INFORMATION TO USERS

The most advanced technology has been used to photograph and reproduce this manuscript from the microfilm master. UMI films the text directly from the original or copy submitted. Thus, some thesis and dissertation copies are in typewriter face, while others may be from any type of computer printer.

The quality of this reproduction is dependent upon the quality of the copy submitted. Broken or indistinct print, colored or poor quality illustrations and photographs, print bleedthrough, substandard margins, and improper alignment can adversely affect reproduction.

In the unlikely event that the author did not send UMI a complete manuscript and there are missing pages, these will be noted. Also, if unauthorized copyright material had to be removed, a note will indicate the deletion.

Oversize materials (e.g., maps, drawings, charts) are reproduced by sectioning the original, beginning at the upper left-hand corner and continuing from left to right in equal sections with small overlaps. Each original is also photographed in one exposure and is included in reduced form at the back of the book. These are also available as one exposure on a standard 35mm slide or as a 17" x 23" black and white photographic print for an additional charge.

Photographs included in the original manuscript have been reproduced xerographically in this copy. Higher quality 6" x 9" black and white photographic prints are available for any photographs or illustrations appearing in this copy for an additional charge. Contact UMI directly to order.

UMI

University Microfilms International
A Bell & Howell Information Company
300 North Zeeb Road, Ann Arbor, MI 48106-1346 USA
313/761-4700  800/521-0600
$L^2$-index theorems for perturbed Dirac operators

Anghel, Nicolae, Ph.D.

The Ohio State University, 1989
L²- INDEX THEOREMS FOR PERTURBED
DIRAC OPERATORS

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of the Ohio State University

By

Nicolae Anghel, B.S., M.S.

* * * * *

The Ohio State University

1989

Dissertation Committee:

Dan Burghelea
Boris Mityagin
Henri Moscovici
Robert Stanton

Approved by

Henri Moscovici
Adviser
Department of Mathematics
V I T A

April 8, 1954 ----------------------------- Born - Valeni de Munte, Rumania
1977 ------------------------------------ B.S., University of Bucharest, Rumania
1978 ------------------------------------ M.S., University of Bucharest, Rumania
1978-1982 ------------------------------ High School Mathematics Teacher
                                        Bucharest, Rumania
1983-1989 ------------------------------ Graduate Teaching Assistant / Lecturer
                                        Department of Mathematics, The Ohio
                                        State University, Columbus, Ohio

P U B L I C A T I O N S

2. A Direct Proof of the Fact that $L^1(X,\mathfrak{B},\mu)$ Satisfies the Strong Maximum Modulus
   MR #82g: 46050
   1982
   MR #84h: 58013
5. The Two Dimensional Magnetic Field Problem Revisited. 1988 Submitted to J.
FIELDS OF STUDY

Major Field: Index Theory

Studies

in:

- Index Theory, Global Analysis  
  Dan Burghelea, Henri Moscovici
- Analysis of Manifolds, Supersymmetry,  
  Differential Geometry  
  Henri Moscovici, Ernst Ruh,  
  Robert Stanton
- Pseudodifferential Operators  
  Boris Mityagin
LIST OF FIGURES

FIGURES

1. The manifolds $M_0$ and $M_1$ are isometric outside compact sets $K_0$ and $K_1$. ................................................................. 48
# TABLE OF CONTENTS

**VITA** ........................................................................................................................................ ii

**LIST OF FIGURES** ................................................................................................................. iv

**INTRODUCTION** ...................................................................................................................... 1

**CHAPTER**  

<table>
<thead>
<tr>
<th>CHAPTER</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>I. GENERALIZED DIRAC OPERATORS — A BRIEF ACCOUNT</td>
<td>9</td>
</tr>
<tr>
<td>II. PERTURBED DIRAC OPERATORS OF FREDHOLM TYPE</td>
<td>15</td>
</tr>
<tr>
<td>III. INDEX-PRESERVING DEFORMATIONS</td>
<td>24</td>
</tr>
<tr>
<td>IV. SEPARATION OF VARIABLES ON WARPED PRODUCTS</td>
<td>32</td>
</tr>
<tr>
<td>V. SOLVING THE INDEX PROBLEM</td>
<td>40</td>
</tr>
</tbody>
</table>

**APPENDICES**

<table>
<thead>
<tr>
<th>APPENDICES</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>A. REMARK ON CALLIAS' INDEX FORMULA</td>
<td>53</td>
</tr>
<tr>
<td>B. THE TWO DIMENSIONAL MAGNETIC FIELD PROBLEM</td>
<td>59</td>
</tr>
</tbody>
</table>

**LIST OF REFERENCES** ........................................................................................................... 66
INTRODUCTION

In 1978 C. Callias was led by physical considerations to the following $L^2$-index theorem (cf. [C]):

**Theorem 0.1.**— *Let $\Sigma$ be the spinor space over $\mathbb{R}^n$, $n$ odd, and $D$ the Dirac operator on $C^\infty(\mathbb{R}^n, \Sigma \otimes \mathbb{C}^m)$. Let $L$ be the perturbation of $D$ by $\sqrt{-1} \text{Id} \otimes \Phi$, where $\Phi \in C^\infty(\mathbb{R}^n, \text{End}(\mathbb{C}^m))$ is Hermitian, asymptotically homogeneous of degree 0, and $\Phi^2$ is positive outside some compact set. Then $L$ is a Fredholm elliptic differential operator, and if $U$ is the unitarization of $\Phi$ at infinity, i.e., $U = |\Phi|^{-1} \Phi$ outside a compact set, one has*

\begin{equation}
L^2\text{-index}(L) = \frac{1}{2 \left(\frac{n-1}{2}\right)!} \left(\frac{\sqrt{-1}}{8\pi}\right)^{\frac{n-1}{2}} \lim_{R \to \infty} \int_{S_R^{n-1}} \text{tr} U(dU)^{n-1}
\end{equation}

In (0.2), $S_R^{n-1}$ stands for the sphere centered at the origin and of radius $R$ in $\mathbb{R}^n$ (see also Appendix A).

This result is considered among the first truly significant $L^2$-index theorems which cannot be "compactified". It is well-known [AS] that the index of any elliptic differential operator on an odd dimensional compact manifold vanishes. For this reason — and many others having to do with its intrinsic beauty, or Callias' way of proving it, or the need to understand its deviation from the Atiyah-Singer index theorem — it attracted a lot of interest. In [BoS], R. Bott and R. Seeley used a topological approach to derive it from a more general index formula in $\mathbb{R}^n$, due to L. Hormander [L] and
B.V. Fedosov [F], M. Stern [S], refined Callias' proof to get $L^2$-index theorems on locally symmetric spaces for geometric operators.

The present thesis attempts a generalization of Theorem 0.1 based on exploring the subtle geometric information encoded in the RHS of (0.2). The nature of this information is obvious. It must involve:

- **a)** the geometry of the \((n-1)\)-dimensional sphere — the "boundary" of $\mathbb{R}^n$, viewed as a direct limit of closed balls, and

- **b)** the spectral properties of the matrix $\Phi$ near this boundary.

Any information at finite distance is made irrelevant for index purposes by the odd dimensionality of the space, just as in the compact case. **Not so obvious, however, is how to bind together a) and b).**

First we notice that the polar coordinates are better suited for understanding (0.2). In fact, deformation techniques and the asymptotic homogeneity of $\Phi$ allow us to assume that $U$ depends only on the solid angle and not the radial distance, outside some compact set. This indicates what class of manifolds is fit for generalization: the manifolds with **warped ends** i.e., manifolds $M$ which outside some compact set are geometrically isometric to warped products $(\mathbb{R},\infty) \times_f N, f \in \mathbb{R}, N$ being some compact manifold, $f \in C^\infty(\mathbb{R})$, $f > 0$. In Callias' case, $\mathbb{R}^n \backslash \{0\} \equiv (0,\infty) \times_0 S_1^{n-1}$, $f(r) = r$, $r \in (0,\infty)$.

As remarked by H. Moscovici and F.B. Wu, the integrand in (0.2) is precisely the part of degree $(n-1)$ in the Chern character of the subbundle $V_+$ of $\mathcal{C}^m$ over $S^{n-1} = S_1^{n-1}$, given by $V_+ = \{U = \text{Id}\}$. Using the Atiyah-Singer index theorem for twisted classical Dirac operators [AS], (0.2) can be rewritten as:

\[(0.3) \quad L^2\text{-index}(L) = \text{index}(\partial_{V_+}^+)\]
where $\partial_{V_+}^+$ is the half-Dirac operator on $S^{n-1}$, with coefficients in $V_+$. Our main result is a formula of type (0.3) for certain perturbed Dirac operators on spinor bundles over odd dimensional spin-manifolds with warped ends.

The proof we give is conceptually simple. Our perturbed Dirac operators are operators $L$ of type $L = D + A$, where $D$ is a (generalized) Dirac operator on some Dirac bundle $S$, as defined by M. Gromov and H.B. Lawson in [GL], and $A$ is a bundle morphism. They have a good Fredholm theory, similar to that in [GL] for Dirac operators. For manifolds $M$ with warped ends these operators can be deformed to yield separation of variables on the end of $M$. That is to say, outside some compact set, a suitable deformation of $L$ is expressible as an operator-valued first order differential equation in $r$ of type

$$(0.4) \quad n \cdot \frac{\partial}{\partial r} + \frac{\partial_N}{f} + \frac{c}{f} \Xi_N + A_N$$

In (0.4) $n \cdot$ denotes the Clifford multiplication on $S$ by $\frac{\partial}{\partial r}$, $\partial_N$ is a selfadjoint first order elliptic operator on a canonically associated Dirac bundle $S_N$ over $N$, and $\Xi_N, A_N$ are bundle morphisms on $S_N$. Operators similar to (0.4) have already been considered in index problems by J. Bruning and R. Seeley in [B], [BrS].

The $L^2$-solution spaces of the restrictions of $L$ and $L^*$ to the end of $M$ can be described in terms of the eigenvalues of $\partial_N$, if we assume that $S$ is a spinor-type [LM] bundle, and $A$ commutes with the Clifford multiplication, as Callias' operator does. Only the 0-eigensections of $\partial_N$ contribute essentially to the index.

The next and final step in the proof is to decide which 0-modes of $L$ and $L^*$ near infinity extend to the whole manifold. For this we use an odd-dimensional variant
of Gromov-Lawson’s relative index theorem [GL]. Two copies of \( L \) are compared, via this theorem, with an operator on a manifold with two warped ends and whose index is computable by direct kernel description.

We now proceed to describe in some detail the content of each chapter.

Chapter I is essentially a review of the (generalized) Dirac operators [GL], and the introduction of their perturbations by bundle morphisms (Definition 1.11). The perturbed Dirac operators are completely characterized by their principal symbol — the Clifford multiplication —, and obey the usual local and global metric properties when compared to their formal adjoints (Proposition 1.13). On complete manifolds they admit a unique closed extension to the \( L^2 \)-space (Theorem 1.14), just as the Dirac operators do; this can happen on some incomplete manifolds as well.

In Chapter II we focus on a particular class of perturbed Dirac operators, namely those operators for which the perturbing endomorphism \( A \) is skew-symmetric and commutes with the Clifford multiplication (Assumption 2.3). The commutator \([D, A]\) is then a bundle morphism (Proposition 2.4), and we have

\[
(0.5) \quad L^\dagger L = D^2 + R_A, \quad LL^\dagger = D^2 + R_{-A}
\]

where \( R_A \) denotes the bundle morphism \([D, A] - A^2\). If \( R_{\pm A} \) is positive at infinity (Assumption 2.9) i.e., \( R_{\pm A} \geq c \text{ Id}, \quad c > 0 \), pointwise, outside some compact set, \( L \) is a Fredholm operator. This fact is explained at length in Theorem 2.11. Corrolary 2.28 provides a sufficient condition for fredholmness: assumption 2.9 holds if \(-A^2\) is positive at infinity and \([D, A]\) decays to 0 at infinity.

In Chapter III we introduce a Callias-type index problem (Definition 3.1), that is, the problem of evaluating the \( L^2\)-index of the perturbed Dirac operator \( L \) con-
sidered in Chapter II, Corollary 2.28. First we remark that this problem is uninteresting on even dimensional manifolds (Remark 3.5). Then we explore various deformations of \( L \) which, while preserving the index, will simplify our task later:

**Deformation 1** — \( L^2\)-index\((L)\) is unchanged by deforming \( A \) continuously into \( U \equiv (-A^2)^{-\frac{1}{2}}A \), outside some compact set \( K \subset M \) (Proposition 3.8). This \( U \) provides a way to split the Dirac bundle \( S\big|_{M-K} \) into a direct sum \( S_+ \oplus S_- \) of Dirac bundles \( S_\pm \) over \( M-K \), according to the eigenvalues \( \pm \sqrt{-1} \) of \( U \) (Lemma 3.18). The restriction of \( L = D+U \) to \( M-K \) can be written in the following matrix form:

\[
(0.6) \quad L|_{M-K} = \begin{pmatrix}
D_+ + \sqrt{-1} & \frac{-\sqrt{-1}}{2} [D, U] |_{S_-} \\
-\frac{\sqrt{-1}}{2} [D, U] |_{S_+} & D_- - \sqrt{-1}
\end{pmatrix}
\]

where \( D_\pm \) is the corresponding Dirac operator on \( S_\pm \).

**Deformation 2** — \( L \) is index-equivalent to a perturbed Dirac operator \( T \) on \( S \) which is diagonal with respect to the splitting \( S\big|_{M-K} = S_+ \oplus S_- \), more exactly (Theorem 3.25)

\[
(0.7) \quad T|_{M-K} = \begin{pmatrix}
D_+ + \sqrt{-1} & 0 \\
0 & D_- - \sqrt{-1}
\end{pmatrix}
\]

**Corollary 3.28**, stating that \( L^2\)-index\((L)\) is 0 if the splitting in Deformation 2 is global, concludes Chapter III.

Chapter IV is devoted to a detailed analysis of the separation of variables formula for Dirac operators on warped products. A warped product \( W \) is a manifold of type \( (\varepsilon, \infty) \times N \), \( N \) compact Riemannian manifold, \( \varepsilon \in \mathbb{R} \), equipped with the Riemannian metric \( dr^2 + f^2(r) \, ds^2 \), \( f \in C^\infty((\varepsilon, \infty), \mathbb{R}_+) \), \( ds^2 \) being the metric on \( N \). Any
Dirac bundle \( S \) over \( W \) induces on the slices \( S_R \equiv S \mid \{R\} \times N \), \( R \in (\varepsilon, \infty) \), canonical structures of Dirac bundles over \( N \), under a mild restriction on the curvature tensor on \( S \). This is done in Theorem 4.13, using the notion of parallel transport along the radial geodesics. Any section \( s \in C^\infty(N, S_R) \) lifts by parallel transport to a section \( s^\sim \in C^\infty(W, S) \) and then the separation of variables formula (formula 4.14) reads:

\[
(0.8) \quad D(s^\sim) = \frac{(\partial R s)^\sim}{f} + \frac{f'}{f}(\Xi_R s)^\sim
\]

where \( D \), respectively \( \partial_R \), is the Dirac operator on \( S \), respectively \( S_R \), and \( \Xi_R \) is a bundle morphism on \( S_R \) depending only on the curvature tensor of \( S \) restricted to \( \{R\} \times N \). For instance, if \( W \) is a spin manifold and \( S \) is the spinor bundle \( \text{Spin}(W) \), then \( \Xi_R = \frac{\dim N}{2} n \cdot n = \frac{\partial}{\partial t} \). When \( S \) is the Clifford bundle of exterior algebras on \( W \),

\[
(0.9) \quad \Xi_R \omega = p n \cdot \omega \quad \omega \text{ p-form on } N
\]

Under the natural identification \( C^\infty(W, S) = C^\infty((\varepsilon, \infty), C^\infty(N, S_R)) \), (0.7) yields (0.4) in the particular case \( A = 0 \).

Chapter V provides the solution to the Callias-type index problem, stated in Chapter III, for spinor bundles over odd dimensional manifolds with warped ends, when the perturbing endomorphism \( U \) is independent of the radial direction.

First the manifold \( M_1 = \mathbb{R} \times f_1 N \) is considered, together with a Dirac bundle \( S_1 \) and a perturbed Dirac operator \( L_1 \) (Proposition 5.8). \( (M_1, S_1) \) has two warped ends isometric to the end of the original pair \( (M, S) \), and a direct calculation gives (Theorem 5.12 and Corollary 5.27):
\( (0.10) \quad L^2\text{-index}(L_1) = 2 \text{index}(\phi_R)_+ \)

In \( (0.10) \) \( (\phi_R)_+ \) is the Dirac operator associated to \( (S_R)_+ \rightarrow N \), and \( (\phi_R)_+^* \) its restriction to \( (S_R)_+^* \) (recall that \( M \) being odd dimensional, \( N \) is even dimensional, so any Dirac bundle \( \xi \) over \( N \) splits with respect to the eigenvalues \( \pm \sqrt{-1} \) of the volume form \( n \) on \( N \) as \( \xi = \xi^+ \oplus \xi^- \).

Next the relative index theorem of Gromov-Lawson, which for odd dimensional manifolds amounts to an index preserving deformation theorem, is used for \( M_1 \) and two copies of \( M \) to yield the main result (Theorem 5.27):

\( (0.11) \quad L^2\text{-index}(L) = \text{index}(\phi_R)_+ \)

In Corollary 5.29, Callias' result \((0.2)\) is derived from \((0.11)\).

The two appendices A and B serve different purposes.

In Appendix A the approach used by Bott and Seeley [BoS] is completed, in the sense that using careful algebraic manipulations we are able to derive Callias' result up to the very last constant.

Appendix B shows in a very concrete situation — the so called two dimensional magnetic field problem [BGGSS] — how interesting a Callias-type problem is in an even dimensional set-up, if the perturbation term \( A \) does not commute with the Clifford multiplication. In particular (Theorem B.11), this \( L^2\)-index is equivalent to an index problem for manifolds with boundary and non-local boundary conditions [APS].

I want to express here my great debt to Henri Moscovici, my advisor, for introducing me to Index Theory, for suggesting the problem, and for many years of help,
encouragement and support. I would also like to thank Dan Burghelea, Alexander Dynin, Boris Mityagin, Robert Seeley, Robert Stanton and Fang Bing Wu who, as professors or colleagues, had a significant impact on my mathematical formation. This thesis, in particular, benefitted a great deal from their comments and suggestions.
CHAPTER I

GENERALIZED DIRAC OPERATORS — A BRIEF ACCOUNT

The generalized Dirac operators and their perturbations form a broad class of first order elliptic differential operators. Their appearance is essentially justified by the modern need of giving an unified treatment to the elliptic operators arising in geometric situations, e.g., the Gauss-Bonnet, signature, Riemann-Roch and classical Dirac operators. Since they are basic in our approach and in order to establish necessary notations we recall briefly here their main properties. For details and proofs we refer to the beautiful references [GL] and [LM].

Let \((M,g)\) be a complete Riemannian manifold of dimension \(n\). For reasons which will become clear a little bit later we will restrict our attention to odd dimensional manifolds only. Most of the statements we are going to make in this chapter are true regardless of the parity of \(n\); nonetheless we assume throughout that \(n\) is odd. Let \(\text{Cl}(M)\) be the Clifford bundle of algebras induced by the tangent bundle \(TM\) and the Riemannian metric \(g\). There is a canonical embedding \(TM \to \text{Cl}(M)\), and then the Riemannian metric and Levi-Civita connection extend from \(TM\) to \(\text{Cl}(M)\). The connection \(\nabla^{\text{LC}}\) on \(\text{Cl}(M)\) preserves the metric and acts as a derivation.

A bundle of left modules over the the bundle of algebras \(\text{Cl}(M)\), say \(S \to M\), will be called a (generalized) Dirac bundle if \(S\) is furnished with a Hermitean metric \(<,>\) and a metric connection \(\nabla^S\) such that
The action on $S$ by unit vectors in $TM \subset \text{Cl}(M)$ is a pointwise isometry.

$$\nabla_e^{S}(\phi \ast s) = \nabla_e^{LC}(\phi) \ast s + \phi \ast \nabla_e^{S}(s) \text{ for all } e \in \mathcal{C}^\infty(TM), \phi \in \mathcal{C}^\infty(\text{Cl}(M))$$

The " $\ast$ " indicates here Clifford multiplication on $S$. (1.2) is simply saying that $\nabla^S$ acts as a derivation with respect to the Clifford action on $S$.

There are two categories of Dirac bundles:

a) the fundamental ones, like $\text{Cl}(M)$ itself, or the spinor bundle $\Sigma$, if $M$ happens to be a spin manifold. To be more specific in this second case, in order that $M$ be a spin manifold, the principal $\text{SO}(n)$-bundle $P_{\text{SO}(M)}$ of oriented frames of $TM$ must lift to a principal $\text{Spin}(n)$-bundle $P_{\text{Spin}(M)}$ equivariantly with respect to $\text{Spin}(n) \rightarrow \text{SO}(n)$. The spinor bundle is then the fibre product $\Sigma = P_{\text{Spin}(M)} \times_{\mu} \Delta$, of $P_{\text{Spin}(M)}$ with a $n$-dimensional spinor space $(\Delta, \mu)$. Recall that the pair $(\Delta, \mu)$ is a spinor space if the complex vector space $\Delta$ is an irreducible module over the algebra $\text{Cl}(\mathbb{R}^n) \otimes \mathbb{C}$ and $\mu$ is the unitary representation $\mu : \text{Spin}(n) \rightarrow \text{U}(\Delta)$ induced by the left multiplication with elements of $\text{Spin}(n) \subset \text{Cl}(\mathbb{R}^n) \otimes \mathbb{C}$. When $n$ is odd there are two inequivalent irreducible $\text{Cl}(\mathbb{R}^n) \otimes \mathbb{C}$-modules but they induce the same group representation $\mu$, which is also irreducible [LM].

Lifting the Riemannian connection on $P_{\text{SO}(M)}$ to $P_{\text{Spin}(M)}$ via the Lie algebra isomorphism $\text{so}(n) \cong \text{spin}(n)$, we get the canonical connection $\nabla^\Sigma$ of $\Sigma$. In fact, any local section $e = \{e_1, ..., e_n\}$ of $P_{\text{SO}(M)}$ can be lifted up to $P_{\text{Spin}(M)}$ and then embedded into the $P_{\Sigma}$ — the principal $\text{SO}(N)$-bundle, $N = 2^2$, of orthonormal bases in $\Sigma$. Doing so we get a local section $s = \{s_1, ..., s_N\}$ in $P_{\Sigma}$, called a spinor basis.
Then

\( \nabla_e \Sigma_{\alpha} s_\alpha = \frac{1}{2} \sum_{i<j} g(\nabla e^L e_i, e_j) e_i e_j s_\alpha \quad e \in C^\infty(TM) \quad \alpha = 1, 2, \ldots, N. \) \hspace{1cm} (1.3)

\textbf{b) the generated ones}, by algebraic operations, out of old ones. For example, if

\[ S \text{ is a Dirac bundle and } E \to M \text{ is any complex vector bundle with Hermitian connection } \nabla^E, \text{ then the tensor product } S \otimes E \text{ is again a Dirac bundle with respect to the tensor product metric and connection} \]

\( \nabla^{S \otimes E} = \nabla^S \otimes \text{Id} + \text{Id} \otimes \nabla^E \) \hspace{1cm} (1.4)

Another example, which will be used later, is \( \text{End}(S) \), the bundle of endomorphisms of a given Dirac bundle \( S \). Here

\( (\nabla^e \text{End}(S) A_j) s = \nabla^e S(As) - A(\nabla^e S s), \quad e \in C^\infty(TM), \ s \in C^\infty(S), \ A \in C^\infty(\text{End}(S)) \) \hspace{1cm} (1.5)

We could have considered the Dirac bundle \( S^* \) dual to \( S \) and then view \( \text{End}(S) \) as \( S^* \otimes S \). Finally the spinor bundle in \textbf{a}) can be generalized, giving up the irreducibility of \( \Delta \). The connection formula (1.3) is preserved. Such a bundle will be referred to as a \textit{spinor-type} bundle and denoted by \( S^* \).

Any Dirac bundle \( S \) generates a distinguished first order differential operator \( D^S = D : C^\infty(S) \to C^\infty(S) \), called the \textit{(generalized) Dirac operator}. Locally it can be expressed by

\[ D = \sum_{i=1}^n e^*_i \nabla e_i \] \hspace{1cm} (1.6)

(1.6) is clearly independent of the local frame \( \{e_1, \ldots, e_n\} \). \( D^{\text{End}(S)} \) will be denoted shortly by \( \mathcal{D} \). \( D \) is \textit{elliptic}. In fact the principal symbol \( \sigma_\xi(D) \in \text{End}(S), \ \xi \in T^*M, \) is
the Clifford multiplication by the tangent vector metric equivalent to $\xi$. This can be seen from the following obvious formula:

\begin{equation}
D(fs) = \text{grad } f \cdot s + fD s, \quad f \in C^\infty(M), s \in C^\infty(S)
\end{equation}

Let $\Omega \subset M$ be any open subset of $M$. The usual inner product in $C^\infty(\Omega, S)$ will be denoted by $(\cdot, \cdot)_\Omega$, i.e.,

\begin{equation}
(s_1, s_2)_\Omega = \int_{\Omega} \langle s_1, s_2 \rangle \, \text{d} \text{vol}, \quad s_1, s_2 \in C^\infty(\Omega, S)
\end{equation}

If $\Omega = M$, we will write $(\cdot, \cdot)$ instead of $(\cdot, \cdot)_M$. The Dirac operator is then seen to be formally selfadjoint, i.e.

\begin{equation}
(Ds_1, s_2) = (s_1, Ds_2), \quad \text{for any } s_1, s_2 \in C^\infty(S), \text{ one of them compactly supported}
\end{equation}

(1.9) is a consequence of the following integration by parts formula for Dirac operators

\begin{equation}
(Ds_1, s_2)_\Omega = (s_1, Ds_2)_\Omega + (n \cdot s_1, s_2)_{\partial \Omega}, \quad s_1, s_2 \in C^\infty(S)
\end{equation}

In (1.10) $\Omega$ is assumed to be any relatively compact open subset of $M$ with piecewise smooth boundary $\partial \Omega$ and $n$ denotes the outward unit normal vector field to $\partial \Omega$. The integration on $\partial \Omega$ is carried out with respect to the measure induced from $\Omega$.

We will be interested in the class of perturbed Dirac operators.

**Definition 1.11.**—An operator $L : C^\infty(S) \to C^\infty(S)$, $S$ any Dirac bundle, will be called a perturbed Dirac operator if $L = D + A$, where $D$ is the Dirac operator associated to $S$ and $A$ is a $0^{th}$ order differential operator on $S$, i.e., $A \in C^\infty(\text{End}(S))$

The properties (1.7), (1.9), and (1.10) can be adjusted to perturbed Dirac ope-
rators. To this end we introduce for any $s_1, s_2 \in C^\infty(S)$, the vector field $V_{s_1, s_2}$ on $M$ defined by

$$<V_{s_1, s_2}, X> = <X \circ s_1, s_2>, \text{ for any tangent vector field } X,$$

and $\text{div } V_{s_1, s_2}$, its divergence. Recall that the divergence of a vector field $V$ is the function $\text{div } V = \sum_i <\nabla_{e_i}^S V, e_i>$. Then we have:

**Proposition 1.13.**— The following statements are equivalent:

(i) $L$ is a perturbed Dirac operator.

(ii) $L$ satisfies (1.7).

(iii) The formal adjoint $L^\dagger$ of $L$ is a perturbed Dirac operator.

(iv) Pointwise, for $s_1, s_2 \in C^\infty(S)$, $<Ls_1, s_2> = <s_1, Ls_2> + \text{div } V_{s_1, s_2}$

(v) $(Ls_1, s_2)_\Omega = (s_1, Ls_2)_\Omega + (n \circ s_1, s_2)_{\partial \Omega}$, under the assumptions of (1.10).

**Proof.**— The proof is identical to the one for Dirac operators (see [GL] and [Ch]).

We now consider extensions of perturbed Dirac operators to $L^2$-sections. Let $C^\infty_0(S) \subset C^\infty(S)$ denote the space of $C^\infty$-sections of $S$ with compact support, and let $L^2(S)$ denote the Hilbert space completion of $C^\infty_0(S)$ in the norm $(, )$. The operator $L_0 = L \big|_{C^\infty_0(S)}$ has two natural extensions to an unbounded operator to $L^2(S)$: a minimal one, $\overline{L}_0$, obtained by taking the $L^2$-closure of the graph of $L_0$, and a maximal
one obtained by taking the domain to be all $s \in L^2(S)$ such that the distributional image $Ls$ is also in $L^2(S)$. In other words, the maximal extension equals $(L_0^\dagger)^*$, the Hilbert space adjoint of $(L_0^\dagger)_0$. By definition these two extensions are closed. Clearly the maximal extension contains the minimal one, and any other closed extension lies in between. The remarkable fact is that they coincide for large classes of manifolds $M$.

**Theorem 1.14.**— Assume that the manifold $M$ is complete. Then any perturbed generalized Dirac operator, regardless of $S$, admits a unique closed extension to $L^2(S)$.

**Proof.**— As in [GL], Theorem 1.17, one uses the completeness of $M$ to reduce the analysis to compactly supported $L^2$-sections of $S$. These in turn can be smoothed out using a *local parametrix* for $L$.

Our manifolds will be taken complete and so there will be no ambiguity when talking about closed extensions of various operators. It is interesting to point out that certain incomplete manifolds yield unique closed extensions for some perturbed Dirac operators as well. It is not hard to prove that if $(M,S,L)$ has this property, and $F \subset M$ is a closed subset of measure zero such that $C_0^\infty(M,S)$ is contained in the closure of $C_0^\infty(M-F,S)$ in the graph norm $\| \cdot \| + \| L \cdot \|$, then $L|_{M-F}$ has this property too. In particular the restrictions of perturbed Dirac operators on $\mathbb{R}^n$ to $\mathbb{R}^n-\{0\}$ are uniquely extendable to closed operators if and only if $n \geq 2$. 
CHAPTER II

PERTURBED DIRAC OPERATORS OF FREDHOLM TYPE

In this chapter we will single out a class of perturbed Dirac operators satisfying the Fredholm property. This class is obtained by imposing a certain positivity condition — assumption 2.9 below — on the endomorphism $A$ near infinity, very similar to that in [GL]. The analytic or Fredholm index for these operators is then introduced in the usual manner.

Let $L = D + A$ be a perturbed Dirac operator on a complete manifold $M$, as introduced in the previous chapter. Let $L^{r,2}(S)$ be the $r$th-Sobolev space, defined as the completion of $C_0^\infty(S)$ in the norm

$$
\|s\|^2 = \int_M \left( <D_s,s> + <D_s,D_s> + \ldots + <D^r_s,D^r_s> \right)
$$

We will be interested in the $C^\infty$- and $L^2$-solution spaces of $L$ and $L^\dagger$. Let us denote these kernel spaces by

$$
\text{ker}(L) = \{ s \in C^\infty(S) \mid Ls = 0 \} \\
L^2\text{-ker}(L) = \{ s \in L^{1,2}(S) \mid Ls = 0 \}
$$

They relate nicely if we make the following assumption on $A$:

ASSUMPTION 2.3. — a) $A$ is pointwise skew-Hermitean i.e. $A^* = -A$, and $A$ and $\mathcal{D}(A)$ are uniformly bounded on $M$ in the pointwise norm.
b) A commutes with the Clifford multiplication on $\mathcal{S}$, i.e., $A(x)\phi = \phi A(x)$, for any $x \in \mathcal{M}$ and any $\phi \in \mathcal{Cl}_x(\mathcal{M})$.

Assumption 2.3 is motivated by the following proposition:

**Proposition 2.4.** — Let $\mathcal{L}$ be a perturbed Dirac operator satisfying Assumption 2.3. Then

a) The domain of the unique closed extension of $\mathcal{L}$ to $L^2(\mathcal{S})$ is $L^{1,2}(\mathcal{S})$ and $\mathcal{L} : L^{1,2}(\mathcal{S}) \to L^2(\mathcal{S})$ is a bounded operator.

b) The commutator $[D, A]$ is a $0$th order differential operator, i.e., $[D, A] \in \mathcal{C}^\infty(\text{End}(\mathcal{S}))$. Moreover, $[D, A]$ is seen to be equal to $\mathcal{D}(A)$; — recall that $\mathcal{D}$ was the generalized Dirac operator on $\mathcal{C}^\infty(\text{End}(\mathcal{S}))$ —

c) $L^\dagger L = D^2 + \mathcal{D}(A) - A^2$

d) $\ker(L) \cap L^2(\mathcal{S}) = L^2 \ker(L) = \ker(L^\dagger L) \cap L^2(\mathcal{S})$

**Proof.** — a) The domain of the closed extension of $\mathcal{L}$ to the $L^2$-space consists of sections $s \in L^2(\mathcal{S})$ such that $Ls \in L^2(\mathcal{S})$. But if $A$ is uniformly bounded on $\mathcal{M}$, $Ls \in L^2(\mathcal{S})$ if and only if $Ds \in L^2(\mathcal{S})$. $\mathcal{M}$ being complete, $s \in L^{1,2}(\mathcal{S})$. The continuity of this extension is obvious.

b) The claim amounts to the linearity of $[D, A]$ with respect to functions $f \in \mathcal{C}^\infty(\mathcal{M})$. If $s \in \mathcal{C}^\infty(\mathcal{S})$,

$$[D, A](fs) = DA(fs) - AD(fs) = D(fAs) - A(\text{grad} f \circ s + fD s) =$$

$$\text{grad} f \circ As + fDAs - \text{grad} f \circ As - f ADs =$$

$$f[D, A].$$
Moreover,
\[
[D,A] = 
\sum_i \left( e_i \nabla e_i S_A - A e_i \nabla e_i S \right) = 
\sum_i e_i \left( \nabla e_i S_A - A \nabla e_i S \right) = 
\sum_i e_i \nabla e_i ^\text{End}(S)(A) = 
\mathcal{D}(A).
\]

c) \[L^\dagger L =
(D + A^*)(D + A) = (D - A)(D + A) = D^2 + [D,A] - A^2 =
D^2 + \mathcal{D}(A) - A^2.\]

d) If \(s \in \ker(L) \cap L^2(S)\), then \(Ds = -As \in L^2(S)\), i.e., \(s \in L^2-\ker(L)\). The opposite inclusion follows from the regularity property of any elliptic system.

Obviously \(\ker(L) \subset \ker(L^\dagger L)\). Let \(s \in \ker(L^\dagger L) \cap L^2(S)\). \(M\) being complete, we can choose compactly supported bump functions \(f \in C^\infty(M)\), \(0 \leq f \leq 1\), \(f = 1\) on any prescribed compact subset of \(M\), such that

\[(2.5) \quad |\text{grad} f|_\infty \overset{\text{def}}{=} \sup_{x \in M} <\text{grad}_x f, \text{grad}_x f>\]

is as small as we wish \([GL]\). Then

\[\|Ls\|^2 = \lim_{f} \|fLs\|^2 = \lim_{f} \langle L^\dagger(fLs), s \rangle =
\lim_{f} 2 (\text{grad}_f fLs, s) \leq \lim_{f} |\text{grad}_f|_\infty (\|fLs\|^2 + \|Ls\|^2) =
= 0.\]

\textbf{Remark 2.6.}— In Proposition 2.4 we can replace \(L\) by \(L^\dagger\). In view of the skew-symmetry of \(A\), any statement about \(L^\dagger\) can be obtained from the corresponding statement for \(L\), simply by replacing \(A\) with \(-A\).
We now proceed to describe sufficient conditions for \( L : L^{1,2}(S) \to L^2(S) \) to be a Fredholm operator. That is, \( L \) is invertible modulo compact operators or equivalently, \( L^2 \)-ker\( (L) \) is finite dimensional, range\( (L) = \{ Ls \mid s \in L^{1,2}(S) \} \) is a closed subspace of \( L^2(S) \), and \( L^2 \)-coker\( (L) = L^2(S) / \text{range}(L) \) — always isomorphic to \( L^2 \)-ker\( (L^\dagger) \) if the closure condition holds — is also finite dimensional.

Let us denote by \( R_A \) the Hermitean, uniformly bounded (on \( M \)), bundle morphism \( \mathfrak{D}(A) - A^2 \). 2.4 e) becomes then
\[
(2.7) \quad L^\dagger L = D^2 + R_A.
\]
on \( \mathcal{C}^\infty(S) \). Notice that
\[
(2.8) \quad \| Ls \|^2 = \| Ds \|^2 + \langle R_A s, s \rangle, \quad s \in L^{1,2}(S)
\]
Certainly (2.8) holds for \( s \in \mathcal{C}^\infty(S) \), from (2.7). However, any element \( s \in L^{1,2}(S) \)
is a \( L^2 \)-limit of some sequence \( s_n \in \mathcal{C}_0^\infty(S) \) such that \( Ds_n \xrightarrow{L^2} Ds \). The claim follows.

The following positivity assumption on \( R_A \) ensures that as a bounded operator \( L \) has finite dimensional kernel and closed range. Thus \( L \) is a semi-Fredholm operator.

Assumption 2.9.— There exists a compact subset \( K \subset M \) and a constant \( c > 0 \) such that \( R_A \geq c \text{Id} \) on \( M - K \), i.e.,
\[
(2.10) \quad \langle R_A v, v \rangle_m \geq c \langle v, v \rangle_m, \quad m \in M - K, \quad v \in S_m
\]
(2.10) will be referred to as the positivity at infinity of \( R_A \).

The main result of this chapter is the following theorem (see also [GL]):
THEOREM 2.11—If the assumptions 2.3 and 2.10 are met, then \( L \) is a semi-Fredholm operator; namely \( L^2\ker(L) \) is finite dimensional and \( \text{range}(L) \) is a closed subspace of \( L^2(S) \).

PROOF.—First we prove the finite dimensionality of \( L^2\ker(L) \). Choose \( k > 0 \) such that \( R_A \geq -k \text{Id} \) on the whole \( M \). Let \( s \in L^2\ker(L), \|s\| = 1 \). If such a \( s \) does not exist, \( L^2\ker(L) = 0 \), and there is nothing to prove. (2.8) yields

\[
\|Ds\|^2 + (R_As,s) = 0
\]

From (2.10) we get

\[
\|Ds\|^2 + (R_As,s)_K + c\|s\|^2_{M-K} \leq 0
\]

Consequently, \( \|Ds\|^2 + c\|s\|^2_{M-K} \leq k\|s\|^2_K \) and since \( \|s\| = 1 \), we have

\[
\frac{1}{c+k}\|Ds\|^2 + \frac{c}{c+k} \leq \|s\|^2_K
\]

We are now going to show that an infinite dimensional \( L^2\)-kernel of \( L \) would contradict (2.14) by making use of the following version of Friedrichs' Lemma [D]:

THEOREM 2.15. (FRIEDRICHS' LEMMA)—For any relatively compact open subset \( \Omega \subset M \), for any elliptic differential operator \( P \) of order \( p > 0 \) on \( \mathcal{C}^\infty(S) \) and for any \( m \in \mathbb{N} \), there exists a positive constant \( C = C(\Omega,P,m) \) such that

\[
\|s\|_{m+p} \leq C (\|Ps\|_m + \|s\|_m), \quad s \in \mathcal{C}^\infty(S)
\]

Taking \( P = L^p \), for \( p \) large enough, and \( m = 0 \), then using Friedrichs' Lemma in conjunction with Sobolev's Lemma, we get:
Theorem 2.17.— For any relatively compact open set \( \Omega \), \( K \subset \Omega \), for any \( q \in \mathbb{N} \), there is a positive constant \( C = C(\Omega, q, L) \) such that

\[
\|s\|_{C^q, K} \leq C \|s\|_{\Omega}
\]

for all \( s \in C^\infty(S) \), \( L(s) = 0 \).

Here \( \|\cdot\|_{C^q, K} \) measures the uniform \( C^q \)-length of \( C^q \)-sections of \( S \), restricted to \( K \).

Assume now that \( \dim L^2(\ker(L)) = \infty \). For \( \epsilon > 0 \) small enough, let \( \{x_1, x_2, \ldots, x_d\} \) be an \( \epsilon \)-dense subset of \( K \), i.e., \( \text{dist}(K, \{x_1, x_2, \ldots, x_d\}) < \epsilon \). Choose \( s \in L^2(\ker(L)) \), \( \|s\| = 1 \) such that \( s(x_i) = 0 \), \( i = 1, \ldots, d \). This is possible since we assumed \( \dim L^2(\ker(L)) = \infty \). Now Theorem 2.17, for \( q = 1 \), and the Mean value theorem, easily yield that \( \sup_K |s| < C\epsilon \), which clearly contradicts (2.14).

In order to prove that \( \text{range}(L) \) is a closed subspace of \( L^2(S) \) we need the following lemma:

Lemma 2.19.— The hypotheses being the same as in Theorem 2.11, there is a \( \alpha > 0 \) such that

\[
\|Ls\| \geq \alpha \|s\|, \quad s \in L^{1,2}(S) \cap (L^2(\ker(L)))^\perp.
\]

(\( L^2(\ker(L)) \)^\perp stands here for the \( L^2 \)-orthogonal complement of the (finite dimensional) \( L^2 \)-kernel of \( L \) in \( L^2(S) \).

Assume for a moment the above lemma true in order to conclude the proof of Theorem 2.11. Let \( s_n \in L^{1,2}(S) \), such that \( Ls_n \rightharpoonup \sigma \), \( \sigma \) being some element in \( L^2(S) \). Since \( L^2(\ker(L)) \subset L^{1,2}(S) \), \( s_n \) can be taken from \( L^{1,2}(S) \cap (L^2(\ker(L)))^\perp \). (2.20) says that \( \{s_n\} \) is a Cauchy sequence in \( L^2(S) \); therefore there is a \( s \in L^2(S) \) such that \( s_n \rightharpoonup s \). However, \( L \) is a closed operator, which implies that \( s \in L^{1,2}(S) \).
and \( Ls = \sigma \). Thus, \( \text{range}(L) \) is a closed subspace of \( L^2(S) \). The proof of Theorem 2.11 is complete.

**Proof of Lemma 2.19.**— By contradiction, suppose that for any \( n \in \mathbb{N} \) there is a section \( s_n \in L^{1,2}(S) \cap (L^2-\ker(L))^\perp \), \( \|s_n\| = 1 \), such that

\[
\|Ls_n\| \leq \frac{1}{n} \|s_n\| = \frac{1}{n}
\]  

(2.21)

The same type of arguments as in the first part of the proof of Theorem 2.11 shows that for \( \delta > 0 \), \( s \in L^{1,2}(S) \), \( \|Ls\| \leq \delta \|s\| \), we have

\[
\|Ds\|^2 + (c-\delta^2)\|s\|^2 \leq (c+k)\|s\|^2
\]  

(2.22)

In particular (2.21) tells us that \( \{s_n\}_n \) is a bounded sequence in the Sobolev norm \((\cdot,\cdot)_1\). If \( \delta << c \), (2.22) also implies

\[
\frac{c-\delta^2}{c+k} \leq \frac{\|s\|^2}{\|s\|^2}
\]  

(2.23)

Thus for \( n \) big enough, there is a constant \( \gamma > 0 \), independent of \( n \), such that

\[
\|s_n\|_K \geq \gamma
\]  

(2.24)

Now, by Rellich's Lemma [D], the restriction map

\[
L^{1,2}(S) \to L^2(K,S)
\]

\( s \to s|_K \)

(2.25)

is compact, so we can subtract a subsequence of \( \{s_n\}_n \), assumed to be \( \{s_n\}_n \) itself, such that \( s_n|_K \) is \( L^2 \)-convergent on \( K \). If \( s' \in L^2(K,S) \) is this limit, (2.24) says that \( \|s\|_K > 0 \), i.e., \( s' \) cannot be identically 0 on \( K \).

The strategy now is to show that \( \{s_n|_{M-K}\}_n \) is \( L^2 \)-convergent as well. If \( s'' \in \)
$L^2(M-K,S)$ is the (would be) limit, then

\[(2.26) \quad s = \begin{cases} s' \text{ on } K \\ s'' \text{ on } M-K \end{cases}\]

is the $L^2$-limit of $\{s_n\}_n$ in $L^2(S)$. $s_n \rightharpoonup L^2 s$, $Ls_n \rightharpoonup 0$, imply $s \in L^{1,2}(S)$ and $Ls = 0$.

On the other hand, $s_n \in (L^2 \ker(L))^{-1}$ implies $s \in (L^2 \ker(L))^{-1}$, i.e., $s \equiv 0$, a contradiction.

Why should $\{s_n\}_{M-K}$ be $L^2$-convergent on $M-K$? The positivity of $R_A$ is responsible for this fact. To make it rigorous, let $\chi \in C^\infty(M)$ be a function such that $0 \leq \chi \leq 1$, $\chi \equiv 1$ on $M-K$ and $R_A \geq \frac{c}{2} \text{Id}$ on $\text{supp} \chi$. Then (2.8) implies

\[(2.27) \quad \|L(\chi(s_m-s_n))\|_{L^2} \geq \frac{c}{2} \|s_m-s_n\|_{M-K}^2, \quad m,n \in \mathbb{N}\]

Using I 1.13 (ii) and (2.21) we see that $\{L(\chi s_n)\}_n$ is a Cauchy sequence in $L^2(S)$. Thus, (2.27) $\{s_n\}_{M-K}$ is a Cauchy, and so convergent, sequence in $L^2(M-K,S)$.

**Corollary 2.28.**— a) If $R_{\pm A}$ is positive at infinity, then $L$ is a Fredholm operator.

b) If $-A^2$ is positive at infinity (assumption 2.9 for $-A^2$ instead of $R_A$) and $\mathcal{D}(A)(x) \to 0$ as $x \to \infty$, then $L$ is a Fredholm operator.

**Proof.**— a) $L$ is a semi-Fredholm operator, by Theorem 2.11. If $R_{-A}$ is also positive at infinity, $L^2 \ker(L^\dagger)$ is finite dimensional as well, thus $L$ is a Fredholm operator.

b) The hypotheses in b) obviously suffice for $R_{\pm A}$ to be positive at infinity, since $R_{\pm A} = \pm \mathcal{D}(A) - A^2$. \hfill \ensuremath{\blacksquare}
The size of the $L^2$-solution spaces for an operator is usually difficult to compute. The correct object to look to is their analytic or Fredholm index, when interested in solution spaces.

**Definition 2.29.**— Let $L$ be a Fredholm operator of the type described in Theorem 2.11 and Corollary 2.28. We define the analytic or Fredholm $L^2$-index of $L$ by the formula

\[(2.30) \quad L^2\text{-index}(L) = \dim L^2\ker(L) - \dim L^2\coker(L) = \dim L^2\ker(L) - \dim L^2\ker(L^\dagger)\]

Our goal in the next chapters will be to evaluate this index for particular classes of perturbed Dirac operators.

We conclude this chapter with a few remarks concerning $L^\dagger L$ and $LL^\dagger$.

**Remark 2.31.**— So far, we viewed $L^\dagger L$ and $LL^\dagger$ as acting on $C^\infty(S)$ only. Just as with $L$, we can consider their closed extensions to the $L^2$-space. For operators satisfying the assumption 2.3, it is an exercise to see that they admit a unique closed extension to $L^2(S)$, with domain $L^{2,2}(S)$. Alternatively, if $L^*$ is the Hilbert space adjoint of $L|_{C_0^\infty(S)}$, these unique extensions equal $L^*L$, respectively $L^*L$. It is a fact [G] that $L$ is a Fredholm operator if and only if $L^*L$ is so.
CHAPTER III
INDEX-PRESERVING DEFORMATIONS

Using deformation theory, in this chapter we will make the first steps toward evaluating the $L^2$-index of a perturbed Dirac operator $L = D + A$, $A$ being subject to the hypotheses of Corollary 2.28 b). We "unitarize" $A$ outside a compact set (Proposition 3.8) and then "diagonalize" $L$ with respect to the bundle splitting induced by the unitarization $U$ of $A$ (Proposition 3.22 and Theorem 3.25), without changing the index.

DEFINITION 3.1.— The problem of evaluating the $L^2$-index of a perturbed Dirac operator $L = D + A$, where $A$ is a skew-Hermitean, uniformly bounded and commuting with the Clifford multiplication endomorphism such that $-A^2$ is positive outside some compact set, and $\partial(A)$ decays to 0 at infinity, will be more simply referred to as a Callias-type index problem. The operator $L$ itself will be called a Callias-type operator.

We now state the basic theorem in index preserving deformation theory [G]:

THEOREM 3.2.— a) (the homotopy invariance) If $T_t$, $0 \leq t \leq 1$, is a continuous homotopy of Fredholm operators between two Hilbert spaces, then

$(3.3) \quad \text{index}(T_0) = \text{index}(T_1)$
b) (the invariance under compact deformations) If $T$ is a Fredholm operator and $C$ is a $T$-compact operator, i.e., $C$ is compact in the graph norm $\|\cdot\| + \|T\cdot\|$, then $T + C$ is a Fredholm operator and

$$\text{(3.4)} \quad \text{index}(T) = \text{index}(T + C)$$

Notice that for our operator $L$, the $L$-compactness is equivalent to the $D$-compactness.

**Remark 3.5.**— The Callias type index problem is uninteresting if $M$ is a) a compact manifold, or b) even dimensional.

a) For $M$ compact, $D$ itself is a Fredholm operator and by Rellich's Lemma, $A$ is $D$-compact. Thus $\text{index}(L) = \text{index}(D) = 0$, since $D$ is selfadjoint.

b) For $M$ even dimensional, the "volume form" on $M$ given by $e = \sqrt{-1} e_1 \ldots e_n \in C^\infty(\text{Cl}(M))$ anticommutes with $D$ and commutes with $A$. As a result $Le = -eL^\dagger$. Thus $e$ is an isometry from $L^2$-ker($L$) to $L^2$-ker($L^\dagger$), which implies $L^2$-index($L$) = 0.

The next proposition shows that only the behavior of $A$ at infinity matters for a Callias-type index problem.

**Proposition 3.6.**— Let $L$ be a Callias-type operator and $\Phi \in C_0^\infty(\text{End}(S))$ such that $\Phi^* = -\Phi$ and $\Phi$ commutes with the Clifford multiplication. Then $L + \Phi$ is a Callias-type operator and

$$\text{(3.7)} \quad L^2\text{-index}(L) = L^2\text{-index}(L + \Phi)$$

**Proof.**— It is clear that if $A$ satisfies the requirements of Definition 3.1, so does $A + \Phi$. In order to prove the index invariance we could use either a) or b) of
Theorem 3.2. For instance, \( L_t = L + t\Phi \), \( 0 \leq t \leq 1 \), is obviously seen to be a continuous homotopy of Fredholm operators linking \( L \) and \( L + \Phi \).

We can simplify \( A \) at infinity by unitarizing it. To this end let us consider the polar decomposition of the skew-Hermitean endomorphism \( A \) outside the compact set \( K \) where \(-A^2\) is strictly positive. Thus on \( M - K \), \( A = PU \), with \( P, U \in C^\infty(M - K, S) \), \( P \) uniformly positive, and \( U \) unitary. Clearly, \( P = (-A^2)^{1/2} \) and \( U = (-A^2)^{-1/2} A \).

PROPOSITION 3.8.—Let \( L = D + A \) be a Callias-type operator and \( \chi \in C^\infty(M) \) a bump function which vanishes on \( K \) and is identically 1 outside some compact set \( \Omega \). Then \( D + \chi U \) is a Callias-type operator and

\[
L^2\text{-index}(L) = L^2\text{-index}(D + \chi U)
\]

PROOF.—\( \chi U \) must satisfy the requirements of Definition 3.1. All of them, except one, are immediate. In particular \( \chi U \) commutes with the Clifford multiplication because any power of \( A \) does so, and \(- (\chi U)^2 \equiv \text{Id} \), outside some compact set. Not so obvious is the fact that \( \mathcal{D}(\chi U)(x) \to 0 \) as \( x \to \infty \), or equivalently \( \mathcal{D}(U)(x) \to 0 \) as \( x \to \infty \). It is clear that \( \mathcal{D} \) acts as a derivation on the subspace of bundle morphisms of \( S \) commuting with the Clifford multiplication, i.e., for \( V, W \in \mathcal{C}^\infty(\text{End}(S)) \), commuting with the Clifford multiplication,

\[
\mathcal{D}(VW) = \mathcal{D}(V)W + V\mathcal{D}(W)
\]

Therefore, at infinity, \( \mathcal{D}(U) = \mathcal{D}((-A^2)^{-1/2})A + A\mathcal{D}((-A^2)^{-1/2}) \), and thus \( \mathcal{D}(U) \) has the required decay property if \( \mathcal{D}((-A^2)^{-1/2}) \) does so. Now a variant of Cauchy's integral formula [K] gives
(3.11) \[
(-A^2)^{-\frac{1}{2}} = \frac{1}{\pi} \int_0^\infty \lambda^{-\frac{1}{2}}(-A^2 + \lambda)^{-1} d\lambda
\]
and so

(3.12) \[
\mathcal{D}((-A^2)^{-\frac{1}{2}}) = \frac{1}{\pi} \int_0^\infty \lambda^{-\frac{1}{2}} \mathcal{D}((-A^2 + \lambda)^{-1}) d\lambda
\]

Using again the derivation property we get

(3.13) \[
\mathcal{D}((-A^2 + \lambda)^{-1}) = (-A^2 + \lambda)^{-1} \mathcal{D}(-A^2 + \lambda)(-A^2 + \lambda)^{-1}
\]

Thus pointwise, outside some compact set where \(|-A^2| \geq c\), we have

(3.14) \[
\|\mathcal{D}((-A^2 + \lambda)^{-1})\| \leq \text{const.} \|(-A^2 + \lambda)^{-1}\|^2 \|\mathcal{D}(A)\| \leq \text{const.} (c + \lambda)^{-2} \|\mathcal{D}(A)\|
\]

Finally

(3.15) \[
\|\mathcal{D}((-A^2)^{-\frac{1}{2}})\| \leq \text{const.} \int_0^\infty \lambda^{-\frac{1}{2}}(c + \lambda)^{-2} d\lambda \|\mathcal{D}(A)\|
\]

shows that \(\mathcal{D}((-A^2)^{-\frac{1}{2}})\) vanishes at infinity.

Now we consider the continuous homotopy \(A_t = tA + (1-t)\chi U\), \(0 \leq t \leq 1\), in \(C^\infty(\text{End}(S))\). \(\mathcal{D}(A_t)\) clearly vanishes at infinity. In order to prove positivity for \(-A_t^2\), notice that

(3.16) \[
-A^2(x) \geq c \text{ Id} \iff \left\{ \begin{array}{c}
\lambda^2 \geq c \text{ for any pure imaginary eigenvalue of } A(x) \\
\sqrt{-1}\lambda \text{ of } A(x)
\end{array} \right.
\]

Then \(A_t(x)\) has eigenvalues \(\sqrt{-1} \lambda + \sqrt{-1} (1-t) \frac{\lambda}{|\lambda|}\), so the elementary inequality

(3.17) \[
(t \lambda + (1-t) \frac{\lambda}{|\lambda|})^2 \geq \min(1,c) \quad 0 \leq t \leq 1
\]
proves that \(-A_t^2 \geq \min (1,c)\), outside some compact set. Now Theorem 3.2 a) applied to the homotopy \(L_t = D + A_t\) yields the desired result. ♦

The Dirac bundle \(S\big|_{M-K}\) splits into a direct sum \(S_+ \oplus S_-\) of Dirac subbundles over \(M-K\). \(L\big|_{M-K}\) can then be written in a \(2 \times 2\) matrix form whose off-diagonal terms are bundle morphisms. Another deformation will wipe out the off diagonal terms and so facilitate, in the next chapters, a separation of variables on manifolds with warped ends.

**Lemma 3.18.**— Let \(U \equiv (-A^2)^{-\frac{1}{2}} A\) on \(M-K\), with \(A\) as in Definition 3.1. Then the bundles \(S_\pm\) over \(M-K\)

\[
(S_\pm)_m = \{ v \in S_m \mid Uv = \pm \sqrt{-1} v \}, \quad m \in M-K
\]

are in a canonical way Dirac bundles.

**Proof.**— Clearly \(S_\pm = (\text{Id} \mp \sqrt{-1} U)S\big|_{M-K}\) and the splitting \(S\big|_{M-K} = S_+ \oplus S_-\) is orthogonal. The functions \(m \to \text{rank}((\text{Id} \mp \sqrt{-1} U)(m)), m \in M-K\) are upper semi-continuous and

\[
\text{rank}((\text{Id} + \sqrt{-1} U)(m)) + \text{rank}((\text{Id} - \sqrt{-1} U)(m)) = \dim S_m \quad \text{(constant)}
\]

Thus these functions must be locally constant on \(M-K\) and this ensures (cf. [A]) that \(S_\pm\) are bundles. Since \(S_\pm = (\text{Id} \mp \sqrt{-1} U)S\), and \(U\) commutes with the Clifford multiplication, \(S_\pm\) are invariant \(\text{Cl}(M)\)-modules. They inherit a Hermitean scalar product from \(S\) and, as already noticed, \(S_+ \perp S_-\).

The condition 1.1 in the definition of a Dirac bundle is trivially satisfied. We endow \(S_\pm\) with the connection
\[ \nabla_e^s \mathcal{S}_\pm \mathcal{S}_\pm \mathcal{S}_\pm = \text{proj}_{\mathcal{S}_\pm} \left( \nabla_e^S \mathcal{S}_\pm \mathcal{S}_\pm \right) = \frac{1}{2} (\text{Id} \mp \sqrt{-1}U) (\nabla_e^S \mathcal{S}_\pm) \]

\[ e \in C^\infty(T(M-K)) , \mathcal{S}_\pm \in C^\infty(M-K, \mathcal{S}_\pm) \]

The check that \( \nabla^\pm \) satisfies 1.2 is immediate. It involves the basic properties of \( \nabla^S \) and \( U \). ♦

Let us denote now by \( D_\pm \) the corresponding Dirac operators on \( (M-K, \mathcal{S}_\pm) \).

**Proposition 3.22.** - Relative to the orthogonal decomposition \( \mathcal{S}|_{M-K} = \mathcal{S}_+ \otimes \mathcal{S}_- \) we have the following matrix representations associated with \( L = D + \chi U \) (Proposition 3.8):

\[ L|_{M-\Omega} = \begin{pmatrix} D_+ + \sqrt{-1} \mathcal{S}(U)|_{\mathcal{S}_-} \\ -\frac{\sqrt{-1}}{2} \mathcal{S}(U)|_{\mathcal{S}_+} & D_- - \sqrt{-1} \end{pmatrix} \]

\[ L^\dagger|_{M-\Omega} = \begin{pmatrix} D_+ - \sqrt{-1} \mathcal{S}(U)|_{\mathcal{S}_-} \\ -\frac{\sqrt{-1}}{2} \mathcal{S}(U)|_{\mathcal{S}_+} & D_- + \sqrt{-1} \end{pmatrix} \]

**Proof.** - Let \( s_\pm \in C^\infty(M-\Omega, \mathcal{S}_\pm) \). Locally, \( Ls_\pm = \sum_i e_i \nabla_{e_i}^s S_\pm + Us_\pm = \sum_i \frac{1}{2} (\text{Id} \mp \sqrt{-1}U) e_i \nabla_{e_i}^s S_\pm \pm \sqrt{-1}s_\pm = \sum_i \frac{1}{2} (\text{Id} \mp \sqrt{-1}U) \nabla_{e_i}^s S_\pm \pm \sqrt{-1}s_\pm = \sum_i e_i \frac{1}{2} (\text{Id} \mp \sqrt{-1}U) \nabla_{e_i}^s S_\pm \pm \sqrt{-1}s_\pm \pm \sqrt{-1}U) s_\pm = \left( D_\pm \pm \sqrt{-1} \right) s_\pm \mp \frac{\sqrt{-1}}{2} [D,U]s_\pm \). ♦
Now we are ready to prove our main deformation result:

**Theorem 3.25.**— Let $L = D + \chi U$, $K$, and $\Omega$ be as in Proposition 3.8. There exists a perturbed Dirac operator of Fredholm type $T : L^{1,2}(S) \to L^2(S)$ such that

\[
T |_{M-\Omega} = \begin{pmatrix} D_+ + \sqrt{-1} & 0 \\ 0 & D_- - \sqrt{-1} \end{pmatrix} \\
T^\dagger |_{M-\Omega} = \begin{pmatrix} D_+ - \sqrt{-1} & 0 \\ 0 & D_- + \sqrt{-1} \end{pmatrix}
\]

and $L^2$-index($L$) = $L^2$-index($T$).

**Proof.**— Clearly, $\Psi \equiv \chi \mathcal{D}(U) = \begin{pmatrix} 0 & \sqrt{-1} \mathcal{D}(U) |_{S_+} \\ -\frac{\sqrt{-1}}{2} \mathcal{D}(U) |_{S_-} & 0 \end{pmatrix}$ belongs to $C^\infty(M,\text{SymmEnd}(S))$. Set $T \overset{\text{def}}{=} L - \chi \mathcal{D}(U)$. $T$ is obviously a perturbed Dirac operator and by Proposition 3.22, $T$ and $T^\dagger$ have the stated matrix representations on $M-\Omega$. We can use a slight variant of Corollary 2.28 to prove that $T$ is a Fredholm operator. In fact the arguments in Theorem 2.11 carry through if we can write $T^\dagger T$ as a sum of a positive operator and a bundle morphism which is positive at infinity. In our case

\[
(3.27) \quad T^\dagger T = (D - \chi \mathcal{D}(U))^2 + ([D - \chi \mathcal{D}(U), \chi U] + \chi^2)
\]

Now $(D - \chi \mathcal{D}(U))^2$ is a positive operator and $[D - \chi \mathcal{D}(U), \chi U] + \chi^2$ is positive at infinity, since on $M-\Omega$, $[D - \chi \mathcal{D}(U), \chi U] + \chi^2 = \mathcal{D}(U) - [\mathcal{D}(U), U] + \text{Id}$, and $\mathcal{D}(U)$ vanishes at infinity. The two indices are seen to be equal, by applying Theorem 3.2 a) to the homotopy $T_t = L - t \chi \mathcal{D}(U)$, $0 \leq t \leq 1$.

We conclude this chapter with the following corollary:
COROLLARY 3.28.— Let us assume that globally on $M$ we have $U^2 = -\text{Id}$ and $\mathcal{D}(U)(x) \to 0$ as $x \to \infty$. Then $L^2$-index$(L) = L^2$-index$(D + U) = 0$.

PROOF.— By Theorem 3.25, $L^2$-index$(L) = L^2$-index $\begin{pmatrix} D_+ + \sqrt{-1} & 0 \\ 0 & D_- - \sqrt{-1} \end{pmatrix}$

As selfadjoint operators $D_\pm$ admit only real eigenvalues. Thus $0$ cannot be an eigenvalue for $\begin{pmatrix} D_+ + \sqrt{-1} & 0 \\ 0 & D_- - \sqrt{-1} \end{pmatrix}$, i.e., $L^2$-ker $\begin{pmatrix} D_+ + \sqrt{-1} & 0 \\ 0 & D_- - \sqrt{-1} \end{pmatrix} = 0$. ♦
CHAPTER IV

SEPARATION OF VARIABLES ON WARPED PRODUCTS

In this chapter we will study Dirac bundles and Dirac operators defined on a special class of manifolds: the warped products of type \((\varepsilon, \infty) \times fN\). The discussion, interesting in its own, serves the purpose of understanding the contribution to the index coming from the end of the manifold \(M\), which end will be taken to be a warped product in the next chapter. Using parallel transport along the radial geodesics, any Dirac bundle can be viewed as a one-parameter family of Dirac bundles over \(N\). Accordingly, for any Dirac operator the variables can be separated; this is particularly insightful for spinor-type bundles and Clifford bundles.

**Definition 4.1.**—Let \((N, ds^2)\) be a compact Riemannian manifold and \(f \in C^\infty((\varepsilon, \infty))\), \(\varepsilon \in \mathbb{R}\), a positive function. The product \((\varepsilon, \infty) \times fN\), equipped with the Riemannian metric \(dr^2 + f^2(r)ds^2\), \(r\) being the coordinate in \((\varepsilon, \infty)\), will be called a warped product and denoted by \(W = (\varepsilon, \infty) \times fN\).

A basic example of a warped product of this type is \((\mathbb{R}^n - \{0\}, \text{Euclidean metric})\) \(\equiv (0, \infty) \times fS^{n-1}_1\); here \(f(r) = r\), and \(S^{n-1}_1\) is the standard \((n-1)\)-dimensional unit sphere in \(\mathbb{R}^n\).

Let \(S\) be any Dirac bundle over a warped product \(W\). Fix an \(R \in (\varepsilon, \infty)\) and denote by \(S_R\) the restriction of \(S\) to \(\{R\} \times N\). Changing the metric on \(N\) from \(ds^2\) to \(f^2(R)ds^2\), we can assume that \(f(R) = 1\) and then identify metrically \(N\) and \(\{R\} \times N\).
The bundle $S_R \to N$ inherits a canonical structure of Dirac bundle, under a mild restriction on the curvature tensor on $S$, which we describe next. The geometric operators are of this type.

For any section $s \in C^\infty(N, S_R)$, define $s^\sim$ in $C^\infty(W, S)$, as being the parallel transport of $s$ along the radial geodesics i.e., $s^\sim$ is the unique section in $C^\infty(W, S)$, subject to

$$s^\sim\big|_{\{R\} \times N} = s, \quad \nabla_{\frac{\partial}{\partial r}} s^\sim = 0$$

Note that for the bundle $T W$, we have $T N \subset (T W)_R$ and if $e \in C^\infty(N, T N)$, then $e^\sim = \frac{E}{f}$, where $E$ is the standard lift of $e$ from $T N$ to $T W$. Also $(\frac{\partial}{\partial r})^\sim \big|_N = \frac{\partial}{\partial r}$.

These are immediate consequences of the properties of the Levi-Civita connection on warped products [O]. Another important fact is

$$\exp(s^\sim) = e^\sim \exp(s^\sim)$$

We assume now that there exists a real valued function $g \in C^\infty((\epsilon, \infty)), g(R) = 1$ such that the curvature tensor $\mathcal{R}_e$ of $S$ satisfies the equation:

$$\mathcal{R}_e \frac{\partial}{\partial r}, e^\sim s^\sim = g(r) (\mathcal{R}_e \frac{\partial}{\partial r}, e^\sim s^\sim), \quad e \in C^\infty(T N), s \in C^\infty(N, S_R)$$

**Proposition 4.5.**—If the above assumption (4.4) holds, then $g(r)$ must satisfy the equation

$$g(r) = \frac{f''(r)}{f(r)f''(R)}$$

**Remark 4.7.**—The equation (4.6) makes sense only if $f''(R) \neq 0$; this is unnecessarily restrictive and artificial. In fact, for the metric cone ($f(r) = r$), it never
holds. As we shall see, however, in many examples of interest (in particular the geometric Dirac bundles) we have

\[(4.8) \quad R_{\frac{\partial}{\partial r}}, e^s = f''(R) Q_e\]

where \(Q_e\) is another tensor having the same properties as \(\mathcal{R}_{\frac{\partial}{\partial r}}, e\) (i.e., it is antisymmetric, a derivation with respect to the Clifford multiplication, etc.) Thus the assumption \((4.4)\) shall be broadened to accommodate a formal cancellation of \(f''(R)\), but we prefer it that way for esthetic reasons.

**Proof of Proposition 4.5.**—Assume \((4.4)\), and \(f''(R) \neq 0\). We will evaluate \(\mathcal{R}_{\frac{\partial}{\partial r}}, e^\sim (e^\sim \circ s^\sim)\) in two different ways using the derivation property of the curvature tensor \(\mathcal{R}_{\frac{\partial}{\partial r}}\). On one hand

\[\mathcal{R}_{\frac{\partial}{\partial r}}, e^\sim (e^\sim \circ s^\sim) = \]

\[\mathcal{R}_{\frac{\partial}{\partial r}}, e^\sim (e^\sim \circ s^\sim) = \mathcal{R}_{\frac{\partial}{\partial r}}, e^\sim (e^\sim \circ s^\sim) = \]

\[= -g(r) \left( \mathcal{R}_{\frac{\partial}{\partial r}}, e (e^\circ s) \right)^\sim = -g(r) \left( \mathcal{R}_{\frac{\partial}{\partial r}}, e (e^\circ s) \right)^\sim = \]

\[= g(r) \left\{ \mathcal{R}_{\frac{\partial}{\partial r}}, e (e^\circ s) + e^\circ \mathcal{R}_{\frac{\partial}{\partial r}}, e (s) \right\}^\sim = \]

\[= -g(r) \frac{f''(R)}{f(R)} n \circ s^\sim + e^\circ \mathcal{R}_{\frac{\partial}{\partial r}}, e^\sim s^\sim \]

Comparing the two results we get the desired identity \((4.6)\).

**Example 4.9.** — \(S = \text{Cl}(W)\), the Clifford bundle of exterior algebras of \(W\).
For Cl(W) we have \([LM]\),
\[ \mathfrak{R}_{\frac{d}{dt}} e^s = \frac{f'(r)}{2f(r)} \text{ad}_{\mathfrak{n}^e} s^\sim, \]
where \(\text{ad}_{\mathfrak{n}^e} s^\sim = \mathfrak{n}^e s^\sim - s^\sim \mathfrak{n}^e^\sim\). Thus Cl(W) satisfies the assumption (4.4) since

\[ (4.10) \quad \mathfrak{R}_{\frac{d}{dt}} e^s = \frac{f'(r)}{2f(r)} \text{ad}_{\mathfrak{n}^e} s^\sim \]

**Example 4.11.** — \(S = \text{Spin}(W)\), the spinor bundle of \(W\).

Again, for Spin(W) the assumption (4.4) holds since

\[ (4.12) \quad \mathfrak{R}_{\frac{d}{dt}} e^s = \frac{f'(r)}{2f(r)} \mathfrak{n}^e^\sim s^\sim \]

Formula (4.12) remains true for any spinor-type bundle \(S\).

**Theorem 4.13.** — Let \(W\), \(S\), and \(S_R\) be as above and assume that (4.4) holds. Then \(S_R \to N\) admits a canonical structure of Dirac bundle, and the associated Dirac operator on \(S\) admits the following separation of variables

\[ (4.14) \quad Ds^\sim = \frac{(\partial_R s)^\sim}{f} + \frac{f'}{f} (\Xi_R s)^\sim, \quad s \in C^\infty(N, S_R) \]

where \(\partial_R\) is the Dirac operator on \(S_R\) and \(\Xi_R\) is an element of \(C^\infty(N, \text{End}(S_R))\). If \(f''(R) \neq 0\), then

\[ (4.15) \quad \Xi = \frac{1}{f''(R)} \sum_{i=1}^{\dim N} e_i \circ \mathfrak{R}_{\frac{d}{dt}} e_i \]

Here \(\{e_i\}_i\) is a local orthonormal basis in \(TN\).

**Proof.** — \(S_R \to N\) will be given the induced structure of Cl(N)-module, as
Cl(N) ⊂ Cl(W) and $S_R ⊂ S$. Thus (1.1) holds trivially. The key point in the proof is the definition of the connection on $S_R$. A careful analysis of $\nabla^S_{dr} e^{-s}$ suggests how such a connection comes around and why assumption (4.4) is necessary.

$$\nabla^S_{dr} e^{-s} =$$

$$\nabla^S_{dr} e^{-s} - \nabla^S_{dr} e^{-s} - \nabla^S_{dr} e^{-s} + \nabla^S_{dr} e^{-s} =$$

$$\nabla^S_{dr} (f \nabla^S_{e^{-s}}) = f \nabla^S_{dr}, e^{-s}$$

Assume for convenience that $f''(R) \neq 0$. Then from (4.4) and Proposition 4.5 we get:

(4.16) $\nabla^S_{dr} (f \nabla^S_{e^{-s}}) = \frac{f''}{f''(R)} (\nabla^S_{dr}, e^s)$

or equivalently

(4.17) $\nabla^S_{dr} (f \nabla^S_{e^{-s}} - \frac{f'}{f''(R)} (\nabla^S_{dr}, e^s)) = 0$

Define the connection $\nabla^R$ on $S_R$ by

(4.18) $\nabla^R_{e^s} \overset{\text{def}}{=} \{ f \nabla^S_{e^{-s}} - \frac{f'}{f''(R)} (\nabla^S_{dr}, e^s) \} |_N = \nabla^S_{e^{-s}} |_N - \frac{f'(R)}{f''(R)} \nabla^S_{dr}, e^s$

Equation (4.17) merely says that

(4.19) $(\nabla^R_{e^s}) = f \nabla^S_{e^{-s}} - \frac{f'}{f''(R)} (\nabla^S_{dr}, e^s)$
The verification that $\nabla^R$ is a metric connection compatible with the Clifford action is a simple exercise. We will check only property (1.2) i.e.,

$$ (4.20) \quad \nabla^R_e (\phi \circ s) = (\nabla^N_e \phi) \circ s + \phi \circ \nabla^R_e s \quad , \quad \phi \in C^\infty(\text{Cl}(N)) \text{, } s \in C^\infty(N, S_R) $$

Indeed

$$ (\nabla^R_e (\phi \circ s))^- = f \nabla^S_e (\phi \circ s)^- - \frac{f'}{f''(R)} (\nabla^N_e \frac{\partial}{\partial r}, e (\phi \circ s))^-$

$$ = f (\nabla^W_e \phi^- \circ s^- - \frac{f'}{f''(R)} (\nabla^N_e \frac{\partial}{\partial r}, e (\phi))^- \circ s^- - \frac{f'}{f''(R)} (\nabla_e \frac{\partial}{\partial r}, e (s))^-) = (f \nabla^W_e \phi^- - \frac{f'}{f''(R)} (\nabla^N_e \frac{\partial}{\partial r}, e (\phi))^-) \circ s^- + \phi \circ (\nabla^R_e s)^- $$

It remains to be checked that

$$ (4.21) \quad f \nabla^W_e \phi^- - \frac{f'}{f''(R)} (\nabla^N_e \frac{\partial}{\partial r}, e (\phi))^- = (\nabla^N_e \phi^-) $$

Since $\nabla^W$, $\nabla^N$, and $\nabla^N$ are derivations with respect to the Clifford multiplication, it suffices to check (4.21) on elements $\phi \in T N$ only. There, it follows because $[O]$, 

$$ (4.22) \quad \nabla^W_e \phi^- = - \frac{\langle e^-, \phi^- \rangle}{f} f' \frac{\partial}{\partial r} + \frac{(\nabla^N_e \phi^-)}{f} \nabla^N_e \frac{\partial}{\partial r}, e (\phi) = - \frac{f''(R)}{f''(R)} < e, \phi > \frac{\partial}{\partial r} $$

Now we prove the separation of variables formula (4.14). For $s$ in $C^\infty(N, S_R)$

$$ D_s^- = \frac{\partial}{\partial r} \circ \nabla^R_s + \sum_{i=1}^{\text{dim } N} e_i^- \circ \nabla^S_{e_i^- s} = \frac{1}{f} \sum_{i=1}^{\text{dim } N} e_i^- \circ (\nabla^R_{e_i^- s})^- + \frac{f'}{f''(R)} \sum_{i=1}^{\text{dim } N} e_i^- \circ (\nabla^N_{e_i^- s})^- = \frac{(\phi_R s)^-}{f} + f' (\Xi_R s)^- $$
where \( \Xi_R s = \frac{1}{f(R)} \sum_{i=1}^{\dim N} e_i \otimes R \frac{\partial}{\partial r}, e_i s \) on \( C^\infty(N,SR) \).

**Remark 4.23.**—A more general curvature assumption of type (4.4), which still makes Theorem 4.13 true, is

\[
(\forall e^R) (4.24) \quad \text{There is a curvature-like tensor } \mathcal{F} \text{ on } C^\infty(W,S) \text{ such that } \nabla^S_{\frac{\partial}{\partial r}} \mathcal{F} = f R. 
\]

Then, \((\nabla^R e) = f \nabla^S e - \mathcal{F} \frac{\partial}{\partial r}, e - s \) and \(D^s = \frac{(\varphi_R s)}{f} + \frac{1}{f} \Xi^s\), where \( \Xi^s = \sum_{i=1}^{\dim N} e_i \otimes \mathcal{F} \frac{\partial}{\partial r}, e_i s \). However, the necessity to control \( \Xi \) makes (4.24) too general.

It is interesting now to trace down the Dirac bundle on \((N,SR)\) and the corresponding endomorphism \( \Xi_R \), when \( S = Cl(W) \) or \( S = Spin(W) \).

(4.25) \( a \) \quad \( S = Spin(W) \). This situation is worked out by Chou [Ch], when \( W \) is even dimensional. If \( \dim W \) is odd, then \( S_R \equiv Spin(N) \), \( \varphi_R \) is the classical Dirac operator on \( N \), and from (4.11) and (4.15) we get \( \Xi_R = \frac{\dim N}{2} n^* \).

(4.25) \( b \) \quad \( S = Cl(W) \). In this case \( S_R \equiv Cl(N) \oplus Cl(N) \), under the identification

\[
(4.26) \quad S_R \otimes \omega = \omega_0 + \omega_1 \otimes \frac{\partial}{\partial r} \to (\omega_0, \omega_1) \in Cl(N) \oplus Cl(N)
\]

The connection \( \nabla^R \) is simply the (direct sum of) Levi-Civita connection(s) and \( \varphi_R \) is a direct sum of two copies of the Gauss-Bonnet operator on \( N \). \( \Xi_R \) then becomes

\[
\frac{1}{2} \sum_{i=1}^{\dim N} e_i \otimes \nabla^R_{n^* e_i} e_i \omega, \quad \text{if } \omega_p \text{ is a Clifford section of degree } p \text{ on } N,
\]

\[
(4.27) \quad \nabla^R_{n^* e_i} (\omega_p) = \begin{cases} 
2 n^* e_i \omega_p & \text{if } e_i = \omega_p \\
0 & \text{if } e_i \neq \omega_p 
\end{cases}
\]
Thus $\Xi_R \omega_p = p \circ \omega_p$. Similarly $\Xi_R (n \circ \omega_p) = (\dim N - p) n \circ (n \circ \omega_p)$. Closely related formulas appear also in [B].

The parallel transport introduced before allows us to trivialize $S$. Precisely, if $\pi : (\varepsilon, \infty) \times N \to N$ is the projection, then $\pi^* (S_R)$ is canonically isomorphic to $S$. When $S$ is viewed as $\pi^* (S_R)$, any section in $C^\infty (W, S)$ can be viewed as an element in $C^\infty ((\varepsilon, \infty), C^\infty (N, S_R))$ and the separation of variables formula (4.14) extends to

(4.28) \[ D_s = n \circ s'(r) + \frac{\partial_{R_s}}{f} + \frac{c}{f} \Xi_{R_s} s, \quad s \in C^\infty ((\varepsilon, \infty), C^\infty (N, S_R)) \]

An easy consequence of (4.18) is that $n$ and $\partial_R$ anticommute.
CHAPTER V
SOLVING THE INDEX PROBLEM

In this section we will solve the Callias-type index problem, stated at the beginning of Chapter III, for a triple \((M, S, L)\) consisting in a complete odd dimensional Riemannian spin manifold \(M\) with a warped end (Definition 5.1), a spinor-type bundle \(S\) (Chapter I, pp. 10-11), and a perturbed Dirac operator \(L = D + A\), for which the potential \(A\) is independent of the radial direction on the warped end. The calculation of the \(L^2\)-index\((L)\) involves two major ingredients:

\textit{a) A direct calculation of the \(L^2\)-index for a particular Callias-type problem on a manifold with two warped ends of type \(\mathbb{R} \times \mathbb{T}\) (Theorem 5.12)}

\textit{b) The use of the relative index theorem of Gromov-Lawson [GL], to relate the index in a) to the index we are interested in.}

The chapter concludes (Corollary 5.29) with a derivation of Callias' original result (Theorem 0.1).

\textbf{Definition 5.1.—} The Riemannian manifold \(M\) is said to have a \textit{warped end} if there is a compact subset \(K \subset M\) and a warped product \(W\) (Definition 4.1), such that \(M - K\) and \(W\) are isometric as Riemannian manifolds. Thereafter we will identify \(M - K\) and \(W\). Any such manifold is complete \([O]\).

For the rest of the chapter we assume that \(M\) is a \((n+1)\)-dimensional spin manifold, \(n\) even, with a warped end \(W\) and \(S\) is a spinor-type bundle over \(M\). As
shown in Chapter IV, $S|_{M-K} = \pi^*(S_R)$, with $S_R = S|_{\{R\} \times N}$ also a spinor-type bundle — a fortiori $N$ is spin too —. The separation of variables for the Dirac operator restricted to sections in $C^\infty(M-K, S)$ yields $\Xi_R = \frac{n}{2} n^\circ$ (see 4.25a). We also assume that $A$ is skew-Hermitean, commutes with the Clifford action and is independent of the radial direction $r$, i.e., $A(r, x) = A(R, x) \equiv A(x)$, $r \geq R$, $x \in N$. Then $-\Delta^2$ is positive at infinity if and only if $-\Delta^2(R, \cdot)$ is positive on $N$, which will also be assumed.

**Proposition 5.2.** — Under the above hypotheses the operator $L = D + A$ is a Fredholm operator if the warping function $f$ on $M-K = W$ has the property that $f(r) \to \infty$, if $r \to \infty$.

**Proof.** — According to Corollary 2.28 b), it is enough to show that $\mathcal{B}(A)\equiv [D, A]$ goes to zero pointwise, as we approach the end of the manifold. Since $D = n^\circ \frac{\partial}{\partial r} + \frac{\partial R}{f^2} n^\circ$ on $M-K$, we see that $[D, A](r, x) = \frac{1}{f(r)} [\partial_R, A](x)$, $r \geq R$, $x \in N$. Thus $[D, A](r, x) \to 0$ as $r \to \infty$, if $f(r) \to \infty$ as $r \to \infty$. $\blacklozenge$

Next we state the relative index theorem ([GL],[Do]).

**Theorem 5.3.** — Let $(M_i, S_i, L_i)$, $i = 0, 1$, be two perturbed Dirac operators which coincide outside compact sets, where manifolds and bundles are assumed to be compatibly isometric ([GL], Assumption III,4.1). Suppose that $L_i^* L_i$ and $L_i L_i^*$, $i = 0, 1$, are strictly positive at infinity (assumptions 2.1 and 2.6 suffice for our needs). Then $L_i$ is a Fredholm operator and

$$L^2\text{-index}(L_0) - L^2\text{-index}(L_1) = \text{ind}_i(L_0, L_1)$$

$$\blacklozenge$$

In equation (5.4) $\text{ind}_i(L_0, L_1)$ is the relative topological index. One way to define it is the following: let $\omega_i$, $i = 0, 1$, be the Atiyah-Singer index form on $M_i$ associated to
L_i [ABP], i.e., $\omega_i$ is the coefficient of $t^0$ in the local asymptotic expansion of the heat kernel $\text{tr} \left[ e^{-\tau L_i} - e^{-\tau L_i} \right], m_i \in M_i$, as $t \to 0$. $\omega_0 = \omega_1$ on the common portion of $M_i$, and so $\int_{M_0} \omega_0 - \int_{M_1} \omega_1 \overset{\text{def}}{=} \text{ind}_i(L_0, L_1)$ makes sense.

**Remark 5.5.**— The relative index theorem is proved in [GL] for generalized Dirac operators on even dimensional manifolds only. However the proof goes through without any change for perturbed Dirac operators. This allows us to consider odd dimensional manifolds as well, in which case Theorem 5.3 amounts to an index-preserving deformation theorem. We also notice that the manifold $M_i$ need not be connected.

**Corollary 5.6.**— In Theorem 5.3 assume that $\dim M_0 = \dim M_1 = \text{odd number. Then}$

\begin{equation}
(5.7) \quad \text{L}^2\text{-index}(L_0) = \text{L}^2\text{-index}(L_1)
\end{equation}

**Proof.**— It is well-known that $\omega_i, i = 0,1$, vanish on odd dimensional manifolds (see [ABP]).

We now return to the $\text{L}^2\text{-index}$ for the Callias-type operator $L$ considered in this chapter. The basic idea is to link, via the relative index theorem, two copies of $M$ with their respective Callias-type operators to an operator $(M_1, S_1, L_1)$ on a manifold with two ends and whose index is easily computable. Next we describe in detail this $(M_1, S_1, L_1)$.

Let us take $M_1 \overset{\text{def}}{=} \mathbb{R} \times_f N$, where $f_1, 0 < f_1 \in C^\infty(\mathbb{R}), f_1 > 0$ and $\lim_{|t| \to \infty} f_1(t) = \infty$. Assume that the portion $(e, \infty) \times_f N$ of $M_1$ is the end of a manifold $M$ equipped with a spinor-type bundle $S$, as we described it at the beginning of this chapter, and $f_1(t)$
let $f(|t|)$ for $|t| \geq R$. If $\pi : \mathbb{R} \times N \to \{R\} \times N \equiv N$, $\pi(t,x) = (R,x)$ is the projection on $N$, we take $S_1 \equiv \pi^*(S_R)$.

**Proposition 5.8.** $S_1 \to M_1$ can be made a Dirac bundle in a canonical way.

**Proof.** The proof is of course similar to the corresponding part in Theorem 4.10. We only sketch it. Fix $(t,x) \in M_1$ and let $\{e_1, \ldots, e_n\}$ be a local orthonormal basis in $T_x N$. Then $\{\frac{\partial}{\partial t}, \frac{1}{f_1} e_1, \ldots, \frac{1}{f_1} e_n\}$ is a local orthonormal basis in $T(t,x)(M_1)$ and we define the Clifford multiplication in $S_1$ by

\[
\text{cliff} \left( \frac{\partial}{\partial t} \right) = n^\circ, \quad \text{cliff} \left( \frac{1}{f_1} e_i \right) = e_i^\circ \quad i = 1, \ldots, n
\]

Also define the connection $\nabla^S_1$ by the formulas:

\[
\nabla^S_1 \frac{\partial}{\partial t} s^\sim = 0, \quad \nabla^S_1 e_i^\circ s^\sim = \frac{1}{f_1} \left( \nabla^S_1 e_i \right) s^\sim + \frac{f'}{2 f_1} \left( \pi^* e_i \right) s^\sim
\]

Here $s$ is a local section in $S_R$ and $s^\sim$ is its parallel lift to $\pi^*(S_R) = S_1$. Now the necessary verifications are identical to those in Theorem 4.13.

Let $D_1$ be the Dirac operator associated to $S_1 \to M_1$ and $L_1 = D_1 + \Psi$, $\Psi(t,x) = \chi(t) A(x)$, $\chi \in C^\infty(\mathbb{R})$, $\chi(t) = \begin{cases} 1 & \text{if } t \geq R \\ -1 & \text{if } t \leq -R \end{cases}$. We can write $L_1 = n^\circ \frac{\partial}{\partial t} + \frac{\partial_R}{f_1} + \frac{f'}{2 f_1} n^\circ + \chi A$ on $C^\infty(M_1, S_1)$. Since $\lim_{|t| \to \infty} f_1(t) = \infty$, $L_1$ is a Fredholm operator (Proposition 5.2). The notations being those of Lemma 3.18, Theorem 3.25, and (4.18), $L_1$ is homotopic to the operator

\[
T_1 = \begin{pmatrix}
(n^\circ \frac{\partial}{\partial t} + \frac{\partial_R}{f_1})^+ + \frac{f'}{2 f_1} n^\circ + \sqrt{-1} \chi & 0 \\
0 & (n^\circ \frac{\partial}{\partial t} + \frac{\partial_R}{f_1})^+ + \frac{f'}{2 f_1} n^\circ - \sqrt{-1} \chi
\end{pmatrix}
\]
globally defined on \( \mathcal{C}^\infty(M_1, S_1) \equiv \mathcal{C}^\infty(M_1, (S_1)_+ \oplus (S_1)_-) \). In writing out (5.11) we used the fact \((S_1)_\pm = \pi^*(S_R)_\pm\), where \((S_R)_\pm \overset{\text{def}}{=} \{ U(R, \cdot) = \pm \sqrt{-1} \text{Id} \} \subset S_R\). \((\partial_R)_\pm\) is the Dirac operator associated to \((S_R)_\pm \rightarrow N\) and therefore \((\partial_R)_\pm = \frac{1}{2} (\text{Id} \mp \sqrt{-1} U) \partial_R \mid (S_R)_\pm\). Set now \((S_R)_\pm^\pm = \{ s \in (S_R)_+ \mid n \cdot s = \pm \sqrt{-1} s \} \) and let \((\partial_R)_\pm^\pm\) be the restriction of \((\partial_R)_\pm\) to \((S_R)_\pm^\pm\). Similarly define \((\partial_R)_\pm^-\) and \((S_R)_\pm^-\).

**Theorem 5.12.**— If \((M_1, S_1, L_1)\) is as above, then

\[
(5.13) \quad L^2\text{-index}(L_1) = \text{index}(\partial_R)_+^+ + \text{index}(\partial_R)_-^-
\]

**Proof.**— Clearly

\[
L^2\text{-index}(L_1) = L^2\text{-index}(T_1) = \\
L^2\text{-index}(n \cdot \frac{d}{dt} + \frac{(\partial_R)_+}{f_1} + \frac{f_1}{f_1} \frac{n \cdot s}{2} + \sqrt{-1} \chi) + \\
L^2\text{-index}(n \cdot \frac{d}{dt} + \frac{(\partial_R)_-}{f_1} + \frac{f_1}{f_1} \frac{n \cdot s}{2} - \sqrt{-1} \chi)
\]

We concentrate next on the \(L^2\)-index of the operator \(n \cdot \frac{d}{dt} + \frac{(\partial_R)_+}{f_1} + \frac{f_1}{f_1} \frac{n \cdot s}{2} + \sqrt{-1} \chi\) defined on \(\mathcal{C}^\infty(M_1, \pi^*(S_R)_+ )\). The obvious Hilbert space isometry

\[
(5.14) \quad \begin{cases}
L^2(M_1, \pi^*(S_R)_+) \rightarrow L^2(\mathbb{R} \times N, \pi^*(S_R)_+)
\
\omega \rightarrow \frac{n}{f_1} \omega
\end{cases}
\]

takes the operator \(n \cdot \frac{d}{dt} + \frac{(\partial_R)_+}{f_1} + \frac{f_1}{f_1} \frac{n \cdot s}{2} + \sqrt{-1} \chi\) into the operator \(n \cdot \frac{d}{dt} + \frac{(\partial_R)_+}{f_1} + \sqrt{-1} \chi\) defined on \(\mathcal{C}^\infty(\mathbb{R} \times N, \pi^*(S_R)_+)\). Let us denote by \(Q\) the closure of the operator \(n \cdot \frac{d}{dt} + \frac{(\partial_R)_+}{f_1} + \sqrt{-1} \chi\) in \(L^2(\mathbb{R} \times N, \pi^*(S_R)_+)\). Then \(Q^* = n \cdot \frac{d}{dt} + \frac{(\partial_R)_+}{f_1} - \sqrt{-1} \chi\), and the task reduced to that of finding \(L^2\text{-index}(Q)\).

As a selfadjoint differential operator on a compact manifold, \((\partial_R)_+\) has a discrete spectrum located on the real line. Since \((\partial_R)_+ n \cdot - n \cdot (\partial_R)_+\), if \(\lambda \neq 0\) is an
eigenvalue corresponding to the eigenvector \( s_\lambda \in C^\infty(N, (S_R)_+) \), then \((-\lambda)\) is also an eigenvalue corresponding to the eigenvector \( n \circ s_\lambda \). Let then \( \{ (\lambda, s_\lambda) \mid \lambda > 0 \} \cup \{ (-\lambda, n \circ s_\lambda) \} \) be the spectral decomposition of \( (\mathcal{Q}_+)^* \). \( \{ s_\lambda, n \circ s_\lambda, s_\alpha \} \) is then a Hilbert basis for \( L^2(N, (S_R)_+) \). \( \{ s_\alpha \}_{\alpha} \) generates the finite dimensional space \( \ker (\mathcal{Q}_+)^* \) and \( \{ s_\lambda, n \circ s_\lambda \} \) is a Hilbert basis for \( [\ker (\mathcal{Q}_+)^*] \perp \), the orthogonal complement of \( \ker (\mathcal{Q}_+)^* \) in \( L^2(N, (S_R)_+) \). Moreover, in the decomposition

\[
L^2(\mathbb{R} \times N, \mathcal{Q}_+^* (S_R)_+) \equiv L^2(\mathbb{R}, L^2(N, (S_R)_+)) = L^2(\mathbb{R}, \ker (\mathcal{Q}_+)^*) \oplus L^2(\mathbb{R}, [\ker (\mathcal{Q}_+)^*] \perp),
\]

\( L^2(\mathbb{R}, \ker (\mathcal{Q}_+)^*) \) and \( L^2(\mathbb{R}, [\ker (\mathcal{Q}_+)^*] \perp) \) are left invariant by \( \mathcal{Q} \) and \( \mathcal{Q}^* \). Thus

\[
(5.15) \quad L^2\text{-index}(\mathcal{Q}) = L^2\text{-index}(\mathcal{Q} \mid C^\infty(\mathbb{R}, \ker (\mathcal{Q}_+)^*)) + L^2\text{-index}(\mathcal{Q} \mid C^\infty(\mathbb{R}, [\ker (\mathcal{Q}_+)^*] \perp))
\]

It is easily seen that the \( L^2\)-isometry of \( [\ker (\mathcal{Q}_+)^*] \perp \) into itself given by

\[
(5.16) \begin{cases}
  s_\lambda \rightarrow n \circ s_\lambda \\
  n \circ s_\lambda \rightarrow s_\lambda
\end{cases}
\]

induces an isometry on \( L^2(\mathbb{R}, [\ker (\mathcal{Q}_+)^*] \perp) \) which identifies

\[
L^2\ker(Q \mid C^\infty(\mathbb{R}, [\ker (\mathcal{Q}_+)^*] \perp)) \quad \text{and} \quad L^2\ker(Q^* \mid C^\infty(\mathbb{R}, [\ker (\mathcal{Q}_+)^*] \perp)).
\]

(5.15) becomes then

\[
(5.17) \quad L^2\text{-index}(\mathcal{Q}) = L^2\text{-index}(\mathcal{Q} \mid C^\infty(\mathbb{R}, \ker (\mathcal{Q}_+)^*))
\]

We can choose \( s_\alpha \in C^\infty(N, (S_R)_+) \), \( (\mathcal{Q}_+)^* \), \( s_\alpha = 0 \), such that \( s_\alpha \in C^\infty(N, (S_R)_+) \) or \( s_\alpha \in C^\infty(N, (S_R)_+) \). This is always possible since \( n \) preserves \( \ker (\mathcal{Q}_+)^* \). Then

\[
(5.18) \quad \mathcal{Q} \mid C^\infty(\mathbb{R}, \ker (\mathcal{Q}_+)^*) = \pm \sqrt{-1} \frac{\partial}{\partial t} + \sqrt{-1} \chi
\]
Any solution \( \sigma \in C^\infty(\mathbb{R}, \ker (\partial_r)_+) \) of the equation \( \sqrt{-1} \sigma' + \sqrt{-1} \chi \sigma = 0 \) is of the form
\[
\sigma(t, x) = e^{-\int_0^t \chi(\tau) \, d\tau} \, s(x), \quad s \in \ker (\partial_r)_+,
\]
thus always in \( L^2(\mathbb{R}, \ker (\partial_r)_+) \), and any solution \( \sigma \in C^\infty(\mathbb{R}, \ker (\partial_r)_-) \) of the equation \( -\sqrt{-1} \sigma' + \sqrt{-1} \chi \sigma = 0 \) is of the form
\[
\sigma(t, x) = e^{\int_0^t \chi(\tau) \, d\tau} \, s(x), \quad s \in \ker (\partial_r)_-,
\]
thus never in \( L^2(\mathbb{R}, \ker (\partial_r)_+) \). As a result
\[
\dim L^2 \ker (Q | C^\infty(\mathbb{R}, \ker (\partial_r)_+)) = \dim \ker (\partial_r)_+.
\]
Similarly
\[
\dim L^2 \ker (Q^* | C^\infty(\mathbb{R}, \ker (\partial_r)_+)) = \dim \ker (\partial_r)_-, \quad \text{i.e.,}
\]
\[
(5.19) \quad L^2 \text{-index} (n \cdot \frac{d}{dt} + \frac{(\partial_r)_+}{f_1} + \frac{f_1}{f_1} \, n \cdot \sqrt{-1} \chi) = \text{index}(\partial_r)_+
\]
The whole argument can be repeated for the operator \( n \cdot \frac{d}{dt} + \frac{(\partial_r)_-}{f_1} + \frac{f_1}{f_1} \, n \cdot -\sqrt{-1} \chi \)
and gives
\[
(5.20) \quad L^2 \text{-index} (n \cdot \frac{d}{dt} + \frac{(\partial_r)_-}{f_1} + \frac{f_1}{f_1} \, n \cdot -\sqrt{-1} \chi) = \text{index}(\partial_r)_-
\]
(5.13) follows. \( \blacksquare \)

**Corollary 5.21.** — Let \( L_1 \) be the Fredholm operator considered in Theorem 5.12. Then
\[
(5.22) \quad L^2 \text{-index}(L_1) = 2 \text{index}(\partial_r)_+
\]

**Proof.** — According to Theorem 5.12, it is enough to show that \( \text{index}(\partial_r)_+ = \text{index}(\partial_r)_- \). The operator \( (\partial_r)_+^*: C^\infty(N, (\mathbb{R})^+) \to C^\infty(N, (\mathbb{R})^-) \) is cobordant to 0.
Let us now introduce the manifold $M_0 = M \sqcup M$, the disjoint union of two copies of $M$ and the Dirac bundle $S_0 = S \sqcup S$. The operator $L_0 = L \sqcup (-L)$ is clearly a Fredholm operator and $L^2$-index$(L_0) = L^2$-index$(L) + L^2$-index$(-L) = 2 L^2$-index$(L)$. Strictly speaking, $(-L)$ and $L_0$ are not perturbed Dirac operators the way we have defined them; however, the results stated so far hold for them as well. If we define $T_0 = T \sqcup (-T)$ on $S_0$, we also have $L^2$-index$(T_0) = 2 L^2$-index$(L)$.

**Proposition 5.23.**— The operators $(M_0, S_0, T_0)$ and $(M_1, S_1, T_1)$ introduced above are isometric outside compact sets i.e., there are compact sets $K_0 \subset M_0$ and $K_1 \subset M_1$ and a manifold isometry $F : M_0 |_{M_0 - K_0} \to M_1 |_{M_1 - K_1}$ covered by a bundle isometry $	ilde{F} : S_0 |_{S_0 - K_0} \to S_1 |_{S_1 - K_1}$ such that

$$T_1 = \tilde{F} T_0 (\tilde{F})^{-1}$$

**Proof.**— The obvious isometry $F$ between $CK_0 \equiv \{(r, x) \in M | r > R\} \sqcup \{(r, x) \in M | r > R\} \subset M_0$ and $CK_1 \equiv \{(t, x) \in M_1 | |t| > R\} \subset M_1$ (Fig. 1) is covered by the bundle isometry $	ilde{F}$ defined by the formula

$$F(m) = \begin{cases} (r, x) & \text{if } m \in 1^{st} M \text{ in } M_0 \text{ and } m = (r, x) \\ (-r, x) & \text{if } m \in 2^{nd} M \text{ in } M_0 \text{ and } m = (r, x) \end{cases}$$
\[ (5.26) \tilde{F}(\nu_m) = \begin{cases} \nu_m & \text{if } m \in \text{1st } M \text{ in } M_0 \text{ and } \nu_m \in S_R \equiv (S_0)_m \equiv (S_1)_m \\ n \circ \nu_m & \text{if } m \in \text{2nd } M \text{ in } M_0 \text{ and } \nu_m \in S_R \equiv (S_0)_m \equiv (S_1)_F(m) \end{cases} \]

The pairs \((M_0, S_0)\) and \((M_1, S_1)\) are isometric outside compact sets \(K_0\) and \(K_1\).

(5.24) is obvious for sections in \(C^\infty((t,x) \mid t > R), S_1)\), since \(\tilde{F} = \text{Id} \) there. Consider now a section \(s\) in \(C^\infty((t,x) \mid t < -R), S_1)\). For simplicity we assume that \(s \in C^\infty((t,x) \mid t < -R), (S_1)_+\). Then \((\tilde{F}^{-1}s)(r,x) = -n \circ s(-r,x)\) and

\[
(T_{0}\tilde{F}^{-1}s)(r,x) = (-L \tilde{F}^{-1}s)(r,x) = \\
-n \circ \frac{\sigma(r,x)}{f(r)} - \frac{n \circ (\partial R)_+ \sigma(r,x)}{f(r)} - \frac{f'(r)}{f(r)} \frac{n \circ \sigma(r,x)}{2} - \sqrt{-1} \sigma(r,x),
\]

where \(\sigma = \tilde{F}^{-1}s\). Thus

\[
(T_{0}\tilde{F}^{-1}s)(r,x) = \\
\frac{\partial s}{\partial r}(-r,x) - \frac{n \circ ((\partial R)_+ s)(-r,x)}{f_1(-r)} + \frac{f'_1(-r)}{f_1(-r)} n \circ (-r,x) + \sqrt{-1} n \circ s(-r,x).
\]
Finally
\[
(F T_0 F^{-1}s)(t,x) = 
\left( \frac{d}{dt} n o \frac{\partial s}{\partial t} (t,x) \right) + \frac{f'(t)}{f_1(t)} n o s(t,x) - \sqrt{-1} s(t,x) = 
(T_1 s)(t,x)
\]

We summarize the results obtained in Corollary 5.6, Theorem 5.12, Corollary 5.21, and Proposition 5.23 in the following main theorem:

**Theorem 5.27.**— Let \( M \) be a odd dimensional Riemannian spin manifold with a warped end \( W = (e, \infty) \times N, f \in C^\infty((e, \infty)), f > 0 \) and \( f(r) \to \infty \) if \( r \to \infty \). Let \( S \) be a spinor-type bundle over \( M \) and let \( A \in C^\infty(M, \text{End}(S)) \) be a skew-Hermitean endomorphism such that \( A \) commutes with the Clifford action on \( S \), \( A \mid W \) is independent of the radial direction \( r \), and \(- A^2\) is positive at infinity. Then the perturbed Dirac operator \( D + A \), where \( D \) is the Dirac operator on \( S \), is a Fredholm operator and

\[
L^2\text{-index}(D + A) = \text{index } (\vartheta_R)^+
\]

Here \((\vartheta_R)^+: C^\infty(N, (S_R)^+) \to C^\infty(N, (S_R)^+)\) is the Dirac operator on the bundle \((S_R)^+ = \{ s \in S \mid \{r\} \times N \subseteq N \mid (-A^2)^{-\frac{1}{2}} As = \sqrt{-1} s, \text{cliff } (\frac{\partial}{\partial r}) s = \sqrt{-1} s \} \) over the compact even dimensional spin manifold \( \{R\} \times N \subseteq N \).

Theorem 5.27 clearly indicates that the \( L^2\text{-index} \) depends only on the spin geometry of the cross section of the manifold \( M \), and on the spectral properties of the
potential \( A \) at infinity. This fact is even better outlined by the following particular case which also leads to an elegant derivation of Callias' index formula (0.2).

**Corollary 5.29.**— a) In Theorem 5.27 assume in addition that \( S = \Sigma \otimes V \) where \( \Sigma \) is the spinor bundle on \( M \) (cf. I a), \( V \) is a trivial bundle over \( M \), and \( A \in C^\infty(M, \text{End}(V)) \). Then

\[
(5.30) \quad L^2\text{-index}(D + A) = \int_N \hat{A}(N) \wedge \text{ch}(V_R)^+ = \int_N \hat{A}(N) \wedge \text{tr} \exp \frac{\sqrt{-1}}{2\pi} \left( \frac{\text{Id} + \Phi}{8} \right) (d\Phi)^2
\]

where \( \hat{A}(N) \) is the total \( \hat{A} \)-class of \( N \), \( \Phi = \frac{1}{\sqrt{-1}} \left( -A^2 \right)^{\frac{1}{2}} A \), \( (V_R)^+ \) is the bundle over \( N \) given by \( (V_R)^+ = \{ v \in V_R \mid v \rangle \langle v = v \} \), and \( \text{ch}(V_R)^+ \) is the Chern character of \( (V_R)^+ \).

b) If \( \hat{A}(N) = 1 \), then (5.30) can be written:

\[
(5.31) \quad L^2\text{-index}(D + A) = \frac{1}{2} \left( \frac{n}{2} \right)! \left( \frac{\sqrt{-1}}{8\pi} \right)^n \int_N \text{tr} \Phi (d\Phi)^n
\]

**Proof.**— Since \( U = \sqrt{-1} \Phi \), we have \( (S_R)^+ = \Sigma \otimes (V_R)^+ \). We prefer to work with \( \Phi \) instead of \( U \) here for esthetical reasons. The first half of equation (5.30) follows now from Theorem 5.27 and the Atiyah-Singer index theorem for twisted Dirac operators on compact manifolds [AS]. The second half is a consequence of the following lemma:

**Lemma 5.32.**— Let \( \xi \to N \) be a trivial bundle equipped with the flat connection with respect to a fixed trivialization. If \( P \in C^\infty(N, \text{End}(\xi)) \) is a projection then the
curvature associated to the induced connection on the subbundle $\xi_+ = \{ P = \text{Id} \}$ is given by $P (dP)^2$.

**Proof of Lemma 5.32.**— Let $\nabla$ denote the flat connection on $\xi$. Then $P \nabla$ is the induced connection on $\xi_+$ whose curvature $r_+$ is given for any $X, Y \in TN$, by

$$r_+(X, Y) = [ P \nabla X, P \nabla Y ] - P \nabla [X, Y] =$$

$$P \nabla X P \nabla Y - P \nabla Y P \nabla X - P \nabla [X, Y] =$$

$$P X(P) \nabla Y - P^2 \nabla X \nabla Y - P Y(P) \nabla X + P^2 \nabla Y \nabla X - P \nabla [X, Y] =$$

$$P X(P) \nabla Y - P Y(P) \nabla X$$

since $\nabla$ is flat and $P$ is a projection. On the other hand

$$P X(P) \nabla Y - P Y(P) \nabla X = P X(P) \nabla Y P - P Y(P) \nabla X P =$$

$$P X(P) Y(P) + P X(P) P \nabla Y - P Y(P) X(P) - P Y(P) P \nabla X =$$

$$P (dP)^2(X, Y) + P X(P) P \nabla Y - P Y(P) P \nabla X = P (dP)^2(X, Y)$$

since $P X(P) P$ and $P Y(P) P$ are $0$, $P$ being a projection. ♦

**Proof of Corollary 5.29 continued:** Put now $\xi = V_R$ and $P = \frac{\text{Id} + \Phi}{2}$ in Lemma 5.32. Then $\xi_+ = (V_R)_+$ and a representative for the Chern character $\text{ch}(V_R)_+$ is $\text{tr} \exp \left( \frac{\sqrt{-1}}{2\pi} r_+ \right) = \exp \frac{\sqrt{-1}}{2p} \left( \frac{\text{Id} + \Phi}{8} \right) (d\Phi)^2$.

**b) If $\hat{A}(N) = 1$ then**

$$L^2\text{-index}(D + A) = \int_N \text{ch}(V_R)_+ = \int_N \text{tr} \exp \frac{\sqrt{-1}}{2p} \left( \frac{\text{Id} + \Phi}{8} \right) (d\Phi)^2$$

Since only the component of top degree of $\text{ch}(V_R)_+$ matters in the above integration and $\text{tr} \exp \frac{\sqrt{-1}}{2\pi} \left( \frac{\text{Id} + \Phi}{8} \right) (d\Phi)^2$ is a form of degree $2$, we get

$$L^2\text{-index}(D + A) =$$

$$\frac{1}{(n/2)!} \int_N \text{tr} \left( \frac{\sqrt{-1}}{2\pi} \left( \frac{\text{Id} + \Phi}{8} \right)(d\Phi)^2 \right)^{n/2} = \frac{1}{(n/2)!} \left( \frac{\sqrt{-1}}{8\pi} \right)^n \int_N \text{tr} \left( \frac{\text{Id} + \Phi}{2} (d\Phi)^2 \right)^{n/2} =$$
\[
\frac{1}{(\frac{n}{2})!} \left(\frac{\sqrt{-1}}{8\pi}\right)^\frac{n}{2} \int_N \text{tr} \, \frac{1}{2} \Phi (d\Phi)^n = \\
\frac{1}{2(\frac{n}{2})!} \left(\frac{\sqrt{-1}}{8\pi}\right)^\frac{n}{2} \int_N \text{tr} (d\Phi)^n + \frac{1}{2(\frac{n}{2})!} \left(\frac{\sqrt{-1}}{8\pi}\right)^\frac{n}{2} \int_N \text{tr} \Phi (d\Phi)^n = \\
\frac{1}{2(\frac{n}{2})!} \left(\frac{\sqrt{-1}}{8\pi}\right)^\frac{n}{2} \int_N \text{tr} \Phi (d\Phi)^n
\]

since \(\text{tr} (d\Phi)^n\) is an exact form. Taking \(M = \mathbb{R}^n\), \(n\) odd, and \(A = \sqrt{-1}\Phi\) outside some compact set, \((5.23)\) becomes exactly Callias' index formula, since \(N \equiv S_1^{n-1}\) and \(\hat{A}(S_1^{n-1}) = 1\). \(\blacklozenge\)
APPENDIX A

REMARK ON CALLIAS' INDEX FORMULA

In this appendix we complete the proof given by Bott and Seeley to formula (0.2). In [BoS], (0.2) is recovered by topological considerations, up to a constant, from a general index formula in $\mathbb{R}^n$ due to Hormander and Fedosov. Essentially we follow their argument; a careful analysis of algebraic nature brings up the missing constant.

Let $L \in \text{Diff}(\mathbb{R}^n, S \otimes_{\mathbb{R}} \mathbb{C}^m)$ be the first order elliptic differential operator whose complete symbol is

$$\sigma(x, \xi) = \delta(\xi) \otimes \text{Id} + \sqrt{-1} \text{Id} \otimes U(x), \quad (x, \xi) \in \mathbb{R}^{2n}$$

$S$ stands here for the spinor space on which the complexified Clifford algebra on $n$ generators $\delta^1, \delta^2, \ldots, \delta^n$, $\delta^i \delta^j + \delta^j \delta^i = 2 \delta^{ij}$ acts, $\delta(\xi) = \sum \xi_i \delta^i \in \text{End}(S)$ is the Clifford multiplication in $S$ by $\xi \in \mathbb{R}^n$ and $U(x)$ is an $m \times m$ Hermitian matrix of $\mathcal{C}^\infty$-functions in $\mathbb{R}^n$ such that $U(x)$ is unitary and homogeneous of degree 0 in $\{|x| \geq 1\}$.

Now a general result of Hormander [H] and Fedosov [F] gives the following formula for the $L^2$-index of $L$:

$$L^2 \text{-index}(L) = -\left(\frac{\sqrt{-1}}{2\pi}\right)^n \frac{(n-1)!}{(2n-1)!} \int_{S^{2n-1}} \text{tr} (\sigma^{-1} d\sigma)^{2n-1}$$
In the above formula the unit sphere $S^{2n-1} \subset \mathbb{R}^{2n}$ is oriented such that if $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$ has coordinates $(x, \xi)$, then $dx_1 \wedge dx_2 \wedge \ldots \wedge dx_n \wedge d\xi_1 \wedge \ldots \wedge d\xi_n > 0$. The matrix of 1-forms $\sigma^{-1} d\sigma$ is raised to a power in the usual way, except that the entries are multiplied by exterior multiplication, and the trace of $(\sigma^{-1} d\sigma)^{2n-1}$ yields a $(2n-1)$-form which can be integrated over $S^{2n-1}$.

The operator $L$ was originally studied by other methods by Callias in [C] and his main result was a much simpler index formula, namely formula (0.2). Following Bott and Seeley [BS], we relate (A.2) and (0.2), using the fact that $S^{2n-1}$ is the topological join of $S^{n-1}$ with itself.

We reorient first $S^{2n-1}$, asking that in $\mathbb{R}^{2n}$ with coordinates $(x, \xi)$, $dx_1 \wedge dx_2 \wedge \ldots \wedge dx_n \wedge d\xi_1 \wedge \ldots \wedge d\xi_n > 0$. This will be considered the standard orientation of $S^{2n-1}$. It will be immediately clear (see also Ch. III, Remark 3.5.6) that only an odd $n$ leads to a non-zero index. For this reason we will carry on the analysis only for $n$ odd, the argument for the other parity being identical. Since

$$\left(\frac{n(n-1)}{2}\right) dx_1 \wedge dx_2 \wedge \ldots \wedge dx_n \wedge d\xi_1 \wedge \ldots \wedge d\xi_n = (-1)^n dx_1 \wedge dx_2 \wedge \ldots \wedge dx_n \wedge d\xi_1 \wedge \ldots \wedge d\xi_n$$

the Hormander-Fedosov formula (A.2) changes by $(-1)^{n-1} / 2$ if we orient $S^{2n-1}$ the standard way. The map

$$j : [0, \pi / 2] \times S^{n-1}_x \times S^{n-1}_\xi \rightarrow S^{2n-1}_{(x, \xi)}$$

$$j(t, x, \xi) = (x \sin t, \xi \cos t) \equiv x \sin t + \xi \cos t$$
is an orientation reversing diffeomorphism outside sets of measure 0. This is the realization of $S^{2n-1}$ as the topological join $S^{n-1} \ast S^{n-1}$. Some standard manipulations involving the Jacobian

(A.5) \[ Dj(t,x,\xi) = \begin{pmatrix} x \cos t & \sin t & \text{Id} & 0 \\ -\xi \sin t & 0 & \cos t & \text{Id} \end{pmatrix} \]

show that indeed the orientation is reversed. Thus

(A.6) \[ \int_{S^{2n-1}} (\sigma^{-1} d\sigma)^{2n-1} = -\int_0^{\pi/2} \int_{S_x^{n-1}} \int_{S_{\xi}^{n-1}} [(j^* \sigma)^{-1} d(j^* \sigma)]^{2n-1} \]

But $(j^* \sigma)(t,x,\xi) = \delta(\xi \cos t) \otimes \text{Id} + \sqrt{-1} \text{Id} \otimes U(x \sin t) = \cos t \delta(x) \otimes \text{Id} + \sqrt{-1} U(x \sin t)$. Since $\text{index}(L)$ is invariant under deformations of $U$ on compact sets (Ch. III, Proposition 3.6), it is possible to prescribe $U$ in a convenient way inside $\{|x| \leq 1\}$, namely

(A.7) \[ U(x \sin t) = \sin t U(x) \quad t \in [0,\frac{\pi}{2}] \quad |x| = 1 \]

This choice of $U$ makes it easy to evaluate the triple integral in (A6). Let $z = j^* \sigma$, i.e., $z = \delta \otimes \cos t + \sqrt{-1} \sin t \otimes U$ on $[0,\frac{\pi}{2}] \times S^{n-1} \times S^{n-1}$. Clearly $z^{-1} = \delta \otimes \cos t - \sqrt{-1} \sin t \otimes U$ and $dz = dt (-\delta \otimes \sin t + \sqrt{-1} \cos t \otimes U) + d\xi \delta \otimes \cos t + \sqrt{-1} \sin t \otimes d_x U$. Then $z^{-1} dz = \sqrt{-1} dt \delta \otimes U + z^{-1}(d\xi \delta \otimes \cos t + \sqrt{-1} \sin t \otimes d_x U)$.

In the remaining we drop the tensor product signs, i.e., $\delta \otimes U$ becomes $\delta U$, etc., the subscripts, i.e. $d\xi \delta = d\delta$, etc., and notice that $\delta$ and $U$ should commute since $U\delta = (\text{Id} \otimes U)(\delta \otimes U) = \delta \otimes U = \delta U$.

We can calculate $(z^{-1} dz)^{2n-1}$ keeping constantly in mind two facts:

a) powers of binomials whose terms commute are easily computable.
b) the final answer should involve only one \( dt \), \((n-1)\) \( d\delta \)'s and \((n-1)\) \( dU \)'s.

This calculation is based on the following obvious commutation relations:

\[
\begin{align*}
(A.8) \quad \begin{cases} 
  z^{-1} \delta &= \delta z^{-1} \\
  (d\delta) \delta + \delta (d\delta) &= 0 \\
  d\delta \, dt &= -dt \, d\delta \\
  dU \, dt &= -dt \, dU \\
  z^{-1}U &= Uz^{-1}
\end{cases}
\end{align*}
\]

and the most important two of them all

\[
(A.9) \quad \begin{align*}
  z^{-1} (d\delta) &= -(d\delta) z , \\
  z^{-1} (dU) &= -(dU) z
\end{align*}
\]

For instance, the second line in (A.8) follows from \( \delta^2 = \text{Id} \) and \( U^2 = \text{Id} \), on the respective unit spheres. Also \( z^{-1}(d\delta) = (\delta \cos t - \sqrt{-1}U \sin t) \, d\delta = -(d\delta) \cos t \, \delta - \delta \sqrt{-1} \sin t \, U = -(d\delta) z \), proves half of (A.9).

Now \( dt \delta U \) commutes with \( z^{-1} \delta \cos t + \sqrt{-1} z^{-1} dU \sin t \). This alone implies

\[
(A.10) \quad (z^{-1}dz)^{2n-1} = (2n-1) \sqrt{-1} \, dt \, \delta U \, (z^{-1}d\delta \cos t + \sqrt{-1} z^{-1} dU \sin t)^{2n-2}
\]

\( z^{-1}d\delta \) commutes with \( z^{-1}dU \) as well, which gives:

\[
(A.11) \quad (z^{-1}d\delta \cos t + \sqrt{-1} z^{-1} dU \sin t)^{2n-2} =
\]

\[
= \frac{(2n-2)!}{(n-1)!} \cos^{n-1} t \, (z^{-1}d\delta)^{n-1} \sin^{n-1} t \, (\sqrt{-1} \, z^{-1} dU)^{n-1}
\]

\( n \) being assumed odd and using (A.9) we have \( (z^{-1}dz)^{2n-1} = [(z^{-1}d\delta)^{2}]^{\frac{n-1}{2}} = (-1)^{\frac{n-1}{2}} (d\delta)^{n-1} \) and similarly \( (\sqrt{-1} z^{-1} dU)^{n-1} = (-1)^{\frac{n-1}{2}} (dU)^{n-1} \).

Putting together (A.10) and (A.11) we see that
and finally

\[(A.13) \quad (z^{-1}dz)^{2n-1} = \sqrt{-1} \frac{(2n-1)!}{(n-1)!^2} (\sin t \cos t)^{n-1} \operatorname{tr} (UdU)^{n-1} \operatorname{tr} (d\delta)^{n-1} \]

Thus the RHS of (A.6) becomes

\[(A.14) \quad \sqrt{-1} \frac{(2n-1)!}{2^{n-1}(n-1)!^2} \int_0^{\pi/2} (\sin 2t)^{n-1}dt \int_\mathbf{S}^{n-1} \operatorname{tr} (UdU)^{n-1} \int_\mathbf{S}^{n-1} \operatorname{tr} (d\delta)^{n-1} \]

But for \(n\) odd

\[(A.15) \quad \int_0^{\pi/2} (\sin 2s)^{n-1}ds = \frac{(n-1)!\pi}{2^n (n-1)^2} \]

Thus the integral in the Hormander-Fedosov formula (A.2) becomes

\[(A.16) \quad \int_{\mathbf{S}^{2n-1}} \operatorname{tr} (\sigma^{-1}d\sigma)^{2n-1} = -\left(\frac{(2n-1)!}{(n-1)!}\right)^n \frac{\pi}{2^{2n-1}} \frac{1}{(n-1)!^2} \int_{\mathbf{S}^{n-1}} \operatorname{tr} (UdU)^{n-1} \int_{\mathbf{S}^{n-1}} \operatorname{tr} (d\delta)^{n-1} \]

The coefficient \(-\left(\frac{(2n-1)!}{(n-1)!}\right)^n \frac{\pi}{2^{2n-1}} \frac{1}{(n-1)!^2}\) is precisely the one left out by Bott and Seeley! The integral \(\int_{\mathbf{S}^{n-1}} \operatorname{tr} (d\delta)^{n-1}\) is computable as well and this gives the final link between the two index formulas (A.2) and (0.2). Indeed using Stokes' theorem

\[(A.17) \quad \int_{\mathbf{S}^{n-1}} \operatorname{tr} (d\delta)^{n-1} = \int_B^n \operatorname{tr} (d\delta)^n \]
where $B^n$ is the unit ball in $\mathbb{R}^n$. However, $d\delta = \sum_i \delta^i d\xi_i$, and since $\delta^i d\xi_i$ commutes with $\delta^j d\xi_j$, we have $(d\delta)^n = n! \delta^1 \wedge \ldots \wedge \delta^n d\xi_1 \wedge \ldots \wedge d\xi_n$. It is easy to check that for odd $n$, $\text{tr} (\delta^1 \wedge \ldots \wedge \delta^n) = (2\sqrt{-1})^{n-1}$. Incidentally for even $n$, $\text{tr} (\delta^1 \wedge \ldots \wedge \delta^n) = 0$, and this alone is responsible for the vanishing of index $(L)$ in even dimensions, a fact referred to earlier (see also Ch.III, Remark 3.5b). As a result

\[(A.18) \quad \int_{S^{n-1}} \text{tr} \delta (d\delta)^{n-1} = n! (2\sqrt{-1})^{n-1} \frac{n-1}{\Gamma(\frac{n}{2})} (2\sqrt{-1})^{n-1} \frac{n-1}{\pi^2} \]

Writing out explicitly $\Gamma(\frac{n}{2})$ we get

\[(A.19) \quad \int_{S^{n-1}} \text{tr} \delta (d\delta)^{n-1} = 2 \frac{3n-1}{2} \frac{n-1}{\sqrt{-1}} \frac{(n-1)!}{(\frac{n}{2})!} \pi^2 \]

Now, from (A.16) and (A.19), we have precisely (0.2). In fact the result of Callias ([C], p. 222) has no minus sign in front of the RHS. However, this is how it should be since Callias takes

\[(A.20) \quad L = \sum_i \sqrt{-1} \delta^i \frac{\partial}{\partial \xi_i} \otimes \text{Id} + \text{Id} \otimes U \]

i.e., $s_L(x,\xi) = -\delta(x) \otimes \text{Id} + \text{Id} \otimes U(x)$. Since $\text{index}(L) = \text{index}(-L)$ and $s_-L(x,\xi) = \delta(x) \otimes \text{Id} + \text{Id} \otimes (-U(x))$, the formula (0.2) changes sign due to

\[(A.21) \quad \int_{S^{n-1}} \text{tr} (-U)[d(-U)]^{n-1} = -\int_{S^{n-1}} \text{tr} U(dU)^{n-1}, \quad n \text{ odd} \]
APPENDIX B

THE TWO DIMENSIONAL MAGNETIC FIELD PROBLEM

What can one say about the index of a perturbed Dirac operator when the manifold $M$ is even dimensional or the potential $A$ does not commute with the Clifford action? Assuming that some type of separation of variables is still possible on the end of the manifold, there is evidence [APS] that the index must contain certain complicated non-local contributions, or eta invariants. In this appendix we see that this is the case in a very elementary set-up taken from Physics [CFKS], the so-called two dimensional magnetic field problem.

Let $L$ be the first order elliptic differential operator

\[(B.1) \quad L = \frac{\partial}{\partial x} + \sqrt{-1} \frac{\partial}{\partial y} + \frac{\partial\phi}{\partial x} + \sqrt{-1} \frac{\partial\phi}{\partial y}\]

defined on $C^\infty(\mathbb{R}^2)$. Here $\phi \in C^\infty(\mathbb{R}^2)$ is real valued and $\phi = \ln r$, for $r = \sqrt{x^2 + y^2}$ big enough.

Despite the fact that the unique closed extension of $L\big|_{C_0^\infty(\mathbb{R}^2)}$, still denoted by $L$, is not a Fredholm operator, $L$ and its formal adjoint $L^\dagger$ do have finite dimensional $L^2$-kernels, and then the analytic $L^2$-index of $L$ is defined as usual (Ch. II, Definition 2.29) by

\[(B.2) \quad L^2\text{-index}(L) = \dim \ker(L) \cap L^2(\mathbb{R}^2) - \dim \ker(L^\dagger) \cap L^2(\mathbb{R}^2)\]
The kernels and this analytic index were analyzed in detail in Refs. [AC] and [CFKS].

In [BGGSS], Simon et al. studied the so-called Witten index associated to $L$ and found that this was precisely the magnetic flux $\frac{1}{2\pi} \int_{\mathbb{R}^2} \Delta \phi = F$, which is also the topological index of Atiyah and Singer. Since the two indices can hardly coincide, it was asked ([BGGSS], [GS]) what was the nature of their difference. The Atiyah-Patodi-Singer index theorem for manifolds with boundary and non-local boundary conditions [APS] provides an elegant way to answer this question: the difference is essentially the $\eta$-invariant of the Dirac operator on the boundary at infinity — the circle—, shifted by $F$, the flux.

Identifying $\mathbb{R}^2$ and $\mathbb{C}$, $(x,y) \leftrightarrow z = x + \sqrt{-1} y$ we can write

$$L = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}} = e^{-\phi} \frac{\partial}{\partial z} e^\phi$$

$$L^* = -\frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}} = -e^{\phi} \frac{\partial}{\partial z} e^{-\phi}$$

(B.3)

Now assume that $\phi = F \ln r$, for $r$ larger than some fixed $R > 0$. It is readily seen that in polar coordinates $(r,\theta)$, we have, for $r > R$,

$$L = e^{\sqrt{-1} \theta} \left( \frac{\partial}{\partial r} + \frac{(\sqrt{-1} \frac{\partial}{\partial \theta} + F)}{r} \right)$$

$$L^* = -e^{-\sqrt{-1} \theta} \left( \frac{\partial}{\partial r} - \frac{(\sqrt{-1} \frac{\partial}{\partial \theta} + F)}{r} \right)$$

(B.4)

Let $u \in L^2(\mathbb{R}^2)$ be such that $Lu = 0$. The restriction $u|_{r > R}$ of $u$ to the complement $\mathbb{C} \setminus B_R = \{ z \in \mathbb{C} \mid |z| > R \}$ is in $L^2((R,\infty) \times S^1, r \, dr \, d\theta$) and admits a Fourier series expansion

$$u|_{r > R} \equiv \sum_{k \in \mathbb{Z}} u_k(r) e^{\sqrt{-1}k \theta}$$

(B.5)
Since \((Lu) \big|_{\geq R} \equiv \sum_{k \in \mathbb{Z}} (u_k' + \frac{(-k+F)}{r} u_k) e^{-1(k+1)\theta} \) is 0, we get \(u_k' + \frac{(-k+F)}{r} u_k = 0\), i.e., \(u_k = c_k r^{k-F}\), \(c_k\) constant, \((\forall) \ k \in \mathbb{Z}\). Thus \(u \big|_{\geq R} \in L^2\) only if \(c_k = 0\), for any \(k \geq F-1\).

This suggests the introduction of the following boundary value problem. Denote by \((L,P \big|_{<F-1})\) the operator whose domain is the set of functions \(v \in C^\infty(B_R)\) such that

\[
(B.6) \quad \frac{1}{2\pi} \int_0^{2\pi} v(R,\theta) e^{-\sqrt{-1}k\theta} \, d\theta = 0 , \quad (\forall) \ k \geq F-1
\]

and on which \(L\) acts according to the first line in \((B3)\). This is a non-local boundary condition. Similarly one defines \((L^+,P \big|_{>F+1})\). With these preparations we have:

**Proposition B.7.**

1. \(L^2\)-\(\ker(L) = \ker(L,P \big|_{<F-1})\)
2. \(L^2\)-\(\ker(L^+) = \ker(L^+,P \big|_{>F+1})\)

**Proof.**

1. From what was said before, the map \(u \to u \big|_{\leq R}\) is 1–1 from \(\ker(L) \cap L^2(R^2)\) into \(\ker(L,P \big|_{<F-1})\). The inverse map is seen to be

\[
(B.8) \quad \ker(L,P \big|_{<F-1}) \ni v \to u = \begin{cases} v(r,\theta) & r < R \\ \sum_{k \in \mathbb{Z}} v_k \left(\frac{r}{R}\right)^{k-F} e^{\sqrt{-1}k\theta} & r \geq R \end{cases} \in L^2\ker(L)
\]

where \(v_k\) is the Fourier coefficient \(\frac{1}{2\pi} \int_0^{2\pi} v(R,\theta) e^{-\sqrt{-1}k\theta} \, d\theta\). The proof of \(b)\) is similar.
In order to bring the Atiyah-Patodi-Singer index theorem into the picture we need to take a different look at the operator (B.1) and at the boundary condition (B.6)

Let $D : C^\infty(\mathbb{R}^2, \mathbb{C}^2) \to C^\infty(\mathbb{R}^2, \mathbb{C}^2)$ be the Dirac operator on $\mathbb{R}^2$, i.e.,

$$(B.9) \quad D = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial x} + \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix} \frac{\partial}{\partial y}$$

Twisting the spinor bundle $\mathbb{R}^2 \times \mathbb{C}$ with the trivial bundle $\mathbb{R}^2 \times \mathbb{C}$ equipped with the connection given by the 1-form $\omega = \sqrt{-1} \frac{\partial \phi}{\partial y} \, dx - \sqrt{-1} \frac{\partial \phi}{\partial x} \, dy$ we get the corresponding Dirac operator

$$(B.10) \quad D_\phi = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial x} + \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix} \frac{\partial}{\partial y} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\partial \phi}{\partial x} + \begin{pmatrix} 0 & -\sqrt{-1} \\ -\sqrt{-1} & 0 \end{pmatrix} \frac{\partial \phi}{\partial y}$$

or equivalently $D_\phi = \begin{pmatrix} 0 & L^* \\ L & 0 \end{pmatrix}$. Now we restrict $D_\phi$ to $B_R$. It is the index of the associated twisted $\frac{1}{2}$-Dirac operator that relates nicely to $L^2$-index($L$). In fact, the boundary conditions described in (B.6) are identical to the spectral boundary condition introduced by Atiyah, Patodi, and Singer in [APS]. To see this, notice that from (B.4) the tangential part of $L$ is the selfadjoint elliptic operator

$\delta = \frac{1}{\sqrt{-1} \frac{\partial}{\partial \theta}} - F$

defined on $C^\infty(\partial B_R)$ and with eigenvalues, up to a scaling factor, $\{ k-F \}$, corresponding to the eigenfunctions $e^{\sqrt{-1}k\theta}$.

Let $P_+$ be the spectral projection on the positive eigenvalues of $\delta$, and take on $C^\infty(\partial B_R)$ the boundary condition $P_+v(R, \cdot)$. Then $(L, P_+)$ is exactly our previously defined $(L, P |_{\leq F})$.

**Theorem B.11.** — a) $\text{index}(L, P_+) = \dim \ker (L, P |_{\leq F}) - \dim \ker (L^+, P |_{> F+1})$

b) $\text{index}(L, P_+) = F + \frac{1}{2} - \frac{1}{2} (\eta_F(0) - \eta)$
where $\eta_F(0)$ is the eta invariant associated to $\delta$, i.e., the value at 0 of the analytic continuation of the $\eta$-function

\begin{equation}
\eta_F(s) = \sum_{k \in \mathbb{Z}, k \neq 0} \frac{\text{sgn}(k-F)}{|k-F|^s} \quad \text{Re}(s) >> 0
\end{equation}

and $h = \text{dim ker } \delta$.

**Proof.**—

a) Since $(L,P) = (L,P|_{<F_1})$, $a)$ is a consequence of the fact that the adjoint $(L,P)^*$ of $(L,P)$ is precisely $(L^t,P|_{>F+1})$. This follows right away from the integration by parts formula (Ch. I, 1.12 (v)), for $\Omega = B_R$ and $n = e^{\sqrt{-1} \theta}$.

b) We can also interpret $(L,P)$ as a twisted version of the $\bar{\partial}$-operator on a compact surface with boundary. Then the Atiyah-Patodi-Singer index for manifolds with boundary [APS], [EGH] gives:

\begin{equation}
\text{index}(L,P) = \frac{1}{2} \int_{B_R} c_1(T_{cB_R}) + \int_{B_R} c_1(\mathbb{R}^2 \times \mathbb{C}, \phi) - \frac{1}{2} (\eta_F(0) - h)
\end{equation}

where $c_1(T_{cB_R})$, respectively $c_1(\mathbb{R}^2 \times \mathbb{C}, \phi)$, is the first Chern class of the complex tangent bundle of $B_R$, respectively the twistor bundle $\mathbb{R}^2 \times \mathbb{C}$. Now the Chern form of a surface coincides with the Euler form, and so by the Gauss-Bonnet theorem, $\int c_1(T_{cB_R})$ is the Euler characteristic of $B_R$, i.e., 1. Also $c_1(\mathbb{R}^2 \times \mathbb{C}) = \frac{\sqrt{-1}}{2\pi} d\omega = \frac{1}{2\pi} \Delta \phi dxdy$. B.11 b) follows.

Note that the RHS of (B.13) should also contain the $\hat{A}$-genus of $B_R$ and a secondary characteristic class of $dB_R$, which accounts for the fact that geometrically $B_R$ in not a product near the boundary. These disappear since the $\hat{A}$-genus of any surface is 0.

Now we are ready to state the basic result:
**Theorem B.14.** — If $L$ is the operator in (B1), then

$$L^2\text{-index}(L) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \Delta \phi - \frac{\text{sgn } F}{2} - \frac{1}{2} [\eta_F(0) - \text{sgn}(F)h]$$

where $\eta_F(0)$ is the eta invariant associated to $\delta = \frac{1}{\sqrt{-1}} \frac{\partial}{\partial \theta} - F$ on $\mathcal{C}^\infty(S^1)$ and $h = \dim \ker \delta$.

**Proof.** — Assume first that $F < 0$. From (B.10) and (B.11,a)) we get

$$L^2\text{-index}(L) = \text{index}(L,P_+) - [\dim \ker (L,P|_{\leq F}) - \dim \ker (L,P|_{<F-1})]$$

Since $\ker (L,P|_{<F-1}) \subset \ker (L,P|_{\leq F})$, the claim is proved if we show that $\ker (L,P|_{\leq F}) = 0$.

Now (B3) says that an element in $\ker (L,P|_{\leq F})$ is essentially a holomorphic function, thus its Fourier series at the boundary must contain only $e^{\sqrt{-1}k\theta}$ with $k \geq 0$. But $P|_{\leq F}$ prevents this, if $F < 0$. We want to stress that $L^2\text{-index}(L) \neq \text{index}(L,P_+)$, if $F \geq 0$. The $F > 0$ case follows interchanging the roles of $L$ and $L^\dagger$ in all we said so far, and $F = 0$ is trivial.

A direct evaluation of the eta invariant $\eta_F(0)$ helps recover Aharonov-Casher’s result [AC].

**Proposition B.16.** — If $\eta_F(s) = \sum_{k \in \mathbb{Z}} \frac{\text{sgn } (k-F)}{|k-F|}^s \quad \text{Re}(s) >> 0$

then

$$\eta_F(0) = \begin{cases} 
2e-1 & \text{if } F > 0 \\
1-2\varepsilon & \text{if } F < 0 \\
0 & \text{if } F \in \mathbb{Z}
\end{cases}
$$

where $F = N + \varepsilon$, $N \in \mathbb{N}$, $0 < \varepsilon < 1$.

**Proof.** — See [G], or use the following regularization of the eta invariant:

$$\beta_F(t) = \sum_{k \in \mathbb{Z}} \text{sgn}(k-F) e^{-t|k-F|}, \quad \eta_F(0) = \lim_{t \to 0} \beta_F(t)$$
**Corollary B.19.**— *If* $L$ *is the operator in* (B1), *then we have* [AC]

\[
\text{L}^2\text{-index}(L) =
\begin{cases}
  N & \text{if } F > 0, \quad F = N + \varepsilon \quad N \in \mathbb{N} \quad 0 < \varepsilon < 1 \\
  N-1 & \text{if } F > 0 \quad F = N \quad N \in \mathbb{N} \\
  0 & \text{if } F = 0 \\
  -N & \text{if } F < 0, \quad -F = N + \varepsilon \quad N \in \mathbb{N} \quad 0 < \varepsilon < 1 \\
  -N+1 & \text{if } F < 0, \quad -F = N \quad N \in \mathbb{Z}
\end{cases}
\]

**Proof.**— Immediate, using Theorem B.14 and Proposition B.16.

**Remark B.21.**— A similar result can be proved for surfaces more general than $\mathbb{R}^2$, namely those with Euclidean ends. A noncompact Riemann surface $M$ is Euclidean at infinity if outside some compact set it is isometric to a disjoint union of finitely many $\mathcal{C}B_R$'s. Here the analog of $L$ is $\overline{\partial} + \partial\phi$. For instance, if $M$ has just one Euclidean end and $F < 0$, then

\[
\text{L}^2\text{-index}(\overline{\partial} + \partial\phi) = F + \frac{1-2g}{2} - \frac{1}{2}(\eta_F(0) - h)
\]

where $g$ is the genus of the compact Riemann surface obtained taking the one-point compactification of $M$. 
LIST OF REFERENCES


