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Robust controller design for large scale systems

İftar, Altuğ, Ph.D.
The Ohio State University, 1988

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ROBUST CONTROLLER DESIGN FOR LARGE SCALE SYSTEMS

A Dissertation
Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

by

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PUBLICATIONS


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CHAPTER I
INTRODUCTION

One of the fundamental issues in controller design is robustness. Since an exact model of a physical plant would be very complicated, if not impossible to obtain, the designer should base the controller design on a nominal model and should require the controller to perform satisfactorily under possible deviations from the nominal model. Here, stability is usually a prerequisite for satisfactory performance. Many other criteria, such as speed of response, low sensitivity to certain disturbances or parameter variations, or low energy consumption, may be used to judge the performance of a controlled system. In addition the plant output may be required to track a specified reference input. The problem of designing a controller for a plant such that the plant output tracks a specified reference input in the presence of certain disturbances and possible modeling errors is usually referred to as the robust servomechanism problem.

There has been two main approaches to the robust controller design problem. One is the frequency domain approach which is to design a controller by meeting certain frequency domain requirements to guarantee stability and good performance. The other approach is the time domain approach which advocates state space methods to design a controller that guarantees the same requirements.

The robust controller design problem is even more challenging for large scale systems (LSS). Large flexible structures (LFS), electric power systems, and socio-
economic systems are a few examples of such systems. These systems may require decentralized controllers for various reasons such as:

- the system may have a decentralized structure which makes it hard or impossible to implement feedback through a centralized location,
- information transfer may be costly so that the designer may wish to impose decentralized feedback,
- on-line centralized feedback calculations may be too time consuming so that simpler local feedback calculations may be preferred.

Furthermore, such systems are, in general, very complex and may be representable only by very high dimensional models. However, it may be possible to obtain much simpler local models for local design of decentralized controllers.

To obtain a local model, a decomposition of the large scale system is needed. In order to obtain a useful decomposition, it is essential to identify the parts of a system that are weakly interconnected. Many LSS may consist of subsystems which are strongly connected through certain dynamics ("the overlapping part"), but weakly connected otherwise. For such systems, it is natural to use an overlapping decomposition rather than a disjoint decomposition.

In this dissertation robust controller design for large scale systems is considered. Particular attention is given to the decomposition of such systems for the purposes of decentralized robust controller design. Both frequency domain and time domain approaches are discussed. Frequency domain methods for decentralized robust controller design are presented. Reduced order representation of uncertain dynamics is discussed. A unified approach to the robust controller design problem for systems which exhibit two classes of uncertainties (uncertain
dynamics and parameter uncertainties) is introduced. It is demonstrated that this state space approach may be useful in local controller design for LSS. Finally, the robust servomechanism problem is discussed. A linear–quadratic (LQ) optimal approach is presented to solve the centralized and decentralized robust servomechanism problems.

1.1 Organization of the Dissertation

Overlapping decomposition within the framework of inclusion principle is discussed in Chapter II. A new inclusion concept, called extension is introduced. The inclusion of cost functions within the same framework is also discussed.

Robust controller design in frequency domain is discussed in Chapter III. Frequency domain methods for decentralized robust controller design are presented.

State space approaches to the same problem are discussed in Chapter IV. It is shown that, under mild conditions, it is possible to obtain a rational transfer function matrix (TFM) to represent the uncertain dynamics. By using such a representation, a unified approach to the robust controller design problem for systems which exhibit two classes of uncertainties is introduced. Application of this approach to the decentralized robust controller design is also presented.

The robust servomechanism problem is considered in Chapter V. A linear–quadratic optimal approach is presented to solve the centralized and decentralized robust servomechanism problems.

Some concluding comments are presented in Chapter VI.
The notation used throughout this dissertation is explained below, where $A$, $B$, and $Q$ denote Hermitian matrices, $X$ denotes a complex vector, and $n$, $m$, and $k$ denote positive integers.

- $C$ : space of complex numbers
- $C^+$ : $\{ \sigma \in C | \Re(\sigma) \geq 0 \}$
- $C^-$ : $\{ \sigma \in C | \Re(\sigma) < 0 \}$
- $I$ : identity matrix of appropriate dimensions
- $I_k$ : $k \times k$ identity matrix
- $\Re(\cdot)$ : real part of $\cdot$
- $\mathbb{R}$ : space of real numbers
- $\mathbb{R}^n$ : $n$-dimensional real vector space
- $\mathbb{R}^{n \times m}$ : space of $n \times m$ real matrices
- $\mathbb{R}^+$ : $\{ \alpha \in \mathbb{R} | \alpha \geq 0 \}$
- $0$ : zero matrix of appropriate dimensions
- $0_{n \times m}$ : $n \times m$ zero matrix
- $\text{blockdiag}(D_1, ..., D_n)$ : block diagonal matrix with blocks $D_1, ..., D_n$ on the main diagonal
- $\det(\cdot)$ : determinant of $\cdot$
- $\text{rank}(\cdot)$ : rank of $\cdot$
- $\text{trace}(\cdot)$ : trace of $\cdot$
- $(\cdot)^{-1}$ : inverse of $\cdot$
- $(\cdot)^T$ : transpose of $\cdot$
- $(\cdot)^*$ : complex conjugate transpose of $\cdot$
- $A^S$ : $A^*A$
\[ ||X||_Q^2 \triangleq X^*QX \]
\[ |X| \triangleq \sqrt{X^*X} \]

\( \rho(\cdot) \) : spectrum of (\cdot)

\( \bar{\sigma}(\cdot) \) : maximum singular value of (\cdot)

\( \sigma(\cdot) \) : minimum singular value of (\cdot)

\( \sigma_i(\cdot) \) : \( i^{th} \) singular value of (\cdot) (in descending order)

\( A > B \) : \( A - B \) is positive definite

\( A < B \) : \( A - B \) is negative definite

\( A \geq B \) : \( A - B \) is positive semi-definite

\( A \leq B \) : \( A - B \) is negative semi-definite

\( / (\cdot) \) : angle of (\cdot)

\( |\cdot| \) : magnitude of (\cdot)

\( \tilde{\cdot} \) : complex conjugate of (\cdot)

\( \dot{\cdot} \) : time derivative of (\cdot)

\( \ddot{\cdot} \) : second time derivative of (\cdot)

\( (\cdot)^{(k)} \) : \( k^{th} \) time derivative of (\cdot)
CHAPTER II
INCLUSION PRINCIPLE AND OVERLAPPING
DECOMPOSITIONS

2.1 Introduction and Background

Many of today's technological and social problems involve so complex systems that it is very costly if not impossible to handle such large dimensional systems as a whole. For the purposes of analysis and control, it is usually necessary to decompose the given system into a number of interconnected subsystems. Once a decomposition is available, each subsystem is to be considered independently and the solutions to the subproblems are to be combined to obtain a solution for the original problem.

Many large scale systems may consist of subsystems which are strongly connected through certain dynamics but weakly connected otherwise. Electric power systems [1], socio-economic systems [2], large flexible structures [3], and freeway traffic regulation [4] are examples of such behaviour. For such systems, disjoint decompositions may easily fail to produce useful results. However, it has been demonstrated that the recently introduced overlapping decompositions [5] may produce useful solutions in such cases [5,6,7,8,9,10,11,12,13,14,15,16].

In the remaining of this section a review of previous work is given. In Section 2.2 a special case of inclusion, which is called extension, is introduced. The motivation behind introducing this new approach is that the presently available
approaches do not, in general, guarantee the contractibility (see Definition 2.9) of the control laws. However, since the control laws are to be designed in the expanded spaces and then contracted for implementation, it is important that any control law designed in the expanded spaces be contractible. The basic advantage of using extension is that any feedback control law designed by this approach is contractible for implementation on the actual system. The inclusion of cost functions in the framework of extension is discussed in Section 2.3. In Section 2.4 these results are applied to overlapping decompositions. The presented design approach is applied to two physical systems in Section 2.5.

2.1.1 Inclusion Principle

The inclusion principle for dynamic systems has been first introduced by Ikeda and Šiljak in the context of overlapping decompositions [5]. The theory of the principle has been elaborated in [9]. Consider the two linear time-invariant (LTI) systems:

\[
\Sigma : \quad \dot{x} = Ax + Bu \\
y = Cx
\]  
(2.1)

and

\[
\tilde{\Sigma} : \quad \dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}u \\
y = \tilde{C}\tilde{x}
\]  
(2.2)

where \( u \in \mathbb{R}^m \) and \( y \in \mathbb{R}^p \) are the input and the output common to both systems, \( x \in \mathbb{R}^n \) and \( \tilde{x} \in \mathbb{R}^{\tilde{n}} \) are the states of the systems \( \Sigma \) and \( \tilde{\Sigma} \) respectively. It is assumed that the order of \( \tilde{\Sigma} \) is larger (i.e., \( \tilde{n} > n \)).

**Definition 2.1** [9] The system \( \tilde{\Sigma} \) includes the system \( \Sigma \), and \( \Sigma \) is included by \( \tilde{\Sigma} \), if there exist transformations \( T : \mathbb{R}^n \rightarrow \mathbb{R}^{\tilde{n}} \) and \( T^\# : \mathbb{R}^{\tilde{n}} \rightarrow \mathbb{R}^n \), satisfying
rank(T) = n and \( T^\# T = I_n \), such that for any initial state \( x_o \in \mathbb{R}^n \) and any fixed input \( u(t) \) we have

\[
x(t; x_o, u) = T^\# \tilde{x}(t; Tx_o, u)
\] (2.3)

and

\[
y[x(t)] = y[\tilde{x}(t)]
\] (2.4)

for all \( t \geq 0 \).

It is also possible to consider certain special cases of inclusion. In particular the following two cases may be of some special interest.

**Definition 2.2** [9] The system \( \Sigma \) is a restriction of the system \( \hat{\Sigma} \), and \( \hat{\Sigma} \) is an unrestriction of \( \Sigma \), if there exist a transformation \( T : \mathbb{R}^n \rightarrow \mathbb{R}^{\hat{n}} \) satisfying \( \text{rank}(T) = n \), such that for any initial state \( x_o \in \mathbb{R}^n \) and any fixed input \( u(t) \) we have

\[
\tilde{x}(t; Tx_o, u) = Tx(t; x_o, u)
\] (2.5)

and

\[
y[x(t)] = y[\tilde{x}(t)]
\] (2.6)

for all \( t \geq 0 \). The system \( \Sigma \) is an aggregation [17] of the system \( \hat{\Sigma} \), and \( \hat{\Sigma} \) is a disaggregation of \( \Sigma \), if there exist a transformation \( T^\# : \mathbb{R}^{\hat{n}} \rightarrow \mathbb{R}^n \) satisfying \( \text{rank}(T^\#) = n \) such that for any initial state \( \tilde{x}_o \in \mathbb{R}^{\hat{n}} \) and any fixed input \( u(t) \) we have

\[
x(t; T^\# \tilde{x}_o, u) = T^\# \tilde{x}(t; \tilde{x}_o, u)
\] (2.7)

and
\( y[x(t)] = y[\tilde{x}(t)] \quad (2.8) \)

for all \( t \geq 0 \).

Under the transformations \( T \) and \( T^\# \), the matrices of the two systems given in (2.1) and (2.2) can be related as:

\[
\begin{align*}
\hat{A} &= TAT^\# + M, \\
\hat{B} &= TB + N, \\
\hat{C} &= CT^\# + L
\end{align*}
\]

where \( M, N, \) and \( L \) are complementary matrices with appropriate dimensions. These matrices must satisfy certain conditions for the inclusion to hold.

**Theorem 2.1** [9] The system \( \tilde{\Sigma} \) given in (2.2) includes the system \( \Sigma \) given in (2.1) if and only if

\[
\begin{align*}
T^\#M^iT &= 0, \\
T^\#M^iN &= 0, \\
LM^{i-1}T &= 0, \\
LM^{i-1}N &= 0
\end{align*}
\]

\( \forall i \in \{1, 2, ..., n\} \). \quad (2.10)

Similar conditions can also be derived for the special cases introduced in Definition 2.2.

**Theorem 2.2** [9] The system \( \Sigma \) given in (2.1) is a restriction of \( \tilde{\Sigma} \) given in (2.2) if and only if

\[
\begin{align*}
MT &= 0, \\
N &= 0, \\
LT &= 0
\end{align*}
\]

and \( \Sigma \) is an aggregation of \( \tilde{\Sigma} \) if and only if

\[
\begin{align*}
T^\#M &= 0, \\
T^\#N &= 0, \\
L &= 0
\end{align*}
\]

(2.12)

So far we have considered only the state inclusion. In fact, the inclusion principle has been extended to the input and output inclusion only recently [14,15]. Consider the system \( \Sigma \) given in (2.1) and the system
\[
\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}\tilde{u}, \\
\tilde{y} = \tilde{C}\tilde{x}
\]  

(2.13)

where \( \tilde{x} \in \mathbb{R}^{\tilde{n}} \), \( \tilde{u} \in \mathbb{R}^{\tilde{m}} \), and \( \tilde{y} \in \mathbb{R}^{\tilde{p}} \) are the state, the input, and the output of the system \( \tilde{\Sigma} \). It is assumed that \( \tilde{n} \geq n \), \( \tilde{m} \geq m \), and \( \tilde{p} \geq p \).

**Definition 2.3** [14] The system \( \tilde{\Sigma} \) includes the system \( \Sigma \), and \( \Sigma \) is included by \( \tilde{\Sigma} \).

If there exist transformations \( T \) and \( T^\# \) as in Definition 2.1 and transformations \( \tilde{T} : \mathbb{R}^m \to \mathbb{R}^{\tilde{m}} \) and \( \tilde{Y} : \mathbb{R}^{\tilde{p}} \to \mathbb{R}^p \) satisfying \( \text{rank}(\tilde{T}) = m \) and \( \text{rank}(\tilde{Y}) = p \), such that for any initial state \( x_0 \in \mathbb{R}^n \) and any fixed input \( u(t) \) we have

\[
x(t; x_0, u) = T^\# \tilde{x}(t; \tilde{T}x_0, \tilde{U}u)
\]

and

\[
y[x(t)] = \tilde{Y}\tilde{y}[\tilde{x}(t)]
\]

for all \( t \geq 0 \).

The necessary and sufficient conditions for the inclusion, as defined above, are given in the following theorem.

**Theorem 2.3** [14] The system \( \tilde{\Sigma} \) given in (2.13) includes the system \( \Sigma \) given in (2.1) if and only if

\[
A^i = T^\# \tilde{A}^i T, \quad A^i B = T^\# \tilde{A}^i \tilde{B} \tilde{U}, \quad \forall i \in \{0, 1, 2, ..., \}.
\]

\[
C A^i = \tilde{Y} \tilde{C} \tilde{A}^i T, \quad C A^i B = \tilde{Y} \tilde{C} \tilde{A}^i \tilde{B} \tilde{U}
\]

(2.16)

Generalizations of restriction and aggregation given in Definition 2.2 are also possible.
Definition 2.4 [14] The system $\Sigma$ is a restriction of $\tilde{\Sigma}$, and $\tilde{\Sigma}$ is an unrestricted version of $\Sigma$, if there exists transformations $T$ and $\tilde{U}$ as above, and $\tilde{Y}^\# : \mathbb{R}^p \to \mathbb{R}^n$ with $\text{rank}(\tilde{Y}^\#) = p$, such that for any initial state $\tilde{x}_0 \in \mathbb{R}^n$ and any fixed input $\tilde{u}(t)$ we have

$$\tilde{x}(t; T\tilde{x}_0, \tilde{U}\tilde{u}) = Tx(t; x_0, u) \tag{2.17}$$

and

$$\tilde{y}[\tilde{x}(t)] = \tilde{Y}^\# y[x(t)] \tag{2.18}$$

for all $t \geq 0$.

Definition 2.5 [14] The system $\Sigma$ is an aggregation of $\tilde{\Sigma}$, and $\tilde{\Sigma}$ is a disaggregation of $\Sigma$, if there exists transformations $T^\#$ and $\tilde{Y}$ as above, and $\tilde{U}^\# : \mathbb{R}^m \to \mathbb{R}^n$ with $\text{rank}(\tilde{U}^\#) = m$, such that for any initial state $\tilde{x}_0 \in \mathbb{R}^\tilde{n}$ and any fixed input $\tilde{u}(t)$ we have

$$x(t; T^\# \tilde{x}_0, \tilde{U}^\# \tilde{u}) = T^\# \tilde{x}(t; \tilde{x}_0, \tilde{u}) \tag{2.19}$$

and

$$y[x(t)] = \tilde{Y} y[\tilde{x}(t)] \tag{2.20}$$

for all $t \geq 0$.

The necessary and sufficient conditions for the restriction and aggregation, as defined above, are given in the following theorems.

Theorem 2.4 [14] The system $\Sigma$ given in (2.1) is a restriction of the system $\tilde{\Sigma}$ given in (2.19) if and only if

$$\tilde{A}T = TA, \quad \tilde{B}\tilde{U} = TB, \quad \tilde{C}T = \tilde{Y}^\# \tilde{C}. \tag{2.21}$$
Theorem 2.5 [14] The system $\Sigma$ given in (2.1) is an aggregation of the system $\hat{\Sigma}$ given in (2.13) if and only if
\[ T^\# \hat{A} = AT^\# , \quad T^\# \hat{B} = B \hat{U}^\# , \quad \hat{Y} \hat{C} = C \hat{T}^\# . \] (2.22)

2.1.2 Optimal Control

Optimal control approach within the framework of inclusion principle and overlapping decompositions has been first considered in [16]. Here we present a brief review of this approach.

Consider the linear feedback control laws:
\[ u = Kx \] (2.23)
for the system $\Sigma$ given in (2.1), and
\[ u = \tilde{K} \tilde{x} \] (2.24)
for the system $\hat{\Sigma}$ given in (2.2).

Definition 2.6 [16] The control law (2.24) is contractible to the control law (2.23) if
\[ Kx(t; x_0, u) = \tilde{K} \tilde{x}(t; T x_0, u) \quad \forall t \geq 0 , \] (2.25)
for any initial state $x_0 \in \mathbb{R}^n$ and any fixed input $u(t)$.

Under the transformations $T$ and $T^\#$ introduced in Definition 2.1, the two feedback gains (2.23) and (2.21) can be related by
\[ \tilde{K} = K T^\# + L_K \] (2.26)
where $L_K$ is a complementary matrix of proper dimension and must satisfy the following condition so that the contractibility holds.
Theorem 2.6 [16] The control law (2.24) is contractible to the control law (2.23) if and only if

\[ L_K M_i^{-1} T = 0, \quad L_K M_i^{-1} N = 0, \quad \forall i \in \{1, 2, \ldots, \bar{n}\} \]  

(2.27)

where \( M \) and \( N \) are the complementary matrices introduced in (2.9).

Consider the two cost functions:

\[ J = \int_0^\infty (x^T Q x + u^T R u) dt \]  

(2.28)

and

\[ \hat{J} = \int_0^\infty (\hat{x}^T \hat{Q} \hat{x} + \hat{u}^T \hat{R} \hat{u}) dt \]  

(2.29)

for the systems \( \Sigma \) and \( \hat{\Sigma} \) given in (2.1) and (2.2) respectively.

Definition 2.7 [16] The pair \((\hat{\Sigma}, \hat{J})\) includes the pair \((\Sigma, J)\) if there exist transformations \( T \) and \( T^\# \) as in Definition 2.1, such that for any initial state \( x_0 \in \mathbb{R}^n \) and any fixed input \( u(t) \) we have

\[ x(t; x_0, u) = T^\# \hat{x}(t; Tx_0, u) \]  

(2.30)

for all \( t \geq 0 \), and

\[ J(x_0, u) = \hat{J}(Tx_0, u). \]  

(2.31)

Let the matrices of the two systems be related as in (2.9) and the weighting matrices of the two cost functions be related by

\[ \hat{Q} = (T^\#)^T QT^\# + M_Q, \quad \hat{R} = R + N_R, \]  

(2.32)

where \( M_Q \) and \( N_R \) are complementary matrices of appropriate dimensions. Then we have the following result.
Theorem 2.7 [16] The pair \((\Sigma, \tilde{J})\) includes the pair \((\Sigma, J)\) if either

(i) \[ MT = 0 , \quad N = 0 , \quad T^T M Q T = 0 , \quad N_R = 0 , \quad (2.33) \]

or

\[ T^# M^i T = 0 , \quad T^# M^{i-1} N = 0 , \]

(ii) \[ M Q M^{i-1} T = 0 , \quad \forall i \in \{1, 2, ..., \tilde{n}\} . \quad (2.34) \]

\[ M Q M^{i-1} N = 0 , \quad N_R = 0 \]

It has to be noted that the conditions (i) and (ii) of the above theorem do not imply each other. Furthermore, if condition (i) is used, then the system \(\Sigma\) is a restriction of the system \(\tilde{\Sigma}\) (see Definition 2.2 and Theorem 2.2).

Let us now reconsider the contractibility of feedback gains. Note that the following result follows from Theorem 2.6.

Corollary 2.1 [16] If

\[ MT = 0 , \quad N = 0 , \quad (2.35) \]

then any control law of the form \((2.24)\) is contractible to a control law of the form \((2.23)\), and \(K\) is given by

\[ K = \tilde{K} T . \quad (2.36) \]

Therefore, if the expansion \((\tilde{\Sigma}, \tilde{J})\) is chosen by using the condition (i) of the Theorem 2.7, then any control law, in particular the optimal control law, designed for the expanded system \(\tilde{\Sigma}\) is contractible. Once such a contraction is obtained, it can be implemented on the actual system \(\Sigma\).
2.2 Control Law Design with Extension

In this section a special case of state and input inclusion is introduced. The basic advantage is that any feedback control law designed by this approach is contractible for implementation on the actual system. Consider the following LTI systems:

\[
\Sigma: \quad \dot{x} = Ax + Bu \tag{2.37}
\]

and

\[
\hat{\Sigma}: \quad \dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}\hat{u} \tag{2.38}
\]

Here \( x \in \mathbb{R}^n \), \( \hat{x} \in \mathbb{R}^{\hat{n}} \), \( u \in \mathbb{R}^m \), \( \hat{u} \in \mathbb{R}^{\hat{m}} \) are the states and the inputs of the systems \( \Sigma \) and \( \hat{\Sigma} \) respectively. It is assumed that \( \hat{n} \geq n \) and \( \hat{m} \geq m \). In the sequel the system \( \Sigma \) is referred to as the original system and \( \hat{\Sigma} \) is referred to as the expanded system. The state and input spaces \( \mathbb{R}^n \) and \( \mathbb{R}^m \) of \( \Sigma \) are called the original state and input spaces, and the spaces \( \mathbb{R}^{\hat{n}} \) and \( \mathbb{R}^{\hat{m}} \) of \( \hat{\Sigma} \) are called the expanded state and input spaces.

Consider the transformations:

\[
T: \mathbb{R}^n \to \mathbb{R}^{\hat{n}}, \quad \text{rank}(T) = n \tag{2.39a}
\]

\[
U: \mathbb{R}^{\hat{n}} \to \mathbb{R}^m, \quad \text{rank}(U) = m \tag{2.39b}
\]

\[
T\#: \mathbb{R}^{\hat{n}} \to \mathbb{R}^n, \quad T\#T = I_n \tag{2.39c}
\]

\[
U\#: \mathbb{R}^m \to \mathbb{R}^{\hat{m}}, \quad UU\# = I_m \tag{2.39d}
\]

**Definition 2.8** The system \( \hat{\Sigma} \) is an extension of the system \( \Sigma \), and \( \Sigma \) is a dis- extension of \( \hat{\Sigma} \), if there exist transformations as in (2.39a)–(2.39d) such that for any initial state \( x_0 \in \mathbb{R}^n \) and any fixed input \( \hat{u}(t) \in \mathbb{R}^{\hat{m}} \), \( 0 \leq t < \infty \), the choice
\[ \dot{x}_o = Tx_0 \quad (2.40a) \]

\[ u(t) = U\hat{u}(t) \quad \forall t \geq 0 \quad (2.40b) \]

implies that

\[ \dot{x}(t; x_o, \hat{u}) = Tx(t; x_o, u) \quad \forall t \geq 0. \quad (2.41) \]

The extension defined above is a special case of inclusion defined in Definition 2.3. In fact, it is a generalization of unrestriction given in Definition 2.2 to the case of state and input inclusion. However, it is different than the unrestriction defined in Definition 2.4. In Definition 2.4 unrestriction is defined for an arbitrary input \( u(t) \) in the original input space, and the input in the expanded space is obtained by a transformation: \( \hat{u}(t) = \tilde{U}u(t) \). Here, on the other hand, the extension is defined for an arbitrary input \( \hat{u}(t) \) in the expanded input space, and the input in the original space is obtained by the contraction given in (2.40b). Therefore, for the unrestriction the allowable set of inputs for \( \hat{\Sigma} \) at any time is only an \( m \)-dimensional subset of \( \mathbb{R}^m \), but for the extension it is \( \mathbb{R}^m \). However, this additional freedom brings in certain additional restrictions in the inclusion conditions which are provided by the following theorem.

**Theorem 2.8** The system \( \hat{\Sigma} \) is an extension of the system \( \Sigma \) if and only if there exist transformations as in (2.39a)-(2.39d) such that

\[ TA = \hat{A}T, \quad TBU = \hat{B} \quad (2.42) \]

**Proof:** The response of the two systems (2.37) and (2.38) are given by

\[ x(t) = e^{At}x_o + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \quad (2.13) \]

and
respectively. By substituting (2.40a) and (2.40b) for \( \tilde{x}_o \) and \( u(\tau) \), and using the series expansion for the matrix exponentials, it can be shown that (2.41) holds for any \( x_o \) and any \( \tilde{u}(t) \) if and only if

\[
\hat{A}^i T = TA^i
\]  

(2.45a)

and

\[
\hat{A}^i \tilde{B} = TA^iBU
\]  

(2.45b)

for all \( i \in \{0, 1, 2, \ldots \} \). For \( i = 0 \) (2.45a) holds trivially. For \( i = 1 \) (2.45a) holds and for \( i = 0 \) (2.45b) holds if and only if (2.42) holds. Furthermore, if (2.42) holds, then (2.45a) holds for \( i \geq 2 \) and (2.45b) holds for \( i \geq 1 \). Hence the result follows.

Under the transformations (2.39a)-(2.39d), the matrices of \( \Sigma \) and \( \tilde{\Sigma} \) can be related as:

\[
\tilde{A} = TA^# + M , \quad \tilde{B} = TBU + N
\]  

(2.46)

where \( M \) and \( N \) are constant complementary matrices. In order \( \tilde{\Sigma} \) to be an extension of \( \Sigma \), these complementary matrices must satisfy certain conditions provided by the following theorem.

**Theorem 2.9** The system \( \tilde{\Sigma} \) is an extension of the system \( \Sigma \) if and only if

\[
MT = 0 , \quad N = 0 .
\]  

(2.47)

**Proof:** By substituting (2.46) into (2.42) and using \( T^#T = I \), we obtain:

\[
TA + MT = TA , \quad TBU + N = TBU ,
\]  

(2.48)
which implies (2.47).

Next consider the linear feedback laws:

\[ u = Kx \tag{2.49} \]

for the system \( \Sigma \), and

\[ \ddot{u} = \ddot{K}\ddot{x} \tag{2.50} \]

for \( \hat{\Sigma} \). The following is a generalization of contractibility given in Definition 2.6 to the case of state and input inclusion.

**Definition 2.9** The control law (2.50) is contractible to the control law (2.49), if there exist transformations as in (2.39a) and (2.39b) such that

\[ Kx(t; x_0, U\ddot{u}) = U\ddot{K}\ddot{x}(t; Tx_0, \ddot{u}), \quad \forall t \geq 0, \tag{2.51} \]

for any initial state \( x_0 \in \mathbb{R}^n \) and any fixed input \( \ddot{u}(t) \in \mathbb{R}^m, 0 \leq t < \infty \).

The two feedback gain matrices in (2.49) and (2.50) can be related by:

\[ \ddot{K} = U^\#K^\# + L \tag{2.52} \]

where \( L \) is a constant complementary matrix, which must satisfy the following conditions so that the contractability holds.

**Theorem 2.10** Given that \( \hat{\Sigma} \) includes \( \Sigma \), the control law (2.50) is contractible to the control law (2.49) if and only if:

\[ UL\ddot{A}^i\dot{T} = 0 \tag{2.53a} \]

and

\[ UL\ddot{A}^i\dot{B} = KT^\#\dddot{A}^i\dot{B}(U^\#U - I) \tag{2.53b} \]
for all } i \in \{0, 1, 2, \ldots\} \).

**Proof:** By pre-multiplying (2.43) by } K \text{ and (2.44) by } U\hat{K}, substituting (2.40a) and (2.40b) for } \hat{x}_0 \text{ and } u(\tau), \text{ using the series expansions for the exponentials, and using the inclusion conditions for the dynamics and input matrices (first line of (2.16) with } \hat{U} = U\#), \text{ one arrives at (2.53a)-(2.53b).} \square

Since the control laws are to be designed in the expanded spaces and then contracted for implementation, it is important that any control law designed in the expanded spaces be contractible. In fact, if } \tilde{\Sigma} \text{ is an extension of } \Sigma \text{ then such a property holds.}

**Theorem 2.11** If } \tilde{\Sigma} \text{ is an extension of } \Sigma \text{ then any control law of the form (2.50) is contractible to a control law of the form (2.49), and } K \text{ is given by:}

\[
K = U\hat{K}T.
\]

**Proof:** By pre-multiplying (2.43) by } U\hat{K}T \text{ and (2.44) by } U\hat{K}, \text{ substituting (2.40a) and (2.40b) for } \hat{x}_0 \text{ and } u(\tau), \text{ using the series expansions for the exponentials, and using the conditions of extension (2.42), it is shown that (2.51) holds with } K = U\hat{K}T. \square

### 2.3 Inclusion of Cost Functions

Consider the cost functions:

\[
J = \int_0^\infty (x^TQx + u^TRu)dt
\]

(2.55)

and

\[
\tilde{J} = \int_0^\infty (\tilde{x}^T\tilde{Q}\tilde{x} + \tilde{u}^T\tilde{R}\tilde{u})dt
\]

(2.56)
for the systems $E$ and $E'$ given in (2.37) and (2.38) respectively. First let us consider a direct generalization of Definition 2.7 to the case of state and input inclusion.

**Definition 2.10** The pair $(\tilde{E}, \tilde{J})$ includes the pair $(E, J)$ if there exist transformations as in (2.39a)-(2.39d), such that $\tilde{E}$ includes $E$ in the sense of Definition 2.3, and

$$J(x_0, u\hat{u}) = \tilde{J}(Tx_0, \tilde{u}) \quad (2.57)$$

for any initial state $x_0 \in \mathbb{R}^n$ and any fixed input $\tilde{u}(t) \in \mathbb{R}^{\tilde{m}}$, $0 \leq t < \infty$.

It can easily be verified that the control weighting matrix in (2.56) must satisfy:

$$\text{rank}(\tilde{R}) \leq m$$

for a valid inclusion as defined above. This requires a non-positive-definite control weighting matrix for the expanded input whenever $\tilde{m} > m$. This property is certainly undesirable since the control law designed in the expanded spaces would be improper. To overcome this difficulty, we introduce a new inclusion concept as follows:

**Definition 2.11** Let $K$ and $\tilde{K}$ be such that $A + BK$ and $\tilde{A} + B\tilde{K}$ are asymptotically stable. Then the pair $(\tilde{E}, \tilde{J})$ includes the pair $(E, J)$ with respect to the feedback control laws $(Kx, \tilde{K}\tilde{x})$ if there exist transformations as in (2.39a)-(2.39d) such that $\tilde{E}$ includes $E$, $\tilde{K}\tilde{x}$ is contractible to $Kx$, and

$$J(x_0, Kx) = \tilde{J}(Tx_0, \tilde{K}\tilde{x}) \quad (2.58)$$

for any initial state $x_0 \in \mathbb{R}^n$. 

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Under the transformations (2.39a)-(2.39d), the matrices of $J$ and $\tilde{J}$ can be related as:

$$\tilde{Q} = (T\#)^TQT\# + M_Q, \quad \tilde{R} = U^TRU + N_R$$

(2.59)

where $M_Q$ and $N_R$ are constant complementary matrices.

Now let $\tilde{\Sigma}$ be an extension of $\Sigma$. Then, by Theorem 2.11, any feedback control law of the form (2.50) is contractible to a feedback control law of the form (2.19), and contracted feedback gains are given by (2.54). Under these assumptions, the necessary and sufficient conditions for the inclusion of cost functions are given by the following theorem.

**Theorem 2.12** If $\tilde{\Sigma}$ is an extension of $\Sigma$, $K = U\tilde{K}T$, and $\tilde{K}$ is such that $\tilde{A} + \tilde{B}\tilde{K}$ is asymptotically stable, then the pair $(\tilde{\Sigma}, \tilde{J})$ includes the pair $(\Sigma, J)$ with respect to $(Kx, \tilde{K}\tilde{x})$ if and only if

$$TT^T(M_Q + \tilde{K}^TN_R\tilde{K})T = 0.$$  

(2.60)

**Proof:** Since $\tilde{A} + \tilde{B}\tilde{K}$ is asymptotically stable the corresponding cost is finite and is given by [18]

$$J(Tx_0, \tilde{K}\tilde{x}) = x_0^TT\tilde{H}Tx_0$$

(2.61)

where $\tilde{H}$ is the solution of

$$(\tilde{A} + \tilde{B}\tilde{K})^T\tilde{H} + \tilde{H}(\tilde{A} + \tilde{B}\tilde{K}) + \tilde{Q} + \tilde{K}^T\tilde{R}\tilde{K} = 0.$$  

(2.62)

Furthermore, since $\tilde{\Sigma}$ is an extension of $\Sigma$, $A + BK$ is also asymptotically stable, and

$$J(x_0, Kx) = x_0^THx_0$$

(2.63)
where $H$ is the solution of

$$(A + BK)^T H + H(A + BK) + Q + K^T R K = 0. \quad (2.64)$$

By (2.61) and (2.63), (2.58) holds for all $x_o \in \mathbb{R}^n$ if and only if

$$T^T H T = H. \quad (2.65)$$

Substitute (2.46) and (2.54) in (2.62), multiply by $T^T$ from left and by $T$ from right, and use (2.47) to obtain

$$(A + BK)^T (T^T H T) + (T^T H T)(A + BK) + T^T QT + T^T K^T R K T = 0. \quad (2.66)$$

By comparing (2.64) and (2.66), (2.65) holds if and only if

$$T^T QT + T^T K^T R K T = Q + K^T R K. \quad (2.67)$$

By substituting (2.59) in (2.67) and noting that $T^T T = I$ and $K = U \tilde{K} T$, (2.60) is obtained.

By using the above theorem, it is always possible to construct a positive definite $\tilde{R}$ and a positive semi-definite $\tilde{Q}$ for given positive definite $R$ and positive semi-definite $Q$. However, the construction depends on the feedback gain matrix $\tilde{K}$. This dependence is in general undesirable, since the feedback gains to be designed depend on the choice of $\tilde{R}$ and $\tilde{Q}$. This difficulty can be resolved if the inclusion is defined with respect to the optimal control.

**Definition 2.12** Suppose that there exists a bounded control of the form (2.50) such that $\tilde{J}(\tilde{x}_o, \tilde{u})$ is minimized for all $\tilde{x}_o \in \mathbb{R}^n$. Then the pair $(\tilde{\Sigma}, \tilde{J})$ includes the pair $(\Sigma, J)$ with respect to the optimal if $\tilde{\Sigma}$ includes $\Sigma$, and

$$\min_{\tilde{u}} \tilde{J}(\tilde{x}_o, \tilde{u}) = \min_{\tilde{u}} \tilde{J}(T \tilde{x}_o, \tilde{u}) \quad (2.68)$$
for any initial state \( x_0 \in \mathbb{R}^n \).

**Theorem 2.13** If \( \hat{\Sigma} \) is an extension of \( \Sigma \) and there exists a bounded control of the form (2.50) such that \( \hat{J}(\hat{x}_0, \hat{u}) \) is minimized for any \( \hat{x}_0 \in \mathbb{R}^{\hat{n}} \), then the pair \( (\hat{\Sigma}, \hat{J}) \) includes the pair \( (\Sigma, J) \) with respect to the optimal if

\[
T^T M Q T = 0 \quad , \quad U(U^T R U + N_R)^{-1} U^T = R^{-1} . \tag{2.69}
\]

**Proof:** The control minimizing \( \hat{J} \) is given by \( \hat{u} = \hat{K} \hat{x} \) [18], where

\[
\hat{K} = -\hat{R}^{-1} \hat{B}^T \hat{P} \quad \tag{2.70}
\]

and \( \hat{P} \) is the positive semi–definite solution of

\[
\hat{A}^T \hat{P} + \hat{P} \hat{A} + \hat{Q} - \hat{P} \hat{R} \hat{R}^{-1} \hat{B}^T \hat{P} = 0 . \tag{2.71}
\]

Furthermore,

\[
\min_{\hat{u}} \hat{J}(T x_0, \hat{u}) = x_0^T T^T \hat{P} T x_0 . \tag{2.72}
\]

On the other hand, the control that minimizes \( J \) is given by \( u = K x \), where

\[
K = -R^{-1} B^T P \quad , \tag{2.73}
\]

\[
A^T P + PA + Q - PBR^{-1} B^T P = 0 , \tag{2.74}
\]

and

\[
\min_u J(x_0, u) = x_0^T P x_0 . \tag{2.75}
\]

By (2.72) and (2.75), (2.68) holds for any \( x_0 \in \mathbb{R}^n \) if and only if

\[
P = T^T \hat{P} T . \tag{2.76}
\]

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Substitute (2.46) in (2.71), multiply by $T^T$ from left and by $T$ from right, and use (2.47) to obtain

$$A^T(T^T\tilde{P}T)+(T^T\tilde{P}T)A + T^T\tilde{Q}T - (T^T\tilde{P}T)B(U\tilde{R}^{-1}U^T)B^T(T^T\tilde{P}T) = 0.$$ (2.77)

By comparing (2.74) and (2.77), (2.76) holds if

$$T^T\tilde{Q}T = Q, \quad U\tilde{R}^{-1}U^T = R^{-1},$$ (2.78)

which, by (2.59), is equivalent to (2.69).

Note that, it is always possible to construct symmetric matrices $M_Q$ and $N_R$ satisfying (2.69) such that $\tilde{Q}$ and $\tilde{R}$ defined by (2.59) are positive semi-definite and positive definite respectively. In fact, one may as well start with positive semi-definite and positive definite matrices $\tilde{Q}$ and $\tilde{R}$ and then construct $Q$ and $R$ by:

$$Q = T^T\tilde{Q}T$$ (2.79)

and

$$R = (U\tilde{R}^{-1}U^T)^{-1}.$$ (2.80)

2.4 Overlapping Decompositions

Consider the system $\Sigma$ given in (2.37). Suppose that the state and the input are partitioned as:

$$x = (x_1^T, x_2^T, x_3^T)^T, \quad x_i \in \mathbb{R}^{n_i},$$ (2.81a)

$$u = (u_1^T, u_2^T, u_3^T)^T, \quad u_i \in \mathbb{R}^{m_i}.$$ (2.81b)
Here it is assumed that \( x_2 \) and \( u_2 \) correspond to the overlapping parts of the state and input spaces respectively. Suppose the system matrices in (2.37) are also partitioned compatibly:

\[
A = \begin{bmatrix}
A_{11} & A_{12} & A_{13} \\
A_{21} & A_{22} & A_{23} \\
A_{31} & A_{32} & A_{33}
\end{bmatrix}, \quad
B = \begin{bmatrix}
B_{11} & B_{12} & B_{13} \\
B_{21} & B_{22} & B_{23} \\
B_{31} & B_{32} & B_{33}
\end{bmatrix}.
\tag{2.82}
\]

Then, an extension of \( \Sigma \) is obtained by choosing

\[
T = \begin{bmatrix}
I_n & 0 & 0 \\
0 & I_n & 0 \\
0 & 0 & I_n
\end{bmatrix}, \quad
U = \begin{bmatrix}
I_m & 0 & 0 & 0 \\
0 & I_m & I_m & 0 \\
0 & 0 & 0 & I_m
\end{bmatrix},
\]

\[
T^\# = \begin{bmatrix}
I_n & 0 & 0 & 0 \\
0 & \frac{1}{2}I_n & \frac{1}{2}I_n & 0 \\
0 & 0 & 0 & I_n
\end{bmatrix}, \quad
M = \begin{bmatrix}
0 & \frac{1}{2}A_{12} & -\frac{1}{2}A_{12} & 0 \\
0 & \frac{1}{2}A_{22} & -\frac{1}{2}A_{22} & 0 \\
0 & -\frac{1}{2}A_{22} & \frac{1}{2}A_{22} & 0 \\
0 & -\frac{1}{2}A_{32} & \frac{1}{2}A_{32} & 0
\end{bmatrix},
\tag{2.83}
\]

\[
N = 0.
\]

The resulting expanded system matrices are:

\[
\tilde{A} = \begin{bmatrix}
A_{11} & A_{12} & 0 & A_{13} \\
A_{21} & A_{22} & 0 & A_{23} \\
\cdots & \cdots & \cdots & \cdots \\
A_{21} & 0 & A_{22} & A_{23} \\
A_{31} & 0 & A_{32} & A_{33}
\end{bmatrix} \triangleq \begin{bmatrix}
\tilde{A}_1 & \tilde{A}_{12} \\
\tilde{A}_{21} & \tilde{A}_2
\end{bmatrix}, \tag{2.84a}
\]

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Next, consider the decoupled subsystems:

\[ \tilde{\Sigma}_i^D : \quad \ddot{x}_i = \tilde{A}_i \dot{x}_i + \tilde{B}_i \ddot{u}_i , \quad i = 1, 2. \] (2.85)

and corresponding cost functions:

\[ \tilde{J}_i = \int_0^\infty (\dot{x}_i^T \tilde{Q}_i \dot{x}_i + \dot{u}_i^T \tilde{R}_i \dot{u}_i) dt , \quad i = 1, 2 \] (2.86)

The total cost function for the expanded system is:

\[ J = J_1 + J_2 = \int_0^\infty (\dot{\tilde{x}}^T \tilde{Q} \dot{\tilde{x}} + \dot{\tilde{u}}^T \tilde{R} \dot{\tilde{u}}) dt \] (2.87)

where

\[ \tilde{Q} = \text{blockdiag}(\tilde{Q}_1, \tilde{Q}_2) , \quad \tilde{R} = \text{blockdiag}(\tilde{R}_1, \tilde{R}_2) . \] (2.88)

Consider the cost function (2.55) for \( \Sigma \). (\( \tilde{\Sigma}, \tilde{J} \)) includes (\( \Sigma, J \)), with respect to the optimal if \( Q \) and \( R \) satisfy respectively (2.79) and (2.80). In particular if

\[ Q = \text{blockdiag}(Q_1, Q_2, Q_3) , \quad R = \text{blockdiag}(R_1, R_2, R_3) \] (2.89)

where \( Q_i \in \mathbb{R}^{n_i \times n_i}, R_i \in \mathbb{R}^{m_i \times m_i} \), then \( \tilde{Q}_i \) and \( \tilde{R}_i (i = 1, 2) \) may be chosen as follows:

\[ \tilde{Q}_1 = \text{blockdiag}(\tilde{Q}_1, \tilde{Q}_2) , \quad \tilde{Q}_2 = \text{blockdiag}(\tilde{Q}_2, \tilde{Q}) \] (2.90)

where \( \tilde{Q}_2 \geq 0, \tilde{Q}_2 \geq 0, \) and \( \tilde{Q}_2 + \tilde{Q}_2 = Q_2 ; \)
\[
\dot{R}_1 = \text{blockdiag}(R_1, \dot{R}_2), \quad \dot{R}_2 = \text{blockdiag}(\dot{R}_2, R_3) \tag{2.91}
\]

where \( \dot{R}_2 > 0, \dot{R}_2 > 0, \) and \( \dot{R}_2^{-1} + \dot{R}_2^{-1} = R_2^{-1} \). In particular one may let:

\[
\dot{Q}_2 = \dot{Q}_2 = \frac{1}{2}Q_2, \quad \dot{R}_2 = \dot{R}_2 = 2R_2. \tag{2.92}
\]

Now, suppose that the control laws

\[
\tilde{u}_i = \tilde{K}^i \tilde{x}_i, \quad i = 1, 2 \tag{2.93}
\]

are designed (say to minimize \( J_i \)) for the decoupled subsystems \( \tilde{\Sigma}_i^D \), for \( i = 1, 2 \), respectively. Partition \( \tilde{K}^i \) as follows:

\[
\tilde{K}^i = \begin{bmatrix}
\tilde{K}^i_{11} & \tilde{K}^i_{12} \\
\tilde{K}^i_{21} & \tilde{K}^i_{22}
\end{bmatrix} \tag{2.94}
\]

where \( \tilde{K}^i_{11} \in \mathbb{R}^{m_1 \times n_1}; \tilde{K}^i_{22}, \tilde{K}^i_{11} \in \mathbb{R}^{m_2 \times n_2}; \tilde{K}^i_{22} \in \mathbb{R}^{m_3 \times n_3}. \) Then the feedback control law:

\[
\tilde{u} = \tilde{K} \tilde{x}, \quad \tilde{K} \triangleq \text{blockdiag}(\tilde{K}^1, \tilde{K}^2) \tag{2.95}
\]

is contracted to the original spaces as:

\[
u = Kx \tag{2.96}
\]

where

\[
K = \begin{bmatrix}
\tilde{K}^1_{11} & \tilde{K}^1_{12} & 0 \\
\tilde{K}^1_{21} & \tilde{K}^1_{22} + \tilde{K}^2_{11} & \tilde{K}^2_{12} \\
0 & \tilde{K}^2_{21} & \tilde{K}^2_{22}
\end{bmatrix} \tag{2.97}
\]

The implementation of this control strategy is summarized in Figure 1.
2.5 Applications

In this section two physical systems are considered. The first of these systems is a mass-spring system, which despite of its simplicity exhibits the general characteristics of interconnected systems. The second one is a large flexible structure consisting of a hub and twelve flexible ribs attached to it.

2.5.1 Mass–Spring System

The *overlapping decomposition approach with extension*, introduced above, is applied to a controller design problem for the mass-spring system shown in Figure 2. The objective is to stabilize the system and to minimize a given quadratic performance index at the same time. Two alternative design approaches are also considered and the results are compared. This system is composed of a *central body* of mass $M$ and two *outer bodies* of each with mass $m$. Forces $u_1$ and $u_2$
can be applied to the respective outer body from the central body and \( u_0 \) can be applied externally to the central body to control the system.

![Mass-spring system](image)

**Figure 2: Mass-spring system**

The dynamic equations governing this system are:

\[
m \ddot{q}_1 + k(q_1 - q_0) + k_0(q_1 - q_2) = u_1 \quad (2.98a)
\]

\[
M \ddot{q}_0 + \sum_{i=1}^{2} k(q_0 - q_i) = u_0 - u_1 - u_2 \quad (2.98b)
\]

\[
m \ddot{q}_2 + k(q_2 - q_0) + k_0(q_2 - q_1) = u_2 \quad (2.98c)
\]

where \( q_0 \) is the displacement of the central body and \( q_1 \) and \( q_2 \) are the displacements of the outer bodies. The system parameters are scaled so that \( M = 1 \) and \( k = 1 \).

The parameters \( \epsilon \) and \( \delta \) are defined such that:

\[
m = \epsilon^2 M \quad \text{and} \quad k_0 = \delta k .
\]

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The above dynamic equations can be written in the state space form as follows (see [19]):

\[ \dot{x}_1 = A_{11}x_1 + A_{12}x_0 + A_{13}x_2 + B_{11}u_1 \]  
(2.99a)

\[ \dot{x}_0 = A_{21}x_1 + A_{22}x_0 + A_{23}x_2 + B_{21}u_1 + B_{22}u_0 + B_{23}u_2 \]  
(2.99b)

\[ \dot{x}_2 = A_{31}x_1 + A_{32}x_0 + A_{33}x_2 + B_{33}u_2 \]  
(2.99c)

where

\[ x_0 = \begin{bmatrix} q_0 \\ \dot{q}_0 \end{bmatrix}, \quad x_1 = \begin{bmatrix} q_1 \\ \dot{q}_1 \end{bmatrix}, \quad x_2 = \begin{bmatrix} q_2 \\ \dot{q}_2 \end{bmatrix}, \]
(2.100a)

\[ A_{11} = A_{33} = \begin{bmatrix} 0 & 1 \\ -(1 + \delta) & 0 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}, \]
(2.100b)

\[ A_{12} = A_{21} = A_{23} = A_{32} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \]
(2.100c)

\[ A_{13} = A_{31} = \begin{bmatrix} 0 & 0 \\ \delta & 0 \end{bmatrix}, \]
(2.100d)

\[ B_{11} = B_{22} = B_{33} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad B_{21} = B_{23} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}. \]
(2.100e)

It is desired to find a controller for this system, such that the value of the following performance index:

\[ J = \int_0^\infty (x^T \dot{x} + u^T \dot{u}) dt, \]
(2.101)

where
\[
x = \begin{bmatrix} x_1 \\ x_0 \\ x_2 \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ u_0 \\ u_2 \end{bmatrix},
\]
is finite and as small as possible.

Three different decentralized controller design strategies are considered:

(i) Overlapping decomposition with extension as presented earlier in this chapter,

(ii) overlapping decomposition with state inclusion only [16], and

(iii) non-overlapping decomposition.

In the first approach, decentralized controllers are designed for each of the two overlapping subsystems:

\[
\dot{x}_i = \tilde{A}_i \tilde{x}_i + \tilde{B}_i \tilde{u}_i
\]
to minimize:

\[
\tilde{J}_i = \int_0^\infty (\tilde{x}_i^T \tilde{Q}_i \tilde{x}_i + \tilde{u}_i^T \tilde{R}_i \tilde{u}_i) dt , \quad i = 1, 2,
\]

where

\[
\tilde{A}_1 = \begin{bmatrix} \frac{1}{\varepsilon} A_{11} & \frac{1}{\varepsilon} A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad \tilde{A}_2 = \begin{bmatrix} A_{22} & A_{23} \\ \frac{1}{\varepsilon} A_{32} & \frac{1}{\varepsilon} A_{33} \end{bmatrix}, \quad (2.104a)
\]

\[
\tilde{B}_1 = \begin{bmatrix} \frac{1}{\varepsilon} B_{11} & 0 \\ B_{21} & B_{22} \end{bmatrix}, \quad \tilde{B}_2 = \begin{bmatrix} B_{22} & B_{23} \\ 0 & \frac{1}{\varepsilon} B_{33} \end{bmatrix}, \quad (2.104b)
\]

\[
\tilde{Q}_1 = \begin{bmatrix} I_2 & 0 \\ 0 & \frac{1}{2} I_2 \end{bmatrix}, \quad \tilde{Q}_2 = \begin{bmatrix} \frac{1}{2} I_2 & 0 \\ 0 & I_2 \end{bmatrix}, \quad (2.104c)
\]
Those controllers are then contracted to the original spaces for implementation as explained in the previous section.

In the second approach, the state space is decomposed as in the first approach. However, a non-overlapping decomposition is employed for the input space: \( u_1 \) and \( u_0 \) are assigned to the first subsystem and \( u_2 \) is assigned to the second subsystem. Hence, the following matrices are changed as indicated:

\[
\tilde{R}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad \tilde{R}_2 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}.
\]  

(2.104d)

In the third approach, a non-overlapping decomposition is employed. Decentralized controllers are designed for each of the decoupled subsystems:

\[
\begin{align*}
\epsilon \dot{x}_1 &= A_{11}x_1 + B_{11}u_1 \\
\dot{x}_0 &= A_{22}x_0 + B_{22}u_0 \\
\epsilon \dot{x}_2 &= A_{33}x_2 + B_{33}u_2
\end{align*}
\]  

(2.106a, 2.106b, 2.106c)

to minimize the corresponding sub-performance indices:

\[
J_i = \int_0^\infty (x_i^T x_i + u_i^T u_i) dt, \quad i = 1, 0, 2.
\]  

(2.107)

The above design approaches are carried out for various values of \( \epsilon \) and \( \delta \). In each case, the expected value of the resulting cost (2.101) over the initial conditions is calculated and compared with the optimal cost. It is assumed that the initial conditions are uniformly distributed on the unit sphere. The results are depicted in Figures 3–20.
Figure 3: $\mathcal{E}(J_{opt}/J)$, With extension, $\delta = 0$.

Figure 4: $\mathcal{E}(J_{opt}/J)$, With state inclusion only. $\delta = 0$. 
Figure 5: $\mathcal{E}(J_{opt}/J)$, With non-overlapping decomposition, $\delta = 0$.

Figure 6: $\mathcal{E}(J_{opt}/J)$, With extension, $\delta = 0.1$
Figure 7: $E(J_{opt}/J)$, With state inclusion only, $\delta = 0.1$

Figure 8: $E(J_{opt}/J)$, With non-overlapping decomposition, $\delta = 0.1$
Figure 9: $\mathcal{E}(J_{opt}/J)$, With extension, $\delta = 1.0$

Figure 10: $\mathcal{E}(J_{opt}/J)$, With state inclusion only, $\delta = 1.0$
Figure 11: $\mathcal{E}(J_{opt}/J)$, With non-overlapping decomposition, $\delta = 1.0$

Figure 12: $\mathcal{E}(J_{opt}/J)$, With extension, $\delta = 10$. 

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Figure 13: $\mathcal{E}(J_{\text{opt}}/J)$, With state inclusion only, $\delta = 10$.

Figure 14: $\mathcal{E}(J_{\text{opt}}/J)$, With non-overlapping decomposition, $\delta = 10$. 
Figure 15: $\mathcal{E}(J_{opt}/J)$, With extension, $\epsilon = 0.001$

Figure 16: $\mathcal{E}(J_{opt}/J)$, With state inclusion only, $\epsilon = 0.001$
Figure 17: $\epsilon(J_{opt}/J)$, With non-overlapping decomposition, $\epsilon = 0.001$

Figure 18: $\epsilon(J_{opt}/J)$. With extension, $\epsilon = 1.0$
Figure 19: $\varepsilon(J_{opt}/J)$, With state inclusion only, $\epsilon = 1.0$

Figure 20: $\varepsilon(J_{opt}/J)$, With non-overlapping decomposition, $\epsilon = 1.0$
From Figures 3-20, it is observed that the first approach, which employs input inclusion, yields better controllers (in the sense that the resulting cost is closer to the optimal cost) than the second approach, which employs only state inclusion. It can also be observed that overlapping decompositions (with or without input inclusion) yield better controllers than non-overlapping decomposition when the direct interactions between the outer systems are weak (i.e. when \( \delta \) is small). However, as such interactions get stronger, non-overlapping decomposition may yield better controllers.

From Figures 3-8, it is apparent that, if \( \delta \) is small, all of the decentralized controllers being considered here perform best whenever the mass of the central body is comparable with the mass of the outer bodies (i.e., whenever \( \epsilon \) is around unity). For the controller designed via input inclusion, degradation from the optimal cost is less than 20% when \( \epsilon = 1 \) and \( \delta = 0 \).

It can be observed from Figures 9-14 that, as \( \delta \) is increased beyond unity, non-overlapping decomposition performs better than overlapping decompositions for small \( \epsilon \). Note that, for large \( \delta \), both of the overlapping decomposition approaches being considered here fail to produce a stabilizing controller when \( \epsilon \) is around unity. However, they yield relatively good controllers for large values of \( \epsilon \). On the other hand, non-overlapping decomposition produces good controllers especially when \( \epsilon \) is around unity.

Figures 15-20 indicate that the performance of the decentralized controllers deteriorate most when \( \delta \) is in the range 1.-1000. If \( \delta \) is increased further, the resulting cost approaches to the optimal cost for small \( \epsilon \). For moderate \( \epsilon \) (0.5 \( \leq \) \( \epsilon \) \( \leq \) 10.), both of the overlapping decomposition approaches yield a destabilizing controller when \( \delta \) is large. However, non-overlapping decomposition approach yields near-optimal controllers in that range as well.
2.5.2 Large Flexible Experiment Structure

A decentralized controller design via overlapping decompositions is considered for the JPL/AFAL Large Flexible Experiment Structure shown in Figure 21. The structure consists of a hub and twelve flexible ribs which are attached to the hub radially. Torques can be applied to the hub externally by the hub actuators to control angular motion of the structure about the two gimbal axes. Rib actuators, located at the root of each rib, can apply forces to the respective ribs relative to the hub. Angular displacement of the hub (and its velocity) can be measured by the hub sensors. Leviation (and velocity) of the individual ribs at two different positions (inner and outer) can be measured respectively by the inner and outer rib sensors.

Here decentralized controllers are designed to control the motions of the two unaxial ribs, namely rib #1 and rib #7, and the motion of the hub in the same plane. The objective is to stabilize the system while minimizing a given cost functional.

The first three dominant modes are considered in the model. These modes together with the mode shape values at relevant locations are shown in Table 1. It can be observed that mode #1 is strongly controllable and observable from the hub actuator and hub sensor, and weakly controllable and observable from the rib actuator and sensors. Thus we use an overlapping decomposition in the following way:

- The model for the hub actuator/sensor pair includes only mode #1.

- The model for the rib actuator #1 and inner and outer rib sensors #1 includes modes #2 and 3.
Figure 21: JPL/AFAL Large Flexible Experiment Structure

Symbols:
- Sensor
- Actuator

HA: Hub Actuator
HS: Hub Sensor
RA: Rib Actuator
LI: Inner Rib Sensor
LO: Outer Rib Sensor
Table 1: Modes and mode shapes of the Large Flexible Experiment Structure

<table>
<thead>
<tr>
<th>Mode No.</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Circular wave no.</td>
<td>1</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>Axis</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Bending Group</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Frequency (rad/s)</td>
<td>0.56656</td>
<td>1.3126</td>
<td>1.5876</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Mode Shape Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rib Actuator # 1</td>
</tr>
<tr>
<td>Rib Actuator # 7</td>
</tr>
<tr>
<td>Hub Actuator</td>
</tr>
<tr>
<td>Inner Rib Sensor # 1</td>
</tr>
<tr>
<td>Inner Rib Sensor # 7</td>
</tr>
<tr>
<td>Outer Rib Sensor # 1</td>
</tr>
<tr>
<td>Outer Rib Sensor # 7</td>
</tr>
<tr>
<td>Hub Sensor</td>
</tr>
</tbody>
</table>

- The model for the rib actuator # 7 and inner and outer rib sensors # 7 includes modes # 2 and 3.

Hence for the hub we have

\[ \dot{x}_0 = A_0 x_0 + B_0 u_0 \]  \hspace{1cm} (2.108a)

\[ y_0 = C_0 x_0 \]  \hspace{1cm} (2.108b)

where \( u_0 \) is the control action of the hub actuator and
\[ y_0 = \begin{bmatrix} \text{hub angle in \# 1-7 plane} \\ \text{hub angle velocity in \# 1-7 plane} \end{bmatrix}. \]

For rib \# i (i = 1, 7) we have:

\[ \dot{x}_i = A_i x_i + B_i u_i \quad \text{(2.109a)} \]

\[ y_i = C_i x_i \quad \text{(2.109b)} \]

where \( u_i \) is the control action of the rib actuator \# i, and

\[ y_i = \begin{bmatrix} \text{Leviation of rib \# i at the inner position} \\ \text{Velocity of rib \# i at the inner position} \\ \text{Leviation of rib \# i at the outer position} \\ \text{Velocity of rib \# i at the outer position} \end{bmatrix}. \]

The overall performance index is chosen to be

\[ J = \int_0^\infty (x^T x + u^T u) dt \quad \text{(2.110)} \]

where \( x \) is the state of the original system that includes all three modes and \( u = (u_0, u_1, u_7)^T \). The subsystem indices for the decomposition given above are

\[ J_0 = \int_0^\infty (x_0^T x_0 + u_0^T u_0) dt \quad \text{(2.111a)} \]

and

\[ J_i = \int_0^\infty (\frac{1}{2} x_i^T x_i + u_i^T u_i) dt \quad \text{(2.111b)} \]

for \( i = 1, 7 \).

The optimal state feedback gains that minimize (2.111a) and (2.111b) with respect to the subsystem models (2.109a)-(2.109b) and (2.109a)-(2.109b) are given by
\[ K_0 = \begin{bmatrix} 0.11208 & 2.0190 \end{bmatrix} \]  

and

\[ K_1 = K_7 = \begin{bmatrix} -8.0963 \times 10^{-3} & 0.88892 & -0.010072 & -0.83567 \end{bmatrix}. \]

Since the output matrices \( C_i \) (\( i = 0, 1, 7 \)) are square and non-singular, an equivalent control is given by

\[ u_i = G_i y_i, \quad i = 0, 1, 7 \]

where

\[ G_i = K_i C_i^{-1}, \quad i = 0, 1, 7. \]

The application of the control law (2.113) results in an easy to implement decentralized output feedback structure. The resulting closed-loop eigenvalues are shown in Table 2 together with the open-loop eigenvalues. A centralized feedback control law that minimizes the index (2.110) is also calculated. The closed-loop eigenvalues under this optimal control are also listed in Table 2. The closeness of these eigenvalues to the closed-loop eigenvalues obtained under the proposed decentralized design approach proves the effectiveness of this design.

**Table 2: Eigenvalues of the Large Flexible Experiment Structure**

<table>
<thead>
<tr>
<th>Open-loop Eigenvalues</th>
<th>Closed-loop Eigenvalues Decentralized Design</th>
<th>Optimal Centralized Design</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \pm j0.5666 )</td>
<td>(-0.087271 \pm j0.56589)</td>
<td>(-0.073571 \pm j0.56899)</td>
</tr>
<tr>
<td>( \pm j1.3126 )</td>
<td>(-0.0036109 \pm j1.3126)</td>
<td>(-0.0036105 \pm j1.3126)</td>
</tr>
<tr>
<td>( \pm j1.5876 )</td>
<td>(-0.0045239 \pm j1.5876)</td>
<td>(-0.0045236 \pm j1.5876)</td>
</tr>
</tbody>
</table>
2.6 Summary

An overview of the previous work on the inclusion principle was given in Section 2.1. The inclusion principle forms the theoretical basis for the overlapping decompositions. A new inclusion concept, called extension, was introduced in Section 2.2. This new concept involves both state and input inclusion. It has been shown that with this approach any control law designed in the expanded spaces is always contractible to the original spaces for implementation.

The inclusion of the cost functions has also been discussed. Necessary and sufficient conditions for the inclusion of quadratic cost functions were derived in Section 2.3. It has been shown that, unlike the case when the two input spaces are identical, the inclusion conditions must, in general, depend on the control laws. However, certain sufficient conditions, which do not involve final control laws, can be found if the inclusion is taken with respect to the optimal control.

In Section 2.4, controller design with overlapping decompositions was discussed within the framework of the new inclusion concept and optimal control. Although systems with only two subsystems were considered for notational simplicity, the extension of the results to systems with more than two subsystems is possible along the same lines.

Possible applications of the presented design approach were illustrated on two physical systems in Section 2.5.
CHAPTER III

ROBUST CONTROLLER DESIGN IN FREQUENCY DOMAIN

3.1 Introduction

The classical trend in robust controller design is to use some frequency domain criterion to come up with a satisfactory controller. Such design methodologies have been widely used for single-input single-output (SISO) systems [20,21]. Frequency domain design approaches for multi-input multi-output (MIMO) systems have also been developed lately.

A quantitative measure of robustness is crucial for a meaningful robust controller design. A well known measure for minimum phase SISO systems is gain and phase margins. Although direct extension to multi-loop margins for MIMO systems is available, such measures do not in general reflect possible simultaneous variations in different loops. However, useful robustness measures can be defined in terms of singular values of certain transfer function matrices (TFMs) [22].

Consider a possibly MIMO system with a nominal TFM $G(s)$. The uncertainties of such a system can be represented at the input by $\Delta_i(s)$, at the output by $\Delta_o(s)$, or additively by $\Delta_a(s)$ as shown in Figure 22. For brevity, let us consider only uncertainties represented at the output, and drop the subscript "a" of $\Delta_a(s)$. Assume that all that is known about the uncertainty is an upper bound $\delta(\omega)$ on its norm:

$$\sigma(\Delta(j\omega)) \leq \delta(\omega) \quad \forall \omega \in \mathbb{R}. \quad (3.1)$$
Furthermore, suppose that the nominal TFM $G(s)$ and the true TFM $(I + \Delta(s))G(s)$ have the same number of poles in $\mathbb{C}^+$. Then, it has been shown that [23], a feedback controller $K(s)$, implemented as shown in Figure 23, guarantees the stability of the closed-loop system if the condition

$$\bar{\sigma}(T(j\omega)) < \frac{1}{\delta(\omega)} \quad \forall \omega \in \mathbb{R}$$

(3.2)

is satisfied. Here

$$T(s) \triangleq G(s)K(s)(I + G(s)K(s))^{-1}$$

(3.3)

is the closed-loop TFM for the nominal system.
The good performance of a feedback control system can also be analyzed by considering certain TFMs of the nominal system such as the return difference matrix $L(s) \triangleq I + G(s)K(s)$. For example, note that the TFM from the plant disturbance (represented at the output) to the plant output is given by $(I + G(s)K(s))^{-1} = L(s)^{-1}$. Hence, to achieve a certain degree of plant disturbance rejection at the output, one may require the condition

$$\sigma(L(j\omega)) \geq \alpha(\omega) \quad \forall \omega \in \Omega,$$

(3.4)

to be satisfied [24]. Here $\alpha(\omega)$ is a positive definite function for all $\omega \in \Omega$, and $\Omega \subseteq \mathbb{R}$ is the set of frequencies where the disturbances are effective.

It has to be pointed out that the above are constraint equations which guarantee robust stability and good performance. The controller design itself may be accomplished in one of many ways. In Section 3.3 some decentralized robust controller design strategies are introduced. However, first we discuss the decomposition of large scale systems for decentralized robust controller design in the next section. An application problem is considered in Section 3.4.

### 3.2 Decomposition of LSS for Decentralized Robust Controller Design

The overlapping decompositions discussed in the previous chapter plays an important role in the design of decentralized controllers for large scale systems (LSS). To avoid notational complexity, let us consider a system with only two decentralized control agents. Such a system may be described by the following model:

$$\dot{x} = Ax + B_1u_1 + B_2u_2,$$  \hspace{1cm} (3.5a)

$$y_1 = C_1x$$ \hspace{1cm} (3.5b)

$$y_2 = C_2x$$ \hspace{1cm} (3.5c)
where \( x \in \mathbb{R}^n \) is the state, \( u_i \in \mathbb{R}^{m_i} \) is the input, and \( y_i \in \mathbb{R}^{p_i} \) is the output of the \( i^{th} \) control agent \((i = 1,2)\). Let the state \( x \) be partitioned as follows:

\[
x = \begin{bmatrix}
x_1 \\
x_0 \\
x_2
\end{bmatrix}
\]  
(3.6)

where \( x_i \in \mathbb{R}^{n_i} \) \((i = 1,0,2)\). The partitioning is usually done in such a way that \( x_i \) corresponds to the part of the state space which is strongly observable and controllable only by the \( i^{th} \) control agent \((i = 1,2)\), and \( x_0 \) corresponds to the part which is strongly observable and controllable by both agents \([11,12]\). Also let the system matrices be partitioned compatibly:

\[
A = \begin{bmatrix}
A_{11} & A_{10} & A_{12} \\
A_{01} & A_{00} & A_{02} \\
A_{21} & A_{20} & A_{22}
\end{bmatrix}, \quad B = \begin{bmatrix}
B_{11} & B_{12} \\
B_{01} & B_{02} \\
B_{21} & B_{22}
\end{bmatrix},
\]

\[
C = \begin{bmatrix}
C_{11} & C_{10} & C_{12} \\
C_{21} & C_{20} & C_{22}
\end{bmatrix}.
\]

Consider the transformation:

\[
\dot{x} = T x = \begin{bmatrix}
x_1 \\
x_0 \\
x_2
\end{bmatrix} \in \mathbb{R}^\hat{n}, \quad \hat{n} \triangleq n + n_0. \tag{3.7}
\]

The expansion of the original system (3.5a)-(3.5c) with respect to the transformation (3.7) is given by:

\[
\dot{x} = \dot{A} \dot{x} + \dot{B} u, \quad \dot{x}(0) = T x(0) \in \mathbb{R}^\hat{n} \tag{3.8a}
\]
\[ y = \hat{C}\hat{x} \] (3.8b)

where

\[
\hat{A} = \begin{bmatrix}
A_{11} & A_{10} & 0 & A_{12} \\
A_{01} & A_{00} & 0 & A_{02} \\
A_{01} & 0 & A_{00} & A_{02} \\
A_{21} & 0 & A_{20} & A_{22}
\end{bmatrix} \triangleq \begin{bmatrix}
\hat{A}_1 & \hat{A}_{12} \\
\hat{A}_{21} & \hat{A}_2
\end{bmatrix},
\] (3.9a)

\[
\hat{B} = \begin{bmatrix}
B_{11} & B_{12} \\
B_{01} & B_{02} \\
B_{01} & B_{02} \\
B_{21} & B_{22}
\end{bmatrix} \triangleq \begin{bmatrix}
\hat{B}_1 & \hat{B}_{12} \\
\hat{B}_{21} & \hat{B}_2
\end{bmatrix},
\] (3.9b)

\[
\hat{C}' = \begin{bmatrix}
C_{11} & C_{10} & 0 & C_{12} \\
C_{21} & 0 & C_{20} & C_{22}
\end{bmatrix} \triangleq \begin{bmatrix}
\hat{C}_1 & \hat{C}_{12} \\
\hat{C}_{21} & \hat{C}_2
\end{bmatrix}.
\] (3.9c)

By using Theorem 2.1, it can be shown that the system (3.8a)-(3.8b) includes the system (3.5a)-(3.5c), and hence the two systems have equivalent input/output descriptions (see Definition 2.1), i.e.,

\[ \hat{G}(s) \triangleq \hat{C}(sI - \hat{A})^{-1}\hat{B} \equiv G(s) \triangleq C(sI - A)^{-1}B. \] (3.10)

Now consider the uncoupled expanded model described by:

\[
\dot{x} = \tilde{A}\tilde{x} + \tilde{B}u
\] (3.11a)

\[ y = \tilde{C}'\tilde{x} \] (3.11b)

where

\[ \tilde{A} = \begin{bmatrix}
\hat{A}_1 & 0 \\
0 & \hat{A}_2
\end{bmatrix}, \] (3.12a)
\[ \hat{B} = \begin{bmatrix} \hat{B}_1 & 0 \\ 0 & \hat{B}_2 \end{bmatrix}, \quad (3.12b) \]
\[ \hat{C} = \begin{bmatrix} \hat{C}_1 & 0 \\ 0 & \hat{C}_2 \end{bmatrix}. \quad (3.12c) \]

The TFM for the uncoupled expanded model is given by:
\[ \hat{G}(s) \triangleq \hat{C}(sI - \hat{A})^{-1}\hat{B} = \text{blockdiag}(G_1(s), G_2(s)) \quad (3.13) \]

where \( G_i(s) \) is the TFM of the local design model:
\[ \begin{align*}
\dot{x}_i &= \hat{A}_i x_i + \hat{B}_i u_i \\
y_i &= \hat{C}_i x_i,
\end{align*} \quad (3.14a) \]

for agent \( i \) (\( i = 1, 2 \)). Here it is assumed that the input and output matrices \( \hat{B}_i \) and \( \hat{C}_i \) (\( i = 1, 2 \)) are of the full rank.

Suppose \( m_i \leq p_i \) (\( i = 1, 2 \)) and let \( \hat{G}(s) \) and \( \tilde{G}(s) \) be related by
\[ \hat{G}(s) = \check{G}(s)(I + E(s)) \quad (3.15) \]

where \( E(s) \) is the multiplicative error matrix between the true TFM \( G(s) \) (or equivalently \( \hat{G}(s) \)) and the uncoupled expanded TFM \( \hat{G}(s) \). Since \( m_i \leq p_i \) and input and output matrices \( \hat{B}_i \) and \( \hat{C}_i \) (\( i = 1, 2 \)) are of the full rank, there exists a TFM \( \check{G}^I(s) \) such that \( \check{G}^I(s) \hat{G}(s) = I \) and \( \sigma(\check{G}^I(s)) = \frac{1}{\sigma(G(s))} \) [25]. Hence from (3.15) we obtain
\[ \sigma(E(s)) = \sigma \left( \check{G}^I(s)(\hat{G}(s) - \hat{G}(s)) \right) \leq \frac{\sigma(\hat{G}(s) - \hat{G}(s))}{\sigma(\hat{G}(s))}. \quad (3.16) \]

Therefore, an upper bound on the norm of \( E(j\omega) \) can be found as:
\[ \sigma(E(j\omega)) \leq \frac{\sigma(\hat{G}(j\omega) - \hat{G}(j\omega))}{\sigma(\hat{G}(j\omega))} \triangleq \epsilon(\omega). \quad (3.17) \]
If $m_i > p_i (i = 1, 2)$, it is possible to represent the multiplicative error at the output as

$$
\hat{G}(s) = (I + E(s))\tilde{G}(s),
$$

(3.18)

rather than at the input as in (3.15). In this case a similar argument would lead to the same result given in (3.17).

Here, one may also choose to represent the error due to interactions by a structured error matrix rather than the unstructured error matrix $E(s)$ considered above (e.g., see [26]). Such an approach would lead to bounds based on structured singular values [27]. Although such bounds may, in general, result in less conservative designs, they are harder to compute.

In practice, model (3.5a)-(3.5c) may not represent the physical system exactly, i.e., there may be some modeling uncertainties present. In such a case, a total error function, $e_m(\omega)$, must be defined, for example, as:

$$
e_m(\omega) = e(\omega)e_c(\omega) + e_0(\omega)
$$

(3.19)

to represent the error between the uncoupled expanded system model and the true system. Here, $e(\omega)$ stands for an upper bound on $\bar{\sigma}(E(j\omega))$, such as the one given in (3.17), $e_c(\omega) \geq 1$ accounts for modeling uncertainties in interactions, and $e_0(\omega) \geq 0$ accounts for uncertainties in subsystem models. Presumably $e_m(\omega)$ satisfies:

$$
e_m(\omega) \geq \bar{\sigma}(E_m(j\omega))
$$

(3.20)

where $E_m(s)$ is the multiplicative error matrix between the true system TFM and the design TFM $\hat{G}(s)$.

In the remaining of this section two special decompositions, which are of some practical importance, are considered.
3.2.1 Decentralized Design with Minimal Local Models

Consider the decentralized control system described by (3.5a)-(3.5c). For technical simplicity, assume that the system does not have any decentralized fixed modes (DFMs) [28]. Let the state $x$ be partitioned as follows:

$$x = \begin{bmatrix} x_{o1} \\ x_{oc1} \\ x_{oc} \\ x_{oc2} \\ x_{o2} & \end{bmatrix}$$  \hspace{1cm} (3.21)

where $x_{oi} \in \mathbb{R}^{noi}$ is observable only by agent–$i$, $x_{oci} \in \mathbb{R}^{noci}$ is observable by both agents, but controllable only by agent–$i$ ($i = 1, 2$), and $x_{oc} \in \mathbb{R}^{nocr}$ is observable and controllable by both agents. Note that the assumption of no DFMs implies that $x_{oi}$ is controllable by agent–$i$ ($i = 1, 2$). Then the matrices of (3.5a)-(3.5c) have the following forms [29]:

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & 0 \\ 0 & A_{22} & 0 & 0 & 0 \\ 0 & A_{32} & A_{33} & A_{34} & 0 \\ 0 & 0 & 0 & A_{44} & 0 \\ 0 & A_{52} & A_{53} & A_{54} & A_{55} \end{bmatrix}$$  \hspace{1cm} (3.22a)

$$B \triangleq \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & 0 \\ B_{31} & B_{32} \\ 0 & B_{42} \\ B_{51} & B_{52} \end{bmatrix}$$  \hspace{1cm} (3.22b)
\[ C \triangleq \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & 0 \\ 0 & C_{22} & C_{23} & C_{24} & C_{25} \end{bmatrix}. \]  

(3.22c)

Consider the transformation:

\[ \dot{x} = Tx = \begin{bmatrix} x_{o1} \\ x_{oc1} \\ x_{oc} \\ x_{oc2} \\ x_{o2} \end{bmatrix} \in \mathbb{R}^\hat{n}, \quad \hat{n} \triangleq n + n_{oc}. \]  

(3.23)

The expansion of the original system (3.5a)-(3.5c) with respect to the transformation (3.23) is given by (3.8a)-(3.8b) with

\[ \hat{A} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & 0 & A_{14} & 0 \\ 0 & A_{22} & 0 & 0 & 0 & 0 \\ 0 & A_{32} & A_{33} & 0 & A_{34} & 0 \\ 0 & A_{32} & 0 & A_{33} & A_{34} & 0 \\ 0 & 0 & 0 & 0 & A_{44} & 0 \\ 0 & A_{52} & 0 & A_{53} & A_{54} & A_{55} \end{bmatrix} \triangleq \begin{bmatrix} \hat{A}_1 & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_2 \end{bmatrix}, \]

\[ \hat{B} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & 0 \\ B_{31} & B_{32} \\ B_{31} & B_{32} \\ 0 & B_{42} \\ B_{51} & B_{52} \end{bmatrix} \triangleq \begin{bmatrix} \hat{B}_1 & \hat{B}_{12} \\ \hat{B}_{21} & \hat{B}_2 \end{bmatrix}. \]
\[
\dot{C} = \begin{bmatrix}
C'_{11} & C'_{12} & C'_{13} \\
0 & C'_{14} & 0 \\
C'_{22} & 0 & C'_{24} & C'_{25}
\end{bmatrix} \triangleq \begin{bmatrix}
\dot{C}'_1 \\
\dot{C}'_2
\end{bmatrix}.
\]

With the above decomposition, the local design models, given by (3.14a)-(3.14b), are indeed the minimal local models for the respective control agents.

An upper bound on the norm of the multiplicative error matrix \(E(j\omega)\) can be calculated by (3.17). If any part of the system (3.5a)-(3.5c) that is observable by agent-\(i\) is also controllable by agent-\(i\), then \(n_{oc_i} = 0\) (\(i = 1, 2\)) and we obtain

\[
\hat{A}_i = \hat{A}_i \triangleq \text{blockdiag} (\hat{A}_1, \hat{A}_2)
\]

and

\[
\dot{C}_i = \dot{C}_i \triangleq \text{blockdiag} (\dot{C}'_1, \dot{C}'_2), \quad i = 1, 2.
\]

Thus \(E(s)\) in (3.15) can be chosen to satisfy

\[
\hat{B} = \hat{B}(I + E(s)),
\]

where \(\hat{B} \triangleq \text{blockdiag} (\hat{B}_1, \hat{B}_2)\), or

\[
\begin{bmatrix}
0 & \hat{B}_{12} \\
\hat{B}_{21} & 0
\end{bmatrix} = \hat{B}E(s).
\]

Premultiplying both sides by \((\hat{B}^T \hat{B})^{-1} \hat{B}^T\), we obtain

\[
\begin{bmatrix}
0 & \left(\hat{B}_1^T \hat{B}_1\right)^{-1} \hat{B}_1^T \hat{B}_{12} \\
\left(\hat{B}_2^T \hat{B}_2\right)^{-1} \hat{B}_2^T \hat{B}_{21} & 0
\end{bmatrix} = E(s).
\]

Therefore, simplified bounds on the norm of \(E(j\omega)\) can be found as:

\[
\hat{\sigma}(E(j\omega)) = \max \left(\hat{\sigma}\left[\left(\hat{B}_1^T \hat{B}_1\right)^{-1} \hat{B}_1^T \hat{B}_{12}\right]\right), \quad \hat{\sigma}\left[\left(\hat{B}_2^T \hat{B}_2\right)^{-1} \hat{B}_2^T \hat{B}_{21}\right]
\]

\[
\triangleq \epsilon_1(\omega)
\]

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Although $\epsilon_1(\omega)$ is a tighter bound, in some cases, one may choose to use $\epsilon_2(\omega)$, since it does not require any matrix inversion. One of these two functions can be used as $\epsilon(\omega)$ in (3.19) to represent the interactions between the local design models.

3.2.2 Decentralized Design for Interconnected Systems

Many large scale systems (e.g., interconnected power systems, some large flexible space structures, socio-economic systems) are formed by the interconnection of a number of subsystems. An interconnected system with $\nu$ subsystems coupled through static and $\mu$ dynamic interconnections can be described by:

\[\begin{align*}
\dot{x}_i &= A_ix_i + \sum_{j=1, j \neq i}^{\nu+\mu} A_{ij}x_j + B_iu_i, & i = 1, \ldots, \nu \\
y_i &= C_ix_i, & i = 1, \ldots, \nu \\
\dot{x}_i &= A_ix_i + \sum_{j=1, j \neq i}^{\nu+\mu} A_{ij}x_j, & i = \nu + 1, \ldots, \nu + \mu.
\end{align*}\] (3.29)

One possible controller design strategy for such a system is to design a local controller for each subsystem considering only that subsystem and the dynamic interconnections that are strongly coupled with it. Examples of such design strategies can be found in many places (e.g., see [1]).

To avoid notational complexity, consider an interconnected system with only two subsystems coupled through a third subsystem which represents dynamic interconnections. Such a system can be compactly described by:

\[\begin{align*}
\dot{x} &= Ax + Bu \quad \text{(3.30a)} \\
y &= Cx \quad \text{(3.30b)}
\end{align*}\]
where

\[
\begin{align*}
x &= \begin{bmatrix} x_1 \\ x_c \\ x_2 \end{bmatrix}, \\
A &= \begin{bmatrix} A_1 & A_{1c} & A_{12} \\ A_{c1} & A_c & A_{c2} \\ A_{21} & A_{2c} & A_2 \end{bmatrix}, \\
B &= \begin{bmatrix} B_1 & 0 \\ 0 & 0 \\ 0 & B_2 \end{bmatrix}, \\
C &= \begin{bmatrix} C_1 & 0 & 0 \\ 0 & 0 & C_2 \end{bmatrix}.
\end{align*}
\]

\[
y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \\
u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.
\]

Here \( x_i \in \mathbb{R}^{n_i} \) is the state, \( u_i \in \mathbb{R}^{m_i} \) is the input and \( y_i \in \mathbb{R}^{p_i} \) is the output of the \( i^{th} \) subsystem \((i = 1, 2)\), and \( x_c \in \mathbb{R}^{nc} \) is the state for the dynamic interconnections.

To achieve a global control objective, a local controller is to be designed for each subsystem by considering the model:

\[
\begin{align*}
\dot{x}_i &= \hat{A}_i \dot{x}_i + \hat{B}_i u_i \quad (3.31a) \\
y_i &= \hat{C}_i \dot{x}_i, \quad i = 1, 2 \quad (3.31b)
\end{align*}
\]

which consists of the \( i^{th} \) subsystem and the dynamic interconnections. Thus:

\[
\begin{align*}
\hat{A}_1 &= \begin{bmatrix} A_1 & A_{1c} \\ A_{c1} & A_c \end{bmatrix}, \\
\hat{A}_2 &= \begin{bmatrix} A_c & A_{c2} \\ A_{2c} & A_2 \end{bmatrix}, \\
\hat{B}_1 &= \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \\
\hat{B}_2 &= \begin{bmatrix} 0 \\ B_2 \end{bmatrix}, \\
\hat{C}_1 &= \begin{bmatrix} C_1 & 0 \end{bmatrix}, \\
\hat{C}_2 &= \begin{bmatrix} 0 & C_2 \end{bmatrix}.
\end{align*}
\]
It is assumed that \((\hat{A}_i, \hat{B}_i, \hat{C}_i)\) represents a stabilizable and detectable system \((i = 1, 2)\).

Consider the transformation:

\[
\dot{x} = Tx = \begin{bmatrix}
  x_1 \\
  x_c \\
  \ldots \\
  x_c \\
  x_2
\end{bmatrix} = \begin{bmatrix}
  \dot{x}_1 \\
  \dot{x}_2
\end{bmatrix}
\]  

(3.32)

The expansion of the system (3.30a)–(3.30b) with respect to the transformation (3.32) is given by (3.8a)–(3.8b) with

\[
\hat{A} = \begin{bmatrix}
  \hat{A}_1 & \hat{A}_{12} \\
  \hat{A}_{21} & \hat{A}_2
\end{bmatrix}, \\
\hat{B} = \begin{bmatrix}
  \hat{B}_1 & 0 \\
  0 & \hat{B}_2
\end{bmatrix},
\]

\[
\hat{C} = \begin{bmatrix}
  \hat{C}_1 & 0 \\
  0 & \hat{C}_2
\end{bmatrix},
\]

where

\[
\hat{A}_{12} = \begin{bmatrix}
  0 & A_{12} \\
  0 & A_{21}
\end{bmatrix}, \\
\hat{A}_{21} = \begin{bmatrix}
  A_{c1} & 0 \\
  A_{21} & 0
\end{bmatrix}.
\]

With the above decomposition, the multiplicative error matrix \(E(s)\) can be chosen to satisfy:

\[
(sI - \hat{A})^{-1}\hat{B} = (sI - \hat{A})^{-1}\hat{B}(I + E(s))
\]  

(3.33)

where

\[
\hat{A} \triangleq \text{blockdiag}(\hat{A}_1, \hat{A}_2).
\]

Premultiply (3.33) by \((sI - \hat{A})\) and rearrange terms to obtain:
\[
\begin{bmatrix}
0 & G_{12}(s) \\
G_{21}(s) & 0
\end{bmatrix} = H(s)E(s)
\] (3.34)

where

\[
G_{ij}(s) \triangleq \hat{A}_{ij}(sI - \hat{A}_j)^{-1}B_j, \quad i = 1, 2, \quad j \neq i
\] (3.35)

is the TFM from the \(j^{th}\) input to the \(i^{th}\) subsystem, and

\[
H(s) \triangleq \begin{bmatrix}
\hat{B}_1 & -G_{12}(s) \\
-G_{21}(s) & \hat{B}_2
\end{bmatrix}.
\] (3.36)

Then we obtain the following bound on the norm of \(E(j\omega)\):

\[
\bar{\sigma}(E(j\omega)) \leq \frac{\max (\bar{\sigma}(G_{12}(j\omega)), \bar{\sigma}(G_{21}(j\omega)))}{\sigma(H(j\omega))} \triangleq c_3(\omega).
\] (3.37)

Note that, from (3.33), we can also obtain

\[
\hat{B}E(s) = (sI - \hat{A})(sI - \hat{A})^{-1}\hat{B} - \hat{B},
\] (3.38)

which leads to

\[
\bar{\sigma}(E(s)) \leq \frac{\bar{\sigma}(sI - \hat{A})\frac{1}{\sigma(sI - \hat{A})} \bar{\sigma}(\hat{B}) + \bar{\sigma}(\hat{B})}{\sigma(\hat{B})}.
\] (3.39)

Therefore,

\[
c_4(\omega) \triangleq \left[ \max_{i=1,2} \left( \bar{\sigma}(j\omega I - \hat{A}_i) + \sigma(j\omega I - \hat{A}) \right) \right] \left[ \max_{i=1,2} \left( \bar{\sigma}(\hat{B}_i) \right) \right] \sigma(j\omega I - \hat{A})_{i=1,2} \left( \sigma(\hat{B}_i) \right)
\] (3.40)

is also an upper bound for \(\bar{\sigma}(E(j\omega))\). Although \(c_3(\omega)\) is, in general, a tighter bound, in many cases it may be easier to compute \(c_4(\omega)\) rather than \(c_3(\omega)\). Either \(c_3(\omega)\) or \(c_4(\omega)\) can be used as \(\epsilon(\omega)\) in (3.19) to represent the interactions between the two subsystems.
3.3 Decentralized Robust Controller Design Strategies for LSS

Many frequency domain approaches to design controllers for multivariable systems have been proposed recently. Quantitative feedback theory (QFT) [30] and $H^\infty$ optimization [31,32] are the two well known examples of such approaches. QFT requires a trial and error design, and hence, designing decentralized controllers with such an approach may require an explosive number of redesigns. The key point in $H^\infty$ optimization is the parametrization of all stabilizing controllers which was developed by Youla et al. [33]. Hence, a direct application of $H^\infty$ optimization would be possible for decentralized systems, if a parametrization of all stabilizing decentralized controllers could be found. Although a constrained parametrization has been reported recently [34], a free parametrization that is suitable for optimization has not been found to date.

In the following, a local design approach is proposed. It is shown that a loop shaping approach can be applied locally. In particular, stability-robustness and good performance requirements can be tested by the local agents. A local design approach, based on the LQG/LTR methodology [23] is also introduced.

3.3.1 Local Design with Loop Shaping

Once a total error function such as (3.19) and a local design model like (3.14a)-(3.14b) are available to each control agent, a local design can be carried out at each local station.

Suppose that a decentralized feedback controller $K_i(s)$ is designed at each local station. Then the overall closed-loop design TFM is given by

$$T(s) = \text{blockdiag}(T_1(s), T_2(s)),$$

where

$$T(s) = \text{blockdiag}(T_1(s), T_2(s)),$$  \hspace{1cm} (3.11)
\[ T_i(s) \triangleq G_i(s)K_i(s)(I + G_i(s)K_i(s))^{-1} \quad (3.12) \]

is the local closed-loop TFM for the \( i^{th} \) agent \((i = 1, 2)\). Due to this block diagonal structure of the design model, stability-robustness condition (3.2) can be written as

\[ \sigma(T_i(j\omega)) < \frac{1}{\epsilon_n(\omega)} \quad \forall \omega \in \mathbb{R}, \quad \forall i \in \{1, 2\}. \quad (3.43) \]

Similarly, the good performance requirement (3.4) can be stated as

\[ \sigma(L_i(j\omega)) \geq \alpha(\omega) \quad \forall \omega \in \Omega, \quad \forall i \in \{1, 2\}, \quad (3.44) \]

where \( L_i(s) \triangleq I + G_i(s)K_i(s) \) is the \( i^{th} \) local return difference matrix.

Therefore, both the stability-robustness condition and the good performance requirement can be tested locally. If the local controllers \( K_i(s) \) are designed to satisfy these local conditions, and the true system in the expanded state space and the design model (3.11a)-(3.11b) have the same number of unstable eigenvalues, then the overall closed-loop system is guaranteed to be stable and satisfy the desired performance criterion.

In designing the local controllers any design approach can be undertaken. One possible approach is presented in the following subsection.

### 3.3.2 Decentralized LQG/LTR Design

It is well known that linear-quadratic (LQ) full state feedback regulators enjoy certain robustness properties [35]. However, it has been shown that such properties may not hold for linear-quadratic-Gaussian (LQG) regulators (with a Kalman filter in the loop) [36]. The linear-quadratic-Gaussian with loop-transfer-recovery (LQG/LTR) design methodology has been recently introduced by Doyle and Stein [23] to recover the robustness properties of LQ regulators.
The LQG/LTR methodology involves two stages. First stage is the design of a target feedback loop (TFL) by considering either a linear-quadratic regulator (LQR) or a Kalman-Bucy filter (KBF) problem. The design parameters in this first stage are chosen to satisfy the stability-robustness requirement (3.2) and the good performance criterion (3.4) for the TFL. In the second stage a KBF (if an LQR problem was considered in the first stage) or an LQR (otherwise) problem is solved to recover the TFL performance. The recovery, and hence robust stability and good performance, are guaranteed if the given system is minimum phase.

Decentralized LQG/LTR controller design approach is a local design methodology which, likewise its centralized counterpart, involves two stages. Assuming that the number of outputs is not larger than the number of inputs at a control station (i.e., \(m_i \leq p_i\)), the first stage is to design a local target feedback loop (local-TFL) as shown in Figure 24.

![Figure 24: Local Target Feedback Loop](image)

The local control gain \(K_i\) is determined by considering a LQR problem and is given by:

\[
K_i = \frac{1}{\rho_i} \hat{B}_i^T P_i ,
\]

where \(P_i = P_i^T > 0\) is the solution of:

\[
\hat{A}_i^T P_i + P_i \hat{A}_i + L_i^T L_i - \frac{1}{\rho_i} P_i \hat{B}_i \hat{B}_i^T P_i = 0 .
\]
The design parameters $\rho_i \in \mathbb{R}$, $\rho_i > 0$ and $L_i \in \mathbb{R}^{m_i \times n_i}$ are chosen such that the stability-robustness requirement (3.43) and a performance criterion, which may be expressed as (3.44) are satisfied for the local TFL. The identity

$$\sigma_j \left( I + G_{K_i}(j\omega) \right) = \sqrt{1 + \frac{1}{\rho_i} \left( \sigma_j \left[ L_i(j\omega I - \hat{A}_i)^{-1}\hat{B}_i \right] \right)^2},$$

for $j = 1, \ldots, n_i$, may be useful in choosing these parameters. Here

$$G_{K_i}(s) \triangleq K_i(sI - \hat{A}_i)^{-1}\hat{B}_i$$

is the loop TFM for the $i^{th}$ local TFL.

The second stage is to design a local Kalman filter with a filter gain:

$$H_i = \frac{1}{\mu_i} \Pi_i \hat{C}_i^T,$$

where $\Pi_i = \Pi_i^T > 0$ is the solution of:

$$\hat{A}_i \Pi_i + \Pi_i \hat{A}_i^T + \hat{B}_i \hat{B}_i^T - \frac{1}{\mu_i} \Pi_i \hat{C}_i^T \hat{C}_i \Pi_i = 0.$$ 

The parameter $\mu_i$ must be chosen small enough to achieve desired performance recovery.

If the number of inputs is larger than the number of outputs (i.e., $m_i > p_i$) at any control station, then the above design procedure must be modified for that station. First a local Kalman filter design must be carried out to come up with a satisfactory local TFL. Then the performance of the TFL can be recovered by solving a LQR problem. If the number of the inputs and the outputs are the same, then any one of the two sequences can be used.

The resulting local controllers are described by

$$u_i = -K_i z_i \quad (3.51a)$$

$$\dot{z}_i = F_i z_i - H_i \epsilon_i \quad (3.51b)$$

$$\epsilon_i = r_i - y_i, \quad i = 1, 2 \quad (3.51c)$$
where $F_i \triangleq \dot{A}_i - \dot{B}_iK_i - H_i\dot{C}_i$, and $r_i$ is the $i^{th}$ command input. The closed loop system is shown in Figure 25.

![Diagram of a decentralized plant controlled by local LQC/LTR controllers.](image-url)

**Figure 25: Decentralized Plant Controlled by Local LQC/LTR Controllers**

The presented approach guarantees overall stability and good performance only when all the local models have only minimum phase zeros. However, the methodology can still be applied to systems with nonminimum phase zeros and may produce very satisfactory results in many cases. Several options for dealing with nonminimum phase problems were suggested in [37]. One of these options (as appropriate) may be adopted for a local design whenever the corresponding local model has nonminimum phase zeros.

### 3.4 Application

In this section decentralized robust controller design examples for a large space structure are considered. The structure under consideration is the COFS Mast Flight System [38]. The basic element of the system is a 60.7 meter long, triangular cross section truss structure. The truss has 54 bays and the linear direct
current motor (LDCM) actuators are distributed along it. A schematic diagram of the structure as deployed on the space shuttle is shown in Figure 26.

In the present example decentralized controllers are designed to control the x–z bending plane dynamics. Since the two bending plane and the torsional dynamics are decoupled, similar controllers can be designed for the y–z bending plane and the torsional dynamics, and these controllers can be implemented without affecting each other. The design objective is to robustly stabilize the system and to introduce a certain amount of damping in lower frequency modes. The modes and mode shapes used in this study are listed in Table 3.

Table 3: Modes and Mode Shapes for the COFS Mast Flight System Example

<table>
<thead>
<tr>
<th>Mode</th>
<th>Mode Shape</th>
<th>Description</th>
<th>Freq. (Hz)</th>
<th>Bay 28</th>
<th>Tip</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) 1st x-z</td>
<td></td>
<td>0.187</td>
<td>.0149</td>
<td>.0486</td>
<td></td>
</tr>
<tr>
<td>(4) 2nd x-z</td>
<td></td>
<td>1.36</td>
<td>.0647</td>
<td>-.0243</td>
<td></td>
</tr>
<tr>
<td>(7) 3rd x-z</td>
<td></td>
<td>3.93</td>
<td>.0111</td>
<td>.0167</td>
<td></td>
</tr>
<tr>
<td>(10) 4th x-z</td>
<td></td>
<td>6.84</td>
<td>-.0567</td>
<td>-.0107</td>
<td></td>
</tr>
</tbody>
</table>

Two LDCM actuators, one at the tip and one at bay–28, and co-located accelerometers are used for control and measurement purposes. Each of the actuators with their associated (inner-loop) compensators can be described by a fourth order model:

\[
\dot{x}_c = A_c x_c + B_{c1} \delta_c + B_{c2} p \\
\dot{f} = C_c x_c + D_{c1} \delta_c + D_{c2} p
\]  

where
Figure 26: COFS Mast Flight System
In the above, $\delta_c$ is the control input, $p$ is the displacement of the structure (in the $x$ direction) at the location of the actuator, and $f$ is the force output of the actuator as applied to the structure. The numerical values for the actuator/inner-loop-compensator parameters are, $\tau = 9.9298 \, \text{s}^{-1}$, $g_1 = 3$, $g_2 = 6$, $K_0 = 83.333$ and $m = 11.4 \, \text{kg}$.

A two-mode model is used to describe the $x$-$z$ bending plane dynamics of the structure:

$$x_s = A_s x_s + B_s f + B_s^{28} f^{28}$$  \hspace{1cm} (3.59)
\[
p^i = \Gamma^i x^i,
\]
\[
y^i = \bar{p}^i = C^i x^i + D^i f^i + D^i_{28} f_{28}
\]

where \( i \) takes on the values "t" and "28" and these denote the quantities at the tip and bay 28 locations respectively.

The equations given above can now be combined to give a state-space model with 12 states. In doing this 0.2 and 0.3% structural damping was assumed for the first and second modes respectively.

The overall dynamics is decomposed into two overlapping portions for the purpose of controller design. The dynamics of the individual actuators are assigned to separate subsystems and the two-mode structural model is kept in the overlapping portion. Thus the subsystem states are described by:

\[
\dot{x}_1 = \begin{bmatrix} x^t_c \\ x^s \\ x^2_{28} \end{bmatrix}, \quad \dot{x}_2 = \begin{bmatrix} x^s \\ x^2_{28} \end{bmatrix}.
\]

An upper bound on the error introduced due to the interactions between the subsystems is determined as

\[
\epsilon(\omega) = \frac{\omega}{(\omega^2 + 1)^{1/2} \left( \left( \frac{\omega}{10} \right)^2 + 1 \right)^{1/2}}.
\]

Furthermore, it is assumed that the modeling uncertainties can be represented by:

\[
\epsilon_c(\omega) = \left( \left( \frac{\omega}{25} \right)^2 + 1 \right)^{1/2}
\]

and

\[
\epsilon_0(\omega) = \frac{\omega \left( \left( \frac{\omega}{10} \right)^2 + 1 \right)^{1/2}}{\left( 50\omega^2 + 1 \right)^{1/2}}.
\]

These functions, together with the final \( \epsilon_m(\omega) \) given by (3.19), are plotted in Figure 27.
Figure 27: Error functions for the COFS Mast Flight System Example
3.4.1 Local LQG Design with Loop Shaping

With the present decomposition, LQG controllers are designed for each subsystem to obtain good performance while satisfying the stability-robustness requirements. Gaussian white noise processes of intensity 1 and 100 are assumed to be present respectively at the input and the output of each subsystem. Local quadratic performance indices are chosen with control weightings $\rho_1 = \rho_2 = 100$, and state weightings $Q_i = L_i^T L_i (i = 1, 2)$, where

$$L_1 = \begin{bmatrix}
10^{-3}C_c & 0 \\
0 & 100\Gamma_s^i
\end{bmatrix}, \quad (3.66a)$$

and

$$L_2 = \begin{bmatrix}
0 & 10^{-3}C_c \\
100\Gamma_s^{28} & 0
\end{bmatrix} \quad (3.66b)$$

The resulting control and filtering gains are:

$$K_1 = \begin{bmatrix}
.0297 & -.0297 & -.0888 & .2329 & -.2171 & .3369 & .1076 & .0103
\end{bmatrix}$$

$$K_2 = \begin{bmatrix}
-.0625 & .0736 & -.4039 & -.0384 & .0147 & -.0147 & -.0439 & .0409
\end{bmatrix}$$

$$H_1 = \begin{bmatrix}
-.0764 \\
-.0272 \\
.0104 \\
.0917 \\
-.0233 \\
-.0509 \\
.0361 \\
.0101
\end{bmatrix}, \quad H_2 = \begin{bmatrix}
-.0033 \\
-.0315 \\
-.1281 \\
-.0862 \\
-.1445 \\
-.0135 \\
.0101 \\
.0182
\end{bmatrix}$$

Thus the individual controllers are of the form:

$$\dot{z}_i = (\dot{A}_i - \hat{B}_i K_i - H_i \hat{C}_i) z_i + H_i (y_i - D_i u_i) \quad (3.67a)$$

$$u_i = -K_i z_i, \quad i = 1, 2. \quad (3.67b)$$
Maximum singular values of the resulting closed loop TFMs are plotted together with the stability bound in Figures 28 and 29. It is observed that the stability-robustness requirement (3.43) is indeed satisfied.

The controllers are also applied to a 4-mode truth model. The results are summarized in Table 4. It is observed that 13% damping in the first and 5% damping in the second mode are achieved. Furthermore this is accomplished with relatively small control and filtering gains. The design is also robust to modeling errors and plant variations. The closed-loop system is guaranteed to remain stable as long as the perturbations are bounded by $\epsilon_m(\omega)$.

**Table 4: Open-Loop and Closed-Loop Eigenvalues for the Local LQG Design**

<table>
<thead>
<tr>
<th>Mode</th>
<th>Structural Eigenvalues (damping, %)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>OPEN-LOOP</td>
</tr>
<tr>
<td></td>
<td>Without actuators</td>
</tr>
<tr>
<td></td>
<td>Freq. (Hz)</td>
</tr>
<tr>
<td></td>
<td>-0.0023 ± j1.1750 (0.2)</td>
</tr>
<tr>
<td></td>
<td>-0.0256 ± j8.5451 (0.3)</td>
</tr>
<tr>
<td></td>
<td>-0.1235 ± j24.693 (0.5)</td>
</tr>
<tr>
<td></td>
<td>-0.2149 ± j42.977 (0.5)</td>
</tr>
</tbody>
</table>

**3.4.2 Decentralized Fixed Order Dynamic Feedback**

Another decentralized controller design is also carried out with the same decomposition as before. However, this time the decentralized controllers are restricted to be of order 2. These controllers are designed by considering the following local performance measures:
Figure 28: Maximum singular value of the closed-loop TFM of the first subsystem for the local LQG design and the stability bound $1/e_m(\omega)$. 

\[ \sigma(T_i(j\omega)) \]

\[ 1/e_m(\omega) \]
Figure 29: Maximum singular value of the closed-loop TFM of the second subsystem for the local LQG design and the stability bound $1/e_m(\omega)$.
where \( z_i \in \mathbb{R}^2 \) is the state of the \( i \)th local controller, which is described by

\[
\dot{z}_i = F_i z_i + G_i (y_i - D_i u_i) \quad (3.69a)
\]

\[
u_i = H_i z_i + K_i (y_i - D_i u_i). \quad (3.69b)
\]

The weights \( Q_i = L_i^T L_i \) (see (3.66a)-(3.66b)) and \( \rho_i (= 100) \) are chosen as before and \( R_i = I_2 \).

The controller parameters are determined by the aid of an optimization program [39] as:

\[
F_1 = \begin{bmatrix} 0.06106 & 0.51984 \\ 0.48981 & -0.23449 \end{bmatrix}, \quad G_1 = \begin{bmatrix} -1.0515 \\ -0.7423 \end{bmatrix}, \quad (3.70a)
\]

\[
H_1 = \begin{bmatrix} 1.5520 \\ 1.3647 \end{bmatrix}, \quad K_1 = -1.3340, \quad (3.70b)
\]

\[
F_2 = \begin{bmatrix} -0.85561 & -0.07107 \\ -0.01983 & -0.42392 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 0.022128 \\ 0.037521 \end{bmatrix}, \quad (3.71a)
\]

\[
H_2 = \begin{bmatrix} 0.91829 \\ 0.82217 \end{bmatrix}, \quad K_2 = -0.53784. \quad (3.71b)
\]

It has to be noted that these numerical values may not be the ones that actually minimize the performance indices (3.68). However, we found them very satisfactory in meeting our performance and robustness requirements.

Maximum singular values of the resulting closed-loop TFMs are plotted in Figures 30 and 31 together with the stability bound. It is observed that the stability-robustness requirement (3.43) is indeed satisfied.
Figure 30: Maximum singular value of the closed-loop TFM of the first subsystem for the fixed order dynamic feedback design and the stability bound $1/\epsilon_m(\omega)$
Figure 31: Maximum singular value of the closed-loop TFM of the second subsystem for the fixed order dynamic feedback design and the stability bound $1/e_m(\omega)$
The controllers are also applied to the 4-mode truth model. The results are summarized in Table 5. The closed-loop system is guaranteed to remain stable as long as the perturbations are bounded by $\epsilon_m(\omega)$.

### Table 5: Closed-Loop Eigenvalues for the Fixed Order Dynamic Feedback Design

<table>
<thead>
<tr>
<th>Mode</th>
<th>Structural Eigenvalues (damping, %)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Freq. (Hz)</td>
<td>CLOSED-LOOP</td>
</tr>
<tr>
<td></td>
<td>Design Model</td>
</tr>
<tr>
<td>0.187</td>
<td>-0.0184 ± j1.1405</td>
</tr>
<tr>
<td></td>
<td>(1.61)</td>
</tr>
<tr>
<td>1.36</td>
<td>-2.6182 ± j10.169</td>
</tr>
<tr>
<td></td>
<td>(24.93)</td>
</tr>
<tr>
<td>3.93</td>
<td>-0.5632 ± j25.677</td>
</tr>
<tr>
<td></td>
<td>(2.19)</td>
</tr>
<tr>
<td>6.84</td>
<td>-2.5095 ± j51.980</td>
</tr>
<tr>
<td></td>
<td>(4.82)</td>
</tr>
</tbody>
</table>

### 3.5 Summary

Frequency domain robust controller design strategies for large scale systems have been considered. Decomposition of LSS for decentralized robust controller design was discussed in Section 3.2. Overlapping decompositions was used to develop local models. Upper bounds on the norm of the multiplicative error matrix, that accounts for the interactions, have been calculated for various possible decompositions. Specific cases considered include decentralized design with minimal local models and decentralized design for interconnected systems. A total error function.
which may represent the error between the design model and the true system, has also been defined.

Decentralized robust design strategies for LSS were investigated in Section 3.3. In particular, local design with loop shaping was discussed and decentralized LQG/LTR design methodology was presented. It has been shown that, once a total error function, such as (3.19), is known to every control agent, stability-robustness and performance requirements can be tested locally. This fact guarantees stability and good performance of the resulting system when the appropriate local controller design procedures are applied.

Although, for notational simplicity, only decentralized systems with two control agents have been considered, the extension of the results to systems with more than two control agents is straightforward. The local models and the error functions introduced in Section 3.2 can be derived along the same lines with some additional care. Once the local models and an error function are determined, the results of Section 3.3 apply directly (with the obvious change of the index set) to systems with any number of control agents.

Decentralized robust controller design examples for a large space structure were considered in Section 3.4. Namely, two design approaches, local (full order) LQG design with loop shaping and decentralized fixed order dynamic feedback, were applied to control the COFS Mast Flight System.
CHAPTER IV

ROBUST CONTROLLER DESIGN IN STATE SPACE

4.1 Introduction

The uncertainties in a mathematical model of a physical system are basically
due to two factors: unknown values of certain system parameters (e.g., resistance
of an electrical component) and totally unmodeled dynamics (e.g., high frequency
modes of a flexible structure). It has been generally accepted that state space mod­
els are more suitable for representing uncertainties due to parameter variations,
while unmodeled dynamics can be represented easier in the frequency domain.

Various methods has been devised to test the stability of a system which is
subject to parameter variations. A method based on vector Liapunov functions
has been proposed by Šiljak [1]. Other Liapunov function based methods have
been introduced by Patel et al. [40], Yedavalli [41], Zhou and Khargonekar [42],
and Keel et al. [43]. A frequency domain approach for systems represented in state
space has been presented by Qiu and Davison [44]. Šiljak [45] has introduced a
parameter space method based on Popov's stability criterion.

A number of approaches to determine whether a characteristic polynomial is
Hurwitz invariant (i.e., whether the roots are retained in the open left half com­
plex plane) under parameter variations have also been studied. Barmish [46] has
utilized the result of Kharitonov [47] to develop a test for perturbed systems. In
his work [47], Kharitonov proved that it is necessary and sufficient to test only four
polynomials to conclude Hurwitz invariance of a set of polynomials with independent coefficient variations. However, if the variations in coefficients of a polynomial are dependent (which is usually the case in control systems), Kharitonov’s result becomes only sufficient and, hence, a degree of conservatism is involved. There has been some work to reduce such conservatism [48]. Alternative approaches have also been proposed. Biernachi et al. [49] have introduced a parameter space method to test Hurwitz invariance. Bartlett et al. [50] have shown that, if the family of all possible characteristic polynomials form a polytope in the coefficient space, then the exposed edges of the polytope determine whether the entire family is Hurwitz or not.

Special controller design methods for systems which are subject to parameter variations have also been proposed. Karmarkar and Šiljak [51] have developed a computer aided design approach to maximize the stability hypercube in parameter space. Ackermann [52] has introduced a pole placement approach. Yedavalli [53] has proposed an optimal controller design approach with a cost function weighting measures of robustness and system performance. Keel et al. [43] have presented an approach which maximizes the radius of the stability hypersphere in parameter space.

Much research has also been undertaken to establish frequency domain design strategies for uncertain systems. However, most of those approaches (e.g., [23,24,22,37]) are designed for systems with truly unstructured uncertainties and can not take advantage of all the available information. More specifically, these methods assume only a known upper bound on the magnitude of the possible perturbations. However, in practice more information is generally available. For example, in the case of flexible structures with co-located actuation and sensing, it is known that the phase of the uncertain dynamics always lies between $0^\circ$ and
$180^\circ$. Such additional information may relax the robust controller design problem considerably and would, in general, lead to less conservative controller design. In the case of unknown parameter values the uncertainty is even more structured.

Many physical systems possess both parameters with unknown exact values and uncertain dynamics. In [54] Wei and Yedavalli proposed a combined frequency domain and state space approach for such systems. However, such an approach would, in general, necessitate many alternating design stages in state space and frequency domain. Some research has also been undertaken to represent structured uncertainty (which may be due to parameter variations) in the frequency domain. Horowitz [30] has proposed an approach based on constructing regions (called templates) on the complex plane to represent the plant variation for some frequency point. Once the templates are determined, stability-robustness and acceptable performance bounds on the loop gain at the corresponding frequencies can be constructed and (if possible) a controller can be designed to satisfy these bounds [30]. An alternative approach proposed by Sideris and Safonov uses a series of conformal mappings of the templates onto the unit disk to transform the structured uncertainty problem to one with unstructured uncertainty which can be solved using $H^\infty$-theory or Nevanlinna-Pick interpolation techniques [55]. However, determination of the templates may be very tedious in many cases. Usually, a large number of frequency points must be considered for a useful representation of the plant parameter uncertainties. Furthermore, the approach proposed in [30] requires a number of trial and error designs and $H^\infty$ minimization suggested in [55] often results in a high dimensional controller.

Here, a unified approach in state space which can be used for systems with both classes of uncertainties and which can utilize any information about the uncertain dynamics is introduced. The main result presented in Section 4.2 demon-
strates that, under mild conditions, it is possible to obtain a rational TFM, possibly
parametrized by a finite dimensional vector, to represent uncertain dynamics and
design a controller accordingly to satisfy stability and desired performance. The
underlying idea here is to represent possibly very high dimensional uncertain dy­
namics in a relatively low order structured form. In Section 4.3, the state space
representation of such dynamics and the robust controller design problem is dis­
cussed. Parameter uncertainty can be combined with such a representation of
uncertain dynamics at this stage. The presented approach is applied to a robust
controller design problem for a large flexible structure in Section 4.4. The struc­
ture under consideration possesses both parameters with unknown exact values
and unmodeled (high frequency) dynamics. Extension of the proposed approach
to decentralized robust controller design is discussed in Section 4.5.

4.2 Reduced Order Models for Uncertain Dynamics

The uncertain dynamics of a system, nominally modeled by a TFM $G(s)$, can
be represented at the input by $\Delta_i(s)$, at the output by $\Delta_o(s)$, or additively by $\Delta_a(s)$
as shown in Figure 22. For brevity, here we consider only uncertainties represented
at the output. Similar results can be proved for other types of representations
along the same lines. We will, in fact, apply those results to an additively modeled
uncertainty in Section 4.4. Henceforth we drop the subscript "o" of $\Delta_o(s)$.

Although $\Delta(s)$ may not be known exactly, some information about it (e.g.,
upper and lower bounds on $\Delta(j\omega)$) is generally available. Note that $\Delta(s)$ may
actually represent very complex (unknown) dynamics, which might require an
unmanageably high dimensional model.

Our purpose is to determine a rational TFM $E(s;p)$ to replace $\Delta(s)$ in the
model, such that a feedback controller $K(s)$ designed for such a model stabi-
lizes the actual plant and achieves desired performance. Here \( E(s; p) \) is possibly parametrized by a vector \( p \) which is taken from a parameter set \( \Pi \). The following theorem demonstrates that once a suitable TFM and an associated parameter set are found to represent the uncertainties, such an approach would, in fact, guarantee the stability of the actual closed-loop system.

**Theorem 4.1 (Absolute Stability)** Suppose that \( \Delta(s) \) is analytic in the closed right half complex plane \( \mathbb{C}^+ \), except possibly at some isolated singular points which constitute a total of \( m \) poles with due count of multiplicity.\(^1\) Let \( E(s; p) \) be a TFM which is rational in \( s \) and continuous in \( p \) and \( \Pi \) be a subset of a finite dimensional real vector space with the properties:

\( a) \) for all \( \omega \in \mathbb{R} \) there exists a \( p_{\omega} \in \Pi \) such that

\[
E(j\omega; p_{\omega}) = \Delta(j\omega)
\]

(4.1a)

if \( \Delta(s) \) is analytic at \( s = j\omega \) or

\[
\lim_{\tau \to 0} E(j\omega + re^{j\theta}; p_{\omega}) = \lim_{\tau \to 0} \Delta(j\omega + re^{j\theta})
\]

(4.1b)

for all \( -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \) if \( \Delta(s) \) is not analytic at \( s = j\omega \).

\( b) \) there exists a \( p_{\infty} \in \Pi \) such that

\[
\lim_{r \to \infty} E(re^{j\theta}; p_{\infty}) = \lim_{r \to \infty} \Delta(re^{j\theta})
\]

(4.1c)

for all \( \frac{\pi}{2} \geq \theta \geq -\frac{\pi}{2} \), and

\( c) \) \( E(s; p) \) has exactly \( m \) poles in \( \mathbb{C}^+ \) for all \( p \in \Pi \).

\(^1\)Note that, one needs to know only the total number of unstable poles of \( \Delta(s) \). Their actual locations need not be known.
For a given rational TFM $G(s)$, suppose that a rational TFM $K(s)$ is chosen such that the closed-loop TFM

$$T_E(s; p) \triangleq (I + E(s; p))G(s)K(s)[I + (I + E(s; p))G(s)K(s)]^{-1} \quad (4.2)$$
does not have any poles in $\mathbb{C}^+$ for all $p \in \Pi$. Also assume that $\det[I + (I + \Delta(s))G(s)K(s)] \neq 0$ on a dense subset of $\mathbb{C}$. Then the true closed-loop TFM:

$$T_\Delta(s) \triangleq (I + \Delta(s))G(s)K(s)[I + (I + \Delta(s))G(s)K(s)]^{-1} \quad (4.3)$$
is analytic in $\mathbb{C}^+$.

**Proof:** Throughout in this proof we drop the arguments $s$ and $p$ for notational brevity. Let $n$ denote the number of poles of $(I + \Delta)GK$ in $\mathbb{C}^+$. Then, since $\Delta$ and $E$ have the same number of poles in $\mathbb{C}^+$, $(I + E)GK$ also has $n$ poles in $\mathbb{C}^+$ for all $p \in \Pi$. Furthermore, since $T_E$ does not have any poles in $\mathbb{C}^+$, by the multivariable Nyquist stability criterion [56], the map $\det[I + (I + E)GK]$ as $s$ is varied on the standard Nyquist contour $\mathcal{D}$ encircles the origin $-n$ times for all $p \in \Pi$. By the hypothesis, $\Delta$ is analytic on $\mathbb{C}^+$, except possibly at isolated singular points. The same is also true for $G$ and $K$, since they are rational TFMs. Therefore, $\det[I + (I + \Delta)GK]$ is analytic on $\mathbb{C}^+$, except possibly at isolated singular points. Furthermore, since $\det[I + (I + \Delta)GK] \neq 0$ on a dense subset of $\mathbb{C}$, $T_\Delta$ is well defined on such a set and we can apply the multivariable Nyquist stability criterion. Note that, the conditions a) and b) imply that the loci of $\det[I + (I + \Delta)GK]$, as $s$ is varied on $\mathcal{D}$, is a subset of $\det[I + (I + E)GK]$ loci as $s$ is varied on $\mathcal{D}$ and $p$ is varied on $\Pi$ (see Figure 32). Furthermore, since $E$ is continuous in $p$, any member of the former set must encircle any point on the complex plane that is encircled by the latter set the same times. Thus, $\det[I + (I + \Delta)GK]$ encircles the origin $-n$
times as $s$ is varied on $D$, which, by the multivariable Nyquist stability criterion, proves that $T_\Delta$ does not have any poles in $C^+$. Hence, the result follows. □

**Remark 4.1** Note that $T_\Delta(s)$ represents a physical system whenever $\Delta(s)$ does, since all other terms that appear in the right hand side of (4.9) are rational TFMs. Furthermore, $T_\Delta(s)$ represents the actual closed-loop system whenever $\Delta(s)$ is the TFM representing the actual unmodeled dynamics. Thus, whenever $T_\Delta(s)$ is analytic in $C^+$, the actual closed-loop system is stable.

In many practical cases, absolute stability may not be sufficient. Instead, for example, one may wish to confine all the closed-loop eigenvalues in a region $R$ of the complex plane. Note that the imaginary axis and the right hand semi-circle of infinite radius together defines the boundary of $C^+$. Hence, if the conditions a) and b) of Theorem 4.1 are met, we say that $E(s,p)$ and $\Delta(s)$ are matched on the boundary of $C^+$. Furthermore, we say that the two TFMs are matched on the boundary of a region $R$, if the obvious generalizations of these conditions hold for the region $R$. The following result can be proved as Theorem 4.1, with an obvious modification of the Nyquist contour.

**Theorem 4.2 (Relative Stability)** Let $R$ be an open subset of $C$. Suppose that $\Delta(s)$ is analytic in $R^c \triangleq C \setminus R$, except possibly at some isolated singular points which constitute a total of $m$ poles with due count of multiplicity. Let $E(s;p)$ be a TFM which is rational in $s$ and continuous in $p$ and $\Pi$ be a subset of a finite dimensional real vector space such that $E(s;p)$ and $\Delta(s)$ are matched on the boundary of $R^c$ and $E(s;p)$ has exactly $m$ poles in $R^c$. For a given rational TFM $G(s)$, suppose that a rational TFM $K(s)$ is chosen such that the closed-loop TFM (4.2) does not have any poles in $R^c$ for all $p \in \Pi$. Also assume that
Figure 32: a) A representative $\operatorname{det}[I + (I + \Delta) GK]$ loci. b) Region of all possible $\operatorname{det}[I + (I + \Delta) GK]$ loci. c) $\operatorname{det}[I + (I + E) GK]$ loci as $s$ is varied on the half Nyquist contour and $p$ is varied on $\Pi$. 
\[ \det[I + (I + \Delta(s))G(s)K(s)] \neq 0 \text{ on a dense subset of } \mathbb{C}. \text{ Then the true closed-loop TFM (4.3) is analytic in } \mathcal{R}^c. \]

As it was discussed in the previous chapter, most of the widely used performance measures (e.g., disturbance rejection and steady state error) can be related to the return difference matrix. For example, to achieve a certain degree of plant disturbance rejection at the output, one may require the return difference matrix to satisfy\(^2\):

\[
[I + (I + \Delta(j\omega))G(j\omega)K(j\omega)]^S \geq Q(\omega) \quad \forall \omega \in \Omega
\]

provided that the left hand side is well defined. Here \(Q(\omega)\) is a positive definite matrix for all \(\omega \in \Omega\) and \(\Omega \subset \mathbb{R}\) is the set of frequencies where disturbances are effective. The following theorem demonstrates that if a controller is designed to satisfy such a performance criterion for the design model (the model in which \(\Delta\) is replaced by \(E\)), then the actual closed-loop system satisfies the same criterion.

**Theorem 4.3 (Good Performance)** Under the conditions of Theorem 4.1, suppose \(K(s)\) is chosen such that

\[
[I + (I + E(j\omega;p))G(j\omega)K(j\omega)]^S \geq Q(\omega) \quad \forall \omega \in \Omega, \; \forall p \in \Pi. \quad (4.5)
\]

Then (4.4) is satisfied.

**Proof:** If the left hand side of (4.4) is well defined, then \(\Delta(s)\) is analytic at \(s = j\omega\). By (4.1a), for such \(\omega\) there exists a \(p_\omega \in \Pi\) such that

\[
I + (I + E(j\omega;p_\omega))G(j\omega)K(j\omega) = I + (I + \Delta(j\omega))G(j\omega)K(j\omega).
\]

\(^2\)The more widely used criterion is defined as a lower bound on the singular values of the return difference matrix. However, note that the condition presented here is more general.
Hence, the result follows.

Under mild conditions on \( \Delta(s) \), the existence of a TFM \( E(s; p) \) and a parameter set \( \Pi \), satisfying the conditions of the above theorems, can be ensured. To avoid notational complexity, we prove this only for the scalar case. The extension to the multivariable case is possible along similar lines (e.g., the two TFMs, \( \Delta(s) \) and \( E(s; p) \), can be considered entry by entry). Furthermore, here we consider only absolute stability. A similar result can be proved for the relative stability case if the region \( \mathcal{R} \) is symmetric about the real axis.

**Theorem 4.4** Let

\[
\Delta(s) = \frac{1}{d_u(s)} \Delta_s(s)
\]

where \( d_u(s) \) is an \( m^{th} \) order polynomial with zeros in \( \mathbb{C}^+ \) and \( \Delta_s(s) \) is analytic in \( \mathbb{C}^+ \) (i.e., factor out the unstable poles of \( \Delta(s) \) as \( \frac{1}{d_u(s)} \)). Furthermore, suppose that \( \Delta(s) \) is such that \( \Delta(j\omega) = \Delta(-j\omega) \) for all \( \omega \in \mathbb{R} \), \( \lim_{\omega \to \infty} \Delta(re^{i\theta}) = c < \infty \) for all \(-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \), and

\[
-k\frac{\pi}{2} \leq \omega(\Delta_s(j\omega)) + n\pi \leq \ell\frac{\pi}{2} \quad \forall \omega \in \mathbb{R}^+
\]

where the equalities can hold only at \( \omega = 0 \) or in the limit as \( \omega \to \infty \). Here \( k \) and \( \ell \) are non-negative integers and \( n \) is an integer such that \( \omega(\Delta_s(0)) + n\pi = 0 \). Let

\[
q \triangleq \max(k + m, \ell)
\]

and

\[
t \triangleq \begin{cases} 
\ell & \text{if } c \text{ is known to be zero} \\
q & \text{otherwise}
\end{cases}
\]

Then there exists a function \( E(s; p) \), continuous in \( p \) and proper and rational in \( s \) with \( q \) poles and \( t \) finite zeros, and a set \( \Pi \subseteq \mathbb{R}^{q+t+1} \) such that the conditions a), b), and c) of Theorem 4.1 hold.

\(^{5}\)The actual value of the constant \( c \) need not be known.
Proof: Let \( p = (\alpha, \beta, \gamma, \epsilon) = (\alpha_1, \ldots, \alpha_t, \beta_1, \ldots, \beta_m, \gamma_1, \ldots, \gamma_{q-m}, \epsilon) \) and

\[
E(s; p) = \frac{\epsilon}{e(s; \beta)} E_s(s; \alpha, \gamma)
\]

where \( e(s; \beta) = s^m + \beta_1 s^{m-1} + \ldots + \beta_{m-1} s + \beta_m \) and

\[
E_s(s; \alpha, \beta) = \frac{s^t + \alpha_1 s^{t-1} + \ldots + \alpha_{t-1} s + \alpha_t}{s^q - \gamma_1 s^{q-1} - \ldots - \gamma_{q-m-1} s - \gamma_{q-m}}
\]

For each \( \omega \in \mathbb{R}^+ \) it is possible to choose real numbers \( \beta_1, \ldots, \beta_m \) such that \( e(s; \beta) \) has \( m \) zeros in \( \mathbb{C}^+ \) and \( l(e(j\omega; \beta)) = l(d_u(j\omega)) \), and to choose real numbers \( \alpha_1, \ldots, \alpha_t, \gamma_1, \ldots, \gamma_{q-m} \) and the sign of \( \epsilon \) such that \( E_s(s; \alpha, \gamma) \) has no poles in \( \mathbb{C}^+ \) and

\[
l(E_s(j\omega; \alpha, \gamma)) = l(\Delta_s(j\omega)).
\]

Hence, if \( d_u(j\omega) \neq 0 \) we already have \( l(E(j\omega; p)) = l(\Delta(j\omega)) \) and the magnitude of \( \epsilon \) can be chosen to satisfy \( |E(j\omega; p)| = |\Delta(j\omega)| \). If \( d_u(j\omega) = 0 \) the analyticity assumption guarantees that the appropriate choice of the parameters \( p \) will satisfy the same conditions in the limit (taken as in (4.1b)).

So far we have satisfied c) and a). Note that if \( c = 0 \) and \( q > t \), then b) is satisfied. If \( q = t \), then \( \epsilon \) can be adjusted to satisfy b).

The parameter set \( \Pi \) is formed by all \( p \) necessary to satisfy above conditions and is a subset of \( q + t + 1 \) dimensional real vector space \( \mathbb{R}^{q+t+1} \).

We note that all the assumptions, possibly except the existence of a lower bound in (4.7), made in the above theorem, are usually met by all physical systems. However, even if there does not exist a lower bound on \( l(\Delta(j\omega)) \), one can still achieve the desired result by including non-rational terms in \( E(s; p) \), such as delay terms \( e^{-s\tau} \).

With regard to choosing a suitable TFM \( E(s; p) \) and an associated parameter set \( \Pi \), we have to note that in certain cases, depending on how much is known about
\( \Delta(s) \), the minimum required dimension of \( \Pi \) (i.e., the number of parameters) can, in fact, be less than the number predicted in Theorem 4.4. On the other hand, in certain other cases, one may prefer to work with a higher order TFM and a higher dimensional parameter set, to reduce the possible conservatism involved in representing \( \Delta(s) \). The following example illustrates the procedure of determining \( E(s; p) \) and \( \Pi \).

**Example 4.1** Suppose all that is known about \( \Delta(s) \) is that it is analytic in \( \mathbb{C}^+ \), \( \Delta(j\omega) = \overline{\Delta(-j\omega)} \), \( \lim_{r \to \infty} \Delta(re^{j\theta}) = 0 \) for all \(-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \),

\[
\left| \frac{\delta_m \omega_1}{j\omega + \omega_1} \right| \leq |\Delta(j\omega)| \leq \left| \frac{\delta_M \omega_1}{j\omega + \omega_1} \right| \quad \forall \omega \in \mathbb{R}^+ ,
\]

and

\[
\angle \left( \frac{1}{j\omega + \omega_1} \right) \leq \angle(\Delta(j\omega)) \leq \angle \left( \frac{1}{j\omega + \omega_2} \right) \quad \forall \omega \in \mathbb{R}^+ ,
\]

where \( \delta_M > \delta_m > 0 \) and \( \omega_2 > \omega_1 > 0 \) are known numbers. The gain and phase of all possible \( \Delta(s) \), together with a representative \( \Delta(s) \) are depicted in Figure 33.

By Theorem 4.4, \( E(s; p) \) can be chosen as

\[
E(s; p) = \frac{\epsilon}{s + \gamma}
\]

where \( p = (\gamma, \epsilon) \). By varying \( \gamma \) within the interval \([\omega_1, \omega_2]\) we can satisfy the phase condition \( \angle(E(j\omega; p)) = \angle(\Delta(j\omega)) \) for all possible \( \Delta(s) \). Furthermore, for a fixed \( \gamma \in [\omega_1, \omega_2] \), we can meet the magnitude condition \( |E(j\omega, p)| = |\Delta(j\omega)| \) for all possible \( \Delta(s) \), by varying \( \epsilon \) within the interval \([\delta_m \omega_1, \delta_M \gamma]\). Hence, we obtain

\[
\Pi = \{(\gamma, \epsilon) | \omega_1 \leq \gamma \leq \omega_2, \delta_m \omega_1 \leq \epsilon \leq \delta_M \gamma \}
\]

which is a bounded subset of \( \mathbb{R}^2 \) and is depicted in Figure 34.
Figure 33: Regions of all possible magnitude and phase of $\Delta(j\omega)$ and a representative $\Delta(j\omega)$ for Example 4.1.
Figure 34: The parameter set II for Example 4.1.
Consider a system modeled by a nominal TFM $G(s; p_G)$, where $p_G \in \Omega_G$ is a vector denoting possible values of some system parameters. Suppose that the dynamics not modeled by $G$ can be represented at the output by an uncertain TFM $\Delta(s)$ satisfying the conditions of Theorem 4.4. Then, one can obtain a rational TFM $E(s; p_E)$ and a corresponding parameter set $\Omega_E$, satisfying the conditions of Theorem 4.1 (or of Theorem 4.2 when appropriate), as discussed in the previous section.

Let $(A(p_G), B(p_G), C(p_G), D(p_G))$ be a realization for $G(s; p_G)$, i.e., let

$$G(s; p_G) = C(p_G)(sI - A(p_G))^{-1}B(p_G) + D(p_G).$$

Furthermore, let $(F(p_E), G(p_E), H(p_E), L(p_E))$ be a realization for $E(s; p_E)$. Then we obtain the following state space model for the uncontrolled system

$$\frac{d}{dt} \begin{bmatrix} x \\ \xi \end{bmatrix} = \begin{bmatrix} A & 0 \\ GC & F \end{bmatrix} \begin{bmatrix} x \\ \xi \end{bmatrix} + \begin{bmatrix} B \\ GD \end{bmatrix} u$$

(4.8)

$$y = \begin{bmatrix} (I + L)C & H \end{bmatrix} \begin{bmatrix} x \\ \xi \end{bmatrix} + (I + L)Du.$$}

Suppose that a feedback controller $K(s)$, which has a realization $(A_K, B_K, C_K, D_K)$, is designed so that the closed-loop dynamics are described by

$$\frac{d}{dt} \begin{bmatrix} x \\ \xi \\ \eta \end{bmatrix} = A_CL \begin{bmatrix} x \\ \xi \\ \eta \end{bmatrix}.$$  

(4.9)
where
\[
A_{CL} = \begin{bmatrix}
A - BU_{DK}(I + L)C & -BU_{DK}H \\
GC - GDU_{DK}(I + L)C & F - GDU_{DK}H \\
-B_{NK}(I + L)C + B_{NK}(I + L)DU_{DK}(I + L)C & -B_{NK}H + B_{NK}(I + L)DU_{DK}H \\
BU_{C_{K}} & GDU_{C_{K}} \\
A_{K} - B_{K}(I + L)DU_{C_{K}}
\end{bmatrix}
\]

and \( U \triangleq (I + D_{K}(I + L)D)^{-1} \). Here \( D_{K} \) should be chosen such that \( U \) exists.

The following result follows from Theorems 4.1, 4.2, and 4.3.

**Corollary 4.1** If \( E(s; p_{G}) \) and \( \Pi \) are chosen to satisfy the conditions of Theorem 4.1 and the closed-loop system (4.9) is stable for all \( p \triangleq (p_{G}, p_{E}) \in \Pi \triangleq \Pi_{G} \times \Pi_{E} \), then the actual closed-loop system, shown in Figure 35, is stable. Furthermore, if the closed-loop system (4.9) satisfies a performance criterion like (4.4) for all \( p \in \Pi \), then the actual closed-loop system also satisfies the same criterion.

If \( E(s; p_{E}) \) and \( \Pi \) are chosen to satisfy the conditions of Theorem 4.2 and the closed-loop system (4.9) has eigenvalues only in the region \( \mathcal{R} \) for all \( p \in \Pi \), then the actual closed-loop system has the corresponding relative stability properties.

**Figure 35: Closed-loop system**

Once a closed-loop system such as (4.9) is designed, one of the many existing state space methods discussed in Section 4.1 can be used to test the robust sta-
bility of this system. Alternatively, one of the robust controller design approaches discussed in the same section can be applied to the open-loop system (4.8) to obtain a stable closed-loop system.

4.4 Application

We consider the application of the presented design approach to the planer truss structure shown in Figure 36. The truss is made of identical uniform aluminum rods that can be displaced in the axial direction. The structure is fixed at nodes 1 and 2. It is controlled by a linear force actuator located at node 16, acting in the horizontal direction. The location of the actuator is chosen such that it has the greatest effect on the first mode relative to the higher modes. Measurements are taken by a co-located linear accelerometer. Ideal sensor and actuator dynamics are assumed. The nominal values of the relevant material parameters for the rods are given in Table 6.

<table>
<thead>
<tr>
<th>Table 6: Material Parameters for the Truss Structure</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rod cross-section area:</td>
</tr>
<tr>
<td>Modulus of elasticity:</td>
</tr>
<tr>
<td>Mass density:</td>
</tr>
<tr>
<td>Length of a vertical or horizontal rod:</td>
</tr>
</tbody>
</table>

The structure has 16 free nodes, each having two degrees of freedom. Hence, it exhibits 32 flexible modes. It is assumed that a certain structural damping is associated with each individual mode. These assumptions lead to a 64th order state space model, which can be represented in modal coordinates as:
Figure 36: Truss structure
\[ \dot{x} = \begin{bmatrix} 0 & I \\ -\Omega^2 & -2Z\Omega \end{bmatrix} x + \begin{bmatrix} 0 \\ \Gamma \end{bmatrix} u \tag{1.10a} \]

\[ y = \Gamma^T \left[ -\Omega^2 - 2Z\Omega \right] x + \Gamma^T \Gamma u \tag{4.10b} \]

where

\[ \Omega = \text{diag} (\omega_1, \omega_2, \cdots, \omega_{32}) \]

\[ Z = \text{diag} (\zeta_1, \zeta_2, \cdots, \zeta_{32}) \]

\[ \Gamma = [b_1, b_2, \cdots, b_{32}]^T. \]

Due to variations of material properties (such as modulus of elasticity and mass density) the exact values of structural frequencies \( \omega_i \) and mode shapes \( b_i \) are uncertain. Nominal values of these quantities (based on the values given in Table 6) for the selected modes are given in Table 7. The actual values are assumed to be within \( \pm 1\% \) of these nominal values. The damping ratios \( \zeta_i \) \((i = 1, 2, \cdots, 32)\) are assumed to be between 0.005 and 0.010.

**Table 7: Modal Frequencies and Mode Shapes for the Truss Structure**

<table>
<thead>
<tr>
<th>( i )</th>
<th>( \omega_i ) (rad/sec)</th>
<th>( b_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>131</td>
<td>0.3927</td>
</tr>
<tr>
<td>2</td>
<td>633</td>
<td>0.1403</td>
</tr>
<tr>
<td>3</td>
<td>843</td>
<td>0.0223</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
</tr>
<tr>
<td>31</td>
<td>15923</td>
<td>-0.1731</td>
</tr>
<tr>
<td>32</td>
<td>16602</td>
<td>-0.3115</td>
</tr>
</tbody>
</table>
It is desired to design a controller to actively dampen the first mode while maintaining total system stability. The modeled dynamics are associated with the first mode only:

\[
\begin{align*}
\dot{x}_m &= \begin{bmatrix} 0 & 1 \\ -\omega_1^2 & -2\zeta_1\omega_1 \end{bmatrix} x_m + \begin{bmatrix} 0 \\ b_1 \end{bmatrix} u \\
y &= \begin{bmatrix} -b_1\omega_1^2 & -2\zeta_1 b_1\omega_1 \end{bmatrix} x_m + b_1^2 u.
\end{align*}
\]

(4.11a) (4.11b)

Note that uncertainties in \(\omega_1, b_1,\) and \(\zeta_1\) are treated here as parameter uncertainties and the part of the system associated with the higher modes is treated as unmodeled dynamics. The transfer function description of the model (4.11a)-(4.11b) is:

\[
G_m(s) = \frac{b_1^2 s^2}{s^2 + 2\zeta_1 \omega_1 s + \omega_1^2}, \quad \Pi_G = (\omega_1, b_1, \zeta_1) \in \Pi_G
\]

(4.12)

where

\[
\Pi_G = \{(\omega_1, b_1, \zeta_1) \mid \omega_{1\text{min}} \leq \omega_1 \leq \omega_{1\text{max}}, b_{1\text{min}} \leq b_1 \leq b_{1\text{max}}, \zeta_{\text{min}} \leq \zeta_1 \leq \zeta_{\text{max}}\}
\]

The bounds on the individual parameters are readily obtained utilizing the previous assumptions:

\[
\begin{align*}
\omega_{1\text{min}} &= 0.99 \omega_{1\text{nom}} = 130, \\
\omega_{1\text{max}} &= 1.01 \omega_{1\text{nom}} = 132, \\
b_{1\text{min}} &= 0.99 b_{1\text{nom}} = 0.3888, \\
b_{1\text{max}} &= 1.01 b_{1\text{nom}} = 0.3966, \\
\zeta_{\text{min}} &= 0.005, \\
\zeta_{\text{max}} &= 0.010,
\end{align*}
\]

where the nominal values \((\cdot)_{\text{nom}}\) are those listed in Table 7.

The transfer function of the actual system, described by (4.10a)-(4.10b) is:
\[
G(s) = \sum_{i=1}^{32} G_i'(s)
\]

where
\[
G_i'(s) = \frac{b_i^2 s^2}{s^2 + 2\zeta_i \omega_i s + \omega_i^2}, \quad i = 1, 2, \ldots, 32.
\]

Therefore if we let
\[
G(s) = G_m(s) + \Delta(s)
\]

we obtain a description of unmodeled additive dynamics as:
\[
\Delta(s) = \sum_{i=2}^{32} G_i(s).
\]

The magnitude and phase of \( \Delta(j\omega) \) for typical \( \omega_i, b_i, \) and \( \zeta_i \) are depicted in Figure 37. Although the exact plots depend on specific parameter values, the general shape of these plots are the same for all possible \( \omega_i, b_i, \) and \( \zeta_i \) values. By Theorem 4.4, the phase plot indicates that a second order model is sufficient to represent these unmodeled dynamics. We let
\[
E(s; p_E) = \frac{K_p s^2}{s^2 + 2\zeta_p \omega_p s + \omega_p^2}, \quad p_E = (K_p, \omega_p, \zeta_p) \in \Pi_E
\]

be such a representation with:
\[
\Pi_E = \{(K_p, \omega_p, \zeta_p) \mid 0 < K_p \leq K_M, \omega_{p_{\text{min}}} \leq \omega_p \leq \omega_{p_{\text{max}}}, \zeta_{\text{min}} \leq \zeta_p \leq \zeta_{\text{max}}\}
\]

where
\[
\omega_{p_{\text{min}}} = 0.90\omega_{2_{\text{nom}}} = 627
\]
\[
\omega_{p_{\text{max}}} = 1.01\omega_{32_{\text{nom}}} = 16768.
\]

An upper bound \( K_M \) for \( K_p \) can be calculated as:

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Figure 37: Magnitude and phase of uncertain dynamics for the truss structure
\[
K_M = \max_{i \in \{2, \ldots, 32\}} \left\{ \frac{b_i^2 + 2\zeta_{\max} \sum_{j=1}^{32} \frac{b_j^2 \omega_j^2}{|\omega_i^2 - \omega_j^2|}}{\sum_{j=1}^{32} \frac{b_j^2 \omega_j^2}{|\omega_i^2 - \omega_j^2|}} \right\} = 0.3275 . \tag{4.18}
\]

Although less conservative bounds on \( K_p \) can be generated at the expense of more involved calculations, we found these bounds satisfactory for our purposes. We note that the transfer function (4.17) together with the set \( \Phi_E \) given following (4.17) satisfies the conditions of Theorem 4.1.

We obtain a design model as a realization of \( G_m(s; p_G) + E(s; p_E) \)

\[
\dot{x} = Ax + Bu \tag{4.19a}
\]
\[
y = Cx + Du \tag{4.19b}
\]

where

\[
A = \begin{bmatrix}
0 & 1 & 0 & 0 \\
-\omega_1^2 & -2\zeta_1 \omega_1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-\omega_p^2 & -2\zeta_p \omega_p & 0 & 0
\end{bmatrix}, \quad B = \begin{bmatrix}
0 \\
0 \\
0 \\
K_p
\end{bmatrix}
\]

\[
C = \begin{bmatrix}
-b_1 \omega_1^2 & -2b_1 \zeta_1 \omega_1 & -\omega_p^2 & -2\zeta_p \omega_p
\end{bmatrix}, \quad D = [b_1^2 + K_p].
\]

Note that this model depends on a parameter vector \( p = (p_G, p_E) \) which can vary over the set \( \Pi \triangleq \Pi_G \times \Pi_E \).

A first order controller:

\[
\dot{x}_c = -ax_c + y \tag{4.20a}
\]
\[
u = -K x_c \tag{4.20b}
\]

is designed based on the model (4.19a)-(4.19b). The nominal parameter values for this model are picked as:

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\[ \omega_1 = \omega_{1\text{nom}} = 131 \]
\[ \omega_p = \sqrt{\omega_{2\text{nom}} \omega_{3\text{nom}}} = 3242 \]
\[ \zeta_1 = \zeta_p = \zeta_{\text{nom}} = 0.0075 \]
\[ b_1 = b_{1\text{nom}} = 0.3427 \]
\[ K_p = \frac{K_M}{2} = 0.1638 . \]

The controller parameters are chosen to be

\[ a = 30 , \quad K = 200 , \]

to satisfy the design goals. The closed-loop system dynamics are described by

\[ \dot{z} = Fz \quad (4.21) \]

where

\[ z = \begin{bmatrix} x \\ x_c \end{bmatrix} , \quad F = \begin{bmatrix} A & -BK \\ C & -a - DK \end{bmatrix} . \]

The closed-loop eigenvalues for the nominal system are shown in Table 8. It is observed that more than 12% damping is achieved in the first mode.

Table 8: Closed-Loop Eigenvalues for the Nominal Design Model

\[
\begin{array}{c|c|c|c}
\hline
\text{Eigenvalue} & \text{Real Part} & \text{Imaginary Part} \\
\hline
-31.72 & -31.72 & 0 \\
-15.55 \mp 126.33i & -15.55 & \mp 126.33 \\
-40.69 \mp 3242i & -40.69 & \mp 3242 \\
\hline
\end{array}
\]

To prove the stability of the actual closed-loop system, it suffices to show that the system \((4.21)\) is stable for all possible parameter variations. To accomplish that, we consider:

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\[ \text{det}(sI - F) = s^5 + \alpha_4 s^4 + \alpha_3 s^3 + \alpha_2 s^2 + \alpha_1 s + \alpha_0 \]

where

\[ \alpha_4 = a + b_1^2 K + K_p K + 2\zeta_1 \omega_1 + 2\zeta_p \omega_p, \]
\[ \alpha_3 = 2\zeta_1 \omega_1 a + 2\zeta_1 \omega_1 K_p K + \omega_1^2 + 2\zeta_p a \omega_p + 2\zeta_p b_1^2 K \omega_p + 4\zeta_1 \zeta_p \omega_1 \omega_p + \omega_p^2, \]
\[ \alpha_2 = a \omega_1^2 + K_p K \omega_1^2 + 4\zeta_p \zeta_1 \omega_1 a \omega_p + 2\zeta_p \omega_1^2 \omega_p + a \omega_p^2 + b_1^2 K \omega_p^2 + 2\zeta_1 \omega_1 \omega_p^2, \]
\[ \alpha_1 = 2\zeta_p a \omega_1^2 \omega_p + 2\zeta_1 \omega_1 a \omega_p^2 + \omega_1^2 \omega_p^2, \]
\[ \alpha_0 = a \omega_1^2 \omega_p^2. \]

The minimum and maximum values of the coefficients \( \alpha_i \) under all possible variations in \( \omega_1, \zeta_1, b_1, \zeta_p, \) and \( K_p \) are:

\[ \alpha_{4\text{min}} = 61.52 + 0.01 \omega_p \]
\[ \alpha_{4\text{max}} = 129.6 + 0.02 \omega_p \]
\[ \alpha_{3\text{min}} = 16824 + 0.6152 \omega_p + \omega_p^2 \]
\[ \alpha_{3\text{max}} = 17722 + 1.2282 \omega_p + \omega_p^2 \]
\[ \alpha_{2\text{min}} = 0.5035 \times 10^6 + 168.2 \omega_p + 61.52 \omega_p^2 \]
\[ \alpha_{2\text{max}} = 1.6685 \times 10^6 + 351.0 \omega_p + 64.10 \omega_p^2 \]
\[ \alpha_{1\text{min}} = 5035 \omega_p + 16824 \omega_p^2 \]
\[ \alpha_{1\text{max}} = 10482 \omega_p + 17549 \omega_p^2 \]
\[ \alpha_{0\text{min}} = \alpha_{0\text{max}} = 0.5035 \times 10^6 \omega_p^2 \]

Here, we take \( \alpha_{i\text{max}} \) and \( \alpha_{i\text{min}} \) as functions of \( \omega_p \) rather than substituting for its maximum and minimum values. This way, we reduce the possible conservatism.

Since the variations in the other parameters are smaller compared to the variations in \( \omega_p \), we choose to ignore the functional dependence of the coefficients on those parameters in order to simplify the following analysis.

We form the four Kharitonov polynomials [47]:
\[ p_1(s) = s^5 + \alpha_{4_{\text{max}}} s^4 + \alpha_{3_{\text{min}}} s^3 + \alpha_{2_{\text{min}}} s^2 + \alpha_{1_{\text{max}}} s + \alpha_{0_{\text{max}}} , \]
\[ p_2(s) = s^5 + \alpha_{4_{\text{max}}} s^4 + \alpha_{3_{\text{max}}} s^3 + \alpha_{2_{\text{min}}} s^2 + \alpha_{1_{\text{min}}} s + \alpha_{0_{\text{max}}} , \]
\[ p_3(s) = s^5 + \alpha_{4_{\text{min}}} s^4 + \alpha_{3_{\text{min}}} s^3 + \alpha_{2_{\text{max}}} s^2 + \alpha_{1_{\text{max}}} s + \alpha_{0_{\text{min}}} , \]
\[ p_4(s) = s^5 + \alpha_{4_{\text{min}}} s^4 + \alpha_{3_{\text{max}}} s^3 + \alpha_{2_{\text{max}}} s^2 + \alpha_{1_{\text{min}}} s + \alpha_{0_{\text{min}}} . \]

By applying Routh's stability criterion [57], it can be shown that each of these four polynomials is stable for all \( \omega_p > 622 > \omega_{p_{\text{min}}} \). Hence, by Kharatinov's Theorem [47], we conclude that the system (4.21) is stable for all parameter values in \( \Pi \). This guarantees, by Theorem 4.1, the stability of the actual controlled system.

The eigenvalues of the closed-loop system with the controller (4.20a)-(4.20b) applied to the 64th order “truth” model (4.10a)-(4.10b) are also calculated for verification purposes. Selected closed-loop eigenvalues are shown in Table 9. It is observed that the desired damping in the first mode and overall stability are both achieved.

<table>
<thead>
<tr>
<th>Table 9: Closed-Loop Eigenvalues of the Truss Structure</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-31.73)</td>
</tr>
<tr>
<td>(-15.56 \pm 126.35i)</td>
</tr>
<tr>
<td>(-6.703 \pm 633.22i)</td>
</tr>
<tr>
<td>:</td>
</tr>
<tr>
<td>(-122.4 \pm 15922i)</td>
</tr>
<tr>
<td>(-134.2 \pm 16601i)</td>
</tr>
</tbody>
</table>

4.5 Decentralized Robust Controller Design

In Chapter III, a frequency domain approach to local robust controller design was presented. Here we discuss a state space approach to the same problem.
Consider an interconnected system described by (3.29a)-(3.29c). For notational simplicity, assume two control agents and dynamic interconnections (i.e., $\nu = 2, \mu = 1$). After applying the expansion (3.32), the expanded system is described by:

$$
\dot{x}_i = \hat{A}_i \dot{x}_i + \hat{A}_{ij} \dot{x}_j + \hat{B}_i u_i \quad (4.22a)
$$

$$
y_i = \hat{C}_i \dot{x}_i, \quad i,j = 1,2, \quad j \neq i. \quad (4.22b)
$$

Suppose a controller, described by:

$$
\dot{z}_2 = F_2 z_2 + G_2 y_2 \quad (4.23a)
$$

$$
u_2 = H_2 z_2 + K_2 y_2, \quad (4.23b)
$$

is designed by the second control agent. Then the overall system dynamics can be described by:

$$
\frac{d}{dt} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} \hat{A}_1 & \hat{A}_{12} & 0 \\ \hat{A}_{21} & \hat{A}_2 + \hat{B}_2 K_2 \hat{C}_2 & \hat{B}_2 H_2 \\ 0 & G_2 \hat{C}_2 & F_2 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ z_2 \end{bmatrix} + \begin{bmatrix} \hat{B}_1 \\ 0 \\ 0 \end{bmatrix} u_1 \quad (4.24a)
$$

$$
y_1 = \hat{C}_1 \dot{x}_1. \quad (4.24b)
$$

Note that the controller or even the subplant dynamics corresponding to the second control agent may not be known to the first agent. However, a certain amount of information, such as upper and lower bounds of the TFMs corresponding to these dynamics, may be available. By defining

$$
\Delta(s) = \begin{bmatrix} \hat{A}_{12} & 0 \end{bmatrix} \left( sI - \begin{bmatrix} \hat{A}_2 + \hat{B}_2 K_2 \hat{C}_2 & \hat{B}_2 H_2 \\ G_2 \hat{C}_2 & F_2 \end{bmatrix} \right)^{-1} \begin{bmatrix} \hat{A}_{21} \\ 0 \end{bmatrix}, \quad (4.25)
$$

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a design model, as depicted in Figure 38, is obtained for the first control agent.

By knowing certain bounds on $\Delta(s)$, a rational TFM $E(s;p)$ parametrized by a vector $p \in \Pi$ and a parameter set $\Pi$ can be determined as explained in Section 4.2. Let $(F(p), G(p), H(p), L(p))$ be a realization for $E(s;p)$. Then

$$\frac{d}{dt} \begin{bmatrix} x \\ \xi \end{bmatrix} = \begin{bmatrix} \dot{A}_1 + L & H \\ G & F \end{bmatrix} \begin{bmatrix} x \\ \xi \end{bmatrix} + \begin{bmatrix} \dot{B}_1 \\ 0 \end{bmatrix} u_1$$  \hspace{1cm} (4.26a)

$$y_1 = \begin{bmatrix} \dot{C}_1 & 0 \end{bmatrix} \begin{bmatrix} x \\ \xi \end{bmatrix}$$  \hspace{1cm} (4.26b)

is a design model for the first control agent. At this stage any uncertainties in the first subsystem dynamics can also be incorporated in this model and a robust controller for the first subsystem can be designed. A similar approach can be taken at the second station as well. This approach guarantees overall stability and desired performance.

**4.6 Summary**

State space robust controller design strategies have been considered. In particular, modeling of uncertain dynamics was discussed in Section 4.2. It has been shown that, under mild conditions on uncertain dynamics, it is possible to obtain
a rational TFM, possibly dependent on a parameter vector which varies over a
subset of a finite dimensional real vector space, to represent uncertain dynamics.
Furthermore, in many practical cases, uncertain dynamics can be represented by
a relatively low order TFM, even if the actual dynamics are of very high order.
The procedure of determining such a TFM has been discussed. It has been shown
that a controller which stabilizes the nominal system, including such a representa­
tion of uncertain dynamics, also stabilizes the actual system. Furthermore, desired
performance or relative stability can also be guaranteed.

State space models for systems with both parameter uncertainties and un­
certain dynamics were developed in Section 4.3. Once such a model is obtained,
already existing methods can be used to design robust controllers. The presented
approach gives a unified framework for the solution to the robust controller design
problem for systems with both parameter uncertainties and uncertain dynamics.

The presented approach was applied to a controller design problem for a large
flexible structure in Section 4.4. Only the lowest frequency mode was modeled for
design purposes and even this model is subject to uncertainties due to unknown
system parameters. All the higher modes were represented by a second order model
which depends on a parameter vector. The range of possible parameter variations
(the set Π) was determined and a first order controller was designed to ensure
stability and desired performance.

Finally, decentralized robust controller design in state space was discussed in
Section 4.5. It has been demonstrated that local controllers which guarantee overall
stability and desired performance can be designed, if certain limited information
about rest of the closed-loop system is available to each local agent. Although,
for notational simplicity, only decentralized systems with two control agents were
considered, it is possible to extend the approach to systems with more control
agents by modifying (4.25) to include the remaining subsystems.
5.1 Introduction and Problem Statement

A basic problem in control engineering is to design a controller for a plant such that the plant output tracks a specified reference input in the presence of certain disturbances. This problem is usually referred to as the servomechanism problem.

The fact that it is not possible in practice to obtain an exact mathematical model of a given plant, necessitates controllers which are robust to plant model variations. Furthermore, the controllers must be designed so that, not only the plant disturbances, but any possible measurement and controller disturbances are compensated as well.

Many researchers have contributed to the solution of the servomechanism problem. Johnson [58,59], Bhattacharrya and Pearson [60,61], Young and Willems [62], Davison and Goldenberg [63], Davison [64,65], Emami-Naeini and Franklin [66], Davison and Özgüner [67], Davison and Ferguson [68], and İftar and Özgüner [69,70] are a few that can be named.

In the remaining of this section centralized and decentralized robust servomechanism problems are formally stated and a brief review of the previous work on the existence of solutions to these problems are given. A linear-quadratic (LQ) optimal solution to the centralized servomechanism problem is presented in Sec-
tion 5.2. The solution for the decentralized case is given in Section 5.3. The decentralized solution also includes the optimal static and fixed order dynamic centralized output feedback as a special case. The frequency domain properties of the proposed controllers are discussed in Section 5.4. Some application problems are presented in Section 5.5.

5.1.1 Centralized Servomechanism Problem

Consider a plant described by the following LTI system model:

\[
\dot{x} = Ax + Bu + Ew \tag{5.1a}
\]
\[
y = Cx + Du + Fw \tag{5.1b}
\]
\[
e = y - y^r \tag{5.1c}
\]

where \(x \in \mathbb{R}^n\) is the state, \(u \in \mathbb{R}^m\) is the input, and \(y \in \mathbb{R}^r\) is the output. It is assumed that the plant, the measurements and the controller may be subject to disturbances \(w \in \mathbb{R}^q\), satisfying

\[
\dot{w} = A_1 w . \tag{5.2}
\]

The error \(e \in \mathbb{R}^r\) is the difference between the output \(y\) and the specified reference input \(y^r\) which is assumed to satisfy:

\[
y^r = G\sigma , \quad \dot{\sigma} = A_2 \sigma , \quad \sigma \in \mathbb{R}^f . \tag{5.3}
\]

It is also assumed that the output

\[
y^m = C^m x + D^m u + F^m w , \tag{5.4}
\]

where \(y^m \in \mathbb{R}^{r^m}\), is available for measurement. All matrices involved are appropriately dimensioned constant matrices.

We need the following definitions in the sequel.
Definition 5.1 The output $y$ is measurable (reconstructable) if there exists a transformation $T$ such that $Ty^m = y^a$ ($T(C^m x + F^m w) = C x + F w$) where $y^a \in \mathbb{R}^r$ is the actual output of the plant corresponding to $y$.

Let

$$\mathcal{R}_1 \triangleq \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m} \times \mathbb{R}^{r \times n} \times \mathbb{R}^{r \times m} \times \mathbb{R}^{n \times q} \times \mathbb{R}^{r \times q} \times \mathbb{R}^{r \times m} \times \mathbb{R}^{m \times m} \times \mathbb{R}^{r \times m} \times \mathbb{R}^{r \times q}$$

and

$$\mathcal{R}_2 \triangleq \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m} \times \mathbb{R}^{n \times q} .$$

By plant perturbations on $\Omega_1$ we mean perturbations of the plant matrices of the form

$$(A, B, C, D, E, F, C^m, D^m, F^m) \rightarrow (A + \delta A, B + \delta B, C + \delta C, D + \delta D, E + \delta E, F + \delta F, C^m + \delta C^m, D^m + \delta D^m, F^m + \delta F^m)$$

for some $(\delta A, \delta B, \delta C, \delta D, \delta E, \delta F, \delta C^m, \delta D^m, \delta F^m) \in \Omega_1 \subset \mathcal{R}_1$. By restricted plant perturbations on $\Omega_2$ we mean perturbations of the $A$, $B$, and $E$ matrices, only, of the form

$$(A, B, E) \rightarrow (A + \delta A, B + \delta B, E + \delta E)$$

for some $(\delta A, \delta B, \delta E) \in \Omega_2 \subset \mathcal{R}_2$.

Definition 5.2 Given the system (5.1a)-(5.1c) and the measurements (5.4) suppose that there exists a controller so that asymptotic tracking takes place (i.e., $e \rightarrow 0$ as $t \rightarrow \infty$) and the resultant controlled system is stable. Let $\Omega_1 \subset \mathcal{R}_1$ ($\Omega_2 \subset \mathcal{R}_2$) be such that, the resultant controlled system remains stable under all
plant perturbations on \( \Omega_1 \) (under all restricted plant perturbations on \( \Omega_2 \)). Then if asymptotic tracking still takes place under all plant perturbations on \( \Omega_1 \) (under all restricted plant perturbations on \( \Omega_2 \)), the controller is said to be a robust (weak robust) controller.

It is desired to find a stabilizing robust controller for the plant \((5.1a)-(5.1c)\) so that \( e \to 0 \) as \( t \to \infty \). Necessary and sufficient conditions for the existence of such a controller were discussed by Davison:

**Lemma 5.1** [63] There exists a robust (weak robust) LTI controller for \((5.1a)-(5.1c)\) such that \( e \to 0 \) as \( t \to \infty \) for all disturbances described by \((5.2)\), for all specified reference inputs described by \((5.3)\), and the overall system is stable if and only if the following conditions all hold:

(a) \((A, B)\) is stabilizable,

(b) \((C^m, A)\) is detectable,

(c) \(m \geq r\),

(d) the transmission zeros of \((C, A, B, D)\) do not coincide with the unstable eigenvalues of \(A_1\) or \(A_2\),

(e) the output \(y\) is measurable (reconstructible).

Note that the conditions (c) and (d) can be combined into

\[
\text{rank } \begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix} = n + r, \quad \forall \lambda \in \left( \rho(A_1) \cup \rho(A_2) \right) \cap \mathbb{C}^+.
\]

If arbitrarily fast tracking is desired (i.e., \( e \to 0 \) arbitrarily fast) then stabilizability/detectability conditions in Lemma 5.1 must be changed to controllability conditions.
bility/observability conditions and the transmission zeros of \((C,A,B,D)\) must not coincide with any eigenvalues of \(A_1\) or \(A_2\).

5.1.2 Decentralized Servomechanism Problem

A decentralized plant with \(\nu\) control agents may be described by the following LTI system model

\[
\dot{x} = Ax + \sum_{i=1}^{\nu} B_i u_i + E w \tag{5.5a}
\]

\[
y_i = C_i x + D_i u_i + F_i w , \quad i = 1, ..., \nu \tag{5.5b}
\]

\[
\epsilon_i = y_i - y_i^r , \quad i = 1, ..., \nu \tag{5.5c}
\]

where \(x \in \mathbb{R}^n\) is the state, and \(u_i \in \mathbb{R}^{m_i}\) and \(y_i \in \mathbb{R}^{r_i}\) are, respectively, the input and the output of the \(i^{th}\) control agent \((i = 1, ..., \nu)\). It is assumed that the plant, the measurements, and the controllers may be subject to disturbances \(w \in \mathbb{R}^q\) satisfying (5.2). The \(i^{th}\) error \(\epsilon_i \in \mathbb{R}^{r_i}\) is the difference between \(y_i\) and a specified reference input:

\[
y_i^r = G_i \sigma , \quad i = 1, ..., \nu \tag{5.6a}
\]

where \(\sigma \in \mathbb{R}^q\) satisfies

\[
\dot{\sigma} = A_2 \sigma . \tag{5.6b}
\]

It is also assumed that the output

\[
y_i^m = C_i^m x + D_i^m u + F_i^m w , \tag{5.7}
\]

where \(y_i^m \in \mathbb{R}^{r_i^m}\), is available for measurement by the \(i^{th}\) control agent \((i = 1, ..., \nu)\). All matrices involved are appropriately dimensioned constant matrices.

Let us define
\[
C \triangleq \begin{bmatrix}
C_1 \\
\vdots \\
C_\nu
\end{bmatrix}, \quad \quad C^m \triangleq \begin{bmatrix}
C_1^m \\
\vdots \\
C_\nu^m
\end{bmatrix}, \quad \quad B \triangleq [B_1, \ldots, B_\nu],
\]

\[
C^* \triangleq \begin{bmatrix}
C_1^m & 0 & 0 & \ldots & 0 & 0 \\
0 & I_{r_1} & 0 & \ldots & 0 & 0 \\
C_2^m & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & I_{r_2} & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & C_\nu^m & 0 \\
0 & 0 & 0 & \ldots & 0 & I_{r_\nu}
\end{bmatrix},
\]

and

\[
D \triangleq \text{blockdiag}(D_1, \ldots, D_\nu).
\]

**Definition 5.3** Consider the system

\[\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx
\end{align*}\]  \hspace{1cm} (5.8a)

\[y = Cx\]  \hspace{1cm} (5.8b)

denoted by \((C, A, B)\), where \(u \in \mathbb{R}^m\) and \(y \in \mathbb{R}^r\). Let \(\kappa \subset \mathbb{R}^{m \times r}\), then \(\lambda\) is a fixed mode of \((C, A, B)\) with respect to \(\kappa\) if

\[
\lambda \in \bigcap_{K \in \kappa} \rho(A + BKC).
\]  \hspace{1cm} (5.9)

Furthermore, for the decentralized system of \((5.5a)-(5.5c)\), which is denoted by \((C', A, B; r_1, \ldots, r_\nu; m_1, \ldots, m_\nu)\). consider
\[
\kappa \triangleq \left\{ K \middle| K = \text{blockdiag} (K_1, \ldots, K_\nu), \ K_i \in \mathbb{R}^{m_i \times r_i}, \ i = 1, \ldots, \nu \right\} . \tag{5.10}
\]

Then the fixed modes of \((C, A, B)\) with respect to \(\kappa\) are called the decentralized fixed modes (DFMs) of the system \((C', A, B; r_1, \ldots, r_\nu; m_1, \ldots, m_\nu)\).

It is desired to find a stabilizing decentralized robust controller for the plant \((5.5a)-(5.5c)\) such that \(e_i \to 0\) as \(t \to \infty\) \(\forall i \in \{1, \ldots, \nu\}\). Necessary and sufficient conditions for the existence of such a controller were previously considered by Davison:

**Lemma 5.2** [65] There exists a decentralized robust (weak robust) LTI controller for the system \((5.5a)-(5.5c)\) such that \(e_i \to 0\) as \(t \to \infty\) \(\forall i \in \{1, \ldots, \nu\}\), for all disturbances described by \((5.5)\), for all specified reference inputs described by \((5.6a)-(5.6b)\) and such that the overall system is stable if and only if the following conditions all hold:

(a) \((C^m, A, B; r_1^m, \ldots, r_\nu^m; m_1, \ldots, m_\nu)\) has no unstable DFMs.

(b) The DFMs of
\[
\begin{pmatrix}
C^*, & \begin{bmatrix}
A & 0 \\
C & \lambda I
\end{bmatrix} \\
B & D
\end{pmatrix}; \ r_1^m + r_2^m + \ldots + r_\nu^m; \ m_1, \ldots, m_\nu
\]
do not contain \(\lambda\), \(\forall \lambda \in (\rho(A_1) \cup \rho(A_2)) \cap \mathbb{C}^+\).

(c) The outputs \(y_i\) \((i = 1, \ldots, \nu)\) are measurable (reconstructable).

If arbitrarily fast tracking is desired, the system in (a) must have no DFMs and the DFMs of the system in (b) must not contain \(\lambda\), \(\forall \lambda \in \rho(A_1) \cup \rho(A_2)\).
5.2 Centralized LQ–Optimal Controller

In this section a LQ approach to solve the centralized robust servomechanism problem is presented. First note that, due to the disturbance and tracking signals, the widely used quadratic cost function

\[ J = \int_0^\infty (x^T Q x + u^T R u) \, dt \]  \hspace{1cm} (5.11)

is unbounded for all possible control inputs \( u \), unless both \( A_1 \) and \( A_2 \) have eigenvalues only in \( \mathbb{C}^- \).

In order to determine a mathematically well behaved, yet physically meaningful cost function, let us define

\[ \Lambda(s) = s^p + a_{p-1} s^{p-1} + \ldots + a_1 s + a_0 \]

\[ \triangleq \prod_{i=1}^{p-q} (s - \lambda_i^r) \cdot \prod_{i=1}^q (s - \lambda_i^d) \] \hspace{1cm} (5.12)

where \( \lambda_i^d \), \( i = 1, \ldots, q \) are the eigenvalues of \( A_1 \) and \( \lambda_i^r \), \( i = 1, \ldots, p - q \) are the eigenvalues of \( A_2 \) that are not the eigenvalues of \( A_1 \) (all multiplicities included), i.e., \( \Lambda(s) \) is the least common multiple of the characteristic polynomials of \( A_1 \) and \( A_2 \). Let us also define the operator:

\[ S \triangleq \frac{d^p}{d\tau^p} + a_{p-1} \frac{d^{p-1}}{d\tau^{p-1}} + \ldots + a_1 \frac{d}{d\tau} + a_0 \] \hspace{1cm} (5.13)

where \( a_0, a_1, \ldots, a_{p-1} \) are the scalars defined in (5.12). Then define

\[ \tilde{x} \triangleq S x \] \hspace{1cm} (5.14a)

\[ \tilde{u} \triangleq S u \] \hspace{1cm} (5.14b)

\[ \epsilon \triangleq [\epsilon_1^T, \epsilon_2^T, \ldots, \epsilon_p^T]^T \] \hspace{1cm} (5.14c)

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where

\[
\varepsilon_1 \triangleq e^{(p-1)} + a_{p-1}e^{(p-2)} + \ldots + a_3 \varepsilon + a_2 \varepsilon + a_1 \varepsilon
\]

\[
\varepsilon_2 \triangleq e^{(p-2)} + a_{p-1}e^{(p-3)} + \ldots + a_3 \varepsilon + a_2 \varepsilon
\]

\[
\vdots
\]

\[
\varepsilon_p \triangleq \varepsilon,
\]

and

\[
z \triangleq \begin{bmatrix} \bar{x} \\ \varepsilon \end{bmatrix}.
\]

(5.14d)

By noting that

\[
S\bar{w} = 0
\]

(5.15a)

and

\[
Sy^r = 0,
\]

(5.15b)

we obtain

\[
\dot{z} = \bar{A}z + \bar{B}\bar{u}
\]

(5.16a)

\[
\varepsilon = \bar{C}z
\]

(5.16b)

where

\[
\bar{A} = \begin{bmatrix}
A & 0 & \ldots & 0 & 0 \\
C & 0 & \ldots & 0 & -a_{r}I_r \\
0 & I_r & \ldots & \ldots & \ldots \\
\vdots & \vdots & \ddots & \ddots & \ddots \\
0 & I_r & \ldots & -a_{p-1}I_r & 0
\end{bmatrix},
\]

\[
\bar{B} = \begin{bmatrix}
B \\
P \\
\vdots \\
0
\end{bmatrix}.
\]
and

\[
\mathcal{C} = \begin{bmatrix}
0 & 0 & \cdots & 0 & I_r
\end{bmatrix}.
\]

The following two lemmas are immediate:

**Lemma 5.3** \((\mathbf{A}, \mathbf{B})\) is controllable (stabilizable) if and only if the following conditions all hold:

(a) \((A, B)\) is controllable (stabilizable),

(b) \(m \geq r\),

(c) the transmission zeros of \((C, A, B, D)\) do not coincide with the (unstable) eigenvalues \(A_1\) or \(A_2\).

**Proof:** \((\mathbf{A}, \mathbf{B})\) is controllable (stabilizable) if and only if

\[
\text{rank} \begin{bmatrix}
\lambda I - \mathbf{A} & \mathbf{B}
\end{bmatrix} = n + pr \quad \forall \lambda \in \mathbb{C}^+
\]

if and only if

\[
\text{rank} \begin{bmatrix}
\lambda I - A & 0 & B \\
- C & \Lambda(\lambda)I_r & D
\end{bmatrix} = n + r \quad \forall \lambda \in \mathbb{C}^+
\] (5.17)

(i) if \(\lambda (\lambda \in \mathbb{C}^+)\) is such that \(\Lambda(\lambda) = 0\) (i.e., \(\lambda\) is an eigenvalue of \(A_1\) or \(A_2\)), then (5.17) holds if and only if

\[
\text{rank} \begin{bmatrix}
\lambda I - A & B \\
- C & D
\end{bmatrix} = n + r
\]

if and only if \(\lambda\) is not a transmission zero of \((C, A, B, D)\) and \(m \geq r\).

(ii) \(\forall \lambda (\forall \lambda \in \mathbb{C}^+)\) such that \(\Lambda(\lambda) \neq 0\) (5.17) holds if and only if

\[
\text{rank} \begin{bmatrix}
\lambda I - A & B
\end{bmatrix} = n
\]
if and only if \( \lambda \) is either not an eigenvalue of \( A \) or is a controllable mode of \((A, B)\).

Furthermore by (i) those \( \lambda (\lambda \in \mathbb{C}^+) \) such that \( \Lambda(\lambda) = 0 \) and \( \lambda \) is an eigenvalue of \( A \) are also controllable. Hence the above statement implies, and is implied by: \((A, B)\) is controllable (stabilizable).

**Lemma 5.4** \((C', A)\) is observable (detectable) if and only if \((C', A)\) is observable (detectable).

**Proof:** \((C', A)\) is observable (detectable) if and only if

\[
\text{rank } \begin{bmatrix} \lambda I - A \\ C \end{bmatrix} = n + pr \quad \forall \lambda (\forall \lambda \in \mathbb{C}^+) 
\]

if and only if

\[
\text{rank } \begin{bmatrix} \lambda I - A \\ C \end{bmatrix} = n \quad \forall \lambda (\forall \lambda \in \mathbb{C}^+) 
\]

if and only if \((C, A)\) is observable (detectable). \(\square\)

Suppose, it is desired to find a controller for the system \((5.1a)-(5.1c)\) such that asymptotic tracking takes place (i.e., \( \epsilon \to 0 \) as \( t \to \infty \)) for all disturbances described by \((5.2)\) and for all reference inputs described by \((5.3)\), the overall system is stable, and the cost function

\[
J = \int_0^\infty (z^T Q z + \bar{u}^T R \bar{u}) \, dt 
\]

where \( Q = Q^T \geq 0, R = R^T > 0 \) is minimized.

**Remark 5.1** If \( Q \) is chosen such that

\[
Q = \text{blockdiag} (Q_0, Q_1, \ldots, Q_p)
\]
where \( Q_0 \in \mathbb{R}^{n \times n} \), \( Q_i \in \mathbb{R}^{r \times r} \), \( i = 1, \ldots, p \), then \( Q_p \) is the weight of the error \( e \) in the cost. \( \{Q_i, Q_{i+1}, \ldots, Q_p\} \), \( i = 1, \ldots, p-1 \) represents the weights of the error and its first \( p-i \) derivatives, and \( Q_0 \) is the weight of \( \bar{x} \), which is a weighted sum of the state \( x \) and its first \( p \) derivatives and quantifies the effort of the system in compensating for the disturbances and achieving tracking. Furthermore, \( R \) is the weight of \( \bar{u} \), which is a measure of the control effort.

Suppose that the state \( x \) is measurable and consider the control

\[
 u = K_1 x + K_2 \eta + \Gamma \omega
\]  

(5.19)

where

\[
 \begin{bmatrix}
 K_1 \\
 K_2
 \end{bmatrix} = -R^{-1} \tilde{B}^T P
\]  

(5.20)

and \( P = P^T \geq 0 \) is the solution of

\[
 \tilde{A}^T P + P \tilde{A} + Q - P \tilde{B} R^{-1} \tilde{B}^T P = 0,
\]  

(5.21)

\( \eta \) is the output of the \textit{servocompensator} which is described by:

\[
 \dot{\xi} = A_\delta \xi + B_\delta e
\]  

(5.22a)

\[
 \eta = C_\delta \xi + F_\delta \omega
\]  

(5.22b)

where

\[
 A_\delta = T \begin{bmatrix}
 0 & I_r & 0 \\
 0 & 0 & 0 \\
 0 & 0 & I_r \\
 -a_0 I_r & -a_1 I_r & \ldots & -a_p I_r \\
 \end{bmatrix} T^{-1}, \quad B_\delta = T \begin{bmatrix}
 0 \\
 0 \\
 0 \\
 I_r
 \end{bmatrix}
\]
and $T$ is a nonsingular transformation matrix. The constant matrices $\Gamma$ and $F_s$ are appropriately dimensioned, but otherwise arbitrary.

Note that both the control input (5.19) and the servocompensator output (5.22b) may be subject to certain disturbances. These disturbances may be same as those applied to the plant, or they may be different. In the latter case, the disturbance process (5.2) must be defined to include these disturbances as well.

We note that, if the output $y$ of the system (5.1a)-(5.1c) is measurable, the input $e$ to the servocompensator (5.22a)-(5.22b) is obtained as

$$e = Ty^m - y^r$$

(5.23a)

where $T$ is the transformation which gives $Ty^m = y^\theta$ (see Definition 5.1). If however, $y$ is not measurable, but reconstructable, $e$ is obtained as

$$e = Ty^m + (D - TD^m)u - y^r.$$

(5.23b)

The closed-loop system is described by

$$\begin{bmatrix}
\dot{x} \\
\dot{\xi}
\end{bmatrix} = \begin{bmatrix}
A + BK_1 & BK_2C_s \\
B_sC + B_sDK_1 & A_s + B_sDK_2C_s
\end{bmatrix}\begin{bmatrix}
x \\
\xi
\end{bmatrix}
+ \begin{bmatrix}
BK_2F_s + B\Gamma + E \\
B_s(DK_2F_s + D\Gamma + F)
\end{bmatrix}w + \begin{bmatrix}
0 \\
-B_s
\end{bmatrix}y^r.$$

(5.24)
Lemma 5.5 The eigenvalues of the closed-loop system (5.24) are the eigenvalues of $\bar{A} - \bar{B}R^{-1}\bar{B}^TP$.

Proof: The system matrix in (5.24) can be written as $U^{-1}(\bar{A} - \bar{B}R^{-1}\bar{B}^TP)U$, where $U = \text{blockdiag}(I_n, C)$, i.e., it is similar to $\bar{A} - \bar{B}R^{-1}\bar{B}^TP$, hence the result follows. □

Now the following result can be proved.

Theorem 5.1 If

(a) $(A, B)$ is stabilizable,
(b) $m \geq r$,
(c) the transmission zeros of $(C, A, B, D)$ do not coincide with the unstable eigenvalues of $A_1$ or $A_2$,
(d) the output $y$ is measurable (reconstrucrable), and
(e) $(Q^{1/2}, \bar{A})$ is detectable,

then the controller described by (5.19), (5.22a)-(5.22b) enjoys the following properties:

(i) Asymptotic tracking takes place for all disturbances described by (5.2) and for all reference inputs described by (5.3).
(ii) The overall system is stable (in the sense that the closed-loop system eigenvalues are in $\mathbb{C}^-$).
(iii) The cost, given by (5.18), is minimized.
(iv) If the disturbance $w$, and the reference input $y^r$ are bounded, then the input $u$ is bounded.
(v) The controller is robust (weak robust).
Proof: Note that the control (5.19) is equivalent to

$$\hat{u} = K_1 \hat{x} + K_2 e = -R^{-1} \tilde{B}^TPz$$  \hspace{1cm} (5.25)

in the modified state and input spaces (5.14d) and (5.14b). By the hypothesis and Lemma 5.3 ($\tilde{A}, \tilde{B}$) is stabilizable. This fact guarantees that the control (5.19) when applied to the system (5.1a)-(5.1c) (or equivalently the control (5.25) when applied to the system (5.16a)-(5.16b)) minimizes the cost (5.18) [18]. Hence, (iii) is proved.

Since $(Q^{1/2}, \tilde{A})$ is detectable, the control (5.25) stabilizes the system (5.16a)-(5.16b), i.e., $\tilde{A} - \tilde{B}R^{-1} \tilde{B}^TP$ has eigenvalues only in $C^-$ [18]. Hence, by Lemma 5.5 (ii) holds.

Property (i) follows from the asymptotic stability of the system (5.16a)-(5.16b) under the control (5.25).

To prove part (iv) note that, whenever $w$ and $y^r$ are bounded, the operator $S$ defined by (5.13) is a stable differential operator. Furthermore, $\bar{u}(t)$ is bounded and $\bar{u}(t) \to 0$ as $t \to \infty$ (it is the solution of the LQR problem). Therefore the solution of

$$Su = \bar{u}(t)$$

is bounded.

To prove part (v) note that, (5.23a) gives $e = y - y^r$ under any plant perturbations on $R_1$, and (5.23b) gives $e = y - y^r$ under any restricted plant perturbations on $R_2$. Furthermore, as long as the input $c$ to the servocompensator (5.22a)-(5.22b) is given by $c = y - y^r$, the closed-loop system matrix in (5.24) is similar to the closed-loop system matrix of (5.16a)-(5.16b) under the control (5.25), under
any plant perturbations on $R_1$ or under any restricted plant perturbations on $R_2$. Hence, the result follows.

Remark 5.2 Properties (i) and (v) of the above Theorem have been proven in [63] for the system (5.1a)-(5.1c) under a controller which includes a disturbance free servocompensator, of the form (5.22a)-(5.22b) with $F_e = 0$, and any stabilizing feedback of the form (5.19) with $\Gamma = 0$. The present approach enables us to extend the proof to noisy controllers (i.e., controllers subject to disturbances satisfying (5.2)), and brings more insight by considering the problem in the modified state-space (5.14d).

Remark 5.3 Necessary and sufficient conditions for the existence of a LTI controller for the robust servomechanism problem have been given in [63] (see Lemma 5.1). By comparing these conditions with the conditions of Theorem 5.1, we observe that, if there exists a solution to the robust (weak robust) servomechanism problem and $Q$ is chosen properly, then there exists a solution to the above stated optimal control problem with properties (i)-(v) of Theorem 5.1.

Remark 5.4 Another optimal control approach to solve the robust servomechanism problem has been presented in [68]. Besides the formulation, the basic difference between this approach and the one presented here is in the choice of the cost function. The cost function defined here allows the designer to assign desired weights to the physically meaningful quantities (see Remark 5.1) and shape the response accordingly. Furthermore, a parameter optimization approach (via non-linear programming) was undertaken in [68]. Here, on the other hand, the necessary conditions of optimality are utilized. The design approach presented
here is computationally much simpler and requires only the solution of an \( n + \tau p \) dimensional matrix Riccati equation.

Weighting only the error \( e \) and the control effort \( \tilde{u} \) in the cost function may be of some special interest.

**Corollary 5.1** If the conditions (a)-(d) of Theorem 5.1 hold and \((C', A)\) is detectable, then the controller described by (5.19), (5.22a)-(5.22b) with \( Q = \tilde{C}^T \tilde{Q} \tilde{C} \) in (5.21) minimizes the cost function

\[
J = \int_0^\infty (e^T \tilde{Q} e + \tilde{u}^T R \tilde{u}) \, dt
\]

where \( \tilde{Q} = Q^T > 0 \). Furthermore asymptotic tracking still takes place, the overall system is stable, the input \( u \) is bounded if \( w \) and \( y^r \) are bounded, and the controller is robust (weak robust).

**Proof:** Since \( \text{rank} (\tilde{Q}^{1/2}) = \text{rank} (\tilde{Q}) = r \), by Lemma 5.4, \((\tilde{Q}^{1/2}, \tilde{A})\) is detectable when \((C', A)\) is detectable, where \( Q^{1/2} = Q^{1/2} \tilde{C} \). Hence, the result follows from Theorem 5.1.

We may also wish to consider the case when only the output \( y^m \) given by (5.4) is available for measurement (i.e., the state \( x \) is not completely measurable). Assuming that \((C'^m, A)\) is detectable, it is possible to design an observer, possibly subject to disturbances satisfying (5.2), with inputs \( u \) and \( y^m \), which produces an estimate \( \hat{x} \in \mathbb{R}^n \) of the state \( x \) [71]. Then the controller

\[
u = K_1 \hat{x} + K_2 \eta + \Gamma w
\]

together with (5.22a)-(5.22b) produces an asymptotically equivalent response to the response of the optimal controller described by (5.19), (5.22a)-(5.22b). Here the gains \( K_1 \) and \( K_2 \) are given by (5.20) as before. Furthermore, if \((C'^m, A)\) is
observable then the difference between the two responses can be made arbitrarily small by choosing arbitrarily fast eigenvalues for the observer.

**Theorem 5.2** If the conditions of Theorem 5.1 hold and \((C^m, A)\) is detectable, then the controller described by (5.27), (5.22a)-(5.22b) enjoys the properties (i), (ii), (iv), and (v) of Theorem 5.1.

**Proof:** Parts (i), (ii), and (iv) can be proved by using the separation principle ([18], pp. 382) and the arguments in the proof of Theorem 5.1. Part (v) is proved as in Theorem 5.1 by using the modified state space \(z = [\bar{x}^T, \bar{\theta}^T, \varepsilon^T]^T\) instead of (5.14d), where \(\bar{\theta} \triangleq S\theta\), and \(\theta\) is the state of the observer. □

In some cases it may not be practical to employ an observer. Instead, one may wish to design simply a static output feedback controller or a dynamic output feedback controller with a pre-specified order. These cases are considered in the following section as special cases of decentralized controllers (with a single control agent).

### 5.3 Decentralized LQ-Optimal Controllers

Consider the decentralized control system (5.5a)-(5.5c). Let

\[
\bar{y}_i \triangleq y_i^m - D_i^m u_i, \quad i = 1, ..., \nu
\]  
(5.28)

where \(y_i^m (i = 1, ..., \nu)\) are the measurements defined in (5.7).

First consider the case when no dynamics other than the local servocompensators (to be defined later) are allowed for control purposes.

#### 5.3.1 Static Feedback

In this case it is desired to find local controllers for the system (5.5a)-(5.5c). when they exist, of the form
\[ u_i = K_i^1 \ddot{y}_i + K_i^2 \eta_i + \Gamma_i w, \quad i = 1, \ldots, \nu \tag{5.29} \]

such that asymptotic tracking takes place (i.e., \( \epsilon_i \to 0 \) as \( t \to \infty, i = 1, \ldots, \nu \)) for all disturbances described by (5.2) and for all reference inputs described by (5.6a)-(5.6b), the overall system is stable, and the cost function

\[ J = \int_{0}^{\infty} \left( z^T Q z + \sum_{i=1}^{\nu} \ddot{u}_i^T R_i \ddot{u}_i \right) dt \tag{5.30} \]

where \( Q = Q^T \geq 0, R_i = R_i^T > 0, i = 1, \ldots, \nu \) is minimized. Here

\[ \ddot{u}_i \triangleq S u_i, \quad i = 1, \ldots, \nu, \tag{5.31} \]

\( z \) is given by (5.14d) with

\[ \epsilon \triangleq [\epsilon_1^T, \ldots, \epsilon_\nu^T]^T, \tag{5.32} \]

and \( \eta_i \) is the output of the \( i^{th} \) local servocompensator described by:

\[ \dot{\xi}_i = A_{s_i} \xi_i + B_{s_i} \epsilon_i \tag{5.33a} \]

\[ \eta_i = C_{s_i} \xi_i + F_{s_i} w, \tag{5.33b} \]

where \( A_{s_i}, B_{s_i}, \) and \( C_{s_i} \) are, respectively, same as \( A_s, B_s, \) and \( C_s, \) shown following (5.22b), with \( I_r \) replaced by \( I_r, \). The constant matrices \( \Gamma_i, \) and \( F_{s_i}, i = 1, \ldots, \nu, \) are appropriately dimensioned, but otherwise arbitrary.

By (5.15a) and \( S y_i^T = 0 \) \((i = 1, \ldots, \nu)\), it follows that \( z(t) \) satisfies

\[ \dot{z} = \bar{A} z + \sum_{i=1}^{\nu} \bar{B}_i \ddot{u}_i \tag{5.34} \]

where \( \bar{A} \) is as in (5.16a), and

\[ \bar{B}_i \triangleq \begin{bmatrix} B_i \\ D_i' \\ 0_{r(p-1)\times m_i} \end{bmatrix}, \tag{5.35} \]
where
\[ D_i^* \triangleq \begin{bmatrix} 0_{m_i \times \sum_{j=1}^{i-1} r_j}, & D_i^T, & 0_{m_i \times \sum_{j=i+1}^{\nu} r_j} \end{bmatrix}^T. \]

Furthermore, if we let
\[ v_i \triangleq \begin{bmatrix} y_i \\ \eta_i \end{bmatrix}, \quad i = 1, \ldots, \nu \] (5.36)

and
\[ \tilde{v}_i \triangleq S v_i, \quad i = 1, \ldots, \nu, \] (5.37)
then we obtain
\[ \tilde{v}_i = \tilde{C}_i z, \quad i = 1, \ldots, \nu \] (5.38)

where
\[ \tilde{C}_i \triangleq \begin{bmatrix} C_i^m & 0 \\ 0 & C_i^* \end{bmatrix}, \] (5.39a)

\[ C_i^* \triangleq \text{blockdiag} \left( \Delta_i, \ldots, \Delta_i \right), \] (5.39b)

\[ \Delta_i \triangleq \begin{bmatrix} 0_{r_i \times \sum_{j=1}^{i-1} r_j}, & I_{r_i}, & 0_{r_i \times \sum_{j=i+1}^{\nu} r_j} \end{bmatrix}, \] (5.39c)

and
\[ \tilde{u}_i = \tilde{K}_i \tilde{v}_i, \] (5.39d)

where
\[ \tilde{K}_i \triangleq \begin{bmatrix} K_i^1 & K_i^2 \end{bmatrix}, \quad i = 1, \ldots, \nu. \] (5.40)

If \( \{\tilde{K}_1, \ldots, \tilde{K}_\nu\} \) is such that
\[ F \Delta A + \sum_{i=1}^{\nu} \bar{p}_i K_i \bar{C}_i \] (5.11)

has eigenvalues only in \( \mathbb{C}^- \), then the cost (5.30) is given by

\[ J = \text{trace} (PZ_0) \] (5.42)

where

\[ Z_o \Delta z_o z_o^T , \] (5.43a)

\[ z_o \Delta \begin{bmatrix} \bar{x}(0) \\ \varepsilon(0) \end{bmatrix} , \] (5.43b)

and \( P = P^T \geq 0 \) satisfies

\[ \hat{F}^T P + P \hat{F} + \bar{Q} = 0 , \] (5.44)

where

\[ \bar{Q} \Delta Q + \sum_{i=1}^{\nu} \bar{C}_i^T \bar{K}_i R_i \bar{K}_i \bar{C}_i . \] (5.45)

To minimize (5.42) over \( \{ \bar{K}_i, ..., \bar{K}_\nu \} \) subject to (5.44), one obtains the necessary conditions

\[ \bar{B}_i^T P L \bar{C}_i^T + R_i \bar{K}_i (\bar{C}_i L \bar{C}_i^T)^{-1} = 0 , \quad i = 1, ..., \nu \] (5.46)

where the matrix of Lagrange multipliers \( L = L^T \geq 0 \) satisfies

\[ \hat{F} L + L \hat{F}^T + Z_o = 0 . \] (5.47)

Assuming \( C_i^m \) is of the full rank, \((\bar{C}_i L \bar{C}_i^T)\) is non-singular, and the feedback gains are found from (5.46) as

\[ \bar{K}_i \Delta \begin{bmatrix} K_i^1 \\ K_i^2 \end{bmatrix} = -R_i^{-1} \bar{B}_i^T P L \bar{C}_i^T (\bar{C}_i L \bar{C}_i^T)^{-1} . \quad i = 1, ..., \nu \] (5.48)
Hence, the solution to the optimal decentralized linear static feedback design problem (if it exists) may be obtained by solving the two Liapunov equations (5.44) and (5.47) and the set of equations (5.48) simultaneously. A number of algorithms have been proposed to solve such equations recently. The descent Anderson-Moore method [72], which is based on the Anderson-Moore algorithm [73], is an effective method. Levine-Athans algorithm [74,75] provides another effective method. A survey of various such methods can be found in a recent paper by Mäkilä and Toivonen [72]. Software packages have also been developed to numerically solve the constrained optimization problem (e.g., [39]).

Also note that the optimal solution depends on the initial conditions. However, in a servomechanism problem, initial conditions are usually not known beforehand. In this case one may choose to minimize the expected value of the cost (5.30) over the initial conditions. This choice allows the designer to redefine \( Z_o \) in (5.47) as:

\[
Z_o \triangleq E[z_0 z_o^T].
\]  

To illustrate the calculation of \( Z_o \), let us consider the following example:

**Example 5.1** Consider the system (5.5a)-(5.5c) which has to be regulated against constant disturbances with \( y^r = 0 \). Hence

\[
z = \begin{bmatrix} \tilde{x} \\ \epsilon \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \epsilon \end{bmatrix}
\]

and
\[
\begin{bmatrix}
    \dot{x}(0) \\
    c_1(0) \\
    \vdots \\
    c_\nu(0)
\end{bmatrix}
= \begin{bmatrix}
    A x(0) + \sum_{i=1}^{\nu} B_i u_i(0) + F w(0) \\
    C_1 x(0) + D_1 u_1(0) + F_1 w(0) \\
    \vdots \\
    C_\nu x(0) + D_\nu u_\nu(0) + F_\nu w(0)
\end{bmatrix}
\]  

(5.51)

where

\[
u_i(0) = K_i^1 (C_i^m x(0) + F_i^m w(0)) + K_i^2 \left(C_i^s \xi_i(0) + F_i^s w(0) \right) + \Gamma_i w(0) .
\]  

(5.52)

Since the overall system is stable, one can assume that the system is at rest before the disturbances are applied (i.e., \(x(0) = 0, \xi_i(0) = 0, i = 1, ..., \nu\)). Therefore (5.51) reduces to

\[
z_o = \bar{E} w_0
\]  

(5.53)

where

\[
\bar{E} \triangleq \begin{bmatrix}
    \sum_{i=1}^{\nu} B_i (K_i^1 F_i^m + K_i^2 F_s + \Gamma_i) + E \\
    D_1 (K_1^1 F_1^m + K_1^2 F_1 + \Gamma_1) + F_1 \\
    \vdots \\
    D_\nu (K_\nu^1 F_\nu^m + K_\nu^2 F_s + \Gamma_\nu) + F_\nu
\end{bmatrix}
\]  

(5.54)

and

\[
w_0 \triangleq w(0)
\]  

(5.55)

which gives

\[
Z_o \triangleq E[z_o z_o^T] = \bar{E} w_0 \bar{E}^T
\]  

(5.56)

where
Next the case, when the dynamics other than the local servocompensators are also allowed, is considered.

5.3.2 Dynamic Feedback

In this case it is desired to find local controllers for the system (5.5a)-(5.5b), when they exist, of the form

\[ u_i = K_i^1 \ddot{y}_i + K_i^2 \eta_i + K_i^3 \psi_i + \Gamma_i w, \quad i = 1, \ldots, \nu \]

such that asymptotic tracking takes place for all disturbances described by (5.2) and for all reference inputs described by (5.6a)-(5.6b), the overall system is stable, and an appropriate cost function (to be defined later) is minimized. Here \( \eta_i \) is the output of the \( i^{th} \) local servocompensator described by (5.33a)-(5.33b) and \( \psi_i \) is the output of the \( i^{th} \) local stabilizing compensator described by:

\[ \dot{\zeta}_i = A_{c_i} \zeta_i + B_{c_i}^1 \ddot{y}_i + B_{c_i}^2 \eta_i + C_{c_i} \psi_i + E_{c_i} w, \quad \zeta_i \in \mathbb{R}^{n_i} \]  

\[ \psi_i = C_{c_i} \zeta_i + F_{c_i} w, \]

where the dimension of the compensator \( n_i \) is chosen \textit{a priori}, and \( \Gamma_i, E_{c_i}, F_{c_i}, i = 1, \ldots, \nu \) are appropriately dimensioned, otherwise arbitrary, constant matrices.

Here, without loss of generality, we may choose

\[ C_{c_i} = I_{n_i}, \quad i = 1, \ldots, \nu \]

and then \( A_{c_i}, B_{c_i} \), and the feedback gains in (5.58) are chosen to minimize the cost function:

\[ W_o \triangleq E[w_o w_o^T]. \]
\[ J = \int_0^\infty \left( z^T Q z + \sum_{i=1}^\nu \bar{u}_i^T R_i \bar{u}_i + \sum_{i=1}^\nu \mathring{\zeta}_i^T N_i \mathring{\zeta}_i \right) dt \]  

(5.61)

where \( Q = Q^T \geq 0, R_i = R_i^T > 0, N_i = N_i^T > 0, i = 1, \ldots, \nu, \) \( z \) and \( \bar{u} \) are as before, and

\[ \mathring{\zeta}_i \triangleq S \zeta_i, \quad i = 1, \ldots, \nu . \]  

(5.62)

**Remark 5.5** \( N_i \) is the weight of \( \mathring{\zeta}_i \), which is a measure of the effort spent by the \( i \)th stabilizing compensator.

To determine the optimal controller parameters let

\[
\begin{bmatrix}
\mathring{z} \\
\mathring{\zeta}_1 \\
\vdots \\
\mathring{\zeta}_\nu
\end{bmatrix} \triangleq \mathring{x} \]  

(5.63a)

and

\[
\begin{bmatrix}
\bar{u}_1 \\
\bar{u}_2 \\
\vdots \\
\bar{u}_\nu
\end{bmatrix} \triangleq \bar{u} \]  

(5.63b)

Then we obtain

\[
\begin{align*}
\dot{\mathring{x}} &= \dot{A} \mathring{x} + \sum_{i=1}^\nu \dot{B}_i \bar{u}_i \\
\mathring{c}_i &= \dot{C}_i \mathring{x}, \quad i = 1, \ldots, \nu
\end{align*}
\]  

(5.64a)

\( \dot{A} = \text{blockdiag} (\bar{A}, 0) , \)  

(5.65a)

\( \dot{B}_i = \text{blockdiag} (\bar{B}_i, H_i^T) . \)  

(5.65b)
\[ \dot{C}_i = \text{blockdiag} \left( \tilde{C}_i, H_i \right), \quad (5.65c) \]

\[ H_i \triangleq \begin{bmatrix} 0_{n_i \times \sum_{j=1}^{i-1} n_j}, I_{n_i}, 0_{n_i \times \sum_{j=i+1}^{\nu} n_j} \end{bmatrix}, \quad (5.65d) \]

and

\[ \dot{u}_i \triangleq \begin{bmatrix} \tilde{u}_i \\ \tilde{\zeta}_i \end{bmatrix}. \quad (5.65e) \]

We also have

\[ \dot{u}_i = \hat{K}_i \tilde{v}_i, \quad i = 1, \ldots, \nu \quad (5.66) \]

where

\[ \hat{K}_i \triangleq \begin{bmatrix} K_i^1 & K_i^2 & K_i^3 \\ B_{c}^1 & B_{c}^2 & A_{c_i} \end{bmatrix}, \quad i = 1, \ldots, \nu. \quad (5.67) \]

If \{\hat{K}_1, \ldots, \hat{K}_\nu\} is such that

\[ \hat{F} \triangleq \hat{A} + \sum_{i=1}^{\nu} \hat{B}_i \hat{K}_i \hat{C}_i \quad (5.68) \]

has eigenvalues only in \( \mathbb{C}^- \), then the cost (5.61) is given by

\[ J = \text{trace} \left( \hat{P} \hat{Z}_o \right) \quad (5.69) \]

where

\[ \hat{Z}_o \triangleq \hat{z}_o \hat{z}_o^T, \quad (5.70a) \]

\[ \hat{z}_o \triangleq \begin{bmatrix} z_o \\ \hat{\zeta}_1(0) \\ \vdots \\ \hat{\zeta}_\nu(0) \end{bmatrix}, \quad (5.70b) \]

and \( \hat{P} = \hat{P}^T \geq 0 \) satisfies
\[
\dot{F}^T \dot{P} + \dot{P} \dot{F} + \dot{Q} = 0, \quad (5.71)
\]

where

\[
\dot{Q} \triangleq \begin{bmatrix}
Q & 0 \\
0 & 0
\end{bmatrix} + \sum_{i=1}^{\nu} \dot{C}_i^T \dot{K}_i^T \dot{R}_i \dot{K}_i \dot{C}_i, \quad (5.72a)
\]

and

\[
\dot{R}_i \triangleq \text{blockdiag } (R_i, N_i), \quad i = 1, \ldots, \nu. \quad (5.72b)
\]

In this case (5.69) is to be minimized over \{\dot{K}_1, \ldots, \dot{K}_\nu\} subject to (5.71). The necessary conditions are:

\[
\dot{B}_i^T \dot{P} \dot{L} \dot{C}_i^T + \dot{R}_i \dot{K}_i (\dot{C}_i \dot{L} \dot{C}_i^T) = 0, \quad i = 1, \ldots, \nu, \quad (5.73)
\]

where the matrix of Lagrange multipliers \(\dot{L} = \dot{L}^T \geq 0\) satisfies

\[
\dot{F} \dot{L} + \dot{L} \dot{F}^T + \dot{Z}_0 = 0. \quad (5.74)
\]

Assuming \(C_i^m\) is of the full rank, \((\dot{C}_i \dot{L} \dot{C}_i^T)\) is non-singular, and the unknown matrices are found as

\[
\dot{K}_i \triangleq \begin{bmatrix}
K_i^1 & K_i^2 & K_i^3 \\
B_i^1 & B_i^2 & A_{ci}
\end{bmatrix} = -\dot{R}_i^{-1} \dot{B}_i^T \dot{P} \dot{L} \dot{C}_i^T (\dot{C}_i \dot{L} \dot{C}_i^T)^{-1}, \quad i = 1, \ldots, \nu. \quad (5.75)
\]

In this case, the solution to the optimal decentralized LTI controller design problem may be obtained by solving the two Liapunov equations (5.71) and (5.74) and the set of equations (5.75) simultaneously. This is analogous to the decentralized optimal static feedback design and the same numerical techniques discussed in the previous subsection can also be applied here.

Since the optimal solution depends on the initial conditions, once again we may choose to minimize the expected value of the cost \((5.61)\) over the initial conditions. In this case \(\dot{Z}_0\) in (5.74) is redefined as
\[ \dot{Z}_o \triangleq \mathcal{E}[\dot{\mathcal{z}}_o \dot{\mathcal{z}}_o^T] \]  

and its calculation must be carried out as for \( Z_o \) in (5.49) with obvious modifications. In particular, for the case of Example 5.1, one obtains

\[ \dot{Z}_o = \dot{E}W_o \dot{E}^T \]  

where

\[ W_o \triangleq \mathcal{E}[w_0 w_0^T] \]

and

\[ E \triangleq \begin{bmatrix} \sum_{i=1}^{\nu} B_i (K_i^1 F_i^m + K_i^2 F_{s_i} + K_i^3 F_{c_i} + \Gamma_i) + E \\ D_1 (K_1^1 F_1^m + K_1^2 F_{s_1} + K_1^3 F_{c_1} + \Gamma_1) + F_1 \\ \vdots \\ D_\nu (K_\nu^1 F_\nu^m + K_\nu^2 F_{s_\nu} + K_\nu^3 F_{c_\nu} + \Gamma_\nu) + F_\nu \\ B_{c_1}^1 F_1^m + B_{c_1}^2 F_{s_1} + E_{c_1} \\ \vdots \\ B_{c_\nu}^1 F_\nu^m + B_{c_\nu}^2 F_{s_\nu} + E_{c_\nu} \end{bmatrix} \]

Finally we note that the static decentralized output feedback case, considered before, can be thought simply as a special case of the dynamic decentralized output feedback case. The former is obtained from the latter by simply setting \( n_i = 0 \), \( i = 1, \ldots, \nu \). Hence, from here on we will refer to the latter case, and set \( n_i = 0 \) if no dynamic stabilizing compensator is employed by the \( j^{th} \) controller.
5.3.3 Existence of Optimal Decentralized Controllers

So far we have derived the necessary conditions for optimality of decentralized controllers, when they exist, for the robust servomechanism problem. In this section we consider the existence of such controllers. First consider the following lemmas:

**Lemma 5.6** ($[C^T, ... , C^T, A, [B_1, ..., B_\nu]; r_1 + pr_1, ..., r_\nu + pr_\nu; m_1, ..., m_\nu]$ does not have any (unstable) DFM if and only if the following conditions all hold:

a) $(C^m, A, B; r_1^m, ..., r_\nu^m; m_1, ..., m_\nu)$ does not have any (unstable) DFM.

b) The DFM of

$$
\left( \begin{array}{c}
C^* \ A \\ C \lambda I_r \\
\end{array} \right) \left[ \begin{array}{c}
0 \\
B \\
0 \\
\end{array} \right]; r_1^m + r_1, ..., r_\nu^m + r_\nu; m_1, ..., m_\nu
$$

do not contain $\lambda$, $\forall \lambda \in \rho(A_1) \cup \rho(A_2)$ ($\forall \lambda \in (\rho(A_1) \cup \rho(A_2)) \cap C^+)$.

**Proof:** $\lambda$ ($\lambda \in C^+$) is not a DFM of

$$([C^T, ..., C^T, A, [B_1, ..., B_\nu]; r_1^m + pr_1, ..., r_\nu^m + pr_\nu; m_1, ..., m_\nu)$$

if and only if

$$\begin{bmatrix}
\lambda - \bar{A} & \bar{B}_{i_1} & \cdots & \bar{B}_{i_\mu} \\
\bar{C}_{j_1} & & & \\
\vdots & \ddots & \ddots & \\
\bar{C}_{j_\nu - \mu} & & 0
\end{bmatrix}
\begin{bmatrix}
1 \\
\vdots \\
\end{bmatrix}
\geq \dim (\bar{A}) = n + rp \quad (5.80)
$$

for all index sets $\{i_1, ..., i_\mu\} \subset \{1, ..., \nu\}$ and $\{j_1, ..., j_{\nu - \mu}\} = \{1, ..., \nu\} \setminus \{i_1, ..., i_\mu\}$.

[76]. Condition (5.80) is equivalent to
Suppose conditions (a) and (b) of the lemma hold. Then

$$\begin{bmatrix}
\lambda I - A & B_i & \ldots & B_i^m \\
C_{j_1}^m & 0 & & \\
0 & \Delta_{j_1} & & \\
\vdots & & \ddots & \\
C_{j_{\nu-\mu}}^m & 0 & & \\
0 & \Delta_{j_{\nu-\mu}} & & \\
\end{bmatrix} \geq n + r.$$  \hspace{1cm} (5.81)

\(\forall \lambda (\forall \lambda \in \mathbb{C}^+), \ \forall \{i_1, \ldots, i_\mu\} \subset \{1, \ldots, \nu\}, \ \{j_1, \ldots, j_{\nu-\mu}\} = \{1, \ldots, \nu\} \setminus \{i_1, \ldots, i_\mu\},\)
and

\(\forall \lambda (\forall \lambda \in \mathbb{C}^+)\)
\[
\begin{bmatrix}
\lambda I - A & 0 & B_{i_1} & \cdots & B_{i_n} \\
-C & 0 & D_{i_1}^* & \cdots & D_{i_n}^* \\
C_{j_1}^m & 0 & 0 & \Delta_{j_1} \\
& \ddots & \ddots & \ddots & \ddots \\
C_{j_{\nu-\mu}}^m & 0 & 0 & \Delta_{j_{\nu-\mu}} \\
& \ddots & \ddots & \ddots & \ddots \\
\end{bmatrix}
\overset{\text{rank}}{\geq n + r}
\]  \hspace{1cm} (5.82b)

\forall \lambda \in \rho(A_1) \cup \rho(A_2) (\forall \lambda \in (\rho(A_1) \cup \rho(A_2)) \cap \mathbb{C}^+), \ \forall \{i_1, \ldots, i_\nu\} \subset \{1, \ldots, \nu\}, \ \{j_1, \ldots, j_{\nu-\mu}\} = \{1, \ldots, \nu\} \setminus \{i_1, \ldots, i_\mu\}.

Hence if \( \lambda (\lambda \in \mathbb{C}^+) \) is such that \( \lambda \in \rho(A_1) \cup \rho(A_2) \), then (5.82b) is equivalent to (5.81), otherwise \( \Lambda(\lambda) \neq 0 \) and (5.82a) implies (5.81). This proves sufficiency.

To prove necessity of conditions (a) and (b), note that whenever (5.81) holds, (5.82a) also holds. Furthermore, if \( \lambda \in \rho(A_1) \cup \rho(A_2) \) then \( \Lambda(\lambda) = 0 \) and (5.81) is equivalent to (5.82b). \( \square \)

**Lemma 5.7** The closed-loop eigenvalues of the system (5.5a)-(5.5b), under the controls described by (5.58), (5.33a)-(5.33b), and (5.59a)-(5.59b), are the same as the eigenvalues of the matrix \( \hat{F} \) given by (5.68).

**Proof:** Let the overall state vector be \( [x^T, \xi_1^T, \ldots, \xi_{\nu-r}^T, \zeta_1^T, \ldots, \zeta_{\nu-r}^T]^T \), then the system dynamics matrix can be written as \( U^{-1} F' U \), where \( U = \text{blockdiag}(I_{n}, \Delta, I_{n-1}, \ldots, I_{n-\nu}) \) and \( \Delta = [C_{i_1}^*, \ldots, C_{i_\mu}^*] \cdot \text{blockdiag}(C_{s_1}, \ldots, C_{s_{\nu}}) \), i.e., the system matrix is similar to \( \hat{F} \), hence the result follows. \( \square \)

Now we are ready to prove the following theorem:
Theorem 5.3 If

(a) $(C^m, A, B; r^m_1, ..., r^m_\nu; m_1, ..., m_\nu)$ has no unstable DFMs,

(b) the DFMs of

\[
\begin{pmatrix}
C^*, 
\begin{bmatrix}
A & 0 \\
C & \lambda I_r
\end{bmatrix}, 
\begin{bmatrix}
B \\
D
\end{bmatrix}; 
B_1^m + r_1, ..., B_\nu^m + r_\nu; m_1, ..., m_\nu
\end{pmatrix}
\]

do not contain $\lambda, \forall \lambda \in (\rho(A_1) \cup \rho(A_2)) \cap \mathbb{C}^+$,

(c) the outputs $y_i (i = 1, ..., \nu)$ are measurable (reconstructable), and

(d) $(Q^{1/2}, \tilde{A})$ is detectable,

then there exists an ordered set of non-negative integers $\{n_1, ..., n_\nu\}$, and decentralized controllers of the form (5.58), (5.33a)-(5.33b), and (5.59a)-(5.59b), satisfying (5.71), (5.73), and (5.74), such that

(i) Asymptotic tracking takes place for all disturbances described by (5.2) and for all reference inputs described by (5.6a) (5.6b).

(ii) The overall system is stable (in the sense that the closed-loop system matrix has eigenvalues only in $\mathbb{C}^-$).

(iii) The cost given by (5.61) (or by (5.30) if $n_i = 0, i = 1, ..., \nu$), is minimized subject to the linear feedback decentralization constraint.

(iv) If the disturbance $w_i$ and the reference inputs $y_i^c, i = 1, ..., \nu$ are bounded, then the inputs $u_i, i = 1, ..., \nu$ are bounded.

(v) The controller is robust (weak robust).
Proof: By the hypothesis and Lemma 5.6 the decentralized system defined by (5.34) and (5.38) does not have any unstable fixed modes. Hence, by Theorem 1 of [28], there exist an ordered set of integers \( \{n_1, \ldots, n_\nu\} \) and controllers of the form

\[
\begin{align*}
\dot{u}_i &= K_i \ddot{v}_i + K_i^3 \zeta_i \\
\dot{\zeta}_i &= A_{ci} \zeta_i + B_{ci} \ddot{v}_i, \quad \zeta_i \in \mathbb{R}^{n_i},
\end{align*}
\]

such that the system (5.34) is stabilized or, equivalently, the system (5.64a)-(5.64b) is stabilizable with a feedback of the form (5.66). Hence, there exist a set \( \{K_1, \ldots, K_\nu\} \) such that the cost (5.61) is minimized subject to the constraint described by (5.66). Furthermore, any set of gains that minimize (5.61) also stabilize the system (5.64a)-(5.64b) (otherwise cost is infinite), and satisfies the conditions (5.71), (5.73), and (5.74) (as derived above). Hence (iii) is proved.

Once the system (5.64a)-(5.64b) is stabilized with the feedback (5.66), the matrix \( \hat{F} \) given by (5.68) has eigenvalues only in \( \mathbb{C}^- \). However, by Lemma 5.7, this implies that the overall system dynamics matrix have eigenvalues only in \( \mathbb{C}^- \). Hence (ii) is proved.

Property (i) follows from the asymptotic stability of the system (5.64a)-(5.64b) under the control (5.66) and the fact

\[
\epsilon_i = \begin{bmatrix} 0_{r_i \times (n+r_p-r)} & \Delta_i & 0_{r_i \times \sum_{j=1}^{\nu} n_j} \end{bmatrix} \hat{z}, \quad i = 1, \ldots, \nu,
\]

where \( \Delta_i \) is given in (5.39c).

Parts (iv) and (v) are proved as in Theorem 5.1 with obvious modifications.

Remark 5.6 Necessary and sufficient conditions for the existence of LTI controllers for the decentralized robust servomechanism problem have been given in
By comparing these conditions with the conditions of Theorem 5.3, we observe that, if there exists a solution to the decentralized robust (weak robust) servomechanism problem and $Q$ is chosen properly, then there exists a solution to the above stated decentralized optimal control problem with properties (i)-(v) of Theorem 5.3.

Remark 5.7 Another optimal control approach to solve the decentralized robust servomechanism problem has been presented in [77] in the framework of [68]. Besides the formulation, the basic difference between this approach and the one presented here is in the choice of the cost function. The cost function defined here allows the designer to assign desired weights to the physically meaningful quantities (see Remarks 5.1 and 5.5) and shape the response accordingly. Furthermore, the solution method suggested in [77] is a non-linear programming approach. Here, on the other hand, the necessary conditions of optimality are derived.

5.4 Frequency Domain Considerations

Recently there has been an interest in the frequency domain LQ design [78]. In this section the frequency domain properties of the LQ controllers developed in this chapter are considered.

For brevity let us consider only the centralized controller of Section 5.2 with the weighting matrix $Q$ chosen as in Remark 5.1. The results are easily extendable to the more general weighting matrices and to the decentralized case.

Using Parseval's identity, the cost function in (5.18) with the above mentioned choice of $Q$ can be written as

$$ J = \frac{1}{\pi} \int_{0}^{\infty} \left\{ |A(j\omega)|^2 \| X(j\omega) \|_{Q_0}^2 + |(j\omega)^{p-1} + a_{p-1}(j\omega)^{p-2} + ... + a_1|^2 \| E(j\omega) \|_{Q_1}^2 + ... \right\} $$
where $X(s)$, $E(s)$, and $U(s)$ are the Laplace transforms of the state $x(t)$, the error $e(t)$, and the input $u(t)$ respectively, and $A(s)$ is as defined in (5.12).

Since the cost $J$ is finite for a stabilizing solution, from (5.85) we can deduce that $|A(j\omega)|^2 \|U(j\omega)\|^2_R$ tends to zero faster than $1/\omega$ as $\omega \rightarrow \infty$. Equivalently $|U(j\omega)|$ tends to zero faster than

$$\frac{1}{\sqrt{\omega}|A(j\omega)|}.$$ 

Since $A(s)$ is a polynomial in $s$, this in turn implies that the solution requires very little control effort, if any, at high frequencies. This property is desirable, since many practical actuators are band-limited. If $Q_0$ is chosen such that $(Q_o^{1/2}, A)$ is observable, then we can draw the same conclusion for $|X(j\omega)|$. That is, the system will have very little high frequency oscillations, if any. This property is important, since in many cases the system model used for design purposes may not be valid at high frequencies. Similar conclusions may also be drawn for $|E(j\omega)|$, if $\{Q_1, \ldots, Q_p\}$ are chosen properly.

5.5 Applications

The design approach developed above is applied to three different physical systems. The first system is a DC servo motor for which centralized servocompensators with state feedback and dynamic output feedback (with a reduced order observer) are designed. The other two systems are typical examples of large flexible structures. Decentralized optimal controllers are designed to control these structures in the presence of constant and sinusoidal disturbances. The results are also compared with the optimal state feedback design in each case.
5.5.1 DC Servo Motor

Let us consider the DC servo motor example given in [68]. The system is modeled as:

\[
\begin{align*}
\dot{x} &= \begin{bmatrix} -B & K_T \\ -K_e & -R \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} u + \begin{bmatrix} -\frac{1}{L} \\ 0 \end{bmatrix} w \\
y^m &= y = \begin{bmatrix} 1 & 0 \end{bmatrix} x \\
e &= y - y^r.
\end{align*}
\] (5.86a)

where, in metric units, \( B = 0.0162 \), \( J = 0.215 \), \( K_T = K_e = 1.11 \), \( R = 1.05 \), and \( L = 0.0053 \). The system is to be regulated against constant disturbances \( w \) such that the output \( y \) can track constant set points \( y^r \).

The cost function is chosen to be of the form (5.26) with unity error weighting. We consider two different control weightings \( R = 1 \) and \( R = 10^{-8} \). The resulting control law is given by

\[ u = K_1 x + K_2 \eta \] (5.87)

and the servocompensator is described by:

\[
\begin{align*}
\eta &= \xi \\
\dot{\xi} &= e.
\end{align*}
\] (5.88a)

The optimal control gains are shown in Table 10.

The simulation results are illustrated in Figures 39–46 for the output and the control input. It is observed that the response becomes faster as the control weighting \( R \) becomes smaller. Furthermore the overshoot due to a step disturbance
Figure 39: Output of the DC motor under state feedback for $R = 1$, $w = 0$, and $y^r = 1$.

Figure 40: Output of the DC motor under state feedback for $R = 1$, $w = 1$, and $y^r = 0$. 
Figure 41: Output of the DC motor under state feedback for $R = 10^{-8}$, $w = 0$, and $y^r = 1$.

Figure 42: Output of the DC motor under state feedback for $R = 10^{-8}$, $w = 1$, and $y^r = 0$. 
Figure 43: Control Input of the DC motor under state feedback for $R = 1$, $w = 0$, and $y^r = 1$.

Figure 44: Control Input of the DC motor under state feedback for $R = 1$, $w = 1$, and $y^r = 0$. 
Figure 45: Control Input of the DC motor under state feedback for $R = 10^{-8}$, $w = 0$, and $y^r = 1$.

Figure 46: Control Input of the DC motor under state feedback for $R = 10^{-8}$, $w = 1$, and $y^r = 0$. 
Table 10: Optimal Control Gains for the DC Motor Example

<table>
<thead>
<tr>
<th>$R$</th>
<th>$K_1$</th>
<th>$K_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$[-0.1688 - 4.390 \times 10^{-3}]$</td>
<td>-1.000</td>
</tr>
<tr>
<td>$10^{-8}$</td>
<td>$[-98.15 - 1.494]$</td>
<td>$-1.000 \times 10^4$</td>
</tr>
</tbody>
</table>

is also much smaller for a smaller $R$. However, if $R$ is too small, an excessive control magnitude may be required in response to a step reference input.

Assuming that only the output speed $\dot{y}$ be measurable, the resulting controller is of the form

$$u = K_1 \dot{x} + K_2 \eta$$

(5.89)

where $K_1$ and $K_2$ are as before, $\eta$ is the output of the servocompensator given by (5.88a)-(5.88b), and

$$\dot{x} = \begin{bmatrix} y \\ \theta + 148.08y \end{bmatrix}$$

(5.90)

where $\theta$ is the state of the observer described by:

$$\dot{\theta} = -962.5\theta + 188.68u - 198.28y.$$  

(5.91)

The time constant of the observer is chosen to be five times faster than the fastest time constant of the system. The simulation results for this case are shown in Figures 47-50. It is seen that in all the cases the response is very close to the corresponding optimal response shown in Figures 39-42.

5.5.2 A Generic LFS Example

The generic model of a LFS given in [79] is considered. The structure has a pair of torque actuators about the $x$ and $y$ axes mounted on the two diagonal
Figure 47: Output of the DC motor under dynamic output feedback for $R = 1$, $w = 0$, and $y^\prime = 1$.

Figure 48: Output of the DC motor under dynamic output feedback for $R = 1$, $w = 1$, and $y^\prime = 0$. 

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Figure 49: Output of the DC motor under dynamic output feedback for $R = 10^{-8}$, $w = 0$, and $y^r = 1$.

Figure 50: Output of the DC motor under dynamic output feedback for $R = 10^{-8}$, $w = 1$, and $y^r = 0$. 
corners of the planer structure (other two pairs in [79] are ignored) and two pairs of
slope sensors co-located with the actuators. The outputs are the slopes measured
by the sensors. Only the flexible part of the structure, which is modeled with nine
modes, is considered. It is required to control the shape of the structure against
constant disturbances.

The cost function is chosen to be of the form (5.26) with unity error weighting.
The control weighting is chosen as $R = 0.01I_4$. The expected value of the optimal
cost under centralized state feedback is calculated by

$$J^* = \text{trace } (PZ_o),$$

where $P$ is the solution of the algebraic matrix Riccati equation $(5.21)$, and $Z_o$
is given by $(5.56)$. Assuming that the initial conditions of the disturbance are
uniformly distributed on the unit ball (i.e., $W_o = I$), the expected value of the
optimal cost is found as

$$J^* = 1.5129 \times 10^{-3}.$$  (5.93)

The eigenvalues of the closed-loop system under optimal state feedback control
are given in Table 11.

Now a decentralized feedback law of the form $(5.29)$ is sought to minimize the
same cost function. The pair of inputs and outputs at each corner is assigned to a
different decentralized control agent. The local servocompensators are described
by $(5.33a)-(5.33b)$ with $A_{s_i} = 0_{2 \times 2}$ and $C_{s_i} = B_{s_i} = I_2$ \,(i = 1, 2). The feedback
gains are found by solving $(5.41)$, $(5.47)$, and $(5.48)$ simultaneously as:

$$
\begin{bmatrix}
0.000 & -0.003 & -0.238 & -9.716 \\
0.007 & 0.003 & 8.149 & -0.115
\end{bmatrix}
$$

and

155
Table 11: Closed Loop Eigenvalues for the Generic LFS Example

<table>
<thead>
<tr>
<th></th>
<th>Under Optimal State Feedback</th>
<th>Under Decentralized Output Feedback</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-0.10552 \pm j20.250$</td>
<td>$-0.07709 \pm j20.250$</td>
<td></td>
</tr>
<tr>
<td>$-0.16533 \pm j33.010$</td>
<td>$-0.15603 \pm j33.010$</td>
<td></td>
</tr>
<tr>
<td>$-0.28591 \pm j57.139$</td>
<td>$-0.27684 \pm j57.140$</td>
<td></td>
</tr>
<tr>
<td>$-0.35183 \pm j70.359$</td>
<td>$-0.34828 \pm j70.359$</td>
<td></td>
</tr>
<tr>
<td>$-0.52313 \pm j104.62$</td>
<td>$-0.51855 \pm j104.62$</td>
<td></td>
</tr>
<tr>
<td>$-0.59676 \pm j119.35$</td>
<td>$-0.59345 \pm j119.35$</td>
<td></td>
</tr>
<tr>
<td>$-0.65305 \pm j130.61$</td>
<td>$-0.65125 \pm j130.61$</td>
<td></td>
</tr>
<tr>
<td>$-0.80921 \pm j161.84$</td>
<td>$-0.80643 \pm j161.84$</td>
<td></td>
</tr>
<tr>
<td>$-0.88660 \pm j177.32$</td>
<td>$-0.88424 \pm j177.32$</td>
<td></td>
</tr>
<tr>
<td>$-7.8106 \times 10^{-2}$</td>
<td>$-6.3689 \times 10^{-2}$</td>
<td></td>
</tr>
<tr>
<td>$-2.9880 \times 10^{-2}$</td>
<td>$-2.4657 \times 10^{-2}$</td>
<td></td>
</tr>
<tr>
<td>$-2.3824 \times 10^{-2}$</td>
<td>$-2.2923 \times 10^{-2}$</td>
<td></td>
</tr>
<tr>
<td>$-9.7869 \times 10^{-3}$</td>
<td>$-9.4094 \times 10^{-3}$</td>
<td></td>
</tr>
</tbody>
</table>

$$K_2 = \begin{bmatrix} 0.000 & -0.003 & -0.145 & -9.666 \\ 0.007 & 0.002 & 8.143 & 0.164 \end{bmatrix}.$$ 

The initial condition matrix $Z_o$ given by (5.56) with $W_o = I$ is used in (5.47). The resulting cost is

$$J = 1.6508 \times 10^{-3}.$$  \hspace{1cm} (5.94)

Comparing this figure with the optimal cost given in (5.93), it is seen that the degradation introduced by the decentralized output feedback is very small. The
eigenvalues of the closed-loop system are given in Table 11, together with the optimal eigenvalues. Closeness of these two sets of eigenvalues indicates closeness in performance.

5.5.3 A Tetrahedral Truss Structure

The tetrahedral truss structure shown in Figure 51 has been previously considered by many researchers as an example for structural control design (e.g., see [80] and [81]). A finite element model, which includes twelve modes, is given in [81].

It is assumed that the structure has to be regulated against sinusoidal disturbances \( w \), satisfying:

\[
\ddot{w} + \omega_0^2 w = 0
\]

where \( \omega_0 = 15 \text{ rad/sec.} \)

A decentralized controller of the form (5.29), (5.33a)-(5.33b) is designed for each co-located actuator/sensor pair to minimize the cost function given in (5.30). Two different state weighting matrices:

\[
Q_1 = \begin{bmatrix} 0 & 0 \\ 0 & I_6 \end{bmatrix}, \quad Q_2 = I_{36},
\]

are considered. Note that \( Q_1 \) weights only the error \( e \) and \( Q_2 \) weights \( e, \dot{e}, \) and \( \ddot{e} \).

The control weights are chosen to be \( R_i = 10^{-2} \) or \( R_i = 10^{-6} \) \( (i = 1, \ldots, 6) \). The resulting costs are shown in Table 12 together with the corresponding optimal costs under state feedback. Note the small degradation introduced by the decentralized output feedback.
Figure 51: Tetrahedral truss structure
Table 12: Costs for the Tetrahedral Truss Structure Example

<table>
<thead>
<tr>
<th>$Q_i$</th>
<th>$R_i$</th>
<th>Optimal Cost Under State Feedback</th>
<th>Cost Under Decentralized Output Feedback</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q_1$</td>
<td>$10^{-2}$</td>
<td>0.0224</td>
<td>0.0281</td>
</tr>
<tr>
<td>$Q_2$</td>
<td>$10^{-2}$</td>
<td>7.2861</td>
<td>13.640</td>
</tr>
<tr>
<td>$Q_1$</td>
<td>$10^{-6}$</td>
<td>$2.1238 \times 10^{-4}$</td>
<td>$2.9020 \times 10^{-4}$</td>
</tr>
</tbody>
</table>

5.6 Summary

The robust servomechanism problem has been considered. A linear–quadratic approach has been presented to solve this problem. Although the cost is defined on modified state and input spaces, it has a physical meaning related to the original system. It is possible to assign desired weights to the error, to its certain derivatives, to the system effort, and to the control effort in the cost (see Remarks 5.1 and 5.5).

The following cases have been considered:

(a) Centralized full-state feedback,

(b) Centralized output feedback with an observer,

(c) Centralized static output feedback (as a special case of (e) below),

(d) Centralized dynamic output feedback (as a special case of (f) below),

(e) Decentralized static output feedback.

(f) Decentralized dynamic output feedback.

It has been shown that for all the cases there exists a linear time-invariant optimal feedback controller, if there exists a solution to the appropriate (centralized or
decentralized) robust servomechanism problem. Those controllers may also be subject to certain disturbances. Furthermore, the resulting controllers enjoy many properties like asymptotic tracking, stability, boundedness and robustness. Here, a servocompensator (or decentralized servocompensators), which is (are) determined by the dynamic characteristics of the disturbance and tracking signals, is always a part of the controller. The term "static feedback" here refers to the case when no dynamics other than servocompensators are employed.

In the cases of (a) and (b), the optimal controller does not depend on the initial conditions of the system nor of the external signals (disturbance and tracking signal). The knowledge of disturbance coupling matrices $E$, $F$ and $F^m$ are also not required. Furthermore, the design of the controller is fairly simple (e.g., compared to the nonlinear programming approach of [68]) and only requires the solution of an $n + r p$ dimensional matrix Riccati equation.

The solution of the optimal controller design problem for the cases (c)-(f) requires solving two Lyapunov equations (5.71), (5.74) and the set of equations (5.75) simultaneously. Some effective methods have been developed to solve such equations recently (see [72]). Furthermore, in these cases the optimal controller depends on the initial conditions through the initial condition matrices $Z_0$ or $\hat{Z}_0$ in (5.47) or (5.74). If the initial conditions are not known, one may choose to optimize the expected value of the appropriate cost function over the initial conditions. In any case, the calculation of $Z_0$ or $\hat{Z}_0$ involves the disturbance coupling matrices. The design is independent of those matrices otherwise. The calculation of $Z_0$ and $\hat{Z}_0$ has been illustrated by an example.

The frequency domain properties of the resulting controllers have also been discussed. It has been shown that the solution does not require much control effort at high frequencies. It has also been shown that, if the weighting matrices are

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chosen properly, the system is guaranteed not to have high frequency oscillations.

Finally, a number of controller design problems were considered in Section 5.5 to illustrate the possible applications of the presented approach.
CHAPTER VI

CONCLUSIONS

6.1 Summary

Robust controller design strategies for large scale systems have been investigated. Basic characteristic of such systems is their complexity. Those systems may usually be representable only by very high dimensional models. However, it may be possible to obtain much simpler local models for local design of decentralized controllers. To obtain a local model, a decomposition of the large scale system is needed. It has been demonstrated that the overlapping decompositions may be useful for that purpose.

An overview of the previous work on the inclusion principle was given in Chapter II. The inclusion principle forms the theoretical basis for the overlapping decompositions. A new inclusion concept, called extension, was introduced in the same chapter. This new concept involves both state and input inclusion. It has been shown that with this approach any control law designed in the expanded spaces is always contractible to the original spaces for implementation. The inclusion of the cost functions has also been discussed. Necessary and sufficient conditions for the inclusion of quadratic cost functions have been derived. It has been shown that, unlike the case when the two input spaces are identical, the inclusion conditions must, in general, depend on the control laws. However, certain sufficient conditions, which do not involve final control laws, can be found if the
inclusion is taken with respect to the optimal control. Finally in Chapter II, controller design with overlapping decompositions was discussed within the framework of the new inclusion concept and optimal control.

Frequency domain robust controller design strategies for large scale systems were considered in Chapter III. Overlapping decompositions was used to develop local models. Upper bounds on the norm of the multiplicative error matrix, that accounts for the interactions, have been calculated for various possible decompositions. A total error function, which may represent the error between the design model and the true system, has been defined. It has been shown that, once such an error function is known to every control agent, stability-robustness and performance requirements can be tested locally. This fact allows local controller design procedures which guarantee stability and good performance. Such design procedures have been investigated. In particular, decentralized LQG/LTR design methodology has been presented.

State space robust controller design strategies were considered in Chapter IV. In particular, modeling of uncertain dynamics was discussed. It has been shown that, under mild conditions on uncertain dynamics, it is possible to obtain a rational TFM, possibly dependent on a parameter vector which varies over a subset of a finite dimensional real vector space, to represent uncertain dynamics. Furthermore, in many practical cases, uncertain dynamics can be represented by a relatively low order TFM, even if the actual dynamics are of very high order. The procedure of determining such a TFM has been discussed. It has been shown that a controller which stabilizes the nominal system, including such a representation of uncertain dynamics, also stabilizes the actual system. Furthermore, desired performance or relative stability can also be guaranteed.

State space models for systems with both parameter uncertainties and uncer-
tain dynamics have also been developed. Once such a model is obtained, already existing methods can be used to design robust controllers. The presented approach gives a unified framework for the solution to the robust controller design problem for systems with both parameter uncertainties and uncertain dynamics.

Finally in Chapter IV, decentralized robust controller design in state space was discussed. It has been demonstrated that local controllers which guarantee overall stability and desired performance can be designed, if certain limited information about rest of the closed-loop system is available to each local agent.

The robust servomechanism problem was considered in Chapter V. A linear-quadratic approach was presented to solve this problem. Both centralized and decentralized cases were considered. Although the cost was defined on modified state and input spaces, it has a physical meaning related to the original system. It is possible to assign desired weights to the error, to its certain derivatives, to the system effort, and to the control effort in the cost. It has been shown that there exists a linear time-invariant optimal feedback controller, if there exists a solution to the appropriate (centralized or decentralized) robust servomechanism problem. Those controllers may also be subject to certain disturbances. Furthermore, the resulting controllers enjoy many properties like asymptotic tracking, stability, boundedness and robustness. A servocompensator (or decentralized servocompensators), which is (are) determined by the dynamic characteristics of the disturbance and tracking signals, is always a part of the controller.

For the centralized servomechanism problem a full state feedback controller can be designed if all the states are available for measurement. Otherwise, an observer can be designed to observe these states. In these cases the optimal controller does not depend on the initial conditions. The knowledge of disturbance coupling matrices $E$, $F$ and $F_m$ are also not required. Furthermore, the design of the
controller is fairly simple (e.g., compared to the nonlinear programming approach of [68]) and only requires the solution of an \( n + rp \) dimensional matrix Riccati equation. If an observer is not practical, fixed order output feedback controllers can be designed. Such controllers were considered as special cases of decentralized controllers.

The solution to the optimal decentralized controller design problem (which includes the fixed order centralized output feedback design problem as a special case) requires solving two Liapunov equations and another set of linear equations simultaneously. Furthermore, in these cases the optimal controller depends on the initial conditions through the initial condition matrices \( Z_0 \) or \( \hat{Z}_0 \). If the initial conditions are not known, one may choose to optimize the expected value of the appropriate cost function over the initial conditions. In any case, the calculation of \( Z_0 \) or \( \hat{Z}_0 \) involves the disturbance coupling matrices. The design is independent of those matrices otherwise. The calculation of \( Z_0 \) and \( \hat{Z}_0 \) has been illustrated by an example.

The frequency domain properties of the resulting controllers have also been discussed. It has been shown that the solution does not require much control effort at high frequencies. It has also been shown that, if the weighting matrices are chosen properly, the system is guaranteed not to have high frequency oscillations.

Throughout the dissertation various practical problems have been considered to illustrate the possible applications of the presented design methodologies.

6.2 Suggestions for Further Research

Further research opportunities along the lines of this dissertation are still open. Static and dynamic output feedback design by using extension can be investigated. A dual approach of extension, which can be used in estimation, may be developed.
In this dissertation, it has been shown that the uncertain dynamics of a system can be represented as a structured uncertainty with arbitrarily small conservatism. However, to increase the effectiveness of the proposed design approach, more effective and less conservative robust controller design methodologies for systems with structured uncertainty are needed.

Recall that the solution of the optimal controller design problem for the centralized or decentralized output feedback cases usually require numerical optimization. A number of algorithms have been proposed by various researchers to solve such problems (see [72]). However, none of these algorithms can guarantee convergence to the optimal. Namely, they may converge to a local minimum rather than the global minimum. Furthermore, they all require an initial set of stabilizing feedback gains and the particular initial set may largely effect the success of the algorithm in converging to the global optimum. Hence, the search is still open for more effective optimization algorithms which can avoid local (but not global) minima and which are less sensitive to the choice of the initial feedback gains.
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