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Nonsymmetric P- and Q-polynomial association schemes and associated orthogonal polynomials

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The Ohio State University, 1989
NONSYMMETRIC P- AND Q-POLYNOMIAL ASSOCIATION
SCHEMES AND ASSOCIATED ORTHOGONAL POLYNOMIALS

DISSERTATION

Presented in Partial Fulfillment of the Requirement for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

by

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* * * * *

The Ohio State University
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ACKNOWLEDGEMENTS

I express sincere appreciation to my adviser, Professor Eiichi Bannai for his suggestions and insight throughout the research. I also would like to thank Professor Tasturo Ito and Dr. Douglas Leonard for their valuable suggestions and encouragement.

The findings in Chapter IV would not have been possible without MAPLE symbolic manipulation system. I would like to thank Dr. Gerald Edger and Dr. Thomas Ralley for teaching me MAPLE. Special thanks go to Dr. Yuji Kodama for providing his office and his computer terminal to let me use MAPLE, and to Ms. Marilyn Radcliff for making funds available to me. I also would like to thank Dr. Warren Sinott with whom I consulted about elliptic curves.
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INTRODUCTION

Distance-transitive digraphs were first introduced by Lam [4]. Damerell [2] defined the distance-regularity for digraphs and proved a fundamental theorem for distance-regular digraphs. There are a few examples of known distance-regular digraphs other than the trivial ones, i.e., the directed cycles. A distance-regular digraph with diameter two and girth three can be constructed from a skew Hadamard matrix, distance-regular digraphs with diameter three and girth four have been constructed by Liebler and Mena [7]. These distance-regular digraphs are short digraphs, namely the diameter is one less than the girth. One can construct a long digraph, namely a distance-regular digraphs in which the diameter is equal to the girth, from a short digraph (see [2]). The short digraphs mentioned above are not only P-polynomial but also Q-polynomial association schemes. Leonard [5,6] has shown that if $\mathcal{S}$ is a nonsymmetric P- and Q-polynomial association scheme then $\mathcal{S}$ is self-dual and $g=g^*=d+1$ or $g=g^*=d$, where $d, g, g^*$, are the diameter, the girth, and the cogirth, respectively. However, it seems unnoticed that no long digraph is Q-polynomial. Consequently, the case
(Theorem 1.1, Chapter II). There does exist a nonsymmetric Q-polynomial association scheme with \( g^* = d \), and we prove in section 2.1 a structure theorem for such association schemes. This result is a dual of [2, Theorem 4].

Next we obtain a necessary condition for the existence of a nonsymmetric P- and Q-polynomial association scheme in terms of its eigenvalues. The eigenvalues are shown to be at most quadratic over the rationals (Theorem 2.2 in Chapter II), unless the association scheme is a directed cycle. By using the fact that the adjacency algebra \( \mathfrak{A} \) becomes a C-algebra, we obtain a number of relations the eigenvalues must satisfy. Indeed, these relations are equivalent to the existence of the associated C-algebra, also to the existence of the associated orthogonal polynomials (Theorem 3.1 in Chapter II).

We discuss the cases where the girth is 5, 6, in Chapter III, IV, respectively. In both cases, a nontrivial solution to the relations obtained in section 2.3 is given, which gives a system of orthogonal polynomials. However, in Chapter III, we prove the nonexistence of nontrivial nonsymmetric P- and Q-polynomial association scheme with girth 5, by using the positivity and integrality of parameters.
CHAPTER I
PRELIMINARIES

1.1. Distance-regular digraphs

A digraph is a pair $\Gamma=(V,E)$ where $V$ is a finite set and $E$ is a subset of $V \times V$. $V$ is called the set of vertices, $E$ is called the set of edges. We use $\alpha \rightarrow \beta$ to denote $(\alpha, \beta) \in E$. By a path from $\alpha$ to $\beta$ we mean a sequence $\alpha = \alpha_0 \rightarrow \alpha_1 \rightarrow \cdots \rightarrow \alpha_s = \beta$. The distance from $\alpha$ to $\beta$, denoted by $d(\alpha, \beta)$, is the smallest length of a path from $\alpha$ to $\beta$. We shall only consider digraphs which are strongly connected, so that $d(\alpha, \beta)$ makes sense for any vertices $\alpha, \beta$. The maximum of $d(\alpha, \beta)$, where $\alpha, \beta \in V$ is called the diameter of the digraph $\Gamma$. We will denote the diameter by $d$. Let $\Gamma_i(\alpha)$ denote the set of vertices $\beta$ at distance $i$ from $\alpha$: $\Gamma_i(\alpha) = \{ \beta \in E \mid d(\alpha, \beta) = i \}$, where $i = 0, 1, \ldots, d$. $\Gamma$ is called distance-regular if for all $i, j$ in $0 \leq i, j \leq d$, the number $|\Gamma_i(\alpha) \cap \Gamma_j(\beta)| = s_{ij}$, say, is the same for all $\alpha, \beta$ such that $d(\alpha, \beta) = j$. The girth of the graph $\Gamma$ is the shortest length
of closed paths. We are interested in directed graphs, thus we assume $s_{01} = 0$, hence the girth is at least three. The following theorem is due to Damerell [2].

Theorem 1.1. Let $\Gamma$ be a distance-regular digraphs with diameter $d$, girth $g$. Then

1. $g = d + 1$

or 2. $g = d$ and the adjacency matrix of $\Gamma$ is of the form $A \otimes J$, where $J$ is the all one $m \times m$ matrix for some $m \geq 2$, and $A$ is the adjacency matrix of a distance-regular digraph with diameter $d - 1$, girth $g$.

1.2. Association schemes

Let $X$ be a nonempty finite set and $(R_i)_{0 \leq i \leq d}$ be a partition of $X \times X$ into $d + 1$ classes. Then $\mathcal{S} = (X, (R_i)_{0 \leq i \leq d})$ is called an association scheme of class $d$ if it satisfies the following properties:

(A1) $R_0 = \{(x, x) | x \in X\}$.

(A2) For any $i \in \{1, 2, \ldots, d\}$, $^{t}R_i = R_{i'}$, for some $i' \in \{1, 2, \ldots, d\}$, where $^{t}R_i = \{(x, y) | (y, x) \in R_i\}$.

(A3) For every pair $(x, y) \in R_K$, the number of $z \in X$ such
that \((x,z)\in R_i, (z,y)\in R_j\) is a constant \(p_{ij}^k\).

If an association scheme satisfies the additional property
\[(A4) \ p_{ij}^k = p_{ji}^k,\]
then it is called a commutative association scheme. If it satisfies the additional property
\[(A5) \ i' = i \text{ for all } i,\]
then it is called a symmetric association scheme.

If \(S = (X, \{R_i\}_{0 \leq i \leq d})\) is an association scheme, the
\(i\)-th adjacency matrix \(A_i\) is defined to be the \(n\) by \(n\)
matrix \((n=|X|)\) whose rows and columns are indexed by the
elements of \(X\) and whose \((x,y)\)-entry is 1 if \((x,y)\in R_i, 0\)
if \((x,y)\notin R_i\). Then \(A_0 + A_1 + \cdots + A_d = J\), the all one matrix, and the
conditions \((A1), ..., (A5)\) are equivalent to the following
\((A1)', ..., (A5)', \) respectively.
\[(A1)' \ A_0 = I\]
\[(A2)' \ A_i^{T} = A_{i'}, \text{ for some } i' \in \{0, 1, \ldots, d\}.
\[(A3)' \ A_i A_j = \sum_{k=0}^{d} p_{ij}^k A_k \text{ for all } i, j.
\[(A4)' \ A_i A_j = A_j A_i \text{ for all } i, j.
\[(A5)' \ A_i^{T} = A_i \text{ for all } i.

In what follows, all association schemes considered in this
paper are assumed to be commutative.
Let \( \mathcal{U} \) be the subalgebra of \( \text{Mat}_n(\mathbb{C}) \) generated by the adjacency matrices, where \( n = |X| \). Then \( \mathcal{U} \) is a commutative algebra of dimension \( d+1 \) over \( \mathbb{C} \). Let \( E_0, E_1, \ldots, E_d \) be the primitive idempotents, write
\[
A_i = \sum_{j=0}^{d} p_j(i) E_j, \quad E_j = \frac{1}{n} \sum_{i=0}^{d} q_j(i) A_i. \tag{1.2.1-2}
\]

Let \( P = (p_j(i)), Q = (q_j(i)) \). Then \( PQ = nI \). The association scheme \( S \) is called self-dual, if \( P = Q \) for a suitable numbering of the primitive idempotents. The Krein parameters are the coefficients \( q_{ij}^k \) appearing in the Hadamard product \( E_i \cdot E_j \) expressed as a linear combination of \( E_k \), \( k = 0, 1, \ldots, d \):
\[
E_i \cdot E_j = \frac{1}{n} \sum_{k=0}^{d} q_{ij}^k E_k. \tag{1.2.3}
\]

It is known that the Krein parameters are nonnegative real numbers. If \( S \) is self-dual, then \( p_{ij}^k = q_{ij}^k \).

\( S \) is said to be \( P \)-polynomial if, with a suitable renumbering of \( A_0, A_1, \ldots, A_d \), there exists a polynomial \( v_i(x) \) of degree \( i \) such that \( A_i = v_i(A_1) \), for each \( i = 0, 1, \ldots, d \). \( S \) is said to be \( Q \)-polynomial if, with a suitable suitable renumbering of \( E_0, E_1, \ldots, E_d \), there exists a polynomial \( v_i^*(x) \) of degree \( i \) such that \( nE_i = v_i^*(nE_1) \) and the multiplication is using the Hadamard product. If \( S \) is \( P \)-polynomial, then \( A_1 \) has \( d \) distinct eigenvalues \( \theta_0 = \ldots = \theta_{d-1} \).
\( p_1(0), \theta_1 = p_1(1), \ldots, \theta_d = p_1(d), \) and we have \( p_j(i) = v_j(\theta_i) \) for all \( i,j \). If \( S \) is \( Q \)-polynomial, then \( \theta_0^* = q_1(0), \theta_1^* = q_1(1), \ldots, \theta_d^* = q_1(d) \) are all distinct and we have \( q_j(i) = v_j^*(\theta_i^*) \).

If \( \Gamma \) is a distance-regular digraph, let \( R_i = \{(\alpha, \beta) \in X \times X \mid d(\alpha, \beta) = i\}, i = 0, 1, \ldots, d \), where \( d(\alpha, \beta) \) is the distance from \( \alpha \) to \( \beta \) and \( d \) is the diameter of \( \Gamma \). Then \( S(\Gamma) = (X, \{R_i\}_{0 \leq i \leq d}) \) is a \( P \)-polynomial association scheme. Because of the convention on distance-regular digraphs, \( S(\Gamma) \) is nonsymmetric. On the other hand for any nonsymmetric \( P \)-polynomial association scheme, \( \Gamma = (X, R_1) \) is a distance-regular digraph. Therefore we do not distinguish nonsymmetric \( P \)-polynomial association schemes from distance-regular digraphs. The girth of a nonsymmetric \( P \)-polynomial association scheme is the girth of the corresponding distance-regular digraph. Note that if \( A_0, A_1, \ldots, A_d \) are the adjacency matrices, then the girth is \( 1 + 1' \), where \( A_1^T = A_1 \). We can define the cogirth of a nonsymmetric \( Q \)-polynomial association scheme with the primitive idempotents \( E_0, E_1, \ldots, E_d \) to be \( 1 + \hat{1} \) where \( E_1^T = E_1 = E_1 \).

Part of the dual of the Damerell's Theorem was proved by Leonard [5].
Theorem 2.1. Let $\mathcal{G}$ be a nonsymmetric $Q$-polynomial association scheme of class $d$, cogirth $g^*$. Then $g^* = d + 1$ or $g^* = d$.

The following theorem of Leonard [5, 6] illustrates that the theory of nonsymmetric $P$- and $Q$-polynomial association schemes seems quite different from that of symmetric ones.

Theorem 2.2. Let $\mathcal{G}$ be a nonsymmetric $P$- and $Q$-polynomial association scheme with diameter $d$, girth $g$, and cogirth $g^*$. Then $\mathcal{G}$ is self-dual, and either $g = g^* = d + 1$ or $g = g^* = d$.

1.3. C-algebras

Let $\mathcal{U}$ be an algebra over $\mathbb{C}$ with a basis $x_0, x_1, \ldots, x_d$ as a linear space. $\mathcal{U}$ together with $x_0, x_1, \ldots, x_d$ is called a C-algebra if the following conditions (i)-(vi) hold:

(i) $\mathcal{U}$ is a commutative algebra, i.e.,

$$x_i x_j = \sum_{k=0}^d p_{ij}^k x_k$$

with $p_{ij} = p_{ji}$.

(ii) $\mathcal{U}$ has the identity element $e = x_0$.

(iii) Every $p_{ij}^k$ is a real number.

(iv) There exists a permutation $i \mapsto i'$ ($i = 0, 1, \ldots, d$)
such that \( p_{ij}^k = p_{ij}^{k'} \), i.e., the mapping \( x_i \to x_i \), \( i=0,1,\ldots,d \) can be extended to an algebra automorphism of \( \mathcal{U} \).

(v) \( p_{ij}^0 = \delta_{ij}, k_i \) with \( k_i > 0 \) for all \( i,j \).

To distinguish the basis \( x_0, x_1, \ldots, x_d \), we use the notation \( \mathcal{U} = \langle x_i \mid 0 \leq i \leq d \rangle \) for the \( C \)-algebra \( \mathcal{U} \).

The adjacency algebra \( \mathcal{U} \) of a commutative association scheme is a \( C \)-algebra with the basis \( A_0, A_1, \ldots, A_d \), where \( A_0, A_1, \ldots, A_d \) are the adjacency matrices. In a \( P \)-polynomial association scheme, there exists a polynomial \( v_i(x) \) of degree \( i \) such that \( A_i = v_i(A_1) \). Then \( \mathcal{U} \) is isomorphic to \( C[x]/(\varphi(x)) \), where \( \varphi(x) \) is the minimal polynomial of \( A_1 \). \( C[x]/(\varphi(x)) \) is a \( C \)-algebra with the basis \( v_0(x), v_1(x), \ldots, v_d(x) \). We refer to [1] for more detail.
CHAPTER II
GENERAL RESULTS

2.1. The girth and the cogirth

In this section we prove the dual of Damerell's theorem, describing the structure of nonsymmetric $Q$-polynomial association scheme in which the diameter is equal to its cogirth. We also show that long digraphs are not $Q$-polynomial.

First we will discuss Damerell's construction in more detail. Let $\mathcal{S}=(X,(R_i)_{0\leq i\leq d})$ be an arbitrary association scheme, $A_0, A_1, \ldots, A_d$, the adjacency matrices, $E_0, E_1, \ldots, E_d$, the primitive idempotents. Let $J$ be the $m\times m$ all one matrix, where $m \geq 2$ is a positive integer. Then the matrices $X_0 = A_0 \otimes I$, $X_1 = A_1 \otimes J$, $\ldots, X_d = A_d \otimes J$, $X_{d+1} = I \otimes (J-I)$, define a commutative association scheme $\mathcal{S}$ of class $d+1$. To see this, simply check that the linear span of $X_0, \ldots, X_{d+1}$ is closed under the multiplication. We can also check that the primitive idempotents are $E_0 = E_0 \otimes \frac{1}{m} J$, $E_1 = E_1 \otimes \frac{1}{m} J$, $\ldots$, $E_d = E_d \otimes \frac{1}{m} J$. 

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\[ E_{d+1} = I \otimes (I - \frac{1}{m}J) \]. Let \( E_{j} = \frac{1}{n} \sum_{i=0}^{d} q_{j}(i) \Lambda_{i} \), \( E_{j} = \frac{1}{n} \sum_{i=0}^{d+1} \tilde{q}_{j}(i) \tilde{\Lambda}_{i} \). The matrices \( Q = (q_{j}(i)) \), \( \tilde{Q} = (\tilde{q}_{j}(i)) \) are called the second eigenmatrices of \( \mathbb{S}, \tilde{\mathbb{S}} \), respectively. It is easy to check that

\[
\tilde{Q} = \begin{bmatrix}
0 & 0 & \cdots & 0 \\
Q & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
q_{0}(0) & q_{1}(0) & \cdots & q_{d}(0)
\end{bmatrix}
\]

Note that no column of \( \tilde{Q} \) has all distinct entries, hence \( \tilde{\mathbb{S}} \) is not \( Q \)-polynomial. According to Theorem 1.1., any distance-regular digraph with \( d = g \) is obtained by the way we constructed \( \tilde{\mathbb{S}} \) from \( \mathbb{S} \). However, \( \tilde{\mathbb{S}} \) cannot be \( Q \)-polynomial even if \( \mathbb{S} \) is. Combining this result with Theorem 1.2., we obtain the following.

**Theorem 1.1.** In any nonsymmetric \( P \)- and \( Q \)-polynomial association scheme, the diameter is one less than its girth and its cogirth.

Therefore, a distance-regular digraph with \( d = g \) is a nonsymmetric \( P \)-polynomial association scheme which is not \( Q \)-polynomial. Dually, a nonsymmetric \( Q \)-polynomial association scheme with \( d = g^{*} \) is not \( P \)-polynomial. We shall describe the structure of such nonsymmetric \( Q \)-polynomial
association schemes.

Let $X^{(1)}, X^{(2)}, \ldots, X^{(m)}$ (m ≥ 2) be commutative
association schemes with the same parameters but not
necessarily isomorphic. Let $A^{(j)}_i$ be the i-th adjacency matrix
of $X^{(j)}$. $E^{(j)}_i$ the i-th primitive idempotent of $X^{(j)}$, where
0 ≤ i ≤ d, 0 ≤ j ≤ d. Define

$$
\tilde{A}_i = \begin{bmatrix}
A^{(1)}_i & 0 \\
0 & \mathbb{I}_d
\end{bmatrix}
$$

(0 ≤ i ≤ d)

$$
\tilde{A}_{d+1} = \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
$$

Then $X_0, X_1, \ldots, X_{d+1}$ define a commutative association
scheme $\tilde{X}$ of class d+1. We write $\tilde{X} = X^{(1)} \oplus X^{(2)} \oplus \cdots \oplus X^{(m)}$.
The primitive idempotents of $\tilde{X}$ are given by

$$
\tilde{E}_0 = \frac{1}{nm} J
$$

$$
\tilde{E}_i = \begin{bmatrix}
E^{(1)}_i & 0 \\
0 & \mathbb{I}_d
\end{bmatrix}
$$

(1 ≤ i ≤ d)
Let \( P = (p_j(i)) \) be the first eigenmatrix of \( \mathcal{S}^{(1)}, \mathcal{S}^{(2)}, \ldots, \mathcal{S}^{(m)} \), i.e., \( A_j = \sum_{i=0}^{d} p_j(i) E_i^{(k)} \). Then the first eigenmatrix \( \widetilde{P} = (\tilde{p}_j(i)) \), where \( \tilde{A}_j = \sum_{i=0}^{d+1} \tilde{p}_j(i) \tilde{E}_i \) is given by

\[
\tilde{P} = \begin{bmatrix}
p_0(0) & p_1(0) & \cdots & p_d(0) & -n
\end{bmatrix}
\]

If \( \mathcal{S}^{(1)}, \mathcal{S}^{(2)}, \ldots, \mathcal{S}^{(m)} \) are nonsymmetric Q-polynomial association schemes with cogirth \( d+1 \), then \( \mathcal{G} \) is a nonsymmetric Q-polynomial association scheme of class \( d+1 \), cogirth \( g^* = d+1 \). Indeed, by a result of Leonard [5], \( F_i^{(j)} = F_j^{(i)} \) with \( i = d+1 - \hat{i} \) for \( 1 \leq i \leq d, 1 \leq j \leq m \). Let \( q^k_{i,j} \) be the Krein parameter of \( \mathcal{S}^{(1)}, \ldots, \mathcal{S}^{(m)} \). Then \( \tilde{E}_1 \circ \tilde{E}_1 = \frac{1}{n} \sum_{k=1}^{i+1} q^k_{i,j} \tilde{E}_k \) for \( 0 \leq i \leq d-1 \). Therefore \( \tilde{E}_1 (0 \leq i \leq d) \) is a polynomial of degree \( i \) in \( \tilde{E}_1 \). Since

\[
\tilde{E}_1 = \frac{1}{n} \begin{bmatrix}
(m-1)J \\
& -1 \\
& & -1 \\
& & & & -1 \\
& & & & & & & & (m-1)J
\end{bmatrix}
\]
\[ E_d \in E = \frac{1}{n} \sum_{k=1}^{d} q_k E_k + \frac{1}{n} E_0 \otimes I \]

(2.1.1)

\[ \tilde{E}_d \in E = \frac{1}{n} q_0 E_0 + \frac{1}{n} \sum_{k=1}^{d} q_k E_k + \frac{1}{n} E_0 \tilde{E}_d + 1 \]

\[ \tilde{E}_{d+1} \]

is a polynomial of degree \( d+1 \) in \( \tilde{E}_1 \). Thus \( \tilde{S} \) is a Q-polynomial association scheme. Since \( \tilde{E}_1^T = \tilde{E}_d \), the cogirth of \( \tilde{S} \) is \( d+1 \).

Now we want to show that every nonsymmetric Q-polynomial association scheme \( \mathcal{S} \) of class \( d+1 \), with cogirth \( g^* = d+1 \) can be obtained by the above construction. Let \( E_0, E_1, \ldots, E_{g^*} \) be the primitive idempotents of \( \mathcal{S} \) such that \( E_i \) is a polynomial of degree \( i \) in \( E_1 \) with respect to the Hadamard product. Let \( E_i \circ E_j = \frac{1}{n} \sum_{k=0}^{g^*} q_{ij}^k E_k \), where \( q_{ij}^k \) is the Krein parameter, \( n \) is the size of \( \mathcal{S} \). Then we have \( \tilde{E}_i = E_i \tilde{E}_i \) with \( \hat{i} = g^* - i \) for \( 1 \leq i < g^* \), \( \hat{g}^* = g^* \). Moreover, the Krein parameters satisfy a number of relations [1, Chap.III, Prop.3.7]. We shall use these relations to obtain the following.

Lemma 1.2. If \( \mathcal{S} \) is a nonsymmetric Q-polynomial association scheme of class \( g^* \), with cogirth \( g^* \), then the Krein parameters of \( \mathcal{S} \) satisfy the following.

1. \( q_{g^*g^*}^i = 0 \) unless \( i = 0 \) or \( g^* \).
2. \( q_{ig^*}^i = 0 \) unless \( i = j \), where \( 1 \leq i \leq g^* - 1 \).
Proof. Step 1. \( q_{ig*}^j = 0 \) unless \( j = 1 \) or \( g* \).

If \( 2 \leq j \leq g*-1 \), then \( 1 \leq j = g* - j \leq g* - 2 \). So \( q_{ig*}^j = 0 \), and \( 0 = q_{ig*}^j = q_{ig*}^j \).

Step 2. \( q_{ig*}^j = 0 \) unless \( j = 1 \)

By step 1, \( E_1 \cdot E_g* = \frac{1}{n} \sum_{j=0}^n g^j E_j = \frac{1}{n} (q_{ig*}^1 E_1 + q_{ig*}^j E_g*) \). So
\[
(nE_1 - q_{ig*}^gJ) (nE_g* - q_{ig*}^1 J) = q_{ig*}^1 q_{ig*}^J nJ. \]
If \( q_{ig*}^1 q_{ig*}^J \neq 0 \), then any entry of the right hand side is nonzero. Thus we have
\[
(nE_g* - q_{ig*}^J)_{ij} \neq 0, \quad (nE_1 - q_{ig*}^gJ)_{ij} \neq 0, \quad \text{for all } i,j. \]
But \( nE_1 - q_{ig*}^J \) is not symmetric, \( nE_g* - q_{ig*}^gJ \) is symmetric, so the left hand side is not symmetric, contradiction. Therefore \( q_{ig*}^1 q_{ig*}^J = 0 \). On the other hand \( q_{ig*}^1 = m_g* q_{i1}^g* / m_1 = m_g* q_{g* - 1}^g* / m_1 \) \( \neq 0 \). Thus \( q_{ig*}^1 = 0 \).

Step 3. \( q_{ig*}^j = 0 \) for \( 1 \leq i \leq g* - 1 \).

For the rest of the proof, \( E_1^i \) will denote the \( i \)-th power of \( E_1 \) with respect to the Hadamard product. By Step 2, \( E_1 \cdot E_g* = \frac{1}{n} q_{ig*}^1 E_1 \), so \( E_1^i \cdot E_g* = \frac{1}{n} q_{ig*}^i E_1^i \) for \( 1 \leq i \leq g*-1 \). Since \( \mathbb{S} \) is \( \mathbb{Q} \)-polynomial, there exists a polynomial \( v_i^*(x) \) of degree \( i \) such that \( nE_i = v_i^*(nE_1) \), where \( n \) is the size of \( \mathbb{S} \). Since the cogirth of \( \mathbb{S} \) is \( g* \), \( q_{i1}^0 = 0 \). By induction on \( i \), it follows that \( v_i^*(x) \) has no constant term. Therefore \( E_i \) is a linear combination of \( E_1, E_1^2, \ldots, E_1^i \).

\[
E_1 \cdot E_g* = (E_1 \cdot E_g*, E_1^2 \cdot E_g*, \ldots, E_1^i \cdot E_g*) \]
\[
= (E_1, E_1^2, \ldots, E_1^i) \]
\[
= (E_1, E_2, \ldots, E_i). \]
Thus \((E_i \circ E_{g^*})E_{g^*} = 0\), \(q_{i g^*}^j = 0\), for \(1 \leq i \leq g^*-1\).

From Step 3, it is immediate to show \(q_{i g^*}^j = 0\) for \(1 \leq i \leq g^*-1\), so (1) is proved.

We shall show (2) by induction on \(i\). The case \(i = 1\) was proved in Step 2. Suppose that the assertion is true up to \(i-1\), for some \(i \geq 2\). If \(j < i\), then \(q_{i g^*}^j = m_i q_{j g^*}^i / m_j = 0\) by induction.

If \(j > i\), then \(i + j = i + g^* - j < g^*\), so \(0 = q_{i g^*}^j = m_j q_{j g^*}^j / m_{g^*} = m_j q_{j g^*}^j / m_{g^*}\). Thus \(q_{i g^*}^j = 0\).

Theorem 1.3. Let \(S = (X, (R_i)_{0 \leq i \leq g^*})\) be a nonsymmetric \(Q\)-polynomial association scheme of class \(g^*\) with cogirth \(g^*\). Then \(S\) is imprimitive. There exist nonsymmetric \(Q\)-polynomial association scheme \(S^{(1)}, S^{(2)}, \ldots, S^{(m)}\) \((m \geq 2)\) of class \(g^*-1\), with cogirth \(g^*\), and the same parameters such that \(S\) is isomorphic to \(S^{(1)} \oplus S^{(2)} \oplus \ldots \oplus S^{(m)}\).

Proof. Let \(A_0, A_1, \ldots, A_{g^*}\) be the adjacency matrices of \(S\), \(E_0, E_1, \ldots, E_{g^*}\) the primitive idempotents, \(n = |X|\). Let \(\mathcal{U} = \langle A_0, A_1, \ldots, A_{g^*} \rangle\) be the adjacency algebra of \(S\), \(\hat{\mathcal{U}} = \langle nE_0, nE_1, \ldots, nE_{g^*} \rangle\) the dual of \(\mathcal{U}\). Then by Lemma 1.2 (1), \(\langle nE_0, nE_1, \ldots, nE_{g^*} \rangle\) is a \(C\)-subalgebra of \(\hat{\mathcal{U}}\). Moreover by Lemma 1.2 (2), \(\langle nE_0 + nE_{g^*}, nE_1, \ldots, nE_{g^*} \rangle\) is the quotient \(C\)-algebra \(\hat{\mathcal{U}} / \langle nE_0, nE_{g^*} \rangle\).

By [1, Theorem 9.9], there exists a \(C\)-subalgebra \(\mathcal{U}_1\) of \(\mathcal{U}\).
such that $\hat{U}_1 \cong \hat{U}/\langle nE_0, nE_g^* \rangle$ and $(\mathcal{U}/\mathcal{U}_1)^\sim \cong \langle nE_0, nE_g^* \rangle$. In particular, $\dim \mathcal{U}_1 = \dim \hat{U}_1 = g*$. We may assume $\mathcal{U}_1 = \langle A_0, A_1, \ldots, A_{g* - 1} \rangle$ with a suitable renumbering of the adjacency matrices. Then by [1, Theorem 9.3], $\bigcup_{i=0}^{g* - 1} R_i$ is an equivalence relation, and every equivalence class has the same size. Let $X_1, X_2, \ldots, X_m$ be the equivalence classes. Then $A_0|_{X_1}, A_1|_{X_1}, \ldots, A_{g* - 1}|_{X_1}$ define an association scheme on $X_i$ whose adjacency algebra is isomorphic to $\mathcal{U}_1$. Since $\hat{U}_1 \cong \hat{U}/\langle nE_0, nE_g^* \rangle = \langle nE_0 + nE_g^*, nE_1, \ldots, nE_g^{* - 1} \rangle$ is a $\mathbb{C}$-algebra of $P$-polynomial type, the association scheme $S(i)$ on $X(i)$ is $Q$-polynomial. Since $R_{g*} = X \times X - \bigcup_{i=0}^{g* - 1} R_i$, it is now easy to see $X \cong S(1) \oplus S(2) \oplus \cdots \oplus S(m)$.

2.2. The splitting field of nonsymmetric $P$- and $Q$-polynomial association schemes

The splitting field of an association scheme is the field obtained by adjoining the all eigenvalues of the adjacency matrices to $\mathbb{Q}$. For example, the splitting field of the directed $n$-cycle, when the directed $n$-cycle is considered as a nonsymmetric $P$- and $Q$-polynomial association scheme, is $\mathbb{Q}(\zeta_n)$, where $\zeta_n$ is a primitive $n$-th root of the unity.

In this section we show that the the splitting field of a nontrivial nonsymmetric $P$- and $Q$-polynomial association scheme is an imaginary quadratic extension of $\mathbb{Q}$. Let
Γ be a distance-regular digraph with diameter d, girth g. We write \( i' = g - i \) for \( i = 1, 2, \ldots, g - 1 \) and \( 0' = 0 \). There corresponds naturally to Γ a nonsymmetric P- and Q-polynomial association scheme \( \Xi \). Let \( p_{ij}^k \) be the intersection numbers of \( \Xi \). Then \( s_{ij} = p_{ij}^1 \) where \( s_{ij} = |\Gamma_i(\alpha) \cap \Gamma_j(\beta)| \) with \( d(\alpha, \beta) = j \) (see section 1.1.). Let \( k_i = p_{ii}^0 = |\Gamma_i(\alpha)| \) be the \( i \)-th valency and write \( b_i = p_{ii}^1 = s_{ii+1} \). As an analogue to the symmetric case (see [1, Chap. III, Prop. 1.2, Prop. 1.4]), we have the following.

Lemma 2.1. (1) \( k_1 = b_0 \geq b_1 \geq \ldots \geq b_{d-1} \).

(2) \( \{k_i\} \) satisfies the unimodality property, i.e.,

\[
1 = k_0 \leq k_1 \leq \ldots \leq k_{\lfloor g/2 \rfloor} \geq \ldots \geq k_{d-1} \geq k_d,
\]

with \( k_i = k_{d+1-i} \) for \( i = 1, 2, \ldots, d \).

Proof. (1) Suppose that \( u, v, x \) are vertices and \( d(u, x) = i, d(v, x) = i-1, d(u, v) = 1 \). Then \( \Gamma_i(x) \cap \Gamma_{i+1}(u) \subseteq \Gamma_1(x) \cap \Gamma_1(v) \), so \( b_i \leq b_{i-1} \).

(2) \( k_i b_i = k_i p_{ii}^1 = k_{i+1} p_{ii+1}^1 = k_{i+1} b_{d-i} \). Thus, if \( 2i \leq d \), then by (1), we have \( k_i \leq k_{i+1} \).

Theorem 2.2. The splitting field of a nontrivial nonsymmetric P- and Q-polynomial association scheme is an
imaginary quadratic extension of $\mathbb{Q}$.

Proof. Let $\theta_0, \theta_1, \ldots, \theta_d$ be the eigenvalues of a nontrivial nonsymmetric $P$- and $Q$-polynomial association scheme $\mathcal{S}$, $\theta_0$ the valency, $\bar{\theta}_i = \theta_{d+1-i}$. If $\theta_1$ is not quadratic over $\mathbb{Q}$, then $\theta_1$ is algebraic conjugate to $\theta_i$ for some $i \neq 1, d$. Let $k_i$ be the $i$-th valency. Then $k_i = k_{d-i}$ by self-duality. By Lemma 2.1(2), we must have $k_1 = k_2$. Since $k_1 b_1 = k_2 b_{d-1}$, Lemma 2.1(1) implies $b_1 = b_2 = \ldots = b_{d-1}$. On the other hand, it is shown in [6] that if $\theta_1$ is not quadratic over $\mathbb{Q}$, then $k_1 = b_1$. Thus $k_1 = b_1 = p_1^i$ for $i = 0, 1, \ldots, d-1$. So $p_1^i = \delta_{i+1,1} k_1$, if $i < d$. In particular, $p_{1d}^i = p_{1,1}^i = \delta_{i0} k_1$, if $i < d$. Also $p_{id}^d = p_{1,1}^1 = k_1 - b_1 = 0$. Therefore $x\nu_d(x) = k_1 \nu_0(x) = k_1$. Setting $x = k_1$, we find $k_1^2 = k_1$, hence $k_1 = 1$. Thus $\mathcal{S}$ is a directed cycle, contradiction.

Since $x\nu_i(x) = \sum_{j=0}^{i+1} p_{ij}^j \nu_j(x)$ with $p_{ij}^j$ being an integer, $\nu_i(x) \in \mathbb{Q}[x]$ for all $i = 0, 1, \ldots, d$. By self-duality, $\theta_1$ is a rational polynomial of degree $i$ in $\theta_1$. Thus all the eigenvalues belong to $\mathbb{Q}(\theta_1)$, hence at most quadratic. Since $\mathcal{S}$ is non-symmetric, the splitting field is an imaginary quadratic extension of $\mathbb{Q}$. 

Remark. The proof of Theorem 2.2 was suggested by Eiichi Bannai. Since the eigenvalues belong to the ring of
integers of \( \mathbb{Q}(\theta_1) \), and the absolute value of the eigenvalues are at most \( k_1 \), it follows that the diameter of nonsymmetric \( P \)- and \( Q \)-polynomial association scheme is bounded by a function of the valency \( k_1 \).

2.3. The \( C \)-algebra of orthogonal polynomials arising from nonsymmetric \( P \)- and \( Q \)-polynomial association schemes

We now investigate nonsymmetric \( P \)- and \( Q \)-polynomial association schemes in the context of \( C \)-algebras. Suppose that there exists a nonsymmetric \( P \)- and \( Q \)-polynomial association scheme with the adjacency matrices \( A_0, A_1, \ldots, A_d \), where \( A_i = \nu_i(A_1) \), and \( \nu_i(x) \) is a polynomial of degree \( i \). Let \( \theta_0, \theta_1, \ldots, \theta_d \) be the eigenvalues of \( A_1 \), with the multiplicities \( m_0 = 1, m_1, \ldots, m_d \), and \( \hat{\theta}_i = \theta_{d+1-i} \). Then \( \text{tr} A_1^j = 0 \) implies that 

\[
\sum_{i=0}^{d} \hat{\theta}_i m_i = n \delta_{j0},
\]

where \( n \) is the size of the association scheme. We may regard this as a system of linear equations with unknowns \( m_0, m_1, \ldots, m_d \). We can actually solve this to find

\[
m_i = \frac{n \theta_0 \ldots \hat{\theta}_i \ldots \theta_d}{\prod_{k \neq i} (\theta_k - \theta_i)}. \tag{2.3.1}
\]
Since \( m_0 = 1 \), we see that \( \theta_1, \ldots, \theta_d \) are nonzero and

\[
\begin{align*}
\frac{\prod_{k \neq 0} (\theta_k - \theta_0)}{\theta_1 \cdots \theta_d} &= n, \\
\theta_0 \prod_{k \neq 0} (\theta_k - \theta_0) &= m_i \prod_{k \neq i} (\theta_k - \theta_i),
\end{align*}
\]  

(2.3.2) (2.3.3)

hence

The polynomials \( v_i(x), i=0,1,\ldots,d \) satisfy the following.

\[
\begin{align*}
\text{deg } v_i(x) &= i, \ i=0,1,\ldots,d \\
v_0(x) &= 1, \ v_1(x) = x \\
v_i(0) &= 0, \ i=1,2,\ldots,d \\
\frac{v_i(\theta_j)}{m_i} &= \frac{v_j(\theta_i)}{m_j}, \ i,j=0,1,\ldots,d
\end{align*}
\]  

(2.3.4) (2.3.5) (2.3.6) (2.3.7)

(2.3.6) follows from \( \text{tr}_i I = 0, \ i=1,\ldots,d \), while (2.3.7) follows from self-duality. By (2.3.5) and (2.3.7), we have \( \theta_0 = m_1 \).

hence by (2.3.3),

\[
\theta_i \prod_{k \neq 1} (\theta_k - \theta_1) = \prod_{k \neq 0} (\theta_k - \theta_0).
\]  

(2.3.8)

Given nonzero distinct complex numbers \( \theta_0, \theta_1, \ldots, \theta_d \) satisfying (2.3.8), \( m_i \) is determined by (2.3.3) and there exist unique polynomials \( v_i(x) \) of degree at most \( i \) satisfying (2.3.5)-(2.3.7). Indeed, \( v_1(x) \) (\( i \geq 2 \)) is
uniquely determined inductively by $v_i(0) = 0$, $v_i(\theta_0) = m_i$.

$v_i(\theta_1) = m_i v_i(\theta_1)/m_1 \ldots v_i(\theta_{i-1}) = m_i v_{i-1}(\theta_i)/m_{i-1}$. We then want to show that $v_i(x)$ is of degree exactly $i$, that is, (2.3.4) holds. Suppose $\text{deg } v_k(x) < k$ for some $k$. We may assume $k$ is the smallest integer such that $\text{deg } v_k(x) < k$. Then $v_0(x), v_1(x), \ldots, v_k(x)$ are linearly dependent. By (2.3.6), $v_1(x), \ldots, v_k(x)$ must be linearly dependent, thus

$$\begin{vmatrix}
    v_1(\theta_0) & \cdots & v_1(\theta_{k-1}) \\
    \vdots & \ddots & \vdots \\
    v_k(\theta_0) & \cdots & v_k(\theta_{k-1})
\end{vmatrix} = 0.$$

Therefore, by (2.3.7),

$$\begin{vmatrix}
    v_0(\theta_1) & \cdots & v_{k-1}(\theta_1) \\
    \vdots & \ddots & \vdots \\
    v_0(\theta_k) & \cdots & v_{k-1}(\theta_k)
\end{vmatrix} = 0.$$

which is a contradiction, since the above is essentially Vandermonde's determinant. Therefore, given nonzero distinct complex numbers $\theta_0, \theta_1, \ldots, \theta_d$, satisfying (2.3.8), there exist unique nonzero complex numbers $m_i$, $i = 0, 1, \ldots, d$, and unique polynomials $v_i(x)$, $i = 0, 1, \ldots, d$ satisfying (2.3.3)-(2.3.7).

Write

$$xv_j(x) = \sum_{k=0}^{d} p_{kj} v_k(x) \pmod{\varphi(x)} \quad (2.3.9)$$
where \( \varphi(x) = \prod_{k=0}^{d}(x-\theta_k) \). Let \( N,N',B' \) be the square matrices of degree \( d+1 \) whose \((i,j)\)-entries are \( v_i(\theta_j) \) respectively, with the convention \( v_i(\theta_d) = v_i(\theta_0) = m_i \), \( p_{1j}^{i+1} = p_{1j}^0 \). Note that \( p_{1j}^0 = 0 \) unless \( j = d \), because of (2.3.6). Thus \( B' \) is upper triangular. Since

\[
\theta_i v_j(\theta_i) = \sum_{k=0}^{d} p_{1j}^k v_k(\theta_i),
\]

we have

\[
\theta_i v_i(\theta_j) = \frac{1}{m_j} \sum_{k=0}^{d} v_{i+1}(\theta_{k+1}) p_{1j}^{k+1} m_{k+1},
\]

with the convention \( m_{d+1} = m_0 = 1 \). Hence \( DN = N'M'B'M^{-1} \), where \( D = \text{diag}(\theta_0, \theta_1, \ldots, \theta_d) \), \( N = \text{diag}(m_0, m_1, \ldots, m_d) \), \( M = \text{diag}(m_0, \ldots, m_d, m_0) \). Write

\[
v_i(x) = \sum_{j=0}^{d} \ell_{ij}[x] = \sum_{j=0}^{d} \ell'_{ij}[x]', \quad i = 0, 1, \ldots, d.
\]

where \( [x]_j = (x-\theta_0)(x-\theta_1)\cdots(x-\theta_{j-1}), [x]'_j = (x-\theta_1)\cdots(x-\theta_j) \). Let \( L = (\ell_{ij}), L' = (\ell'_{ij}), U = ([\theta_j]_i), U' = ([\theta_{j+1}]_i) \). Then \( L, L' \) are lower triangular, \( U, U' \) are upper triangular. Moreover \( N = LU, N' = L'U' \), so \( L'^{-1}DL = U'M'B'M^{-1}U^{-1} \). The left hand side is lower triangular and the right hand side is upper triangular, hence both are diagonal. Since \( L \) and \( L' \) have the same nonzero diagonal entries, we have \( L'^{-1}DL = D \), hence

\[
DUM = U'M'B'.
\]

This formula is due to Leonard [5], who obtained the essentially same formula prior to proving
self-duality. Note that we have derived (2.3.13) solely from (2.3.3)-(2.3.8), without assuming the existence of the association scheme or the C-algebra. Comparing entries of (2.3.13), we find

\[ p_{i+1}^{i+1} = \frac{\theta_i \theta_{i+1}}{\theta_i \theta_{i+1}} \]

(2.3.14)

\[ p_{i}^{i+1} = \frac{(\theta_i - \theta_0)(\theta_i - \theta_{i+1} - \theta_i)}{(\theta_i - \theta_{i+1})(\theta_{i+1} - \theta_i)} \]

(2.3.15)

Theorem 3.1. Let \( \theta_0, \theta_1, \ldots, \theta_d \) be nonzero distinct complex numbers, satisfying \( \theta_d = \theta_0, \theta_i = \theta_{d+1-i}, (i=1, \ldots, d) \) and (2.3.8). Let \( m_i, v_i(x) \) be defined by (2.3.3)-(2.3.7). Then the following are equivalent.

1. \( \mathcal{U} = \mathbb{C}[x]/(\prod_{i=0}^{d} (x - \theta_i)) = \langle v_i(x) \mid 0 \leq i \leq d \rangle \) is a C-algebra
2. \( m_i > 0 \) for \( i=0,1, \ldots, d \). \( v_i(x) \in \mathbb{R}[x] \) for \( i=0,1, \ldots, d \).
3. \( m_i > 0 \) for \( i=0,1, \ldots, d \), and \( \{v_i(x)\}_{0 \leq i \leq d} \) is a system of orthogonal polynomials with respect to the inner product \( \langle f,g \rangle = \sum_{i=0}^{d} f(\theta_i)g(\theta_i)^{m_i} \).

Proof. First we show that (1) implies (2) and (3). Suppose that \( \mathcal{U} \) is a C-algebra. Since \( v_0(x) = 1, v_1(x) = x, v_0(x) \in \mathbb{R}[x], v_1(x) \in \mathbb{R}[x] \). Since for any \( i=1,2, \ldots, d-1, \)
\[ xv_i(x) = \sum_{j=0}^{i+1} p_{1i}^j v_j(x) \]  
(2.3.16)

and \( p_{1i}^j \in \mathbb{R} \), we see \( v_i(x) \in \mathbb{R}[x] \) by induction. The leading
of \( v_i(x) \) is \( (p_1^2 p_1^3 \ldots p_1^{i+1})^{-1} \). Write

\[ v_i(x) v_j(x) = \sum_{k=0}^{d} p_{ij}^k v_k(x) + c_{ij}(x) \prod_{k=0}^{d} (x - \theta_k). \]  
(2.3.17)

Assume \( i \neq 0 \) and set \( x = 0 \). Then

\[ 0 = p_{ij}^0 + c_{ij}(0)(-1)^{d+1} \theta_0 \theta_1 \ldots \theta_d. \]  
(2.3.18)

\[ \delta_{i', i} = (-1)^d c_{ij}(0) \theta_0 \theta_1 \ldots \theta_d. \]  
(2.3.19)

If we take \( j = d+1-i \), then \( c_{i, d+1-i} = c_{i, d+1-i}(x) \) is a nonzero
constant. This forces \( i' = d+1-i \). Indeed, \( c_{i, d+1-i} \) is the
product of the leading coefficients of \( v_i(x) \) and
\( v_{d+1-i}(x) \). By (2.3.14), the leading coefficient of \( v_i(x) \) is

\[
\frac{\prod_{j=1}^{i} (\theta_j - \theta_0)}{m_i[	heta_i]_i}
\]

and the leading coefficient of \( v_{i'}(x) \) is

\[
\frac{\prod_{j=1}^{i'} (\theta_j - \theta_0)}{m_i'[	heta_i'][i']}
\]

\[
= \frac{\overline{m}_i[	heta_i']_i'}{\theta_0 \theta_1 \ldots \theta_{i'-1} \prod_{j=1}^{i'} (\theta_j - \theta_0)}
\]
since it is real. Thus

\[ c_{i,d+1-i} = c_{ii}, \]

\[
=-\frac{m_i^2 \prod_{i \neq k} (\theta_i - \theta_k)}{\theta_0 \cdots \theta_d \prod_{j=1}^d (\theta_j - \theta_0)} = \frac{(-1)^d m_i}{\theta_0 \cdots \theta_d}. \tag{2.3.21}
\]

By the definition of C-algebra, 0 < ki = (-1)\( ^d c_{ii}, \theta_0 \theta_1 \cdots \theta_d = m_i. \) Therefore, (1) implies (2). Also, \( v_i(x) \mapsto v_i(\theta_j), \) 0 ≤ j ≤ d are the distinct linear representations of \( \mathcal{U}. \) Thus (3) is a consequence of the orthogonality relations for C-algebras (see [1]).

Suppose that (3) holds. Note that if \( f(x), g(x) \in \mathbb{R}[x], \) then \( \langle f, g \rangle \in \mathbb{R}. \) If we orthonormalize the polynomials \( 1, x, \ldots, x^d, \) by the Gram-Schmidt process, we obtain orthonormal polynomials with real coefficients. Since \( v_i(\theta_0) = m_i > 0, \) \( v_i(x) \) must be a real constant multiple of the orthonormal polynomial of degree \( i, \) hence \( v_i(x) \in \mathbb{R}[x]. \) Therefore (3) implies (2).

Finally, assuming \( v_i(x) \in \mathbb{R}[x], m_i > 0, \) we verify that \( \mathcal{U} \) is a C-algebra. Clearly \( \mathcal{U} \) is a commutative algebra with the identity \( v_0(x) = 1, \) so (i), (ii) of the definition hold.
The following propositions will be used in Chapter 3.4.

Proposition 3.2. Let \( \theta_0, \theta_1, \ldots, \theta_d \) be nonzero distinct complex numbers, satisfying \( \theta_0 = \bar{\theta}_0, \theta_i = \bar{\theta}_{d+1-i}, (i=1, \ldots, d) \) and (2.3.8). Let \( m_i, p_{i+1}^{d+1-i} \) be as in (2.3.3), (2.3.14), respectively. Then we have

\[
m_{i+1}^{p_{i+1}^{d+1-i}} = m_i^{p_{i+1}^{d-i}}
\]

(2.3.22)

Proof. Straightforward. \( \blacksquare \)
Proposition 3.3. Under the same assumption as Proposition 3.2, if $v_i(x) \in \mathbb{R}[x]$ for $i=0,1,\ldots,d-1$, then $v_d(x) \in \mathbb{R}[x]$.

Proof. Let $V(x) = v_0(x) + v_1(x) + \cdots + v_d(x)$. Then for $j=1,2,\ldots,d$,

$$V(\theta_j) = \sum_{i=0}^{d} v_i(\theta_j) = \frac{1}{m} \sum_{i=0}^{d} m_i v_j(\theta_i) = 0. \quad (2.3.23)$$

Moreover $V(\theta_0) = m_0 + m_1 + \cdots + m_d \in \mathbb{R}$, since $m_0 + m_1 \in \mathbb{R}$. Therefore $V(x)$ is the uniquely determined polynomial of degree $d$ under the above $d+1$ conditions, which force $V(x) \in \mathbb{R}[x]$.

Hence the assertion follows.  \[\blacksquare\]
CHAPTER III THE CASE WHERE THE GIRTH IS 5

3.1. Specialization of the results in section 2.3.

In this section we specialize Theorem 3.1 in Chapter II to obtain explicit conditions for the existence of the C-algebra.

Theorem 1.1. For real numbers \( x_1, x_2 \), the following are equivalent.

(1) There exist nonzero distinct complex numbers \( \theta_0, \theta_1, \theta_2, \theta_3, \theta_4 \) satisfying (2.3.8) with \( d=4, \theta_0=\bar{\theta}_0, \text{Re}\theta_1=\text{Re}\theta_4=x_1, \text{Re}\theta_2=\text{Re}\theta_3=x_2, \theta_2=\bar{\theta}_3 \), such that \( \mathcal{A}=\mathbb{C}[x]/(\prod_{i=0}^{4}(x-\theta_i)) \)

\( =\langle v_i(x) | 0 \leq i \leq 4 \rangle \) is a C-algebra, where \( v_i(x) \) \( 0 \leq i \leq 4 \) are defined by (2.3.3)-(2.3.7).

(2) \( 4(2x_1+1)(x_1-x_2+2x_2)(x_2^2+x_2-x_1)=x_2(6x_1x_2+2x_1-2x_2^2+x_2)^2 \)

\( \text{(3.1.1)} \)

\( x_2 \geq 0 \) and \( x_1<\min(x_2-2x_2^2, -\frac{1}{2}) \),

\( \text{(3.1.2)} \)

or \( x_2 < 0 \) and \( x_1>\max(x_2+\frac{1}{2}, x_2+x_2^2, -\frac{1}{2}) \).

Proof. Suppose that (1) holds. Write \( \theta_0=k, \theta_1=\bar{\theta}_0=x_1+\sqrt{-1}y_1, \theta_2=\bar{\theta}_3 \)

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=x_2+\sqrt{-1}y_2. k,y_1,y_2 \in \mathbb{R}. By Theorem 3.4, we have k=m_1>0, m_2>0.

Thus \( m_1=m_4, m_2=m_3 \), and
\[
\begin{align*}
\theta_0 + m_1(\theta_1 + \bar{\theta}_1) + m_2(\theta_2 + \bar{\theta}_2) &= 0 \quad (3.1.3) \\
\theta_0^2 + m_1(\theta_1^2 + \bar{\theta}_1^2) + m_2(\theta_2^2 + \bar{\theta}_2^2) &= 0 \quad (3.1.4)
\end{align*}
\]
i.e.,
\[
\begin{align*}
k + 2x_1 m_1 + 2x_2 m_2 &= 0 \quad (3.1.5) \\
k^2 + 2(x_1^2 - y_1^2) m_1 + 2(x_2^2 - y_2^2) m_2 &= 0 \quad (3.1.6)
\end{align*}
\]
Eliminating \( m_2 \), we obtain
\[
\begin{align*}
k(x_2^2 - y_2^2 - x_2 k) &= 2m_1 (x_2^2 - y_2^2) - x_1 (x_2^2 - y_2^2) \quad (3.1.7) \\
x_2^2 - y_2^2 - x_2 k &= 2x_2 (x_1^2 - y_1^2) - 2x_1 (x_2^2 - y_2^2) \quad (3.1.8)
\end{align*}
\]
By (2.3.14), (2.3.15), we have
\[
p_{11}^2 = \frac{\theta_1 (\theta_1 - \theta_0) m_1}{(\theta_2 - \theta_1) m_2} \quad (3.1.9)
\]
\[
p_{11}^1 + p_{12}^2 = \theta_1 - \frac{\theta_2 - \theta_0 \theta_3}{\theta_3 - \theta_2} \quad (3.1.10)
\]
Since \( p_{11}^2, p_{11}^1, p_{12}^2 \) are real, we obtain
\[
(x_1 - k)(x_1 y_2 - x_2 y_1) - (x_1^2 - x_2 y_1 + y_1)(y_2 - y_1) = 0 \quad (3.1.11) \\
x_2(x_2 - k) - y_2^2 = 2y_1 y_2 \quad (3.1.12)
\]
By (2.3.3), \( m_2 \) is real if and only if
\[
(x_2^2 - y_2^2)((x_1 - x_2)^2 - y_2^2 + y_1^2) + 2y_2^2(x_1 - x_2)(2x_2 - k) = 0 \quad (3.1.13)
\]
Eliminating \( k \) from (3.1.11), (3.1.12), we have
Eliminating \( k \) from (3.1.12), (3.1.13), we have

\[
(\mathbf{IC}_{r} \mathbf{X}) y^2 + (2x_r^2 + x_1^2 - 3y_1^2)x_2 y_2 + x_2 y_1 ((x_1 - x_2)^2 + y_1^2) = 0
\]  

(3.1.14)

Eliminating \( k \) from (3.1.12), (3.1.13), we have

\[
(x_1 - x_2) y_2^3 + (2x_1 - 3x_2) y_1 y_2^2 + (x_1^2 - x_2^2)x_2 y_2 + x_2 y_1 ((x_1 - x_2)^2 + y_1^2) = 0
\]  

(3.1.15)

If \( x_2 = 0 \), then (3.1.12) implies \( y_2 = -2y_1 \). Then (3.1.11) becomes \( x_1^2 - 3y_1^2 - 2kx_1 = 0 \) which gives (3.1.13) becomes \( -x_1^2 + 3y_1^2 - 2kx_1 = 0 \). Since \( k \neq 0 \), we must have \( x_1 = y_1 = 0 \), contradiction. Therefore \( x_2 \neq 0 \). From (3.1.14), (3.1.15), by using the fact \( x_2 y_2 \neq 0 \), we have

\[
y_2^2 + 2y_1 y_2 + (x_1 - x_2)^2 - 3y_1^2 = 0
\]  

(3.1.16)

From (3.1.14), (3.1.16), we have

\[
((3x_1 - x_2)y_1^2 - x_1(x_1 - x_2)(x_1 - 2x_2))y_2 + 2x_2 y_1 ((x_1 - x_2)^2 - y_1^2) = 0
\]  

(3.1.17)

Eliminating \( k \) from (3.1.8), (3.1.12), we have

\[
x_1 y_2^2 - y_1 y_2 + x_2 (x_1 y_2^2 - x_1 x_2^2) = 0
\]  

(3.1.18)

Eliminating \( y_2 \) from (3.1.16), (3.1.18),

\[
(2x_1 + 1)y_1 y_2 - (3x_1 - x_2)y_1^2 + x_1(x_1 - x_2)(x_1 - 2x_2) = 0
\]  

(3.1.19)

From (3.1.17), (3.1.19), by using the fact \( y_1 \neq 0 \),

\[
(2x_1 + 1)y_2^2 + 2x_2 ((x_1 - x_2)^2 - y_1^2) = 0
\]  

(3.1.20)

From (3.1.16), (3.1.20), we have

\[
2(2x_1 + 1)y_1 y_2 + (2x_1 + 1)((x_1 - x_2)^2 - 3y_1^2)
\]

\[-2x_2 ((x_1 - x_2)^2 - y_1^2) = 0
\]  

(3.1.21)

From (3.1.19), (3.1.21).
\[3y_1^2 = (x_1 - x_2)(x_1 - x_2 + 2x_2^2). \quad (3.1.22)\]

Since \(y_1 \neq 0\), we have \(x_1 \neq x_2\). Substituting (3.1.22) into (3.1.17), we find
\[(6x_1 x_2 + 2x_1 - 2x_2 + x_2) y_2 + 4y_1 (x_1 - x_2 - x_2^2) = 0. \quad (3.1.23)\]

Substituting (3.1.23) into (3.1.19), we have
\[3(2x_1 + 1) y_1 y_2 - x_2 (x_1 - x_2 - 6x_1 x_2 + 2x_1 - 2x_2 + x_2) = 0. \quad (3.1.24)\]

If \(x_1 = -\frac{1}{2}\), then \(6x_1 x_2 + 2x_1 - 2x_2 + x_2 = 0\) by (4.18), so \(x_2^2 + x_2 + \frac{1}{2} = 0\).

This is a contradiction since \(x_2\) is real. Therefore \(x_2 \neq -\frac{1}{2}\), and
\[y_1 y_2 = \frac{x_2 (x_1 - x_2) (6x_1 x_2 + 2x_1 - 2x_2^2 + x_2)}{3(2x_1 + 1)}. \quad (3.1.25)\]

By (3.1.23), (3.1.25),
\[y_2^2 = \frac{4x_2 (x_1 - x_2) (x_2^2 + x_2 - x_1)}{3(2x_1 + 1)}. \quad (3.1.26)\]

From (3.1.22), (3.1.25), (3.1.26), we find (3.1.1). Also from (3.1.12), (3.1.25), (3.1.26), we find
\[k = x_2 (2x_2 - 2x_1 + 1) > 0. \quad (3.1.27)\]

Since \(k = m_1 > 0\) and \(m_2 > 0\), (3.1.5) implies
\[x_2 (2x_1 + 1) < 0 \quad (3.1.28)\]

By (3.1.22), (3.1.26), (3.1.28), we have
\[(x_1 - x_2) (x_1 - x_2 + 2x_2^2) > 0. \quad (3.1.29)\]
\[(x_1 - x_2) (x_1 - x_2 - x_2^2) > 0. \quad (3.1.30)\]

Now it is straightforward to check that (3.1.2) is equivalent to the inequalities (3.1.27)-(3.1.30).
Conversely, suppose that \( x_1, x_2 \) satisfy (3.1.1), (3.1.2). Then the inequalities (4.21)-(4.24) hold. In particular, \( x_1 \neq -\frac{1}{2}, x_2 \neq 0, x_1 \neq x_2 \). Then the right hand sides of (3.1.22), (3.1.26) are positive, so we can find nonzero real numbers \( y_1, y_2 \) satisfying (3.1.22), (3.1.26). \( y_1, y_2 \) are determined up to sign, but we may choose their signs so that \( y_1, y_2 \) satisfy (3.1.27). This is possible by (3.1.1).

Define \( k \) by (3.1.27). Reversing the argument in the first part of the proof, we can recover (3.1.8) and (3.1.11)-(3.1.26). By (3.1.16), we have \( \text{Re}(\theta_1 - \theta_2)^2(\theta_1 - \theta_3) = 0 \). Since (3.1.11) is equivalent to \( \theta_1(\theta_1 - \theta_0)/(\theta_2 - \theta_1) \in \mathbb{R} \), we see that 
\[
\theta_1(\theta_1 - \theta_0)(\theta_1 - \theta_2)(\theta_1 - \theta_3)(\theta_1 - \theta_4)
\]
is real, hence \( m_1 \in \mathbb{R} \). Also, we have \( m_2 \in \mathbb{R} \) by (3.1.13). Therefore, as in the first part of the proof, we obtain (3.1.7). Thus \( m_1 = k \) by (3.1.8).

Hence (2.3.8) holds and \( m_1 > 0 \). Also,
\[
m_2 = -\frac{k(2x_1 + 1)}{2x_2} > 0. \tag{3.1.31}
\]

By Theorem 3.1 in Chapter II, it now suffices to show \( v_i(x) \in \mathbb{R}[x], i = 0, 1, \ldots, 4 \). By the definition, \( v_0(x), v_1(x) \in \mathbb{R}[x] \). By (3.1.11), we have \( p_{11}^2 \in \mathbb{R} \), thus the leading coefficient of \( v_2(x) \) is real. Since \( v_2(0) = 0, v_2(\theta_0) = m_2 \in \mathbb{R} \), we see \( v_2(x) \in \mathbb{R}[x] \). By (2.3.14).
which is real since $\text{re} \, \theta_2 = 0$. Since

$$x v_1(x) = p_{11}^2 v_2(x) + p_{11}^1 v_1(x)$$

and $p_{11}^2 v_2(x) \in \mathbb{R}[x]$, we see $p_{11}^1 \in \mathbb{R}$. Then by \((3.1.12)\), we have $p_{12}^2 \in \mathbb{R}$. Setting $x = k$ in

$$p_{12}^3 v_3(x) = x v_2(x) - p_{12}^2 v_2(x) - p_{12}^1 v_1(x),$$

we find $p_{12}^1 \in \mathbb{R}$, hence $v_3(x) \in \mathbb{R}[x]$. Then $v_4(x) \in \mathbb{R}[x]$ by Proposition 3.3 in Chapter II. Therefore $\mathcal{U}$ is a C-algebra by Theorem 3.1.

**Remark.** The set of $(x_1, x_2)$ satisfying \((3.1.1), (3.1.2)\) is not empty. For example, one can take $x_1 = -(21 + 5\sqrt{33})/32$, $x_2 = 21/2$. However the C-algebra obtained cannot be the adjacency algebra of a nonsymmetric P- and Q-polynomial association scheme, since $\theta_1$ is not an algebraic integer for the above $x_1$.

### 3.2. The nonexistence theorem

By Theorem 2.2. in Chapter 2 and Theorem 1.1, if there exists a nontrivial nonsymmetric P- and Q-polynomial association scheme with girth 5, then there exists a solution $(x_1, x_2)$ of \((3.1.1)\) such that $2x_1, 2x_2$ are rational integers. Write $X_1 = 2x_1, Y = 2x_2$, and rewrite \((3.1.1)\) as
\[ 8X^3 + (9Y^3 + 16Y^2 - 12Y + 8)X^2 - (10Y^4 + 2Y^3 - 16Y^2 + 16Y)X \\
+ Y^5 - 6Y^4 - 3Y^3 + 8Y^2 = 0. \tag{3.2.1} \]

It turns out that the equation (3.2.1) has only finitely many integer solutions, and none of them corresponds to a nonsymmetric \( P \)- and \( Q \)-polynomial association scheme. We omit the details of determination of all integer solutions of (3.2.1), which is rather complicated. Instead, in the proof of Theorem 2.1, we use the integrality of the intersection numbers to obtain a bound for \( Y \).

Theorem 2.1. The only nonsymmetric \( P \)- and \( Q \)-polynomial association scheme with girth 5 is the directed five cycle.

Proof. If there exists a nontrivial nonsymmetric \( P \)- and \( Q \)-polynomial association scheme with girth 5, there exists an integer solution \((X,Y)\) of (3.2.1) such that \( x_1 = X/2, x_2 = Y/2 \) satisfy the conditions of Theorem 3.1. By (2.3.14),

\[ p_{11}^2 = \frac{\theta_1 (\theta_1 - \theta_0) m_1}{(\theta_2 - \theta_1) m_2}. \tag{3.2.2} \]

Hence \( (x_2 - x_1) m_2 p_{11}^2 = (x_1 (x_1 - k) - y_1^2) m_1 \). By (3.1.22), (3.1.27), and (3.1.31), we find

\[ p_{11}^2 = - \frac{Y(Y^2 - Y - 3XY - 2X)}{6(X+1)}. \tag{3.2.3} \]
Since \( \mathcal{U} \) is the adjacency algebra of a nontrivial non-symmetric \( P \)- and \( Q \)-polynomial association scheme with girth 5, \( p_{11}^2 \) is an integer. Thus

\[
\frac{Y(Y^2 + 2Y + 2)}{X+1}
\]

is an integer. Since \( Y = 2x_2 \neq 0 \), and \( Y^2 + 2Y + 2 > 0 \), we have

\[
|Y|(Y^2 + 2Y + 2) \geq |X+1|. \tag{3.2.4}
\]

Moreover,

\[
k = p_{10}^1 + p_{11}^1 + p_{12}^1 + p_{13}^1 + p_{14}^1 = 1 + 2p_{11}^1 + 2p_{12}^1.
\tag{3.2.5}
\]

so \( k \) is odd. The trace of the intersection matrix \( B_1 = (p_{ij}) \) is

\[
0 + 0 + 0 + 0 = k + X + Y. \quad \text{Also } \text{tr}B_1 = p_{10}^0 + p_{11}^1 + p_{12}^2 + p_{13}^3 + p_{14}^4 = 2p_{11}^1 + 2p_{12}^1.
\]

which is even. Thus \( X + Y \) is odd. By (3.1.27), \( Y \) must be odd. Let \( f(X,Y) \) be the left hand side of (3.2.1).

Case 1. \( Y > 0 \).

By (3.1.2), (3.2.4), we have \(-1 > X < Y^3 - 2Y^2 - 2Y - 1\). Then

\[
f(X,Y) > X^2(Y^3 - 28Y) - 2XY(5Y^3 + Y^2 - 8Y + 8) + Y^2(Y^3 - 6Y^2 - 3Y + 8).
\]

If \( Y \geq 7 \), then \( f(X,Y) > 0 \), contradiction. Thus \( Y \leq 5 \).

Case 2. \( Y < 0 \).

By (3.1.2), (3.2.4), we have \( 0 \leq X \leq Y^3 - 2Y^2 - 2Y - 1 \). Then

\[
f(X,Y) \leq X^2(Y^3 - 28Y) - 2XY(5Y^3 + Y^2 - 8Y + Y) + Y^2(Y^3 - 6Y^2 - 3Y + 8).
\]

If \( Y \leq -7 \), then \( f(X,Y) < 0 \), contradiction. Thus \( Y \geq -5 \).
Now we conclude \( Y \in \{ \pm 1, \pm 3, \pm 5 \} \). The only integer solutions of (3.2.1) with \( Y \in \{ \pm 1, \pm 3, \pm 5 \} \) are \((X,Y) = (-2, -1), (0, 1)\). However, neither satisfies (3.1.2). This is a contradiction. \( \blacksquare \)
CHAPTER IV
THE CASE WHERE THE GIRTH IS 6

4.1. Specialization of the results in section 2.3.

Specializing Theorem 3.1. to the case \( d = 5 \), we obtain the following.

Theorem 1.1. Let \( \theta_0, \theta_1, \ldots, \theta_5 \) be nonzero distinct complex numbers, satisfying \( \theta_0 = \overline{\theta}_0, \overline{\theta}_1 = \theta_6-1 \). (i=1, ..., 5) and (2.3.8).
Let \( m_i, v_i(x) \) be defined by (2.3.3)-(2.3.7). Let \( p_{ij}^k \) be the coefficients defined by (2.3.9). Then the following are equivalent.

1. \( \mathcal{A} = \mathbb{C}[x]/(\prod_{i=0}^{5} (x-\theta_i)) = \langle v_i(x) \mid 0 \leq i \leq 5 \rangle \) is a \( C \)-algebra
2. \( m_1 = \theta_0 > 0, m_2 > 0, p_{11}^2 \in \mathbb{R}, p_{12}^3 \in \mathbb{R}, p_{12}^2 \in \mathbb{R} \).

Proof. It suffices to show that (2) implies (1). \( \theta_0 > 0, \theta_3 < 0 \) and (2.3.3) imply \( m_3 > 0 \). Also, we have \( m_4 = m_2 > 0, m_5 = m_1 = \theta_0 = \theta_0 \) > 0. We have \( v_0(x) = 1, v_1(x) = x, x^2 = p_{11}^2 v_2(x) + p_{11}^1 v_1(x), p_{11}^2 \in \mathbb{R} \). setting \( x = \theta_0, \theta_2 = p_{11}^2 m_2 + p_{11}^1 \theta_0 \), we find \( p_{11}^1 \in \mathbb{R} \). Thus
\[ v_2(x) \in \mathbb{R}[x]. \] Since \[ xv_2(x) = p_{12}^3 v_3(x) + p_{12}^2 v_2(x) + p_{12}^1 v_1(x), \]
we find \( p_{12}^1 \in \mathbb{R} \) by setting \( x = \theta_0 \). This implies \( v_3(x) \in \mathbb{R}[x] \). By Proposition 3.2 in Chapter 2, we have \( p_{13}^4 = m_3 p_{12}^3 / m_4 \in \mathbb{R} \). Also, \( p_{13}^3 \in \mathbb{R} \) by (2.3.15). Since
\[
\begin{align*}
v_3(x) &= p_{13}^4 v_4(x) + p_{13}^3 v_3(x) + p_{13}^2 v_2(x) + p_{13}^1 v_1(x) \quad (4.1.1)
\end{align*}
\]
we have \( p_{13}^2 v_4(x) + p_{13}^1 v_2(x) + p_{13}^1 v_1(x) \in \mathbb{R}[x] \). Setting \( x = \theta_0 \), we find \( p_{13}^1 m_2 + p_{13}^1 m_1 \in \mathbb{R} \). Since \( \theta_3 \in \mathbb{R} \), it follows that \( v_2(\theta_3) \in \mathbb{R} \).
\[
\begin{align*}
v_4(\theta_3) &= m_4 v_3(\theta_3) / m_3 = m_4 v_3(\theta_2) / m_3 = m_4 v_2(\theta_3) / m_3 \in \mathbb{R}.
\end{align*}
\]
Therefore, \( p_{13}^2 v_2(\theta_3) + p_{13}^1 \theta_3 \in \mathbb{R} \). If we let \( \alpha \) be the imaginary part of \( p_{13}^2 \), \( \beta \) the imaginary part of \( p_{13}^1 \), then
\[
\begin{align*}
&\begin{cases}
m_2 \alpha + m_1 \beta = 0 \\
v_2(\theta_3) \alpha + \theta_3 \beta = 0
\end{cases} \quad (4.1.2)
\end{align*}
\]
We may regard this as a system of equations with respect to \( \alpha \) and \( \beta \). The determinant is
\[
\begin{align*}
m_2 \theta_3 - m_1 v_2(\theta_3) & = m_2 \theta_3 - m_1 (\frac{1}{p_{11}} (\theta_3^2 - p_{11}^1 \theta_3)) \\
& = - \frac{m_2 p_{11}^2 + m_1 p_{11}^1}{p_{11}^2} \theta_3 - \frac{m_1}{p_{11}^2} \theta_3^2
\end{align*}
\]
\[
\begin{align*}
&= - \frac{\theta_3}{p_{11}^2} (\theta_3^2 - m_1 \theta_3) \\
&= - \frac{\theta_3}{p_{11}^2} (\theta_3 - \theta_3) \neq 0 \quad (4.1.3)
\end{align*}
\]
Therefore $\alpha=\beta=0$, so that $p_{13}^2, p_{13}^1 \in \mathbb{R}$. Hence $v_4(x) \in \mathbb{R}[x]$.

Finally, $v_5(x) \in \mathbb{R}[x]$ by Proposition 3.3 in Chapter II.

4.2. The set of solutions of the homogeneous equations obtained in section 4.1.

In this section we investigate the equations obtained by the conditions $m_1 \in \mathbb{R}$, $m_2 \in \mathbb{R}$, $p_{11}^2 \in \mathbb{R}$, $p_{12}^3 \in \mathbb{R}$, $p_{12}^2 \in \mathbb{R}$. Let $\theta_0, \theta_1, \ldots, \theta_5$ be nonzero distinct complex numbers satisfying $\theta_0 = \bar{\theta}_0, \bar{\theta}_1 = \theta_6 - i$. (i=1,...,5). Note that the numerator of (2.3.3) is always real under these assumptions. Thus $m_1 \in \mathbb{R}$ if and only if

\[ \text{Re } \theta_1(\theta_0-\theta_1)(\theta_2-\theta_1)(\theta_3-\theta_1)(\theta_4-\theta_1)=0. \quad (4.2.1) \]

Also $m_2 \in \mathbb{R}$ if and only if

\[ \text{Re } \theta_2(\theta_0-\theta_2)(\theta_1-\theta_2)(\theta_3-\theta_2)(\theta_5-\theta_2)=0. \quad (4.2.2) \]

$p_{11}^2 \in \mathbb{R}$ if and only if

\[ \text{Im } \theta_1(\theta_1-\theta_0)(\theta_4-\theta_5)=0. \quad (4.2.3) \]

$p_{12}^3 \in \mathbb{R}$ if and only if

\[ \text{Im } \theta_2(\theta_2-\theta_0)(\theta_2-\theta_1)(\theta_3-\theta_5)(\theta_3-\theta_4)=0. \quad (4.2.4) \]

$p_{12}^2 \in \mathbb{R}$ if and only if

\[ \text{Im } (\theta_1^2-\theta_1\theta_2+\theta_0\theta_2-\theta_2^2)(\theta_3-\theta_4)=0 \quad (4.2.5) \]
The equations (4.2.1)-(4.2.5) do not imply \( m_1 = \theta_0, \ m_1 > 0, \) \( m_2 > 0. \) However, it is important to make the following remark. Under (4.2.1), \( m_1 = \theta_0 \) is equivalent to

\[
\text{Re} \left\{ \theta_1 (\theta_0 - \theta_1) (\theta_2 - \theta_1) (\theta_3 - \theta_1) (\theta_4 - \theta_1) (\theta_5 - \theta_1) \right\} \\
+ (\theta_1 - \theta_0) (\theta_2 - \theta_0) (\theta_3 - \theta_0) (\theta_4 - \theta_0) (\theta_5 - \theta_0) = 0 \quad (4.2.6)
\]

Since (4.2.1)-(4.2.5) are all homogeneous, the set of solutions is a union of rays. If \( \theta_0 = \lambda \alpha_0, \ \theta_1 = \lambda \alpha_1, \ \theta_2 = \lambda \alpha_2, \ \theta_3 = \lambda \alpha_3, \ \theta_4 = \lambda \alpha_4, \ \theta_5 = \lambda \alpha_5, \ \lambda \in \mathbb{R}, \) is a ray of solutions, then there exists a unique nonzero \( \lambda \in \mathbb{R} \) such that \( \theta_0 = \lambda \alpha_0, \ \theta_1 = \lambda \alpha_1, \ \theta_2 = \lambda \alpha_2, \ \theta_3 = \lambda \alpha_3, \ \theta_4 = \lambda \alpha_4, \ \theta_5 = \lambda \alpha_5 \) is a solution of (4.2.6). Therefore, we will only consider the equations (4.2.1)-(4.2.5), whose set of solutions can be regarded as an algebraic set of the real projective space of dimension 5. Let \( \theta_0 = k, \ \theta_1 = x + \sqrt{-1} y, \ \theta_2 = u + \sqrt{-1} v, \ \theta_3 = z. \) Let \( f_1, f_2, f_3, f_4, f_5 \) be the left hand side of (4.2.1)-(4.2.5), respectively. Then \( f_1, f_2, f_3, f_4, f_5 \in \mathbb{Z}[x, y, u, v, z, k]. \) Let \( Z \) be the set of zeros of \( f_1, f_2, f_3, f_4, f_5: Z = \{(x, y, u, v, z, k) | f_i(x, y, u, v, z, k) = 0 \text{ for } i = 1, 2, 3, 4, 5\}. \) We are not interested in the set \( Z \) itself, rather, the subset \( Z_0 \) which consists of elements of \( Z \) for which \( \theta_0, \theta_1, \ldots, \theta_5 \) are nonzero distinct.
Lemma 2.1. If \((x,y,u,v,z,k)\in Z_0\), then \(x \neq u\).

Proof. If \(x = u\), then \(f_3 = (x^2 - y^2 - xk)(v - y)\). Since \(\theta_1 \neq \theta_2\), we must have \(y^2 = x^2 - xk\). Substituting this into \(f_2\), we find \(f_2 = x(2x-k)(x^2 - xk - v^2)\). But \(0 < y^2 = x^2 - xk\) implies \(x \neq 0, x-k \neq 0, 2x-k \neq 0\). Thus \(v^2 = x^2 - xk = y^2\), \(\theta_1 = \theta_2\) or \(\theta_4\), contradiction. \(\blacksquare\)

Lemma 2.2. If \((x,y,u,v,z,k)\in Z_0\), then \(xv \neq uy\).

Proof. If \(xv = uy\), then by Lemma 2.1, \(x\) is nonzero. Thus

\[
f_3 = \frac{(x^2 + y^2)(x-u)}{x} \tag{4.2.7}
\]

which can not be zero. \(\blacksquare\)

By Lemma 2.2, we can solve (4.2.3) for \(k\):

\[
k = \frac{(x^2 + y^2)y + (x^2 - y^2)v - 2xyu}{xv - yu} \tag{4.2.8}
\]

Solving (4.2.5) for \(k\), we have

\[
k = -\frac{yz^2 + (y+v)(u^2 + v^2 - 2uz)}{vz} \tag{4.2.9}
\]

Substituting (4.2.9) into \((v^2 + (u-z)^2)f_3 + f_4 = 0\), we obtain

\[
z = u - \frac{v((x-u)^2 + v^2 - y^2)}{2y(x-u)} \tag{4.2.10}
\]

Moreover, substituting (4.2.8), (4.2.10) into (4.2.1), (4.2.5), we obtain
Suppose that \( x, y, u, v \) satisfy (4.2.11), (4.2.12), \( x \neq u \), and that \( \theta_1, \theta_2, \theta_4, \theta_5 \) are nonzero distinct. One can verify that if \( k, z \) are defined by (4.2.8), (4.2.10), respectively, then \( (x, y, u, v, z, k) \) satisfies (4.2.1)-(4.2.5).

The directed 6-cycle gives a solution of (4.2.11), (4.2.12):

\[
(x, y, u, v) = (x, \pm \sqrt{3} x, -x, \pm \sqrt{3} x), \tag{4.2.13}
\]

where \( x \) is an arbitrary real number. It is easy to verify that the following is also a solution of (4.2.11), (4.2.12):

\[
(x, y, u, v) = (x, \pm \sqrt{3} x, 3x, \pm \sqrt{3} x), \tag{4.2.14}
\]

which gives a solution of (4.2.1)-(4.2.5):

\[
(x, y, u, v, z, k) = (x, \pm \sqrt{3} x, 3x, \pm \sqrt{3} x, 4x, 2x). \tag{4.2.15}
\]

However, this solution violates the condition \( m_2 > 0 \).

Note that (4.2.1) is quadratic in \( (x-u)^2 \), and its discriminant is

\[
16y^2(v-2y)(v^3+2v^2y-4y^3). \tag{4.2.16}
\]

If there exists a nonsymmetric \( P \) - and \( Q \) - polynomial association scheme with girth 6, then \( (x-u)^2 \) is rational.
so \((v-2y)(v^3+2v^2y-4y^3)\) is a square. Moreover, \(v^2,v,y,y^2\) are rational by Theorem 2.2 in Chapter II. If \(v=2y\), then \(g_1=4y((x-u)^2+3y^2)^2\neq 0\), contradiction. Thus \(X=y/(v-2y)\) is rational. Therefore the elliptic curve

\[ Y^2=12X^3+20X^2+8X+1 \quad (4.2.17) \]

must have a rational point. It can be checked that the elliptic curve (4.2.17) has infinitely many rational points, by exhibiting at least 17 rational points (Mazur's theorem, see for example [8]).

Since \((x-u)^2\) is positive, it follows from (4.2.11) that

\[ (v+2y)(v^3+2v^2y+vy^2-6y^3)<0. \quad (4.2.18) \]

Solving this inequality, we have

\[-2 < \frac{v}{y} < \sqrt[3]{\frac{82 + \sqrt{83}}{27}} + \sqrt[3]{\frac{82 - \sqrt{83}}{27}} \frac{2}{3}. \quad (4.2.19)\]

We have obtained various necessary conditions for the existence of a nontrivial nonsymmetric P- and Q-polynomial association scheme with girth 6. Unfortunately we are unable to prove the nonexistence theorem.

Conjecture. There exist no nontrivial nonsymmetric P- and Q-polynomial association scheme with girth 6.

On the other hand, there exist \(\theta_0,\theta_1,\ldots,\theta_5\) satisfying the conditions of Theorem 1.1 which do not correspond to
any nonsymmetric P- and Q-polynomial association scheme. The point \((X,Y)=(-2/3,1)\) is a rational point of the elliptic curve (4.2.17) which gives a solution of the equations (4.2.11),(4.2.12):

\[(x,y,u,v) = (x,y, \frac{35-\sqrt{889}}{48} x, \frac{1}{2} y)\]  

with \(y^2 = \frac{529+13\sqrt{889}}{4320} x^2\).

Set

\[x = \frac{2713+91\sqrt{889}}{60}\]  

(4.2.21)

and define \(k\) by (4.2.9), \(z\) by (4.2.10). Let

\[\theta_0 = k = \frac{432943+14521\sqrt{889}}{2700} \]  

(4.2.22)

\[\theta_1 = \frac{2713+91\sqrt{889}}{60} + \sqrt{-1} y.\]  

(4.2.23)

\[\theta_2 = \frac{1757+59\sqrt{889}}{360} + \frac{1}{2} y.\]  

(4.2.24)

\[\theta_3 = -\frac{239+8\sqrt{889}}{75}.\]  

(4.2.25)

\[\theta_4 = \frac{\theta_2}{\theta_1}.\]  

(4.2.26)

\[\theta_5 = \frac{\theta_5}{\theta_1}.\]  

(4.2.27)

where

\[y^2 = \frac{210851341+7071727\sqrt{889}}{243000}.\]  

(4.2.28)

Then \((\theta_0, \theta_1, \ldots, \theta_5)\) satisfies the conditions of Theorem 1.1. Therefore, there exists a system of orthogonal polynomials \((v_i(x))_{0 \leq i \leq 5}\) defined by (2.3.4)-(2.3.7).
LIST OF REFERENCES


6. ________, Nonsymmetric, metric, cometric association schemes are self-dual, preprint.
