{\infty} 2m \left\{ \frac{H_m^{(2)}(\beta u_{\rho_n})}{H_m^{(2)}(\beta u_{\rho_n})} \right\} \times \left\{ J_m(\beta u_{\rho_n}) + b_m H_m^{(2)}(\beta u_{\rho_n}) \right\} \sin m \Delta \phi \]  

(A.72)

\[ dE_{\rho}^{\rho, TM} = \frac{I_0 \Delta s \beta Z_c}{4D_z} \sum_{n=-\infty}^{\infty} e^{-j \kappa_z \Delta z} \frac{1}{u_{\rho_n} \beta \rho_o} \sum_{m=1}^{\infty} 2m \left\{ \frac{H_m^{(2)}(\beta u_{\rho_n})}{H_m^{(2)}(\beta u_{\rho_n})} \right\} \times \left\{ J_m(\beta u_{\rho_n}) + a_m H_m^{(2)}(\beta u_{\rho_n}) \right\} \sin m \Delta \phi \]  

(A.73)

\[ dE_{\rho}^{\phi, TM} = \frac{I_0 \Delta s \beta Z_c}{4D_z} \sum_{n=-\infty}^{\infty} e^{-j \kappa_z \Delta z} \frac{1}{u_{\rho_n} \beta \rho_o} \sum_{m=0}^{\infty} \epsilon_m \cdot H_m^{(2)}(\beta u_{\rho_n}) \left[ J_m(\beta u_{\rho_n}) + b_m H_m^{(2)}(\beta u_{\rho_n}) \right] \cos m \Delta \phi \]  

(A.74)

\[ dH_{\rho}^{\rho, TM} = \frac{I_0 \Delta s \beta}{4D_z} \sum_{n=-\infty}^{\infty} e^{-j \kappa_z \Delta z} \frac{1}{u_{\kappa_n}^2(\beta \rho)(\beta \rho_o)} \sum_{m=1}^{\infty} 2m^2 H_m^{(2)}(\kappa \rho_o) \left[ J_m(\kappa \rho_o) + a_m H_m^{(2)}(\kappa \rho_o) \right] \cos m \Delta \phi \]  

(A.75)

\[ dH_{\phi}^{\phi, TM} = \frac{I_0 \Delta s \beta}{4D_z} \sum_{n=-\infty}^{\infty} e^{-j \kappa_z \Delta z} u_{\kappa_n} \sum_{m=0}^{\infty} \epsilon_m \cdot H_m^{(2)}(\beta u_{\rho_n}) \left[ J_m(\beta u_{\rho_n}) + b_m H_m^{(2)}(\beta u_{\rho_n}) \right] \cos m \Delta \phi \]  

(A.76)
\[
\begin{align*}
\frac{dH_{\phi}^{\phi,TE}}{d\phi} &= -\frac{I_0 \Delta s \beta}{4Dz} \sum_{n=-\infty}^{\infty} e^{-j\kappa z \Delta z} \frac{u_{m,n}}{u_{m,n}^2 \beta(\beta \rho)} \sum_{m=1}^{\infty} 2m^2 H_m^{(2)}(\kappa \rho) \cos m \Delta \phi \\
&\quad \left[ J_m(\kappa \rho) + a_m H_m^{(2)}(\kappa \rho) \right] \cos m \Delta \phi
\end{align*}
\]
\[(A.77)\]

\[
\begin{align*}
\frac{dH_{\rho}^{\phi,TE}}{d\phi} &= -\frac{I_0 \Delta s \beta}{4Dz} \sum_{n=-\infty}^{\infty} e^{-j\kappa z \Delta z} \frac{u_{m,n}}{u_{m,n}^2 \beta(\beta \rho)} \sum_{m=1}^{\infty} 2m^2 \left\{ \begin{array}{l}
H_m^{(2)}(\beta u_{m,n} \rho) \\
H_m^{(2)}(\beta u_{m,n} \rho) 
\end{array} \right\} \\
&\quad \left\{ \begin{array}{l}
J_m(\beta u_{m,n} \rho) + b_m H_m^{(2)}(\beta u_{m,n} \rho) \\
J_m(\beta u_{m,n} \rho) + b_m H_m^{(2)}(\beta u_{m,n} \rho)
\end{array} \right\} \sin m \Delta \phi
\end{align*}
\]
\[(A.78)\]

\[
\begin{align*}
\frac{dH_{\rho}^{\phi,TE}}{d\phi} &= -\frac{I_0 \Delta s \beta}{4Dz} \sum_{n=-\infty}^{\infty} e^{-j\kappa z \Delta z} \frac{u_{m,n}}{u_{m,n}^2 \beta(\beta \rho)} \sum_{m=1}^{\infty} 2m^2 \left\{ \begin{array}{l}
H_m^{(2)}(\beta u_{m,n} \rho) \\
H_m^{(2)}(\beta u_{m,n} \rho)
\end{array} \right\} \\
&\quad \left\{ \begin{array}{l}
J_m(\beta u_{m,n} \rho) + a_m H_m^{(2)}(\beta u_{m,n} \rho) \\
J_m(\beta u_{m,n} \rho) + a_m H_m^{(2)}(\beta u_{m,n} \rho)
\end{array} \right\} \sin m \Delta \phi
\end{align*}
\]
\[(A.79)\]

**A.7 Free Space Fields**

The fields due a periodic array radiating in a homogeneous space (cylinder absent) may be obtained from the foregoing formulas by setting the cylinder scattering coefficients \(a_m\) and \(b_m\) equal to zero wherever they appear.

**A.8 Periodic Electric Green's Dyadic**

The preceding fifteen equations for the TM and TE components of the electric field due to an periodic array of arbitrarily oriented Hertzian dipoles can be tabulated concisely in the form of a dyadic function. This electric dyadic Green's function is a modification of the dyadic presented by Tai [6] for the case of a single element radiating in the presence of an infinite PEC cylinder. This modification entails (1) the replacement of the axial spectral integral with a summation and (2) the substitution of the discrete axial and radial eigenvalues \(-\kappa_z\) and \(\kappa_\rho\) for...
the continuous values wherever they appear in Tai's dyadic (the minus sign is due to the fact that Tai uses the \(e^{-jwt}\) time convention), and (3) replace the Hankel function of the first kind with the Hankel function of the second kind. In addition, there is a new proportionality constant which may be determined by comparing solutions for an arbitrary field component with those derived in the preceding section.

Thus, for a periodic linear array with arbitrarily oriented elements, with the reference element terminals located at \(\vec{R}_0 = (\rho_o, \phi_o, z_o)\) (with \(\rho_o \geq a\) where \(a\) is the cylinder radius), the electric field radiated in the presence of an infinite PEC cylinder at a field point \(\vec{R} = (\rho, \phi, z)\), \(\rho \geq a\) is given by

\[
\vec{E} = j\beta Z_o \int_{\text{element}} \Gamma_\pi(\vec{R}, \vec{R}') \cdot I(\vec{R}') \, d\vec{R}'
\]  

(A.80)

The periodic electric Green’s dyadic is given by: (note \(e^{+jwt}\) time convention)

\[
\Gamma_\pi(\vec{R}, \vec{R}') = \frac{j}{4D_z} \sum_{n=-\infty}^{\infty} \sum_{m=0}^{\infty} \epsilon_m \kappa_\rho^2
\]

\[
\left\{ \tilde{M}_{em}^{(4)}(\kappa_z, \vec{R}) \left[ \tilde{M}_{em}^{(4)}(-\kappa_z, \vec{R}') + a_m \tilde{M}_{em}^{(4)}(-\kappa_z, \vec{R}') \right] +
\right. 
\left. + \tilde{M}_{om}^{(4)}(\kappa_z, \vec{R}) \left[ \tilde{M}_{om}^{(4)}(-\kappa_z, \vec{R}') + a_m \tilde{M}_{on}^{(4)}(-\kappa_z, \vec{R}') \right] +
\right. 
\left. + \tilde{N}_{em}^{(4)}(\kappa_z, \vec{R}) \left[ \tilde{N}_{em}^{(4)}(-\kappa_z, \vec{R}') + b_m \tilde{N}_{en}^{(4)}(-\kappa_z, \vec{R}') \right] +
\right. 
\left. \tilde{N}_{om}^{(4)}(\kappa_z, \vec{R}) \left[ \tilde{N}_{om}^{(4)}(-\kappa_z, \vec{R}') + b_m \tilde{N}_{on}^{(4)}(-\kappa_z, \vec{R}') \right] \right\}
\]

for \(\rho > \rho_o\) and

\[
\Gamma_\pi(\vec{R}, \vec{R}') = \frac{j}{4D_z} \sum_{n=-\infty}^{\infty} \sum_{m=0}^{\infty} \epsilon_m \kappa_\rho^2
\]

\[
\left\{ \left[ \tilde{M}_{em}^{(1)}(\kappa_z, \vec{R}) + a_m \tilde{M}_{en}^{(4)}(\kappa_z, \vec{R}) \right] \tilde{M}_{em}^{(4)}(-\kappa_z, \vec{R}') +
\right. 
\left. + \left[ \tilde{M}_{om}^{(1)}(\kappa_z, \vec{R}) + a_m \tilde{M}_{on}^{(4)}(\kappa_z, \vec{R}) \right] \tilde{M}_{om}^{(4)}(-\kappa_z, \vec{R}') +
\right. 
\left. + \tilde{N}_{em}^{(1)}(\kappa_z, \vec{R}) + b_m \tilde{N}_{en}^{(4)}(\kappa_z, \vec{R}) \right\} \tilde{N}_{em}^{(4)}(-\kappa_z, \vec{R}') +
\]

149
\[ + \left[ N^{(1)}_{\text{en}}(\kappa_z, \vec{R}) + b_m N^{(4)}_{\text{en}}(\kappa_z, \vec{R}) \right] N^{(4)}_{\text{en}}(-\kappa_z, \vec{R}') \right\} + \\
\]

for \( \rho < \rho_o \)

where

\[ a_m = - \frac{J'_m(\kappa_\rho a)}{H^{(2)}_m(\kappa_\rho a)} \quad \text{and} \quad b_m = - \frac{J_m(\kappa_\rho a)}{H^{(2)}_m(\kappa_\rho a)} \]

and \( M \) and \( N \) are cylinder vector wave functions given by

\[
\begin{align*}
\widetilde{M}^{(j)}_{\frac{j}{6}}(\kappa_z, \vec{R}) &= e^{-j\beta \kappa_z z} \left[ \frac{m}{\rho} Z^{(j)}_{\frac{j}{6}}(\kappa_\rho \rho) \right] \frac{\sin m\phi}{\cos m\phi} - \\
&\quad - \frac{\partial Z^{(j)}_{\frac{j}{6}}(\kappa_\rho \rho)}{\partial \rho} \frac{\cos m\phi}{\sin m\phi} \tag{A.83}
\end{align*}
\]

\[
\begin{align*}
\widetilde{N}^{(j)}_{\frac{j}{6}}(\kappa_z, \vec{R}) &= \frac{1}{\beta} e^{-j\beta \kappa_z z} \left[ -j \kappa_z \frac{\partial Z^{(j)}_{\frac{j}{6}}(\kappa_\rho \rho)}{\partial \rho} \right] \frac{\cos m\phi}{\sin m\phi} + \\
&\quad + j m \kappa_\rho \rho Z^{(j)}_{\frac{j}{6}}(\kappa_\rho \rho) \tag{A.84}
\end{align*}
\]

for \( j = 1 \) or \( 4 \) and \( Z^{(1)}_m(x) = J_m(x), \ Z^{(4)}_m = H^{(2)}_m(x) \). The axial wavenumber \( \kappa_z = \beta u_z \), where \( u_z = S_z + \frac{\eta}{D_z} \). The radial wavenumber \( \kappa_\rho = \sqrt{\beta^2 - \kappa_z^2} \), where the negative square root is chosen for \( |\kappa_z| > \beta \).
APPENDIX B

PROPERTIES OF THE PATTERN FUNCTION

The element pattern function (as defined by equation (2.21) and evaluated explicitly in equation (2.24)) incorporates information about the element pitch angle, length and curvature. The pattern function for a particular element configuration strongly influences the rate of convergence of the field representations (2.27) and (2.28). In this section, we focus on the concept of the pattern function as a bandpass filter function in m-space, i.e., the spectrum of azimuthal eigenvalues. By exploiting well-known properties of the Fourier transform, we gain insight into the convergence properties of the sum over the m index.

There are several reasons for focusing attention on the sum on m. First, the sum over the Floquet wave modes (n index) converges quickly (20 terms or less is usually adequate for cylinders less than two wavelengths in diameter). While the Floquet sum is affected in general by the pattern function (and even dominated by it for vertical elements), it is relatively insensitive to the fine structure of the pattern function. In contrast, the character of the azimuthal modal series is strongly dependent on the local features of the pattern function envelope. The m sum generally converges slowly; the number of significant terms ranges from about 20 to 30 for transverse elements to several hundred terms for near vertical elements. Also, the $R_{nm}$ Fourier coefficients involve complicated algebraic combinations of Bessel functions. It is desirable from a numerical standpoint to compute a large
“batch” of Bessel functions, sum on $m$, then increment $n$. It is desirable to know in advance what range of orders contribute significantly.

In the case of infinite periodic planar arrays [17], an analogous pattern function comes into play in the plane wave expansion of the radiated fields from the array structure. For planar arrays, the element pattern function amounts to a modal filter which suppresses all but the lowest order bundle of discrete inhomogeneous plane waves radiated by the array. The pattern function $P = P(u^{(i)}, v^{(j)})$ is a function of the transverse components of the $(i, j)^{th}$ propagation vector. If we replace $u^{(i)}$ and $v^{(j)}$ by continuous values, the resultant function $P = P(u, v)$ is precisely the far-field pattern of a single array element (excluding the cosine obliquity factor) radiating in a homogeneous space. Although the pattern factor weights the plane wave spectra emanating from the planar array, the summations representing the electromagnetic fields are convergent even if the pattern function is not included (provided that the test element is not in the plane of the array).

This is not necessarily the case with periodic linear arrays if the fields are expanded in cylindrical wave functions. For example, the series representation for the mutual impedance between two collinear arrays with vertical non-overlapping elements (with or without a cylinder present) is divergent because the two arrays, though offset in the z-direction, occupy the same radial position so that the radial Green's function becomes singular. However, this divergence, a purely mathematical peculiarity of the cylindrical wave functions, is independent of the pattern function.

The fact that we express the fields radiated from a periodic linear array as a sum of cylindrical waves rather than plane waves constitutes a fundamental point of departure which distinguishes the character of the field series representations. (While it is possible to express the field radiated by a column array as a sum of
plane waves, there would be no point in doing so as we could no longer conveniently apply boundary conditions at cylindrical surfaces). The pattern function as defined in section 2.3 is not related to the array element far-field pattern in any obvious way (excluding the special case of vertical dipoles for which the pattern function is a function of the axial wavenumber $\kappa_z$ only and not the angular mode index $m$). One problem is that the wave modes associated with the $m$ index do not lend themselves to simple physical interpretation; the $m$ eigenvalues are simply angular frequencies, mathematical artifacts of the Fourier series expansion of the fields in the $\phi$ coordinate.

The index $n$ can be associated with the elevation angle of a cylindrical wave propagating along a cone centered about the $z$ axis. This is so because the normalized axial wavenumber $u_z = S_z + \frac{n\lambda}{D_z}$, so that for $n = 0$ (normally the only propagating mode), $u_z = S_z = \cos \theta_0$ where $\theta_0$ is the angle between the cone and the $z$ axis. For $|n| > 0$, corresponding to evanescent wave modes, the cone angle associated with $u_z$ becomes complex. For a given $n$, we sum on $m$ to obtain the azimuthal variation of the $n^{th}$ Floquet cylinder wave.

In spite of the differences cited above, one should not lose sight of the fact that the element pattern functions as defined for the periodic planar array and the periodic linear array both originate as Fourier transforms of the element current distribution, and therefore share the same fundamental mathematical structure.

Let us then assess the structure of the element pattern factor viewed as a Fourier transform in $m$-space. We may rewrite (2.23) with $s$, the arc length, as the integration variable, through the change of variables $s = \rho_o \sqrt{1 + \zeta^2 \phi'}$, where $\rho_o$ is the radial distance from the $z$-axis to the array terminals. The "plus" pattern
factor then reads

\[ P_n^+(m) = \frac{1}{I(0)} \int_{-L}^{L} I(\beta s) \exp \left( -j \frac{m - m_n}{\rho_0 d} s \right) ds \]  \hspace{1cm} (B.1)

\[ = \frac{1}{I(0)} \mathcal{F}\{I(\beta s)\} \]

where \( \mathcal{F} \) denotes the Fourier transform and \( d = \sqrt{1 + \zeta^2} \). The shift parameter \( m_n \), a function of the Floquet index \( n \), is given by

\[ m_n = \kappa_z \rho_0 \zeta = \beta u_z \rho_0 \zeta = \beta \rho_0 \zeta \left( S_z + \frac{n \lambda}{D_z} \right) \] \hspace{1cm} (B.2)

For a given Floquet mode \( n \), the constant \( m_n \) is the translational shift of the pattern beam peak in \( m \)-space. (The “minus” pattern functions behave similarly except a shift-right becomes a shift-left and vice versa). The scaling factor \( d \) is the relative beamwidth in \( m \) space for some element pitch angle compared to the beamwidth when the pitch angle is zero, i.e., when \( \zeta = 0 \). From equation (2.24), we find that when \( \zeta \) equals zero (assume zero scan for simplicity), the first null occurs approximately at \( m = \frac{3\pi \rho_0}{2L} \), resulting in a null-to-null beamwidth \( \Omega_0 \simeq 10^{\rho_0 \rho / L} \). Therefore, the beamwidth for an arbitrary element pitch angle is given by

\[ \Omega(\zeta) = \text{null-to-null beamwidth in } m \text{-space} \]

\[ = \Omega_0 \, d \simeq 10^{\rho_0 \rho / L} \sqrt{1 + \zeta^2} \] \hspace{1cm} (B.3)

Figures 43 — 47 show the element pattern function for the first few Floquet modes as the element pitch angle is increased from 0 deg. (transverse elements) to 90 deg. (vertical elements). In all cases, the radial distance \( \rho_o \) to the arrays is \( 0.475 \lambda_o \). For the transverse element case one observes in figure 43 that the pattern function is the same for all \( n \) as one infers from equation (2.26).

As we increase the element pitch from zero, we observe the emergence of the pattern replicas corresponding to successive Floquet wave modes (figure 1.5).
For a pitch angle of 30 degrees (figure 44) the replica spacing is, according to (B.2), \( \frac{6c_{\rho_0}^2}{D_2} \) or about 2.8 units at center frequency. The null-to-null beamwidth is, according to equation (B.3), about 11.0 units. The peak amplitude \( P_{\text{max}} \) is to a first approximation equal to near the center frequency. Figure 48 shows that the variation in peak amplitude with frequency is about twice as small as the change when the dipole length \( L \) varies by the same percentage (see figure 49). For interelement spacings sufficiently small so that no grating lobes propagate, the central pattern function is associated with the \( n = 0 \) propagating mode, whereas the translated patterns correspond to evanescent cylinder wave modes. Evidently, the higher order evanescent modes exhibit a higher average azimuthal frequency content (i.e., more rapid angular variations).

As the element pitch is increased (figure 45), the pattern replicas get fatter and more widely separated. For pitch angles near but not exactly 90 degrees (see figure 46 and note the revised scale) a problem arises in that the \( m \) sums corresponding to even the lowest order Floquet modes entails the computation of several thousand terms. As a result, there is a practical upper bound on the element pitch angle (about 85 degrees for the range of parameters considered in this paper) before our numerical code becomes unstable and/or the calculations become excessively time consuming.

The behavior of the pattern function when the element pitch angle is exactly 90 degrees (vertical dipole) is quite interesting (figure 47). From equation (2.25) we already know that for a given \( n \), the pattern function is independent of \( m \). For a given \( n \), we might imagine “pinning” the \( m = 0 \) value of the pattern corresponding to a pitch angle in an epsilon neighborhood of 90 degrees and then stretching out the pattern so that the \( m \)-coordinate of the beam peak tends to infinity. Notice that the (constant) pattern values for each \( n \) do not correspond to the beam peak.
values, which would be the same, about .67 for this example, for every \( n \). It would appear at first that the vertical dipole case is uncomputable since we can never sum out to the first nulls of the \( n = 0 \) pattern, much less reach the beam peaks for the \( |n| > 0 \) patterns. Fortunately, this is not the case. In fact, the vertical dipole case is the easiest case to compute. The reason is that for vertical dipoles the sums on \( m \) may be evaluated in closed form by invoking the addition theorem for the Hankel function of order zero.

We observe from equation (B.3) that to a first approximation the \( m \)-space pattern beamwidth is independent of frequency. Indeed, figure 48 confirms that the beamwidth is fairly insensitive to frequency over a 40 percent bandwidth about center frequency. The reason is that as the frequency is increased, the electrical length of \( \rho_0 \) grows at the same rate as \( L \), the element half-length. That is, \( \phi_L = \frac{L}{\rho_0 \sqrt{1 + \xi^2}} \), the angular span of the dipole, is (obviously) invariant with frequency. But the \( m \)-space beamwidth is roughly proportional to \( \phi_L \) as may be seen most easily from equation (2.23).

Unless noted otherwise we will assume that the obliquity factor \( S_z \) is zero for simplicity. \( S_z \) appears in the displacement parameter \( m_n \). One effect of a non-zero scan is to shift the entire ensemble of \( m \)-space patterns by an amount \( \beta \rho_0 \xi S_z \), as illustrated in figure 50.
Figure 43: Pattern function for transverse (zero pitch) elements as a function of the angle mode index $m$, for the first few Floquet modes.
Figure 44: Pattern function for elements with 30 deg. pitch angle as a function of the angle mode index $m$, for the first few Floquet modes.
Figure 45: Pattern function for elements with 45 deg. pitch angle as a function of the angle mode index $m$, for the first few Floquet modes.
Figure 46: Pattern function for elements with (steep) 89.9 deg. pitch angle as a function of the angle mode index \( m \), for the first few Floquet modes.
Figure 47: Pattern function for vertical elements as a function of the angle mode index $m$, for the first few Floquet modes.
Figure 48: Pattern function as a function of the angle mode index $m$, with frequency as a parameter; element pitch $\alpha = 55$ deg.; first few Floquet modes are shown.
Figure 49: Pattern function as a function of the angle mode index $m$, with the dipole length as a parameter; element pitch $\alpha = 55$ deg.; first few Floquet modes are shown.
Figure 50: Pattern function as a function of the angle mode index \( m \), with the obliquity factor \( S_z \) as a parameter; element pitch \( \alpha = 55 \text{ deg.} \); first few Floquet modes are shown.
APPENDIX C
THIN WIRE THEORY

In computing the array self-impedance, one encounters a singularity in the integral representation of the electric field when the source and field points coincide. That is, equation (1.2) must be interpreted as a Cauchy principal value integral, which, except for the vertical element case radiating in free-space, cannot be evaluated in closed form and is difficult to evaluate numerically.

Therefore, it is desirable in calculating the array self-impedance to approximate the integral with the value of the integrand at one point. Then we are left with the well-behaved line integral (1.7), for which an exact series representation can be derived for the class of array elements considered in this dissertation.

In several texts on antenna theory [9][18][19] a thin-wire approximation is introduced such that the dipole current tube is replaced by a current filament located on the wire axis, while the electric field is evaluated a wire radius away from the axis. (It is assumed that the dipole is straight and radiates in free space). We will refer to this as the axis filament model. This model is equivalent to the current tube model in the sense that for some equivalent wire radius, the two models yield the same numerical value, at a given frequency, for the self-impedance.

For very thin wires, the equivalent wire radius $r_{eq}$ is approximately equal to the physical wire radius $r_w$. For the case of a single linear dipole radiating in free
space, Imbriale [7] derives a first order correction, with the result that

\[
\frac{r_{eq}}{r_w} = 1.0 - 0.40976 \frac{r_w}{\Delta L}
\]

where \( \Delta L \) is the subsectional element length. Thus the percentage difference between the physical wire radius and the equivalent wire radius is proportional to the element aspect ratio to a first approximation.

In a recent paper, Collin [20] discusses the fundamental limitations of the axis filament model. He considers an equivalent filament current \( \tilde{I}(z) \) whose distribution gives the same vector potential at the wire surface as the true wire surface current \( I(z) \) (and thus radiates the same fields outside the wire). This concept is not the same as the axis filament model discussed above, which involves replacing the current tube with a concentrated filament whose total current at each value of \( z \) is the same as that of the current tube. In contrast, the axial distribution of \( \tilde{I}(z) \) is in general different than the "true" tube current density \( I(z) \) (particularly near the terminal region). Collin shows that the spatial Fourier series coefficients \( \tilde{I}_n \) of \( \tilde{I}(z) \) behave asymptotically like

\[
\tilde{I}_n \sim 2\pi r_w I_n \exp\left(\frac{n\pi r_w}{\sqrt{n\pi^2 r_w^2 + \Delta L}}\right), \quad n \text{ large}
\]

where the \( I_n \) are the Fourier coefficients of the true surface distribution \( I(z) \). Thus, the \( \tilde{I}_n \) diverge exponentially from the \( I_n \) more rapidly as the dipole length is decreased, for a given wire thickness. One infers that as more current modes are added (and thereby subdividing the element length of each modal subarray) the moment method solution (using the axis filament model) for the input impedance will tend to diverge as \( M \) becomes comparable to \( \frac{2\Delta L}{\pi r_w} \).

Though the studies cited above specifically address the isolated linear dipole problem, one expects trends to carry over to periodic arrays of curved dipoles. However, only the simplest array configurations are amenable to exact analysis.
In the following two sections, we consider two special cases: a periodic linear array of axial dipoles (1) radiating in free space and (2) radiating in the presence of a perfectly conducting infinite ground plane. Our objectives are to (i) establish a range of validity for the thin filament model, (ii) to determine the associated equivalent wire radius, (iii) gauge the effect of nearby inhomogeneities.

C.1 Equivalent Wire Radius for Linear Array of Axial Elements

We consider an infinite periodic linear array of vertical dipoles radiating in free-space. We wish to compute the array self-impedance $Z_{self}$, which is the terminal emf induced due to a unit current impressed at the reference element terminals. First, we compute $Z_{self}$ according to the formal definition, which entails evaluating the surface integral in equation (1.2) exactly. Next we consider the thin filament model and determine the equivalent wire radius which will yield the same result.

In this appendix we take the viewpoint that the array under consideration is one of $M$ subsectional arrays (recall that an array modelled with $M$ current modes “generates” $M$ subsectional arrays) It will be understood that the terms array, element, reference terminals and their attributes refer to subsectional arrays, unless otherwise stated.

Our present interest is in computing the diagonal elements $Z_{kk}$ (i.e., the self-terms) of the array impedance matrix. The results are also applicable to the superdiagonal terms (corresponding to adjacent overlapping PWS current modes) if more than one PWS mode per array is used. Each subsectional array is envisioned as having its own set of reference terminals, which, except for the center set (corresponding to the actual physical array terminals), are normally short-circuited. They may be open-circuited, however, for the purpose of computing the
impedance matrix elements $Z_{nm}$ and so are a useful mathematical artifice.

The array parameters are as follows: the interelement spacing is $D_z$, the wire radius is $a$, and the element length is $2\Delta L$. The array linear phase slope is $\beta S_z$. The array reference element terminals arrays are located at the origin of a cylindrical coordinate system. In this chapter (and throughout this paper) all linear dimensions are given in wavelengths referred to the local medium (here: free-space) at a given but otherwise arbitrary design frequency which we designate as $f_0$. All frequencies are normalized with respect to $f_0$.

C.1.1 Exact Integration Method

For vertical elements, one may show (see Appendix A) that the axial component of the electric field obtained from equation (1.2) reduces to

$$E_z(z) = \frac{Z_o}{j2\pi\beta} \int_{-\Delta L}^{\Delta L} \int_{0}^{2\pi} f(z') \left[ \frac{\partial^2}{\partial z'^2} + \beta^2 \right] G_{\pi_o}(R) \, dz' \, d\phi'$$

(C.2)

where $Z_o = \sqrt{\frac{\mu_o}{\epsilon_o}}$ is the normalized PWS current distribution and where $G_{\pi_o}$ is the free-space periodic Green's function. We assume that the tube current density is of the form

$$I(z) = \frac{1}{2\pi r_w} \frac{\sin \beta (\Delta L - |z|)}{\sin \beta \Delta L} = \frac{1}{2\pi r_w} f(z)$$

(C.3)

so that $I(0) = (2\pi r_w)^{-1}$, corresponding to a one ampere total terminal current.

The free-space periodic Green's function evaluated at the wire surface is given by (see Appendix A):

$$G_{\pi_o}(R) = \frac{1}{4jD_z} \sum_{n=-\infty}^{\infty} e^{-j\kappa_z (z-z')} H_0^{(2)}(\kappa_\rho |\bar{\rho} - \bar{\rho}'|)$$

(C.4)

Here, $\kappa_z = \beta u_z$. $u_z$, the normalized axial wavenumber is given by

$$u_z = \left( S_z + \frac{n\lambda}{D_z} \right)$$

168
for \( n = 0, \pm 1, \pm 2, \ldots \). Also, \( \kappa_{\rho n} = \beta u_\rho \), where the normalized radial wavenumber \( u_\rho = \sqrt{1 - u_z^2} \). For \( u_z > 1 \), \( u_\rho \) is taken to be negative imaginary in order that the fields vanish as \( \rho \) approaches infinity (\( e^{jwt} \) time convention). Lastly, \( \vec{\rho} \) and \( \vec{\rho}' \) are the radial position vectors of the field and source points on the wire surface with

\[
R = |\vec{\rho} - \vec{\rho}'| = \sqrt{2r_w} \sqrt{1 - \cos(\phi - \phi')} = 2r_w \sin \left( \frac{\phi - \phi'}{2} \right)
\]  
(C.5)

Hence,

\[
E_z(z) = -\frac{1}{2\pi} \frac{\beta Z_0}{4D_z} \sum_{n=-\infty}^{\infty} u_\rho^2 \int_{-\Delta L/2}^{\Delta L/2} e^{-j\kappa_z(z-z')} f(z') \sin \left( \frac{\phi - \phi'}{2} \right) d\phi' dz'
\]  
(C.6)

Since \( E_z \) cannot depend on \( \phi \), we have set \( \phi' \) equal to zero in writing equation (C.6). We have also used the fact that

\[
\left[ \frac{\partial^2}{\partial z^2} + \beta^2 \right] G_{\pi_0}(R) = (\beta u_\rho)^2 G_{\pi_0}(R)
\]  
(C.7)

We consider first the integral on \( \phi' \). Let

\[
\Omega_n = \int_0^{2\pi} H_0^{(2)}(2\kappa_\rho r_w \sin \frac{\phi'}{2}) d\phi'
\]  
(C.8)

Making the change of variables \( x = \sin \phi' \) and using the identities [21]:

\[
\int_0^1 \frac{J_0(bx)}{\sqrt{1 - x^2}} dx = \frac{\pi}{2} \left[ J_0 \left( \frac{b}{2} \right) \right]^2
\]  
(C.9)

\[
\int_0^1 \frac{Y_0(bx)}{\sqrt{1 - x^2}} dx = \frac{\pi}{2} \left[ J_0 \left( \frac{b}{2} \right) Y_0 \left( \frac{b}{2} \right) \right]
\]  
(C.10)

\[
\int_0^1 \frac{K_0(bx)}{\sqrt{1 - x^2}} dx = \frac{\pi}{2} \left[ J_0 \left( \frac{b}{2} \right) K_0 \left( \frac{b}{2} \right) \right]
\]  
(C.11)

where \( K_0(x) = \frac{\pi}{2j} H_0^{(2)}(-jx) \), for \( x > 0 \).
we find that (C.8) can be expressed in closed form with the result that

\[ \Omega_n = \frac{\pi}{2} J_0(\kappa \rho r_w) H_0^{(2)}(\kappa \rho_n r_w), n = 0, \pm 1, \pm 2, \ldots \]  

(C.12)

\[ \begin{cases} 
\frac{\pi}{2} J_0(\kappa \rho r_w) H_0^{(2)}(\kappa \rho r_w), & n = 0 \\
I_0(-j \kappa \rho r_w) K_0(-j \kappa \rho r_w), & n = \pm 1, \pm 2, \ldots 
\end{cases} \]  

(C.13)

Equation (C.13) follows from (C.12) on the assumption that the element spacing \( D_z \) is sufficiently small such that no grating lobes propagate, i.e., \( \kappa \rho_n \) is negative imaginary for \( |n| > 0 \).

Returning now to equation (C.6), one finds after performing the \( z' \) integration that

\[ E_z(z) = -\frac{Z_o \beta \rho_o}{4 D_z} \sum_{n=-\infty}^{\infty} e^{-j \kappa z z'} u_p^2 P_n J_0(\kappa \rho r_w) H_0^{(2)}(\kappa \rho r_w) \]  

(C.14)

where we define the element pattern factor

\[ P_n = \frac{1}{\rho_o} \int_{-\Delta L/2}^{\Delta L/2} f(z) e^{i \kappa z z'} dz = \frac{2}{\beta \rho_o(1 - u_2^2)} \left[ \frac{\cos \beta u_2 \Delta L - \cos \beta \Delta L}{\sin \beta \Delta L} \right] \]  

(C.15)

The self-impedance is found by integrating the product of the tangential electric field (C.14) with the normalized PWS current distribution \( f(z) \) over the element length and multiplying by -1:

\[ Z_{\text{self}}^o = -\int_{\Delta L/2}^{\Delta L/2} E_z(z) f(z) \, dz \]  

(C.16)

with the result that

\[ Z_{\text{self}}^o = \frac{Z_o (\beta \rho_o)^2}{4 \beta D_z} \sum_{n=-\infty}^{\infty} u_p^2 P_n J_0(\kappa \rho r_w) H_0^{(2)}(\kappa \rho r_w) \]  

(C.17)

Equation (C.17) gives the array self-impedance found by exact integration. (The superscript "o" refers to the fact that the array is radiating in free space).
C.1.2 Equivalent Filament Method

We now replace the current tube with a current filament. We imagine an identical filament array, which we designate as the test array, positioned parallel with and displaced from the axis of original array (the reference array) by \( r_{eq} \), the equivalent wire radius. The reference element of the test array will be referred to as the test element. The axial position of the test element is the same as that of the reference element of the reference array. We call this model the axial filament model.

The equivalent wire radius is that displacement for which the mutual reactance equals the array self-reactance at some frequency, say, \( f_0 \). In the context of the axial filament model, the array self-impedance is taken to be the open-circuit voltage induced across the terminals of the test element when one ampere of current is impressed at the reference terminals of the reference array. The axial component of the electric field evaluated at the test element is

\[
E_z(z) = -\frac{Z_0 \beta \rho_0}{4D_z} \sum_{n=-\infty}^{\infty} u_\rho P_n H_n^{(2)}(\beta u_\rho r_{eq})
\]  

(C.18)

The open-circuit voltage induced at the test element is the negative of the integral of the electric field (C.18) times the PWS current distribution (C.3) Evaluation of the integral essentially adds a multiplicative factor \(-\rho_0 P_n\) to the previous expression (C.18). The self-impedance, according to the axial filament model, is then given by

\[
\tilde{Z}_{self}^o = \frac{Z_0 (\beta \rho_0)^2}{4\beta D_z} \sum_{n=-\infty}^{\infty} (u_\rho P_n)^2 H_n^{(2)}(\beta u_\rho r_{eq})
\]  

(C.19)

The tilde notation distinguishes the axial filament model impedance expression from the expression obtained by exact integration (equation (C.17)). As before, the “o” superscript means the array is radiating in free space.
C.1.3 Solution for the Equivalent Wire Radius

It is obvious upon comparing equations (C.17) and (C.19) that there is no equivalent wire radius which will result in an equality in both the real and imaginary parts of each expression. There will be in fact two equivalent radii, corresponding to the real and imaginary parts, respectively.

First, consider the self-resistance: We make the usual assumption that the array element spacing is sufficiently small so that no grating lobes propagate, i.e., $u_p$, the normalized radial wavenumber, is real for $n = 0$ and negative imaginary for $n = \pm 1, \pm 2, \ldots$, where $n$ is the Floquet mode index. From equations (C.8) and (C.17) we have

$$R_{\text{self}}^0 = \Re \left\{ \tilde{Z}_{\text{self}}^0 \right\} = \frac{Z_0 (\beta \rho_0)^2}{4 \beta D_z} u_p^2 P_0^2 \left[ J_0 (\beta u_p r_w) \right]^2 \quad (C.20)$$

The real part of (C.19) is equal to

$$\hat{R}_{\text{self}}^0 = \Re \left\{ \tilde{\varphi}_{\text{self}}^0 \right\} = \frac{Z_0 (\beta \rho_0)^2}{4 \beta D_z} (u_p P_0)^2 J_0 (\beta u_p r_{eq}) \quad (C.21)$$

Using the small argument approximation $J_0(x) \simeq 1 - \frac{x^2}{4}$ in (C.20) and (C.21) and equating the resulting expressions, one finds that

$$\frac{r_{eq}}{r_w} \simeq \sqrt{2} \left[ 1 - \left( \frac{\kappa r_w}{4} \right)^2 \right] \quad (C.22)$$

Thus, for thin wires, the equivalent radius for the self-resistance is about 1.4 times larger than the physical wire radius. This is the same result obtained by Imbriale for a single linear dipole. The self-resistance is insensitive to wire radius so that the large percentage difference between the physical and equivalent wire radii for self-resistance is of no consequence. In contrast, the self-reactance is sensitive to wire radius (except near resonance). It is acceptable to determine the equivalent
wire radius for self-reactance and use this value in computing the self-resistance as well.

Let us now consider the self-reactance. The self-reactance $X_{\text{self}}^0$ is given by the Floquet wave series

$$X_{\text{self}}^0 = \Im \{Z_{\text{self}}^0\}$$

$$= - \frac{Z_o(\beta \rho_o)^2}{4\beta D_z} \sum_{n=-\infty}^{\infty} u_\rho^2 P_n^2 J_0(\kappa_{\rho r_w}) Y_0(\kappa_{\rho r_w})$$

$$= \frac{Z_o(\beta \rho_o)^2}{4\beta D_z} \frac{2}{\pi} \left[ -\frac{\pi}{2} u_\rho^2 P_0^2 J_0(\kappa_{\rho r_w}) Y_0(\kappa_{\rho r_w}) \right]$$

$$+ \sum_{n=-\infty}^{\infty} u_\rho^2 P_n^2 I_0(-j\kappa_{\rho r_w}) K_0(-j\kappa_{\rho r_w})$$

Note that in the last equation, each Bessel and modified Bessel function has a positive real argument. For the equivalent filament model, one finds from (C.19) that

$$\tilde{X}_{\text{self}}^0 = \Im \{\tilde{Z}_{\text{self}}^0\}$$

$$= - \frac{Z_o(\beta \rho_o)^2}{4\beta D_z} \sum_{n=-\infty}^{\infty} u_\rho^2 P_n^2 Y_0(\kappa_{\rho r_{eq}})$$

$$= \frac{Z_o(\beta \rho_o)^2}{4\beta D_z} \frac{2}{\pi} \left[ -\frac{\pi}{2} u_\rho^2 P_0^2 Y_0(\kappa_{\rho r_{eq}}) + \sum_{n=-\infty}^{\infty} u_\rho^2 P_n^2 K_0(-j\kappa_{\rho r_{eq}}) \right]$$

The series (C.25) is the same as the series (C.23) except for an additional factor $J_0(\kappa_{\rho r_w})$ multiplying each term of the latter. Clearly, $r_{eq}$ tends to $r_w$, as $r_w$ tends to zero, since $J_0(0) = 1$. Setting the difference $X_{\text{self}}^0 - \tilde{X}_{\text{self}}^0$ equal to zero, one may solve for the root $r_{eq}$ numerically, using the bisection method for example.

The equivalent wire radius $r_{eq}$ approaches the physical wire radius $r$ from below with increasing element length $\Delta L$ until $\Delta L$ gets close to a half-wavelength as
Figure 51: Equivalent wire radius as a function of dipole length for linear array of vertical dipoles radiating in free space
shown in figure 51. Near resonance, \( r_{eq} \) becomes unstable. Evidently, the equivalent wire radius approaches the physical wire radius from above, above resonance, and from below, below resonance. The sensitivity of the self-reactance to wire radius tends to zero at resonance (see derivative curve, figure 51) and so the wild behavior of \( r_{eq} \) near resonance is plausible. Since the self-reactance is insensitive to the wire radius near resonance, this behavior is of no physical consequence. As seen in figure 51, Imbriale's two-term approximation for \( r_{eq} \) essentially gives an "average" value near resonance. Notice the close agreement between Imbriale's two-term \( r_{eq} \) approximation for a single dipole and our own results based on a periodic linear array at least below resonance.

For short elements, we observe in figure 52 a significant discrepancy between the self-reactance (based on one PWS current mode) obtained by exact integration and the axial filament model, respectively. (The curve for the axial filament model data was obtained by setting the equivalent wire radius equal to the physical wire radius). This discrepancy demonstrates that for very short elements, a significant error in the self-reactance may result if one assumes \( r_{eq} = r_w \).

Next, let us turn our attention to figures 53 and 54 which show the self-impedance as a function of frequency with the number of PWS current modes as a parameter with the wire radius set at .001\( \lambda \) and .015\( \lambda \), respectively. The figures show that the magnitude of the error

\[
\epsilon_Z = Z_{in}^o - \tilde{Z}_{in}^o
\]

increases with increasing wire radius. This occurs because the difference between the optimal wire radius and the physical wire radius grows as the wire radius increases (see equation (C.22)).

175
Figure 52: Comparison of self-reactance using the axial filament approximation versus the exact integration method

\[ r_w = 0.015 \lambda \]
\[ D_z = 0.6 \lambda \]
\[ S_z = 0. \]
Figure 53: Self impedance of periodic linear array of vertical dipoles as a function of frequency; comparison of exact integration and axial filament methods; number of PWS current modes as a parameter; wire radius = .001\(\lambda\) at \(f_0\).
Figure 54: Self impedance of periodic linear array of vertical dipoles as a function of frequency; comparison of exact integration and axial filament methods; number of PWS current modes as a parameter; wire radius = .015\lambda at f_0
We also observe that at a given frequency, $|\xi_Z|$ increases as the number of PWS current modes is increased. As the number of current modes is increased, the length of each current subsection decreases so that the error incurred by setting $r_{eq} = r_w$ model to fill the impedance matrix is magnified. (For the curves shown in figures 53 and 54, the axial filament model was employed in computing all impedance matrix elements including the self-terms.)

Lastly, we note that for a given number of PWS modes, the error $\xi_Z$ is greater in the higher frequency range (above resonance). In general, more PWS current modes are needed at frequencies above resonance due to the fact that the current distribution has a more complicated distribution with a rapidly varying phase (especially near the terminal region). The magnitude and phase of the current distribution on the reference element with the number of PWS current modes as a parameter for the low, mid and high frequency cases is plotted in figures 55, 56 and 57. For these curves, the wire radius was set at $.001\lambda$ at $f_o$. The contrasting fat ($r_w = .015\lambda$) wire case exhibits a larger phase variation along the length of the dipole (see figures 58, 59 and 60). In employing the axial filament model, the benefit of adding more current modes is at some point (depending on the wire radius) offset by the error incurred by setting $r_{eq} = r_w$. Furthermore, the result obtained by the axial filament model is bound to diverge, according to Collin, once the number of modes exceeds a threshold value, which may be very low for element aspect ratios significantly less than 100. We do not offer any quantitative threshold value here; Collin’s analysis was based on a trigonometric (Fourier series) expansion of the current distribution rather than a PWS expansion.

If one attempts to add more than 5 PWS current segments in the thick-wire case, the impedance begins to diverge. This divergence is inferred from the observation that the seven mode impedance curves (not shown in figures) are
Figure 55: Magnitude and phase of current distribution on reference element of a periodic linear array of axial dipoles; 1, 3 and 5 PWS current modes; freq=0.8f₀; wire radius = .001λ
Figure 56: Magnitude and phase of current distribution on reference element of a periodic linear array of axial dipoles; 1, 3 and 5 PWS current modes; freq = f₀; wire radius = 0.001λ
Figure 57: Magnitude and phase of current distribution on reference element of a periodic linear array of axial dipoles; 1, 3 and 5 PWS current modes; freq=1.2f₀; wire radius = .001λ
Figure 58: Magnitude and phase of current distribution on reference element of a periodic linear array of axial dipoles; 1, 3 and 5 PWS current modes; 
freq=0.8f₀; wire radius = .015λ
Figure 59: Magnitude and phase of current distribution on reference element of a periodic linear array of axial dipoles; 1, 3 and 5 PWS current modes; freq = $f_0$; wire radius = 0.015λ
Figure 60: Magnitude and phase of current distribution on reference element of a periodic linear array of axial dipoles; 1, 3 and 5 PWS current modes; freq=$1.2f_0$; wire radius = $0.015\lambda$
further from the five mode values than the five mode values are from the three mode values. In contrast, there is no discernable difference between the equivalent filament model values and those obtained by the exact integration method for the thin-wire case \( r_w = 0.001\lambda \). Although there is a significant difference between the one segment and three segment input impedances, the three and five segment solutions are in good agreement, an indication of satisfactory convergence.

C.2 Equivalent Wire Radius for a Linear Array of Vertical Dipoles with Ground Plane

Our objective in this study is two-fold. First, we show that the equivalent wire radius is not significantly altered by the presence of a conducting body in close proximity to the array (even though the array self-impedance is obviously affected). Secondly, we introduce and justify the concept of impedance averaging in the context of periodic linear arrays radiating in an inhomogeneous space. For example, the field radiated by an periodic linear array of axial dipoles in the presence of an off-axis inhomogeneity no longer exhibits azimuthal symmetry about the array axis. If we replace the element current tube with the axial filament model the coupling between the reference array and the test element will clearly depend on the azimuthal position of the test element. This problem is resolved by averaging the coupling as the test element is rotated about the reference array.

The more general case involving linear arrays of curved dipoles radiating in the presence of a dielectric layered cylinder is too difficult to analyse exactly in terms of thin wire theory. Instead, we examine a much simpler case which still incorporates the essential issues: a periodic linear array of axial dipoles radiates in the presence of an infinite ground plane, displaced \( b \) meters from the array. The dipoles have a wire radius \( a \). We will tentatively (and incorrectly) assume that the
wire is sufficiently thin so that the tube current has only an axial component with no azimuthal variation.

For the moment, let us imagine that the ground plane is not present. One may show that the expression for self-impedance (equation (C.17)) using the exact integration method (current tube model) may be obtained from an alternative viewpoint: we decompose the current tube into an infinite ensemble of current filament arrays each of length $\Delta L$ center fed with a current density equal to $\frac{1}{2\pi r_w}$, so that the total array input current is one ampere. Let $V$ be the voltage across the array terminals (the same for every filament array). For a particular filament, say filament $k$, this potential is equal to the superposition of voltages induced at the reference terminals of the $k^{th}$ filament array due to the currents flowing on each ensemble member (including the $k^{th}$ array). This potential is equal to the integral of the total electric field $E_z$ times $f(z)$, the normalized PWS current distribution on the $k^{th}$ filament, along the element. This electric field then is the sum of field contributions from each ensemble member evaluated at the $k^{th}$ filament. One concludes after some thought that this field is given by equation (C.6). Since we assumed a unit total input current, the self-impedance $Z_{self}^o$ equals $V$ and is given by

$$Z_{self}^o = \int_{-\Delta L/2}^{\Delta L/2} E(z)f(z) \, dz$$

(C.26)

which leads to the previous result (C.17). Through an interchange of summations and integrals in equations (C.6), (C.16) and (C.17) we may rewrite $Z_{self}^o$ in the form

$$Z_{self}^o = \frac{1}{2\pi} \int_0^{2\pi} Z(C) \, d\phi$$

(C.27)

where $C = 2r_w \sin \left( \frac{\Delta \phi}{2} \right)$ is the length of the chord joining two filament elements separated by an angular distance $\Delta \phi$ on the wire surface and where according to
(C.17) together with (C.6) and (C.16)

\[ Z(C) = \frac{(\beta \rho_0)^2 Z_0}{4\beta D_2} \sum_{n=-\infty}^{\infty} (u_n P_n)^2 H_0^{(2)}(\kappa_n C) \]  

(C.28)

Thus, we may view the array self-impedance as the average mutual impedance between filament arrays comprising the element current tube. This viewpoint was originally introduced by Schellkunoff [22] in the context of a single linear dipole radiating in free-space and was also adopted by Imbriale [7] in his article on the applications of the method of moments to thin-wire dipoles. Returning now to the situation at hand, we replace the array plus ground plane combination with the imaged equivalent configuration as shown in figure 61. \( R' \) and \( R'' \) denote the positions of of corresponding filament arrays on the wire surface and its image, respectively. \( R \) is the position of a field point on the surface of the wire. \( C' \) and \( C'' \) are the the distances between \( R \) and the two filament arrays at \( R' \) and \( R'' \). The coupling to a filament at \( R \) due to all the filaments (including the image filaments) on the wire surface is obtained according to (C.27) by integrating on \( \phi' \), the azimuthal position of \( R' \). (We imagine that as \( \phi' \) varies, the corresponding point \( R'' \) on the image wire also moves so that \( R' \) and \( R'' \) maintain the same position relative to their respective wire axes.)

\[ Z(R) = \frac{1}{2\pi} \int_0^{2\pi} [Z(C') - Z(C'')] \, d\phi' \] 

(C.29)

(The minus sign on the second term is due to the fact that the current on the image array is oppositely signed).

The expression in equation (C.29) cannot be the array self-impedance since \( Z(R) \) varies with \( R \) (unless the ground plane is infinitely far away, in which case the term \( Z(C'') \) vanishes and (C.29) reduces to (C.27)).

The difficulty here stems from our contradictory assumptions that each filament member has the same terminal voltage and the same input current. But
Figure 61: Current tube geometry with image current tube; equivalent axial filament model with image
the image wire introduces an asymmetry which requires that the filament currents are unequal about the wire perimeter. For thin wires, the filament currents are approximately equal (i.e., the current is approximately uniform) and we take the self-impedance to be the average value

\[
Z_{self} = \frac{1}{2\pi} \int_0^{2\pi} Z(R) \, d\phi
\]

\[
= \left(\frac{1}{2\pi}\right)^2 \int_0^{2\pi} \int_0^{2\pi} \left[Z(C') - Z(C'')\right] \, d\phi' \, d\phi
\]

\[
= Z_{self}^0 - \left(\frac{1}{2\pi}\right)^2 \int_0^{2\pi} \int_0^{2\pi} Z(C'') \, d\phi' \, d\phi
\]  

(C.30)

where \(Z_{self}^0\) is the self-impedance of the array radiating in free-space as given by equation (C.27):

\[
Z_{self}^0 = \frac{Z_0(\beta \rho_0)^2}{4\beta D_z} \sum_{n=-\infty}^{\infty} u_\rho^2 P_n^2 J_0(\kappa \rho a) H_0^{(2)}(\kappa \rho r_w)
\]

(C.31)

and \(Z(C'')\) is given by

\[
Z(C'') = \frac{Z_0(\beta \rho_0)^2}{4\beta D_z} \sum_{n=-\infty}^{\infty} u_\rho^2 P_n^2 H_0^{(2)}(\beta u_\rho C'')
\]

(C.32)

(see equation (C.19) where the right side of (C.19) is interpreted now as the mutual impedance between two infinitesimally thin filament arrays). In summary, equation (C.30) represents the self-impedance of an array of axial directed elements backed by an infinite ground plane using the current tube model. This expression is based on the exact integration method in conjunction with the current tube model.

We now consider an equivalent thin filament model. We replace the wire and its ground plane image by two infinitesimally thin filaments (figure 61) located at the wire axis and its image axis, respectively. In terms of this model, the array self-impedance is approximated as the average sum of the coupling between the filament (located on the wire axis) and a test array located at \(R\) and between the
image filament (located on the axis of the image wire) and the test array:

\[
\tilde{Z}_{\text{self}} = \tilde{Z}_{\text{self}}^0 - \frac{1}{2\pi} \int_0^{2\pi} Z(C) \, d\phi \tag{C.33}
\]

where \(\tilde{Z}_{\text{self}}^0\) is the self-impedance (using the thin filament model, see equation (C.19)) of the array radiating in free space.

\[
\tilde{Z}_{\text{self}}^0 = \frac{Z_\alpha \beta}{4D_z} \sum_{n=-\infty}^{\infty} (u_\rho P_n)^2 H_0^{(2)}(\beta u_\rho r_{eq}) \tag{C.34}
\]

The equivalent wire radius, \(r_{eq}\), is that value for which expressions (C.30) and (C.33) are equal (it is sufficient to find \(r_{eq}\) such that the imaginary parts are equal and use that value for the self-resistance as well). From figure 49b, it is clear that for \(\frac{r_{eq}}{2b} \ll 1\)

\[
\frac{1}{2\pi} \int_0^{2\pi} Z(C''') \, d\phi' \simeq Z(C)
\]

so that

\[
\left(\frac{1}{2\pi}\right)^2 \int_0^{2\pi} \int_0^{2\pi} Z(C''') \, d\phi' \, d\phi \simeq \frac{1}{2\pi} \int_0^{2\pi} Z(C) \, d\phi \tag{C.35}
\]

But for thin dipoles

\[
\frac{1'}{2\pi} \int_0^{2\pi} Z(C) \, d\phi \simeq Z(2b) \tag{C.36}
\]

as is apparent from a study of figure 49b.

Substituting result (C.36) into (C.33) and also (C.35) into (C.30), we see that

\[
Z_{\text{self}} - \tilde{Z}_{\text{self}} \simeq Z_{\text{self}}^0 - \tilde{Z}_{\text{self}}^0 \tag{C.37}
\]

Therefore the equivalent wire radius is unaffected by the presence of the ground plane (unless the ground plane separation is on the order of a wire radius), as claimed. Of course, the self-impedance is significantly changed from the free space value since the coupling \(Z(2b)\) from the image array may be large even when \(2b\) is several wavelengths.
C.3 Generalizations

Suppose now the array is radiating in the presence of an infinite perfectly conducting cylinder. The scattered field at the array due to induced currents on the cylinder (including creeping wave contributions) will be not be significantly greater in magnitude than that scattered by a ground plane at the same distance (indeed it may be weaker). Therefore, we do not expect any significant change in the equivalent radius from the free space value.

If the array is embedded in a cylindrical dielectric layer enclosing the cylinder we do not expect any change in equivalent wire radius for essentially the same reason (provided that the array is not too close to the air-dielectric interface), except that all electrical lengths must be scaled by the local index of refraction.

In summary, each element in the array behaves like an isolated dipole radiating in a homogeneous medium provided that all scattering bodies and dielectric discontinuities are at least several wire radii away and provided that the vertical interelement spacing $D_z$ is much greater than a wire radius. If these conditions are satisfied, then one might reasonably infer that the equivalent filament model can be used for the general case of an arbitrarily oriented curved array element provided that the radius of curvature of the wire is much greater than the wire radius. If the array element (i) is curved and/or non-axial or (ii) is radiating in an inhomogeneous environment one needs to "impedance average".

An explicit description of the impedance averaging procedure is described in appendix D; the basic steps are as follows: One constructs a circle lying in the plane perpendicular to the tangent to the dipole at the reference terminals. The radius of this circle is the equivalent wire radius, which, for an arbitrary element, one sets equal to the physical wire radius as a first approximation. The test array
is rotated about this circle. At pre-selected points one computes the array mutual impedance between the reference and test arrays. The array self-impedance is taken to be the average of these values.
APPENDIX D
MECHANICS OF IMPEDANCE AVERAGING

In appendix C, we show that for thin-wire elements, the array self-impedance may be approximated as the average mutual impedance between two arrays ("test" and "reference" arrays) of filament elements as the test array is moved around a circle whose radius equals the physical wire radius. In this section, we describe the mechanics of this averaging process as well as the effects of the dielectric and array scan on the coupling, as a function of the test array position on the averaging circle.

Suppose the (sub)array reference element terminals are located at \( \bar{R}_o = (\rho_o, \phi_o, z_o) \). In order to define the plane containing the averaging circle, it is convenient to define a local cartesian coordinate system whose origin is at \( \bar{R}_o \). The basis vectors for this local coordinate system are comprised of \( \hat{p} \), the unit vector tangent to the element at the terminals and two other vectors \( \hat{n}_1 \) and \( \hat{n}_2 \) which lie in a plane perpendicular to \( \hat{p} \). From section 2.2, we recall that

\[
\hat{p} = \frac{1}{\sqrt{1 + \zeta^2}} [\hat{\phi} + \zeta \hat{\xi}] \quad \text{(D.1)}
\]

We define the unit vector

\[
\hat{n}_1 = -\frac{1}{\sqrt{1 + \zeta^2}} [\zeta \hat{\phi} - \hat{\xi}] \quad \text{(D.2)}
\]

so that \( \hat{n}_1 \cdot \hat{p} = 0 \). We also define

\[
\hat{n}_2 = \hat{p} \times \hat{n}_1 = \hat{\rho}. \quad \text{(D.3)}
\]

194
Then \((\hat{p}, \hat{n}_1, \hat{n}_2)\) comprise an orthonormal vector set. If \(r_w\) is the equivalent wire radius (which, for thin wires, we may assume equals the physical wire radius) then the averaging circle \(C\) may be parameterized by

\[
\bar{R}_C = \bar{R}_o + \bar{R}_w(t) \tag{D.4}
\]

where

\[
\bar{R}_w(t) = r_w(\cos t \hat{n}_2 + \sin t \hat{n}_1), \quad 0 \leq t \leq 2\pi \tag{D.5}
\]

Converting to rectangular coordinates, we have

\[
\bar{R}_o = \rho_o [\cos \phi_o \hat{x} + \sin \phi_o \hat{y} + z_o \hat{z}] \tag{D.6}
\]

\[
\hat{n}_1 = -\frac{1}{\sqrt{1 + \zeta^2}} \left[ \zeta \hat{x} - \hat{z} \right] \tag{D.7}
\]

\[
\hat{n}_2 = \hat{\rho} = \cos \phi_o \hat{x} + \sin \phi_o \hat{y} \tag{D.8}
\]

so that the position vector \(R_C\) of a point on the averaging circle is given by

\[
\bar{R}_C = R_x \hat{x} + R_y \hat{y} + R_z \hat{z} \tag{D.9}
\]

\[
= \left[ \rho_o \cos \phi_o + r_w \frac{\zeta}{\sqrt{1 + \zeta^2}} \sin t \sin \phi_o + r_w \cos t \cos \phi_o \right] \hat{x} + \left[ \rho_o \sin \phi_o - r_w \frac{\zeta}{\sqrt{1 + \zeta^2}} \sin t \cos \phi_o + r_w \cos t \sin \phi_o \right] \hat{y} + \left[ z_o + \frac{r_w}{\sqrt{1 + \zeta^2}} \sin t \right] \hat{z}
\]

Finally, returning to cylindrical coordinates we have

\[
\bar{R}_C = R_\rho \hat{\rho} + R_\phi \hat{\phi} + R_z \hat{z} \tag{D.10}
\]

195
where

\[ R_\rho = \sqrt{R_x^2 + R_y^2} \]  \hfill (D.11)

\[ R_\phi = \tan^{-1} \frac{R_y}{R_x} \]  \hfill (D.12)

\[ R_z = z_0 + \frac{r_w}{\sqrt{1 + \zeta^2}} \sin t \]  \hfill (D.13)

Equations (D.11), (D.12), and (D.13) give the radius, azimuthal position and axial displacement, respectively, of the reference terminals of the test array on the averaging circle. If we construct a horizontal line \( \ell \) passing through the cylinder axis and the reference terminals of the reference array, then the point corresponding to the parameter value \( t = \{0, \pi\} \) is a point on \( \ell \) at a radius \( \rho = \rho_0 \pm r_w \), for every element pitch \( \zeta \). Thus, the points corresponding to \( t = 0 \) and \( t = \pi \) are fixed points defining an axis about which the averaging circle is rotated as the element pitch \( \zeta \) is varied. Increasing \( \zeta \) corresponds to clockwise (counterclockwise) motion on the averaging circle as viewed from above, for positive (negative) values of \( \zeta \).

(In the limiting case, \( \zeta \to 0 \) for directed elements, the circle lies in a vertical plane intersecting the \( z \) axis and we move up from the \( t = 0 \) position).

Figures 62 through 65 show the self-impedance of single array with various element pitches (from vertical to transverse) radiating in the presence of the PEC cylinder. The impedance is plotted as a function of the position of the test array on the averaging circle. The angle parameter \( t \) ranges from 0 to 360 degrees in 20 degree steps.

Let us consider the vertical element case first. We observe that for thin axial dipoles, the locus of impedance values lies on a simple open arc which, to a first approximation, is a straight line segment in the complex \( Z \)-plane. It is clear that in the limiting case as \( a \), cylinder radius tends to zero the arc must degenerate to a single point (assuming the dielectric layer is also absent) because the fields
Figure 62: Self-impedance as a function of position of test array on the averaging circle; vertical element case ($\alpha = 90$ deg.)
Figure 63: Self-impedance as a function of position of test array on the averaging circle; element pitch \( \alpha = 75 \text{ deg.} \)
Figure 64: Self-impedance as a function of position of test on the averaging circle; element pitch \( \alpha = 30 \) deg.
Figure 65: Self-impedance as a function of position of test on the averaging circle; transverse elements (α = 0 deg.)
are azimuthally symmetric with respect to the element axis when the cylinder and dielectric layer are absent.

The arc must be an open arc (i.e., not closed) since for every parameter value \( t_1 \) there is a complementary point \( t_2 = 2\pi - t_1 \) such that the fields at points corresponding to \( t_1 \) and \( t_2 \) are equal; this is obvious from the symmetry of the set-up. Since the impedance arc is quasi-linear and since the point \( t = 90 \) degrees lies roughly at the midpoint of that arc, we may take the self-impedance for vertical elements to be the mutual impedance between the reference and test arrays with the test array positioned at \( t = 90 \) degrees. Figure 62 shows that such a one-point average is reasonable even if the array is scanned and a dielectric layer is present. We note from figures 62—65 that the predominant effect of the dielectric layer is to lower the resonant frequency as expected.

Next, we consider two sloping element cases (figures 63 and 64). If \( S_z = 0 \) (no scan), then the self-impedance trajectory on the averaging circle again lies on a simple open quasi-linear arc, as required by symmetry considerations. Indeed, as shown in section E, we may, for the unscanned case, replace the reference and test arrays by their respective reference elements alone insofar as a determination of impedance symmetries are concerned. Consider two points \( t_1 \) and \( t_2 = 2\pi - t_1 \) on the averaging circle. The relative axial and azimuthal displacement \((\Delta z, \Delta \phi)\) of a test array with reference element at \( t = t_1 \) is equal and opposite to the relative displacement of a test array at \( t = t_2 \). But we have shown in section E that under these conditions, the corresponding mutual impedances are equal. Therefore, to each point \( t_1 \) on the averaging circle there is an image point \( t_2 = 2\pi - t_1 \) at which we obtain the same impedance value. This is true whether or not the dielectric layer is present as proven in section E. However, if the array is scanned, this symmetry is lost and mutual impedance trajectory traces a simple closed curve as
$t$ ranges over 360 degrees.

In the numerical case studies we have undertaken, we find without exception that the impedance trajectory on the averaging circle lies on a smooth ellipse-like curve in the complex plane (which may degenerate to a line segment as noted above). Therefore, for the general sloping element configuration with dielectric present, it is acceptable to take a two-point impedance average evaluated at $t = 0$ and $t = 180$ degrees for example.
APPENDIX E

SYMMETRY RELATIONS

It is an interesting fact that for an arbitrary element orientation the array mutual impedance $Z_{1,2}$ is not in general equal to $Z_{2,1}$, unless the obliquity factor $S_z$ is zero (in which case the equality holds for all element orientations). The consequences of this asymmetry are discussed in section F.

$Z_{1,2}$ may be obtained from $Z_{2,1}$ by replacing $\Delta z$ and $\Delta \phi$ by $-\Delta z$ and $-\Delta \phi$ in equation (2.30). In fact, if the elements of either array are "obliquely" slanted ($\zeta$ is greater than zero and bounded above) and $S_z$ is not zero (scan case) then the equality generally does not hold. However, if the elements of arrays 1 and 2 are both vertically or both horizontally oriented, then $Z_{1,2} = Z_{2,1}$, even for a non-zero scan, provided that $\Delta z$, the relative axial displacement of the reference terminals is zero (or a multiple of $D_z$). The presence or absence of the dielectric layer has no bearing on these symmetry relations.

Indeed, if $\zeta_1 = \zeta_2 = \zeta = \infty$ (vertical element case), equation (2.30) reduces to

$$Z_{2,1} = -\frac{(\beta p_1)(\beta p_2)Z_d}{4\beta D_z} \sum_{n=-\infty}^{\infty} p_n e^{-j\kappa_z \Delta z} \sum_{m=0}^{\infty} \epsilon_m R_{nm}^z e^{-j\zeta} \cos m\Delta \phi \quad (E.1)$$

where $\epsilon_m = \left\{ \begin{array}{ll} 1, & m = 0 \\ 2, & m > 0 \end{array} \right.$ As noted earlier, for $\zeta = \infty$, the "plus" and "minus" functions are equal and the pattern function is a function of $n$ but not $m$. Also, the pattern functions for arrays 1 and 2 are the same. For this reason we have used a simpler notation in (E.1) for the pattern factors. In section 3.3 of chapter III,
we show that $R_{nm}^{\hat{z} \rightarrow \hat{z}}$ and $R_{nm}^{\hat{\phi} \rightarrow \hat{\phi}}$ are even functions of $n$ if and only if the array scan is zero. Therefore if $S_z = 0$, then $\kappa_z = \beta D_z$ and we obtain the still simpler expression

$$Z^{2,1} \mid \zeta = \infty \quad \zeta = 0 = \frac{(\beta \rho_1)(\beta \rho_2)}{4\beta D_z} \sum_{n=0}^{\infty} \epsilon_n P_n^2 \cos \frac{2\pi n \Delta z}{D_z} \sum_{m=0}^{\infty} \epsilon_m$$

$$\cdot R_{nm}^{\hat{z} \rightarrow \hat{z}}(\rho) \cos m \Delta \phi$$

(E.2)

Since this expression is even in $\Delta z$ and $\Delta \phi$, it follows that $Z^{2,1} = Z^{1,2}$. For the non-scan case, we see from (E.1) that $Z^{1,2} \mid \zeta = \infty = Z^{2,1} \mid \zeta = \infty$ only if $\Delta z = 0$.

By a similar analysis, one may show that $Z^{2,1} \mid \zeta = 0$ exhibits the same symmetry behavior under the same conditions as that of $z$ directed elements.

Finally, we observe that for the general case for which $0 < \zeta, \leq \infty$, the terms involving $R_{nm}^{\hat{z} \rightarrow \hat{z}}$ and $R_{nm}^{\hat{\phi} \rightarrow \hat{\phi}}$ will come into play. These coefficients are in general neither even nor odd in $n$. Furthermore, $P_n^+(m) \neq P_n^-(m)$, so that no symmetry obtains for the general slant case unless the scan is zero. In the latter case, we might expect that $Z_{2,1} = Z_{1,2}$ based on physical reasoning; for the zero scan case, two arrays may be replaced by their respective reference elements alone insofar as a determination of symmetry relations are concerned. We can show this directly from equation (2.30). An examination of equation (2.24) reveals that if $S_z = 0$ then $P_n(m) = P_{-n}(m)$ for any $\zeta$. We also need the facts that $R_{nm}^{\hat{z} \rightarrow \hat{z}}$ and $R_{nm}^{\hat{\phi} \rightarrow \hat{\phi}}$ are odd functions of $n$ for the no-scan case, as shown in section 3.3. ($R_{nm}^{\hat{z} \rightarrow \hat{z}}$ and $R_{nm}^{\hat{\phi} \rightarrow \hat{\phi}}$ are even in $n$ for no-scan, as previously noted). We may therefore, rewrite (2.30), summing on $n$ from zero to $\infty$:

$$Z^{2,1} \mid S_z = 0 = -\frac{Z_d}{\sqrt{1 + \zeta^2}} \sum_{n=0}^{\infty} \epsilon_n \cos \kappa_z \Delta z \sum_{m=0}^{\infty} \epsilon_m \left[ \zeta_1 \zeta_2 R_{nm}^{\hat{z} \rightarrow \hat{z}} + R_{nm}^{\hat{\phi} \rightarrow \hat{\phi}} \right]$$

204
\begin{equation}
\cdot \left[ P_n^{(1)+}(m)P_n^{(2)+}(m) + P_n^{(1)-}(m)P_n^{(2)-}(m) \right] \cos m\Delta \phi + \\
\sum_{n=0}^{\infty} \epsilon_n \sin \kappa \Delta z \sum_{m=0}^{\infty} \left[ \zeta_2 R_{nm}^{\phi-\phi} + \zeta_1 R_{nm}^{\phi-\phi} \right] \\
\cdot \left[ P_n^{(1)+}(m)P_n^{(2)+}(m) + P_n^{(1)-}(m)P_n^{(2)-}(m) \right] \sin m\Delta \phi \right] \quad (E.3)
\end{equation}

where

\[ C = \frac{(\beta \rho_1)(\beta \rho_2)}{4\beta D_z} \]

Since we obtain the same expression if we replace \( \Delta z \) and \( \Delta \phi \) by \( -\Delta z \) and \( -\Delta \phi \), it follows that \( Z^{1,2} = Z^{2,1} \), as claimed.

For these and other reasons also related to loss of symmetry the numerical calculations for the general scanned slant element case are considerably more time consuming than the vertical or horizontal element cases.

In order to obtain an explicit series representation for the mutual impedance, we need to know the \( R_{nm} \) coefficients introduced in the general expression for the electric field in section 2.3. This in turn entails that we solve two related but independent electromagnetic boundary value problems, the subject of chapter III.
APPENDIX F

STRUCTURE OF THE IMPEDANCE MATRIX

It is desirable to introduce a systematic array element indexing scheme, such that the symmetries inherent in the cylinder-plus-arrays geometry are reflected to advantage in the impedance matrix structure. Accordingly, a particular current mode is classified according to its mode position, tier and array group. These terms are defined as follows: On a particular (physical) array element, the (PWS current) mode position is counted up from the left to the right ends of the element (or from bottom to top in the case of vertical elements) as illustrated in the example shown in figure 66.

Successive arrays (as we move CCW around the cylinder) may be staggered in their axial offset positions in a repeating modulo $J$ sequence, assuming there are $J$ distinct possible offset positions. The first array in each subset of $Y$ arrays is arbitrarily assigned an axial offset of zero. Each such $J$-member array subset is called an array group and is assigned a array group index which increases in order of increasing CCW angle position about the cylinder. The $j^{th}$ member of each array group (ordered CCW) is said to occupy the $j^{th}$ tier position. Thus, in the example shown in figure 66, there are 3 array groups, each comprised of two tiers. There are 3 PWS current modes per element. In figure 66, these current modes are represented as disjointed segments for clarity, though they are in reality overlapping, as in equation (2.5).
Figure 66: Coupling diagram for slant set-up. In this example, there are 3 array groups, 2 tiers, and 3 PWS current modes.
Figure 67: Structure of impedance matrix; 3 array groups, 2 tiers, 3 PWS modes.
Additional levels of structure must be added in the case that distinct arrays differ in other attributes, such as element pitch or length, in addition to the reference element axial offset. The possibilities are endless; only the three level hierarchy described above will be considered explicitly here.

Let us rewrite the governing matrix equation (2.8) in terms of the triple index set described above. Let us further assume that each array has an identical load $Z_{load}$ placed across terminals located at the center of each element (corresponding to mode number $\frac{M+1}{2}$, where $M$ must be an odd integer). The voltage induced across the reference terminals of the subarray corresponding to current mode $m$ of the array on the $j^{th}$ tier in the $q^{th}$ array group is given by

$$qV_{m}^{j} = \sum_{q'=1}^{Q} \sum_{j'=1}^{J} \sum_{m'=1}^{M} q_{q'j'}Z_{mm'}^{j'} q_{j'm'}^{j} + Z_{load} q_{j'm} \delta(m, \frac{M}{2} + 1) \quad (F.1)$$

where $q = 1, \ldots, Q$, $j = 1, \ldots, J$, and $m = 1, \ldots, M$, and $\delta(m, n) = \begin{cases} 1, & m = n \\ 0, & m \neq n \end{cases}$

The impedance matrix (figure 67) corresponding to the set-up in figure 66 exhibits a block-block matrix structure. The interactions between all sub-elements on two (not necessarily distinct) arrays are represented by elementary partition matrices (bounded by dotted lines in figure 67) called primitives. The collective set of interactions between arrays on all possible tiers in a given array group pair is manifested in the block matrices (bounded by solid lines in figure 67) called array blocks and is comprised of a set of primitives. The impedance matrix is comprised of blocks of array blocks, each of which accounts for all interactions between a certain array group pair.

In the current example, each primitive consists of a 3 by 3 matrix corresponding to 3 PWS current modes. Each array block will be a 2 by 2 matrix of primitives corresponding to two tiers for this example. Finally, the composite
impedance matrix is a 3 by 3 matrix of array blocks corresponding to three array groups.

The utility of this indexing system may be demonstrated by considering how various symmetries manifest themselves in the resultant matrix structure. The reader can verify these symmetries through a study of figure 66 in which several coupling terms are illustrated. Note that the arrows point from the reference (source) array to the test array. We will continue the ongoing example of 3 array groups with 2 tiers and 3 current modes per element. Thus, there are six arrays altogether; the impedance matrix consists of 324 elements.

In the present example we will not only assume that each array group is equally spaced in angle about the cylinder but that every individual array is as well. Although it is not necessary to make this assumption, maximum symmetry obtains from this configuration.

CASE 1: VERTICAL ELEMENTS, NO SCAN

This is the case exhibiting maximum symmetry; only 16 of 324 elements in this example are non-redundant (see figure 68). Each array block is Toeplitz; moreover the ensemble of array blocks is block-block Toeplitz. The former result follows from the fact that given two array groups, the set of interactions between elements of the $j^{th}$ and $j''^{th}$ tiers are indistinguishable from the set involving the $(j + q)^{th}$ and the $(j' + q)^{th}$ tiers, respectively. That the impedance matrix exhibits block-block Toeplitz symmetry is obvious since any two array groups differing by the same angular displacement will couple identically. Furthermore, each primitive matrix is also Toeplitz, while each diagonal primitive (corresponding to interactions amongst elements of the same array) is symmetric as well.

CASE 2: VERTICAL ELEMENTS, SCANNED

A loss of redundancy is incurred due to the fact that the mutual coupling
between two subarrays with different axial offsets is not the same if the roles of the reference and test arrays are reversed. Reverting back to single subscript notation for a moment, this means that $Z_{mn} \neq Z_{nm}$ if the arrays are scanned (i.e., $S_z \neq 0$) and if the axial positions of arrays $m$ and $n$ differ. This, of course, does not constitute a violation of reciprocity; the impedances under consideration are array impedances. The claimed result merely reflects the fact that the field radiated by a scanned periodic column array is not axially symmetric. Consequently, the array blocks, while still Toeplitz, are no longer symmetric (see figure 69). The number of non-redundant elements has increased to 30 from 16 due to scan.

CASE 3: TRANVERSE ELEMENTS, NO SCAN

This case is similar to case 1 in terms of symmetry relations (see figure 70). One difference, which becomes significant if the number of arrays exceeds 15 or 20, is the loss of symmetry in the diagonal primitives in the non-diagonal array blocks. This is due to the fact that adjacent current modes in transverse elements are displaced in angle with respect to each other.

CASE 4: TRANVERSE ELEMENTS, SCAN

Again, this case is similar to the corresponding vertical elements case 2. In contrast to the no scan case, there is a slight increase in symmetry over the scanned vertical element case (see figure 71). The increase only affects the diagonal primitive of the diagonal array block (i.e., interactions between elements on same array); thus, the improvement is insignificant as it is independent of the number of tiers or array blocks.

CASE 5: SLANTED ELEMENTS, NOSCAN

A significant loss of redundancy occurs in the more general case in which the array elements are neither purely vertical nor transverse (see figure 72). This comes about as a result of the azimuthal asymmetry in the radiated fields. If we denote the
angular and axial displacement dependence of the mutuals as

\[ Z^{2,1} = Z^{2,1}(\Delta \phi, \Delta z) \]

then we note that for unscanned slant elements,

\[ Z^{2,1}(-\Delta \phi, -\Delta z) = Z^{2,1}(\Delta \phi, \Delta z) \quad \text{(F.2)} \]

\[ Z^{2,1}(0, -\Delta z) = Z^{2,1}(0, \Delta z) \quad \text{(F.3)} \]

\[ Z^{2,1}(-\Delta \phi, 0) = Z^{2,1}(\Delta \phi, 0) \quad \text{(F.4)} \]

but

\[ Z^{2,1}(-\Delta \phi, \Delta z) \neq Z^{2,1}(\Delta \phi, \Delta z) \quad \text{if } \Delta z \neq 0 \quad \text{(F.5)} \]

We will need the material in chapter III to prove this, but the reader can verify that these symmetry relations are true through a study of the example shown in figure 66. The resultant field asymmetry due to relation (F.5) is reflected in the loss of symmetry at both the array block level and the block-block level. The partition matrices are still Toeplitz at all block levels however.

CASE 6: SLANTED ELEMENTS, SCAN

This most general case exhibits the least matrix redundancy, but only slightly less than the previous case. In figure 73 it is observed that only the "self" primitive (diagonal primitive of the diagonal array block) is affected. Little further loss of redundancy is suffered for the slant element case when the array is scanned. Thus, the impedance matrix is block Toeplitz (but not necessarily symmetric) for all cases, based only on the original assumption of equiangularly spaced arrays, each equidistant from the cylinder axis.

Other special symmetries arise for particular configurations. For example, array blocks involving two array groups exactly 180 degrees apart exhibit additional internal symmetry. There are many possibilities to consider; we found it
useful to employ a Karnaugh map in order to systematically extract all possible redundancies.
Figure 68: Non-redundant impedance matrix entries for vertical elements, unscanned case.
Figure 69: Non-redundant impedance matrix entries for vertical elements, scanned case.

<table>
<thead>
<tr>
<th></th>
<th>Z_{11}</th>
<th>Z_{12}</th>
<th>Z_{13}</th>
<th>Z_{11}</th>
<th>Z_{12}</th>
<th>Z_{13}</th>
<th>Z_{11}</th>
<th>Z_{12}</th>
<th>Z_{13}</th>
<th>Z_{11}</th>
<th>Z_{12}</th>
<th>Z_{13}</th>
<th>Z_{11}</th>
<th>Z_{12}</th>
<th>Z_{13}</th>
</tr>
</thead>
<tbody>
<tr>
<td>Z_{21}</td>
<td>Z_{21}</td>
<td>Z_{21}</td>
<td>Z_{21}</td>
<td>Z_{21}</td>
<td>Z_{21}</td>
<td>Z_{21}</td>
<td>Z_{21}</td>
<td>Z_{21}</td>
<td>Z_{21}</td>
<td>Z_{21}</td>
<td>Z_{21}</td>
<td>Z_{21}</td>
<td>Z_{21}</td>
<td>Z_{21}</td>
<td>Z_{21}</td>
</tr>
<tr>
<td>Z_{31}</td>
<td>Z_{31}</td>
<td>Z_{31}</td>
<td>Z_{31}</td>
<td>Z_{31}</td>
<td>Z_{31}</td>
<td>Z_{31}</td>
<td>Z_{31}</td>
<td>Z_{31}</td>
<td>Z_{31}</td>
<td>Z_{31}</td>
<td>Z_{31}</td>
<td>Z_{31}</td>
<td>Z_{31}</td>
<td>Z_{31}</td>
<td>Z_{31}</td>
</tr>
<tr>
<td>Z_{11}</td>
<td>Z_{11}</td>
<td>Z_{11}</td>
<td>Z_{11}</td>
<td>Z_{11}</td>
<td>Z_{11}</td>
<td>Z_{11}</td>
<td>Z_{11}</td>
<td>Z_{11}</td>
<td>Z_{11}</td>
<td>Z_{11}</td>
<td>Z_{11}</td>
<td>Z_{11}</td>
<td>Z_{11}</td>
<td>Z_{11}</td>
<td>Z_{11}</td>
</tr>
<tr>
<td>Z_{21}</td>
<td>Z_{21}</td>
<td>Z_{21}</td>
<td>Z_{21}</td>
<td>Z_{21}</td>
<td>Z_{21}</td>
<td>Z_{21}</td>
<td>Z_{21}</td>
<td>Z_{21}</td>
<td>Z_{21}</td>
<td>Z_{21}</td>
<td>Z_{21}</td>
<td>Z_{21}</td>
<td>Z_{21}</td>
<td>Z_{21}</td>
<td>Z_{21}</td>
</tr>
<tr>
<td>Z_{31}</td>
<td>Z_{31}</td>
<td>Z_{31}</td>
<td>Z_{31}</td>
<td>Z_{31}</td>
<td>Z_{31}</td>
<td>Z_{31}</td>
<td>Z_{31}</td>
<td>Z_{31}</td>
<td>Z_{31}</td>
<td>Z_{31}</td>
<td>Z_{31}</td>
<td>Z_{31}</td>
<td>Z_{31}</td>
<td>Z_{31}</td>
<td>Z_{31}</td>
</tr>
</tbody>
</table>
Figure 70: Non-redundant impedance matrix entries for transverse elements, unscanned case.
Figure 71: Non-redundant impedance matrix entries for transverse elements, scanned case.
Figure 72: Non-redundant impedance matrix entries for slanted elements, unscanned case.
Figure 73: Non-redundant impedance matrix entries for slanted elements, scanned case.
REFERENCES


