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On cubic graphs that are edge critical for the torus

Fiedler, Joseph Robert, Ph.D.
The Ohio State University, 1988
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UMI
ON CUBIC GRAPHS THAT ARE EDGE CRITICAL FOR THE TORUS

DISserTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By

Joseph Robert Fiedler, A.B., M.Sc.

*****

The Ohio State University
1988

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Advisor
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For Rebecca

To Whom it Mattered
ACKNOWLEDGMENTS

My mathematical debts I owe are many. First amongst them is that to A. E Ross who introduced me, as he has so many, to the beauties of mathematics. I owe especial thanks to Professor Don Miller of the University of Victoria for teaching a young and stubborn graduate student that graphs were much more than degenerate CW-complexes and to Neil Robertson for undertaking my education. Most recently I owe an irredeemable debt to Phil Huneke for his friendship and his quiet encouragement. I wish to thank Dr. Robert Wooddell Weaver for the use of his program Graph Analyzer and his kindness in updating it to meet my needs. Without the bookkeeping capabilities that this supplied I could not have undertaken the necessary explorations. I must also thank Dr. Richard Vitray for introducing me -- over my objections -- to the notion of representativity, that made sense out of what had previously been simply mounds of calculations. Finally I must thank my wife, Janet Meyer, for her patience and support.
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Introduction

The study of embedding of graphs into surfaces begins with the 1930 paper of Kuratowski [7] in which he proves that a graph $\Gamma$ may be embedded in the plane unless it has one of two subgraphs, now known as $K_5$, the complete graph on 5 vertices, and $K_{3,3}$ the complete bipartite graph on "three plus three" vertices. Kuratowski's paper also inaugurated the class of "forbidden minor" results. The next result of Kuratowski-type for embeddings was the independent discovery in 1972 by Milgram [9] and Glover & Huneke [4] of a list of six cubic graphs (the five that are connected are given in Chapter VI under the names $G, F_{11}, F_{12}, F_{13} \& F_{14}$) with the property a cubic graph fails to embed in the projective plane if and only if it contains a homeomorph of some graph in the list. In the language of the current paper such graphs are edge-critical for the projective plane. (The term "edge-critical" was chosen as more descriptive than the more usual "minimal"; a term that currently enjoys at least three distinct usages). Following the Glover, Huneke paper came a series of works by Glover, Huneke and their students, most notably Wang [6] and Archdeacon [1]. The end result of these works was to establish a list of 103 graphs as the complete collection of edge-critical graphs for the projective plane.

Recently Archdeacon and Huneke [2] have managed to extend the results of Archdeacon's thesis to show that for each nonorientable surface $\Sigma$ there is a finite
list L of graphs with the property that a graph G fails to embed in \( \Sigma \) if and only if it contains a subgraph homeomorphic to some graph in L. The techniques used seem to strongly depend upon the nonorientability of \( \Sigma \). The present work was undertaken not only in the hope of supplying a cubic list for the torus mimicking that of [4] and [9] for the projective plane but to gain some insight into the nature of orientable surfaces. While the generation of a complete list of edge-critical cubic graphs for the torus seems not to have been realized, it is hoped that the results of Chapter IV, in particular the relationship between embedibility on one surface and representativity in another will shed light the relationship between the orientable and nonorientable genus of a graph.

The existence of a complete and finite list for each surface is now assured by a series of papers by Robertson and Seymour (see especially [10]). These papers supply very powerful and deep results about the nature of graphs and "forbidden minor" results in general. It will take considerable time before the significance of the papers can be fully realized. It is the author's hope that examples of such "forbidden minor" theories will be of use in explicating this more general theory.

After listing in Chapter I, the major definitions and results from graph theory and the topology of surfaces, we establish, in Chapter II, some useful facts about embeddings of particular graphs. In Chapter III, we quickly review the results of Richard Decker [3] that are germane to our program and produce seven cubic graphs that contain a nontrivial 3-separation and are edge-critical for the torus (hereafter simply edge-critical). We establish that all edge-critical cubic graphs in the list of sixteen graphs of Appendix A are 3-connected and contain no 3-separation
that does not arise from a vertex bond — such graphs are called "internally 4-connected".

In Chapter IV we establish a general structure theorem for 3-connected graphs that embed in the projective plane. Such a graph decomposes into a Mobius ladder $M$ and a planar graph $P$. Exploiting this decomposition we are able to show that a 3-connected projective planar graph embeds in the torus provided it has some projective planar embedding of representativity 3 or less. Richter, Huneke, and Robertson [oral communication] have extended this to show that a projective planar graph embeds in the orientable surface with $g \ (> 0)$ handles precisely if it has a projective planar embedding of representativity at most $2g - 1$. In Chapter V we proceed to the grubby work of producing the five cubic graphs — collected in Appendix B for easy reference — that embed in the projective plane but are edge-critical for the torus.

Chapter VI displays the results of a partly successful attempt to capture the remaining edge-critical graphs by exploiting the results of [4] and [9]. After proving a lemma needed to justify the procedure, we examine the extensions of the four connected edge-critically nonprojective planar graphs on 12 vertices. It is seen that any edge-critical graph for the torus that contains a homeomorph of one of $F_{11}$, $F_{12}$, $F_{13}$, $F_{14}$ must also contain a homeomorph of the graph $G$. Such graphs must contain at least 16 vertices. A list of cubic graphs on sixteen vertices that contain $G$ and are edge-critical for the torus is given in Appendix C.

That the graphs in appendices A and C are distinct has been verified using a program
supplied by Dr. R. W. Weaver (documented in [13]). We depart from past custom and do not present evidence for the nonisomorphism of the listed graphs in the belief that the recent proliferation of microcomputer programs for analysis of graphs makes the documentation of nonisomorphism of graphs unnecessary.

Appendices A, B, and C contain 35 cubic graphs that are edge-critical—a 36th, on eighteen vertices, from the folklore is shown to contain G and displayed at the end of Chapter VI.

A final word about the nomenclature of Appendices A, B and C. Common given names male in Appendix B, female in Appendices A and C, were used to designate the graphs for reference. Such a procedure is unusual and interpretable as being flip. Such is not the author's intention. It is our belief that the graphs displayed fall into intrinsically meaningful categories—the class of "fragmented k-graphs" for example—but that those categories are not at present apparent. The designators used are to be thought of as nonce usages pending a fuller understanding of their relationships.
§ I.1 Graphs

By a graph $G$ we shall mean an ordered pair $(\Gamma, V(G))$, where $\Gamma$ is a nonempty compact Hausdorff space and $V(G)$ is a finite collection of points of $\Gamma$ that satisfies the condition that each component of $\Gamma - V(G)$ is homeomorphic to the open unit interval $(0, 1)$. We refer to the elements $V(G)$ as the vertices of $G$. The components of $\Gamma - V(G)$ are called the edges of $G$; we write $(\Gamma - V(G)) = E(G)$.

Given a vertex $x$ and an edge $e$, we say $x$ and $e$ are incident provided $x$ is in the closure of $e$ (The topological closure of a set $S$ is denoted $\text{cl}(S)$). For any edge $e$ $\text{cl}(e)$ is either a homeomorph of the unit circle in which case $e$ is called a loop, or is a homeomorph of the closed unit interval $[0, 1]$ in which case we say that $e$ is a link. If $e$ is a loop $e$ is incident with a single vertex. If $e$ is a link $e$ is incident with exactly 2 vertices, called the ends of $e$. Vertices are said to be adjacent if they are the ends of a common link. For a link $e$ of $G$ a choice of homeomorphism $\chi: [0, 1] \rightarrow \text{cl}(e)$ is called an orientation of $e$. The pair $(\text{cl}(e), \chi) = e$ is called an oriented edge, and we say the $e$ is orientated from $\chi(0)$ to $\chi(1)$. A choice of orientation for every edge of $G$ is called an orientation of $G$. If two links of $G$ have the same ends we say that they are parallel edges in $G$. A graph without loops or parallel edges is said to be simple.
Our graphs will be assumed finite i.e. $E(G)$ will be assumed to be of finite cardinality. The valence (or degree) of a vertex $x$, is computed as the sum of twice the number of loops incident with $x$ with the number of links incident with $x$. A vertex is isolated if it is of degree zero; pendant (or monovalent) if it is of degree 1; divalent if it is of degree 2; and cubic if of degree 3. In a graph $G$, vertices of degree greater than two are the nodes of $G$. A graph is cubic if each of its vertices is cubic.

A path in a topological space $T$ is a continuous map $\chi: [0, 1] \rightarrow T$. If $G = (\Gamma, V(G))$ is a graph and $\chi: [0, 1] \rightarrow \Gamma$ we say $\chi$ is a path in $G$ from $x$ to $y$ provided $\chi(0) = x$ and $\chi(1) = y$ are vertices of $G$ and for each edge $e$ of $G$ either $e \cap \chi([0, 1]) = e$ or $e \cap \chi([0, 1])$ is empty. The elements of the set $X = \chi([0, 1]) \cap V(G)$ are called the vertices of $\chi$. If $\{0 = t_0 < t_1 < t_2 < \ldots < t_n = 1\}$ $= \chi^{-1}(X)$ we say that $\chi$ is a path of length $n$. If $\chi(t_0) = \chi(t_n)$ we say that $\chi$ is a closed path (or circuit). If $\chi$ is a homeomorphism we say $\chi$ is a simple path. If $\chi([0, 1])$ is homeomorphic to the circle and $\chi$ is a homeomorphism on $(0, 1)$ we say that $\chi$ is a simple closed path (or polygon). The length of the shortest polygon in $G$ is the girth of $G$. Setting $x_i = \chi(t_i)$ we denote the set $\chi([0, 1])$ as $[x_0, x_1, x_2, \ldots, x_n]$. We shall regularly abuse our notation by failing to maintain the distinction between $\chi$ and its image and by describing a simple closed path as "the cycle $[x_0, x_1, x_2, \ldots, x_{n-1}]$". While there are several ambiguities inherent in the notation and usages described above, in practice they seem to engender little confusion.

Two graphs $G = (\Gamma, V(G))$ and $L = (\Lambda, V(L))$ are said to be homeomorphic if there is a homeomorphism $\rho: \Gamma \rightarrow \Lambda$. If further $\rho(V(G)) = V(\Lambda)$ we say that
G and L are isomorphic. An isomorphism of graph G onto itself is called an automorphism. The collection of all automorphisms of G is denoted \( \text{Aut}(G) \).

If \( G = (\Gamma, V(G)) \) and \( L = (\Lambda, V(L)) \) are graphs with \( \Gamma \supseteq \Lambda \) and \( V(G) \supseteq V(L) \) we say that \( G \) is a topological supergraph of \( L \) or that \( L \) is a topological subgraph of \( G \). If further \( E(G) \supseteq E(L) \) we say that \( G \) is a supergraph of \( L \) or that \( L \) is a subgraph of \( G \).

Suppose \( B \) is a subgraph of \( G \), then define

\[ W(G, B) = \{ x \mid x \in V(B) \text{ and } x \text{ is incident with some edge in } E(G) - E(B) \} \]

the elements of \( W(G, B) \) are called the vertices of attachment of \( B \) in \( G \). If \( B \) and \( L \) are subgraphs of \( G \) and \( B \) satisfies: i) \( B \) is not a subgraph of \( L \); and ii) \( W(G, B) \) is a subset of \( V(L) \) we say that \( B \) attaches to \( L \). If further \( B \) is minimal with respect to properties i) and ii) we say that \( B \) is a bridge mod \( L \).

If \( \Gamma \) is a connected topological space we call \( G \) a connected graph. If \( e \) is an edge of \( G \), then \( (G - e) = (\Gamma - e, V(G)) \) is the subgraph of \( G \) obtained by deleting \( e \). If \( S \) is a collection of edges of \( G \) then \( G - S = (G - \cup S, V(G)) \). If \( G - e \) has more components than does \( G \) we say that \( e \) is an isthmus of \( G \). In general if \( S \) is a collection of edges in \( G \) and \( G - S \) has more components than does \( G \) then \( S \) is said to disconnect \( G \). If \( S \) disconnects \( G \) and no proper subset of \( S \) disconnects \( G \) then \( S \) is a bond of \( S \). If \( S \) is a bond of cardinality \( n \) then \( S \) is called an \( n \)-bond in \( G \). If the collection of edges incident with a vertex \( x \) form a bond, it is called the vertex bond of \( x \).

If \( W = (\Omega, V(W)) \) and \( L = (\Lambda, V(L)) \) are subgraphs of \( G \), then \( W \cup L = (\Omega \cup \Lambda, V(W) \cup V(L)) \) is also a subgraph of \( G \). An \( n \)-separation on \( G \) is a pair \( \{W, L\} \) of subgraphs of \( G \) satisfying the three conditions: i) \( W \cup L = G \);
ii) \( \Omega \cap \Lambda \) consists of exactly \( n \geq 0 \) vertices of \( G \); and, iii) \( E( W ) \) and \( E( L ) \) are each of cardinality at least \( n \). If \( x \) is a vertex of valence \( n \) in a simple graph \( G \) and \( \{ W, L \} \) is an \( n \)-separation of \( G \) with \( W \) formed by the vertex \( x \) and its incident edges, we call \( \{ W, L \} \) a trivial \( n \)-separation of \( G \). A graph is \( n \)-separated if it has an \( n \)-separation. A graph is \( k \)-connected provided it has no \( n \)-separation for \( n < k \). If \( G \) has a 1-separation it is said to be separable. The maximal 2-connected subgraphs of \( G \) are called the blocks of \( G \). If \( G \) is itself 2-connected then it is said to be nonseparable.

For cubic graphs the notions of \( n \)-separation and \( n \)-bond are closely related, we assert the following as easy consequences of the definitions:

1. If \( G \) is a connected cubic graph and \( G \) is 1-separated then \( G \) has an isthmus.
2. If \( G \) is a nonseparable cubic graph and \( G \) is 2-separated then \( G \) has a 2-bond.
3. If \( G \) is a 3-connected cubic graph and \( G \) is 3-separated then \( G \) has a 3-bond.

Finally if \( L = ( A, V(L) ) \) is a connected subgraph of \( G \) then \( G /_L \) shall denote the graph \( ( \Gamma /_A, ( V(G) - V(L) ) \cup \{ A \} ) \).
§ 1.2 Surfaces

A 2-manifold $M$ is a Hausdorff space in which each element has a neighborhood homeomorphic to the open 2-cell $D^2 = \{ (x, y) \mid x^2 + y^2 < 1 \}$. We will refer to any subset of a manifold $M$ that is homeomorphic to $D^2$ as either a disk or an open 2-cell. A 2-manifold with boundary $N$ is a Hausdorff space in which each element has a neighborhood homeomorphic to $D^2$ or to $D^2_+ = \{ (x, y) \mid x^2 + y^2 < 1 \text{ and } x \geq 0 \}$. Those points of a 2-manifold with boundary $N$, for which no neighborhood is homeomorphic to $D^2$ are boundary points of $N$. A surface $\Sigma$ is a compact connected 2-manifold. The 2-sphere $S^2 = \{ (x, y, z) \mid x^2 + y^2 + z^2 = 1 \}$ is a manifold. Note also that for any point $p$, $S^2 - p$ is homeomorphic to $D^2$. We shall have occasion to discuss manifolds with and without boundaries with reference to their mapping diagrams.

![Mapping diagrams for the annulus and Mobius band](image)

Figure I.01

Figure I.01 above illustrates mapping diagrams for the annulus and Mobius band (also known as a cross-cap) respectively. These spaces are formed from the indicated 2-cells by identifying, ("pasting together") their left and right boundaries so as to bring the four indicated points into coincidence. The torus and the projective plane -- denoted by $T$ and $\Pi$ in the sequel -- we take to be the surfaces formed from the mapping diagrams of Figure I.02, as shown below.
We note here that \( \Pi \) is homeomorphic to the surface formed by identifying the boundaries of \( S^2 - D \), where \( D \) is homeomorphic to \( D^2 \), and the mobius band. The process described above is known as taking the connected sum of a mobius band and the sphere. The **Classification Theorem for Surfaces** states that each surface is homeomorphic either to the sphere, or the connected sum of \( n \geq 1 \) toruses or of \( S^2 \) with \( m \geq 1 \) Mobius bands. If the latter the surface is said to be nonorientable; otherwise the surface is said to be orientable. The reader is referred to **Massey [8]** for a full and readable discussion of mapping diagrams, surfaces and the Classification Theorem.

A path \( \alpha: [0, 1] \rightarrow \Sigma \) is a closed path in \( \Sigma \) provided \( \alpha(0) = \alpha(1) \). Then \( \alpha \) is a simple closed path in \( \Sigma \) provided \( \alpha([0, 1]) \) is homeomorphic to a circle and \( \alpha \) is a homeomorphism on \((0,1)\). Two closed paths \( \alpha \) and \( \beta \) are homotopic in \( \Sigma \) provided there is a continuous map \( H: [0, 1] \times [0, 1] \rightarrow \Sigma \) such that \( H(x, 0) = \alpha(x) \), \( H(x, 1) = \beta(x) \), and \( H(0, y) = H(1, y) \). A path is inessential if it is homotopic to a constant map. If a path that is not inessential it is an essential path. For any manifold an operation (essentially concatenation) may be defined on \( \pi_1(\Sigma) \) closed paths which gives rise to the fundamental group of the surface, usually denoted \( \pi_1(\Sigma) \). For such an operation the equivalence class of inessential maps plays the role of the neutral element. In
Chapter V we have occasion to refer to this operation and violate tradition by writing it additively. Again the reader is directed to Massey [8] for a discussion of \( \pi_1(\Sigma) \).

For our purposes it more than suffices to note that:

i) \( \pi_1(\Pi) = \mathbb{Z}_2 \) the group of two elements and any pair of essential cycles in \( \Pi \) meet; and

ii) \( \pi_1(\mathbb{T}) = \mathbb{Z} * \mathbb{Z} \) the free group on two generators, and essential cycles, not multiples of the same generator, intersect.

For \( R \) any subset of a manifold \( M \) the boundary of \( R \), denoted \( \text{Bdry}(R) \) is

\[ \{ x \mid x \in M \text{ and each neighborhood of } x \text{ meets both } R \text{ and } (M - R) \} \].

That inessential cycles are the boundaries of disks in \( \Sigma \) is the content of the Theorem below (See Vitray [12]):

**Theorem:** Let \( \beta : [0, 1] \rightarrow \Sigma \) be a simple closed path that is inessential in a surface \( \Sigma \). Then \( \Sigma - \beta([0, 1]) \) consists of two connected regions \( R_1 \) and \( R_2 \) for which \( \text{Bdry}(R_1) = \text{Bdry}(R_2) = \beta([0, 1]) \). If \( \Sigma \) is the sphere then both of \( R_1 \) and \( R_2 \) are homeomorphic to \( D^2 \). If \( \Sigma \) is not a sphere then precisely one of \( R_1 \) and \( R_2 \) is a disk.
§ 1.3 Surface Embeddings of Graphs

An embedding of a graph \( L = (\Lambda, V(G)) \) into a manifold \( M \), is a continuous injective mapping \( \eta: \Lambda \longrightarrow M \). (We write \( \eta: L \longrightarrow M \) and let \( \eta(L) \) represent the point set \( \eta(\Lambda) \)). The components of \( M - \eta(L) \) are the regions of \( \eta \). If each region of \( \eta \) is homeomorphic to \( D^2 \) we say that the embedding is an open 2-cell embedding. If \( M \) is a surface and \( \eta \) is open 2-cell we refer to the regions of \( \eta \) as the faces of \( \eta \). The collection of faces of \( \eta \) is denoted by \( F(\eta) \). If \( \eta \) is any open 2-cell embedding of a graph \( L \) into a surface \( \Sigma \) then \( L \) is a connected graph and the value of the following sum (for finite set \( X \) we denote its cardinality by \( |X| \) ) depends only on the surface \( S \) and is called the Euler characteristic of \( \Sigma \):

\[
|F(\eta)| - |E(L)| + |V(L)| \quad .
\]

The figures below illustrate embeddings of the complete bipartite graphs \( K_{2,3} \) and \( K_{3,3} \) into the torus and into the projective plane, respectively. See Massey [81] for proof of the above assertions.

Each embedding of Figure I.03 is an open 2-cell embedding and thus allows a computation of the Euler characteristic.

For \( T \) :

\[
|F(\eta)| - |E(K_{2,3})| + |V(K_{2,3})| = 1 - 6 + 5 = 0
\]
For $\Pi$:

$$|F(\eta)| - |E(K_{3,3})| + |V(K_{3,3})| = 4 - 9 + 6 = 1.$$  

We remark that if e is an edge of a cubic graph L, and e is in some triangle of L then any embedding of L - e in a surface $\Sigma$ may be extended to an embedding of L into $\Sigma$.

If $\eta: L \to \Sigma$ is an open 2-cell embedding and F is a face of $\eta$, then the elements of $V(L) \cap \eta^{-1}(\text{cl}(F))$ are the vertices incident with F; the elements of $E(L) \cap \eta^{-1}(\text{cl}(F))$ are said to be the edges incident with F. Note that the image under $\eta$ of the union of the edges and vertices incident with F is equal to the boundary of F. Let $\xi: D^2 \to F$ be a homeomorphism of the open disk onto F. Let $\xi^*: \text{cl}(D^2) \to \text{cl}(F)$ be the continuous extension of $\xi$. The restriction of $\xi^*$ to $S^1$, the boundary of $D^2$ may be viewed --- take $\xi^*(\gamma(t)) = \xi^*(\cos(2\pi t), \sin(2\pi t))$ --- as a closed curve in L. This curve is the oriented boundary cycle of F. We remark that each boundary cycle has two possible orientations corresponding to the choice of maps $\gamma(t)$ or $\gamma(1-t)$ from $[0, 1]$ above. Let an embedding $\eta$ be fixed, and x and y be adjacent vertices of L with e an edge joining them. If e is in the image of the boundary cycle $\gamma$ of a face F of $\eta$ with $\gamma^1(e) = [\gamma^1(x), \gamma^1(y)]$ we say that e is oriented from x to y in $\eta$. It is possible to simultaneously choose orientations for each boundary cycle so that for every edge e with ends x and y, e appears in two (not necessarily distinct) boundary cycles once with orientation from x to y and once with orientation from y to x precisely if $\Sigma$ is an orientable surface. Given an embedding $\eta$ it is possible to canonically reconstruct $\eta$ from the collection of its boundary cycles.
Embeddings \( v, \) and \( \mu \) of graph \( L \) into a surface \( \Sigma \) are said to be equivalent provided there is a homeomorphism \( \xi: \Sigma \rightarrow \Sigma \) and an isomorphism \( \zeta: L \rightarrow L \) such that \( \xi(v(x)) = \mu(\zeta(x)) \). Two embeddings \( \mu \) and \( v \) are seen to be equivalent if and only if there is an isomorphism \( \zeta: L \rightarrow L \) that maps the boundary cycles of \( \mu \) onto the boundary cycles of \( v \).

In chapters III, IV and V we shall need the following definitions:

**Definition:** If \( \eta: L \rightarrow \Sigma \) is an embedding of a graph \( L \) into a surface \( \Sigma \), then by an **O-arc** of \( \eta \) we shall mean a simple closed path \( \gamma: [0, 1] \rightarrow S \) satisfying:

i) \( \gamma \) is essential in \( \Sigma \).

ii) \( \gamma([0, 1]) \cap \eta(L) = W(\gamma) \) is a finite set of points.

**Definition:** The **weight** of an O-arc \( \gamma \) is the cardinality of \( W(\gamma) \).

**Definition:** An embedding \( \eta \) of \( L \) is said to be **n-representative** provided \( n \) is the minimum of \( W(\gamma) \) for all O-arcs \( \gamma \) of \( \eta \).
§ II.1: $K_{3,3}$

The first and fundamental graph in the study of embeddings is $K_{3,3}$, one of Kuratowski's nonplanar graphs. We take $K_{3,3}$ to have vertex set \{1, 2, 3, 4, 5, 6\} with edges \([x, y]\) exactly when \(x\) and \(y\) are of different parity. Thus
\[
|V(K_{3,3})| = 6 \quad \text{and} \quad |E(K_{3,3})| = 9.
\]
Henceforth, for any graph $G$, take
\[
V = |V(G)| \quad \text{and} \quad E = |E(G)|.
\]
We shall need to know the embeddings of $K_{3,3}$ into both the projective plane, $\Pi$, and the Torus, $T$. We treat the projective plane first.

Any embedding $\xi$ of $K_{3,3}$ into $\Pi$ will divide $\Pi$ into $|R(\xi)| = R$ regions. Each of the regions of $\xi$ is either a Mobius band or a 2-cell. If one of the regions were a Mobius band then performing a simple surgery on $\Pi$ to replace the Mobius band with a 2-cell produces an embedding of $K_{3,3}$ into the sphere, a contradiction. Thus any embedding $\xi$ is an open 2-cell embedding and satisfies $V - E + R = 1$.

Substituting $V = 6$ and $E = 9$ we obtain $R = 4$. Let the face-degrees of $\xi$ be $f_1, f_2, f_3, f_4$ then $f_1 + f_2 + f_3 + f_4 = 2E = 18$. Since $K_{3,3}$ is simple and bipartite each $f_i$ is even and at least 4. The only face degree sequence consistent with these requirements is $(4, 4, 4, 6)$. The figure below realizes such an embedding $\xi$. To see that the indicated embedding is the only embedding of $K_{3,3}$ into $\Pi$, up to equivalence, we need to use the following observations about embeddings on cubic graphs into surfaces.
1. The sequence \([..., x, y, x, ...]\) never occurs in the boundary cycle of a region \(F\).

2. The sequence \([..., x, y, z, ...]\) cannot occur twice in the boundary cycles of an single embedding.

![Figure II.01](image)

Observe that condition 1 implies that for any face \(F\) with face degree less than 8 of an embedding of \(K_{3,3}\) into some surface \(\Sigma\), then \(F\) has a bounding cycle that is a simple cycle in \(K_{3,3}\) containing 4 or 6 vertices. Since the smallest polygon in \(K_{3,3}\) is a 4-gon, the face of \(\zeta\) of degree 6 must be bounded by a 6-gon in \(K_{3,3}\). The 6-gon of \(\zeta\) may be taken as \([1, 2, 3, 4, 5, 6, 1]\). The edge \([1, 2]\) must appear in some degree 4 face, say \(F\), of \(\zeta\) with boundary cycle \([1, 2, x, y]\). By condition 1, \(x\) is one of 3 or 5. If \(x = 3\) then \(\zeta\) violates condition 2 above and vertex 2 is a divalent vertex in the embedding but not in the graph; a contradiction. By condition 1, \(y\) must be one of 4 or 6; by condition 2, \(y\) is 4. Thus the face \(F\) on \(\zeta\) has boundary cycle \([1, 2, 5, 4]\). Similarly the boundary cycles containing \([2, 3]\) and
[3, 4] must be [2, 3, 6, 5] and [3, 4, 1, 6] respectively. This completes the proof of:

**Proposition:** There is exactly one embedding of $K_{3,3}$ into the projective plane.

Consider now an embedding $\xi$ of $K_{3,3}$ into the torus $T$. Suppose there were an essential cycle $\alpha$ of $T$ contained in $T - \xi(K_{3,3})$. Then $\xi(K_{3,3})$ is contained in the annulus $T - \alpha$. This supplies a planar embedding of $K_{3,3}$. Therefore for any embedding $\xi$, $\xi(K_{3,3})$ contains nonhomotopic essential cycles $\beta$ and $\gamma$ meeting at a vertex $x$ of $\xi(K_{3,3})$. So $\xi(K_{3,3})$ contains a mapping diagram for $T$ with $T$ represented as a 2-cell attached to a theta-graph in $\xi(K_{3,3})$. Since the faces of $\xi$ are the components of $T - \xi(K_{3,3})$ and since $K_{3,3}$ is 3-connected, $\xi$ is an open 2-cell embedding. Recalling that the euler characteristic of $T$ is zero; we get

$$0 = V - E + R(\xi) = 6 - 9 + R,$$

hence $R = 3$. Taking $f_1$, $f_2$, $f_3$ as the face degrees of $\xi$, each $f_i$ is even and at least 4, and $f_1 + f_2 + f_3 = 2E = 18$. The possible face degree sequences are (4, 4, 10), (4, 6, 8), and (6, 6, 6). Embeddings that realize the face degree sequences (4, 4, 10) and (6, 6, 6) are shown below.

![Figure II.02]
Since $T$ is an orientable surface, in addition to conditions 1. and 2., the boundary cycles of $\xi$ must also satisfy:

3. If an edge $[x, y]$ occurs in the (not necessarily distinct) regions $F$ and $F'$ then it occurs once with orientation $[x, y]$ and once with orientation $[y, x]$.

To see that the sequence $(4, 6, 8)$ cannot be realized as the face degree sequence of an embedding of $K_{3,3}$ note that WOLOG the bounding cycle of the face of degree six may be taken to be $[1, 2, 3, 4, 5, 6]$ -- with this orientation -- and the face $F$ of degree four in $\xi$ may be taken to contain the edge $[1, 2]$. As above the boundary cycle of $F$ must be $[1, 2, 5, 4]$ with one of the two orientations $[1, 2, 5, 4]$ or $[2, 1, 4, 5]$. However both of these violate condition 3, which supplies a contradiction to the existence of an embedding of $K_{3,3}$ into $T$ of type $(4, 6, 8)$.

We now verify that the embeddings shown above are the only embeddings on $K_{3,3}$ into $T$ that realize their face degree sequences.

Consider an embedding $\xi$ of $K_{3,3}$ with faces $F_1$, $F_2$, $F_3$ all of degree 6. WOLOG we may take $F_1$ to be the 6-gon $[1, 2, 3, 4, 5, 6]$ with that orientation. Let $F_2$ be the face $[2, 1, w, x, y, z]$; then:

i) $w = 4$; else $F_2$ with $F_1$ violates condition 2.

ii) $x = 3$; else $[4, 5]$ violates condition 3.

iii) $y = 6$; and

iv) $z = 5$; else $F_2$ violates condition 1.

The boundary cycle of $F_3$ similarly is $[1, 6, 3, 2, 5, 4]$ as in Figure II.02.
Turning our attention now to an embedding $\xi$ with faces $F_1, F_2, F_3$ of degrees 4, 4, 10 respectively, we take $F_1$ to be $[1, 2, 5, 6]$ with the given orientation and -- since $F_2$ must share at least one edge with $F_1$ -- we take $F_2$ as $[5, 2, y, z]$:  

i) $y = 3$; else $F_3$ violates condition 1.

ii) $z = 4$; else $F_1$ and $F_2$ condition 2.

Consider $F_2 = [1, 6, q, r, s, t, u, v, w, x]$: 

i) $q = 3$; else $F_3$ violates condition 1 ($q = 1$) or $F_3$ with $F_2$ violate condition 2.

ii) $r = 2$; else $[3, 4]$ violates condition 3.

iii) $s = 1$; by conditions 1 and 2.

iv) $t = 4$; by conditions 1 and 2.

v) $u = 3$; else $[4, 5]$ violates condition 3.

vi) $v = 6$

vii) $w = 5$

viii) $x = 4$

This is exactly the embedding indicated in the left figure and we have completed the proof of:

**Proposition:** There are exactly two embeddings of $K_{3,3}$ into the torus; one has face degree sequence $(4, 4, 10)$ the other $(6, 6, 6)$. 
§ II.2 Theta Graphs:

In this section we take $\Theta$ to have vertex set $\{ 1, 2 \}$ and oriented edge set $\{ a, b, c \}$. Further we give each of the edges an orientation from 1 to 2.

Remark: There are exactly two embeddings of a Theta graph $\Theta$ into $\Pi$.

![Type A and Type B embeddings of Theta graphs](image)

Figure II.03

Proof: If no cycle $\Theta$ is essential then $\Theta$ is embedded in a disk in $\Pi$. Since $\Theta$ is 3-connected there is but one embedding on $\Theta$ into a disk -- shown above as type A. Otherwise there is an essential cycle of the embedding and the embedding is an open 2-cell embedding with two faces (since the Euler characteristic of $\Pi$ is 1 and $\Theta$ has two vertices and three edges the number of faces must satisfy $2 - 3 + F = 1$). Since $\Theta$ is bipartite any embedding may have only even face degrees satisfying $f_1 + f_2 = 2 \times 3 = 6$. Thus we obtain $f_1 = 2$ and $f_2 = 4$.

One face may be taken as having boundary $ab^{-1}$. The second face cannot have consecutive occurrences of edge $c$ and WOLOG may be taken to start with edge $a^{-1}$. 
This suffices to force the boundary of the second face to be $a^{-1}cb^{-1}c$. Thus the mapping diagram of the second face is:

![Diagram of the mapping diagram of the second face]

**Figure II.04**

**Lemma:** There are exactly three embeddings $v$ of $\Theta$ into the torus $T$.

![Diagrams of Type I, II, and III embeddings]

**Figure II.05**

**Proof:** There are three simple cycles in $\Theta$. If each cycle is contractible in $T$ then $v(\Theta)$ is contained in a disk of $T$. Since 3-connected graphs have a unique planar embedding $v$ is of Type I. Conversely, if $ab^{-1}$ is essential and $ac^{-1}$ contractible
then $bc^{-1} = (ab)^{-1}(ac)$ is essential and homototic to $ab^{-1}$, so that $v$ is of Type II. If $ab^{-1}$ is essential in $T$ and homotopic to $ac^{-1}$ then $bc^{-1} = (ab^{-1})^{-1}(ac^{-1})$ is contractible and $v$ is of Type II. If $ab^{-1}$ and $ac^{-1}$ are both essential but not homototic then $v$ is of Type III.

Let us record the observation that only embeddings of Types II and III contain a region which meet all three edges of $\Theta$. The form in which we will use it is:

**Lemma:** If a graph $\Gamma$ contains disjoint subgraphs $\Theta_1$ and $\Theta_2$ each homeomorphic to $K_{2,3}$ and $\lambda$ is any embedding of $\Gamma$ into the torus $T$, then either:

i) One of $\lambda|\Theta_1$ or $\lambda|\Theta_2$ is of Type I. or

ii) Both of $\lambda|\Theta_1$ or $\lambda|\Theta_2$ are of Type II.

**Proof:** If $\lambda|\Theta_1$ (or $\lambda|\Theta_2$) is an embedding of Type III then $\lambda|\Theta_2$ (or $\lambda|\Theta_1$ respectively) is contained in a 2-cell of $T - \lambda(\Gamma)$. 


§ II.3 The Mobius Ladders:

Definition: By $M_k$ for $k \geq 3$ we shall mean the cubic graph on $2k$ vertices 1, 2, ..., $2k$ that is formed from the cycle [1, 2, ..., 2k] by joining vertices $i$ and $j$ for $i$ and $j$ congruent mod $k$.

The following are elementary observations about $M_k$.

1. $M_k$ contains $K_{3,3}$ as a topological subgraph.
   Proof: Delete all but three of the edges of type [i, i+k].

2. Aut($M_k$) is transitive on vertices.
   Proof: The map sending $i$ into $i+1$ modulo $2k$ is an automorphism of $M_k$.

3. For $k \geq 4$ there are two orbits of edges under Aut($M_k$). Each edge may be distinguished as being in one or two 4-gons. Those edges in exactly one 4-gon are the edges joining vertex $i$ to $i+1$ in the above definition. (The cycle formed by such edges we shall refer to as $S$, the spine of the ladder.) The edges joining vertices $i$ and $j$ congruent mod $k$ are contained in two 4-gons and will be referred to as the rungs of the ladder.
   Proof: Clear.

4. Deleting any rung from $M_{k+1}$ produces a graph homeomorphic to $M_k$.
   Proof: By 2 above we may delete the rung [2k+2, k+1] and renumber $k+2$ through $2k+1$ as $(k+1)'$ through $(2k)'$. 
5. Every pair of rungs of $M_k$ form a pair of overlapping bridges of the spine.

Proof: Clear.

6. For $k \geq 4$ there are exactly two embeddings $v$, of $M_k$ into $\Pi$:

![Diagram showing two types of embeddings](image)

Figure II.06

Proof: In extending the embedding of $K_{3,3}$ to an embedding of $M_4$ the fourth rung can split the face of degree 6 or not. Since each pair of rungs are overlapping bridges (mod S) for $k > 4$ there is at most one rung interior to $v(S)$.

We turn our attention to embeddings, $v$ of $M_k$ into the torus. For $k = 3$ we have shown that there are precisely two embeddings. So assume that $k \geq 4$. Let $S$ be the spine of the ladder and suppose that $v(S)$ is null in $T$. Since for any three rungs $R_1, R_2, R_3 \cup \{R_1, R_2, R_3\}$ is a homeomorphic of $K_{3,3}$ two of the three rungs together with $S$ have the following image under $v$: 
This configuration divides $T$ into a pair of open 2-cells each containing the spine as part of their boundary. Clearly, since each pair of rungs overlap, at most one rung can cross each of the 2-cells. Thus we obtain the figure below as the unique drawing of $M_4$ in $T$ with a contractible spine.

We have established the following.
**Proposition:** If $v : M_k \rightarrow T$ is an embedding of an Mobius ladder with $k \geq 4$ and $S$ as its spine into the torus then either:

1. $v(S)$ is essential in $T$, or
2. $k = 4$ and $\mu$ is the embedding $(5, 5, 7, 7)$ of $M_4$ in the figure above.

**Definition:** If $v : M_k \rightarrow T$ for $k \geq 4$, and $v(S)$ is an essential cycle in $T$ then we say that $v$ is a typical embedding of $M_k$.

There are a number of typical embeddings of $M_4$ and consequently $M_k$ for $k \geq 4$.

---

**Figure II.09** $M_4$ as $(4, 5, 5, 10)$

**Figure II.10** $M_4$ as $(4, 4, 5, 11)$
That each of these embeddings is different it is easy to check because they have distinct face degree sequences: \((4, 5, 5, 10)\); \((4, 4, 5, 11)\); \((4, 4, 4, 12)\); and \((4, 6, 6, 8)\).
Definition: Suppose that $\nu : M_k \rightarrow T$ for $k \geq 4$ is a typical embedding. For each rung $R$, $S \cup \{R\}$ forms a theta-graph and $\nu|_{S \cup \{R\}}$ may be of type II or III:

- If $\nu|_{S \cup \{R\}}$ is of type III, $R$ is said to be **essential** in the embedding $\nu$.
- If $\nu|_{S \cup \{R\}}$ is of type II, $R$ is said to be **null** in the embedding $\nu$.

Consider a typical embedding $\mu$ of $M_k$ into $T$. The orientability of $T$ allows us to observe that an $\epsilon$-neighborhood $N$ of $\mu(S)$ is an annulus divided into two (annular) regions by $\mu(S)$. These two regions may be arbitrarily denoted as "U" and "D". Note further that if $R$ is a null rung of $\mu$ then $\mu(R)$ may be taken to lie entirely within one of the two regions $U$ or $D$. Since the union of $S$ and any three rungs of $M_k$ is a homeomorph of $K_{3,3}$ no embedding of $M_k$ can contain more than two null rungs. The preceding drawings of $M_4$ as $(4, 4, 5, 11)$ and $(4, 5, 5, 10)$ can clearly be extended to single null rung and double null rung embeddings respectively of $M_k$ for any $k > 4$. Indeed we have the following.

**Proposition:** For each $k \geq 4$ there is exactly one single null rung embedding and exactly one double null rung embedding of $M_k$.

**Proof:** Let $R = [1, k + 1]$ be a null rung of $\mu$. Consider an $\epsilon$-neighborhood, $N$ of $\mu(S)$ and designate the side of $\mu(S)$ which contains $\mu(R)$ as $U$. Then all of the rungs $[2, k + 2],...[k, 2k]$ must exit the arc $\mu([1, 2, ..., k])$ though the region $D$. If any rung except $[k, 2k]$ were to be null the remaining essential rungs would be confined to a planar region, contradicting the definition of essential. Thus each of the intermediate rungs $[i, i + k]$ cross the annulus $T - N$ from one bounding cycle to the other. This completes the embedding uniquely.
Suppose that $\mu$ contains no null rungs. Since all of the vertices of $M_k$ lie on the spine, each face of is $\mu$ formed by two rungs of the ladder and a pair of arcs of the spine of $M_k$. Yet further, since $T$ is orientable, condition 3 of §II.1 requires that if the oriented edge $[i, j]$ occurs in the boundary of a face the second face meeting both $i$ and $j$ must contain the oriented edge $[j, i]$. If $R$ is an essential rung in a typical embedding then one end of $\mu(R)$ meets $\mu(S)$ through the region $U$ and the other through the region $D$. Thus any typical embedding of $M_k$ without null rungs gives rise to a unique -- up to cyclic order -- sequence of $k$ $U$'s and $k$ $D$'s. Such sequences must satisfy the property that the entries in positions $i$ and $(i + k)$ are different. As an example consider the embeddings $(4, 4, 4, 12)$ and $(4, 6, 6, 8)$ of $M_4$. The information necessary to describe these embedding is contained in the sequences $(U, U, U, U, D, D, D, D)$ and $(U, U, D, U, D, D, U, D)$ respectively.

Conversely, consider the sequence $(U, U, D, U, D, D, U, D)$ as shown below: 

![Figure II.13](image-url)
The face structure of an embedding realizing such a sequence is completely determined as:

\[ [1, 2, 7, 6] ; [2, 3, 4, 9, 8, 7] ; [4, 5, 6, 7, 8, 3, 2, 1, 10, 9] ; [8, 9, 10, 5, 4, 3]; [10, 1, 6, 5] \].

Generally, let \( \Sigma = (s_1, s_2, \ldots, s_k, s_{k+1}, \ldots, s_{2k}) \) be any cyclic sequence of \( U \) and \( D \) subject only to the requirement that \( s_i \) and \( s_{i+k} \) (subscripts are interpreted mod \( 2k \)) are of opposite type. Then \( \Sigma \) determines uniquely (up to a cyclic permutation of the vertices of \( M_k \)) the faces of an embedding \( \mu : M_k \rightarrow T \) as follows:

For each \( U \) at position \( i \), in the sequence let \( j_i \) be the next (order is that of increasing index) position in which \( U \) occurs form a face with boundary as follows: \( [i, i+1, \ldots, j_i, j_i+k, j_i+k-1, \ldots, i+k] \).

Note that the string \( [j_i+k, j_i+k-1, \ldots, i+k] \) runs between consecutive occurrences of \( D \). One cell is obtained for each \( U \) in the sequence. Each pair \( \{i, i+1\} \) occurs once as \( [i, i+1] \) between occurrences of \( U \) and once as \( [i+1, i] \) between occurrences of \( D \). Each pair \( \{i, i+k\} \) appears once in the order \( (U, D) \) and once in the order \( (D, U) \).

The surface formed by assembling these faces is orientable and satisfies; \( V = 2k, E = 3k, \) and \( F = k \) thus has Euler characteristic 0. By the classification theorem for surfaces the assembled surface is \( T \). Clearly the graph of this embedding is \( M_k \).
We may now classify all embeddings of $M_k$ into $T$ as of the following types:

i) If $k = 4$ there is a unique embedding with $\mu(S)$ contractible.

ii) For each $k \geq 4$ there is a unique typical embedding with two null rungs. It has faces degree sequence:

$$[4, \ldots, 4, k + 1, k + 1, 10]$$

$$\begin{array}{c}
\hline
\text{k - 3} \\
\end{array}$$


iii) For each $k \geq 4$ there is a unique typical embedding with one null rung. It has face degree sequence:

$$[4, \ldots, 4, k + 1, k + 7]$$

$$\begin{array}{c}
\hline
\text{k - 2} \\
\end{array}$$


iv) For $k \geq 4$ there are a number of typical embeddings with each rung essential. These are in one-to-one correspondence to cyclic sequences of U's and D's as described above. In particular there is a unique combed embedding corresponding to the sequence $(U, U, \ldots, U, D, D, \ldots, D)$. This embedding has face degree sequence:

$$[4, \ldots, 4, 2k + 4]$$

$$\begin{array}{c}
\hline
\text{k - 1} \\
\end{array}$$
Chapter III
Low Connectivity

Throughout "T" will represent a cubic graph without triangles, loops or 2-gons that is edge-critically nontoroidal.

§ III.1 Review of Decker's Results
In his doctoral dissertation The Genus of Certain Graphs [8] Richard W. Decker has shown that any graph with the property that it:

1) contains a subgraph K homeomorphic to one of the two Kuratowski graphs -- $K_5$ and $K_{3,3}$ -- and
2) contains, disjoint from K, one of three configurations known as k-graphs,

does not embed in T. Further he showed by explicit construction that there are exactly 259 such graphs that are edge-critically nontoroidal. Of these 259 graphs ten are cubic. We begin this chapter by categorizing these ten.

0-Separated Graphs: There is exactly one disconnected $\Gamma$; the disjoint union of two $K_{3,3}$'s.

From Chapter II every embedding of $K_{3,3}$ in the torus is an open 2-cell embedding.
Thus if $K_{3,3} \cup K_{3,3}$ were to be embedded in the torus one copy of $K_{3,3}$ would be embedded in a 2-cell; which is impossible. Thus $K_{3,3} \cup K_{3,3}$ is nontoroidal. On the other hand, deleting an edge of $K_{3,3} \cup K_{3,3}$ leaves a graph homeomorphic to the disjoint union of $K_{3,3}$ and $K_4$, which may clearly be embedded in $T$. To obtain that $K_{3,3} \cup K_{3,3}$ is the only disconnected $\Gamma$ let the components of $\Gamma$ be $C_1, C_2, \ldots, C_n$, with $n \geq 2$. Since $\Gamma$ is edge-critically nontoroidal at least two of $\Gamma$'s components must be nonplanar, WOLOG $C_1$ and $C_2$. But by Kuratowski's Theorem each of $C_1$ and $C_2$ contains a copy of $K_{3,3}$. So by minimality $\Gamma = K_{3,3} \cup K_{3,3}$

1-Separated Graphs: No $\Gamma$ contains an isthmus.

Suppose $\Gamma$ had an isthmus $e = [x, y]$ with end graphs $H_1$ and $H_2$. If both of $H_1$ and $H_2$ are nonplanar then $\Gamma - e$ contains $K_{3,3} \cup K_{3,3}$ and is not minimal. Thus one of $H_1$ and $H_2$ must be planar. WOLOG $H_2$ is planar. Since $\Gamma$ is edge-critically nontoroidal $H_1$ embeds on the torus. Let $\eta$ be an embedding of $H_1$ and let $A$ denote the face of $\eta$ that meets the vertex $x$. Take some planar embedding $\mu$ of $H_2$ in which $y$ is on the infinite face. Now take $B$ to be a disk in the plane that contains $\mu(H_2)$ on its interior. To embed $\Gamma - e$, excise a disk $C$ from $A$, form $\nu = \eta \mid_{T - C} \cup \mu$ by identifying the boundaries of $B$ and $C$ to produce an embedding $\Gamma - e$ into $T$. We complete $\nu$ to an embedding of $\Gamma$ by choosing a simple path $\alpha$ in $A - C$ from $\nu(x)$ to $\nu(y)$ and mapping $e$ onto $\alpha$.

2-Separated Graphs: There are precisely three 2-separated $\Gamma$'s.

Suppose that $\{ e_1 = [x_1, y_1], e_2 = [x_2, y_2] \}$ is a 2-bond of $\Gamma$. Write $\Gamma = H_1 \cup H_2 \cup \{ e_1, e_2 \}$ with $x_1$ and $x_2$ in $H_1$ and $y_1$ and $y_2$ in $H_2$ with $H_1$ and
$H_2$ disjoint. Note that both of $H_1$ and $H_2$ must contain at least two vertices and hence an edge.

Not both of $H_1$ and $H_2$ can be nonplanar as then $\Gamma$ contains $K_{3,3} \cup K_{3,3}$.

Further not both of $H_1$ and $H_2$ can be planar. If each is planar an embedding of $\Gamma$ into the torus may be constructed as follows. Take embeddings $\lambda_1$ and $\lambda_2$ of $H_1$ and $H_2$, respectively, into the sphere $S$. Let $F$ and $G$ be faces of $\lambda_1$ and $\lambda_2$ that contain $x_1$ and $y_1$, respectively on their boundaries. Interior to $F$ and $G$ excise disks $F'$ and $G'$ and form $\lambda = \lambda_1|_{S-F'} \cup \lambda_2|_{S-G'}$, an embedding of $H_1 \cup H_2$ into $S$, by identifying $\lambda_1$ and $\lambda_2$ on the boundaries of $F'$ and $G'$. Extend $\lambda$ to $\mu$, an embedding of $\Gamma - e_2$ into $S$, by choosing a simple arc $\alpha$ across $(F - F') \cup (G - G')$ from $\lambda_1(x_1)$ to $\lambda_2(y_1)$ and mapping $e_1$ onto $\alpha$.

In $\mu$ the vertices $\mu(x_2)$ and $\mu(y_2)$ occur in the boundaries of distinct faces $K$ and $L$. Extend $\mu$ to $\mu': \Gamma - e_2 \rightarrow T$ as follows. In the interior of $K$ and $L$ excise disks...
$K'$ and $L'$ respectively. Let $A$ be an annulus with boundary cycles $C_1$ and $C_2$. Then form $T$ as the connected union of $(S - (K' \cup L'))$ and $A$ by identifying $C_1$ with the boundary of $K'$ and $C_2$ with the boundary of $L'$. Finally complete $\mu'$ to an embedding $v$: $\Gamma \rightarrow T$ by choosing a simple path $\alpha$ in $(K - K') \cup A \cup (L - L')$ from $x_2$ to $y_2$ and mapping $e_2$ onto $\alpha$. Thus in any $\Gamma$ with a 2-separation $H_1$ may be taken as nonplanar and $H_2$ planar. $H_2$ is tightly constrained however.

**Remark:** With $\Gamma$ as above; $\Gamma / H_1 = H_2 \cup \{ [y_1, y_2] \}$ is nonplanar.

If $\Gamma$ is 2-separated as above with $H_1$ nonplanar and $H_2$ planar then suppose for contradiction that $\Gamma / H_1$ is planar. Let $e$ be any edge in $H_2$, then by the minimality of $\Gamma$ there is an embedding $\lambda: (\Gamma - e) \rightarrow T$. Since $H_1$ is nonplanar, it contains $K_{3,3}$. Since any toroidal embedding of $K_{3,3}$ is a 2-cell embedding, so must be the embedding of its supergraph $H_1$. Thus there is a face $D$ of $\lambda[H_1]$ that contains $\lambda( (H_2 - e) \cup \{e_1, e_2\})$. Take $D'$ as a disk interior to $D$ that contains $\lambda(H_2 - e)$. Then $D$ may be chosen so that $\text{Bdry}(D)$ meets $\lambda(e_1)$ at a single point, $z_1$ and $\lambda(e_2)$ at a single point $z_2$.

Now let $\mu$ be a planar embedding of $\Gamma / H_1$ and take $E$ as a disk in the plane that contains $\mu[H_1]$ and whose boundary meets $\mu(e_i)$ at single points $\mu(\lambda^{-1}(z_i))$. 
We may construct an embedding $v$ of $\Gamma$ into $T$ as:

$$v = \lambda[H_1 \cup \{ [x_i, \lambda^{-1}(z_i)] \} \cup \mu[H_2 \cup \{ [\lambda^{-1}(z_i), y_i] \}]$$

by excising $D'$ from $T$ and identifying $\text{Bdry}(D)$ with $\text{Bdry}(E)$ (while identifying $z_i$ and $\lambda(\mu^{-1}(z_i))$).

Summarizing our observations, if $\Gamma$ has a 2-separation then:

i) $H_1$ must be nonplanar, hence contain a homeomorph of $K_{3,3}$

ii) $H_2$ must be planar

iii) $\Gamma/H_1 = H_2 \cup \{y_1, y_2\}$ must be nonplanar.

The third condition requires that $H_2$ have a subgraph as in the figure below:
Up to symmetry, there are three ways of attaching the $K_4$ above to $K_{3,3}$ with a pair of disjoint arcs. A: Both vertices on one edge. B: Vertices on adjacent edges. C: Vertices on nonadjacent edges. (The resulting graphs are II B 2.6, II B 2.5, II B 2.4 of Decker's catalogue.)

Suppose that one of the above did have embedding $v$ into $T$. Then $v(K_4 \cup e_1 \cup e_2)$ would lie interior to a disk $F$ of $T - v(K_{3,3})$. Taking an arc of $\partial F$ joining $v(x_1)$ to $v(x_2)$ produces an embedding of $K_{3,3}$, a contradiction.
3-separated Graphs: Let \( \{ e_i = [x_i, y_i] ; 1 = 1, 2, 3 \} \) be a 3-bond of a 3-connected \( \Gamma \) and write \( \Gamma = H_1 \cup H_2 \cup \{ e_1, e_2, e_3 \} \) for disjoint \( H_1 \) and \( H_2 \) with \( H_1 \) containing vertices \( x_1, x_2, x_3 \) and \( H_2 \) containing \( y_1, y_2, \) and \( y_3 \). Suppose further that \( \{ e_1, e_2, e_3 \} \) is not a vertex bond.

![Figure III.05](image)

Since the \( H_i \)'s are each more than a vertex neighborhood all of the \( x_i \)'s and \( y_i \)'s are distinct and each \( H_i \) contains a cycle. In as much as \( \Gamma \) is 3-connected the \( H_i \)'s cannot be 1-, or 0-separated. As before at most one of the \( H_i \)'s say \( H_1 \), is nonplanar and hence contains a homeomorph \( K \) of \( K_{3,3} \).

Suppose \( H_1 \) is nonplanar and that \( \Gamma/H_1 \) is planar. Let \( e \) be an edge of \( H_2 \). Take \( \lambda:( \Gamma - e ) \rightarrow T \) to be an embedding. Since \( H_1 \) contains a homeomorph of \( K_{3,3} \), and \( \Gamma \) is 3-connected the faces of \( \lambda \) are 2-cells. One of them, \( D \), contains \( \lambda((H_2 - e) \cup e_1 \cup e_2 \cup e_3) \). Interior to \( D \) choose a disk \( D' \) that contains \( H_2 \) on its interior and meets each \( \lambda(e_i) \) at a single point \( z_i \). Then \( \Gamma/H_1 \) is the graph formed
from \( H_2 \) by adjoining a vertex \( X \) and three edges \([ X, y_i ] \) \( i = 1, 2, 3 \). Let \( \mu \) be an embedding of \( \Gamma/H_1 \) into the sphere. Take \( E \) to be a disk on the sphere that contains \( X \), meets \( \mu(e_i) \) at \( \mu(\lambda^{-1}(z_i)) \) for each \( i = 1, 2, 3 \) and is disjoint from \( \mu(H_2) \).

Extend \( \lambda \) to an embedding
\[
v = \lambda(H_1 \cup [x_1, \lambda^{-1}(z_1)] \cup [x_2, \lambda^{-1}(z_2)] \cup [x_3, \lambda^{-1}(z_3)]) \cup \mu(H_2 \cup [y_1, \lambda^{-1}(z_1)] \cup [y_2, \lambda^{-1}(z_2)] \cup [y_3, \lambda^{-1}(z_3)])
\]
by identifying the boundaries of \( D' \) and \( E \) while mapping \( \mu(\lambda^{-1}(z_i)) \) to \( z_i \). This last is always possible in one of the two orientations of \( \text{Bdry}(E) \).

If \( H_1 \) is nonplanar and \( \Gamma/H_1 \) is nonplanar, then \( H_2 \) contains one of the two subgraphs shown below.

![Figure III.06](image)

If \( H_2 \) contains the first configuration with a nonplanar \( H_1 \) then \( \Gamma \) contains one of the three 2-separated graphs listed above. The subgraph on the right is the motivating
Definition: Suppose that $G$ is a cubic graph and suppose that a subgraph $L$ of $G$ is a homeomorph of $K_{2,3}$ with arcs $\alpha$, $\beta$, and $\gamma$. Then $L$ is said to be a $k$-graph in $G$ provided there is a bridge of $H$ with vertices of attachment on each of $\alpha$, $\beta$, and $\gamma$.

Constructing the minimal 3-connected graphs $\Gamma$ containing a 3-bond between a $k$-graph and a $K_{3,3}$ amounts to determining the distinct ways in which one may choose either a pair of edges -- two ways, the edges are adjacent or not -- or a triple of edges inside $K_{3,3}$ to serve as "sites of attachment" for the $e_i$'s. Below are displayed the four ways of choosing a of triple edges in $K_{3,3}$.

![Figure III.07](image)

The six graphs we have described are known in Decker's catalogue as:

- II C 3.14 Two adjacent edges
- II C 3.12 Two nonadjacent edges
- II C 3.17 3-claw

That each of these three fails to embed follows as before. $L \cup e_1 \cup e_2 \cup e_3$ being
complementary to K, a homeomorph of \( K_{3,3} \) would be embedded in a face of \( v|\kappa \). Any such embedding together with an arc of the bounding circuit would supply a planar embedding of \( K_{3,3} \).

Representations of the ten Decker Graphs described above are given in Appendix A.
§ III.2 3-separations without $K_{3,3}$

Decker's catalogue of edge-critical graphs provides a powerful restriction on the remaining $\Gamma$'s when coupled with a theorem of Glover and Huneke.

**Theorem:** Glover and Huneke [5].

Let $K$ be a graph without valency 2 vertices; let $e$ be an edge of $K$; let $L_1$ be a 2-connected subgraph of $K$ not containing $e$; let $L_2$ be a connected component of $K - st(L_1)$ which does not intersect $e$ and let $M$ be a compact 2-manifold. If $K$ does not embed in $M$ but $\varphi: (K - e) \rightarrow M$ is an embedding such that $\varphi(L_1)$ is contractible in $M$, then

(i) $K/L_2$ is nonplanar, and

(ii) there is a $k$-graph of $K$ disjoint from $L_2$.

Suppose $\Gamma$ is an edge-critically nontoroidal graph, and suppose $\Gamma$ contains a subgraph $\Lambda_2$ that is a homeomorph of $K_{3,3}$. Suppose further $\Gamma - \Lambda_2$ contains a subgraph $\Theta$ homeomorphic to $K_{2,3}$; write $\Theta$ as the union of a polygon $\Lambda_1$ and an arc $\alpha$ and select an edge $e$ in $\alpha$. Since $\Gamma$ is assumed edge-critically nontoroidal there is an embedding $\varphi: (\Gamma - e) \rightarrow T$. Since $T - \varphi(\Lambda_2)$ is a connection of 2-cells $\varphi(\Lambda_1)$ is contractible. Thus taking $M = T$; $K = \Gamma$; $L_1 = \Lambda_1$; $L_2 = \Lambda_2$; and $e = e$ in the above theorem we are guaranteed the existence of a $k$-graph of $\Gamma$ disjoint from $\Lambda_1$. As an immediate consequence we have following:

**Lemma:** If $\Gamma$ is a cubic edge-critically nontoroidal graph and $\Gamma$ contains the disjoint union of $K_{3,3}$ and $K_{2,3}$ then $\Gamma$ is one of the ten 0-, 2-, or 3-separated graphs described in Decker's dissertation.
We use this to complete the census of 3-separated $\Gamma$'s. Suppose $\Gamma$ is not one of Decker's ten graphs and $H_1$ is nonplanar, then from the above remark we have that $H_2$ is a cactus, i.e. each block of $H_2$ is a polygon or an edge. Such graphs are not only planar but have the property that for every planar embedding each vertex of the cactus is on the infinite face. Clearly then $\Gamma/H_1$ is planar and by the results of § III.1, $\Gamma$ embeds in $T$. This suffices to show that if $H_1$ is nonplanar then $\Gamma$ is one of the six 3-connected examples of Decker.

**Condition:** In the sequel we may assume that both of the $H_j$'s are planar.

**Claim:** Under the conditions above, each of $\Gamma/H_j$ is nonplanar.

**Proof of claim:** $\Gamma/H_1$ is the graph formed from $H_2$ by adjoining a vertex $X$ and three edges $[X, y_i]$ $i = 1, 2, 3$. Suppose now that $\Gamma/H_1$ is planar, and take $\lambda$ as an embedding of $\Gamma/H_1$ into the sphere. Note that $\Gamma/H_2$ is the graph formed from $H_1$ by adjoining a vertex $Y$ and three edges $[Y, x_i]$ $i = 1, 2, 3$. Further $\Gamma/H_2$ is isomorphic to a proper subgraph of $H_1$. By the proceeding remark and the assumption of edge-criticality for $\Gamma$, there is an embedding $\mu: \Gamma/H_2 \to T$. In the sphere, choose a disk $D$ that contains $\lambda(H_2)$ but not $\lambda(X)$, whose boundary meets each $\lambda(e_i)$ at a single point $z_i$. In $T$ choose a disk $E$ containing $Y$ on its interior and meeting each $\mu(\lambda^{-1}(e_i))$ only at $\mu(\lambda^{-1}(z_i))$. Extend $\mu$ to an embedding $\nu = \mu|_{H_1 \cup \{x_1, \mu^{-1}(z_1)\} \cup \{x_2, \mu^{-1}(z_2)\} \cup \{x_3, \mu^{-1}(z_3)\}} \cup \lambda|_{H_2 \cup \{y_1, \mu^{-1}(z_1)\} \cup \{y_2, \mu^{-1}(z_2)\} \cup \{y_3, \mu^{-1}(z_3)\}}$

by identifying the boundaries of $D$ and $E$ while mapping $\lambda(\mu^{-1}(z_i))$ to $z_i$. This is
always possible in one of the two orientations of $\text{Bdry}(D)$.

**Condition:** In the remainder of this chapter we may take $\Gamma/H_1$ to be nonplanar.

This, incidentally requires that both of the $H_j$ contain a $k$-graph of $\Gamma$. Such graphs are known not to embed into the projective plane $\Pi$ and to contain one of five connected graphs as topological subgraphs.

The following construction supplies a further restriction on the remaining $\Gamma$'s.

With $\Gamma$ as described above suppose that $\lambda$ and $\mu$ are embeddings of the graphs $H_1 \cup [x_1, x_2]$ and $H_2 \cup [y_1, y_2]$ (the graphs formed from $H_1$ and $H_2$ by adjoining a single edge with the indicated endpoints) respectively, into the sphere. Choose $D$ to be a disk in the sphere that meets $\lambda(H_1 \cup [x_1, x_2])$ in a simple arc of $\lambda([x_1, x_2])$. Let $u_1$ and $u_2$ be the two points of intersection of $\lambda([x_1, x_2])$ and $\text{Bdry}(D)$. (Choose $u_1$ to lie between $\lambda(x_1)$ and $u_2$ along $\lambda([x_1, x_2])$).

Symmetrically take $v_1$ and $v_2$ as points of intersection of a disk $E$ meeting $\mu(H_2 \cup [y_1, y_2])$ in a simple arc interior to $\mu([y_1, y_2])$. The embedding:

$$v = \lambda([H_1 \cup [x_1, \lambda^{-1}(u_1)] \cup [x_2, \lambda^{-1}(u_2)]) \cup 
\mu([H_2 \cup [y_1, \mu^{-1}(v_1)] \cup [y_2, \mu^{-1}(v_2)])$$

formed by identifying $\text{Bdry}(D)$ and $\text{Bdry}(E)$ with $u_i$ mapped to $v_i$, $i = 1, 2$ is an embedding of $\Gamma - e_3$ into the sphere. Now, as in § III.1, $v$ may be extended to an embedding $v'$ of $\Gamma$ into the torus by excising disks $K'$ and $L'$ interior to faces $K$ and $L$ of $v$ that contain $x_3$ and $y_3$ on their respective boundaries, identifying the
boundary cycles of an annulus $A$ with $\text{Bdry}(D')$ and $\text{Bdry}(E')$ and choosing a simple arc $\alpha$ in $(D - D') \cup A \cup (E - E')$ from $v(x_3)$ to $v(y_3)$ to be the image of $e_3$.

**Condition:** If $\Gamma = H_1 \cup H_2 \cup e_1 \cup e_2 \cup e_3$ is as above and $H_1 \cup e_i$ is planar then $H_2 \cup e_i$ is nonplanar.

Suppose $H_1$ has the property that $H_1 \cup [x_1, x_2]$ and $H_1 \cup [x_1, x_3]$ are nonplanar, while $H_1 \cup [x_2, x_3]$ is planar. The above condition requires that $H_2 \cup [x_2, x_3]$ also be nonplanar, else the construction described above produces a toroidal embedding of $\Gamma$.

Suppose $H_2 \cup [y_1, y_2]$ and $H_2 \cup [y_1, y_3]$ are both planar graphs with $H_2 \cup [y_2, y_3]$ nonplanar. Suppose further that $H_1 \cup [x_1, x_2]$ and $H_1 \cup [x_1, x_3]$ are nonplanar while $H_1 \cup [x_2, x_3]$ is planar.

Since $H_2 \cup [y_2, y_3]$ is nonplanar there is a subgraph of $H_2$, $K$, that is homeomorphic to $K_4$ in which $y_2$ and $y_3$ are on noncoincident edges. Take the nodes of $K$ to be $\{a, b, c, d\}$ and let $y_1$ and $y_2$ be on edges $[a, d]$ and $[b, c]$ respectively. Since $K_4$ is 3-connected there is a unique planar embedding of $K_4$. The faces of any planar embedding $\lambda$ of $H_2 \cup [y_1, y_j]$ are a refinement of the four faces of $\lambda|_K$. By hypothesis both of $H_2 \cup [y_1, y_2]$ and $H_2 \cup [y_1, y_3]$ are planar so that there is some planar $\lambda$ embedding of $H_2$ in which the image of $y_3$ is -- WOLOG -- in the region bounded by $[a, b, y_2, c]$; likewise, then for some $\lambda'$ the image of $y_3$ is in -- WOLOG -- the region bounded by $[a, b, d, y_3]$. Either $y_3$ is positioned on $[b, d]$ as below or else $H_2$ has a 2-separation and by rechoosing $K$
we obtain the figure below.

![Figure III.08](image)

In either case any graph satisfying the conditions imposed on \( H_2 \) above contains a homeomorph of \( K_4 \) with the indicated divalent vertices.

Note that since \( H_1 \cup [x_1, x_2] \) is nonplanar; as in the case of \( H_2 \) above there is a subgraph \( L \) of \( H_1 \) that is homeomorphic to \( K_4 \) with \( x_1 \) and \( x_2 \) as indicated. Further letting \( \mu \) be a planar embedding of \( H_1 \); \( \mu(L) \) divides the plane into four regions. If \( x_3 \) is a vertex of \( L \) then it may not lie on the edge arc \([b, c]\) as then \([x_2, x_3]\) completes a homeomorph of \( K_{3,3} \) with \( H_2 \cup e_2 \cup e_3 \) disjoint from a k-graph on \{a, b, c, d, x_1\}. Since \( H_1 \cup [x_2, x_3] \) is planar \( x_3 \) cannot lie on arc \([a, d]\). The other arc -- up to symmetry -- of \( L \) is \([b, d]\). Suppose that the arc \([b, d]\) cannot be rechosen so as to avoid \( x_3 \). Since \( H_1 \cup [x_1, x_3] \) is nonplanar \( \mu(x_1) \) and \( \mu(x_3) \) are in different faces of every planar embedding. If \( x_3 \) were on \([a, d]\) then the
boundary cycle on the face subdividing the region bounded by $\mu([a, b, d, x_1])$ that contains $x_1$ together with the cycle $[a, d, c]$ supplies a $k$-graph disjoint from the homeomorph of $K_{3,3}$ on $H_1 \cup e_2 \cup e_3 \cup [x_2, b, x_3]$.

Thus there is a choice of $L$ for which $x_3$ is not a vertex of $L$. Let $T$ be a planar embedding of $H_1 \cup [x_2, x_3]$. WLOG the faces that contain $\mu'([x_2, x_3])$ may be taken to be interior to the region bounded by the cycle $\mu([b, c, d])$. Let $E$ be the face of $\mu'$ that meets the arc $[b, x_2]$. If $\text{Bdry}(E) - \mu'( [x_2, x_3])$ is disjoint from $\mu'([b, d, c])$ then there is a path in $H_1$ from $x_2$ to $x_3$ disjoint from the $k$-graph on $\{a, b, c, d, x_1\}$. Symmetrically $\text{Bdry}(F) - \mu'( [x_2, x_3])$ meets $\mu'([b, d, c])$. The two subgraphs that result are illustrated below.

The graph on the left satisfies the conditions we have imposed on $H_1$, it gives rise to the irreducible graph $G_{ena}$ as shown below. The graph on the left admits of a planar
embedding in which \( x_3 \) is accessible to \( x_1 \).

![Diagram](image)

*Figure III.10*

If \( H_j \) contains the right hand graph either:

i) there is a path from \( x_2 \) to \( x_3 \) in \( H_j \) disjoint from the boundary of the face of some embedding \( \mu \) that contains \( x_1 \)

--- in which case \( \Gamma \) contains a k-graph disjoint from a \( K_{3,3} \) or,

ii) there is a 2-bond of \( H_j \) consisting of one edge from each arc \([b, e]\) and \([d, f]\)

--- in which case \( \mu( [e, x_3, f]) \) is on the boundary of the face of \( \mu[H_j] \) that contains \( x_2 \) there is an embedding of \( H_j \) in which \( x_1 \) and \( x_3 \) are on the boundary of a common face

--- in which case \( H_j \cup [x_1, x_3] \) is planar.

The graph of the figure below, Gena is our first example of a "fragmented k-graph". Observe that the induced subgraph on \{ 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16 \} is a
homeomorph of $K_{3,3}$ disjoint from the cycle $[1, 2, 3, 4, 5, 1]$. Thus, in any toroidal drawing of Genoa $[1, 2, 3, 4, 5, 1]$ is an inessential cycle. Likewise the subgraph induced on $\{1, 5, 6, 7, 10, 11, 12, 13, 14, 15, 16\}$ is a homeomorph of $K_{3,3}$ disjoint from the cycle $[2, 3, 4, 8, 9, 2]$. Again any toroidal drawing of Genoa must contain $[2, 3, 4, 8, 9, 2]$ as an inessential cycle. These two observations require that the theta-graph formed by the union of these two cycles contain no essential cycles in any toroidal drawing of Genoa. However, the subgraph induced on $\{1, 2, 3, 4, 5, 8, 9\}$ is a k-graph in Genoa and thus must contain at least one essential cycle in any toroidal drawing of its supergraph Genoa. Thus Genoa does not embed in $T$.

![Figure III.11 Genoa](image-url)
§ III.3 $\lambda$-graphs

Before proceeding we introduce a definition.

**Definition.** A graph $S$ is called a $\lambda$-graph provided:

1. $S$ is homeomorphic to a nonseparable, planar cubic graph and contains exactly three divalent vertices $x_1$, $x_2$, and $x_3$.
2. Each of the graphs $S \cup [x_i, x_j]$, $i \neq j$ is nonplanar.
3. No connected subgraph of $S$ containing all three of $x_1$, $x_2$, and $x_3$ is disjoint from a homeomorph of $K_{2,3}$.
4. $S$ is minimal with respect to properties 1 and 2 above.

Continuing the notation of section III.2 we suppose that for each $i \neq j$, $H_1 \cup [x_i, x_j]$ is nonplanar. The conditions that $\Gamma$ not contain one of the six previously listed nontoroidal graphs and $H_1$ satisfy the above hypotheses require that $H_1$ be a $\lambda$-graph. Below we catalogue the seven distinct $\lambda$-graphs. The graphs formed by joining an $\lambda$-graph to a k-graph with a 3-bond will all be seen to be edge-critically nontoroidal. Suppose that $S$ is an $\lambda$-graph, since $S \cup [x_1, x_2]$ is nonplanar $S$ contains a subgraph homeomorph to $K_4$ as below:

![Figure III.12](attachment:image.png)
Symmetrically $S$ contains a homeomorph of $K_4$ and contains both of $x_2$ and $x_3$. Since $S$ is homeomorphic to a nonseparable cubic graph and $x_2$ and $x_3$ are divalent there is a cycle in $S$ containing $x_2$ (or $x_1$) and $x_3$. Such a cycle must meet $K$ in at least two vertices. By taking the first vertex in each direction from $x_3$ along this cycle one produces a simple arc $A$ that contains $x_3$ on its interior and meets $K$ exactly at its endpoints, $\alpha$ and $\beta$. Below we display -- up to symmetry -- the two points of attachment, $\alpha_1$ and $\alpha_2$, for $\alpha$.

![Diagram](image)

Figure III.13

With the choice of $\alpha_1$ as the site of attachment we list -- up to symmetry, including interchange of $x_1$ and $x_2$ -- the possible attachments $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6$ for $\beta$. 
Except for the attachment $\beta_6$ there is a cycle in $K \cup A$ that contains all three of $x_1$, $x_2$, and $x_3$. Continuing, we display the three attachment sites for $\beta$ if $A$ attaches at $\alpha = \alpha_2$.

For each of attachments sites $\beta_1$ and $\beta_2$ there is cycle in $K \cup A$ that contains all of $x_1$, $x_2$, and $x_3$. An arc from $\alpha_2$ to $\beta_3$ completes a $K_{3,3}$. Since $S$ is planar, $S$
cannot contain a subgraph homeomorphic to $K_{3,3}$.

Consider the subgraph formed when the arc $A$ attaches to $\alpha_1$ and $\beta_6$. Either there is a cycle in $S$ that contains the three $x_i$'s, or all of the subgraphs of $S$, homeomorphic to $K_4$ and containing $x_3$ and $x_1$ on nonadjacent edges of the $K_4$, meet each subgraph of $S$ homeomorphic to $K_4$ and containing $x_2$ and $x_1$ on nonadjacent edges of the $K_4$ only on an arc of each that lies wholly on the edge of each $K_4$ that contains $x_1$. By hypothesis there are two such homeomorphs of $K_4$. So if $S$ contains no cycle meeting is three $x_i$'s $S$ is the graph $\Sigma$ below:

![Figure III.16 The graph $\Sigma$](image)

Remark: If a graph $S$ satisfies conditions 1. and 2. of the definition of $\lambda$-graph then either:

i. There is a cycle in $S$ that contains all three of $x_1, x_2,$ and $x_3$, or

ii. $S$ contains the graph $\Sigma$ shown above.
Taking the graph $\Sigma$ as $H_1$ gives rise to no new $\Gamma$'s as attaching a vertex 14 to vertices 3, 6, 9 in the figure above produces the Decker graph II C 3.17. (Find a $K_{3,3}$ in the left half of the figure on vertices \{1, 2, 3, 4, 5, 6\} and the k-graph on \{7, 8, 9, 10, 11\} with 3-bond formed by [7, 13], [9, 14] and [11, 12]).

It is clear that in a planar embedding of a $\lambda$-graph the vertices $x_1, x_2, x_3$ must all lie in separate regions. Thus any disk bounded by a cycle through all of the $x_i$'s must be subdivided into three regions in one of the two ways:

![Diagram](attachment:image.png)

Figure III.17

The remaining $\lambda$-graphs can be constructed from their planar embeddings by identifying a cycle $C$ through the $x_i$'s and a subdivisions of the interior and exterior regions formed by the image of $C$ under a planar embedding of $S$. Below we catalogue such graphs.

First consider the situation in which there is at least one Y-bridge -- shown below as
drawn in the interior region -- such a bridge divides the circuit into three symmetric arcs. Specifically assume there is a vertex \( v \) (= vertex 10) in a region formed by a cycle through \( x_1, x_2, \) and \( x_3 \) such that each \( x_i \) is in the boundary of a region \( R_i \) containing \( v \) on its boundary.

As depicted above we take \( C \) to be the cycle in the boundary of \( R_1 \cup R_2 \cup R_3 \) that contains \( v \). A second Y-bridge may have its vertices of attachment in the three distinct arcs determined by the first Y or not.

Since a pair of ears have four ends one of the three arcs determined by the Y-bridge must contain the ends of two ears. They may lie on the same half arc determined by the divalent vertex or not.
Ears that start from the same half-arc may both be short or one may be long. The two diagrams above are labeled so as to display an isomorphism.

If the ears cannot be chosen to originate from the same half arcs then by choice of C ears can be chosen as in Figure III.20 above to complete the λ-graph. If all the
bridges of C are ears -- i.e. if no vertex v of S as described above lies interior to the image of C in any planar embedding of S -- we catalogue the graphs S by the length of chains of overlapping bridges.

**Pairs of overlapping bridges**

Figure III.21

Graphs S₅ and S₆

In the first each pair contains one of the x₁'s in its induced theta. In the second one pair contains two of the x₁'s in the induced theta.

**All four bridges may overlap to form either a 4-chain or a 4-cycle:**
In both cases deleting the neighborhoods of A and B leaves a cycle so that these may be analyzed as two Y-bridges — already catalogued above.

This gives us a list of at six distinct λ-graphs, $S_1$ through $S_6$. The restriction on producing a theta disjoint from a $K_{3,3}$ outlaws chains on bridges in the λ-graph. (The reader is asked to verify that each of the graph above is minimal with respect to conditions 1. and 2. of the definition.) There is an additional property of the six λ-graphs that is needed to show that the graphs $S_i^{**}$ described below are edge-critically nontoroidal.

Lemma: If S is one of the λ-graphs above and $S^*$ is the graph formed from S by adjoining a twelfth vertex adjacent to each of the divalent vertices of S then $S^*$ embeds in the Torus T.

Proof:
Figure III.23

Figure III.24
Lemma: If $\mu$ is any toroidal embedding of $S^*$ for a $\lambda$-graph $S$ then $\mu|_{S^*}$ is an open 2-cell embedding.

Proof: Suppose not. Since $S$ contains a $k$-graph, $\mu[S]$ contains an essential cycle. Thus $T - \mu[S]$ is contained in a disk or an annulus. If an annulus, then there is an annulus that contains $\mu[S]$ and the two cycles that form the boundary of this sector are accessible to all three of $x_i$. Hence two say, $x_1$ and $x_2$ of them must be accessible to the same cycle. This, however supplies a planar embedding of $S \cup \{x_1, x_2\}$ contradicting the definition of $\lambda$-graph.

Definition: Take $\Theta$ to be the graph depicted below:
For each $S_j$ let $S_j^{**}$ be the cubic graph $S_j \cup \Theta \cup [c, x_1] \cup [d, x_2] \cup [e, x_3]$.

**Lemma:** Each $S_j^{**}$ is nontoroidal.

**Proof:** Since $\Theta$ is a $k$-graph of $S_j^{**}$ for any embedding $\mu: S_j \rightarrow T$, $\mu(\Theta)$ contains an essential cycle. But $(S_j^{**} - a)$ is a homeomorph of $S_j^*$, and so by the above lemma the regions of $T - \mu(S_j)$ are 2-cells. Thus $\mu(\Theta)$ is contractible.

That each $S_j^{**}$ is edge critical is an easy check. This completes the census of $\Gamma$'s that contain a nontrivial 3-separation.
Chapter IV

Structure of Projective Planar Graphs

Throughout this chapter $\Gamma$ will represent a simple graph satisfying:

1. $\Gamma$ is nonplanar.
2. $\Gamma$ contains no triangles.
3. $\Gamma$ has an embedding $v$ into the projective plane, $\Pi$.
4. $\Gamma$ is has no 2-separations.
5. $\Gamma$ has no nontrivial 3-separations.

In this chapter and in Chapter V the embedding $v: \Gamma \rightarrow \Pi$ will be fixed. As a result we shall abuse our notation and make no distinction between the graph $\Gamma$ and its image under $v$ in the projective plane.

§ IV.1 The Structure Theorem

**Structure Theorem:** If $\Gamma$ and $v$ are as above, then $\Gamma$ may be written as the edge-disjoint union of $M$, a homeomorph of a Mobius $k$-ladder, for $k \geq 3$, and a graph $P$. Further $M \cap P$ is contained on the spine $S$ of $M$, with $S \cup P$ planar, further if $k = 3$, then $\Gamma$ embeds in the torus.

The proof comes as a series of Lemmas.
Consider $C$, the collection of closed disks $\delta$ in $\Pi$ which may be written as the union of closed regions of $v$. That $C$ is nonempty follows since $\Gamma$ contains a homeomorph of $K_{3,3}$. Clearly $C$ is finite and partially ordered under set inclusion.

**Lemma:** If $\Delta$ is a maximal element of $C$ then $\Delta$ meets each vertex of $\Gamma$.

**Proof:** Suppose not. Let $x$ and $y$ be vertices of $\Gamma - \Delta$ and $\Delta$, respectively; by Menger's Theorem (See Wilson [14]) there are three internally vertex disjoint paths from $x$ to $y$ in $\Gamma$. Since $\Delta$ is contractible, $\Pi/\Delta$ is homeomorphic to $\Pi$ and may be seen to contain a drawing of $\Gamma/\left(\Gamma \cap \Delta\right) = \Gamma^*$. In $\Gamma^*$ vertices $x$ and $\Delta$ are joined by three internally vertex disjoint paths derived from those in $\Gamma$ joining $x$ and $y$. Thus $\Gamma^*$ contains a subgraph $\Theta$ homeomorphic to $K_{2,3}$ with nodes $x$ and $\Delta$.

From Chapter II there is a cycle of $\Theta$ contractible in $\Pi$. Let this cycle be composed of arcs $\alpha$ and $\beta$ from $x$ to $\Delta$ and $\Delta$ to $x$, respectively. Since $\alpha \beta$ is homotopically null there is a $O$-arc $\omega$ in $\Pi/\Delta$ (hence $\omega'$ in $\Pi$) essential in $\Pi$ and disjoint from $\alpha \beta$ (from $\alpha \cup \beta \cup \Delta$). The edges of $\alpha$ and $\beta$ and all of the vertices save $\Delta$ are edges and vertices of $\Gamma$, so that $\alpha$ and $\beta$ give rise to paths $\alpha'$ and $\beta'$ in $\Gamma$. Since $\Gamma$ is cubic the paths $\alpha'$ and $\beta'$ terminate at distinct vertices of $\Gamma$, $a$ and $b$ respectively, on $\text{Bdry}(\Delta)$. Vertices $a$ and $b$ separate $\text{Bdry}(\Delta)$ into two arcs. Take $\gamma$ to be one of these two arcs oriented from $a$ to $b$. The path $\alpha \gamma \beta$ is a cycle in $\Gamma$ disjoint from $\omega$.

Any two essential cycles in $\Pi$ must intersect; so $\alpha \gamma \beta$ is a contractible cycle bounding a region $R$ in $\Pi$. $R$, being bounded by a path in $\Gamma$, may be written as a union of faces of $v$. Since $\text{cl}(R)$ meets $\Delta$ along $\gamma$, $\text{cl}(R) \cup \Delta = \Delta'$ is a disk in $\Pi$. Since $\Delta'$ is disjoint from $\omega$ and any pair of essential cycles in $\Pi$ meet $\Delta'$ is in $C$, contradicting the maximality of $\Delta$. 
**Lemma:** Let $\Delta$ be a maximal element of $C$ and set $S = \text{Bdry}(\Delta)$ and let $e_1, e_2, \ldots, e_k$ be the edges of $\Gamma$ not contained in $\Delta$. For each $e_i$ let the ends of $e_i$ divide $S$ into the arcs $S_i'$ and $S_i''$; then for each $e_i$, $S_i' \cup v(e_i)$ and $S_i'' \cup v(e_i)$ are essential cycles of $\Pi$.

**Proof:** It suffices to show that $S \cup e_i$ contains an essential cycle. $S \cup e_i$ is a theta-graph and by hypothesis $S$ is contractible. If $S \cup e_i$ is a theta-graph of type $B$, the conclusion follows immediately. Suppose $S \cup e_i$ is contractible for some $e_i$, then $S \cup e_i$ is a theta-graph of type $A$ in $\Pi$. The edge $e_i$ is not in the disk bounded by $S$, therefore one of $e_i \cup S_i'$ or $e_i \cup S_i''$ bounds a contractible region $R$ properly containing $S$ and hence $\Delta$. The existence of such $R$ contradicts the maximality of $\Delta$.

**Remark:** Since by hypothesis $\Gamma$ is nonplanar, $k \neq 0$.

**Lemma:** Let $\Delta$ be a maximal element of $C$ and set $S = \text{Bdry}(\Delta)$, then $S \cup \{ e_i : 1 \leq i \leq k \}$ is homeomorphic to:

1) A $\Theta$-graph, if $k = 1$
2) $K_4$, if $k = 2$
3) $K_{3,3}$, if $k = 3$
4) $M_k$, for $k \geq 4$.

**Proof:** 1) is obvious.

Let the ends the $e_i$'s be labeled cyclically around the circuit $S$ as $x_1, x_2, \ldots, x_{2k}$. To establish 2), 3) and 4) we must show that $x_i$ is joined to $x_{i+k}$. Write $e_i = [x_i, x_{i+k}]$ for some $r \leq k + 1$. So that $S_i'$ (continuing the notation of the preceding Lemma) contains the vertices $x_2, x_3, \ldots, x_{r-1}$ and $S_i''$ contains the vertices $x_{r+1}, x_{r+2}, \ldots, x_{2k}$. No edge $e_i$ may join vertices $x_p$ and $x_q$ on $S_i'$ since then $v(e_i) \cup S_i''$ and
v( e_1 ) \cup [ x_p , x_q ] are disjoint essential cycles of \( \Pi \). Similarly no \( e_i \) has both ends on \( S_1'' \). This gives a matching between vertices \( x_2, ..., x_r \) on \( S_1' \) and \( x_{r+1}, ..., x_{2k} \) on \( S_1'' \) hence \( r = k + 1 \). Thus \( e_1 = [ x_1, x_{k+1} ] \). Symmetrically for each \( j \leq k \), \( e_j = [ x_j, x_{k+j} ] \) under a suitable labeling of the edges. This completes the proof of the Lemma.

**Notation:** Let \( \Delta \) be a maximal element of \( C \), and let \( S = \text{Bdry}( \Delta ) \). Set \( \Gamma - \Delta = \{ e_1, ..., e_k \} \). Take \( P = \text{int}( \Delta ) \cap \Gamma \) and \( M = S \cup \{ e_1, ..., e_k \} = \Gamma - P \).

**Lemma:** Writing \( \Gamma = M \cup P \) as above; then \( k \geq 3 \).
If \( k = 0 \) then \( \Gamma = \Gamma \cup \Delta \) is planar. If \( k = 1 \), a planar embedding is constructed by embedding \( \Delta \) in the plane with \( x_1 \) and \( x_2 \) on the infinite face and extending this with a simple arc across the infinite region joining \( x_1 \) and \( x_2 \) face to \( P \cup e_1 = \Gamma \).
If \( k = 2 \) or \( k = 3 \) the figure below shows toroidal embeddings of \( M \). They may be extended to embeddings of \( \Gamma \) by pasting \( \Delta \) into the "circular" disk shown.

![Figure IV.01](image-url)
This completes the proof of the Structure Theorem.

*Note:* In the sequel we shall refer to the nodes of $M$ as the vertices $1, 2, \ldots, 2k$; a notation consistent with the labeling of the vertices in the definition of $M_k$. The reader needs to note that the representation of $\Gamma$ as the union of $M$ and $P$ is not unique. Below are two representations of a graph on 20 vertices with different planar parts. Note that in one representation $P$ is a tree, in the other $P$ contains a 5-cycle.

![Figure IV.02](image-url)
When in the following we have to refer to faces of an embedding \( v \), the nomenclature we employ will frequently be relative to a particular decomposition \( \Gamma = M \cup P \). As such the contractible region bounded by \( S \) will be denoted by \( D \).

Each face of \( \Pi - \Delta \) -- referred to as an "infinite face" of \( M \) -- is bounded by edges \( e_i \), \( e_{i+1} \) and two arcs of \( S \) and will be denoted by \( F_i \).

We list some immediate consequences of the Structure Theorem:

**Corollary 0:** With \( \Gamma \) as above, the components of \( P \) are precisely bridges (mod \( S \)) in \( \Gamma \).

**Proof:** \( \Gamma \) is connected.
Corollary 1. With $\Gamma$ as above, if $B$ is a component of $P$ then the vertices of attachment of $B$ in $S$ do not lie between $i$ and $i+1$.

Proof: Such a component would supply a 2-separation of $\Gamma$.

Corollary 2. With $\Gamma$ as above, no component of $P$, $B$, has vertices of attachment restricted to the arc $[i, i + 2]$.

Proof: Letting $x$ be the vertex of attachment of $B$ nearest $i$, and $y$ the vertex of attachment of $B$ nearest $i + 2$; the set of edges joining $x$ and $y$ to $B$ together with the edge $[i, i + k + 1]$ supplies a nontrivial 3-separation of $\Gamma$.

Corollary 3. No pair of components of $P$ are skew (mod $S$).

Proof: $P \cup S$ is planar.
§ IV.2 Embeddings into $T$.

**Construction A:** Write $\Gamma = M \cup P$ and let $X$ and $Y$ be points on arcs $[i, i+1]$ and $[i+k, i+k+1]$ respectively. If there is a simple path $\tau$ in $\Delta$ between $X$ and $Y$ that intersects at most one edge $e = [u, v]$ of $P$ then there exists an embedding $\mu: \Gamma \rightarrow T$ with $\mu|_M$ a combed embedding of $M_k$.

![Figure IV.04](image)

$\Delta - \tau$ consists of a pair of 2-cells $E'$ and $F'$. Let $P_{\text{left}}$ and $P_{\text{right}}$ be the components of $P - e$ (or $P$) contained in $E'$ and $F'$ respectively. In $(\text{cl}(\Delta - \tau))$ take $E$ to be a 2-cell containing $P_{\text{left}}$ whose closure is disjoint from $\tau$; and, $F$ to be a 2-cell containing $P_{\text{right}}$ whose closure is disjoint from $\tau$. Take $\lambda: M_k \rightarrow T$ to be the combed embedding of $M_k$ into the torus. Extend $\lambda$ to $\lambda'$ an embedding of $\Gamma - e$ by excising disks $E''$ and $F''$ with disjoint closures that meet $\lambda(S)$ along...
\( \lambda( S \cap \text{cl}(E) ) \) and \( \lambda( S \cap \text{cl}(F) ) \) respectively, and by pasting in disks \( E \) and \( F \) identifying their boundaries with those of \( E'' \) and \( F' \). If necessary we complete the embedding to \( \lambda'' : \Gamma \rightarrow T \) by choosing a simple arc from \( \lambda'(u) \) in \( E \) across \( T - ( \lambda(M) \cup E'' \cup F' ) \) to \( \lambda'(v) \) in \( F \).

![Figure IV.05](image)

**Nomenclature:** Given \( \Gamma = P \cup M \) a path \( \tau \) as described in the statement of Construction A will be called a true diagonal (of \( \Delta \)).

**Corollary to Construction A.** \( P \) has no component \( B \) with vertices of attachment confined to \([i, i+1] \cup [i+k, i+k+1]\).

**Proof:** By Corollary 1, \( B \) cannot attach solely to one of the arcs. Suppose \( B \) is a component of \( P \) with vertices of attachment confined to but occurring on both of the arcs \([i, i+1]\) and \([i+k, i+k+1]\), then \( \Gamma \) embeds in \( T \) with an combed embedding. Such a component supplies a true diagonal \( \tau \) as indicated below.
Construction B: Let X and Y be points on arcs \([i, i + 1]\) and \([i + k - 1, i + k]\) respectively, if there is a path \(\tau\) in \(\Delta - P\) between X and Y, then there exists an embedding \(\mu: \Gamma \to T\) with \(\mu|_{M_k}\), and embedding, of \(M_k\) of type iii.

\(\Delta - \tau\) consists of a pair of 2-cells \(E'\) and \(F'\). Let \(P_{\text{left}}\) and \(P_{\text{right}}\) be the components of \(P\) contained in \(E'\) and \(F'\) respectively. In \((\text{cl}(\Delta) - \tau)\) take \(E\) to be a 2-cell containing \(P_{\text{left}}\) whose closure is disjoint from \(\tau\); and, \(F\) to be a 2-cell containing \(P_{\text{right}}\) whose closure is disjoint from \(\tau\). Take \(\lambda: M_k \to T\) to be the embedding of \(M_k\) into the torus of type iii. Extend \(\lambda\) to \(\lambda'\), an embedding of \(\Gamma\), by excising disks \(E''\) and \(F''\) with disjoint closures that meet \(\lambda(S)\) along \(\lambda(S \cap \text{cl}(E))\) and \(\lambda(S \cap \text{cl}(F))\) respectively, and by pasting in disks \(E\) and \(F\) identifying their boundaries with those of \(E''\) and \(F''\).
Nomenclature: Given $\Gamma = P \cup M$ a path $\tau$ as described in the statement of Construction B will be called a short diagonal (of $\Delta$).
Embedding Theorem: If $\Gamma$ has a 2-, or 3-representative embedding in $\Pi$ then $\Gamma$ embeds in the torus.

Proof: From Chapter II recall that each embedding of $K_{3,3}$ is 2-representative. Since $\Gamma$ is nonplanar each embedding of $\Gamma$ is at least 2-representative. Suppose that $\Gamma$ has a 2-representative embedding $\nu$, and take $\omega$ to be an O-arc $\Pi$ of weight 2.

Write $\Gamma = M \cup P$ with $k \geq 3$ as above. If $k = 3$ then the Structure Theorem assures a toroidal embedding. Then $\omega$ cannot be contained in $\Delta$ so $\omega$ crosses one or more of the $F_i$'s. If $\omega$ crosses a single $F_i$ then it meets one face of $\nu$ in $\Delta$ and hence crosses no edges. Now $\omega\Delta$ supplies path $\tau$ of Construction A. Suppose next that $\omega$ is an essential cycle of weight 3. Then $\omega$ crosses one or two of the $F_i$'s. If exactly one then $\omega$ crosses two faces in $\Delta$ and $\omega\Delta$ meets one edge $e$, of $\Gamma$. Again $\omega\Delta$ supplies the path $\tau$ of Construction A. If $\omega$ crosses precisely two of the $F_i$'s, they must be adjacent; so that $\omega$ crosses $\Delta$ interior to a single face from $[i, i+1]$ to $[i+k-1, i+k]$. Taking $\omega\Delta = \tau$ satisfies the conditions of Construction B.

The converse of the Embedding Lemma requires that we examine the structure imposed on $\Gamma$ by the existence of toroidal embeddings of $\Gamma$. Our first order of business is to restrict the number of embeddings that need be considered.

Combing Lemma: With $\Gamma$ as above, if $\mu: \Gamma \rightarrow T$ where $\mu|_M$ is a typical embedding of $M_k$ without null rungs, then there is an embedding $\mu'$ of $\Gamma$ into $T$ with $\mu'|_M$ a combed embedding of $M_k$.

Proof of Lemma: Resuming the notation of Chapter II, if the sequence of U's and
D's arising from \( \mu |_M \) contains a string of \( k \) D's (or equivalently U's) there is nothing to prove. Thus the sequence defined by \( \mu |_M \) has a string of at most \( k-2 \) U's or D's. We show that \( \mu |_M \) may be modified to produce an embedding with a longer string of U's and D's. By induction on the length of strings of D's the claim is then established. Suppose that the longest string of D's is of length \( n \). WOLOG vertex 1 is labeled U, vertices 2, 3, ..., \( n+1 \) are labeled D and vertex \( n+2 \leq k \) is labeled U. Then \( \mu |_M \) has \( n-1 \) faces of degree 4 each bounded by a pair of rungs \([ i, i+k \], [ i+1, i+k+1 \] for \( 2 \leq i \leq (n+1) \) and two edges of \( S \). Observe that \( \mu |_M \) also has a face \( F \) of degree \( 2n+4 \) with boundary cycle:

\([ 1, 2, ..., n+2, n+k+2, n+k+1, ..., k+1 ]\).

Take \( B = P \cap \mu^{-1}( \text{cl}(F) ) \). The vertices of attachment of \( B \mod S \) lie interior to the arcs \([ 1, n+2 ]\) and \([ k+1, n+k+2 ]\). Choose \( x_1 \) on \([ 1, 2 ]\) and \( x_2 \) on \([ n+1, n+2 ]\) such that all vertices of attachment of \( B \) that lie on \([ 1, n+2 ]\) are
contained on \([x_1, x_2]\). Symmetrically choose \(y_1\) and \(y_2\) on \([k+1, k+2]\) and 
\([n+k+1, n+k+2]\) so that all vertices of attachment of \(B\) on \([k+1, n+k+2]\)
are on \([y_1, y_2]\).

Since no component of \(P\) outside \(B\) overlaps \(R_2\) \((\text{mod } S)\), \(x_1\) may be chosen so
that no component of \(P\) outside \(B\) has a vertex of attachment on \([x_1, 2]\). Similarly
there is no component of \(P - B\) with vertices of attachment on \([x_2, n+2]\), 
\([k+1, y_1]\) or \([y_2, n+k+1]\). We now construct two arcs that avoid all
components of \(P\) by following rungs \(R_1\) and \(R_{n+2}\) along the interior of \(F\) and
Rungs \(R_2\) and \(R_{n+1}\) on the outside of the squares. These arcs bound an annulus
containing the components in \(B\) and the squares determined by rungs \(R_2\) through
\(R_{n+1}\). Further this annulus is divided into two disks by \(S\), one interior to \(F\), the

\[\text{Figure IV.10}\]
other containing rungs $R_2$ through $R_{n+1}$.

Figure IV.11

Excising the region interior to $F$ and interchanging it with the other region, we create a new embedding $\mu'$ in which vertices 1 through $n + 2$ are all labeled U and vertices $k + 1$ through $n + k + 2$ are labeled D. This completes the proof of the claim.

Lemma: Write $\Gamma = M \cup P$ as above, and suppose there exists an embedding $\eta: \Gamma \rightarrow T$ for which $\eta|_M$ is of type i. Then there exists an embedding $\lambda$ of $\Gamma$ into $T$ for which either $\lambda|_M$ is a combed embedding or $\lambda|_M$ is of type iii.

Proof: The faces of partial embedding $\eta|_M$ impose a partition on the components of $P$. Let $A_1, A_2, A_3, A_4$ be the subgraphs of $P$ mapped by $\eta$ into the regions of $T - \eta(M)$ as indicated below:
If $A_3$ and $A_4$ are both empty or if no element of either is skew (mod $S$) to the rung $R_1$, then we may find a short diagonal across $\Delta$ from a point near 1 on $[1,2]$ to a point near 5 on $[5,6]$. Thus by Construction A we obtain a combed embedding $\lambda$. Such is illustrated in the figure below:
Thus we may assume that $A_3$ is nonempty and contains a component skew (relative to $S$) to the rung $R_4$. By corollaries 1 and 2, no component of $A_3$ has vertices of attachment confined to $[8, 2]$; hence we may assume the existence of a component with vertices of attachment on $[3, 4]$ and also along $[1, 8]$ and/or $[6, 7]$.

Suppose there is a component, $B$, in $A_3$ with vertices of attachment along both of $[3, 4]$ and $[6, 7]$. By corollaries 1 and 2, no component of $A_4$ may have vertices of attachment only along $[4, 6]$. Likewise corollary 1 confines the components of $A_1$ to lie along the arc $[1, 3]$ and the components of $A_2$ to have vertices of attachment only along an arc of $[6, 8]$. This allows us to conclude that $[4, 5]$ and $[5, 6]$ are edges of $T$. The configuration is illustrated below:

![Figure IV.14](image)

Since $[4, 5]$ and $[5, 6]$ are edges of $T$, there is a face $E$ of $v$ that contains both of them in its boundary. The boundary $E$ meets both $[3, 4]$ and $[6, 7]$ so that the
interior $E$ supplies a path for a minor diagonal across $\Delta$ and construction $B$ supplies an embedding $\lambda$ of type iii. The remaining configuration requires that there be a bridge in $A_3$ that meets both of $[3,4]$ and $[1,8]$ but that all such bridges have no vertex of attachment on $[6,7]$. Let $B$ be such a bridge, $a$ its vertex of attachment on $[3,4]$ nearest to 4, and $b$ its vertex of attachment on $[1,8]$ nearest to 8. There is a simple path $\alpha$ in $B$ from $a$ to $b$. Further $\alpha \cup [b, a]$ bounds a disk $R$ in $\Delta$. Let $R'$ be a maximal disk containing $R$ and with boundary of type $\alpha' \cup [b, a]$ where $\alpha'$ is a simple path in $B$. Take the boundary of $R'$ as $\beta \cup [b, a]$.

\[\text{Figure IV.15}\]

$\beta$ separates $\Delta$ into disks $R'$ and $R''$ and no arc exits $\beta$ into the region $R''$ since such would have to terminate:

- on $\beta$: contradicting the maximality of $R'$
- on $[3,4]$: contradicting the choice of $a$
- on $[8,1]$: contradicting the choice of $b$, or
on \([6, 7]\) contradicting the definition of \(B\).

Therefore there is a face of \(v\) contained in \(R''\) with \(a\) as an arc of its boundary. Such face supplies a short diagonal needed for construction \(B\) and hence yields an embedding of \(\Gamma\) of type iii. This completes the proof of the lemma.
§ IV.3 Cyclic Orders

We extend the notation of Section II.3 as follows.

**Definition:** Write $\Gamma = M \cup P$, and suppose that $\eta: \Gamma \rightarrow T$, $\eta|_M$ is of type ii or is a combed embedding. Take $N$ to be an $\varepsilon$-neighborhood of $\eta(S)$ in $T$ and with the two regions of $N - \eta(S)$ labeled as $U$ and $D$. There is exactly one face $E$, of $\eta|_M$ whose closure contains an essential cycle. Let $A = [i, i+1]$ be an arc of $S \cap E$, if the boundary cycle of $E$ meets the $[i, i+1]$ exactly once (in either orientation) we say that $A$ is **one-sided in $\eta$**. Further if $A$ is one-sided in $\eta$ and every $\varepsilon$-neighborhood of $A$ meets $U$ (meets $D$) then we say $A$ is labeled as $U$ (as $D$).

**Twisted Lemma:** Write $\Gamma = M \cup P$, and suppose that for $\eta: \Gamma \rightarrow T$, $\eta|_M$ is of type ii, or is a combed embedding with $E$ the face of $\eta|_M$ whose closure contains an essential cycle in $T$. If $[i, j]$ is an interval on $S$ all of whose links are labeled $U$, and $[s, t]$ is a second interval all of whose links are labeled $D$, then the number of disjoint paths in $\eta(P)$ crossing $E$ that join $[i, j]$ to $[s, t]$ is at most one.

**Proof of Lemma:** Suppose not. In the proof we will take $i < j$ and $s < t$. Take $\alpha$ and $\beta$ to be two disjoint simple paths in $E$ joining $a_1$ and $b_1$ on $[i, j]$ to $a_2$ and $b_2$ on $[s, t]$ respectively. We may arrange our notation so that $a_1$ and $b_1$ occur in the order $(i, a_1, b_1, j)$. Considering the paths in $A$, a contractible region of $\Pi$, we find that the order of $b_1$ and $b_2$ in $S$ must be $[s, b_2, b_1, t]$. 

If the embedding $\eta|_{M}$ is of type ii then the boundary cycle of $E$ is
$[1, 2, k + 2, k + 1, 1, 2k, 2k - 1, k - 1, k, 2k]$. WOLOG $[1, 2]$ and
$[k - 1, k]$ are the two intervals labeled U while $[k + 1, k + 2]$ are labeled D.

If the embedding is a combed embedding, then $E$'s boundary cycle is
$[1, 2, ..., k, k + 1, 1, 2k, 2k - 1, ..., k + 1, k, 2k]$. WOLOG the edges of $S$
in $[1, 2, ..., k]$ are labeled U and those in $[k + 1, k + 2, ..., 2k]$ labeled D.

But now in all circumstances, the natural order of $s$ and $t$ is reversed in the
boundary of $E$ so that $b_1$ and $b_2$ must occur in the order $[s, b_1, b_2, t]$; a
contradiction that establishes the lemma.

**Claim:** Suppose $\Gamma = M \cup P$ is as above, with $v: \Gamma \longrightarrow \Pi$ and suppose there
exists an embedding $\eta: \Gamma \longrightarrow T$ for which $\eta|_{M}$ is of type ii, then there exists an
embedding $\lambda$ of $\Gamma$ into $T$ for which $\lambda|_{M}$ is either a combed embedding or $\lambda|_{M}$ is
of type iii.

**Proof:** Let the partial embedding $\eta|_{M}$ be labeled as shown below:

![Figure IV.16](image-url)
Adopting the notation of the Lemma above we label \([1, 2]\) and \([k-1, k]\) as U while \([k+1, k+2]\) and \([2k-1, 2k]\) are labeled D. The faces of the embedding \(\tau|_M\) impose a partition on the components of \(P\) into \(k\) parts. From corollaries 0, 1, and 2 we may assume that each component of \(P\) has vertices of attachment on \(S\); that no component attaches only on an arc of type \([i, i+1]\); and that no component attaches only along an arc of type \([i, i+2]\). If there is a bridge embedded in any of the 4-gons of \(\tau|_M\), then its vertices of attachment must lie on both of the arcs \([i, i+1]\) and \([i+k, i+k+1]\). But the existence of such a bridge implies the existence of a 2-representative embedding of \(\Gamma\) into \(\Pi\) and hence an embedding of \(\Gamma\) into \(T\). So we may take the partition to have only (possibly) nonempty parts:

\[A_1 = \{\text{components of } P \text{ embedded into face } [1, 2, ..., k, k+1]\}\]

\[A_2 = \{\text{components of } P \text{ embedded into face } [k, k+1, ..., 2k-1, 2k]\}\]

\[A_3 = \{\text{components of } P \text{ embedded into the face of valency } 10\}\]

Suppose that \(A_1\) contains a component with vertices of attachment in the arc \([k, k+1]\). Then this component must also have vertices of attachment along the arc \([1, k-1]\). Since none of the components of \(P\) are skew (mod \(S\)) we may conclude that no bridge of \(A_3\) attaches to the arc \([k-1, k]\). This implies that there are points \(X\) and \(Y\) on arcs \([1, 2]\) and \([k, k+1]\) respectively such that all bridges of \(P\) attach to the interior of either \([X, 2, ..., k, Y]\) or \([Y, k+1, ..., 2k, X]\) as indicated below:
Such a division supplies the short diagonal \( \tau \) required for Construction B and hence an embedding of type iii of \( \Gamma \) into \( T \). Symmetrically, we see that if \( A_2 \) has a bridge that attaches to the arc \([ k, k + 1 \]) we may construct an embedding of type iii. We conclude that neither \( A_1 \) nor \( A_2 \) has vertices of attachment along the arc \([ k, k + 1 \]). More pointedly, the vertices of attachment of components in \( A_1 \) are restricted to lie on the arc \([ 1, 2, \ldots, k - 1, k \]) and those of components in \( A_2 \) along the arc \([ k + 1, k + 2, \ldots, 2k \]). Further, the arc \([ k, k + 1 \]) is an edge of \( \Gamma \). Consider \( \Gamma \) in \( \Pi \). Take \( Y \) on \([ k + 1, k + 2 \]) and \( X \) on \([ 1, 2 \]) to be such that all the vertices of attachment of components in \( A_3 \) are contained on the arc \([ Y, X \]). If there is a path \( \tau \) across \( \Delta \) from \( X \) to \( Y \) that meets two or fewer faces, then the embedding is 3-representative or less and there is an embedding of type iii or a combed embedding of \( \Gamma \) into \( T \). Thus any path across the interior of \( \Delta \) must meet three faces of \( v \) so that \([ k - 1, k \]) must have at least two vertices of attachment from \( A_3 \). (Symmetrically so must \([ k + 1, k + 2 \]).)
In fact there must be an arc joining the last vertex on \([ k - 1, k ]\) to the first vertex on \([ k + 1, k + 2 ];\) or else there is a face that spans the disk \(\Delta\) either from \([ k - 1, k ]\) to \([ 2k-1, 2k ]\) (symmetrically \([ 1, 2 ]\)) or \([ k , k + 1 ]\) to \([ 2k, 1 ]\). Either case supplies a path \(\tau\) separating the components of \(\Gamma\) as required for Construction A.
Consider now the path from the last vertex on \([k - 1, k]\) to the first on \([k + 1, k + 2]\). Let the faces that meet it be \(E_1, E_2, \ldots, E_m\) in order. The boundary of the union of these faces is a circuit, one of whose components \(C\) contains an arc of \([k - 1, k]\) and an arc \(\alpha\) joining \([k - 1, k]\) and \([k + 1, k + 2]\).
Additionally C must contain a second arc \( \beta \) in \( \text{int}(\Delta) \) from \([k - 1, k]\) that terminates on \( S \) and an arc \( \gamma \) in \( \text{int}(\Delta) \) from \([k + 1, k + 2]\) that terminates on \( S \). If \( \beta = \gamma \) then \( \alpha \) and \( \beta \) supply two disjoint arcs from a U edge to a D edge, a contradiction. So \( \beta \) attaches to \([2k - 1, 2k]\), \([2k, 1]\), or \([1, 2]\). Any face \( E_i \) that meets \( \beta \) and \( \gamma \) cannot contain an arc of its boundary joining \( \beta \) and \( \gamma \) unless such arc is in \( S \) -- such an arc completes a second path between U and D edges.

**Case 1:** If \( \beta \) attaches to \([2k - 1, 2k]\) then so must \( \gamma \). As indicated in the figure below there is a face extending from \([2k - 1, 2k]\) \( \alpha \). Taking a path as indicated provides a long diagonal as needed for Construction A and embeds \( \Gamma \) in \( T \) with a combed M.
Case 2: If $\beta$ attaches along $[2k, 1]$ then $\gamma$ attaches along $[2k-1, 2k]$ or $[2k, 1]$. If the former then the path of case 1 is available. If the latter then the path from $X$ to $Y$ as indicated below supplies the path $\tau$ required in Construction A and guarantees a combed embedding.
Case 3: If $\beta$ attaches to $[1, 2]$ the only attachment of $\gamma$ not covered by symmetry above is to $[2k - 1, 2k]$ but then the long diagonals of either of the above figures are available for use in Construction A. In all events we obtain a combed embedding of $\Gamma$.

The immediate consequence of the above Lemmas is:

**Theorem:** If $\Gamma = M \cup P$ is as above and there is an embedding $\eta: \Gamma \rightarrow T$, then either $M$ is a homeomorph of $K_{3,3}$ or there is an embedding $\lambda: \Gamma \rightarrow T$ for which either $\lambda|_M$ is a combed embedding or an embedding of type iii.

We are now prepared to state and prove the converse to the Embedding Theorem.

**Theorem:** If $\Gamma = M \cup P$ is as above and there exists an embedding $\eta: \Gamma \rightarrow T$, then there is some embedding $\nu: \Gamma \rightarrow \Pi$ that is 3-representative or less.
Proof: If $M$ is a homeomorph of $K_{3,3}$ then $v: \Gamma \rightarrow \Pi$ contains an O-arc of weight 3 that avoids $P$. We may in the rest of the proof assume that $M$ is a homeomorph of $M_k$ for $k \geq 4$.

Case 1. Suppose $\eta|_M$ is of type iii. The faces of $\eta$ impose a partition into $k$ parts on the components of $P$. If any of the parts arising from the 4-gons of $\eta|_M$ is nonempty then, from the corollary to Construction A, there is an embedding of $\Gamma$ with $\eta|_M$ a combed embedding; we shall consider $\Gamma$ there. Therefore the only (possibly nonempty) parts are those arising from the $(k+1)$-face and the $(k+7)$-face. Denote the parts as $A_1$ and $A_2$ respectively. We may take $A_1 = \{ B_1, B_2, \ldots, B_n \}$ as nonempty. For each $B_i$ take $x_i$ to be its vertex of attachment on $S$ closest to 1 and $y_i$ its vertex of attachment closest to $k$. For each $B_i$ there is a simple path $\alpha_i$ in $B_i$ from $x_i$ to $y_i$; and $\alpha_i \cup [ y_i, x_i ]$ bounds a disk $D_i$ in $\Delta$. For each $i$ take $\alpha'_i$ to be a path in $B_i$ that bounds a maximal disk $D'_i$ containing $D_i$. Denote the union of all the $D'_i$'s by $D'$ and $\Delta - \text{cl}(D')$ by $D''$. No arc may exit $\text{Bdry}(D')$ into $D''$ since such would have to:

A) terminate on $S$ which either
   A1) contradicts the definition of $x_i$ or $y_i$ for some $i$; or
   A2) constitutes a new bridge contradicting the definition of $A_1$ or violating the partition of the components of $P$; or

B) terminate on some $\alpha'_j$ thus
   B1) contradicting the definition of component, if $i \neq j$; or
   B2) the maximality of $D_j$, if $i = j$.

But this implies that the face $E$ of $D''$ that meet $x_1$ must meet all of the $x_i$'s and $y_i$'s;
indeed, the boundary of $E$ meets both $[1, 2]$ at say $X$ and $[k, k + 1]$ at $Y$. By taking a simple cycle $\omega$ that crosses $E$ from $X$ to $Y$ and traverses $F_k$ from $Y$ to a point $Z$ on $R_1$ thence across $F_1$ back to $X$ we obtain an $O$-arc of weight 3.

Before proceeding with case 2 we make the following observation. Write $\Gamma = M \cup P$ as above and suppose that $v: \Gamma \rightarrow \Pi$ has no $O$-arc of weight less than four; this requires that $M$ be homeomorphic to $M_k$ for some $k \geq 4$. Consider any pair of opposite arcs on $M$, $[i-1, i]$ and $[i+k-1, i+k]$. In $\Delta$ take $D_1$ as the union of all faces of $v$ that meet $[i-1, i]$ and take $D_2$ as the union of all faces of $v$ that meet $[i+k-1, i+k]$. The closures of $D_1$ and $D_2$ are necessarily disjoint; otherwise we could construct an $O$-arc in $\Pi$ passing through $F_{i-1}$ and at most two faces in the $D_i$'s. In particular the boundaries of the $D_i$'s are disjoint and each contains a path from a vertex on the arc $[i, i+k-1]$ to some vertex on $[i+k, i-1]$. 

Figure IV.23
Case 2. Suppose now that $\lambda: \Gamma \rightarrow T$ is an embedding of $\Gamma$ into the torus and that for some decomposition of $\Gamma = M \cup P$, $\lambda|_M$ is a combed embedding of $M_k$ for $k \geq 4$. By the Twisted Lemma, $P$ contains at most one arc joining $[1, k]$ to $[k + 1, 2k]$. From the above observation this requires that $\lambda: \Gamma \rightarrow T$ contain an $O$-arc of weight 3 or less.

This completes the proof of the Theorem.
Chapter V

Five Projective Planar Graphs

Throughout this chapter \( \Gamma \) will represent a simple cubic graph satisfying:

1. \( \Gamma \) is nonplanar.
2. \( \Gamma \) has an embedding \( v \) into the projective plane, \( \Pi \).
3. \( \Gamma \) contains no subgraph homeomorphic to the disjoint union of \( K_{3,3} \) and \( K_{2,3} \).
4. \( \Gamma \) has no 2-separations.
5. \( \Gamma \) has no nontrivial 3-separations.
6. \( \Gamma \) is edge-critically nonrotroidal.

§ V.1 Establishing the Planar configurations

The results of Chapter IV imply that each embedding \( \eta: \Gamma \rightarrow \Pi \) is at least 4-representative. That has the following consequence:

**Proposition:** For any \( \eta: \Gamma \rightarrow \Pi \) and any face \( E \) of \( \eta \) there is a decomposition of \( \Gamma = M \cup P \) with \( \text{Bdry}(E) \) disjoint from \( M \).

**Proof:** Let \( E_1, E_2, ..., E_n \) be the closed faces of \( \eta \) that share an edge with \( E \). Take \( R = E \cup E_1 \cup E_2 \cup ... \cup E_n \). It suffices to show that \( R \) contains no essential O-arc of \( \Pi \); since if \( R \) is contractible in \( \Pi \), taking \( \Delta \) as a maximal closed disk containing
R as in the first Lemma of the proof of the Structure Theorem then $M = \Gamma - \text{int}(\Delta)$ gives the required decomposition.

Since $\Gamma$ has no 2-separations and no pair of faces of $\eta$ contain an essential $O$-arc in their union, $\text{Bdry}(E) \cap \text{Bdry}(E_i) = a_i$, an edge of $\Gamma$ for each $i$. Since $\eta$ is has no $O$-arcs of weight less than four, $E \cup E_i \cup E_j$ contains no essential $O$-arc of $\Pi$ for all $i$ and $j$. Define $R_i = E \cup E_i$ and inductively for $1 < i \leq n$ define $R_i = R_{i-1} \cup E_i$.

Suppose for some $i$, $R_i$ contains an essential $O$-arc. Choose $i$ to be the least such index; by the above, $3 \leq i$. Take $\omega$ to be an simple essential $O$-arc in $R_i$ of least weight. Such $\omega$ traverses any face of $R_i$ at most once and $\omega$ must traverse $E_i$ else $\omega$ is in $R_{i-1}$. Write $\omega = \omega_1 + \omega_2$ where $\omega_1$ traverses $E_i$ from point $A$ to $B$ on $E_i \cap R_{i-1}$ and $\omega_2$ traverses $R_{i-1}$. Suppose $A$ is on $E_j$ and $B$ is on $E_k$. Take $\omega_2'$ to be a path in $E_j \cup E_k \cup E$ from $B$ to $A$. Since $R_{i-1}$ contains no essential $O$-arcs of $\Pi$, $\omega_2 + \omega_2'$

\[ \text{Figure V.01} \]
is null homotopic and \( \omega - (\omega_2' + \omega_2) = \omega' = \omega_1 - \omega_2' \) is essential and has weight at most 4. Take \( \omega' \) to traverse \( E \cup E_j \) from A to C with C on \( E \cap E_k \). Consider \( E \cup E_i \cup E_j \) and denote by \( \alpha \) the arc of \( \omega' \) that traverses \( E \cup E_i \cup E_j \). Take \( \alpha' \) to be a path traversing \( E \cup E_i \) from C to B. Since \( E \cup E_i \cup E_j \) contains no essential \( O \)-arcs of \( \Pi \), \( \alpha + \alpha' \) is null homotopic. Thus \( \omega'' = \omega' - (\alpha + \alpha') \) is an essential \( O \)-arc of weight at most 3. This contradiction completes the proof.

Consider \( E \) and \( F \), a pair of faces of \( v \) that meet on an edge \( a \). \( \text{Bdry}(E) \cup \text{Bdry}(F) \) is a theta-graph in \( \Gamma \). Let the faces of \( v \) that meet the cycle \( C = \text{Bdry}(E \cup F) \) be \( E_1, ... E_m \) cyclically around \( C \), and let \( N(E, F) = E \cup F \cup E_1 \cup E_2 \cup ... \cup E_m \). We shall refer to \( N(E, F) \) as the neighborhood of \( E \cup F \).

Since \( v \) has no essential \( O \)-arcs of weight less than four \( E \cup F \cup E_i \) does not contain an essential \( O \)-arc of \( \Pi \). The situation is quite different for \( N(E, F) \).
Theta Lemma: If $\Gamma$ and $v$ are as above and $E$ and $F$ are faces of $v$ meeting on an edge $a$ then $N(E, F)$ contains an essential $O$-arc.

Proof: If $N(E, F)$ does not contain an essential $O$-arc of $\Pi$ then $\text{Bdry}(N(E, F))$ bounds a disk $D$ which may be extended as in the first lemma of the proof of the Structure Theorem to a maximal disk $\Delta$. Taking $M = \Gamma - v^{-1}(\text{int}(\Delta))$, $M$ is isomorphic to a Möbius ladder $M_k$ with $k \geq 4$. But such $M_k$ contains a homeomorphism of $K_3$, now disjoint from the $0$-graph $\text{Bdry}(E \cup F)$ contradicting condition 3.

Proposition: If $\Gamma$ and $v$ are as above there is a face of $v$ of degree 4 or 5.

Proof: Let $v$ be the number of vertices of $\Gamma$; and $e$ the number of edges. Since $\Gamma$ is cubic: $3v = 2e$. Take $f_i$ to represent the number of faces of $v$ of face degree $i$ and let $f = \sum f_i$. As each edge meets some face twice: $2e = \sum if_i$. Since $\Gamma$ is nonplanar, $v$ is a 2-cell embedding and the Euler characteristic of $\Pi$ is 1: $v - e + f = 1$.

Multiplying the last equation by 6 and substituting $3v = 2e = \sum if_i$; $6f = \sum 6f_i$; and $2e = \sum if_i$; we get

$$2(\sum if_i) - 3(\sum if_i) + \sum 6f_i = 6$$  \hspace{1cm} [1]$$

whence:

$$\sum (6 - i)f_i = 6$$  \hspace{1cm} [2]$$

Since $f_0 = f_1 = f_2 = f_3 = 0$

$$4f_4 + 5f_5 = 6 + \sum (i - 6)f_i \text{ for } i \geq 7. \hspace{1cm} [3]$$
Since the right hand side of equation [3] is positive, so must the left hand side be.
In particular one of \( f_4 \) or \( f_5 \) is positive.

Let \( \Gamma \) and \( v \) be as above and let \( E \) be a face of smallest degree. By the above \( E \) is of degree 4 or 5. In the sequel we shall take a decomposition of \( \Gamma = M \cup P \) so that the circuit bounding \( E \) is disjoint from \( M \). Recall that \( M \) is a homeomorph of \( M_k \) for \( k \geq 4 \). We enumerate the faces meeting \( E \) as \( E_1, E_2, E_3, E_4, (E_5) \) cyclically around \( E \). Since \( \text{Bdry}(E \cup E_1) \) is a \( 3 \)-graph and \( M \) contains a homeomorph of \( K_{3,3} \), each \( E_i \) must meet \( S \), the spine of \( M \). Since \( \Gamma \) is cubic each \( E_i \) meets \( S \) at two points which we denote as \( L_i \) and \( R_i \) (referred to as the left and right attachments respectively). Finally we note that \( R_i \) and \( L_{i+1} \) may or may not be the same vertex. If they are, then we say that \( E_i \) and \( E_{i+1} \) are separated by a whisker. If not then \( E_i \) and \( E_{i+1} \) are separated by a split end. If \( E_i \) and \( E_{i+1} \) are separated by a split end we take the edge meeting both \( E_i \) and \( E_{i+1} \) as \( a = [p_i, q_i] \) with \( q_i \) incident with \( E \).

Figure V.03 illustrates the configurations if \( \Gamma \) contains a 4-face or a 5-face. The rest of the embedding of \( \Gamma \) is comprised of the k "infinite faces" \( F_1, F_2, ..., F_k \) of the mobius ladder \( M \).
For any infinite face $F_j$, $F_j$ meets $S$ in two disjoint arcs $[i, i + 1]$ and $[i + k, i + k + 1]$. If $[i, i + 1]$ meets some face $E_j$ then $[i + k, i + k + 1]$ may not meet a face $F$ adjacent to $E_j$ (in particular $E_{j+1}$ or $E_{j-1}$) as such would allow an essential $O$-arc through $F_j$, $E_j$ and $F$ of weight at most 3. Since $T$ is assumed to be nontoroidal all essential $O$-arcs of $v$ have weight at least 4.

**Proposition**: Suppose $\Gamma = P \cup M$ with $M$ homeomorphic to $M_k$ for some $k \geq 4$ as above. Suppose further that for some $i$, $R_i \neq L_{i+1}$ let $T$ be the region of $\Delta$ bounded by the cycle $[R_i, a_i, L_{i+1}]$. Then if $T$ is not a face of $v$ then $k \leq 5$.

**Proof**: Denote by $F_1$ that infinite face that meets $R_i$ and let $[1, 2]$ be the end of $F_1$ meeting $R_i$. By the above remark $[k + 1, k + 2]$ cannot meet $[L_{i+1}, R_{i+1}]$ or $[L_{i-1}, R_{i-1}]$. Thus $[k + 1, k + 2]$ meets $S$ along the arc $[R_{i+1}, L_{i-1}]$ or along
Case 1: If \([ R_p, L_{i+1} ] \supseteq [ k+1, k + 2 ]\), then one end of each infinite face \(F_j\) is contained in \([ R_p, L_{i+1} ]\). Consider the pair of adjacent faces \(E\) and \(E_{i+2}\).

\(N( E, E_{i+1} )\) is disjoint from \(\text{int}( T )\). If \(E_{i+2}\) is not adjacent to \(F_1\) then there is an essential cycle in \(\Pi\) (through \(T\) and \(F_1\)) disjoint from \(N( E, E_{i+2} )\). Since \(\Pi\) contains no pair of disjoint essential cycles this contradicts the Theta Lemma above.

But now the interval \([1, 2]\) must include \([ L_{i-1}, R_{i-1} ]\); in particular \(E_{i-1}\) does not meet \(F_2\). Considering adjacent pair \(E\) and \(E_{i-1}\); we note that \(T \cup F_2\) is disjoint from \(N( E, E_{i-1} )\) and contains an essential cycle of \(\Pi\), again contradicting the Theta Lemma.

Figure V.04
Case 2: [R\(_{i+1}\), L\(_{i-1}\)] \supseteq [k + 1, k + 2]. Denote by F\(_j\) that infinite face that meets L\(_{i+1}\). The arc [j, j + 1] meets both T and E\(_{i+1}\) so that [j + k, j + k + 1] meets S along the arc [R\(_{i+2}\), L\(_i\)]. Since k + 2 \leq j + k + 1, [k + 1, k + j + 1] is contained on the arc [R\(_{i+1}\), L\(_i\)]. If T is not a face, then T contains at least two adjacent faces G and H. N(G, H) is contained in the union of T and T's neighbors. The neighbors of T consist of E\(_i\), E\(_{i+1}\), and the infinite faces F\(_3\), ..., F\(_j\). Such a collection of regions contains an essential cycle only if j = k. But then the interval [1, 2] contains [L\(_i\), R\(_i\)] and symmetrically [k, k + 1] contains [L\(_{i+1}\), R\(_{i+1}\)]. Considering the adjacent faces F\(_3\) and F\(_4\) if k > 5 there is an O-arc across E\(_i\), E\(_{i+1}\), and F\(_k\) disjoint from N(F\(_3\), F\(_4\)), hence N(F\(_3\), F\(_4\)) cannot contain an O-arc of \(\Pi\). This contradiction establishes the Lemma.
§ V.2 Central 4-gon and Frizzies I

In this section we shall assume that:

1. \( v \) has a face \( E \) of degree 4. Thus we write \( \Gamma = P \cup M \) with \( \text{Bdry}(E) \cap M = \emptyset \).

2. \( E_1 \) and \( E_2 \) are separated by a split end for which the region \( T \) bounded by \( \left[ R_1, L_2, a_1 \right] \) is not a face of \( v \).

3. \( M \) is isomorphic to \( M_4 \).

Figure V.05 below illustrated the configuration and fixes the labeling of fifteen of the vertices of \( \Gamma \).

The rungs of \( M \) are edges \([ 1, 5 ]; [ 2, 6 ]; [ 3, 7 ]; \) and \([ 4, 8 ]; \).
The region $T$ bounded by the cycle $[2, 3, 4, 15, 13, 14]$ must have at least two faces that prevent a weight three O-arc across $T, E_1$ and $E_4$. Such faces require the existence of a path -- vertex 16 to vertex 17 -- with 16 situated on $[13, 15]$ and 17 on $[4, 3]$ or $[3, 2]$. In either case there exists a region $F$, contained in $T$, that is the union of faces of $v$, meets faces $F_4$, $F_3$ and $E_2$ but meets none of $E$, $E_1$, $E_3$, $E_4$ or $F_1$. And there exists a region $G$, contained in $T$, that is the union of faces of $v$, meets faces $F_2$, $F_1$ and $E_1$ but meets none of $E$, $E_2$, $E_3$, $E_4$ or $F_4$.

Before proceeding to cases we can make some further observations:

A. If vertex 1 lies on the arc $[L_4, R_4]$, then $L_1 = R_4$ and vertices 5 & 6 must lie on $[R_2, L_3]$.

B. If vertex 5 lies on the arc $[L_3, R_3]$, then $R_2 = L_3$ and vertices 1 & 2 must lie on $[L_1, R_4]$.

**Case 1:** Suppose that vertex 5 lies on $[L_3, R_3]$. Then:

i) $R_2 = L_3 = \text{vertex } 18$.

ii) Since $F_4$ meets both of $E_2$ and $E_3$, vertices 1 and 8 must lie between $L_1 = \text{vertex } 19$ and $R_4 = \text{vertex } 20$; we set $a_4 = \text{vertex } 21$.

iii) $F_1$ meets $E_1$, so vertex 6 lies on $[5, L_4]$.

iv) Let $F'$ be a face of $F$ meeting $F_3$. If vertex 7 does not lie on $[6, R_3]$ there is an O-arc through $F_1, E_1, E$ and $E_3$ disjoint from $N(F_3, F')$. Since any two O-arcs in $\Pi$ intersect this contradicts the Theta Lemma, thus vertex 7 is on $[6, R_3]$ so that $R_3 = L_4 = \text{vertex } 22$.

This forces the configurations shown in Figure V.06.
The configuration shown is completed to a graph in two ways by the alternative placements of vertex 17. In either placement there is an O-arc through $F_1, F_4, F, G$, $F_1$ disjoint from $N(E, E_4)$, contradicting the Theta Lemma. The Theta Lemma is satisfied by any $\Gamma$ so Case 1 gives rise to no $\Gamma$'s.

**Case 2:** Vertex 1 lies on $[L_4, R_4]$. Then:

i) $L_1 = R_4 = \text{vertex 21}$

ii) Since $F_1$ meets $E_1$ and $E_4$, vertices 5 and 6 lie between $R_2 = \text{vertex 18}$ and $L_3 = \text{vertex 19}$; we set $a_2 = \text{vertex 20}$.

iii) Since $F_4$ meets $E_2$, vertex 8 lies on $[R_3, 1]$.

The resulting configuration is illustrated in Figure V.07 below.
In the configuration above there is an O-arc through \( F_2 \), the face bounded by the cycle \([ 18, 5, 6, 19, 20 ]\), \( F_4 \), \( F_3 \), and \( F_2 \) that is disjoint from \( N(E, E_1) \). Since placement of vertices 7, 8, 17 does not affect either the O-arc or neighborhood Case 2 gives rise to no \( \Gamma \)'s.

**Case 3:** Vertex 1 lies on \([ R_4, L_1 \)) and vertex 5 lies on \([ R_2, L_3 \)). Then:

i) Label \( a_2 = \text{vertex } 18; a_4 = \text{vertex } 19; R_2 = \text{vertex } 20; L_3 = 21; L_1 = \text{vertex } 22; \) and \( R_4 = \text{vertex } 23 \).

ii) If vertex 6 lies on \([ 5, 21 \)) then the resulting configurations contain (delete edge \([ 19, 23 \)) that of Figure V.06 from Case 2. So vertex 6 lies on \([ 21, 23 \)).

iii) If vertex 8 lies on \([ 23, 1 \)) then the resulting configurations contain (delete edge
iv) If vertex 7 lies on \([R_3, 8]\), then there is an O-arc through through \(F_1\), \(E_3\), \(E\), \(E_1\), and \(F_1\) disjoint from \(N(F_3, F')\) where \(F'\) is the previously chosen face in \(F\) that meets \(F_3\). Placement of vertex 17 either on \([3, 4]\) or \([2, 3]\) gives rise to the graphs of Figures V.08 and V.09.

The reader is referred to Appendix B in which the graph Kyle is displayed. The graph of Figure V.08 is isomorphic to the graph Kyle under the mapping below:

(Vertex of Figure V.08 --- vertex of Kyle)

1---13 2---18 3---24 4---23 5---9 6---14 7---19 8---8 9---21
10---15 11---20 12---5 13---6 14---2 15---16 16---22 17---7
18---11 19---1 20---12 21---10 22---17 23---24 24---4
Again with reference to the representations of Appendix B, in the graph of Figure V.09, deletion of $[13, 14]$ leaves a homeomorph of the graph George. An isomorphism from the nodes of the deleted graph to the vertices of George is:

$1 \rightarrow 3 \ 2 \rightarrow 4 \ 3 \rightarrow 5 \ 4 \rightarrow 6 \ 5 \rightarrow 7 \ 6 \rightarrow 8 \ 7 \rightarrow 1 \ 8 \rightarrow 2$

$9 \rightarrow 11 \ 10 \rightarrow 12 \ 11 \rightarrow 9 \ 12 \rightarrow 10 \ 15 \rightarrow 20 \ 16 \rightarrow 15 \ 17 \rightarrow 19$

$18 \rightarrow 16 \ 19 \rightarrow 14 \ 20 \rightarrow 21 \ 21 \rightarrow 22 \ 22 \rightarrow 18 \ 23 \rightarrow 17 \ 24 \rightarrow 13$

Thus Case 3 gives rise to one graph $\Gamma$.

**Case 4**: Vertex 5 lies on $[R_3, L_4]$. Then:

i) $R_2 = L_3 = \text{vertex } 18$

ii) Take $R_3 = \text{vertex } 19; \ L_4 = \text{vertex } 20; \ a_3 = \text{vertex } 21.$
iii) If vertex 1 lies on [ 19, 20 ] one may find an O-arc across F_1 and the region bounded by the cycle [ 20, 21, 19, 5, 6, 7, 8, 1 ] disjoint from N( E, E_2 ); so vertex 1 lies on the arc [ R_4, L_1 ].

iv) Take a_4 = vertex 22; L_1 = vertex 23; R_4 = vertex 24.

v) Since F_4 meets E_3, vertex 8 lies on [ 24, 1 ].

vi) Since F_1 meets E_1, vertex 6 lies on [ 5, 20 ].

The resulting configuration is illustrated in Figure V.10

But now, if F_3 fails to meet the region R bounded by [ 5, 6, 20, 21, 19 ], there is an O-arc through F_1, E_1, E, E_3 and R disjoint from N( F, F_3 ). So vertex 7 lies on the arc [ 6, 20 ]. But then F_2 fails to meet E_4 and there is an essential cycle through F_4, E_3, E, E_4 and the region bounded by [ 24, 8, 1, 23, 22 ] that is disjoint from
N( G', F_2 ), where G' is a face of G that meets F_2. Thus Case 4 gives rise to no graphs \( \Gamma \).

**Case 5:** Suppose 1 lies on \([ R_3, L_4 ]\). Then:

i) \( R_4 = L_1 = \text{vertex } 18 \)

ii) Take \( R_3 = \text{vertex } 19; L_4 = \text{vertex } 20; a_3 = \text{vertex } 21 \).

iii) If vertex 5 lies on \([ 19, 20 ]\), one may find an O-arc across \( F_1 \) and the region bounded by the cycle \([ 20, 21, 19, 5, 6, 7, 8, 1 ]\) disjoint from \( N( E, E_2 ) \); so vertex 5 lies on the arc \([ R_2, L_3 ]\).

iv) Take \( a_2 = \text{vertex } 23; L_3 = \text{vertex } 24; R_2 = \text{vertex } 22 \).

v) Since \( F_1 \) meets \( E_4 \), vertex 6 lies on \([ 22, 24 ]\).

vi) Since \( F_4 \) meets \( E_2 \), vertex 8 lies on \([ 19, 1 ]\).

The resulting configuration is illustrated in Figure V.11.
But in the configuration of Figure V.11 one may find an O-arc through $F_2$, the region bounded by the cycle \([22, 5, 6, 24, 23] \), $F_4$, $F$, $G$ and $F_2$ that is disjoint from $N(E, E_4)$. Thus Case 5 produces no $\Gamma$'s.

This exhausts the investigation of this section.
In this section we shall assume that:

1. \( v \) has a face \( E \) of degree 4. Thus we write \( \Gamma = P \cup M \) with \( \text{Bdry}(E) \cap M = \emptyset \).

2. \( E_1 \) and \( E_2 \) are separated by a split end for which the region \( T \) bounded by
   \[ [R_1, L_2, a_1] \] is not a face of \( v \).

3. \( M \) is isomorphic to \( M_5 \).

Figure V.12 below illustrates the configuration and fixes the labeling of seventeen of
the vertices of \( \Gamma \).

The rungs of \( M \) are edges \([1, 6]; [2, 7]; [3, 8]; [4, 9]; \) and \([5, 10]\).
As in the preceding section there must be a path from vertex 18 to vertex 19 splitting the region $T$ bounded by $[15, 16, 2, 3, 4, 5, 17]$. Vertex 18 lies on arc $[17, 15]$ with 19 on one of $[4, 5]; [3, 4];$ or $[2, 3]$. As before in each of these cases one may find a region $F$ in $T$ that is the union of faces of $V$ that meets $E_2$ and $F_5$ and that is disjoint from $F_1, E, E_1, E_3,$ and $E_4$.

**Case 1.** Suppose vertex 1 lies on $[L_4, R_4]$. Then:

i) $L_1 = R_4 =$ vertex 20.

ii) $F_1$ and $F_5$ both meet $E_4$, thus vertices 6 and 7 lie on $[R_2, L_3]$. Let $a_2 = 21$; $R_2 = 22; L_3 = 23$.

iii) If $F_3$ does not meet the arc $[22, 23]$, one may find an O-arc through $F_1, E_1, E_2$ and the region bounded by the cycle $[22, 6, 7, 23, 21]$ disjoint from $N( F_4, F_3 )$.

iv) Similarly, $F_4$ must meet the arc $[L_4, R_4]$ or else allow an O-arc through $F_5, E_2, E$, and $E_4$ disjoint from $N( F_3, F_2 )$. Thus $L_4 = R_3 = 24$

The resulting configuration is shown in Figure V.13 below.
In the configuration of Figure V.13, there is an O-arc through faces $F_3$, the region bounded by the cycle $[21, 22, 6, 7, 8, 23]$, $F_5$, $F_4$ and $F_3$, disjoint from $N(E, E_1)$. Thus the configuration of Figure V.13 gives rise to no $\Gamma$'s.

Case 2. Vertex 5 lies on $[R_2, L_3]$. Then:

i) $L_3 = R_2 = \text{vertex 20}$.

ii) $F_1$ and $F_5$ both meet $E_3$, thus vertices 10 and 1 lie on $[R_4, L_1]$. Let $a_4 = 21$; $R_4 = 22$; $L_1 = 23$.

iii) If $F_3$ does not meet the arc $[22, 23]$, one may find an O-arc through $F_5$, $E_2$, $E_1$ and the region bounded by the cycle $[22, 10, 1, 23, 21]$ disjoint from $N(F_2, F_3)$.

iv) Similarly, $F_4$ must meet the arc $[L_3, R_3]$ or else allow an O-arc through $F_1$, $E_1$, $E$, and $E_3$ disjoint from $N(F_3, F_2)$. Thus $L_4 = R_3 = 24$. 
The resulting configuration is shown in Figure V.14 below.

Figure V.14

In the configuration of Figure V.14 one may find an O-arc through $F_3$, the region bounded by the cycle $[21, 22, 9, 10, 1, 23]$, $F_2$ disjoint from $N(E, E_2)$. Thus Case 2 gives rise to no $\Gamma$'s.

Case 3. Suppose that vertex 1 lies on $[R_4, L_4]$ and vertex 6 on $[R_2, L_3]$. Then:

i) Let $R_2$ = vertex 20; $L_3$ = vertex 21; $a_2$ = vertex 22; $R_4$ = vertex 23; $L_1$ = vertex 24; and $a_4$ = vertex 25.

ii) $F_3$ must meet the arc $[R_4, L_1]$ else there is an O-arc through $F_5$, $E_2$, $E$, $E_1$ and the region bounded by the cycle $[23, 1, 24, 25]$ disjoint from $N(F_2, F_3)$. Thus
vertices 10 and 9 lie on $[23, 1]$. 

iii) Symmetrically, $F_3$ must meet the arc $[20, 21]$ else there is an O-arc through $F_1$, $E_1$, $E$, $E_2$ and the region bounded by the cycle $[20, 6, 21, 22]$ disjoint from $N(F_3, F_4)$. Thus vertices 7 and 8 lie on $[6, 21]$. 

iv) Now, however, there is an O-arc through $F_1$, $E_1$, $E$, $E_3$ and the region bounded by the cycle $[20, 6, 7, 8, 21, 22]$ disjoint from $N(F_4, F')$ for $F'$ a face of $F$ meeting $F_4$.

Case 4: Vertices 1 lies on the arc $[R_3, L_4]$. Then:

i) Set $L_1 = R_4 = \text{vertex } 20$; $a_3 = \text{vertex } 21$; $R_3 = \text{vertex } 22$; and $L_4 = \text{vertex } 23$.

ii) If vertex 6 lies on $[22, 1]$, then the union of region bounded by the cycle $[23, 21, 22, 6, 7, 8, 9, 10, 1]$ with $F_1$ contains an O-arc disjoint from $N(E_2, E)$. Thus vertex 6 lies on $[R_2, L_3]$. Set $a_2 = \text{vertex } 24$; $R_2 = \text{vertex } 25$; $L_3 = \text{vertex } 26$.

iv) Since $F_5$ meets $E_2$, vertex 10 lies on $[22, 1]$.

v) Since $F_1$ meets $E_4$, vertex 7 lies on $[6, 26]$.

The resulting configuration is displayed in Figure V.15.
vi) If $F_3$ fails to meet the region $R$ bounded by the cycle $[21, 22, 10, 1, 23]$ there is an O-arc through $F_5, E_2, E_3, E_4$ and $R$ disjoint from $N(F_2, F_3)$. This places vertex 9 on the arc $[22, 10]$.

vii) Similarly, if $F_3$ fails to meet the region $R'$ bounded by $[24, 25, 6, 7, 26]$ there is an O-arc through $F_1, E_1, E_2$ and $R'$ disjoint from $N(F_3, F_4)$. Thus vertex 8 lies on the arc $[7, 26]$.

Consequently, $E_3$ is not a neighbor of $F_4$ and we may find an essential cycle through $F_1, E_1, E_3$ and the region bounded by $[24, 25, 6, 7, 8, 26]$ that is disjoint $N(F_4, F')$ for $F'$, a face in $F$ that meets $F_4$.

Case 5: Vertex 6 lies on the arc $[R_3, L_4]$ and vertex 1 lies on $[R_4, L_1]$.

i) Set $R_2 = L_3 = \text{vertex } 20; a_3 = \text{vertex } 21; R_3 = \text{vertex } 22; L_4 = \text{vertex } 23;
a_4 = vertex 24; R_4 = vertex 25; and L_1 = vertex 26.

ii) Since F_1 meets E_1, vertex 7 lies on [6, 23].

iii) Since F_5 meets E_3, vertex 10 lies on [25, 1].

The resulting configuration is illustrated in Figure V.16.

In the configuration of Figure V.16 one may find an O-arc through F_1, the region bounded by the cycle [25, 10, 1, 26, 24], E_4, and the region bounded by the cycle [22, 6, 7, 23, 21] disjoint from N(E_2, F'') for F'' a face in F that meets E_2.

This completes the investigation of this section.
§ V.4 4-gons With Two Split Ends

In this section we consider \( \Gamma \)'s for which \( v \) contains a faces bounded by a 4-gon \( C \). We write \( \Gamma = M \cup P \) with \( C \) disjoint from \( M \). Further we may assume that no split end has the frizzies, i.e. if \( E_i \) and \( E_{i+1} \) are separated by a split end then the region \( G_j \) bounded by arcs of \( M, E_i, \) and \( E_{i+1} \) is a face of \( v \). Considering any whisker separating \( E_i \) and \( E_{i+1} \) some infinite face \( F_j \) meets both of \( E_i \) and \( E_{i+1} \) at the end of the whisker. The other end of \( F_j \) cannot meet any face neighboring either of \( E_i \) or \( E_{i+1} \) thus \( F_j \) can attach only to a split end face between \( E_{i-1} \) and \( E_{i+2} \). Thus \( P \cup S \) is one of the three graphs displayed below. Since \( M \) has at least 4 rungs we start the labeling of the vertices of \( P \) with 9. This does not constitute a claim that there are precisely 4 rungs. In the figures below note that each of the labeled regions is a face of \( v \) and that the remaining vertices of \( \Gamma \) are exactly the 2k ends of the rungs of \( M \).

Figure V.17 Two Split Ends
In the remainder of this section we explore the graphs deriving from Figure V.17.

Let $F_1$ be the infinite face that meets vertex 16. Let 1 and 2 be the vertices of the end
of $F_1$ that contains vertex 16. By the above both $k + 1$ and $k + 2$ lie interior to $[17, 18]$. Note that $F_1$ cannot overlap 13; so vertex 2 is on $[16, 13]$. Take $F_3$ to be the infinite face that meets 13. The opposite ends of $F_3$ must lie on $[19, 20]$ and so $F_3$ cannot share any edges with $F_1$. This forces the configuration of Figure V.20 below. Vertex 1 may lie on $[k + 4, 20]$ or $[20, 16]$; while 4 must lie on $[13, 17]$ or $[17, k + 1]$.

![Figure V.20](image)

There are -- up to symmetry -- three distinct graphs that complete the configuration. These are shown below in Figures V.21, V.22 and V.22. Further, one of these graphs is a subgraph of any $\Gamma$ whose planar part is that of Figure V.17.
Figure V.21

Figure V.22
The graph of Figure V.21 is listed in Appendix B as Harold, one of the five edge-critically nontoroidal graphs that embed in $\Pi$. In the graphs of Figures V.22 and V.23 one may find an O-arc through $F_4$, $G_1$, $F_2$, and $F_3$ disjoint from $N( E_4, E )$. Since this is a property that would be shared by any supergraph no $\Gamma$ contains these as subgraphs.
§ V.5 Central 4-gon with Three Split Ends

In this section we shall assume that $P$ is the graph of Figure V.17; that that involves three split ends. Take $F_1$ to be the face that meets the vertex 13 as shown below. Then vertices 1, 2, 5, and 6 must be as shown in Figure V.24 below:

Vertices $i$ and $j$ are on the infinite faces that meet vertex 17 and vertex 22, respectively. The existence of vertices $i$ and $j$ is assured since $\Gamma$ may have no triangles. Let $F_2$ denote the infinite face sharing the rung [2, 6] with $F_1$, and let $F_k$ denote the infinite face sharing the rung [1, 5] with $F_1$. 

[Diagram of Figure V.24]
Case 1: Suppose that vertex 3 = vertex i. If vertex 7 lies on the arc [6, 20] then $F_3, G_2, F_k, \ldots, F_4$ contains an O-arc disjoint from $N(E_1, E)$; as shown in Figure V.25. Thus, if vertex 7 ≠ vertex j, then vertex 7 lies on [20, 21] or on [21, j].

If vertex 7 = vertex j, then letting [x, y] represent the second rung bounding $F_3$, the arc [22, 1] contains the vertex x (See Figure V.26). If y lies on [3, 18] the resulting graph is a supergraph of Harold (delete edge [16, 22]). If y lies on [18, 19] the graph produced is George, one of the edge-critical graphs of Appendix B. Finally y cannot lie on [19, 4] for such a graph contains a weight three O-arc through $F_2, G_2, \text{ and } F_3$. Since these are assumed to be faces on $\nu$ no $\Gamma$ can contain this configuration.
Suppose 7 lies on [21, j], then vertex 4 cannot lie on [3, 18] since such a graph contains an O-arc through $F_1$, $G_2$, $E_2$, and $E_1$ disjoint $N(F_3, G_3)$. Nor does vertex 4 lie on [18, 19], as then $\Gamma$ contains the graph of Figure V.27 below as a subgraph. But the graph of Figure V.27 is a supergraph (deleting [14, 17]) of the graph Harold from Appendix B. Finally vertex 4 cannot lie on [19, 5] for then $\Gamma$ would contain a weight three O-arc through $F_2$, $G_2$, and $F_3$. 
Finally vertex 7 must lie on \([ 20, 21 ]\) and since \(F_3\) meets \(E_3\), vertex 4 lies on \([ 3, 18 ]\). Thus \(G\) contains the graph of Figure V.28 as a subgraph. However, the graph of Figure V.28 is seen to be a supergraph (delete \([ 16, 22 ]\)) of Harold.

**Case 2:** Vertex 8 = j. Symmetrically with the Case 1 we find no minimal graphs \(\Gamma\) except George.
Case 3: In the final case vertex $3 \neq i$ and vertex $8 \neq j$. By the choice of vertices $i$ and $j$ vertices 3 and 8 fall on $[2, 17]$ and $[22, 1]$, respectively. The configuration is illustrated in Figure V.29. If the face $F_3$ meets the face $E_4$ then $\Gamma$ contains a weight 3 O-arc through $F_3, E_4$ and $E_1$ and thus embeds in $T$. If, on the other hand, $F_3$ does not $E_4$ then none of $F_k, F_{k-1}, ..., F_5, E_2, E$ nor $E_4$ are contained in $N(F_2, F_3)$, so one may find an O-arc disjoint from $N(F_2, F_3)$. 
This contradiction completes the examination of the three split end configuration.
§ V.6 Central 4-gon and Four Split Ends

We turn our attention now to the case of four split ends. Let Figure V.30 below fix the notation for the vertices of $P$. Note that each face $G_i$ must have at least one rung attached to its boundary and thus is incident with at least two distinct infinite faces. For each of the vertices 17, 18, ..., 24 we will denote by $f_j$ that infinite face of $\nu$ that contains $j$ on its boundary. Note, that while $f_{18} \neq f_{19}$, it may well be true that $f_{19} = f_{20}$.

\[ \text{Figure V.30} \]

**Case 1:** For each $j$, $f_j \neq f_{j+1}$.

In particular we may take $F_1 = f_{17} \neq f_{18}$. Let $F_1$ be bounded by rungs $[1, k+1]$ and $[2, k+2]$; then 2 must lie on $[17, 18]$ as shown in Figure V.31 below.
In addition to $F_1$ there are at least three distinct infinite faces: $f_{18}$, $f_{19}$, $f_{20}$. Since $F_1$ meets $E_1$, the vertices $k + 1$ and $k + 2$ on the interval $[20, 23]$. If $F_1$ meets $[22, 23]$ then $G_4$, $E_4$, $G_1$ and $F_1$ contain an O-arc of $\Pi$ disjoint from $N(E_2, G_2)$. Thus $k + 1$ and $k + 2$ lie on the interval $[20, 22]$. Symmetrically, $f_{19}$ must attach along the interval $[22, 24]$; $f_{20}$ along $[23, 25]$; etc. Not both ends of $F_1$ can lie on $[20, 21]$, since then the $N(F_1, G_1)$ contains none of $f_{19}$, $G_4$, $E_3$, $E_2$ and $G_2$ whose union contains an O-arc of $\Pi$. Thus one of $k + 1$ and $k + 2$ lies on $[21, 22]$. Either $F_1 = f_{21}$ or both of $k + 1$ and $k + 2$ lie on $[21, 22]$. Suppose the latter. Then two vertices of $f_{21}$ must be lie $[24, 1]$. This, however, is impossible (taking $f_{21}$ as a new "$F_1$" in the argument above). Thus $F_1 = f_{17} = f_{21}$. The
argument is completely general; successively taking $F_1 = f_{18}$, $F_1 = f_{19}$, and $F_1 = f_{20}$ establishes the identities; $f_{18} = f_{22}$; $f_{19} = f_{23}$ and $f_{20} = f_{24}$.

The subgraph of $\Gamma$ so produced is the graph Kyle of Appendix B.

**Case 2:** Suppose there is at least one infinite face meeting consecutive $G_i$'s.

WOLOG $F_1 = f_{17} = f_{18}$, as illustrated in Figure V.32.

![Figure V.32](image)

Vertices $k+1$ and $k+2$ of $F_1$ must lie on the interval $[20, 23]$. Not both may lie on $[20, 21]$ else there is an O-arc through $F_2$, $F_3$, $F_k$, $G_3$ disjoint from $N(E, E_1)$. Symmetrically not both may be on $[22, 23]$. Neither may both of $k+1$ and $k+2$ lie on $[21, 23]$. If so then $N(E_2, G_3)$ is disjoint from $F_1$, $G_4$, $E_4$. 
and $G_1$ whose union contains an O-arc. Symmetrically, $k + 1$ and $k + 2$ may not both be confined to $[20, 22]$. This forces the subgraph of Figure V.33 below:

If $f_{19} = f_{20} = F_1'$ then $F_1' = f_{23} = f_{24}$ and all other infinite faces meet opposite pairs of $G_i$'s. WOLOG let $F$ be such a face meeting $G_2$ and $G_4$. If $F_1$ and $F_1'$ are adjacent along $[1, k+1]$ then $\Gamma$ contains an O-arc of weight 3 through $F_1$, $F_1'$ and $E_2$. Thus there is an infinite face $F'$ meeting $G_1$ and $G_2$, thus $G$ contains as a subgraph the graph of Figure V.34 below. The graph of Figure V.34 is however a supergraph of Harold as may be seen by deleting $[1, 8]$ and $[2, 3]$. 
Thus \( f_{19} \neq f_{20} \) and \( f_{23} \neq f_{24} \). Denote by \( F_2 \) that infinite face that shares the edge \([ 2, k + 2 ]\) with \( F_1 \). Suppose \( F_2 \neq f_{19} \). Then \( N(F_2, G_3) \) does not include any of \( F_k, G_3, E_2, E_1, \) and \( G_1 \); which contains an O-arc of \( \Pi \), so that \( F_2 = f_{19} \).

Symmetrically, \( F_2 = f_{23} \) and \( F_k = f_{20} = f_{24} \). Since \( k \geq 4 \), \( F_2 \) and \( F_k \) are not adjacent and \( \Gamma \) contains the graph Fred.
§ V.7 Central 5-gons with Frizzies

In the remaining two sections we assume that $\nu: \Gamma \rightarrow \Pi$ has no faces whose boundary is a 4-gon.

We start with some observations:

**Proposition:** No infinite face meets three consecutive $E_i$'s.

**Proof:** Suppose for some $j$, $F_j$ meets three consecutive faces $E_{i-1}, E_i$, and $E_{i+1}$ i.e. that the interval $[R_{i-1}, L_i, R_i, L_{i+1}]$ is contained on an arc $[j, j + 1]$ of $F_j$. Since $F_j$ meets all of $E_{i-1}, E_i$, and $E_{i+1}$ the arc $[k + j, k + j + 1]$ is contained in the interval $[R_{i+2}, L_{i-2}]$. However $F_2, F_3, \ldots F_k$ and the planar region $T$ containing the cycle $[R_{i+1}, k + j + 1, k + j + 2, L_{i-2}, a_{i+2}]$ contains an O-arc of $\Pi$ disjoint from $N(E, E_i)$. 
Proposition: Suppose for some $i$, $R_i \neq L_{i+1}$. Let $T$ be the region in $\Delta$ bounded by the cycle $[R_i, L_{i+1}, a_i]$, then $T$ meets at least two infinite faces.

Proof: If $T$ meets only one infinite face $F$ then $F$ meets both of $E_i$ and $E_{i+1}$. For some face $T'$ of $v$ in $T$ that meets $F$, $N(F, T')$ is contained a contractible neighborhood of $F$, contradicting the Theta Lemma.

We proceed with the main business of this section with the following conventions. Suppose that for some $i$, $R_i \neq L_{i+1}$. Suppose further that the region bounded by the cycle $[R_i, L_{i+1}, a_i]$ is not a face of $v$. By the results of § V.1 each infinite face $F_1, F_2, ...F_k$ of $\Gamma$ meets $T$ exactly once and $k = 4$ or $5$. The case of $k = 5$ is illustrated below.

Figure V.36
If $E_i$ for $i = 2, 3, 4$ meets only the infinite faces $F_2, F_3,$ and $F_4,$ then there is an $O$-arc through $F_1, F_5$ and $T$ disjoint from $N(E, E_i)$. If $L_{i+1} \neq R_i$ for any further choice of $i,$ the resulting region is a face. Since by hypothesis all faces have at least five sides, any such region must contain the ends of two rungs of $M$. We now consider the placement of vertex 1.

**Case 1:** Suppose vertex 1 lies on the arc $[R_4, L_5]$ then:

i) So must vertex 10 or $v$ has a face of degree 4. Take $v_5 = v$ vertex 21, $L_5 = v$ vertex 20, and $R_4 = v$ vertex 19.

ii) If vertex 9 is not on $[19, 10]$ then one may find an O-arc through $F_5, E_1, E_5,$ and the region bounded by the cycle $[19, 10, 1, 20, 21]$ disjoint from $N(F_2, F_3)$. The resulting configuration is illustrated in Figure V.37 below.

![Figure V.37](image-url)
But now $E_4$ can at most meet $F_2$ and $F_3$ which contradicts the observation immediately preceding case 1. Hence forth we shall note that $R_4 = L_5 = \text{vertex 19}.$

**Case 2:** Symmetrically vertex 6 cannot lie on $[R_1, L_2]$. Henceforth we shall note that $R_1 = L_2 = \text{vertex 20}.$

**Case 3:** Suppose vertex 1 lies on $[L_4, 19]$, Then:

i) Vertex 9 lies on $[L_4, 1]$ else there is an O-arc through $F_5, E_1, E_5,$ and $E_4$ disjoint from $N(F_2, F_3)$.

ii) Since vertex 10 lies between vertices 1 and 9, vertex 10 lies on $[L_4, 1]$. Thus, as shown in Figure V.38 below neither $F_1$ nor $F_5$ can meet $E_3$ and we have a contradiction as in Case 1.

![Figure V.38](image-url)
Case 4: Symmetrically with case 2 vertex 6 cannot lie on \([20, R_2]\).

Case 5: If vertex 1 lies on \([R_3, L_4]\) and vertex 6 lies on \([R_2, L_3]\). Clearly then neither \(F_1\) nor \(F_2\) can meet \(E_3\) and we have our contradiction as in Case 1.

Suppose now that \(k = 4\).

We note that each of the faces \(E_2, E_3, E_4\) must meet one of \(F_1\) or \(F_4\). Further that none of these faces \(E_i, i = 2, 3, 4\) can meet only \(F_1\) (or \(F_4\)) as this permits an O-arc through \(F_4, T, F_2,\) and \(F_3\) (or through \(F_1, T, F_3,\) and \(F_2\)) disjoint from \(N(E, E_i)\). To index our cases we consider the position of vertex 1.

Case 1: Suppose vertex one lies on \([R_4, L_5]\).

i) Set \(a_4 = \text{vertex } 19, L_5 = \text{vertex } 20,\) and \(R_4 = \text{vertex } 21.\)

ii) If the region bounded by \([19, 21, 1, 20]\) is not a face then vertices 8 and 7 must lie on \([8, 7]\). If the region is a face then vertex 8 must lie on \([21, 1]\) since \(v\) has no faces of valence 4. The resulting configuration is displayed below.
Note that neither of faces $F_1$ or $F_4$ can meet $E_4$ since both meet a neighboring face. But $E_4$ must meet one of them. Thus vertex 1 does not lie on $[R_4, L_5]$. We note that $L_5 = R_4 = \text{vertex } 19$.

Case 2: Symmetrically, 5 does not lie on $[R_1, L_2]$. We note that $R_1 = L_2 = \text{vertex } 20$.

In the remaining case, vertex 1 lies on $[L_4, R_4]$ and vertex 5 on $[L_2, R_2]$. (Note that $E_4$ cannot meet only $F_1$, and $E_2$ cannot meet only $F_4$.)
Looking at vertex 1, one sees that vertex 6 must lie on \([5, L_3]\) lest opposite ends of \(F_1\) meet \(E_4\) and \(E_3\). Looking at vertex 5, one sees that vertex 2 must lie on \([R_3, 1]\) lest opposite ends of \(F_4\) meet \(E_2\) and \(E_4\). But now \(E_3\) meets only \(F_2\) and \(F_3\), supplying the final contradiction to \(T\) containing two faces.
§ V.8 Central 5-gons With More Than One Split Ends

We may hence forth assume that the only faces of $v$ are the infinite faces $F_i$, $1 \leq i \leq k$, the central face $E$, the five faces adjacent to $E$, and possibly as many as five faces $G_i$ formed if $R_i \neq L_{i+1}$. Suppose that for some $i$, $R_i \neq L_{i+1}$. Since the smallest face of $v$ has valence 5, on each such interval $[R_i, L_{i+1}]$ there must be two additional vertices $j$ and $j+1$ forming the end of some infinite face $F_j$. In the diagram below such a region $G$ is drawn. $F'$ and $F''$ denote the infinite faces that meet both $G$ and $E_1$ or $G$ and $E_2$, respectively. They may or may not share an edge with $F_1$.

![Figure V.41](image-url)
In order that $F_1$ have at least 5 edges, $[k + 1, k + 2]$ must meet one of the $L_i$'s or $R_i$'s. If $[k + 1, k + 2]$ meets the interval $[R_{2}, L_{2}]$ $([R_{5}, R_{1}])$, then there is an $O$-arc of $\Pi$ disjoint from $N(E_4, E_5)$ ($N(E_4, E_3)$). Thus the interval $[k + 1, k + 2]$ contains one of $R_3, L_4, R_4, L_5$. From the above observations, $[k + 1, k + 2]$ could not contain both of $R_3$ and $L_4$ ($R_4$ and $L_5$) if distinct. Further $[k + 1, k + 2]$ could not contain both of $L_4$ and $R_4$ without allowing an $O$-arc of $\Pi$ disjoint from $N(E, E_4)$. By symmetry we may assert that $F_1$ contains exactly one of $R_3$ or $L_4$. Suppose in $\Gamma$ there are two triangular faces, $G$ and $G'$. If $G'$ meets $E_2$ and $E_3$ and $F_1$ contains $L_4$ then $\Gamma$ contains the following configuration.

![Diagram](image_url)

Figure V.42

The neighbors of $F_1$ and $F_k$ are $G, G', F_2, F_{k-1}, E_4$ and $E_1$. In order for this
collection to contain an O-arc, \( k = 4 \). This requires that \( R_4 = L_5 \), \( R_5 = L_1 \), \( R_2 = L_3 \) and that there be a fifth vertex, vertex 7 on the boundary of \( E_5 \). Vertex 3 may be placed on the boundary of \( E_2 \) or \( E_3 \); either choice yields an isomorphic graph.

If \( G' \) meets \( E_2 \) and \( E_3 \) and \( F_1 \) contains \( R_3 \), then \( G \) contains the configuration illustrated in Figure V.44. Since \( [2k, 1] \) meets \( G \) note that \( [k -1, k] \) cannot meet \( E_2 \). Note also that since \( E_5 \) must have five edges vertex \( 2k \) cannot lie on the boundary of \( E_4 \). Now if \( k > 5 \) we may find an O-arc through \( F_3, E_2, E, E_4 \) and
possible $E_{k-2}, \ldots, E_4$ disjoint from the neighbors of $F_1$ and $F_k$. If $k = 4$ then $L_1 = R_5; L_5 = R_4$; and vertex 8 must lie on $[L_5, R_5]$; however there is no legal placement of vertex 4. If $k = 5$, $R_2 = L_3; R_4 = L_5; R_5 = L_1$ and vertex 10 must lie on $E_5$. In order to prevent the O-arc of the preceding argument $F_3$ cannot meet $E_4$, and so vertex 9 must lie on $[8, L_4]$. But now there is no satisfactory placement of vertex 5.

![Figure V.44](image)

In the third of our cases $G' \neq G$ and $\Gamma$ contains the graph of the following figure as a subgraph. We show the known rungs and other vertices. It is the graph known as Harold of Appendix B.
§ V.9 Central 5-gons With at Most One Split End

We now assume that the representation $\Gamma = P \cup M$ has at most one face not adjacent to $E$. Consider then a graph with exactly one such region $G$. Any such graph has the graph of the diagram below as a subgraph. (In the remainder of this chapter we shall violate our previous conventions and label opposite ends of a rung with consecutive integers.)

![Diagram](figure-v.46)

Figure V.46

There is an infinite face $F$ that meets vertex 5. Let its rungs be $[17, 18]$ and $[x, x']$, with vertex $x$ on $[4, 5]$ and vertex 17 on $[5, 6]$. Note that $x$ need not be distinct from vertex 16. Since $F$ meets $E_5$ and $E_4$ with one of its ends, the other end
may not meet $E_1$ or $E_2$. Thus both vertices 17 and $x'$ must lie on $[3, 13]$. Again note that $x'$ need not be distinct from 15.

Case 1: Vertex 18 lies on $[15, 2]$. The configuration is illustrated in Figure V.47.

Consider the infinite face $F'$ that meets vertex 3. It meets $E_2$ and $E_3$ so each of its rungs must have a vertex that lies on $E_5$. The rung $[2, 17]$ could be a rung of $F$, but $[13, 14]$ could not. This implies the existence in $\Gamma$ of a fourth rung $[19, 20]$ with vertex 19 on $[14, 3]$ and vertex 20 on $[17, 6]$. In the resulting embedding of $\Gamma$ in $\Pi$, we may identify:
\( \alpha \) The face \( F_1 \), which meets \( G \) on \([ 13, 15 ]\) and contains vertex 4 in its boundary.

\( \beta \) A face \( F_2 \) that meets \( G \) in a subarc of \([ 15, 2 ]\) and contains vertex 5.

\( \gamma \) A face \( F_3 \) that contains vertex 2 in its boundary and meets \( E_5 \) on a subarc of \([ 17, 20 ]\). and,

\( \delta \) A face \( F_4 \) that contains vertex 1 on its boundary and meets \( E_3 \) on a subarc of \([ 19, 14 ]\).

The resulting configuration is illustrated in Figure V.48 below.

If \( F_3 \) and \( F_4 \) are adjacent there is a weight three O-arc through \( F_4 \), \( G \), and \( F_3 \). Since the only additional edges in \( \Gamma \) are further rungs of \( M \) there must be an additional
rung splitting $F_3$ (or symmetrically splitting $F_4$). This however reduces Case 1 to Case 2 below.

**Case 2:** Vertex 18 lies on $[2, 3]$. The configuration is illustrated in Figure V.49.

Consider the infinite face $F'$ that meets vertex 3. It meets $E_2$ and $E_3$ so each of its rungs must have a vertex that lies on $E_2$, which as before forces the existence of a rung $[19, 20]$ with vertex 19 on $[3, 14]$ and vertex 20 on $[17, 6]$. The resulting graph (illustrated in Figure V.50) is an isomorph of the graph $Ia$ of Appendix B. Using the pentagon $[3, 18, 17, 20, 19]$ as the central face an isomorphism from the graph of Figure V.50 to $Ia$ is:
Finally consider a girth five graph $\Gamma$ for which $v$ has no faces other than the infinite faces $F_j$, the central face $E$, and the faces $E_1, E_2, E_3, E_4, E_5$ that are adjacent to $E$. Figure V.51 shows $P \cup S$ and fixes the labeling of ten vertices of $\Gamma$. As earlier in this section we abandon our previous conventions on naming rungs and label opposite vertices with consecutive integers.
Consider the infinite face $F_1$ that meets vertex 1. Let its rungs be $[11, 12]$ and $[13, 14]$ with vertex 1 contained in $[11, 13]$. Vertex 11 must lie on the interval $[5, 1]$ since $E_1$ must have at least five vertices. Likewise vertex 13 lies on $[1, 2]$. Since $F_1$ meets $E_1$ and $E_2$ both of vertices 12 and 14 lie on $[3, 4]$.

Let $f_2$ be the infinite face that meets vertex 2. The two rungs must have one vertex on the boundaries of $E_2$ and $E_3$ and the other on the boundary of $E_5$. Since vertex 14 lies on the boundary of $E_4$, $f_2$ must be bounded by two rungs $[15, 16]$ with vertex 15 on $[13, 2]$ and $[17, 18]$ with vertex 17 on $[2, 3]$. The resulting configuration is shown in Figure V.52. Finally let the infinite face containing vertex 3 be $f_3$. The rungs bounding $f_3$ must each have one vertex on the boundary of $E_1$. This suffices to guarantee the existence of a rung $[19, 20]$ with vertex 19 on $[17, 3]$ and vertex 20 on $[5, 11]$. The subgraph of $G$ on these twenty vertices is the graph $I_{an}$ of Appendix B.
This completes the generation of graphs $\Gamma$ that embed in $\Pi$ and fail edge-critically to embed in the torus $T$. 
Chapter VI
Non Projective Planar Graphs

It follows from the results of Chapters III, IV and V that, if $\Gamma$ is an edge-critically nontoroidal cubic graph that is not included in the catalogues of Appendices A and B then $\Gamma$ satisfies the following properties:

1. $\Gamma$ is triangle free.
2. $\Gamma$ has no nontrivial 3-separation.
3. $\Gamma$ contains no subgraph homeomorphic to the disjoint union of $K_{3,3}$ and $K_{2,3}$.
4. $\Gamma$ does not embed in the Projective Plane.
5. $\Gamma$ does not embed (edge-critically) in the Torus.

§ VI.1 Recursive Enumeration

The fourth condition allows us to conclude that $\Gamma$ is a supergraph of one of the five connected cubic irreducibly non-projective planar graphs (See Glover and Huneke [4]). Each of the five non-projective planar graphs is cubic, triangle-free, and 2-connected. Except for $F_{11}$, each is 3-connected. We treat $F_{11}$ separately. Since for the other four graphs $A$, both of $A$ and $\Gamma$ are 3-connected and cubic we may construct $\Gamma$ from $A$ by successively adding edges to form intermediate graphs $B_0 = A, B_1, B_2, ..., B_n = \Gamma$. (See Tutte [11]). Each of the graphs $B_i$ may be taken to be 3-connected. It is convenient to suppose each of the intermediate graphs
to be triangle free; a convenience justified by the lemma below.

**Extension Lemma:** Suppose $A$ is a 2-connected topological proper subgraph of a connected triangle-free cubic graph $\Gamma$. Suppose further that $\Gamma$ has no nontrivial 3-separations, then there exists a topological subgraph $A'$ of $\Gamma$ that is a homeomorph of $A$ and an arc $P$ of $\Gamma - A'$ such that $A' \cup P$ has girth at least four.

**Proof:** Assume not. Since $A \neq \Gamma$, so there is at least one arc, $\alpha$, contained in $A - \Gamma$ with end points in $A$. Further one $\alpha$ attaches to different edges of $A$ else $\Gamma$ is 2-separated. Since by hypothesis $A \cup \alpha$ contains a triangle, $\alpha$ meets $A$ on adjacent edges $[u, x]$ and $[w, x]$ of $A$; as shown below. Let the third vertex of $\Gamma$ adjacent to $x$ in $A$ be denoted by $v$, and recall that $St_A(x)$ equals the union of $[x, u]$, $[x, v]$ and $[x, w]$.

![Figure VI.01](image)
There must be some bridge of \( G - A \) that has vertices of attachment on \( St_A(x) \) and off \( St_A(x) \) else \( \{ u, v, w \} \) is a nontrivial 3-separation of \( \Gamma \). So there is an arc \( \beta \) in \( \Gamma \) with one end on \( St_A(x) \) and one end elsewhere. One end of \( \beta \) lies on one of \([ x, u]\), \([ x, v]\), or \([ x, w]\). We denote this vertex by \( c \). Since by hypothesis, \( \Gamma \cup \beta \) has girth at most 3, the other end of \( \beta \) denoted by \( d \) lies on an edge of \( A \) coincident with \( u, v, \) or \( w \) as necessary.

Case 1: Suppose \( c \) lies along the arc \([ x, b]\) as in the figure below:

![Figure VI.02](image)

Forming \( A - [v, b, c] \cup \beta = A' \), and \( P = \alpha \cup [v, b] \) we obtain a homeomorph of \( A \) and an arc of \( \Gamma - A' \) for which \( A' \cup P \) contains no triangles.

Note that any vertex of attachment of the bridge containing \( \alpha \) that lies off \([ x, u]\) and \([ x, w]\) supplies an arc \( P \) such that \( A \cup P \) contains no triangles, contrary to our
Suppose \([ y, z ]\) to be an edge of \(A\) and let an arc of \(\Gamma - A\) that attaches to \([ y, z ]\) and a second edge of \(A\) be called \(y\)-phillic or \(z\)-phillic as its other end attaches to \(St_A(y)\) or \(St_A(z)\). Along \([ x, u ], [ x, v ], [ x, w ]\) let \(u', v', w'\) be the last (from \(u, v, w\) respectively) \(u\)-, \(v\)-, \(w\)-phillic vertex and \(u'', v'', w''\) be the first (from \(u, v, w\) respectively) \(x\)-phillic vertex. By hypothesis \(A \neq \Gamma\) so there is an \(x\) for which \(x \neq u''\) and \(u \neq u'\). If any of \(u', v'\) or \(w'\) separate \(x\) and \(u''\) \(v''\) or \(w''\) respectively then we may find arcs \(a\) and \(b\) as in Case 1 above. Otherwise \([ u', v' w' ]\) supplies a nontrivial 3-separation of \(\Gamma\); contrary to hypothesis. This contradiction completes the proof of the Extension Lemma.

We now proceed to examine the extensions of the five cubic edge-critically non-projective planar graphs: \(F_{11}, F_{12}, F_{13}, F_{14}\), and \(G\). The graph \(G\) plays a central role in the process as most extensions of the other four graphs contain \(G\) as a subgraph. We display a representation of \(G\):

![Figure VI.03](image-url)
§ VI.2 Extensions of $F_{11}$.

Two representations of $F_{11}$ are shown below. In the right hand representation the three orbits of the edges of $F_{11}$ are indicated by the labels A, B, C.

![Diagram](image)

Figure VI.04

Single edge extensions of $F_{11}$ may be specified by detailing two edges of $F_{11}$ to be "split". Thus $F_{11} + 1,2;6,7$ will denote the graph formed from $F_{11}$ by deleting edges $[1,12]$ and $[6,7]$, adding two new vertices 13 and 14 and new edges $[12,13]$, $[1,13]$, $[13,14]$, $[6,14]$ and $[7,14]$. This graph is shown below.
Note that $F_{11} + 1, 2; 6, 7$ contains a homeomorph of $K_{3,3}$ -- the induced subgraph on \{ 1, 2, 3, 4, 5, 6, 13, 14 \} -- disjoint from the theta on \{ 7, 8, 9, 10, 11 \}; so that no $\Gamma$ contains $F_{11} + 1, 2; 6, 7$ as a subgraph. Our interest in extensions of $F_{11}$ is to find those that may be intermediate to some $\Gamma$ as described above. Since each such $\Gamma$ is 3-connected and triangleless by the Extension Lemma the extensions of $F_{11}$ of interest to us are the 3-connected triangleless ones. The 3-connected extensions of $F_{11}$ that do not contain $F_{11} + 1, 12; 6, 7$ must either include a path from the induced subgraph on \{ 1, 2, 3, 4, 5, 6 \} or be homeomorphic to the graph below:
That this graph contains $G$ may be seen by deleting the edges $[9, 10], [3, 4]$ and $[13, 15]$. Up to homeomorphism the remaining 3-connected triangleless extensions of $F_{11}$ are shown below. Three contain $G$ while the fourth, $F_{11}+ 1,2; 7,10$, formed by splitting a class B and a distant class B edge, contains a homeomorph of $K_{3,3}$ induced on vertices $\{1, 7, 8, 9, 10, 11, 12, 13, 14\}$ that is disjoint from theta induced on vertices $\{2, 3, 4, 5, 6\}$.

<table>
<thead>
<tr>
<th>Representative</th>
<th>Description</th>
<th>G obtained by deletion</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_{11} + 1,2; 9,12$</td>
<td>Class B and nearby class B edges.</td>
<td>$[1, 13]$ and $[12, 14]$</td>
</tr>
<tr>
<td>$F_{11} + 1,2; 8,11$</td>
<td>Class B and class C edges</td>
<td>$[1, 13]$ and $[9, 10]$</td>
</tr>
<tr>
<td>$F_{11} + 2,5; 8,11$</td>
<td>Class C and class C edges</td>
<td>$[3, 4]$ and $[9, 10]$</td>
</tr>
</tbody>
</table>

Below are shown embeddings of each of the three pertinent graphs.
Figure V.07

Figure VI.08
§ VI.3 Extensions of $F_{12}$

Two representations of $F_{12}$ are shown below. In the right hand representation the three orbits of the edges of $F_{12}$ are indicated by the labels A, B, C.

![Diagram](image)

Figure VI.09

Below are listed the topologically distinct single edge extensions of $F_{12}$ that contain neither a triangle nor a subgraph homeomorphic to the disjoint union of $K_{3,3}$ and $K_{2,3}$:

<table>
<thead>
<tr>
<th>Representative</th>
<th>Description</th>
<th>$G$ obtained by deletion</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_{12}+1,2;11,12$</td>
<td>Class A edges within a 4-gon</td>
<td>Not over $G$</td>
</tr>
<tr>
<td>$F_{12}+1,2;7,8$</td>
<td>Class A pair from distinct 4-gons</td>
<td>[1, 9] and [3, 7]</td>
</tr>
<tr>
<td>$F_{12}+1,2;5,8$</td>
<td>Class A pair from distinct 4-gons</td>
<td>[1, 9] and [4, 5]</td>
</tr>
<tr>
<td>$F_{12}+1,2;5,6$</td>
<td>Class A pair from distinct 4-gons</td>
<td>[1, 9] and [4, 5]</td>
</tr>
<tr>
<td>$F_{12}+1,2;4,12$</td>
<td>Class A and class B</td>
<td>[2, 3] and [9, 10]</td>
</tr>
<tr>
<td>$F_{12}+1,2;3,7$</td>
<td>Class A and class B</td>
<td>[1, 9] and [11, 12]</td>
</tr>
</tbody>
</table>
An embedding of each of these eleven extensions of $F_{12}$ is given below:

$F_{12}+2,3;4,12$ Class B pair on one 4-gon $[1,9]$ and $[2,11]$

$F_{12}+2,3;4,5$ Class B pair on distinct 4-gons $[1,9]$ and $[2,11]$

$F_{12}+2,3;6,10$ Class B pair on distinct 4-gons $[2,11]$ and $[3,13]$

$F_{12}+2,3;9,10$ Class B and class C

Vertex 12

$F_{12}+3,4;9,10$ Class C pair Not over G

Figure VI.10
§ VI.4 Extensions of $F_{13}$.

Two representations of $F_{13}$ are shown below. In the right hand representation the two orbits of the edges of $F_{13}$ are indicated by the labels A and B.

Below are listed the topologically distinct single edge extensions of $F_{13}$ that contain neither a triangle nor a subgraph homeomorphic to the disjoint union of $K_{3,3}$ and $K_{2,3}$.

<table>
<thead>
<tr>
<th>Representative</th>
<th>Description</th>
<th>$G$ obtained by deletion</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_{13}+1,2;5,6$</td>
<td>Class A pair at distance 1</td>
<td>$[6, 7]$ and $[11, 12]$</td>
</tr>
<tr>
<td>$F_{13}+1,2;6,11$</td>
<td>Class A pair at distance 2</td>
<td>$[6, 7]$ and $[11, 12]$</td>
</tr>
<tr>
<td>$F_{13}+1,2;11,12$</td>
<td>Class A pair at distance 3</td>
<td>$[6, 7]$ and $[12, 14]$</td>
</tr>
<tr>
<td>$F_{13}+1,2;9,10$</td>
<td>Classes A and B in 5-gon</td>
<td>$[8, 12]$ and $[10, 14]$</td>
</tr>
<tr>
<td>$F_{13}+1,2;4,5$</td>
<td>Classes A and B in 5-gon</td>
<td>$[2, 3]$ and $[6, 11]$</td>
</tr>
</tbody>
</table>
Below are shown an embedding for each of the nine graphs listed.

\begin{itemize}
\item $F_{13}+1,2;10,11$ Classes A and B off 5-gon $[4,5]$ and $[6,7]$
\item $F_{13}+1,2;4,12$ Classes A and B off 5-gon $[1,10]$ and $[2,7]$
\item $F_{13}+2,7;9,10$ Class B pair $[1,5]$ and $[8,9]$
\item $F_{13}+6,7;9,10$ Class B pair $[3,4]$ and $[10,14]$
\end{itemize}

Figure VI.13
Figure VI.14
§ VI.5 Extensions of $F_{14}$.

Two representations of $F_{14}$ are shown below. In the right hand representation the three orbits of the edges of $F_{14}$ are indicated by the labels A, B, and C.

![Diagram of two representations of $F_{14}$](image)

Figure VI.15

Below are listed the topologically distinct single edge extensions of $F_{14}$ that contain neither a triangle nor a subgraph homeomorphic to the disjoint union of $K_{3,3}$ and $K_{2,3}$.

<table>
<thead>
<tr>
<th>Representative</th>
<th>Description</th>
<th>G obtained by deletion</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_{14}+9,11;10,12$</td>
<td>Type A pair</td>
<td>Not over G</td>
</tr>
<tr>
<td>$F_{14}+8,12;9,11$</td>
<td>Types A and B</td>
<td>$[2,3]$ and $[6,10]$</td>
</tr>
<tr>
<td>$F_{14}+7,8;9,11$</td>
<td>Types A and C</td>
<td>$[1,9]$ and $[4,5]$</td>
</tr>
<tr>
<td>$F_{14}+1,9;3,11$</td>
<td>Type B pair from one H</td>
<td>$[1,2]$ and $[6,7]$</td>
</tr>
<tr>
<td>$F_{14}+2,10;3,11$</td>
<td>Type B pair from distinct H's</td>
<td>$[5,9]$ and $[7,11]$</td>
</tr>
<tr>
<td>$F_{14}+2,10;7,11$</td>
<td>Type B pair from distinct H's</td>
<td>$[5,9]$ and $[7,14]$</td>
</tr>
</tbody>
</table>
$F_{14} + 1,2;3,11$ Types B and C at distance 1
$F_{14} + 1,8;3,11$ Types Band C at distance 2
$F_{14} + 1,8;3,4$ Type C pair at distance 2
$F_{14} + 1,8;4,5$ Type C pair at distance 3

Below is shown an embedding for each of the ten graphs listed

Figure VI.16
Figure VI.17
§ VI.6 The four exceptional graphs and the graph G.

In the foregoing catalogues of extensions of $F_{11}$, $F_{12}$, $F_{13}$, and $F_{14}$, with four exceptions, each graph that fails to contain a disjoint union of $K_{3,3}$ and $K_{2,3}$ was also an extension of the graph G. Before reporting the status of supergraphs of G we deal with these four exceptional graphs.

$E_{14}+9,11;10,12$

The reader is asked to observe that the embedding given above for $F_{14}+9,11;10,12$ consists of seven hexagons and establishes $F_{14}+9,11;10,12$ as the toroidal dual of $K_7$. As such, there are two extensions of $F_{14}+9,11;10,12 = K_7^*$ that have girth greater than three -- one of girth four, e.g. $K_7^*+3,4;7,8$, and one of girth five, e.g. $K_7^*+9,13;12,14$ -- and that each embeds in the torus. G is obtained from $K_7^*+9,13;12,14$ by deleting edges $[1,8]$; $[6,10]$; and $[7,1]$. G is obtained from $K_7^*+3,4;7,8$ by deleting edges $[1,8]$; $[6,10]$; and $[7,1]$. G is obtained from $K_7^*+9,13;12,14$ by deleting edges $[1,2]$; $[5,6]$; and $[13,15]$.

$E_{14}+1,8;4,5$:

In the interest of brevity we will denote $F_{14}+1,8;4,5$ by $E_3$. There are six topologically distinct extensions of $E_3$ that are not clearly over one of the other extensions of $F_{14}$. Each of them contains G as a subgraph as indicated below:

<table>
<thead>
<tr>
<th>Extension</th>
<th>Obtain G by deleting</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_3+3,4;7,8$</td>
<td>$[5,14]$; $[3,15]$; and $[9,11]$</td>
</tr>
<tr>
<td>$E_3+3,4;5,14$</td>
<td>$[2,10]$; $[6,7]$; and $[8,12]$</td>
</tr>
<tr>
<td>$E_3+4,14;8,13$</td>
<td>$[2,3]$; $[6,7]$; and $[13,16]$</td>
</tr>
<tr>
<td>$E_3+1,13;4,14$</td>
<td>$[5,9]$; $[4,12]$; and $[13,15]$</td>
</tr>
</tbody>
</table>
The embedding given above for $E_3 = F_{14} + 1,8;4,5$ may be extended to embed the first four graphs in the list. $E_3 + 4,12;5,14$ embeds by reembedding the edge $[13,14]$ in the embedding of $F_{14} + 1,8;4,5$ to split the face $[1,2,10,6,5,14,4,3,11,8,13,1]$. An embedding for $E_3 + 2,3;6,7$ is shown below.

The graph $E_3 + 1,2;11,12$ contains a homeomorph of $K_{3,3}$ on

$\{3,4,5,6,7,8,9,10,13,14\}$ disjoint from the $\theta$-graph on vertices $\{1,2,11,12,15,16\}$.
The graph $E_{2+3,13;9,14}$ contains $G$ as is seen by deleting edges $[1, 12]; [6, 7]; [13, 15]$.

The graph $E_{2+3,13;10,14}$ contains $G$ as is seen by deleting edges $[1, 2]; [5, 6]; [13, 15]$.

$E_{12+1.2;11,12}$:
There are six graphs that arise as extensions of $F_{12+1,2;11,12} = E_1$ that are not obviously over one of the other extensions of $F_{12}$. Each of these six contains a subgraph isomorphic to the disjoint union of $K_{3,3}$ and $K_{2,3}$. The list of extensions of $E_1$ is:

<table>
<thead>
<tr>
<th>Representative</th>
<th>$K_{3,3}$ induced on vertices</th>
<th>$K_{2,3}$ induced on vertices</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_1+1,13;12,14$</td>
<td>${2,3,4,5,6,7,8,9,10,11}$</td>
<td>${1,12,13,14,15,16}$</td>
</tr>
<tr>
<td>$E_1+1,13;11,14$</td>
<td>${1,2,3,4,11,12,13,14,15,16}$</td>
<td>${5,6,7,8,9,10}$</td>
</tr>
<tr>
<td>$E_1+1,12;3,11$</td>
<td>${1,2,9,10,11,12,13,14,15,16}$</td>
<td>${3,4,5,6,7,8}$</td>
</tr>
<tr>
<td>$E_1+5,6;7,8$</td>
<td>${1,2,3,4,9,10,11,12,13,14}$</td>
<td>${5,6,7,8,15,16}$</td>
</tr>
<tr>
<td>$E_1+5,8;6,7$</td>
<td>${3,4,5,6,7,8,9,15,16}$</td>
<td>${1,2,11,12,13,14}$</td>
</tr>
<tr>
<td>$E_1+1,12;13,14$</td>
<td>${2,3,4,5,6,7,8,9,10,11}$</td>
<td>${1,12,13,14,15,16}$</td>
</tr>
</tbody>
</table>

Summarizing the results of the preceding catalogue we find that every remaining edge-critically nontoroidal graph must contain a homeomorph of that graph $G$. The readers might wish amuse themselves by discovering a homeomorph of $G$ in each of the 3-separated graphs in Chapter III. The preceding observations are summarized in the following:
Proposition: Every edge-critically non-toroidal graph which is not projective planar contains a subgraph homeomorphic to \( G \).

There are a plethora of graphs that contain \( G \) topologically. All distinct simple triangle free supergraphs of \( G \) which contain no subgraph homeomorphic to the disjoint union of \( K_{3,3} \) and \( K_{2,3} \) on 12 and 14 vertices (they number 4 and 48 respectively) embed in the torus. An undirected search through the upper ideal of topological supergraphs of \( G \) to determine all remaining edge-critically nontoroidal graphs might be possible but would be surely uninformative. An attempt has been made to recover all of the remaining sixteen vertex edge-critically nontoroidal graphs by exploiting some of the structure imposed by the known restrictions.

Any edge-critically nontoroidal graph \( \Gamma \) that lies over \( G \) must contain a pair of disjoint \( k \)-graphs. In those graphs \( \Gamma \) of interest to us there is no subgraph homeomorphic to the disjoint union of \( K_{3,3} \) and \( K_{2,3} \). The \( k \)-graphs in \( G \) are separated by a bond of at least 3 edges. Those seven graphs in which there is a bond of exactly three edges are described in Chapter II and catalogued in Appendix A. Appendix C catalogues the eleven graphs found by examining those graphs that result from connecting small disjoint graphs with 4-, 5-, and 6-bonds. Appendix A supplies a complete catalogue of the remaining edge-critically nontoroidal graphs on 16 vertices.

It is of interest to note that the "well known" example of an edge-critically nontoroidal graph on 18 vertices formed by adjoining an edge to the unique girth six graph on 16 vertices may be viewed as a pair of \( \theta \)-graphs joined by a 7-bond.
Figures IV.18 and IV.19 give two representations of this graph.
Appendix A

The ensuing figures give a representation of each of the seventeen distinct edge-critically nontoroidal cubic graphs that have either:

1. a 0-separation: one graph on 12 vertices or

2. a 2-separation: three graphs on 14 vertices or

3. a nontrivial (i.e. not a vertex bond) 3-separation: six graphs containing $K_{3,3} \cup K_{2,3}$ on 14 vertices and seven graphs 16 vertices.

A microcomputer analysis (see Weaver [13]) has shown all seventeen to be distinct.
0\,-\text{Separated Graph:}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure_a.01}
\caption{Decker IIB2.4}
\end{figure}

\textbf{2\,-\text{Separated Graphs:}}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure_a.02}
\caption{Decker IIB2.5}
\end{figure}
3 - Separated Graphs:

Decker IIB2.6

Figure A.04

Decker IIC3.12

Figure A.05

Decker IIC3.14

Figure A.06
Non-Decker Graphs:

Figure A.10

Decker IIC3.19

Figure A.11
Graph $S_1^{**}$

Figure A.12

Graph $S_2^{**}$

Figure A.13
Graph $S_{3}^{**}$

Figure A.14

Graph $S_{4}^{**}$

Figure A.15
Graph $S_{5}^{**}$

Figure A.16

Graph $S_{6}^{**}$

Figure A.17
Appendix B

The following pages contain a projective planar drawing of each of the five projective planar graphs that are edge-critically nontoroidal. The drawing is given so as to display either a central 4-gon or central 5-gon in the planar part \( P \) attached to an exterior Mobius ladder. The labeling of the Mobius ladders is consistent with the conventions of Chapter IV. Thus in the drawing of Ian the rungs of the exterior 5-ladder are \([ 1, 6 ]; [ 2, 7 ]; [ 3, 8 ]; [ 4, 9 ]; [ 5, 10 ]\).

After each drawing is a listing of faces and edges crossed by weight four \( O \)-arcs in the drawing that suffices to show that each of these drawings is edge-critically 4-represented. If an edge is used in more than one \( O \)-arc its second and subsequent occurrences are printed in nine point type. In order to see that these graphs do not have 3-representative drawings it suffices to note the for each of the following graphs the embedding of the Mobius ladder is unique since the planar part is skew (mod the spine) to each of the rungs; and that the union of the planar part and the spine of the Mobius ladder is 3-connected hence has a unique planar embedding.
Name: Ian  Vertices: 20  4-gons: 0  Planar Part: Simple 5-gon

Figure B.01

<table>
<thead>
<tr>
<th>Faces</th>
<th>Weight Four O-arcs</th>
<th>Edges</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_1 \ E_1 \ E \ E_4$</td>
<td>[ 1, 16 ]</td>
<td>[ 11, 15 ]</td>
</tr>
<tr>
<td>$F_5 \ E_1 \ E_5 \ E_4$</td>
<td>[ 1, 10 ]</td>
<td>[ 15, 20 ]</td>
</tr>
<tr>
<td>$F_1 \ E_2 \ E_3 \ F_5$</td>
<td>[ 1, 6 ]</td>
<td>[ 2, 16 ]</td>
</tr>
<tr>
<td>$F_2 \ E_2 \ E_5 \ E_5$</td>
<td>[ 2, 3 ]</td>
<td>[ 11, 12 ]</td>
</tr>
<tr>
<td>$F_1 \ E_1 \ E_5 \ F_2$</td>
<td>[ 1, 16 ]</td>
<td>[ 15, 20 ]</td>
</tr>
<tr>
<td>$F_2 \ F_3 \ E_3 \ E_4$</td>
<td>[ 3, 8 ]</td>
<td>[ 4, 17 ]</td>
</tr>
<tr>
<td>$F_3 \ E_2 \ E_1 \ F_4$</td>
<td>[ 3, 17 ]</td>
<td>[ 11, 16 ]</td>
</tr>
<tr>
<td>$F_4 \ E_3 \ E_5 \ E_5$</td>
<td>[ 4, 5 ]</td>
<td>[ 12, 13 ]</td>
</tr>
<tr>
<td>$F_4 \ F_5 \ E_4 \ E_5$</td>
<td>[ 5, 10 ]</td>
<td>[ 6, 18 ]</td>
</tr>
<tr>
<td>$F_3 \ E_5 \ E \ E_3$</td>
<td>[ 8, 9 ]</td>
<td>[ 14, 15 ]</td>
</tr>
</tbody>
</table>
Name: Harold  Vertices: 20  4-gons: 1  Planar Part: Two split ends

Figure B.02

<table>
<thead>
<tr>
<th>Faces</th>
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</thead>
<tbody>
<tr>
<td>$F_1$</td>
<td>[1, 5]</td>
</tr>
<tr>
<td>$F_2$</td>
<td>[2, 6]</td>
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<tr>
<td>$F_3$</td>
<td>[3, 7]</td>
</tr>
<tr>
<td>$F_4$</td>
<td>[4, 8]</td>
</tr>
<tr>
<td>$F_1 E_4 E_3 G_1$</td>
<td>[1, 16]</td>
</tr>
<tr>
<td>$F_4 E_4 E_2$</td>
<td>[1, 20]</td>
</tr>
<tr>
<td>$F_1 E_1 E_2 G_1$</td>
<td>[2, 16]</td>
</tr>
<tr>
<td>$F_2 E_1 E_3$</td>
<td>[2, 3]</td>
</tr>
<tr>
<td>$F_3 E_1 E_4 G_2$</td>
<td>[3, 13]</td>
</tr>
<tr>
<td>$F_3 E_2 E_3 G_2$</td>
<td>[4, 13]</td>
</tr>
<tr>
<td>$F_4 E_2 E_4$</td>
<td>[4, 17]</td>
</tr>
<tr>
<td>$F_4 G_1 F_2 G_2$</td>
<td>[5, 17]</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>[2, 6]</th>
<th>[3, 7]</th>
<th>[4, 8]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_4$</td>
<td>[11, 15]</td>
<td>[14, 18]</td>
<td>[5, 6]</td>
</tr>
<tr>
<td>$F_3 E_1 E_4 G_2$</td>
<td>[11, 12]</td>
<td>[9, 10]</td>
<td>[4, 17]</td>
</tr>
<tr>
<td>$F_1 E_2 E_3 G_1$</td>
<td>[9, 13]</td>
<td>[14, 17]</td>
<td>[5, 6]</td>
</tr>
<tr>
<td>$F_2 E_1 E_3$</td>
<td>[9, 12]</td>
<td>[10, 11]</td>
<td>[18, 19]</td>
</tr>
<tr>
<td>$F_3 E_1 E_4 G_2$</td>
<td>[12, 16]</td>
<td>[15, 20]</td>
<td>[7, 8]</td>
</tr>
<tr>
<td>$F_3 E_2 E_3 G_2$</td>
<td>[10, 14]</td>
<td>[15, 19]</td>
<td>[7, 8]</td>
</tr>
<tr>
<td>$F_4 E_2 E_4$</td>
<td>[9, 10]</td>
<td>[11, 12]</td>
<td>[1, 20]</td>
</tr>
<tr>
<td>$F_4 G_1 F_2 G_2$</td>
<td>[6, 18]</td>
<td>[7, 19]</td>
<td>[8, 20]</td>
</tr>
</tbody>
</table>
Name: George  Vertices: 22  4-gons: 3  Planar Part: Three split ends

Figure B.03

Weight Four O-arcs

<table>
<thead>
<tr>
<th>Faces</th>
<th>Edges</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_1$</td>
<td>1, 5</td>
</tr>
<tr>
<td>$F_2$</td>
<td>1, 13</td>
</tr>
<tr>
<td>$F_3$</td>
<td>1, 8</td>
</tr>
<tr>
<td>$F_4$</td>
<td>2, 13</td>
</tr>
<tr>
<td>$E_1$</td>
<td>2, 17</td>
</tr>
<tr>
<td>$E_2$</td>
<td>3, 17</td>
</tr>
<tr>
<td>$E_3$</td>
<td>3, 18</td>
</tr>
<tr>
<td>$E_4$</td>
<td>7, 18</td>
</tr>
<tr>
<td>$E_5$</td>
<td>5, 19</td>
</tr>
<tr>
<td>$E_6$</td>
<td>7, 21</td>
</tr>
</tbody>
</table>
Name: Kyle  Vertices: 24  4-gons: 5  Planar Part: Four split ends

Figure B.04

Weight Four O-arcs

<table>
<thead>
<tr>
<th>Faces</th>
<th>Edges</th>
</tr>
</thead>
<tbody>
<tr>
<td>F₁</td>
<td>[1, 5]</td>
</tr>
<tr>
<td>F₂</td>
<td>[1, 24]</td>
</tr>
<tr>
<td>F₃</td>
<td>[1, 17]</td>
</tr>
<tr>
<td>F₄</td>
<td>[5, 20]</td>
</tr>
<tr>
<td>G₁</td>
<td>[5, 21]</td>
</tr>
<tr>
<td>G₂</td>
<td>[3, 18]</td>
</tr>
<tr>
<td>G₃</td>
<td>[3, 19]</td>
</tr>
<tr>
<td>G₄</td>
<td>[7, 22]</td>
</tr>
<tr>
<td>E₁</td>
<td>[7, 23]</td>
</tr>
<tr>
<td>E₂</td>
<td>[8, 24]</td>
</tr>
<tr>
<td>E₃</td>
<td>[2, 17]</td>
</tr>
</tbody>
</table>
Name: Fred  Vertices: 24  4-gons: 6  Planar Part: Four split ends

Figure B.05

Weight Four O-arcs

<table>
<thead>
<tr>
<th>Faces</th>
<th>Edges</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_1$</td>
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</tr>
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</tr>
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</tr>
<tr>
<td>$F_4$</td>
<td>[4, 8]</td>
</tr>
<tr>
<td>$G_1$</td>
<td>[1, 17]</td>
</tr>
<tr>
<td>$E_3$</td>
<td>[13, 17]</td>
</tr>
<tr>
<td>$E_2$</td>
<td>[10, 14]</td>
</tr>
<tr>
<td>$F_1$</td>
<td>[19, 20]</td>
</tr>
<tr>
<td>$G_2$</td>
<td>[2, 17]</td>
</tr>
<tr>
<td>$E_4$</td>
<td>[6, 7]</td>
</tr>
<tr>
<td>$E_1$</td>
<td>[12, 16]</td>
</tr>
<tr>
<td>$E_2$</td>
<td>[19, 20]</td>
</tr>
<tr>
<td>$E_3$</td>
<td>[10, 14]</td>
</tr>
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<td>$G_3$</td>
<td>[2, 17]</td>
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<td>$G_4$</td>
<td>[6, 7]</td>
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<td>[12, 16]</td>
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<td>[23, 24]</td>
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<td>$E_1$</td>
<td>[19, 20]</td>
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<td>$G_2$</td>
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<tr>
<td>$E_3$</td>
<td>[19, 20]</td>
</tr>
<tr>
<td>$E_4$</td>
<td>[19, 20]</td>
</tr>
</tbody>
</table>

Edges
Appendix C

Graphs on 16 Vertices

The ensuing pages give a list of thirteen graphs on 16 vertices that are edge-critically nontoroidal. Each is a supergraph of the non-projective planar graph G. Following each representation is a short argument that the displayed graph fails to embed in the torus. The first six graphs are examples of the "fragmented k-graph" phenomenon as observed in Chapter III with the graph Gena. The remaining graphs at present require ad hoc arguments to show their nontoroidal character. Unfortunately these arguments retain much of their original flavor. That these graphs are all distinct has been verified with a microcomputer program (see Weaver [13]).
Let $K$ be the homeomorph of $K_4$ induced on $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

$K \cup \{3, 14, 13, 7\}$ is a homeomorph of $K_{3,3}$ and disjoint from the cycle $C = \{10, 11, 12, 13, 16, 15\}$, hence any drawing of $\Gamma$ in $T$ must have $C$ null.

$K \cup \{5, 11, 10, 9\}$ is a homeomorph of $K_{3,3}$ and disjoint from the cycle $C' = \{12, 13, 14, 15, 16\}$, hence any drawing of $\Gamma$ in $T$ must have $C'$ null.

Since $C \cup C'$ is a $k$-graph in $\Gamma$, not both of $C$ and $C'$ may be null.
Let $K$ be the homeomorphic of $K_4$ induced on $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

$K \cup [3, 13, 14, 7]$ is a homeomorphic of $K_{3,3}$ and disjoint from the cycle $C = [10, 11, 12, 13, 16, 15]$, hence any drawing of $\Gamma$ in $T$ must have $C$ null.

$K \cup [5, 10, 11, 9]$ is a homeomorphic of $K_{3,3}$ and disjoint from the cycle $C' = [12, 13, 14, 15, 16]$, hence any drawing of $\Gamma$ in $T$ must have $C'$ null.

Since $C \cup C'$ is a $k$-graph in $\Gamma$, not both of $C$ and $C'$ may be null.
Let $K$ be the homeomorph of $K_4$ induced on \{1, 2, 3, 4, 5, 6, 7, 8, 9\}.

$K \cup \{3, 14, 13, 7\}$ is a homeomorph of $K_{3,3}$ and disjoint from the cycle $C = \{10, 11, 12, 13, 16, 15\}$, hence any drawing of $\Gamma$ in $T$ must have $C$ null.

$K \cup \{5, 11, 10, 1\}$ is a homeomorph of $K_{3,3}$ and disjoint from the cycle $C' = \{12, 13, 14, 15, 16\}$, hence any drawing of $\Gamma$ in $T$ must have $C'$ null.

Since $C \cup C'$ is a $k$-graph in $\Gamma$, not both of $C$ and $C'$ may be null.
Let $K$ be the homeomorph of $K_4$ induced on \{1, 2, 3, 4, 5, 6, 7, 8, 9\}.

$K \cup \{3, 13, 14, 7\}$ is a homeomorph of $K_{3,3}$ and disjoint from the cycle $C = \{10, 11, 12, 13, 16, 15\}$, hence any drawing of $\Gamma$ in $T$ must have $C$ null.

$K \cup \{5, 11, 10, 1\}$ is a homeomorph of $K_{3,3}$ and disjoint from the cycle $C' = \{12, 13, 14, 15, 16\}$, hence any drawing of $\Gamma$ in $T$ must have $C'$ null.

Since $C \cup C'$ is a $k$-graph in $\Gamma$, not both of $C$ and $C'$ may be null.
Let $K$ be the homeomorph of $K_4$ induced on $\{1, 2, 10, 11, 12, 13, 14, 15, 16\}$.

$K \cup \{1, 8, 7, 13\}$ is a homeomorph of $K_{3,3}$ and disjoint from the cycle $C = \{3, 4, 5, 6, 9\}$, hence any drawing of $\Gamma$ in $T$ must have $C$ null.

$K \cup \{2, 3, 9, 14\}$ is a homeomorph of $K_{3,3}$ and disjoint from the cycle $C' = \{4, 5, 6, 7, 8\}$, hence any drawing of $\Gamma$ in $T$ must have $C'$ null.

Since $C \cup C'$ is a $k$-graph in $\Gamma$, not both of $C$ and $C'$ may be null.
Let $K$ be the homeomorph of $K_4$ induced on $\{1, 2, 10, 11, 12, 13, 14, 15, 16\}$.

$K \cup \{1, 8, 7, 14\}$ is a homeomorph of $K_{3,3}$ and disjoint from the cycle $C = \{3, 4, 5, 6, 9\}$, hence any drawing of $\Gamma$ in $T$ must have $C$ null.

$K \cup \{2, 3, 9, 13\}$ is a homeomorph of $K_{3,3}$ and disjoint from the cycle $C' = \{4, 5, 6, 7, 8\}$, hence any drawing of $\Gamma$ in $T$ must have $C'$ null.

Since $C \cup C'$ is a k-graph in $\Gamma$, not both of $C$ and $C'$ may be null.
The \( K_{2,3} \) induced on \{ 10, 11, 12, 13, 14, 15, 16 \}, \( \Theta \), is a k-graph of \( \Gamma \). Any toroidal embedding \( \mu \) of \( \Gamma \) must embed \( \Theta \) with an essential cycle. For any \( \mu \), the vertices 1, 3, 5, 7, 9 of the \( K_4 \) induced on \{ 1, 2, 3, 4, 5, 7, 8, 9 \}, \( \Omega \), must lie on one region of \( T - \mu(\Omega) \). Unless \( \mu(\Omega) \) has an essential cycle, no region of \( T - \mu(\Omega) \) meets four edges of \( \Omega \). Since \( \Theta \) and \( \Omega \) are disjoint \( \mu(\Theta) \) and \( \mu(\Omega) \) must contain homototic essential cycles. Any \( \mu \) induces the drawing below:
Vertices 13 and 14 (adjacent to 7 and 3 respectively) lie on one arc of $\Theta$. This arc must be $C$ since neither of arcs $A$ or $B$ lies on a region with both of vertices 3 and 7. Symmetrically $C$ must contain vertices 10 and 11.
The $K_{2,3}$ induced on $\{10, 11, 12, 13, 14, 15, 16\}$, $\Theta$, is a $k$-graph of $\Gamma$. Any toroidal embedding $\mu$ of $\Gamma$ must embed $\Theta$ with an essential cycle. For any $\mu$, the vertices $1, 3, 5, 7, 9$ of the $K_4$ induced on $\{1, 2, 3, 4, 5, 7, 8, 9\}$, $\Omega$, must lie on one region of $T - \mu(\Omega)$. Unless $\mu(\Omega)$ has an essential cycle no region of $T - \mu(\Omega)$ meets four edges of $\Omega$. Since $\Theta$ and $\Omega$ are disjoint $\mu(\Theta)$ and $\mu(\Omega)$ must contain homototic essential cycles. Any $\mu$ induces the drawing below:
Vertices 13 and 14 (adjacent to 5 and 7 respectively) lie on one arc of $\Theta$. This arc must be C since neither of arcs A or B lies on a region with both of vertices 5 and 7. Symmetrically C must contain vertices 10 and 11.
The graph $M$ induced on \{ 1, 2, 3, 4, 5, 7, 8, 10, 11, 13, 14 \} is a homeomorph of $M_4$ with rungs $R_1 = [11, 3, 13]$; $R_2 = [5, 1, 7]$; $R_3 = [8, 2, 10]$; and $R_4 = [16, 4, 14]$ disjoint from the cycle $C = [12, 6, 9, 15]$. The edges connecting $C$ to $M$ joining the vertices of $C$ to the center of edges in $M$ in cyclic order. The only embedding of $M_4$ in which all four rungs are on the boundary of a single face $F$, is the embedding with two null rungs. $C$ must embed in $F$ as a contractible cycle so that $C \cup \text{Bdry}(F)$ is the boundary of an annulus.
The vertices \{1, 2, 3, 4\} occur in the order \[3, 1, 4, 2\] on \text{Bdry}(F). In order to attach to C the imposed order is \[3, 1, 2, 4\].
The graph M induced on \{ 1, 2, 3, 4, 5, 6, 8, 9, 11, 13, 14, 15, 16 \} is a homeomorph of \( M_4 \) with rungs \( R_1 = [6, 1, 5] \); \( R_2 = [9, 2, 8] \); \( R_3 = [11, 3, 13] \); and \( R_4 = [16, 14, 4] \). M's complement is an H and the edge [3, 12]. The H attaches to three rungs -- \( R_1 \) at 1, \( R_2 \) at 2, and \( R_4 \) at 14 -- and to the spine of M adjacent to the fourth rung. Only the embedding with two null rungs allows this attachment. Any embedding \( \mu \) of \( \Gamma \) into the torus induces the drawing below:

The H contains an arc \( \alpha \) connecting the centers of nonconsecutive rungs -- shown in
the diagram above. Disjoint from $\alpha$ are a pair of paths -- one in the H subgraph, the
other the edge $[3, 12]$ -- joining the centers of the remaining pair of rungs to the
spine of M. Attachments on the rungs are indicated by the hollow square and circle
respectively. The possible attachments for the other ends of the two paths are
indicated by smaller squares and circles respectively. The sites of attachment must
be on adjacent arcs of the spine. The reader should note that there is no way to
attach both arcs simultaneously.
The graph induced on \{ 1, 2, 3, 4, 5, 6, 7, 9 \} \Theta, is a k-graph in \Gamma as is the disjoint graph \Omega induced on \{ 10, 11, 12, 13, 14, 15, 16 \}. As such, for any embedding m of \Gamma m on \Theta \cup \Omega must induce the following drawing:

Vertex 8 must be in one of the two annuli of figure C.16 and vertices 3 and 9 -- indicated with small squares -- must be on the boundary of the other annulus since
they each attach to the same arc on $\Omega$. Vertex 7 must lie on the third arc of $\Theta$.

Vertices 1 and 6 lie on the same arc of $\Theta$ as does 7 and flank vertex 7. Since vertex 7 attaches to some arc of $\Omega$, the path from 1 to 8 to 6 must be as indicated in Figure C.17 below.

Figure C.17

Vertex 5 however occupies one of the two positions indicated with an asterisk and cannot be joined to $\Omega$. 
Let $K$ be the homeomorph of $K_4$ induced on $\{7, 8, 9, 11, 12, 13, 14, 15\}$.

$K \cup [9, 10, 6, 5, 14]$ is a homeomorph of $K_{3,3}$ disjoint from the cycle $C = [1, 16, 4, 3, 2]$, hence any drawing of $\Gamma$ in $T$ must have $C$ null.

$K \cup [9, 10, 2, 3, 12]$ is a homeomorph of $K_{3,3}$ disjoint from the cycle $C' = [1, 16, 4, 5, 6]$, hence any drawing of $\Gamma$ in $T$ must have $C'$ null.

Since $C \cup C'$ is a $k$-graph in $\Gamma$ not both of $C$ and $C'$ may be null.
The $\mathbb{K}_{2,3}$ induced on $\{9, 10, 13, 14, 15\}$ is a $k$-graph of $\Gamma$.

The $\mathbb{K}_{2,3}$ induced on $\{1, 2, 3, 4, 5, 6\}$ is a $k$-graph of $\Gamma$.

These two $k$-graphs are joined by a 4-bond comprised of the edge $[4, 15]$ and the 2-arcs $[5, 12, 13], [6, 8, 9]$, and $[3, 11, 10]$. In any toroidal embedding of $\Gamma$, the disjoint $k$-graphs must embed as parallel essential cycles separating $\mathcal{T}$ into two annuli. Since each of the 2-arcs contains a vertex adjacent to 7 they must all lie in one of these two regions. But they divide this annulus into three disks none of which may contain 7.
### Bibliography


