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The discovery of a second rotational wave in an infinite fluid-filled porous material

Liu, Qing-Rui, Ph.D.
The Ohio State University, 1989
THE DISCOVERY OF A SECOND ROTATIONAL WAVE IN AN INFINITE FLUID-FILLED POROUS MATERIAL

DISSER TATION

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

By

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* * * * *

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To my wife
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NOMENCLATURE

a : Characteristic length
b(1) : Body force of the solid part
b(2) : Body force of the fluid part
C_1 : Amplitude of the solid part
C_2 : Amplitude of the fluid part
D : Dissipation energy function
e : Solid dilatation
e_{ij} : Averaged solid strain
e_{ij}^{(1)} : Averaged solid strain
e_{ij}^{(2)} : Averaged fluid strain
f : Cyclic Frequency
f_c : Reference frequency
h : Heat flux
K̃ : Permeability
k_s : Bulk modulus of the solid
k_f : Bulk modulus of the fluid
p^{(1)} : The symmetric part of the stress associated with the solid
p^{(2)} : The symmetric part of the stress associated with the fluid
\[ \begin{align*}
  r & : \text{Heat supply} \\
  s & : \text{Fluid stress} \\
  S & : \text{Specific surface area} \\
  s_1 & : \text{The solid constituent} \\
  s_2 & : \text{The fluid constituent} \\
  T & : \text{The kinetic energy function} \\
  t_s^{(1)} & : \text{Surface traction over the solid part} \\
  t_s^{(2)} & : \text{Surface traction over the fluid part} \\
  U & : \text{Internal energy} \\
  u_s & : \text{Solid displacement} \\
  u_s^{(1)} & : \text{Solid displacement} \\
  u_s^{(2)} & : \text{Fluid displacement} \\
  V & : \text{Total volume} \\
  V_f & : \text{Fluid volume} \\
  v_s^{(1)} & : \text{Solid velocity} \\
  v_s^{(2)} & : \text{Fluid velocity} \\
  V_L & : \text{Dilatational wave velocity in rock} \\
  V_F & : \text{Dilatational wave velocity in water} \\
  V_1 & : \text{Wave velocity of the first dilatational wave} \\
  V_2 & : \text{Wave velocity of the second dilatational wave} \\
  \dot{V}_1 & : \text{Wave velocity of the first rotational wave}
\end{align*} \]
\( \tilde{V}_2 \) : Wave velocity of the second rotational wave

\( V_C \) : Reference velocity of the dilatational wave

\( V_R \) : Reference velocity of the rotational wave

\( V_S \) : Shear wave velocity in rock

\( w_i \) : Fluid displacement

\( \alpha \) : Radial Frequency

\( \varepsilon \) : Fluid dilatation strain

\( \mu^s \) : Shear modulus of the solid

\( \mu^f \) : Kinematic fluid viscosity

\( \mu_{11} \) : Coefficient of viscosity due to rotational interaction

\( \mu_2 \) : Coefficient of viscosity due to velocity gradient

\( \pi \) : Diffusion force

\( \rho_f \) : Fluid density

\( \rho_s \) : Solid density

\( \sigma^{(1)}_{ij} \) : Stress of the solid

\( \sigma^{(2)}_{ij} \) : Stress of the fluid

\( \sigma_{ij} \) : Stress of the solid

\( \phi \) : Porosity

\( \psi_1 \) : Phase angle of the solid constituent

\( \psi_2 \) : Phase angle of the fluid constituent
CHAPTER I
INTRODUCTION

Problems related to fluid-filled porous materials have wide applications in geophysics and material science. In ocean sediment studies, the marine seismologist [30-32] is particularly interested in gathering information about the composition and the history of the ocean bottom through the use of propagating waves. In the petroleum industry, engineers locate oil reservoirs by examining reflected-wave data from underground discontinuities. Both marine sediments and geological materials are porous. They are filled with fluids such as air, water and oil. In some cases, the assumption that geological materials are a single continuum may be appropriate. However, when the pores are relatively large, the effects of the pores and of the fluid inside the pores on the mechanical behavior of the geological materials are significant. Therefore, the investigation of wave phenomena in a fluid-filled porous material is of great importance.

In the investigation of wave phenomena, it is important to have constitutive equations which describe the overall mechanical response of a fluid-filled porous material, including interactions between fluid and solid constituents. Traditionally, the interaction force between fluid and solid constituents is considered through Darcy's Law. The pointwise stress distribution exerted from the viscous fluid to the pore walls induces an overall interaction force. Since this stress
distribution on the pore walls is not generally constant, it also causes an interaction torque between the solid and fluid constituents in addition to the interaction force. In the well-known theory by Biot, however, the effect of the interaction torque is ignored. Also, in this traditional theory, the overall viscous stress is not considered even though the fluid inside the pores is assumed to be Newtonian viscous.

In this work, we will examine the effects of the interaction torque and the overall viscous stress on the wave phenomena in a fluid-filled porous material. The obtained results will be compared with the traditional theory [5, 6].

1.1 Review of Literature

The study of elastic wave propagation in a fluid-saturated porous material was initiated by Biot in the 1950s [5-6]. The porous material was assumed to be statistically isotropic and linearly elastic, and the pores were assumed to be interconnected and fully saturated with a Newtonian viscous fluid. His theory was based on the theory of linear elasticity and Darcy's Law. He obtained the equations of motion governing the wave propagation on the basis of Lagrange equations through a proposed kinetic energy function. Biot predicted the existence of two dilatational waves in a fluid-filled porous material, as opposed to one dilatational wave in a non-porous solid material. The dispersion of the dilatational wave of the first kind is practically negligible. However, the dilatational wave of the second kind was shown to be highly attenuated and thus is not easy to detect. In addition to the two dilatational waves, Biot only predicted the existence of one rotational wave. The dispersion of this rotational wave is also shown to be negligible.
The existence of a second bulk compressible wave in a fluid-filled porous material was experimentally confirmed by Plona [29] in 1980. In his experiment, he used sintered glass beads saturated with water. Subsequently, the dilatational wave of the second kind was also observed in human bones [20]. Experimentally, this dilatational wave of the second kind is also shown to be very attenuated and it can only be detected under certain conditions [8, 29], namely

(a) Continuity of the solid matrix and interconnection of the pores.
(b) Adequately high incident wave frequency (ultrasonic).
(c) Sufficiently small pore size in order to avoid scattering.
(d) Significant differences between the bulk moduli of the fluid and solid for distinguishing between the two bulk waves.

The dilatational wave of the second kind can be neglected in some application problems because of its attenuated nature. However, in the application of Biot's theory to layered porous materials [8], a significant amount of energy may be lost due to the energy partition to the second dilatational wave through the mode conversion at the interface. In this case, the effect of the dilatational wave of the second kind on other types of waves cannot be disregarded.

There are many articles published on the applications of Biot's theory. Stoll [30-32] studied acoustic waves in saturated ocean sediment. Berryman [4] has compared Biot's theory with experiments. The effect of permeability [8] on the surface wave as well as the reflection and refraction mechanism [8, 10] have been examined. The attenuation of the dilatational waves was shown to be "insensitive" to changes in permeability for any interface condition; in contrast, the
attenuation of the shear wave was shown to be sensitive to variations in permeability. Motivated by Biot's work, Feng et al [13] have shown that a slow surface wave (compared to Rayleigh surface wave) may propagate at a fluid-porous solid interface; this wave phenomenon also has been verified experimentally [26].

As indicated by Biot [6], the Poiseuille flow assumption is no longer valid when the frequency is beyond a critical value. When Poiseuille flow breaks down, the drag and virtual mass coefficients depend on frequency. Many simple models such as pores of flat and circular shapes have been established to simulate these coefficients as a function of frequency. For example, Bedford et al [3] have studied the oscillatory motion of a cylindrical wall, and closed-form solutions have been obtained for the cases of cylindrical pores parallel to, perpendicular to, and at an arbitrary angle to the motion.

In Biot's theory, it is implicitly assumed that the outer boundary of a sample consists of solid particles when the deformation of a porous material is considered. However, the outer boundary of an actual sample consists of both solid and fluid particles. From this point of view, the micro-mechanical constitutive theory has been recently re-examined by Katsube [19]. The volume average stress tensors were shown to have skew-symmetric parts, which are related to the interaction torque between solid and fluid constituents. The skew-symmetric stress tensor of the solid constituent was shown to be equal in magnitude and opposite in sign to that of the fluid constituent. In addition, the overall viscous stress was also shown to exist in a porous material saturated with a Newtonian viscous fluid.

From a continuum mechanics point of view, mixture theories [1, 2] have been developed for multi-phase materials, such as fluid-fluid
mixtures, fluid-gas mixtures, composite materials and fluid-filled porous materials. Each point of the mixture is assumed to be occupied by two or more distinct particles. At the same time, because of this assumption, the interactions between distinct constituents have been well described within the theoretical framework of mixture theories. When mixture theories are applied to problems related to fluid-filled porous materials [14], they predict the interaction torque term in the form of skew-symmetric stress tensors and the overall viscous stress. These skew-symmetric stress tensors satisfy the same special property as in the micro-mechanical theory. The interaction torque has been shown to be proportional to the relative vorticity tensor under the assumption of isotropy. While there is no proof available for this constitutive relation from the micro-mechanical point of view, the constitutive equations have a physical basis because the interaction torque becomes larger if the overall fluid rotates faster relative to the overall solid.

1.2 Research Objectives and Outline

The goal of this study is to investigate plane waves in a fluid-filled porous material based on mixture theory and to examine the effect of these additional terms on dispersion and attenuation mechanisms. Development of this thesis proceeds as follows.

Biot's theory of elastic wave propagation in a fluid-filled porous material, including the constitutive equations as well as equations of motion, is presented in Chapter II.

In Chapter III, the basic results in the mixture theory are summarized. The linear constitutive equations and the equations of
motion obtained in the modified mixture theory are presented. In the modified mixture theory, the material constants are expressed in more physically meaningful constants. The virtual mass coefficient $\rho_{12}$, which was first introduced by Biot in his phenomenological theory but ignored in the mixture theory, is shown to have physical significance. We further modify the equations of motion in the mixture theory by including the virtual mass coefficient terms.

In Chapter IV, using Biot's results for wave propagation in a porous material saturated with an ideal fluid and also Mackenzie's effective moduli, we examine the phase velocities as well as amplitude ratios of the three types of waves as a function of porosity.

In Chapter V, the plane wave phenomenon in a fluid-filled porous material is studied based on the modified mixture theory. Four types of waves, two dilatational and two rotational, are shown to exist in a fluid-filled porous material. The dispersion, attenuation and phase differences of these four types of waves are studied and compared with those obtained by Biot's theory. In addition, the porosity dependence of the wave velocities of these four types of waves is studied in both low and high frequency ranges. In the high frequency range, the results obtained are compared with the experimental and the theoretical investigations reported in the literature.

Finally, research conclusions are summarized in Chapter VI.
CHAPTER II
BIOT'S THEORY

Biot's theory [5-6] for a fluid-filled porous material including the equations of motion and constitutive equations is presented. The basic assumptions associated with the material properties of a fluid-filled porous material are summarized. In order to obtain equations of motion, Biot employed the Lagrange equation, postulating the kinetic energy and the dissipation function for a representative sample. These postulations are briefly explored.

2.1 Constitutive Equations for a Porous Solid Filled with a Fluid

We consider a representative volume element of a fluid-filled porous material. On one hand, this representative element is considered to be large enough so that it contains the details of the microstructure (e.g., several pores or grains). On the other hand, the sample must be small enough for the average response to be representative of the local response of a porous material idealized as a continuum. Instead of considering pointwise stress and strain distribution in this representative volume element, we introduce volume average kinetic and kinematic quantities with volumetric porosity and attempt to apply the principles of continuum mechanics to this inhomogeneous media.
The volume average stress may be decomposed into the solid part, \( \sigma_{ij} \), and fluid part, \( s \delta_{ij} \), respectively. The magnitude of the fluid stress, \( s \), is proportional to the fluid pressure \( p \) as follows:

\[
s = -\phi p,
\]

where

\[
\phi = \frac{V_f}{V},
\]

\( V_f \) and \( V \) are, respectively, the volume of the fluid and that of the representative sample.

The average strain tensor corresponding to the solid and the fluid (dilatation strain) are denoted by

\[
e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}),
\]

and

\[
\epsilon = \omega_{k,k'}
\]

where \( u_i \) and \( \omega_k \) are, respectively, the components of the average solid and fluid displacement vectors*.

To establish the stress-strain relation for a fluid-filled porous material, Biot made the following physical assumptions:

(1) The porous material is statistically isotropic and linearly elastic.

---

* This is Biot's notation, though the velocity vector of the fluid is generally considered to be the primary variable.
(2) The pores are all interconnected and fully saturated with a fluid.

(3) The fluid inside the pores is Newtonian viscous.

(4) The porous material is a conservative system, i.e., all dissipative forces are neglected.

The potential energy $U$ per unit volume of a aggregate is given by

$$U = \frac{1}{2} \sigma_{ij} e_{ij} + \frac{1}{2} s \epsilon,$$  \hspace{1cm} (2.5)

and the stresses can be obtained from the relations,

$$\sigma_{ij} = \frac{\partial U}{\partial e_{ij}},$$  \hspace{1cm} (2.6)

$$s = \frac{\partial U}{\partial \epsilon}.$$  \hspace{1cm} (2.7)

Due to the linear isotropic assumption and the existence of a potential energy function, Eqs. (2.5) through (2.7) lead to

$$\sigma_{ij} = 2N e_{ij} + A e_\delta_{ij} + Q \epsilon \delta_{ij},$$  \hspace{1cm} (2.8)

$$s = Q \epsilon + R \epsilon,$$  \hspace{1cm} (2.9)

where

$$e = e_{xx} + e_{yy} + e_{zz},$$  \hspace{1cm} (2.10)

$N$, $A$, $Q$ and $R$ are four material constants of a fluid-filled porous
material. These material constants depend on the bulk moduli of the solid and fluid, the shear modulus of the solid and the porosity of the porous material.
2.2 Equations of Motion Governing a Porous Material

Introducing the kinetic and dissipation function, Biot has established the equations of motion for a porous material filled with a Newtonian viscous fluid by means of Lagrange equations. When the material is assumed to be statistically isotropic, the kinetic energy $T$ for a unit volume of porous media can be postulated as follows.

$$T = \frac{1}{2} \rho_{11} \ddot{u}_i \ddot{u}_i + \rho_{12} \dot{\omega}_i \dot{\omega}_i + \frac{1}{2} \rho_{22} \dot{\omega}_i \dot{\omega}_i, \quad \text{(summation over } i) \quad (2.11)$$

where coefficients $\rho_{11}$, $\rho_{12}$, $\rho_{22}$ are the mass coefficients. They are related as follows:

$$\rho_{11} + \rho_{12} = (1-\phi) \rho_s, \quad (2.12)$$

$$\rho_{22} + \rho_{12} = \phi \rho_f, \quad (2.13)$$

where $\rho_s$ and $\rho_f$ are the mass densities of the solid and the fluid respectively. In the above kinetic energy postulation, the so-called virtual mass coefficient $\rho_{12}$ has been controversial over the years. In the mixture theory formulation [14], for instance, this term is not considered. The term associated with $\rho_{12}$, however, plays an important role in the investigation of wave propagation in a porous material. We motivate the existence of $\rho_{12}$ by considering a cylinder moving in an infinite fluid in Appendix B. It is made clear that this term takes into account of the fact that the relative fluid flow through a porous
material is not uniform.

The dissipation, which results from the viscous fluid flow, is assumed to depend only on the relative motion between the solid and the fluid. Due to statistical isotropy, the orthogonal directions are uncoupled and the dissipation function can subsequently be postulated as

$$2D = b \sum d_i d_i$$  \hspace{1cm} (summation over i) \hspace{1cm} (2.14)$$

where

$$d_i = \dot{u}_i - \dot{w}_i.$$ \hspace{1cm} (2.15)$$

The constant $b$ depends on the porosity, the kinematic fluid viscosity and Darcy's coefficient of permeability.

We note that Biot has neglected the fluid viscosity in establishing the stress-strain relations. However, the fluid viscosity is included in the postulation of the dissipation function. To be exact, the fluid viscosity should also be considered in the constitutive equations.

If we denote the total force acting on the solid and the fluid per unit volume in the $i$-direction by $Q_i^s$ and $Q_i^f$, respectively, Lagrange's equations then can be written as

$$\frac{\partial}{\partial t} \left( \frac{\partial T}{\partial \dot{u}_i} \right) + \frac{\partial D}{\partial \dot{u}_i} = Q_i^s,$$ \hspace{1cm} (2.16)$$

$$\frac{\partial}{\partial t} \left( \frac{\partial T}{\partial \dot{w}_i} \right) + \frac{\partial D}{\partial \dot{w}_i} = Q_i^f.$$ \hspace{1cm} (2.17)$$

On the other hand, the components of the force per unit volume corresponding to the solid and fluid can be expressed as stress
gradients. Hence, in terms of the stresses and displacements, the dynamic equations (2.16) and (2.17) are written as

\[ \sigma_{ij,j} - \frac{\partial^2}{\partial t^2} (\rho_{11}u_i + \rho_{12}w_i) + b \frac{\partial}{\partial t} (u_i - w_i), \] \hspace{1cm} (2.18)

\[ \frac{\partial s}{\partial x_i} = \frac{\partial^2}{\partial t^2} (\rho_{12}u_i + \rho_{22}w_i) - b \frac{\partial}{\partial t} (u_i - w_i). \] \hspace{1cm} (2.19)

To investigate wave propagation in the medium, Biot obtained the displacement equations of motion by combining Eqs. (2.18) and (2.19) with the constitutive equations as follows.

\[ N \nabla^2 u_i + (A+N) \frac{\partial e}{\partial x_i} + Q \frac{\partial \varepsilon}{\partial x_i} = \frac{\partial^2}{\partial t^2} (\rho_{11}u_i + \rho_{12}w_i) + b \frac{\partial}{\partial t} (u_i - w_i), \] \hspace{1cm} (2.20)

\[ Q \frac{\partial e}{\partial x_i} + R \frac{\partial \varepsilon}{\partial x_i} = \frac{\partial^2}{\partial t^2} (\rho_{12}u_i + \rho_{22}w_i) - b \frac{\partial}{\partial t} (u_i - w_i). \] \hspace{1cm} (2.21)

The displacement Eqs. (2.20) and (2.21) govern the wave propagation for a linearly elastic porous solid filled with a Newtonian viscous fluid. The term \( \rho_{12} \), which appears in both Eqs. (2.20) and (2.21), essentially is a coupling term between the solid and fluid.
CHAPTER III
MODIFIED MIXTURE THEORY

The basic results in the mixture theory, i.e., the energy balance, the continuity equations and the equations of motion [14] are summarized in this chapter. The linear constitutive equations and the equations of motion obtained in the modified mixture theory [17] are presented. In the modified mixture theory, the material constants, which appear in the original mixture theory and cannot be measured, are expressed in terms of more physically meaningful constants. The virtual mass coefficient \( \rho_{12} \) was first introduced by Biot in his phenomenological theory but ignored in the mixture theory. We show the existence of this term by considering an infinitely long cylinder moving in an ideal fluid field. The equations of motion obtained in the mixture theory are modified by including the virtual mass coefficient.

3.1 Mixture Theory

As discussed in the extensive review of the mixture theory by Atkin and Craine [1, 2], the basic governing equations have been presented by many researchers. In this thesis, we will follow the notation used by Green and Naghdi [14]. A brief summary of the mixture theory following Katsube [17] is presented below.

Let \( S_1 \) and \( S_2 \) denote the two continua. An energy balance at time \( t \) for a fixed volume \( V \) with a closed surface \( A \) is postulated as follows:
\[
\frac{\partial}{\partial t} \int_{\mathbf{v}} \left[ (\rho_1 + \rho_2) \mathbf{v} + \frac{1}{2} \rho_1 \mathbf{v}^{(1)} \cdot \mathbf{v}^{(1)} + \frac{1}{2} \rho_2 \mathbf{v}^{(2)} \cdot \mathbf{v}^{(2)} \right] d\mathbf{v} \\
+ \int_{A} \left[ \mathbf{n} \cdot (\rho_1 \mathbf{v}^{(1)} + \rho_2 \mathbf{v}^{(2)}) + \frac{1}{2} \rho_1 (\mathbf{n} \cdot \mathbf{v}^{(1)}) (\mathbf{v}^{(1)} \cdot \mathbf{v}^{(1)}) + \right. \\
+ \frac{1}{2} \rho_2 (\mathbf{n} \cdot \mathbf{v}^{(2)}) (\mathbf{v}^{(2)} \cdot \mathbf{v}^{(2)}) \right] dA = \\
- \int_{\mathbf{v}} \left[ (\rho_1 + \rho_2) \mathbf{r} + \rho_1 \mathbf{b}^{(1)} \cdot \mathbf{v}^{(1)} + \rho_2 \mathbf{b}^{(2)} \cdot \mathbf{v}^{(2)} \right] d\mathbf{v} + \int_{A} (\mathbf{t}^{(1)} \cdot \mathbf{v}^{(1)}) + \\
+ \mathbf{t}^{(2)} \cdot \mathbf{v}^{(2)} \right] dA - \int_{A} h dA,
\]

where \( U, h \) and \( r \) are internal energy, heat flux and heat supply per unit mass of the mixture, respectively. \( \mathbf{v}^{(1)} \) and \( \mathbf{v}^{(2)} \), \( \rho_1 \) and \( \rho_2 \), \( \mathbf{t}^{(1)} \) and \( \mathbf{t}^{(2)} \), \( \mathbf{b}^{(1)} \) and \( \mathbf{b}^{(2)} \) are, respectively, the velocities, mass densities, surface tractions per unit area and body forces of \( s_1 \) and \( s_2 \).

We denote the partial stresses for the solid and fluid constituents by \( \sigma_{ki}^{(1)} \) and \( \sigma_{ki}^{(2)} \), respectively. The subscript "k" associated with \( \sigma_{ki}^{(1)} \) and \( \sigma_{ki}^{(2)} \) refers to the surface normal direction and the subscript "i" refers to the direction of the stress components on that surface.

Based on the energy balance Eq. (3.1) and the invariance conditions
under superposed rigid body motion, it has been shown in general that
the partial stress tensors are not symmetric. The skew-symmetric part
of \( \sigma_{ki}^{(1)} \) is equal to the negative of the skew-symmetric part of \( \sigma_{ki}^{(2)} \). We
now denote the symmetric part of the partial stresses corresponding to
the solid and the fluid, respectively, by \( P^{(1)} \) and \( P^{(2)} \), i.e.,

\[
P^{(1)} = \{ \sigma_{ki}^{(1)} \},
\]

\[
P^{(2)} = \{ \sigma_{ki}^{(2)} \},
\]

and designate the skew-symmetric part of the stresses by

\[
\psi = \{ \sigma_{[ki]}^{(1)} \} = \{- \sigma_{[ki]}^{(2)} \}.
\]

Using the energy balance law Eq. (3.1) together with the invariance
conditions, we obtain the continuity equations and the equations of
motion [1, 2, 14] for a mixture of two constituents.

**Continuity Equation:**

\[
\frac{\partial \rho_{\alpha}}{\partial t} + \text{grad}(\rho_{\alpha} v^{(\alpha)}) = 0 \quad (\alpha = 1, 2; \text{ no summation over } \alpha),
\]

**Equations of Motion:**

\[
\text{div} P^{(1)} + \text{div} \psi^T + \pi + \rho_1 b^{(1)} = \rho_1 \psi^{(1)}.
\]
To obtain the constitutive equations, we denote the displacements of the solid and the fluid, respectively, by \( u_i^{(1)} \) and \( u_i^{(2)} \). We introduce the solid infinitesimal strain

\[
e_{ij}^{(1)} = \frac{1}{2}(u_{i,j}^{(1)} + u_{j,i}^{(1)})
\]

and the fluid volumetric strain

\[
e_{mm}^{(2)} = u_{m,m}^{(2)}
\]

In addition, the stresses and the diffusive force can be decomposed into two parts.

\[
\sigma_{ij}^{(1)} = \sigma_{ij}^{(1)'} + \sigma_{ij}^{(1)''},
\]

\[
\sigma_{ij}^{(2)} = \sigma_{ij}^{(2)'} + \sigma_{ij}^{(2)''},
\]

\[
\pi_i = \pi_i' + \pi_i'',
\]

where the single prime represents the part due to deformation and the double prime represents the part due to rate of deformation. We obtain the linear constitutive equations for fluid flow through an elastic
solid in an anisotropic case as follows:

\[ \sigma^{(1)}_{ij} = A_{ijkl} e^{(1)}_{kl} + D_{ij} e^{(2)}_{mm}, \] (3.11)

\[ \sigma^{(2)}_{ij} = (\lambda_7 e^{(2)}_{mm} + D_{mn} e^{(2)}_{mm}) \delta_{ij}, \] (3.12)

\[ \pi_i = 0, \quad \sigma^{(1)}_{[ij]} = \sigma^{(2)}_{[ij]} = 0, \] (3.13)

\[ \sigma^{(1)}_{ij} = G_{ijkl} d^{(1)}_{kl} + H_{ijkl} d^{(2)}_{kl}, \] (3.14)

\[ \sigma^{(2)}_{ij} = J_{ijkl} d^{(1)}_{kl} + L_{ijkl} d^{(2)}_{kl}, \] (3.15)

\[ \sigma^{(1)}_{[ij]} = -\sigma^{(2)}_{[ij]} = -S_{ijkl} \Lambda_{kl} + T_{ijk} \nu_k, \] (3.16)

\[ \pi_i = Q_{ij} \nu_j + R_{ijk} \Lambda_{jk} \] (3.16a)

where

\[ \Lambda_{ij} = \Gamma^{(1)}_{ij} - \Gamma^{(2)}_{ij}, \] (3.17)

\[ \Gamma^{(\alpha)}_{ij} = 1/2(\nu^{(\alpha)}_{i,j} - \nu^{(\alpha)}_{j,i}), \quad \alpha = 1, 2 \] (3.18)
\[
v_i = v_i - v_i^{(2)},
\]

\[
d_{ij}^{(\alpha)} = \dot{e}_{ij}^{(\alpha)}, \quad \alpha = 1, 2.
\]

The quantities \( A_{ijkl}, D_{ij}, G_{ijkl}, H_{ijkl}, J_{ijkl}, L_{ijkl}, Q_{ij}, R_{ijk} \), \( S_{ijkl} \) and \( T_{ijk} \) denote tensors associated with the material constants which depend upon the geometric structure, fluid viscosity and mechanical properties of the solid and fluid, such as the elastic moduli, etc.
3.2 Modified Mixture Theory

The conventional mixture theory has some difficulties in the application to fluid saturated materials. By studying a small representative volume element [17], Katsube and Carroll have clarified that the two different constituents are in fact an equivalent homogenous fluid and an equivalent homogeneous solid instead of a fluid and a solid in the sense of a usual single continuum. The linear constitutive equations obtained in the modified mixture theory can be summarized as follows:

\begin{align}
\sigma_{ij}^{(1)'} &= \lambda_1^{(1)} \epsilon_{ij}^{(1)} + 2\mu_1 \epsilon_{ij}^{(1)} + \lambda_6^{(2)} \epsilon_{ij}^{(2)} \\
\sigma_{ij}^{(2)'} &= (\lambda_7 \epsilon_{mm}^{(2)} + \lambda_6 \epsilon_{mm}^{(1)}) \delta_{ij} \\
\pi' &= 0, \quad \sigma_{[ij]}^{(1)'} = \sigma_{[ij]}^{(2)'} = 0,
\end{align}

and the constants \( \lambda_1, \lambda_6, \lambda_7 \) and \( \mu_1 \) can be further expressed as

\begin{align}
\lambda_1 + 2/3\mu_1 &= \frac{(1-\phi) k_F^s \left( \phi k_F^{*}(1) + (1-\phi) k_F^f \right)}{\phi k_F^{*}(1) + \phi (1-\phi) k_F^s + (1-\phi) k_F^f} \\
\lambda_6 &= \frac{\phi (1-\phi) k_F^s k_F^f}{\phi k_F^{*}(1) + \phi (1-\phi) k_F^s + (1-\phi) k_F^f} \\
\lambda_7 &= \frac{\phi k_F^f \left( \phi k_F^{*}(1) + \phi (1-\phi) k_F^s \right)}{\phi k_F^{*}(1) + \phi (1-\phi) k_F^s + (1-\phi) k_F^f}
\end{align}
\[ \mu_1 = \frac{(1-\phi)\mu^s \mu^* \star(1)}{\mu^* \star(1) + (1-\phi)\mu^s} \quad (3.26) \]

where

\[ \mu^* \star(1) = \frac{\mu^1 (1-\phi)\mu^s}{(1-\phi)\mu^s - \mu(1)} \quad (3.27) \]

\[ k^* \star(1) = \frac{k^1 (1-\phi)k^s}{(1-\phi)k^s - k^1} \quad (3.28) \]

\( \phi, k^s, k^e, k^1, \mu^1 \) and \( \mu^s \) are, respectively, the porosity, solid bulk modulus, fluid bulk modulus, effective bulk modulus, effective shear modulus and shear modulus of the solid.

The stresses due to velocity gradient and rate of deformation can be expressed as follows:

\[ \sigma^{(1)}(ij) = 0, \quad (3.29) \]

\[ \sigma^{(2)}(ij) = \lambda_2 d^{(2)}_{mm} \delta_{ij} + 2\mu_2 d^{(2)}_{ij}, \quad (3.30) \]

\[ \sigma^{(1)}[ij] = -\sigma^{(2)}[ij] = -2\mu_{11} A_{ij}, \quad (3.31) \]

and

\[ \pi_1'' = \lambda_9 v_i. \quad (3.32) \]
The material constants $\lambda_i$ appearing in the mixture theory can not be directly measured in the laboratory. In the modified mixture theory, however, these material constants have been expressed in terms of more physically meaningful constants.
3.3 Equations of Motion with $\rho_{12}$ Term

The so-called virtual mass coefficient $\rho_{12}$ term, which was first introduced in Biot's theory, has been controversial over the years. In Biot's postulation for the kinetic energy, $\rho_{12}$ is a term associated with the coupling effect arising from the solid and fluid interaction. This effect in essence measures the dynamic coupling between the solid and fluid constituents.

In contrast to Biot's theory, $\rho_{12}$ is assumed to be zero in the postulation of the energy balance law in the classical conventional mixture theory [14]. According to this mixture theory formulation, the total kinetic energy is considered to be the sum of the kinetic energy associated with the solid and fluid, i.e.,

$$T = \frac{1}{2} \rho_{11} \ddot{u}_1 \dot{u}_1 + \frac{1}{2} \rho_{22} \ddot{w}_1 \dot{w}_1. \quad (3.33)$$

However, the displacement fields $u_1$ and $w_1$ considered in both Biot's and mixture theories are average quantities over the representative sample. Therefore, we cannot conclude that the total kinetic energy is merely a superposition of the associated contributions. To verify our argument, we consider a cylinder of radius $R$ moving in an ideal fluid field [see appendix B]. By calculating the total kinetic energy, we show that $\rho_{11}$, $\rho_{12}$ and $\rho_{22}$ are given below:

$$\rho_{11} = \left[ \pi \left( \frac{R}{L} \right)^2 - (2\pi + 4) \left( \frac{R}{L} \right)^4 \right] \rho_f + \pi \left( \frac{R}{L} \right)^2 \rho_s. \quad (3.34)$$

$$\rho_{12} = \left[ (4 + 2\pi) \left( \frac{R}{L} \right)^4 - \pi \left( \frac{R}{L} \right)^2 \right] \rho_f, \quad (3.35)$$
\[ \rho_{22} = [1 - (4+2\pi)\left(\frac{R}{L}\right)^4] \rho_f. \]  \hspace{1cm} (3.36)

where \( L \) is the length of the volume considered [see appendix B]. Based on Eq. (3.34) through Eq. (3.36), we have

\[ \rho_{11} + \rho_{12} = (1-\phi)\rho_s, \]  \hspace{1cm} (3.37)

\[ \rho_{22} + \rho_{12} = \phi \rho_f, \]  \hspace{1cm} (3.38)

which are exactly what Biot has concluded in Eqs. (2.12) and (2.13).

Introducing \( \rho_{12} \) in the equations of motion, we obtain

\[ \text{div}\!p^{(1)} - \pi + \rho_1 b^{(1)} = \rho_{11}^* + \rho_{12}^* \]  \hspace{1cm} (3.39)

\[ \text{div}\!p^{(2)} - \pi + \rho_2 b^{(2)} = \rho_{12}^* + \rho_{22}^*. \]  \hspace{1cm} (3.40)

The above newly modified equations of motion will be employed to investigate the wave phenomena in porous media.
In this chapter, the basic results of the wave propagation in a porous material saturated with an ideal fluid obtained by Biot are summarized. Based on Biot's work and Mackenzie's results on effective moduli, we study the phase velocities as well as amplitude ratios of the three types of waves as a function of porosity.

4.1 Dilatational Waves.

In an ideal fluid case, the mixture theory is exactly identical to Biot's theory, if the $\rho_{12}$ term is included in the mixture theory. Thus, the results for wave propagation obtained from both theories should also be the same. A brief summary of Biot's results [5] follows.

We introduce

$$\text{div } \underline{u} = \epsilon, $$

$$\text{div } \underline{w} = \epsilon. $$

where $\underline{u}$ and $\underline{w}$ are, respectively, the displacement vectors of the solid and fluid. Setting $b=0$ and applying the divergence operator to both Eqs. (2.20) and (2.21), we obtain
\[ \nabla^2 (P \varepsilon + Q \varepsilon) = \frac{\partial^2}{\partial t^2} \left( \rho_{11} \varepsilon + \rho_{12} \varepsilon \right), \]  
\tag{4.3} 

\[ \nabla^2 (Q \varepsilon + R \varepsilon) = \frac{\partial^2}{\partial t^2} \left( \rho_{12} \varepsilon + \rho_{22} \varepsilon \right), \]  
\tag{4.4} 

where

\[ P = 2A + N, \]  
\tag{4.5} 

Equations (4.3) and (4.4) are the coupled equations governing the volumetric strains of the solid and fluid. These two equations govern the propagation of dilatational waves. We introduce \[5\]

\[ a_{11} = \frac{P}{H}, \quad a_{22} = \frac{R}{H}, \quad a_{12} = \frac{Q}{H}, \]  
\tag{4.6} 

\[ b_{11} = \frac{\rho_{11}}{\rho}, \quad b_{22} = \frac{\rho_{22}}{\rho}, \quad b_{12} = \frac{\rho_{12}}{\rho}, \]  
\tag{4.7} 

where

\[ H = P + R + 2Q \]  
\tag{4.8a} 

\[ \rho = \rho_{11} + \rho_{22} + 2\rho_{12}, \]  
\tag{4.8b} 

\[ \rho \] represents the mass per unit volume of the fluid-solid aggregate.

Thus, Eqs. (4.6) through (4.8) lead to

\[ a_{11} + a_{22} + 2a_{12} = b_{11} + b_{22} + 2b_{12} = 1. \]  
\tag{4.9} 

In addition, it is convenient to define a reference velocity $V_c$ by

$$\frac{V_c^2}{c_p} = \frac{H}{\rho}.$$  \hspace{1cm} (4.10)

By adding Eqs. (4.3) and (4.4), we note that $V_c$ is the velocity of the dilatational wave when the relative motion between the solid and the fluid are completely prevented. Inserting Eqs. (4.6), (4.7) and (4.10) into Eqs. (4.3) and (4.4), we obtain

$$V^2(a_{11}e + a_{12}Z) = \frac{1}{V_c^2} \cdot \frac{\partial^2}{\partial t^2} (b_{11}e + b_{12}Z),$$ \hspace{1cm} (4.11)

$$V^2(a_{12}e + a_{22}Z) = \frac{1}{V_c^2} \cdot \frac{\partial^2}{\partial t^2} (b_{12}e + b_{22}Z).$$ \hspace{1cm} (4.12)

To investigate plane waves, we assume the dilatation of the solid and fluid in the following form

$$e = C_1 \exp i(\lambda x + \alpha t),$$ \hspace{1cm} (4.13)

$$\epsilon = C_2 \exp i(\lambda x + \alpha t),$$ \hspace{1cm} (4.14)

where $C_1$ and $C_2$ are, respectively, the amplitudes of the solid and fluid constituents. $\lambda$ is the wave number and $\alpha$ is the radial frequency.

The velocity $V$ of these waves is given by
We introduce $Z$ defined by

$$Z = \frac{\nu^2}{c^2}. \quad (4.16)$$

Substituting Equations (4.13) through (4.16) into Equations (4.11) and (4.12), we obtain

$$(B - ZA)X = 0, \quad (4.17)$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}, \quad (4.18)$$

$$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{bmatrix}, \quad (4.19)$$

$$X = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}. \quad (4.20)$$

According to Eq. (4.17), the characteristic equation of the dilatational waves is a quadratic equation of $Z$. As was pointed out by Biot [5], the fact that Eq. (4.17) has two roots, $Z_1$ and $Z_2$, shows the
existence of two dilatational waves. Furthermore, the two dilatational waves are shown to propagate with in-phase and out-of-phase motion between the solid and the fluid, respectively. The in-phase dilatational wave is called the dilatational wave of the first kind and the out-of-phase dilatational wave is called the dilatational wave of the second kind.

Denoting the velocities of the dilatational waves of the first and the second kind by \( V_1 \) and \( V_2 \), respectively. From Eq. (4.16), we obtain

\[
V_1 = \sqrt{\frac{v_c^2}{Z_1}}, \quad (4.21)
\]

\[
V_2 = \sqrt{\frac{v_c^2}{Z_2}}. \quad (4.22)
\]

Biot has shown that the dilatational wave of the first kind propagates faster than that of the second kind, and \( Z_1 \) thus represents the smaller eigenvalue obtained from Eq. (4.17).

The amplitude ratio, obtained from Eq. (4.17), is

\[
\frac{C_1}{C_2} = \frac{(Z_{a12} - b_{12})}{(b_{11} - Z_{a11})}. \quad (4.23)
\]

By substituting \( Z_1 \) and \( Z_2 \), respectively, into Eq. (4.23), we obtain the amplitude ratios of the dilatational waves of the first and second kind.
4.2 Rotational Wave

The rotational wave in a porous material filled with an ideal fluid can be investigated in the same manner as in the dilatational wave case.

We introduce

\[ \hat{\omega} = \text{curl} \ \hat{u}, \]  

\[ \hat{\Omega} = \text{curl} \ \hat{\omega}. \]  

Applying the curl operation to both sides of Eqs. (2.20) and (2.21) and eliminating \( \hat{\Omega} \), we obtain

\[ N.\nabla^2 \hat{\omega} = \rho_{11}\left[1 - \rho_{12}^2/(\rho_{11} \cdot \rho_{22})\right] \frac{\partial^2}{\partial t^2} \hat{\omega}. \]  

According to Eq. (4.26), only one type of rotational wave is predicted. The velocity of propagation of this wave is given by

\[ V_s = \sqrt{\frac{N}{\rho_{11}\left[1 - \rho_{12}^2/(\rho_{11} \cdot \rho_{22})\right]}}. \]

In addition, the rotation \( \omega \) of the solid is coupled proportionally to the rotational \( \Omega \) of the fluid according to
Thus, even with an ideal fluid in a porous medium, Biot predicts the existence of a rotational wave associated with the fluid. This is due to the fact that the fluid displacement considered is an averaged quantity.

Assuming the rotation of the solid and the fluid to be

\[ \omega = C_1 \exp(i(\ell x + \alpha t)), \]  
\[ \Omega = C_2 \exp(i(\ell x + \alpha t)), \]

we obtain the amplitude ratio of the solid and the fluid

\[ \frac{C_1}{C_2} = -\frac{\rho_{22}}{\rho_{12}}. \]

Eq. (4.31) characterizes the relative deformation of the solid and fluid constituents as the rotational wave propagates.
4.3 Phase Velocities and Amplitude Ratios

4.3.1 In-phase Dilatational Wave Velocity and Amplitude Ratio

The phase velocity and amplitude ratio are important in the investigation of wave phenomena. In this section, we study the wave velocity and the amplitude ratio of the dilatational wave of the first kind as a function of porosity.

Mackenzie [25] has studied the effective elastic constants, $\mu^{(1)}$ and $k^{(1)}$, of a solid with spherical pores. Based on his work, the shear and bulk moduli of a porous material can be expressed as a function of the shear modulus of the solid $\mu^s$, the bulk modulus of the solid $k^s$ and the porosity as follows:

$$\mu^{(1)} = \mu_1 = \mu^s - 5\mu^s \frac{3k^s + 4\mu^s}{9k^s + 8\mu^s} \phi,$$

$$\frac{1}{k^{(1)}} = \frac{1}{k^s (1-\phi)} + \frac{3\phi}{4\mu^s (1-\phi)}.$$

Katsube has defined the elastic constants, $\mu^*(1)$ and $k^*(1)$, based on the differential straining of the pore space and the solid matrix. In the investigation of the constitutive laws of a fluid-filled porous material, Katsube [17] has shown that $\mu^*(1)$ and $k^*(1)$ are related to the shear and bulk modulus of the solid, shear and bulk modulus of a porous solid and porosity as follows:

$$\frac{1}{\mu^*(1)} = \frac{1}{\mu^{(1)}} - \frac{1}{(1-\phi)\mu^s},$$
Comparing the constitutive equations of the mixture theory with these of Biot's theory, we obtain

\[ \frac{1}{k^*(1)} - \frac{1}{k^{(1)}} - \frac{1}{(1-\phi)k^s} \]  

(4.34b)

As an example, consider a porous rock saturated with water with the effects of the viscosity neglected.

To calculate the amplitude ratio and the phase velocity of the propagating wave, first, we evaluate \( a_{ij} \) and \( b_{ij} \) from Eqs. (4.6) and (4.7) by using Eqs. (2.12), (2.13), (3.53), (3.23) through (3.26), (4.8), and (4.32) through (4.35d). Second, we solve the quadratic Equation (4.17) to get \( Z_1 \) and \( Z_2 \). Third, we evaluate \( V_c \) by using Eqs. (4.8) and (4.10). With the constants in Table (4.1), the phase velocity and amplitude ratio of the in-phase dilatational wave are calculated, respectively, from Eqs. (4.21) and (4.23). The results are shown in Figures (4.1) and (4.2).

Discussion:

Based on the material constants in Table 4.1, the dilatational wave velocity in the solid without pores and that in the fluid are given by
\[ V_L = \sqrt{\frac{\lambda^s + 2\mu^s}{\rho_s}} = 3989 \text{ m/s}, \]  

(4.36)

\[ V_F = \sqrt{\frac{k_f}{\rho_f}} = 1507 \text{ m/s}. \]  

(4.37)

As the porosity approaches zero, Fig. (4.1) shows that the velocity of the dilatational wave of the first kind approaches that of the dilatational wave in a non-porous solid. Since the dilatational wave propagates faster in the solid than it does in the fluid, we predict that the in-phase dilatational wave velocity decreases as porosity increases, and Fig. (4.1) verifies our prediction.

Fig. (4.2) shows that the amplitude ratio \( C_1/C_2 \) increases with porosity. Physically, this makes sense because the solid matrix is easier to deform when the porosity is larger. While the porosity is very small, Fig. (4.2) indicates that the solid and fluid constituents deform at essentially the same magnitude.
Table 4.1

Selected Material Constants for a Water-Filled Rock.

\[ \rho_s = 2650 \, \text{kg/m}^3 \]
\[ \rho_f = 1000 \, \text{kg/m}^3 \]
\[ k^s = 34.5 \times 10^9 \, \text{Pa} \]
\[ k^f = 2.27 \times 10^9 \, \text{Pa} \]
\[ \mu^s = 5.75 \times 10^9 \, \text{Pa} \]
Fig. 4.1. First dilatational wave velocity as a function of porosity.
Fig. 4.2. Solution for the first dilatational wave amplitude ratio as a function of porosity.
4.3.2 Out-of-phase Dilatational Wave Velocity and Amplitude Ratio

Following the same procedure described in section (4.3.1), the phase velocity and the amplitude ratio of the out-of-phase dilatational wave are calculated based on Eqs. (4.22) and (4.23). The results are plotted in Fig. (4.3) and Fig. (4.4), respectively.

As shown in Fig. (4.3), the out-of-phase dilatational wave velocity increases as the porosity increases. As the porosity decreases to zero, the second dilatational wave disappears. The out-of-phase dilatational wave velocity propagates slower than the dilatational wave in the fluid without pores.

Fig. (4.4) shows that the amplitude ratio $C_1/C_2$ increases as the porosity increases.

As shown in Figs. (4.3) and (4.4), the change of the porosity has a significant influence on both the second dilatational wave velocity and the amplitude ratio. According to our calculated results, the dilatational wave of the second kind becomes more important as the porosity increases.
Fig. 4.3. Second dilatational wave velocity as a function of porosity.
Second dilatational wave amplitude ratio

\[ \frac{c_1}{c_2} \]

\[ 0 \quad 0.02 \quad 0.04 \quad 0.06 \quad 0.08 \quad 0.1 \]

\[ 0 \quad 0.1 \quad 0.2 \quad 0.3 \quad 0.4 \]

porosity, \( \phi \)

**Fig. 4.4.** Solution for the second dilatational wave amplitude ratio as a function of porosity.
4.3.3 Rotational Wave Velocity and Amplitude Ratios

We combine Eqs. (2.12), (2.13), (3.53), (4.7) and (4.32) to evaluate $\rho_{11}$, $\rho_{12}$, $\rho_{22}$ and $\mu_1$. Using Eqs. (4.27) and (4.31) and the material properties given in Table (4.1), we then evaluate $V_s$ and $C_1/C_2$. The calculated results are shown in Fig. (4.5) and Fig. (4.6).

The rotational wave velocity in the solid without pores is given by

$$V_s = \sqrt{\frac{\mu_s}{\rho_s}} = 1473 \text{ m/s}. \quad (4.38)$$

There is no rotational wave propagation in an ideal fluid. Therefore, as shown in Fig. (4.5), the larger the porosity, the slower the rotational wave propagates. Similar to the case of the dilatational wave of the first kind, Fig. (4.5) shows that the rotational wave velocity approaches its velocity in the solid without pores as the porosity approaches zero. The amplitude ratio $C_1/C_2$ increases as porosity increases and the amplitude of the solid and the fluid are virtually the same when the porosity is small. These results are similar to the results obtained for the dilatational wave of the first kind and the same physical explanation can be given.
Rotational wave velocity (m/s)

Porosity, $\phi$

Fig. 4.5. Rotational wave velocity as a function of porosity.
Fig. 4.6. Solution for the rotational wave amplitude ratio as a function of porosity.
CHAPTER V

PLANE WAVES IN AN INFINITELY FLUID-FILLED POROUS MATERIAL

In this chapter, we examine the plane wave phenomena in an elastic porous material filled with a Newtonian viscous fluid in the low frequency range [see Ref. 5] by using the modified mixture theory. We show that four types of waves, i.e., two dilatational waves and two rotational waves, may propagate in a fluid-filled porous material. The dispersion, attenuation and phase differences of these four types of waves are studied and compared with those obtained by using Biot’s approach. The porosity dependence of the wave velocities of these four types of waves is studied in the low and high frequency ranges. The evaluated phase velocities are compared with the experimental results as well as with the results obtained from Biot’s theory [4].

5.1 Waves Predicted by Biot’s Theory

For the purpose of comparison, we now summarize Biot’s results [5] for acoustic wave propagation in a porous material filled with a Newtonian viscous fluid as follows:

Applying the curl operator on both sides of Eqs. (2.20) and (2.21), we obtain the following equations governing rotational wave propagation
\[ N \nu^2 \omega - b \frac{\partial^2}{\partial t^2}(\omega - \Omega) = \frac{\partial^2}{\partial t^2}(\rho_{11}\omega + \rho_{12}\Omega), \quad (5.1a) \]

\[ b \frac{\partial}{\partial t}(\omega - \Omega) = \frac{\partial^2}{\partial t^2}(\rho_{12}\omega + \rho_{22}\Omega). \quad (5.1b) \]

Inserting the plane wave form equations (4.29) and (4.30) into Eqs. (5.1a) and (5.1b) and eliminating the constants \( C_1, C_2 \), we obtain

\[ \frac{N \nu^2}{(\rho a^2)} = E_r - iE_i, \quad (5.2) \]

where

\[ E_r = \frac{1 + b_{22}(b_{11}b_{22} - b_{12}^2)(f/f_c)^2/(b_{12} + b_{22})^2}{b_{22}.(f/f_c)^2/(b_{12} + b_{22})^2 + 1}. \quad (5.3) \]

\[ E_i = \frac{(b_{12} + b_{22}).f/f_c}{b_{22}.(f/f_c)^2/(b_{12} + b_{22})^2 + 1}. \quad (5.4) \]

\[ f = \alpha/(2\pi). \quad (5.5) \]

The characteristic frequency \( f_c \) in Equations (5.3) and (5.4) is defined by

\[ f_c = b/(2\pi \rho_2), \quad (5.6) \]

where
Biot [5] has pointed out that the Poiseuille flow is no longer valid when the frequency exceeds a critical value $f_c$. When the frequency is less than $f_c$, it is defined as the low frequency range [5]. For pores of circular shapes, Biot has shown $f_c$ is given by Eq. (5.7a)

$$
\dot{f}_c = 0.15 f_c.
$$

(5.7a)

Similarly, when the frequency exceeds $\dot{f}_c$, it is defined as the high frequency range.

By decomposing the complex wave number into a real and an imaginary part, i.e.,

$$
\ell = \ell_r + i \ell_i,
$$

(5.8)

we obtain the rotational wave phase velocity

$$
V_r = \alpha/|\ell_r|.
$$

(5.9)

We now introduce a rotational reference velocity

$$
V_R = (N/\rho)^{1/2},
$$

(5.10)
where \( N \) and \( \rho \) are defined in Eqs. (2.8) and (4.8b), respectively. This reference velocity physically represents the rotational wave velocity of a porous material when the relative motion between the solid and the fluid are completely prevented. Thus, Eq. (5.2) yields

\[
\frac{V_r}{V_R} = \sqrt{\frac{2}{\sqrt{(E_{\perp}^2 + E_{\|}^2)^{1/2} + E_r}}}. \tag{5.11}
\]

This implies that only one rotational wave exists in a porous media.

We next consider the dilatational waves. Applying the divergence operator to both sides of Eqs. (2.20) and (2.21), we obtain the equations governing the dilatational waves

\[
\nabla^2 (P_\epsilon + Q_\epsilon) - b \frac{\partial}{\partial t} (e - \epsilon) = \frac{\partial^2}{\partial t^2} (\rho_{11}^e + \rho_{12}^e), \tag{5.12}
\]

\[
\nabla^2 (Q_\epsilon + R_\epsilon) + b \frac{\partial}{\partial t} (e - \epsilon) = \frac{\partial^2}{\partial t^2} (\rho_{12}^e + \rho_{22}^e). \tag{5.13}
\]

Inserting the plane wave form Equations (4.13) and (4.14) into Eqs. (5.12) and (5.13), we obtain the characteristic equation as follows:

\[
(a_1 a_{22} - a_{12}^2)Z^2 - (a_{11} b_{22} + a_{22} b_{11} - 2a_{12} b_{12})Z + (b_{11} b_{22} - b_{12}^2) + i(Z - 1)b/(\alpha \rho) = 0, \tag{5.14}
\]

where

\[
Z = \ell^2 v_c^2 / a^2. \tag{5.15}
\]
Eq. (5.14) is a quadratic function of $Z$ and this implies that two types of dilatational waves may propagate in a porous medium. By choosing some specific material constants, Biot has studied both the nondimensional phase velocities and the attenuation of the dilatational waves as a function of the nondimensional frequency. He has concluded that the in-phase dilatational wave propagates faster than the out-of-phase wave, and that the out-of-phase wave propagation is more attenuated.
5.2 Dilatational Waves Predicted by Modified Mixture Theory

Following the standard procedure of Biot [5], we now examine wave phenomena in a porous medium based on the modified mixture theory. The equations of motion (3.39) and (3.40) together with the stress strain relations lead to

\[
(\lambda_1 + \mu_1)\nabla(\nabla \cdot \mathbf{u}^{(1)}) + \mu_1 \nabla^2 \mathbf{u}^{(1)} + \lambda_6 \nabla(\nabla \cdot \mathbf{u}^{(2)}) - \\
- \mu_{11}[\nabla(\nabla \cdot \mathbf{u}^{(1)}) - \nabla^2 \mathbf{u}^{(1)} + \nabla^2 \mathbf{u}^{(2)} - \nabla(\nabla \cdot \mathbf{u}^{(2)})] - \\
- \lambda_9 (\mathbf{u}^{(1)} - \mathbf{u}^{(2)}) = \rho_{11} \ddot{\mathbf{u}}^{(1)} + \rho_{12} \ddot{\mathbf{u}}^{(2)},
\]

\[(5.16)\]

\[
\lambda_7 \nabla(\nabla \cdot \mathbf{u}^{(2)}) + \lambda_6 \nabla(\nabla \cdot \mathbf{u}^{(1)}) + (\lambda_2 + \mu_2 - \mu_{11})\nabla(\nabla \cdot \mathbf{u}^{(2)}) + \\
(\mu_2 + \mu_{11}) \nabla^2 \mathbf{u}^{(2)} + \mu_{11} \nabla(\nabla \cdot \mathbf{u}^{(1)}) - \mu_{11} \nabla^2 \mathbf{u}^{(1)} - \lambda_9 (\dot{\mathbf{u}}^{(1)} - \\
- \dot{\mathbf{u}}^{(2)}) = \rho_{12} \ddot{\mathbf{u}}^{(1)} + \rho_{22} \ddot{\mathbf{u}}^{(2)},
\]

\[(5.17)\]

where \(\mathbf{u}^{(1)}\) and \(\mathbf{u}^{(2)}\) are the displacement vectors of the solid and fluid constituents, respectively. Applying the divergence operator on both sides of Eqs. (5.16) and (5.17), we obtain

\[
\nabla^2 [((\lambda_1 + 2\mu_1)\varepsilon + \lambda_6 \varepsilon) - \lambda_9 \frac{\partial}{\partial t} (\varepsilon - \varepsilon) = \frac{\partial^2}{\partial t^2} (\rho_{11} \varepsilon + \rho_{12} \varepsilon),
\]

\[(5.18)\]
\[ \nabla^2 (\lambda_6 \varepsilon + \lambda_7 \varepsilon) + \lambda_9 \frac{\partial}{\partial t} (e - \varepsilon) + (\lambda_2 + 2\mu_2) \nabla^2 \frac{\partial}{\partial t} \varepsilon = \frac{\partial^2}{\partial t^2} (\rho_{12} \varepsilon + \rho_{22} \varepsilon), \]

(5.19)

where

\[ e = \text{div} u^{(1)}, \quad \varepsilon = \text{div} u^{(2)}. \]

(5.20)

Comparing Eqs. (5.12) and (5.13) with Eqs. (5.18) and (5.19), we obtain Eqs. (4.35), (5.21) and (5.22)

\[ P = \lambda_1 + 2\mu_1 \]

(5.21)

\[ b = \lambda_9 \]

(5.22)

The term associated with \( \lambda_2 + 2\mu_2 \) in Eq. (5.19), however, is an additional term which does not appear in Biot's wave Equations (5.12) and (5.13). We note that \( \lambda_2 \) and \( \mu_2 \) can be approximated [18] by

\[ \lambda_2 = \phi \lambda^f \]

(5.23a)

\[ \mu_2 = \phi \mu^f \]

(5.23b)

In the above equations, \( \mu^f \) and \( \lambda^f \) are, respectively, the viscous constants related to the rate of deformation and the rate of the volumetric strain of the fluid.

We seek plane wave solutions in the form of Equations (4.13) and (4.14). Substituting Eqs. (4.13) and (4.14) into Eqs. (5.18) and (5.19), we obtain

\[ C_1 \left[ - \left( \lambda_1 + 2\mu_1 \right) t^2 - \lambda_9 \alpha + \rho_{11} \alpha^2 \right] + \]
\[ C_2[- \lambda_6 \ell^2 + i \alpha \lambda_9 + \rho_{12} \alpha^2] = 0, \quad (5.24) \]

\[ C_1[- \lambda_6 \ell^2 + i \alpha \lambda_9 + \rho_{12} \alpha^2] + \]

\[ C_2[- \lambda_7 \ell^2 - (\lambda_2 + 2 \mu_2) \ell^2 \alpha - i \alpha \lambda_9 + \rho_{22} \alpha^2] = 0. \quad (5.25) \]

For non-zero amplitudes, the characteristic equation associated with Eqs. (5.24) and (5.25) can be simplified as follows:

\[ Z^2(w_1 + iw_2) + Z(w_3 + iw_4) + w_5 - iw_6 = 0, \quad (5.26) \]

where

\[ w_1 = \left( a_{11} a_{22} - a_{12} \right) \alpha, \]

\[ w_2 = a_{11} d_1 \alpha^2, \]

\[ w_3 = \left( - a_{11} b_{22} - b_{11} a_{22} - d_1 d_2 + 2 a_{12} b_{12} \right) \alpha, \quad (5.27) \]

\[ w_4 = d_2 - \alpha^2 d_1 b_{11}, \]

\[ w_5 = \alpha (b_{11} b_{22} - b_{12}^2), \]

\[ w_6 = d_2, \]

and

\[ d_1 = \frac{\lambda_2 + 2 \mu_2}{\lambda_1 + 2 \mu_1 + \lambda_7 + 2 \lambda_6}, \quad (5.28a) \]

\[ d_2 = \frac{\lambda_9}{\rho}, \quad (5.28b) \]

\[ \nu_c^2 = \frac{\lambda_1 + 2 \mu_1 + \lambda_7 + 2 \lambda_6}{\rho}, \quad (5.29a) \]
In Eq. (5.27), \( a_{ij} \) and \( b_{ij} \) are material constants defined in Eqs. (4.6) and (4.7).

In the above equations, terms associated with the parameter \( d_1 \) are the only terms which do not appear in Biot's derivation. The parameter \( d_1 \) relates the stress to the rate of deformation tensor of the fluid. Since the fluid considered is Newtonian, the fluid velocity gradient terms may not be neglected.

The radial frequency \( \alpha \) can be further expressed as a function of the non-dimensional frequency \( f/f_c \), i.e.,

\[
\alpha = \frac{d_2}{b_{22} + b_{12}} \frac{f}{f_c},
\]

where \( f \) and \( f_c \) are defined in Eqs. (5.5) and (5.6), respectively.

We note that \( \lambda_0 \) here considered is assumed to be independent of the frequency. As was pointed out by Biot, however, this is only true if the frequency is in the low frequency range. When the frequency is in the high frequency range, \( \lambda_0 \) must be modified by a frequency correction function \( F(\alpha) \) [see appendix A].

To investigate the dispersion and the attenuation of the dilatational waves, we obtain the two roots \( Z_1 \) and \( Z_2 \) from Eq. (5.26) as follows:

\[
Z_{1,2} = \frac{-w_3 - iw_4 \pm [w_3^2 - w_4^2 - 4w_1 w_6 - 4w_2^2 w_6 + i(2w_3 w_4 + 4w_1 w_6 - 4w_2 w_6)]^{1/2}}{2(w_1 + i w_2)}
\]

\[(5.31)\]
Let $V_1$ and $V_2$ denote the dilatational wave velocities associated with $Z_1$ and $Z_2$, respectively. The non-dimensional velocities and attenuation are given by

$$\frac{V_\beta}{V_c} = \frac{1}{|\text{Re}(\sqrt{Z_\beta})|}, \quad (\beta = 1, 2)$$

and

$$\frac{L_c}{X_\beta} = \frac{|\text{Im}(\sqrt{Z_\beta})|}{\frac{f_c}{f}}, \quad (\beta = 1, 2)$$

where

$$X_\beta = \frac{1}{|\text{Im}(\kappa(\beta))|}, \quad (\beta = 1, 2)$$

$$L_c = \frac{V_c}{2\pi f_c}.$$  

In Eq. (5.34), $\kappa(\beta)$ ($\beta = 1, 2$) are the complex wave numbers associated with the dilatational wave of the first and second kind.

As was pointed out by Katsube [17], the resistance coefficient measuring the interaction force between the solid and the fluid $\lambda_9$ may be approximated by the permeability $\kappa$ as follows:

$$\lambda_9 = \frac{\mu^2 f}{\kappa}.$$  

(5.36)
As an example, we consider a porous rock saturated with water. The selected material constants are given in Table (5.1).

To calculate the phase velocities and the attenuation of the dilatational waves, first, we evaluate $a_{ij}$ and $b_{ij}$ by following the same procedure discussed in Section (4.3.1). Second, we calculate $d_1$ and $d_2$ from Eqs. (5.28a) and (5.28b) by using Eqs. (3.23) through (3.28), (4.32), (4.33), (5.29d) and (5.36). Third, we evaluate $\alpha$ from Eq. (5.30) by using Eqs. (4.7) and (5.28). Fourth, we extract the two roots $Z_1$ and $Z_2$ from Eq. (5.26) by combining Eq. (5.27), (5.28a), (5.28b) and (5.30). Finally, we calculate the nondimensional velocities and the attenuation based on Eqs. (5.32) through (5.33).

Figs. (5.1) through (5.4) show that the numerical results obtained from both the modified mixture theory and Biot's theory are essentially identical. This implies that the term $d_1$, which is ignored by Biot, is negligible.

As indicated by Figs. (5.1) and (5.2), the first dilatational wave travels faster than the second dilatational wave and is less attenuated. When the frequency approaches zero, the first dilatational wave velocity approaches $V_c$ while the second dilatational wave velocity approaches zero. Figs. (5.3) and (5.4) show that the dilatational wave of the second kind is more attenuated than the first one.
Table 5.1

Selected Material Constant for a Porous Rock Saturated with Water

- \( \rho_s = 2650. \text{ kg/m}^3 \)
- \( \rho_f = 1000. \text{ kg/m}^3 \)
- \( k^s = 34.5 \times 10^9 \text{ Pa} \)
- \( k^f = 2.27 \times 10^9 \text{ Pa} \)
- \( \mu^s = 5.75 \times 10^9 \text{ Pa} \)
- \( a = 0.003 \text{ m} \)
- \( \phi = 0.1 \)
- \( \mu^f = 0.001 \text{ Pa.s} \)
- \( k_o = 9.87 \times 10^{-14} \text{ m}^2 \)
Fig. 5.1. Solution for the nondimensional first dilatational wave velocity as a function of nondimensional frequency.
Fig. 5.2. Solution for the nondimensional second dilatational wave velocity as a function of nondimensional frequency.
Fig. 5.3. The nondimensional attenuation of the dilatational wave of the first kind vs. nondimensional frequency.
Fig. 5.4. The nondimensional attenuation of the dilatational wave of the second kind vs. nondimensional frequency.
We now investigate the phase difference between the solid and fluid constituents. From Eqs. (5.24) and (5.25), we have

\[
\frac{c_1}{c_2} = \frac{\sqrt{E_1^2 + E_2^2}}{\sqrt{F_1^2 + F_2^2}} \ e^{i(\psi_1 - \psi_2)},
\]

(5.37)

where

\[
E_1 = -a_{12} \ \text{Re}(Z) + b_{12}, \quad (5.38a)
\]
\[
E_2 = -a_{12} \ \text{Im}(Z) + w_6/\alpha, \quad (5.38b)
\]
\[
F_1 = -b_{11} + a_{11} \ \text{Re}(Z), \quad (5.39a)
\]
\[
F_2 = a_{11} \ \text{Im}(Z) + w_6/\alpha. \quad (5.39b)
\]

In Eq. (5.37), \(\psi_1\) and \(\psi_2\) are the phase angles of the solid and fluid constituents, respectively.

With the material constants given in Table 5.1, the phase difference between the solid and the fluid is calculated based on Eqs. (5.37) through (5.39b). At low frequencies, Figs. (5.5) and (5.6) show that the phase shifts for the dilatational waves of the first and second kind essentially correspond to in-phase and out-of-phase motion, respectively. Physically speaking, when the frequency is very low, the wave length is so long that the wave does not recognize the microstructure of the porous material. Thus, the wave propagates as if it is were in a porous material filled with an ideal fluid [see Ref. 5 for the results in the ideal fluid case].
Fig. 5.5. Calculated phase difference of the dilatational wave of the first kind as a function of nondimensional frequency.
Fig. 5.6. Calculated phase difference of the dilatational wave of the second kind as a function of nondimensional frequency.
5.3 Rotational Waves Predicted by the Modified Mixture Theory

The rotational waves can be studied in a manner similar to the dilatational wave case. Applying the curl operator on both sides of Eqs. (5.16) and (5.17), we obtain

\[ \mu_1 \frac{\partial^2 \omega}{\partial t^2} - \lambda_9 \frac{\partial}{\partial t} (\omega - \Omega) + \mu_{11} \frac{\partial^2}{\partial t^2} (\omega - \Omega) = \frac{\partial^2}{\partial t^2} (\rho_{11} \omega + \rho_{12} \Omega), \]

(5.40)

\[ \lambda_9 \frac{\partial}{\partial t} (\omega - \Omega) + \mu_2 \frac{\partial^2}{\partial t^2} \Omega - \mu_{11} \frac{\partial^2}{\partial t^2} (\omega - \Omega) = \frac{\partial^2}{\partial t^2} (\rho_{12} \omega + \rho_{22} \Omega), \]

(5.41)

where

\[ \omega = \text{c}_{u_{11}} u^{(1)}, \]

(5.42)

\[ \Omega = \text{c}_{u_{12}} u^{(2)}. \]

(5.43)

Comparing Eqs. (5.40) and (5.41) with Eqs. (5.1a) and (5.1b), we note that the terms associated with \( \mu_2 \) and \( \mu_{11} \) do not occur in Biot's theory. The constants \( \mu_2 \) and \( \mu_{11} \) are the material constants related to overall fluid viscous stress and the interaction torque, respectively, between the solid and fluid constituents.

Once again, we look for the plane wave in the form of Eqs. (4.29) and (4.30). Inserting Eqs. (4.29) and (4.30) into Eqs. (5.40) and (5.41), we obtain

\[ C_1 \left[ - \mu_1 \ell^2 + \rho_{11} \alpha^2 - (\mu_{11} \ell^2 \alpha + \lambda_9 \alpha) \right] + C_2 \left[ \mu_{11} \ell^2 \alpha^2 + \alpha \lambda_9 + \rho_{12} \alpha^2 \right] = 0, \]

(5.44)
For non-zero amplitudes, we obtain the characteristic equation associated with (5.44) and (5.45) as follows:

\[ Z^2(P_1 + iP_2) + Z(P_3 + iP_4) + P_5 - iP_6 = 0, \]  

(5.46)

where

\[ P_1 = -q_1 q_2 \alpha^2, \]  
\[ P_2 = q_2 \alpha + q_1 \alpha, \]  
\[ P_3 = -b_{22} - d_2 q_2, \]  
\[ P_4 = d_2 / \alpha - \alpha q_2 b_{11} - \alpha q_1, \]  
\[ P_5 = b_{11} b_{22} - b_{12}^2, \]  
\[ P_6 = d_2 / \alpha, \]  

(5.47)

\[ Z = (l^2 v_R^2) / \alpha^2, \]  
\[ v_R = \sqrt{\mu_1 / \rho}, \]  
\[ q_1 = \mu_{11} / \mu_1, \]  
\[ q_2 = \mu_2 / \mu_1. \]  

(5.48)

(5.49)

Since the quadratic equation (5.46), in general, has two roots \( Z_1 \) and \( Z_2 \), there exist rotational waves of the first and second kind corresponding to \( Z_1 \) and \( Z_2 \), respectively.
The interaction torque between the fluid and solid constituents and the overall fluid viscous stress, both of which are related to the fluid velocity gradient terms, are not considered in Biot's theory. Katsube and Carroll [17] obtained the fluid velocity profile for a steady state fluid flow through a porous material bounded by two impermeable infinite plates, as well as that for an axial-symmetrical fluid flow through a porous material. Due to these additional terms, the zero fluid velocity conditions at the boundaries are shown to be satisfied as opposed to Biot's theory. The fluid velocity profile is shown to depend on a constant "a" defined as follows:

\[ a = \sqrt{\frac{\mu_{11} + \mu_2}{\lambda_9}}. \]  

(5.50)

This constant has the dimension of length and can be considered as the characteristic length of the micro-structure of the porous materials.

In Eqs. (5.46) through (5.49c), the effect of the additional terms on wave propagation appears through the material constants \( \mu_{11} \) and \( \mu_2 \) in Eqs. (5.49b) and (5.49c). If we set \( \mu_{11} \) and \( \mu_2 \) to zero, Eq. (5.46) is no longer a quadratic in \( Z \), and we recover Eq. (5.2) in Biot's theory.

To evaluate \( \mu_{11} \) for a given value of porosity, we note that \( \mu_2 \) and \( \lambda_9 \) can be evaluated from Eqs. (5.23b) and (5.36), respectively. In addition Katsube has pointed out that the characteristic length \( a \) can be estimated from the boundary characteristics of the velocity profile. Therefore, \( \mu_{11} \) can be evaluated from Eq. (5.50).

Let \( V_1 \) and \( V_2 \) denote the rotational wave velocities associated with \( Z_1 \) and \( Z_2 \), respectively. The non-dimensional velocities are given by
\[ \frac{V_\beta}{V_R} = \frac{1}{|\text{Re}(\sqrt{Z_\beta})|}, \quad (\beta = 1, 2) \]  

(5.51)

where \( V_R \) is defined in Eq. (5.10).

The attenuation is given similarly by

\[ \frac{L_R}{X_\beta} = |\text{Im}(\sqrt{Z_\beta})| \cdot \frac{f}{f_c} \]  

(5.52)

where

\[ L_R = \frac{V_R}{2\pi f_c} \]  

(5.52a)

To calculate the phase velocities and the attenuation of the rotational waves, first, we evaluate \( b_{ij} \) following the same procedure discussed in Section (4.3.1). Second, we calculate \( q_1 \) and \( q_2 \) from Eqs. (5.49b) and (5.49c) by using (5.50), (5.36), (5.23b), (4.32). Third, following the procedure used in section 5.2 we evaluate \( d_2 \) and \( \alpha \). Fourth, we calculate \( p_1 \) through \( p_6 \) based on Eqs. (5.47a) through (5.47f). Fifth, we extract the two roots \( Z_1 \) and \( Z_2 \) from Eqs. (5.46). Finally, we evaluate the phase velocities and attenuation from Eqs. (5.51) and (5.52).

As shown in Fig. (5.7), at a constant frequency, the larger the value of \( a \), the closer \( V_1 \) becomes to \( V_R \). The physical explanation of this phenomenon is given as follows.

In comparison with Biot's theory, we have additional viscous terms such as interaction torque between solid and fluid constituents and the fluid viscous stress. Therefore, the fluid and the solid tend to move as a composite medium. Since the characteristic length \( a \) is proportional to \( \mu_{11} \) and \( \mu_2 \), the fluid and solid thus act as more like a
"viscoelastic" aggregate as \( a \) increases. Because \( V_R \) represents the phase velocity when the relative rotation between the fluid and the solid are completely prevented. Hence, \( V_1 \) approaches \( V_R \) as \( a \) increases. Fig. (5.7) confirms this physical behavior.

As expected, the velocity of the rotational wave of the first kind coincides with that of Biot as \( a \) approaches zero.

Fig. (5.8) shows that the change of the characteristic length \( a \) has a significant effect on the the velocity of the second rotational wave. The larger the value of \( a \), the higher the velocity of the second rotational wave. If we set \( \mu_1 \) and \( \mu_2 \) equal to zero, the second rotational wave vanishes, as revealed in Fig. (5.8).

As shown in Fig. (5.9), the study of attenuation of the first rotational wave illustrates that it becomes less attenuated as \( a \) increases. This is also due to the fact that the solid and the fluid constituents tends to move together as \( a \) increases. As expected, the attenuation is the same as that of Biot if \( a \) is zero. The attenuation of the second rotational wave is shown in Fig. (5.10) and it indicates that this wave becomes very attenuated as \( a \) becomes very small. In general, the second rotational wave is more attenuated than the first kind.

An alternative way to evaluate the value of \( a \) is to compare the experimental results with the numerical evaluation in Fig. (5.7). In Fig. (5.7), we obtain \( \frac{V_1}{V_R} \) from Biot's theory. Due to the ignorance of the viscous terms in Biot's theory, the results for \( \frac{V_1}{V_R} \) obtained from Biot's theory should differ from the experiment data. By gradually increasing the value of \( a \) from 0 (Biot's theory) to a certain value in our numerical calculation until the results matches the experiment results. We thus obtain the exact value of \( a \).
Fig. 5.7. Nondimensional velocity of the rotational wave of the first kind vs. nondimensional frequency.
Fig. 5.8. Nondimensional velocity of the rotational wave of the second kind vs. nondimensional frequency.
Fig. 5.9. Calculated nondimensional attenuation of the rotational wave of the first kind vs. nondimensional frequency.
Fig. 5.10. Calculated nondimensional attenuation of the rotational wave of the second kind vs. nondimensional frequency.
To investigate the phase shifts between the solid and fluid displacement vectors of the rotational waves, we have from Eq. (5.44)

\[
\frac{C_1}{C_2} = \frac{\sqrt{E_1^2 + E_2^2}}{\sqrt{F_1^2 + F_2^2}} \, e^{i(\psi_1 - \psi_2)}, \quad (5.53)
\]

where

\[E_1 = -q_2\alpha \text{Im}(Z) - q_1\alpha \text{Im}(Z) - b_{22}, \quad (5.54a)\]

\[E_2 = q_2\alpha \text{Re}(Z) + q_1\alpha \text{Re}(Z) + d_2/\alpha, \quad (5.54b)\]

\[F_1 = -q_1\alpha \text{Im}(Z) + b_{12}, \quad (5.54c)\]

\[F_2 = d_2/\alpha + q_1\alpha \text{Re}(Z), \quad (5.54d)\]

Fig. (5.11) is the plot of the phase shift for the rotational wave of the first kind. In Fig. (5.11), we can also observe the effect of \(\mu_1\) and \(\mu_2\) on the phase shift of the rotational wave of the first kind. At any given frequency, the larger the value of \(a\), the smaller the phase shift of the rotational wave of the first kind. This means that the solid and the fluid constituents tends to move together. Hence, the physical interpretation given for Fig. (5.7) is consistent with the phenomena observed in Fig. (5.11). In addition, the first rotational wave is shown to correspond virtually to in-phase motion at very low frequencies.
Figs. (5.12) is the plot of the phase shift for the rotational wave of the second kind. The second dilatational wave becomes virtually an out-of-phase motion at very low frequencies and the phase shift decreases very significantly as the frequency increases.
Fig. 5.11. Phase difference of the rotational wave of the first kind as a function of nondimensional frequency.
Fig. 5.12. Phase difference of the rotational wave of the second kind as a function of nondimensional frequency.
5.4 The Effect of Change in Porosity on Wave Velocities in the Low Frequency Range

5.4.1 Effect of Variation in Frequency

We now investigate the effect of porosity change on wave velocities in a porous material saturated with a viscous fluid in the low frequency range by using the modified mixture theory. We will make use of Mackenzie's effective modulus equations (4.32) and (4.33) for \( \mu^{(1)} \) and \( k^{(1)} \), and Eqs. (5.23a) and (5.23b) for fluid viscosity \( \lambda_2 \) and \( \mu_2 \).

To express the material constant \( \lambda_9 \) and the characteristic length \( a \) in terms of porosity, we need to make the following assumptions about the porous material.

1. All the pores are spherical with equal diameter.
2. The number of pores per unit representative volume element remains the same, but the diameter may change.

Suppose we have \( N \) spherical pores with radius \( r \) in a unit volume of porous material. The volumetric porosity \( \phi \) and the specific surface area \( S \) can be written, respectively, as follows.

\[
\phi = \frac{4}{3} \pi r^3 N, \quad (5.55a)
\]
\[
S = 4\pi r^2 N. \quad (5.55b)
\]

Eliminating \( r \) from the above equations, we have

\[
S = N^{1/3}(4\pi)^{1/3}(3\phi)^{2/3}. \quad (5.56)
\]

When \( \phi = 0.1 \), the specific surface area of the porous material in Eq. (5.56) is given by
\[ S_o = N^{1/3}(4\pi)^{1/3}(0.3)^{2/3}. \]  

(5.57)

Eliminating \( N \) by combining Eqs. (5.56) and (5.57), we get

\[ S = (10\phi)^{2/3} S_o. \]  

(5.58)

Kozeny-Carmon experimentally obtained the permeability equation

\[ \hat{k} = \frac{\phi^3}{(1 - \phi)^2 k_o S^2}. \]  

(5.59)

where \( k_o \) and \( S \) are the shape factor and the specific surface area, respectively. Using the permeability \( \hat{k}_o \) given in Table 5.1 when \( \phi = 0.1 \) and combining Eqs. (5.58) and (5.59), we obtain \( k_o S^2 \) as follows.

\[ k_o S^2 = 1.25 \times 10^{10} (10\phi)^{4/3}. \]  

(5.60)

Using Eqs. (5.36), (5.59) and (5.60), we obtain \( \lambda_g \) as a function of porosity.

\[ \lambda_g = \frac{(1 - \phi)^2 \mu^f}{\phi} \times 1.25 \times 10^{10} (10\phi)^{4/3}. \]  

(5.61)

As discussed previously, the parameter \( a \) measures the characteristic length of the microstructure of a porous material. Assuming spherical pores with equal diameter, the diameter is proportional to the characteristic length of the microstructure of a
porous material. When $\phi = 0.1$, we estimate the value of $a$ to be $a_0 = 0.003 \text{ [m]}$ from the sharpness of the velocity profile [17]. Then the value of $a$ for porosity $\phi$ other than 0.1 is given by

$$a = a_0 \frac{r}{r_o}. \quad (5.62)$$

From Eq. (5.55a), we have

$$r = \left(\frac{3\phi}{4\pi N'}\right)^{1/3}. \quad (5.63)$$

When $\phi = 0.1$, the radius $r$ is given by

$$r_o = \left(\frac{3}{40\pi N}\right)^{1/3}. \quad (5.64)$$

Eqs. (5.62), (5.63) and (5.64) lead to

$$a = a_0 (10\phi)^{1/3}. \quad (5.65)$$

To calculate the phase velocities of the four types of waves as a function of porosity, we use Eqs. (5.61) and (5.65) to evaluate $\lambda_0$ and $a$. Following the procedure discussed in Sections (5.2) and (5.3) in calculating the nondimensional velocities, Eqs. (5.32) and (5.51) are evaluated based on the material constants given in Table (5.1).

As shown in Fig. (5.13), the first dilatational wave velocity approaches that of the solid material as porosity vanishes. As the porosity increases, the velocity of the dilatational wave of the first
kind becomes smaller. This is due to the fact that the dilatational wave velocity in the fluid is slower than that in the solid.

Fig. (5.14) shows that the second dilatational wave disappears if the porosity is zero. Since there exists only one rotational wave in a non-porous solid material, the obtained results are realistic. In addition, the velocity of the second dilatational wave increases as the frequency increases.

Fig. (5.15) indicates that the velocity of the first rotational wave also approaches that of the solid material as the porosity approaches zero, as expected. Similar to the dilatational waves of the first kind, the velocity of the first rotational wave decreases as the porosity increases.

In Fig. (5.16), the second rotational wave vanishes as the porosity goes to zero and increases very rapidly as the porosity increases from 0 to 0.2. The frequency change does affect the velocity of the second rotational wave; in contrast, it has virtually no effect on the velocities of the first dilatational wave and first rotational wave as the frequency varies within the low frequency range.
Fig. 5.13. Velocity of the dilatational wave of the first kind as a function of porosity.
Fig. 5.14. Velocity of the dilatational wave of the second kind as a function of porosity.
Fig. 5.15. Velocity of the rotational wave of the first kind as a function of porosity.
Fig. 5.16. Velocity of the rotational wave of the second kind as a function of porosity.
5.4.2. **Effect of variation in bulk moduli, permeability and fluid viscosity.**

To study the effect of the change of the fluid and solid bulk moduli on the phase velocities, we define $k$ as follows:

$$k = \frac{k_f}{k_s} \quad (5.65a)$$

Following the same derivation described in Section 5.4.1, the phase velocities of the four types of waves are calculated. As shown in Fig. (5.17) and Fig. (5.18), the phase velocities of the two types of dilatational waves increases with $k$. This is due to the fact that the effect of fluid on the phase velocities of the dilatational waves becomes larger as $k_f$ increases. In addition, our calculations show that the variation of $k$ has virtually no effect on the velocities of the two types of rotational waves. Physically, the phase velocities of the rotational waves are not affected by the change of the bulk moduli. Thus, these results are also expected.

Permeability is also an important parameter in the study of the wave propagation in a fluid-filled porous material. Once again, following the same steps described in Section 5.4.1 in the calculation of the phase velocities, the effect of the change of permeability on phase velocities are studied and shown in Fig. (5.19) and Fig. (5.20). Fig. (5.19) shows that the velocity of the second dilatational wave increases as the permeability increases. We note that the same conclusion has been drawn from experimental studies [8]. Fig. (5.20) shows that the velocity of the rotational wave of the second kind decreases as permeability increases.
As fluid viscosity varies, the phase velocities of the four types of waves are calculated following the procedures in Section 5.4.1. Fig. (5.21) shows that the phase velocity of the dilatational wave of the second kind decreases as fluid viscosity increases. We also note that this is consistent with experimental works [8]. Fig. (5.22) indicates that the phase velocity of the rotational wave of the second kind increases with fluid viscosity increase. Finally, our calculations show that the phase velocities of the first dilatational wave and of the first rotational wave are virtually not affected by the variation of fluid viscosity.
Fig. 5.17. Effect of bulk moduli on the phase velocity of the dilatational wave of the first kind.
Fig. 5.18. Effect of bulk moduli on the phase velocity of the dilatational wave of the second kind
Fig. 5.19. Effect of permeability on the phase velocity of the dilatational wave of the second kind.
Fig. 5.20. Effect of permeability on the phase velocity of the rotational wave of the second kind.
Fig. 5.21. Phase velocity of the dilatational wave of the second kind.
Fig. 5.22. Phase velocity of the rotational wave of the second kind.
5.5 The Effect of Change in Porosity on the Wave Velocities in the High Frequency Range

5.5.1. Velocities of the Dilatational Waves in the High Frequency Range

As was pointed out by Biot, the coefficient \( \lambda_9 \) must be modified with the frequency correction function in the high frequency range. The characteristic equations will be exactly the same as those given in Section (5.2) with \( \lambda_9 \) replaced by \( \lambda_9 F(\alpha) \), where \( F(\alpha) \) is a frequency correction function (see appendix). This frequency correction function can be written as

\[
F(\alpha) = F_r + i F_i. \tag{5.66}
\]

In Eq. (5.66), \( F_r \) and \( F_i \) are the real and imaginary part of \( F(\alpha) \).

Consequently, Eqs. (5.31) through (5.33) remain the same provided that \( w_3 \) through \( w_6 \) in Eq. (5.27) are replaced by

\[
w_3 = (-a_{11}b_{22} - b_{11}a_{22} + 2a_{12}b_{12} - d_1d_2F_r)\alpha - d_2F_i, \tag{5.67a}
\]
\[
w_4 = -\alpha^2d_1b_{11} + d_2F_r - d_1d_2F_i, \tag{5.67b}
\]
\[
w_5 = b_{11}b_{22}\alpha - \alpha b_{12}^2 + d_2F_i, \tag{5.67c}
\]
\[
w_6 = d_2F_r. \tag{5.67d}
\]

The material constants in Table (5.2) are the same as those used by Plona [20] and Berryman [4]. Based on these material constants in Table (5.2), the velocities of the dilatational waves are calculated as a function of porosity in the high frequency range.
Fig. (5.23) shows that the velocity of the dilatational wave of the first kind decreases as porosity increases. This result is similar to the result in the low frequency range. Fig. (5.24) indicates that the second rotational wave velocity increases very rapidly as porosity increases and decreases to zero as the porosity approaches zero.
Table 5.2
Selected Material Constants for Sintered Glass Beads
Saturated with Water

\[ \rho_s = 2480 \text{ kg/m}^3 \]

\[ \rho_f = 1000 \text{ kg/m}^3 \]

\[ k^s = 40.7 \times 10^9 \text{ Pa} \]

\[ k^f = 2.2 \times 10^9 \text{ Pa} \]

\[ \mu^s = 29.7 \times 10^9 \text{ Pa} \]

\[ \mu^f = 0.001 \text{ Pa.s} \]

\[ a = 0.00006 \text{ m} \]

\[ k_o = 2.31 \times 10^{-14} \text{ m}^2 \]
Fig. 23. Velocity of the dilatational wave of the first kind as a function of porosity at high frequency.
Fig. 5.24. Velocity of the dilatational wave of the second kind as a function of porosity at high frequency.
5.5.2. Velocities of the Rotational Waves in the High Frequency Range

In the high frequency range, the frequency significantly influences both \( \lambda_9 \) and \( \mu_{11} \). In addition to the role of \( \lambda_9 \) in the analysis of the dilatational wave phenomena, \( \mu_{11} \) also plays an important role in the analysis of the rotational waves. Therefore, we need to consider the frequency dependence of \( \mu_{11} \). In order to obtain the exact frequency dependence of \( \mu_{11} \), a detailed micro-mechanical analysis of the porous material will be necessary. However, in this work, we assume that the frequency dependence of \( \mu_{11} \) is similar to \( \lambda_9 \) and it can be approximated by \( \mu_{11} \cdot F(\alpha) \).

Following the derivations in Section 5.5.1, the rotational wave velocities as a function of porosity can be obtained from Eqs. (5.51) and (5.52), except \( p_1 \) through \( p_6 \) are replaced by the following:

\[
P_1 = -q_1q_2\alpha^2F_r - aq_1F_i, \tag{5.68a}
\]
\[
P_2 = q_2\alpha + aq_1F_r - q_1q_2\alpha^2F_i, \tag{5.68b}
\]
\[
p_3 = -b_{22} - d_2q_2F_r - d_2F_i/\alpha, \tag{5.68c}
\]
\[
p_4 = d_2F_i/\alpha - aq_2b_{11} - aq_1F_r - a_2q_2F_i, \tag{5.68d}
\]
\[
p_5 = b_{11}b_{22} - b_{12}^2 - d_2F_i/\alpha, \tag{5.68e}
\]
\[
p_6 = d_2F_i/\alpha. \tag{5.68f}
\]

Figs. (5.25) and (5.26) show the velocities of the rotational waves of the first and second kind versus porosity, respectively. It is seen
from Fig. (5.25) that the velocity of the rotational wave of the first kind decreases as the porosity increases. Following the trend of the velocity of the dilatational wave of the second kind, Fig. (5.26) shows that the velocity of the rotational wave of the second kind also increases as the porosity increases. As the porosity approaches zero, the velocity of the second rotational wave decreases to zero. This behavior is expected, since the rotational wave of the second kind disappears in a non-porous solid.
Fig. 5.25. Velocity of the rotational wave of the first kind as a function of porosity at high frequency.
Fig. 5.26. Velocity of the rotational wave of the second kind as a function of porosity at high frequency.
5.5.3 **Comparison with Experimental Results as Well as Biot's Theory**

Plona has performed experiments on the effect of the porosity change on the velocities of the two dilatational waves and the one rotational wave, using small glass beads saturated with water. His results are shown in Fig. (5.27). In Fig. (5.27), $V_{L}^{1}$, $V_{L}^{2}$ and $V_{s}^{1}$ are, respectively, the phase velocities of the first dilatational wave, of the second dilatational wave and of the first rotational wave. In addition, Berryman also studied the velocities of the three types of waves as a function of porosity by using Biot's theory and the self-consistent theory of composite materials. His results are also plotted in Fig. (5.27).

By using the same material constants used by Plona and Berryman, we have studied the velocities of the four types of waves as a function of porosity based on the modified mixture theory and Mackenzie's effective moduli. It is seen that the computed velocities of the first rotational wave and of the second dilatational wave agree well with Berryman's results. The velocity of the first dilatational wave is, however, slightly different from the results obtained in both experiment and Biot's theory.

In section 5.2, the results for the dilatational waves obtained from the modified mixture theory and Biot's theory have been shown to be the same. The slight variation on the phase velocity of the dilatational wave of the first kind shown in Fig. (5.27) is due to the use of different models for the evaluation of the effective moduli, $k^{(1)}$ and $\mu^{(1)}$. 
On one hand, Berryman has estimated the bulk and shear moduli of the porous material, \( k^{(1)} \) and \( \mu^{(1)} \), by considering the material to be composed of \( N \)-Phases of granular constituents and a phase of vacuum. Therefore, \( k^{(1)} \) and \( \mu^{(1)} \) are governed by a set of coupled non-linear equations with known constants \( k^S, \mu^S, \) and \( \phi \). Berryman evaluated \( k^{(1)} \) and \( \mu^{(1)} \) by numerical iteration. On the other hand, Mackenzie has studied the effective bulk and shear moduli of a porous solid with spherical pores. As a result, \( K^{(1)} \) and \( \mu^{(1)} \) have been explicitly obtained as a function of porosity, bulk and shear moduli of the solid. His results are summarized in Eqs. (4.32) and (4.33). In our calculations, we have used Mackenzie's results in Eqs. (4.32) and (4.33) to evaluate \( k^{(1)} \) and \( \mu^{(1)} \).

In addition, our calculation shows that the phase velocity of the second rotational wave is very sensitive to the variation of the characteristic length \( a \). The problem is that the material constant \( \mu_{11} \) can not be estimated exactly, and so neither can the value of \( a \).

As a final remark, we note that the frequency is chosen as \( f=500 \) kHz in Plona's experiment. In this case, \( f/f_c \) is calculated to be 0.73 at porosity \( \phi = 0.1 \). Clearly, the frequency is in the high frequency range.
Fig. 5.21. Phase velocities obtained from Biot's theory, modified mixture theory and experiment as a function of porosity when f = 500 kHz.
CHAPTER VI
CONCLUSIONS

The plane wave phenomena in a fluid-filled porous material by using the modified mixture theory is investigated by considering the interaction torque effect between the solid and fluid constituents and the fluid viscous stress, not previously included in Biot's theory. The effect of the associated additional terms on the phase velocities, phase differences, attenuation, and amplitude ratios are studied and the results are compared with Biot's theory as well as with the experimental work [4]. A summary of the major results is presented below:

1. The virtual mass coefficient \( \rho_{12} \) is shown to have significance in the mixture theory, and the governing equations of motion are modified.

2. In the ideal fluid case, the governing equations of wave propagation in the mixture theory are shown to be identical to Biot's theory. By using Mackenzie's model to evaluate the effective bulk and shear modulus, the phase velocities and the amplitude ratios are studied as a function of porosity. As expected, the phase velocity of the dilatational wave of the first kind and that of the rotational wave of the first kind are shown to decrease with increasing porosity. On the other hand, the velocity of the
dilatational wave of the second kind increases with the porosity increase. The amplitude ratio $C_1/C_2$ is shown to increase as the porosity increases for all types of waves. As the porosity approaches zero, the amplitude ratio of the dilatational wave of the first kind tends to unity. This implies that the solid and fluid displacements propagate in essentially with the same amplitude.

3. In the Newtonian viscous fluid case, the phase velocities, attenuation and phase differences of the dilatational and rotational waves are studied.

For dilatational waves, we conclude:

a) The phase velocity of the dilatational wave of the first kind is shown to decrease as the porosity decreases in low and high frequency ranges. The converse is true for the dilatational wave of the second kind. As the porosity approaches zero, the velocity of the dilatational wave of the second kind approaches zero. This corresponds to the disappearance of the dilatational wave of the second kind in a non-porous solid material. For a given porosity, the phase velocity of the dilatational wave of the first kind increases slightly as frequency increases, whereas the velocity of the dilatational wave of the second kind increases significantly with frequency.

b) Physically, the dilatational waves of the first and second kind are shown to correspond virtually to in-phase and out-of-phase motions of solid and fluid aggregates, respectively, at low frequencies.
c) The study of dispersion and attenuation shows that the results obtained in the modified mixture theory are essentially the same as those obtained from Biot's theory. Thus if one is only interested in the study of the dilatational waves, Biot's theory should be appropriate.

For rotational waves, we conclude:

a) A second rotational wave is shown to exist. Physically, this second rotational wave exists due to the interaction torque between the solid and fluid constituents and the overall fluid viscous stress. The nature of the second rotational wave is highly attenuated. This rotational wave of the second kind is shown to be the counterpart of the dilatational wave of the second kind.

b) The phase velocity of the rotational wave of the first kind is shown to decrease as the porosity decreases in both low and high frequency ranges. As the porosity approaches zero, the velocity of the rotational wave of the second kind approaches zero. This corresponds to the disappearance of the rotational waves of the second kind in a non-porous solid material. For a given porosity, frequency has little effect on the phase velocity of the rotational wave of the first kind, whereas the velocity of the rotational wave of the second kind may increase as frequency increases.

c) The second rotational wave is shown to be very attenuated. The smaller the characteristic length $a$, the higher the attenuation. As $a$ approaches zero, the rotational wave of the second kind becomes extremely attenuated. This also corresponds
to the disappearance of the rotational wave of the second kind, and hence no second rotational wave is predicted in Biot's theory.

d) Physically, the rotational waves of the first and second kind are shown to correspond virtually to in-phase and out-of-phase motions of solid and fluid aggregates at very low frequencies.
APPENDIX A

FREQUENCY CORRECTION FUNCTION IN A 3-D DUCT

To obtain the frequency correction function $F(\alpha)$, Biot [6] has considered the flow of a viscous fluid in a circular tube subjected to an oscillatory motion at the boundary. Through calculating the ratio of the total friction force to the average velocity, $F(\alpha)$ is shown to be as follows:

$$F(\alpha) = F[\beta(\alpha)] = \frac{1}{4} \frac{\beta T(\beta)}{1 - 2T(\beta)/(i\beta)}, \quad (A.1)$$

where $T(\beta) = \frac{\text{ber}\beta + i\text{bei}\beta}{\text{ber}\beta + i\text{bei}\beta}, \quad (A.2)$

$$\beta = R(\alpha/\nu)^{1/2}, \quad (A.3)$$

$$\nu = \mu^f/\rho^f. \quad (A.4)$$

$R$ is the radius of the circular duct. ber$\beta$ and bei$\beta$ are, respectively, the real and imaginary part of the Bessel Function $J_0(\sqrt{i^3\beta}),$ i.e.,

$$J_0(\sqrt{i^3\beta}) = \text{ber}\beta + i\text{bei}\beta. \quad (A.5)$$
APPENDIX B

JUSTIFICATION OF THE $\rho_{12}$ TERM

To verify the existence of $\rho_{12}$, we consider the superposition of the following two motions:

First, as shown in Fig. B.1, we consider an infinitely long rigid cylinder of radius "R" in an infinite ideal fluid field. This cylinder is moving with velocity $U$ perpendicular to its axis and the fluid is at rest at infinity. Second, we consider this cylinder moving together with all the fluid in the negative x-direction with velocity $V$.

Superpose the above two motions. The velocity potential and its complex conjugate associated with this motion are given by [21]

$$\Phi = Vr \cos \theta + \frac{u_2}{r} \cos \theta$$

(B.1)

$$\bar{\psi} = Vr \sin \theta - \frac{u_2}{r} \sin \theta$$

(B.2)

These velocity potentials characterize the motion of an infinitely long rigid cylinder moving with a velocity $W = U - V$ perpendicular to its length in an infinite fluid field. In this fluid field, the fluid moves at a speed of $V$ in the negative x-direction at infinity.

To calculate the total kinetic energy of a representative volume, we consider a cube of length "L" as shown in Fig. (B.2). In this representative sample, the center of the cylinder coincides with the center of the cube. The total kinetic energy of this cube is composed
Fig. B.1. Long cylinder moving in x-direction.
Fig. B.2. A representative porous material.
of two parts, the kinetic energy due to the fluid $T_f$, and that due to the solid $T_s$. The pointwise velocities of the fluid particles in the x, y and z directions are denoted, respectively, by $V_x$, $V_y$, and $V_z$. The kinetic energy of the fluid in this cube can be written as

$$T_f = \frac{1}{2} \rho_f \iiint_{\Omega_f} \left( V_x^2 + V_y^2 + V_z^2 \right) dx dy dz, \quad (B.3)$$

where

$$v_x = -\frac{\partial \Phi}{\partial x}, \quad (B.4)$$

$$v_y = -\frac{\partial \Phi}{\partial y}, \quad (B.5)$$

$$v_z = -\frac{\partial \Phi}{\partial z}, \quad (B.6)$$

Similarly, the kinetic energy of the solid part in this volume is given by

$$T_s = \frac{1}{2} \rho_s \iiint_{\Omega_f} w^2 dx dy dz. \quad (B.7)$$

Combining Eqs. (B.3) through (B.7), the total kinetic energy of the sample is shown to be

$$T = T_f + T_s = \frac{1}{2} \rho_{11} \ddot{w}^2 + \rho_{12} \ddot{\dot{w}} + \frac{1}{2} \rho_{22} \ddot{v}^2, \quad (B.8)$$

where
In Eq. (B.8), $\dot{W}$ and $\dot{V}$ are, respectively, the average velocities of the solid and of the fluid.

We also have

$$\rho_{11} + \rho_{12} = (1-\phi)\rho_{s}, \quad \text{(B.12)}$$

$$\rho_{22} + \rho_{12} = \phi\rho_{f}, \quad \text{(B.13)}$$

which are exactly what Biot has concluded in Eqs. (2.12) and (2.13).
REFERENCES


