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Characterizing levels of understanding of functions and their graphs

Browning, Christine Annette Cheadle, Ph.D.
The Ohio State University, 1988

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CHARACTERIZING LEVELS
OF UNDERSTANDING
OF FUNCTIONS AND THEIR GRAPHS
DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By

Christine Cheadle Browning, B.S., M.A.

*****
The Ohio State University
1988

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To my family, David, Joshua, and David
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CHAPTER I
INTRODUCTION

A high percentage of students entering colleges and universities with a background of three to four years of high school mathematics are unable to begin their study of collegiate mathematics with calculus. A concern here is that a majority of students then will be essentially locked out of many career choices since calculus tends to be a prerequisite course for many collegiate majors, e.g. science, mathematics, or engineering (Waits & Demana, 1988c). A means of improving this situation is to focus on and enhance the students' mastery of essential topics in the precalculus curriculum.

Precalculus should provide a conceptual base for the three principal components of calculus: limits, derivatives and integrals. A graph provides the mode of introduction and definition for these components in the calculus curriculum. Therefore, an understanding of functions and their graphs is assumed prior to the study of calculus. Research shows this "assumed" understanding is lacking.

The focus of this study, then, is on the student's understanding of functions and graphs within a precalculus curriculum. As a means of investigating and improving the understanding of functions and graphs, an assessment instrument will be developed. The intent of the instrument and the study is to determine and characterize levels of graphical understanding based on the students' responses to the individual items on the instrument and student interviews. The study will attend to questions of reliability and validity of the instrument. If these levels of understanding can be characterized, they can provide guidelines for improved instruction related to graphing in the precalculus classroom.
1.1 Motivation for Intensified Study of Graphing

The 1985 Report of the Higher Education Research Institute at University of California (Astin, et al, 1983) found that 92% of the freshman in public universities took three or more years of college preparatory mathematics in high school. The corresponding figure for Autumn Quarter 1986 freshman at The Ohio State University was 87% , with 63% of the freshman having four or more years (Waits & Demana, 1988a). Waits and Demana (1988c) have also determined from Albers et al. (1981) that "Ohio State enrollments in freshman and sophomore mathematics courses are within 2% of the figures given for all 4-year colleges and universities." Thus, the population and characteristics of students at Ohio State should be fairly similar to those populations of other state-supported colleges and universities. Of the 63% with four or more years of college preparatory mathematics attending Ohio State University, approximately one in six were able to begin their mathematics studies with calculus. Thus, the number of years of mathematics in high school is not a strong predictor for calculus readiness. Why not?

Two possibilities, of many, may explain why students with three or four years of high school mathematics are not ready for calculus. One is related to the type of mathematics courses taken; not all are college preparatory courses. Three years of mathematics consisting of prealgebra I, algebra I, and consumer mathematics does not prepare one for calculus. Yet many students who do take college preparatory courses are still not ready for calculus. Waits and Demana (1988a) found more than 60 percent of the students at Ohio State with less than four years of college preparatory mathematics and about 54 percent of the students with exactly three years of college preparatory mathematics have remedial mathematics placement. The students' placement is based on
the results of a placement exam developed at The Ohio State University. Hence, the college preparatory or precalculus curriculum itself may be the fault.

The precalculus curriculum should provide a conceptual base for the three principal components of calculus: limits, derivatives, and integrals. What provides a base for discussion of these components? A graph! Many, if not all, calculus textbooks appeal to a graph for an understanding of limits of functions. Derivatives and differentiation generally begins with an investigation of tangent lines on graphs of functions. A full understanding of functions and their graphs is assumed prior to the study of calculus, but the students' intuition is lacking when it comes to interpreting the immense amount of information conveyed in a graph (Carpenter, et al 1975, 1980, 1983; Brown, et al, 1988; Kerslake, 1981; Barclay, 1987; Ponte, 1984). Hence, this study will "zoom in" on this area of curricular concern for precalculus, by investigating whether an understanding of functions and graphs can be characterized into various levels.

1.2 Graphing in the Curriculum

The importance of graphing and graphing skills within the mathematics curriculum is well documented. The National Council of Supervisors of Mathematics identified graphing skills as one of the ten basic mathematical skills necessary for all students (NCSM, 1977). The National Council of Teachers of Mathematics publication, An Agenda for Action: Recommendations for School Mathematics of the 1980's, recommends an increased emphasis on high level graphing skills such as organizing and presenting data, interpreting data, and drawing inferences and predicting from data. Weintraub (1967) states "graphs... have assumed an increasingly important role in our society. They present concepts in a concise manner or give at a glance information which would require a great deal of descriptive writing. They often distill a wealth of information into a small amount of space." The importance of understanding and
interpreting graphs has certainly increased over the past twenty years! Indeed, teaching students to graph is "obviously important for the development of scientifically literate individuals" (Padilla, et al 1986). The Concepts in Secondary Mathematics and Science Programme (CSMS) (Hart, 1980) investigated several areas within the secondary mathematics curriculum including that of graphing. The project developed tests for each topic with "the items for the test "Graphs" selected in order to investigate the important underlying ideas which are necessary components of the understanding of graphs today" (Kerslake, 1977).

Most of us believe a picture (or graph) is a reliable means of presenting information effectively versus the written word or other symbolic representation because they can display "global trends as geometric patterns that visual systems encode easily" (Pinker, 1983). Kosslyn (1987) believes the way graphs function to communicate information is well-structured as compared to charts, maps, or diagrams. Educators need to be careful of students' preconceptions and how these preconceptions influence the students' interpretation of the information (encoding) (Goldenberg, 1986). Cleveland and McGill (1985) believe "graphs provide powerful tools... for communicating quantitative information" but also suggest we need to understand the students' "graphical perception," the visual decoding process, in order to make the graphical method of presentation successful, i.e. understandable. Barclay (1987) believes most students remain novices in understanding and interpreting graphs because students do not understand the "syntax" of the graphing symbolism. He would argue the traditional paper and pencil means of teaching mathematical graphing should give way to options provided by the microcomputer, thus empowering students "to become active investigators in the mathematics world" (1987). Results of the First, Second, Third, and Fourth National Assessments of Educational Progress (Carpenter, et al 1975, 1980, 1983; Brown, et al
1988) agree with Barclay; students were able to read simple graphs, but could not perform related skills such as interpreting, generalizing, integrating, or extending the information in the graph.

Others argue the graphing curriculum does not provide time for problem solving, where the student must decide what type of graph to use or whether to use a graph at all. In other words, no "real world" problems are presented in the graphing curriculum, problems that do arise in the science curriculum. The students themselves claimed to have graphing skills but were "almost never required to decide which type of graph to make" (continuous vs. discrete) in their mathematics class (Silberstein, 1986). Other research indicates students lack the ability to interpret underlying relationships contained in graphs (Ponte, 1984) and successful use of graphs, particularly with accurate construction and interpretation, is correlated with understanding the purpose of graphing in the curriculum (Patterson, 1984), i.e. why are we graphing anyway? Students view graphing as just another hurdle in the obstacle course of mathematics. Kerslake (1981) found a "large gap between the relatively simple reading of information from a graph and the appreciation of an algebraic relationship" with 13 to 15 year old students. This suggests the problems and difficulties associated with the concepts related to graphing begin early and continue throughout the secondary curricula.

It is assumed for a standard calculus curriculum that one has a strong background and understanding about graphing. The data from the Second International Mathematics Study (SIMS) (Travers, 1985) show little knowledge or growth for United States precalculus twelfth graders on these topics. The United States posttest average score on topics related to elementary functions and calculus for precalculus students was 25%, a score in the bottom quartile of the participating 15 countries.
McKenzie and Padilla (1986) believe, along with the author, the importance of graphs coupled with the deficiencies in graphing skills provide a fertile area for research. The author believes, too, research is pointing towards a curriculum change.

The focus of this study is on the student's understanding of functions and graphs within a precalculus curriculum. As a means of improving the understanding of functions and graphs, an assessment instrument will be developed. The intent of the instrument and the study is to determine and characterize levels of understanding, based on the students' responses to the items and interviews with the students. The study will attend to questions of reliability and validity of the instrument. If these levels of graphical understanding can be characterized (based on ease of understanding to the student and not necessarily the "logical" order of the subject) they could provide improved instructional guidelines for the educator; one step towards improving the precalculus graphing curriculum.

1.3 An Assessment Instrument

At The Ohio State University, a project group called the Calculator and Computer Precalculus (C^2PC) Project (Demana & Waits, 1988b; Osborne & Foley, 1988) is moving toward increasing the attention given to graphing in the precalculus curriculum. The goal of the C^2PC project is to improve student understanding of functions and graphs by enhancing the precalculus curriculum through the use of computer and calculator based graphing. They are modifying the typical precalculus curriculum and applying computers and calculators to emphasize the correspondence between the numerical and algebraic representations of functions with their graphical counterpart. The number of examples, or the base for generalizing about functions and graphs, is increased by the use of computers/calculators. Students participating in the program, along with one precalculus class not in the C^2PC program, will provide the sample for this present investigation.
In order to determine the hypothesized increased understanding of functions and graphs provided by this modified curriculum, a measure of assessment needs to be developed specific to these areas of concern. Thus, the C²PC project provided a motivation for the author to develop an instrument to assess student understanding of functions and graphs. This study will focus on developing a "Graphing Levels" instrument to determine if levels of understanding exist and can be characterized. The Graphing Levels instrument will then be used to determine any growth in graphical "ability" in an attempt to assess any growth in understanding.

The title of the instrument indicates the author's belief that the understanding of functions and their graphs will occur in levels, perhaps hierarchical. This belief is based on research pertaining to the van Hiele levels of geometric understanding (van Hiele, 1958; Hoffer 1981, 1983; Usiskin 1982; Mayberry 1983; Fuys and Geddes 1984; Shaughnessy and Burger 1985, 1986; Crowley 1987). Although this research is based on levels of understanding related to geometric learning, a later work of van Hiele (1986) extended his ideas about levels to all mathematical learning. Very little research to date has focused on extending van Hiele's ideas to other areas. In characterizing graphing levels found in this study, level characteristics ala van Hiele will be "cross referenced" and similarities/differences discussed.

Van Hiele describes five levels a student progresses through in learning; recognition, analysis, ordering, formal deduction, and rigor. The levels provide a guide for the teacher in presenting material to the students. Since the levels are hierarchical, material of a high level should not be presented to the student until he is functioning at that level. The theory is appealing because it is not as dependent on the age of the learner as is the Piagetian theory of learning. The stages van Hiele describes within the levels are information, guided orientation, discussion, free orientation, and
integration. This provides a guide for encoding information and thus facilitates the learning process.

Another major research project related to levels of understanding is the Concepts in Secondary Mathematics and Science (CSMS) program in England (Hart, 1980). Statistical analyses of the present instrument will be similar to those of the CSMS project (see Chapter 3). Results of the CSMS study demonstrate the presence of levels of understanding in many mathematical topics (ratio and proportion, fractions, positive and negative numbers, graphs, vectors, algebra, etc).

1.4 Research Technique

Based on prior results from a pilot study (Appendix B) a "Graphing Levels Test" was redesigned (Appendix A). The test consists of 25 5-response items. The items are all related to graphs of functions and graphing skills. The items were designed to require different levels of graphical understanding. The selection of items was based on a previous pilot study, prior research of van Hiele levels of learning and the CSMS group, and suggestions from mathematicians and mathematics educators. Not all questions included a graph; some required an ability for visual processing (Bishop, 1983). Since the Graphing Levels instrument is in the development stage, questions of validity and reliability will be attended to in this present study.

The population of this study consists of precalculus students enrolled in Central Ohio. The present sample is enrolled at two suburban and two city high schools in central Ohio in grades eleven and twelve. The majority of these students are involved in the C^PC Project; a subsample is enrolled in a typical precalculus course to discuss possible differences vis-a-vis test results and to further validate level structure found using the C^PC subsample.
This study will take place in one school year with a pretest administered in September and a posttest given in April. The posttest will be administered before June thereby imposing a slight time restriction on the study. Upon analysis of pretest data and student interviews, the posttest items may be altered to improve interpretability of results. Posttest results from the \( C^2PC \) group and the standard precalculus group, the control, will be compared with the pretest results. Similarities and differences between pre- and posttest results will be examined and discussed.

1.5 Research Questions

The main objective is to develop an instrument that, once validated, could be used for assessment of graphical understanding in a precalculus or calculus curriculum. With this objective in mind, this study will focus on the following question:

Can a graphing instrument be designed that will describe levels of learning for functions and their graphs?

Specific hypotheses to be attended to are:

1) \( H_0: \) Students enrolled in the \( C^2PC \) project will have the same group posttest mean as students enrolled in a typical precalculus course vs.  
\( H_a: \) Students enrolled in the \( C^2PC \) project will have a significantly greater group posttest mean as compared to students in a typical precalculus course.

2) \( H_0: \) The number of "high level" items answered correctly is independent of the number of low level items answered correctly vs.  
\( H_a: \) Students will not be able to answer an average number of "high level" items correctly without being able to answer more than an average number of "low level" items, implying a hierarchy.
Topics for discussion based on assessment instrument results will include:

1) If there is a significant difference between posttest results from C2PC students versus "control" students, those who have no calculator or computer enhancement with the precalculus material, what does this difference suggest?

2) If levels of understanding can be described, do they suggest a modification of the present precalculus curriculum?

3) Is the level structure found in the pretest analysis consistent with the posttest analysis from both groups in the sample?

The study will also provide more insights into how students learn as related to levels of understanding.

1.6 Summary

In search of a "better" curriculum for precalculus, the author has decided to focus her attention on the understanding about functions and their graphs. In particular, there is an interest in identifying and defining levels of graphical understanding. The author believes the teaching and understanding of graphs can be improved if the presentation of the material is done in a particular fashion as related to the order of the levels. At present, very little research has been done in this particular area, that of identifying and defining levels of graphical understanding. This present research study will build on the previous pilot study in the attempt to provide useful information pertaining to graphical understanding for the classroom precalculus and calculus teachers.
CHAPTER II
REVIEW OF THE LITERATURE

2.1 Introduction

There is much literature pertaining to the benefits of using graphical displays to convey and retain information (Cleveland, 1985; Cleveland & McGill, 1985; Kosslyn, 1987; Pinker, 1983). Yet a limited amount of literature exists related to how one first learns to use a graph and then progresses to a fuller or higher understanding. The literature demonstrates or suggests that graphing is a valuable tool for communicating ideas in the mathematics classroom. Other areas in the school curriculum, science, history, English, also have documented the value of using graphs for ease of concept attainment (Barclay, 1987; Graeber, 1986; Harper, 1986; Hittleman, 1985, Arons, 1983). This study will focus on the students' understanding of functions and their graphs to determine if this understanding occurs in levels and if these levels can be characterized based on student performance.

The review of the literature will begin with what is known about "graphical learning" followed by the research related to the van Hiele levels of understanding, the germ of the idea for the present study, and other sources related to learning hierarchies. A primary literature resource is the CSMS project, as this study is a "partial" replication of that project; the CSMS project goals and results will be discussed as related to this present study. Literature related to using computers for improved understanding of graphing concepts along with research related to visual information processing will also be discussed.
2.2 Mathematical connections and graphical learning

The Cartesian system developed by Descartes in 1637 appears to have dominated conceptions of what graphs were and what they should be used for, "i.e. the depiction of the mathematical functions governing the behavior of objects in space and time" (Biderman, 1978). Wainer and Thissen (1981) note Descartes' "integration of the geometric and the algebraic systems established what until the present day remains the most intellectually important and useful of diagrammatic graphic systems."

The importance of this integration of the geometric and algebraic systems is also reflected in one of the standards for the 9-12 mathematics curriculum (NCTM, 1987). The standards refer to integration as "making a mathematical connection" between two equivalent representations, such as an algebraic equation and a graph. The commission on standards states "probably the most important connections within high school mathematics can be drawn between algebra and geometry."

Research shows, however, that students are not making these connections. NAEP and SIMS results suggest that students can do little beyond plotting points and reading information from simple graphs. Kerslake (1981) drew similar conclusions in her research that students fail to appreciate the ties between an equation and a graph.

The author suggests these connections are not made due to a lack of understanding; it is not fully understood how one learns to use and interpret graphs nor why a graph conveys more information to a reader than other formats such as diagrams or pictographs. Hence, the use of graphs within the mathematics curriculum is very limited and trivial.

Research related to the advantages of graphical displays and how one interprets or encodes the display is still new. Pinker (1983) believes our visual system can encode information easily when presented with a visual display. To this end, Pinker (1981)
proposed a theory of graph comprehension that tried to explain this supposed advantage of graphical displays. His theory made three claims. First, our visual system has the ability to identify a great number of two-dimensional shape attributes quickly and easily. These attributes include length, shape, orientation, height, smoothness, continuity, parallelism, and others. Second, different aspects of a data set will be translated into different types of visual patterns. For example, part of the data set could be viewed as parallel lines and another portion presented as intersecting lines. Third, efficient or experienced graph readers know the correspondence between quantitative trends and visual patterns for different graph formats.

Based on his theory, Pinker made the following statement: "The ease of reading a certain type of information from a certain graph format will depend on the extent to which that graph format translates that trend into a single visual pattern that the visual system can automatically extract, and the extent to which the reader knows that the correspondence in that format between the quantitative trend and the visual pattern holds" (Pinker, 1983).

Pinker (1983) conducted three experiments which tested the hypothesis that graphs convey information effectively since they can display global trends as geometric patterns that visual systems encode easily. Results, based on subjects' response times to answering questions, showed that questions about single values of a variable in a graph were easier to answer when the variable was displayed as a segment length versus a segment angle. However, questions related to global trends of a variable were easier to answer when the variable was displayed as a segment angle. Pinker concluded that graph formats and types of information conveyed by graphs are not uniformly easy or difficult, but that, given particular information, it will be conveyed efficiently in a graph format to the extent that it corresponds to a naturally perceivable visual pattern. The difficulty a
person has in extracting information from a given graph format was found to depend on three things: how the format translates the information into a visual pattern, how easily the visual system can encode that pattern, and whether the reader knows the correspondence between the conceptual information and visual pattern. Within this third precondition, the author believes there is a major factor contributing to the lack of student graphical understanding and appreciation. Students generally do not understand this needed correspondence between the conceptual information and the visual pattern and hence have difficulty in interpreting or extrapolating information from a graph. This lack of understanding is due to insufficient attention to graphs within the mathematics curriculum.

Graphs can be powerful tools for communicating information if students are equipped with the necessary understanding and skills to make "connections". Vonder Embse (1987) found that the experience of a graph reader played a major role in the reader being able to interpret information from mathematical graphs of polynomial functions. He recommended that student experience with using and interpreting information from graphs be substantially increased throughout the entire mathematics curriculum. It is the intent of the author to show that an increased amount of time spent on using and interpreting graphs within the precalculus curriculum using technology will increase student understanding and appreciation of graphical displays by i) significantly higher test means on the Graphing Levels instrument as compared to a standard precalculus class mean, and ii) by showing C²PC students to be at higher levels of understanding on posttest results than a standard precalculus class.

The increased attention spent on graphing within the curriculum will provide students the opportunity to make the necessary mathematical connections between algebraic and geometric representations. But increasing the time spent on graphing is not
enough to ensure the connections are being made. The method of instruction must also be considered.

2.3 Levels of understanding

The problems students face related to the understanding of graphs are varied and extensive. Is there another way of presenting the topic to precalculus students to prepare them better for the understanding of calculus, i.e. to make them calculus ready? If alternative curricula are possible, how can teachers determine whether or not a fuller understanding of graphs is being attained? Learning theories related to hierarchies or levels of understanding may provide some insight into the problems mentioned.

The van Hiele model of development in geometry had its origins in the late 1950's but was not brought to the attention of American mathematics educators until the 1970s (Wiszup, 1976). Van Hiele believes student learning progresses through five hierarchial levels of learning. The labeling of these levels differs somewhat from paper to paper and author to author but the characteristics of each remain the same. The following provides a brief description of each level as related to geometric learning which includes capabilities of the students.

Level 1: Recognition

Students recognize and can name geometric figures but do not know the properties of the figures. Initial vocabulary is developed. The objects of thought are the figures themselves (Fuys and Geddes, 1986). The student wonders whether a drawing fits his/her conceptions of a member of a class of figures (Usiskin, 1982).

Level 2: Analysis

Students begin to focus on components of the figure and properties of the figure yet they do not see any relationship between properties. Students wonder whether the properties for a particular figure hold true only for each individual case or if they hold for
all such figures (Usiskin, 1982). Objects of thought are the classes of figures which were products of the level 1 activities; students discover the properties of figures.

Level 3: Ordering

Students recognize interrelationships between properties of figures and between the figures. Class inclusions are now formed based on the interrelationships. The students order the properties, needing to know whether one statement always follows from another (Usiskin, 1982). The objects of thought are the properties yielding an ordering of the properties (Fuys and Geddes, 1986).

Level 4: Formal deduction

Students are aware of the significance of deduction as a means of developing geometric theory; they understand the role of axioms, definitions, theorems, and the structure of proof. The objects of thought are the ordering of relations (Fuys and Geddes, 1986).

Level 5: Rigor

Students can manipulate symbols without reference to concrete models. The objects of thought are the ordering relations (Fuys and Geddes, 1986). Many researchers of the van Hiele levels question the testability of this level; discussion of this level is minimal, if any is given at all. Most researchers believe few students are at level 5 understanding while in secondary school (most samples for van Hiele studies were drawn from the secondary school population).

Usiskin mentions several properties of these levels in his study: adjacency, distinction, separation, and attainment. The levels exhibit a property of "adjacency"; what was intrinsic in the preceding level becomes extrinsic in the current level. Each level has its own linguistic symbols and its own network of relationships connecting those symbols, a property Usiskin labels "distinction." "Separation" indicates that two
persons who reason at different levels cannot understand each other. Finally, "attainment", the learning process leading to complete understanding at the next higher level has five phases. The five phases that Usiskin describes are distinct from the five levels. The phases, also described by van Hiele, provide a guide to instruction for the educator. They are briefly:

- **Information**: students become acquainted with the selected topic. The teacher learns what prior knowledge the students have about the topic.
- **Guided orientation**: students become involved in exercises carefully selected and sequenced by the teacher. Short tasks are designed to elicit specific responses. The students learn what direction further study will take.
- **Explicitation (discussion)**: the students are allowed an opportunity to express and exchange their newly formed ideas with a newly learned vocabulary.
- **Free orientation**: students are given a variety of tasks. These tasks can be completed in several ways and many are open-ended.
- **Integration**: students review and summarize what they have learned; no new material is presented.

These levels and phases describe how a student progresses through "geometrical" learning; one should not present certain material to the students until they are at the appropriate level. Research of Mayberry (1983), Shaughnessy and Burger (1985), Fuys and Geddes (1984), and Usiskin (1982) have clearly demonstrated the learning levels for geometry exist and are hierarchical in nature.

Van Hiele believes his theory describes how all learning takes place, not just that related to geometry; his present focus is on structure as it pertains to didactics (van Hiele, 1986). Hoffer (1983) suggests an interpretation of van Hiele's thought levels as categories, where category theory is a study of structure. "A category consists of a
collection of elements, called objects, and a collection of relations between the objects, called morphisms, that satisfy a list of postulates." Thus, Hoffer suggests van Hiele's levels can be described in the following manner:

- **Level 1**: Objects are the base elements of the study.
- **Level 2**: Objects are the properties that analyze the base elements.
- **Level 3**: Objects are statements that relate the properties.
- **Level 4**: Objects are partial orderings (sequences) of the statements.
- **Level 5**: Objects are properties that analyze the partial orderings.

Hoffer provides an example for organizing the objects of geometric transformations.

- **Level 1**: The objects are changes in figure, such as horizontal and vertical shifting, stretching, and rotating.
- **Level 2**: The objects are properties of the changes in terms of what they do to figures, such as preserve lengths, change orientation, or distort shape.
- **Level 3**: The objects are statements that relate the properties, such as the composition of two reflections being equal to a rotation or translation.
- **Level 4**: The objects are sequences of statements, such as proofs that reflections generate the isometries.
- **Level 5**: The objects are properties that analyze groups of transformations relative to different geometries (Hoffer, 1983).

The morphisms (or relations) are chosen in a "natural" way. For the objects of the van Hiele levels for geometry, Hoffer chose transformations to be the morphisms for each of the levels as an example. Note that transformations as morphisms in the geometry categories are the objects in the transformation categories.
Hoffer explains in terms of category theory a functor puts the objects and morphisms of one category into a correspondence with the objects and morphisms of another category to satisfy certain postulates. Thus, van Hiele's phases of learning form a functor. Hoffer compares the van Hiele functor with what Dienes refers to as a learning cycle, consisting of a free play stage, a structured play stage, and stages of abstraction, representation, and description (see Resnick & Ford, 1981).

Research, to date, in the United States, has not focused on this "general" aspect of van Hiele's levels but has been limited to examination within the setting of geometric learning. The pilot project for this study was an attempt to describe levels for graphical understanding but was not as successful as the author had hoped. However, enough evidence was found to support continuation of the idea of levels of learning within the context of graphing and functions.

2.4 The Concepts in Secondary Mathematics and Science (CSMS) Programme

The Concepts in Secondary Mathematics and Science (CSMS) Programme had as one of its goals to characterize levels of understanding for various topics of mathematics, including that of graphing, a goal similar to the idea behind van Hiele levels of learning. Since the objective of this research was to characterize levels of understanding for graphing, the author chose the CSMS project design as a base for the present research process.

This research program was based at Chelsea College, University of London from 1974-1979. "The research was firmly based in schools and on the school curriculum so that the topics investigated would be recognizable to teachers as the type of mathematics they taught" (Hart, 1980). Two of the primary aims of the CSMS research were 1) to identify order of difficulty throughout the presentation of individual topics and 2) to
provide evaluatory procedures designed to help teachers to identify the stages reached in their students' thinking. These aims fit well into my belief of levels of learning.

The program realized the traditional route to assessing a student's understanding was via a test, a test designed to examine only algorithmic skills. Very little attention was given to the idea of levels or a hierarchy of understanding in the development of these tests. Those interested in testing concept acquisition have often been the research psychologists whose test items are not necessarily related to the present mathematics curriculum. One aim of CSMS was to amend these conditions (Hart, 1980).

CSMS believed the literature related to learning hierarchies, fell into three categories:

1) a learning sequence or sequence of understanding that is essentially in the learner,

2) a teaching sequence that is in the teacher, textbook, or teaching machine program,

3) a logic sequence that is in the topic.

Suppes (1966) has pointed out that though similar these three aspects are not isomorphic.

For anyone interested in the psychological foundations of mathematical concept formation it is natural to ask what is the sort of connection that holds between the logical structure of mathematical concepts and the psychological processes of acquisition of the concept. As far as I know, not very much has been written in the psychological literature about this kind of question. My present view, based partly on our experiments and partly on conjecture, is that the psychological stratification of mathematical concepts will seldom, if ever, do violence to the logical structure of these concepts; but it will markedly deviate from the mathematical analysis of the same concepts with respect to the amount of detail that must be considered.
CSMS believed Piagetian theory fell into category 1. The author believes van Hiele's theory extends across categories 1 and 2, allowing some control to the teachers for the learning process.

The aim of CSMS(M) in trying to form hierarchies of understanding in a number of mathematical topics resulted in an amalgam of a topic sequence and a sequence of understanding for each topic. Items were then written to embody the key concepts and the performance of children on these items provided the basis for the determination of a hierarchy of understanding. It is recognized that the research was assessing understanding in the sense of 'what can a child do?', rather than 'what does the child know?' (Hart, 1980).

Kerslake (1981) states "the items for the test 'Graphs' were selected in order to investigate the important underlying ideas which are necessary components of the understanding of graphs in schools today." Key concepts tested included interpretation of scattergrams, time-distance graphs, continuity, the effect of changing axes on scattergrams, plotting points, gradient (slope) and parallelism, etc. A final version of the graphing test was administered to approximately 1800 students, ranging from ages thirteen to fifteen years. Analysis of test results identified three groups of items, each group being a level. The criterion of success on two-thirds of the items within a level assigns a child to the next highest level. The levels are described as follows:

Level 1 includes items which involve plotting points, interpretation of block graphs, recognition that a straight line represents a constant rate, and simple interpretation of a scattergram.
Level 2 includes simple interpolation from a graph, recognition of the connection between rate of growth and gradient, use of scales shown on a graph, interpretation of simple travel graphs and awareness of the effect of changing the scale of a graph.

Level 3 includes items that require an understanding of the relation between a graph and its algebraic expression, for example $y = 2x$, $y = 2$, $x = 2$, and $x + y = 2$.

Kerslake (1981) concludes from the test results that "there appears to be a large gap between the relatively simple reading of information from a graph and the appreciation of an algebraic relationship" and "while many children will be able to read information from a graph or to plot given data, it seems that only a few will be able to understand the connection between an equation and a graph." This demonstrates a lack of graphical understanding begins early on; we cannot assume that the "connections" are made simply because the students are enrolled in a precalculus course.

Kerslake continues that it is "important that one should look carefully at the order (emphasis mine) in which graphical ideas are presented..." This present study has implications for identifying an order of presentation of graphical ideas within a precalculus curriculum in order to facilitate student learning.

The CSMS research process involved the following components:

i) development of written tests for several topics;

ii) student interviews for test modification and finding student methods and errors;

iii) statistical analyses to obtain hierarchies based on what the children appeared to understand rather than on a logical analysis of the topics in mathematical terms.

The present research will incorporate the same components as a "partial" replication of the CSMS project in that it only involves the topic of graphing. Hart concluded that "similar studies using different items but on the same topics need to be done" (Hart, 1980).
2.5 Computers and Graphing

This is a relatively new area of research which is rapidly growing due to the increased accessibility of computers in schools. Microcomputer based laboratories in the science curriculum are increasing and research shows graphing skills and understanding are improving with the use of the computer (Barclay, 1985). Mokros (1985) makes several encouraging remarks in her paper: "the study suggests that by linking the concrete and the abstract, the computer may serve as an important "carrier" of problem solving skills... It is possible the intensive juxtaposition of these concrete and formal operations could facilitate the development of formal operational thinking." One would hope these same claims should also hold true within the mathematics classroom. Barclay (1987) believes the "microcomputer empowers students to become active investigators in the mathematics world." Gallitano (1983) further confirmed Barclay's statement in her study. She found students became more involved with their own learning when using the computer in their mathematics classroom.

If the computer provides an increased knowledge base, perhaps one could expect an increase in the intuitive understanding of abstract concepts. Demana and Waits make several arguments for using technology to improve and increase student understanding of graphs.

Computer graphing permits students to solve quickly and effectively complex problems whose solutions are not accessible by traditional methods. By analyzing numerous graphs of functions in a short period of time, students are able to build strong intuition and understanding about functions. Computer graphing does not remove the need for students to think about mathematics. Quite the contrary - it can serve to motivate and enable students to think about more mathematics! (Demana & Waits, 1988a).
A graphical approach eliminates the need for specialized standard numerical techniques used to solve complex problems (Demana & Waits, 1987a, 1987b). For example, the solutions to \( x^3 + 2x - 1 = 0 \), accurate to hundredths, can be determined quickly using a "zoom-in" process. "Microcomputers make the use of contrived problems obsolete and open the door for students to confront realistic and interesting applications" (Waits & Demana, 1987). "Computers encourage generalization because students are able to investigate many special cases quickly and gain the necessary evidence to make these generalizations" (Waits & Demana, 1988a). For example, students can investigate the differences between the graphs of \( y = x \), \( y = x + 2 \), \( y = x + 5 \), \( y = x - 3 \) in a few minutes. The generalization that \( y = x \) differs from \( y = x + a \) by a vertical shift of \( a \) units can be made. Then when presented with \( y = x^2 \) and \( y = x^2 + 2 \) or \( y = \sin(x) \) and \( y = 3 + \sin(x) \), generalizations are made quickly due to former experience and ease of experimentation of hypotheses using technology.

Tall (1985a, 1985b; 1986a, 1986b) has written several articles arguing the use of computer graphing "offers a resource in the mathematical classroom undreamt of not long ago. It enables a cognitive approach to the calculus." He believes the graphic display allows for a wider range of examples and non-examples to be considered, furthering and developing student insight of topics related to calculus.

But much like the calculator in the elementary classroom, the computer is still facing resistance in the secondary mathematics curriculum. SIMS (IEA, 1985) provides evidence for a typical precalculus curriculum from teacher surveys. The study referred to precalculus as a course dealing with topics from the areas of trigonometry, college algebra, analytic geometry, and some introductory ideas of calculus; a course designed to prepare one for concepts related to calculus. The use of technology within the precalculus classroom, based on teacher responses, is still minimal. The standard use for
the calculator was to check exercises. "Despite the advent of new technologies, there appears to have been little movement in the mathematics curriculum away from the past reliance on teacher and textbook" (Dossey, et al, 1988). The rigidity of the curriculum is due to several reasons, among which are a lack of understanding the demonstrated benefits of the use of the computer, a fear of not knowing enough about the machine itself, and an incorrect assumption that programming skills are necessary to teach the mathematics utilizing a computer.

Changes within the mathematics curriculum are necessary to make full use of technology. Suggestions for change of the graphing curriculum will be made based on results from this study. "The rise of the personal computer makes this a particularly opportune time to rethink the mathematics curriculum, and technology's role within it. Not only does the computer introduce an entirely new subject area, but it also creates opportunities for revitalizing traditional subject matter and instructional methods" (Murname and Raizen, 1988). "Computers (and calculators) change both what is feasible and what is important in the mathematics curriculum" (Steen, 1987). This study will determine if the use of technology within the precalculus classroom has a positive effect on student understanding of functions and graphs by comparing posttest scores of both the C²PC and control groups.

2.6 Visual information processing and display design

"It has often been said that graphs capitalize on many of the virtues of pictures, being worth the numerical equivalent of a thousand words. This is without question true of some graphs, but unfortunately it is not true of many - perhaps most - graphs.." (Kosslyn, 1985). The author argued earlier that one reason for graphs not communicating information effectively was a lack of "making connections" on the graph reader's part (see Section 2.2). Kosslyn would argue the construction of the visual
display itself may be mostly at fault due to failure to follow certain guidelines as related to visual information processing. Although his guidelines are intended for statistical graphs, many of Kosslyn's suggestions could be used for construction of mathematical graphs intended for use in a textbook or classroom lesson, where mathematical graphs are graphs that represent a functional or mathematical relationship between two or more variables.

In his review of five books related to graphics, Kosslyn (1985) provides a brief overview of visual information processing.

Figure 1 presents an overview of the contemporary "canonical theory" of human visual information processing... This scheme divides processing into three general phases. At each phase information can be difficult to process if the limitations of the system are not respected.

When you first look at a display, the light reflected from it (or emitted by it, if it is on a cathode-ray tube (CRT)) strikes the retinas and is immediately converted into a pattern of neural impulses... When reaching the brain, they (the neural impulses) first are processed to detect patterns of lines, blobs, colors, and textures... These visual properties are not "semantically interpreted" initially; they are simply seen as visual patterns... This ("pre-semantic") processing leads to the formation of what we can call a "visual sketch."

Alternatives to Kosslyn's view are suggested by Cleveland and McGill (1985) and Julesz (1981). Cleveland and McGill define visual decoding for elementary graphical-perception tasks as "the instantaneous perception of the visual field that comes without apparent mental effort" where Julesz uses the phrase preattentive vision. Kosslyn continues,

The output from the first phase of processing is organized into "perceptual units," which are operated upon in the second phase of processing... The perceptual
units are held in a "short-term memory," which is also sometimes called "working memory..."

Finally, the input ultimately must access the relevant information stored in "long-term memory." This information constitutes one's knowledge about how charts and graphs serve to convey information. Without accessing such stored knowledge, one would not know how to relate points on two axes... In addition, part of this comprehension process involves drawing inferences about the input...

Figure 1. Three levels of human visual information processing (Kosslyn, 1985)

Kosslyn states a number of factors affect getting information into the "system" and must be considered in display design: adequate discriminability, visual properties, processing priorities, and perceptual distortion.

Adequate discriminability refers to variations in marks. For instance, there must be a certain minimal size or the mark will not be seen; two or more marks to be discriminated from each other must differ by a minimal proportion.

"The eye and brain are sensitive to a number of visual properties (e.g., size, orientation, darkness, hue, intensity, texture, etc.), variations of which can be used to convey information. In many cases, variations of one property can be combined with variations of others, allowing a single mark to convey a large amount of information" (Kosslyn, 1985). Kosslyn does not provide any clear directives to follow in relating
visual properties to display design other than differences in quantities should not be represented by differences in color.

The visual system also detects differences in line weight, orientation and length, shading, colors, and other visual properties. Larger differences are more easily detected than smaller ones; heavier lines are detected before lighter ones. "Graphs should be constructed so that one notices the more important things first."

Humans have a tendency to distort input and err in perceiving patterns. It is not clear from Kosslyn’s review how to correct for perceptual distortion, but that one must be aware of such a problem in interpreting information from a graphic display.

Kosslyn also outlines short-term memory constraints. When we first observe a display, we do not tend to view each separate mark or little dot but rather we see patterns that are formed (according to Gestalt theory).

The Gestalt Laws of Organization must be respected if parts of a framework are to group together. Proximity: marks that are close together will tend to be grouped together. Good continuation: marks that suggest a continuous line will tend to be grouped together. Similarity: marks that have similar shapes, orientations, colors, and so on will tend to be grouped together. Common fate: marks or lines that seem to be going the same way will be grouped together. (Kosslyn, 1985).

Display designers must also recall the limited capacity of the short-term memory. One should not require the reader to hold more than the capacity limit of seven chunks of the short-term memory. A chunk is defined to be a meaningful piece of data (Waldrop, 1987).

The third phase of visual information processing requires one to access long-term memory to locate stored information about the display. Thus, the display designer must
create ways of display construction so that each part will easily make contact with the stored relevant information. As Kosslyn suggests, the displays should not be ambiguous nor should they lead one to access inappropriate information.

Research in the area of visual information processing of mathematical graphs has been performed recently at The Ohio State University. Vonder Embse (1987) examined eye movement data from expert and novice graph readers to determine variables critical to mathematical graph interpretation.

The eye movement data allows one to examine empirically the first phase of the visual information process, that of perceptual image. The perceptual image, or visual decoding, provides the reader with a visual sketch of the graph. The eye movement data allows the researcher to examine where the graph reader focuses his/her attention in creating such a visual sketch. Vonder Embse wanted to determine if differences existed between expert (graduate students and professors in mathematics and mathematics education) and novice (remedial mathematics undergraduates) graph readers. Eye tracking equipment provided data on the position and duration of eye fixations when viewing mathematical graphs. His results indicated that there were both differences and similarities between the groups. One of the differences found was for the average fixation duration on important blocks, a block being a portion of the plane. Experts had longer average fixations on the blocks that contained important information, i.e. an x- or y-intercept, a maximum or minimum value of the graph, et cetera; novices did not alter their fixation durations in the same manner. However, Vonder Embse's data did not support the idea that experts read graphs more effectively because they spend more time looking at the important information. Both groups allocated total viewing time equally for important blocks. Viewing time for the experts was made up of a small number of
long fixations. Viewing time for the novices was made up of a large number of short fixations.

A performance task was also done in the study. Subjects were asked to pick out the graph just viewed on a screen from among several distractors. This portion of the study addressed the second phase of visual information processing, that of short-term or working memory. Since the experts had long average fixation durations in important blocks they scored significantly higher ($p < 0.0005$) than the novices.

Vonder Embse states the long fixation durations of the experts were due to their experience and background. The graphs had the "power" to draw both groups attention to important blocks but a cognitive decision was made by the experts to remain focused for longer durations on these blocks than the novices. Hence, Vonder Embse concluded the experience of a graph reader played a major role in decoding a graph. This supports Pinker's theory that the power of a graph to transmit information depends, in part, on the experience of the graph reader.

In addition, then, to Kosslyn's suggestions for display design, one needs to account for the experience of the graph reader. This present research will also attend to the effects of experience in understanding graphs. The C2PC group had far more experience in reading, using, and interpreting graphs than the control group. This increased experience should be reflected in a higher posttest mean score than the control.

2.7 Summary

The author believes the research available indicates the importance of graphs for effective communication of information and the existence of levels of graphical understanding. If these levels of understanding can be identified and defined within the precalculus curriculum, they could provide a means towards improving student learning.
A "Graphing Levels" instrument would be a valuable tool for both identification of levels and assessment of student understanding.

The instrument developed in the present study attempted to incorporate key concepts within the precalculus curriculum and provided items related to the predetermined levels of the CSMS study. As in the CSMS study, no preconceived level descriptions were made, allowing the item results and student interviews to dictate the characterization of levels.

Research has also demonstrated the benefits of the use of technology within the mathematics curriculum. Graphing technology within the precalculus classroom would provide an increased number of examples of functions and their graphs, thus providing a larger base for generalization and improving the student's "graphical intuition." A comparison of test results were made between students utilizing the technology and those not which indicated an improvement in graphical understanding and intuition, thus demonstrating the benefits of the use of technology within the precalculus classroom.

Problems relating to visual information processing and graphical display design were discussed when characterizing levels of understanding using the instrument items.
CHAPTER III
METHODS AND PROCEDURES

3.1 Introduction

The research took place during Autumn Quarter, 1987 and Winter and Spring Quarters, 1988 at The Ohio State University, Columbus, Ohio. A 25 item instrument was designed, titled "Graphing Levels Test," and given to precalculus students at two city and two suburban high schools in central Ohio. The majority of the students were enrolled in the C2PC precalculus classes with the remainder enrolled in a standard precalculus class. The Graphing Levels Test was used as both a pre- and posttest with only minor item modification on the posttest. Students interviews took place after both pre- and posttest administration. Pretest interviews were used as an aid in determining item modifications along with suggestions from mathematicians and mathematics educators. Results from posttest interviews aided in determining level characteristics. Pretest data were used in the cluster analysis to determine levels. These results were then compared with the posttest data from both the C2PC and control groups. Discussion of the findings from these comparisons are in Chapter 6.

This chapter includes a further description of the population, instrument design and construction, the statistical analyses and interview protocols.

3.2 Population

The population for this study involved students enrolled in precalculus courses, grades 11 and 12, in central Ohio. The sample for this study included over 200 students enrolled at four central Ohio high schools, Centennial High School, Upper Arlington High School, Walnut Ridge High School, and Franklin Heights High School. Thirty-two of these students were enrolled in a precalculus class making no use of graphing
calculators or computer graphing utilities, while the remainder was involved in the C²PC course.

The group of 32 students served as a "control" group for posttest analyses. For pretest analyses, the random sample was drawn from all students participating in the study. As Duncan's multiple range test showed no significant difference in group mean scores, see Section 3.3, the author believed the groups were initially comparable. A control group was used for contrasting the performance of students in different curriculum settings. The contrast was important in that if the level structure was invariant across settings, this would provide additional support for the validity of the structure. This control group would also serve as another means of measuring the reliability of the instrument and for assessing any increased benefit in the use of technology. There was reason to believe the C²PC group would perform better than the control group based on results from a pilot study.

Two hundred eleven students from the four school districts took the pretest. Due to normal absences and course withdrawals, only 185 took the posttest. For pretest statistical analyses, a random sample of 125 was chosen from the 211. This 125 included students who would later serve as part of the control group. In posttest analyses, the sample was split into two groups, the C²PC group and the "control" group of 32 students. From the C²PC group, a random sample of 125 was chosen again and used for analyses. Independent posttest analyses were done on the control group and compared with pre- and posttest analyses from the random samples.

3.3 Instrument construction and implementation

If learning does progress through hierarchical levels, a means of determining characteristics of these levels is vital. A pilot study performed Spring Quarter of 1987 at The Ohio State University involved the development of an instrument that would aid in
characterizing van Hiele levels of graphical understanding. The items from this pilot instrument were designed with the van Hiele learning level descriptions in mind. These descriptions leaned heavily towards geometrical learning and were somewhat difficult to use in determining graphical items. The data from the student test results were used in a factor analysis. Resulting factors did not group items into their predetermined levels. It was determined then that further study using dichotomous data would not utilize factor analysis as the principal statistical analysis. Factor analysis provides more reliable results when dealing with continuous data. Statistical analyses are provided in Appendix B, along with the pilot version of the instrument.

A modified version of the pilot testing instrument was developed containing 25 items from many parts of the precalculus curriculum, items intended to address more than just algorithmic skills and directly related to the precalculus curriculum. A multiple choice format with five responses per item was chosen for two reasons: 1) the format is relatively easy for classroom teachers to use and interpret results and 2) the format lends itself to statistical analyses of results. The choices provided for each of the 25 items were based on the author's past experiences relative to precalculus student misunderstandings, other instruments related to graphing, pilot test results, colleague input, suggestions of other precalculus teachers, et cetera. Development of appropriate items was not an easy task as precalculus concepts are typically written without reference to a graphical representation. Thus, no "item bank" of graphical questions existed.

Each item was given equal weight and had only one correct response. No previous assignment of "item to learning level" was given other than a general agreement that an item was more difficult than another (similar to the CSMS project). The test, titled "Graphing Levels Test" (see Appendix A), was given in September, 1987 to 211 precalculus students and in April, 1988 to 185 precalculus students enrolled in four
school districts in central Ohio. Students were given 35-40 minutes to complete the test. Student interviews took place after the pretest given in the fall to aid in 1) possible modification needed in item terminology and 2) assessing validity of item responses. Upon item analysis of the pretest and student interviews, one item was replaced and the wording in two other items was modified. Both versions of the Graphing Levels Test are in Appendix A.

The pretest item analysis, using a statistical package called Item-A, of a random sample of 125 students provided an internal consistency measure (Kuder-Richardson 20) of 0.665, mean item difficulty of 0.458, and mean item discrimination of 0.346 (with 40% of the items in the 0.41 to 0.60 range). The average point biserial correlation, a correlation of the item to total score, was 0.329 with 14 of the 25 items above 0.3. The mean test score was 12.55 with a standard deviation of 3.59.

The posttest item analysis of 125 randomly chosen C²PC students showed an increased internal consistency (KR-20 = 0.739), a drop in mean item difficulty to 0.239 (which was expected), and a slightly increased mean item discrimination of 0.356. The average point biserial was 0.359 with 18 of the 25 items above 0.3. The mean test score was 19.02 with S.D. of 3.52. Posttest item analyses of the "control" group had a slightly decreased internal consistency to 0.636, a mean item difficulty of 0.460, and a mean item discrimination of 0.317. The average point biserial was 0.309 with 15 items greater than 0.3. The mean test score was 13.5 with S.D. of 3.28.

These statistics suggest the instrument to be reliable and valid for measuring graphical understanding.

Duncan's multiple range test was used to detect any significant (p<0.05) differences between group means. In the pretest analyses, the random sample was divided into six groups of the classes of students corresponding to the six teachers involved in the C²PC
project in the four school districts. Table 1 shows no significant differences between group test means except for Group 2 (asterisks indicate significant difference). This particular group of students was enrolled in an advanced precalculus class and their high group mean was anticipated.

Table 1  
Duncan’s Multiple Range Test

PRETEST: All Data

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<thead>
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<th>Mean</th>
<th>Group</th>
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</thead>
<tbody>
<tr>
<td>11</td>
<td>6</td>
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<tr>
<td>11.375</td>
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</tr>
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<td>11.5</td>
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</tr>
<tr>
<td>16.3514</td>
<td>2</td>
</tr>
</tbody>
</table>

Duncan’s multiple range test for posttest analyses again showed Group 2 outperforming the other five groups (Table 2). The posttest analysis was performed with all 185 students. Note five groups did significantly better \( p < 0.05 \) than Group 3; this group was the posttest control. These differences between the \( C^2PC \) groups and the
control provides some evidence that the technology probably makes a difference in the students' graphical understanding.

Final analysis began after the April posttest. Further interviews also took place to aid in assigning levels to items (see Section 3.5 Interview Protocols).

3.4 Analysis

In order to characterize levels of understanding, analyses similar to that of the CSMS data was used, as this study was an attempt to replicate their results. "Facility Ordered Complete Linkage" was the phrase the project used to describe the means of assessing levels. This process involves the following:

A criterion value of phi is first chosen. Starting with the easiest item, the next most easy item with a phi value greater than the criterion is joined to it. The third item to join the group will be the next most easy item with phi values greater than the criterion with both items already included. Subsequent items are joined similarly, provided the phi values with items already included are all greater than the criterion. This continues until there are no further items achieving the criterion and the process is then repeated, starting with the second most easy item. And so on, to form a chain of items attached to each item (Hart, 1980).

The phi value mentioned in the above description refers to the item to item (Pearson) correlation.

The project mentions that some of the chains created might be subsets of others and could be ignored. They suggest further that the maximum value property of phi would alter the difficulty range spanned by the chains and that the analysis be repeated with different criterion values "to balance the desire for reasonable sized chains and suitable difficulty range with the need to have the chains as homogeneous as possible" (Hart, 1980).
Facility ordered complete linkage is essentially cluster analysis. An objective of cluster analysis is to partition a set of objects "into a hierarchial arrangement of homogeneous subgroups" (Lorr, 1983), hence the reason for use of such an analysis. Close and Murtagh (1986) argue, too, that "cluster analysis should be among the repertoire of techniques used to identify sequences of mathematical skills." CSMS modified the analysis by considering facility levels and inter-item correlations together in determining the clusters.

In addition to this complete linkage analysis, items had to satisfy the following criteria before they were considered to form a group:

i) they should be of approximately the same level of difficulty;

ii) the values of the inter-item correlations should be at an acceptable level;

iii) there should be some measure of mathematical coherence to the items;

iv) the groups should be scalable in the sense that a student's success on a group entailed success on all easier groups (Hart, 1980).

The analysis outlined above was performed on the random sample of the pretest data. Results of this analysis were compared with the results from the clustering of the posttest random sample and control group to determine if hierarchial levels existed.

A factor analysis was performed on both pre- and posttest data using SPSS to see if the factor structure corresponds to the level structure as determined by the facility ordered complete linkage. Factor analysis is used to describe, if possible, the covariant relationships among many variables, in this case, the items of the test, in terms of a few basic underlying factors. Recall clustering refers to the formation of homogeneous subgroups. That is, all items related to a particular level should be highly correlated
among themselves but have relatively small correlations with items that characterize other levels.

Means and standard deviations were compiled for each item. An item by item correlation matrix was also compiled in preparation for a factor analysis. The principal component method of extracting the unknown factors was used. The initial statistics provided by the extraction include:

1) the eigenvalue associated with the item which corresponds to the variance equivalent to that of one item. Factors with eigenvalues less than 1 will not be considered viable factors in the analysis;

2) the percent of variance accounted for by a factor. A cutoff value of 4-5% of the variance accounted for was set for the analysis;

3) the cumulative percent of variance accounted for by the combined factors.

The factor matrix provides a listing of factors with their corresponding items and correlations. The oblique rotation was used to produce a rotated factor matrix; an oblique rotation was used since the levels are assumed to be hierarchial and therefore correlated.

Further estimates of internal consistency reliability on the clusters and subscales created was also done.

A Guttman scalogram analysis will be performed to determine if the levels, determined by the cluster analysis, form a scale or hierarchy. This analysis is a means of analyzing the underlying operating characteristics of the items to determine if their interrelationships, the levels formed in this case, meet the properties which define a Guttman scale (SPSS, 1970). These properties are that the scale be unidimensional, items measure movement toward or away from the same underlying object, and cumulative, items can be ordered by degree of difficulty.
3.5 Interview Protocols

There were two sets of interviews during the study. The first set took place after the pretest in September and the second set after the posttest in April. The interviews were used to 1) identify modifications needed in the terminology of the items, 2) determine the validity of the distractors for the items, and 3) determine the suitability of the items for the precalculus students. The information gained from the interviews was also used to provide further guidance in the identifying and defining of graphing levels in addition to the statistical processes outlined. As suggested by the CSMS project, the "child-method" used for particular items could be used as a guide in determining a cut-off between two groups (or levels). "The general description given to the levels in each of the hierarchies could then take into account the methods used by children rather than merely relying on the mathematical demand of particular items" (Hart, 1980).

The interviewer in all cases was the author. The interviewees were the students from the six different precalculus classes, five involved in the C^2PC Project and one (the control) not using technology. The students were asked by their respective teachers if they wanted to participate in an interview and, if so, to provide the dates and times of their study halls. A list of these students was given to the author. Students were then randomly selected by the author and interviewed at the schools. The interview consisted of the students taking the "Graphing Levels Test" again but this time they needed to talk their way through the items, i.e. explain what they were doing and why. No indication was given to the student whether he/she was right or wrong but just general encouragement to provide all steps in their solution process.

Each interview was taped and later transcribed (see Appendix C). The interviews were limited to the class period so as not to interrupt the student's daily schedule. The
students were informed of the purpose of the interview at the beginning, i.e. to help teachers in furthering student understanding of functions and their graphs.

3.6 Procedures

The initial stage of this project was the development of the Graphing Levels test.

In September, 1987 and April, 1988, the 25 item test was administered to over 200 students in grades eleven and twelve attending two city and two suburban high schools in central Ohio. Students were enrolled in two "versions" of a precalculus course; the C²PC version and a "typical" (control) version. Students were assigned by the school districts to these classes. The test involved 35-40 minutes of class time and was administered by personnel involved in the C²PC project. Students were encouraged to do their best and to answer as many questions as possible in the allowable time. Students were permitted to return to unanswered items if they finish before the end of the allotted time.

Student interviews took place after the pretest to determine if item modification was necessary. After the posttest in April, interviews took place again, involving students from both groups in the sample. These interviews were used as an aid in level characterization.

Analysis of student test results began following administration of the posttest in April, 1988. Facility ordered complete linkage was used to determine levels from a random sample of the pretest data. Factor analysis was also used as a check of the level structure produced by the cluster analysis. Characterization of levels was determined using the results from the analyses and student interviews.
4.1 Introduction

Cluster analyses similar to that performed in the CSMS project was performed on pretest results from a random sample of 125 students from the original 211 to determine levels of graphical understanding. Four clusters were suggested from the analyses. A factor analysis was performed to compare factored subscales with clusters for validation of levels.

The clusters were compared with posttest results from the split sample, that of the C²PC group (n=125) and the control group (n=32) to determine if level structure was preserved for both groups to further the validity of the level construct.

Results from a Guttman scalogram analysis using the pretest data was used to determine if the levels formed a hierarchy.

Level attainment criteria similar to that of the CSMS project was used in a crosstabulation of both pre- and posttest data to compare growth of graphical understanding between the two groups.

4.2 Clustering of items into levels

Using the process described in the CSMS project, the facility ordered complete linkage, four clusters were found using the pretest results from the random sample. Briefly, the facility ordered complete linkage process involves starting with the easiest item and linking it with the next easiest item with an inter-item correlation greater than or equal to a preselected criterion. The third item to join the cluster will be the next easiest item with a correlation greater than the previous linking correlation. This process
continues until the correlation criteria can no longer be met. The routine is repeated with the second easiest item starting the link. Table 3 lists the item numbers within the clusters and their corresponding facility (per cent correct). Since four of the items from the Graphing Levels test (items 4, 17, 22, and 23) did not meet the criterion phi value (inter-item correlation) of 0.2 they were not clustered. CSMS chose 0.35 or greater for their criterion phi value, but the Graphing Levels pretest results did not provide many inter-item correlations greater than 0.35. Phi values less than 0.2 produced a large number of small chains where criterion phi values greater than 0.2 created few chains, hence 0.2 was chosen for this study. (See Appendix C for pretest inter-item correlations, linkage, and clustering.)

Items were then grouped based on facility. Following CSMS criteria, the clusters were limited to span a band of facility of not more than twenty per cent. Once grouped, the items were then examined within each cluster to check for mathematical coherence. Characterization of levels was based on the nature of the items themselves and student problem solving strategies revealed in the posttest interviews. Chapter Five provides level descriptions.

4.3 Pretest clusters

Cluster (Level) 1 consisted of five items, 1, 2, 5, 9, and 21 (see Appendix C for clusters) ranging in facility from 78.4 per cent (item 2) to 96.8 per cent (item 21), with a mean corrected phi value of 0.464. The mean corrected phi value provided by the item analysis is a measure of the item's contribution to the test reliability and of the item's ability to detect performance differences between the upper and lower 27.5 per cent of the sample. Interpretation of the levels in terms of associated skills is in Chapter Five.
Table 3 Pretest Clustering compared to Posttest Facilities

<table>
<thead>
<tr>
<th>Question Number</th>
<th>Pre</th>
<th>Facility Post (N=125)</th>
<th>Post (N=32)</th>
</tr>
</thead>
<tbody>
<tr>
<td>21</td>
<td>96.8</td>
<td>92.8</td>
<td>93.8</td>
</tr>
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<td>1</td>
<td>96</td>
<td>100</td>
<td>96.9</td>
</tr>
<tr>
<td>5 Level 1</td>
<td>79.2</td>
<td>92</td>
<td>84.4</td>
</tr>
<tr>
<td>9</td>
<td>79.2</td>
<td>90.4</td>
<td>75</td>
</tr>
<tr>
<td>2</td>
<td>78.4</td>
<td>74.4</td>
<td>87.5</td>
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<td>14</td>
<td>70.4</td>
<td>94.4</td>
<td>87.5</td>
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<td>15</td>
<td>69.6</td>
<td>84.8</td>
<td>62.5</td>
</tr>
<tr>
<td>6 Level 2</td>
<td>66.4</td>
<td>83.2</td>
<td>68.8</td>
</tr>
<tr>
<td>8</td>
<td>62.4</td>
<td>91.2</td>
<td>37.5</td>
</tr>
<tr>
<td>18</td>
<td>61.6</td>
<td>89.6</td>
<td>59.4</td>
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<tr>
<td>20</td>
<td>60</td>
<td>91.2</td>
<td>59.4</td>
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<td>11</td>
<td>52</td>
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<td>96.9</td>
</tr>
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<td>19 Level 3</td>
<td>44.8</td>
<td>80</td>
<td>53.1</td>
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<td>44</td>
<td>77.6</td>
<td>46.9</td>
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</tr>
<tr>
<td>16</td>
<td>28</td>
<td>43.2</td>
<td>28.1</td>
</tr>
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<td>24 Level 4</td>
<td>17.6</td>
<td>24</td>
<td>6.3</td>
</tr>
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<td>15.2</td>
<td>88</td>
<td>21.9</td>
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<tr>
<td>25</td>
<td>9.6</td>
<td>68.8</td>
<td>15.6</td>
</tr>
</tbody>
</table>

Cluster 2 consists of six items, 6, 8, 14, 15, 18, and 20, ranging in facility from 60 per cent (item 20) to 70.4 per cent (item 14), with a mean corrected phi of 0.652.

Cluster 3 consists of six items, 3, 7, 10, 11, 12, and 19, ranging in facility from 36.8 per cent (item 12) and 52 per cent (item 11), with a mean corrected phi of 0.693.

Cluster 4 consists of items 13, 16, 24, and 25, ranging in facility from 9.6 per cent (item 25) to 28 per cent (item 16), with a mean corrected phi of 0.496.
Reliability coefficients for the four clusters (Cronbach's alpha and KR-20) are listed in Table 4, along with mean item difficulty and discrimination.

A factor analysis was performed on the pretest random data using a four factor model for validation of the cluster model. An oblique rotation was employed since the factors were assumed to be correlated. All previous criteria for the factor analysis were met, that of factors having eigenvalues greater than one and accounting for variance of more than 5 per cent. Only items with loadings of 0.4 or greater were considered within each created subscale. (See Table 5).

Table 4 Cluster Statistics

<table>
<thead>
<tr>
<th>Cluster</th>
<th>Item #s</th>
<th>Cronbach's $\alpha$</th>
<th>KR-20</th>
<th>Mean item difficulty</th>
<th>Mean item discrimination</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1,2,5,9, 21</td>
<td>0.2748</td>
<td>0.286</td>
<td>0.141</td>
<td>0.474</td>
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<tr>
<td>2</td>
<td>6,8,14,15,18,20</td>
<td>0.408</td>
<td>0.403</td>
<td>0.349</td>
<td>0.501</td>
</tr>
<tr>
<td>3</td>
<td>3,7,10,11,12,19</td>
<td>0.4766</td>
<td>0.494</td>
<td>0.551</td>
<td>0.669</td>
</tr>
<tr>
<td>4</td>
<td>13,16,24,25</td>
<td>0.3911</td>
<td>0.378</td>
<td>0.824</td>
<td>0.587</td>
</tr>
</tbody>
</table>

Listed beside the item number in Table 5 is the corresponding cluster (or level) number from the cluster analysis. Note, items 22 and 23 were not included in the cluster analysis. A discussion of the correspondence of clustered levels to factor analytic subscales is in Chapter Five.
### Table 5  Factor Subscale Statistics

<table>
<thead>
<tr>
<th>Item #s (level #)</th>
<th>Item loadings</th>
<th>KR-20</th>
<th>Mean item difficulty</th>
<th>Mean item discrimination</th>
<th>% of variance explained</th>
</tr>
</thead>
<tbody>
<tr>
<td>8 (2)</td>
<td>0.687</td>
<td>0.539</td>
<td>0.520</td>
<td>0.625</td>
<td>12.1</td>
</tr>
<tr>
<td>13 (4)</td>
<td>0.573</td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>24 (4)</td>
<td>0.567</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9 (1)</td>
<td>0.516</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>11 (3)</td>
<td>0.409</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>18 (2)</td>
<td>0.407</td>
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<tr>
<td>25 (4)</td>
<td>0.660</td>
<td>0.388</td>
<td>0.512</td>
<td>0.615</td>
<td>7.5</td>
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<tr>
<td>2 (1)</td>
<td>0.541</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>16 (4)</td>
<td>0.528</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5 (1)</td>
<td>0.412</td>
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<td></td>
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<tr>
<td>7 (3)</td>
<td>-0.719</td>
<td>0.468</td>
<td>0.518</td>
<td>0.649</td>
<td>6.8</td>
</tr>
<tr>
<td>23 -</td>
<td>-0.536</td>
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<tr>
<td>20 (2)</td>
<td>-0.517</td>
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<td>6 (2)</td>
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<td>1 (1)</td>
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<td>0.398</td>
<td>0.798</td>
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<td>22 -</td>
<td>0.560</td>
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<tr>
<td>14 (2)</td>
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<td>19 (3)</td>
<td>0.427</td>
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</tbody>
</table>

#### 4.4 Cluster Hierarchy

A Guttman scalogram analysis was performed using the pretest random data to determine the validity of a hierarchy. Each student's responses to the items from their respective clustered levels were scored right or wrong. If the student met the criterion for a level he/she received a 1; if not, a zero. Thus, each student's score formed an ordered quadruple (see Appendix E). For example, 1010 means that the student met the criterion on Levels 1 and 3 but not for Levels 2 or 4. Of the 16 possible response patterns, only five should appear if the hierarchy is valid: 0000, 1000, 1100, 1110, 1111. A response pattern that differs from these five is termed an error. The number of errors is the
number of zeros preceding the last score of one. Thus the example, 1010, has one error; 0001 has three errors, and so on.

The coefficient of reproducibility, a measure of the extent to which a student's score is a predictor of his response pattern, was determined for the scale. It is given by the formula:

\[
R = 1 - \frac{\text{total number of errors}}{\text{total number of responses}}
\]

\[
= 1 - \frac{6}{21 \times 114} = 0.9975
\]

A coefficient of reproducibility higher than 0.9 is considered to indicate a valid scale. The coefficient of reproducibility for the pretest random sample is 0.9975 indicating a hierarchy.

Since the levels were found to be hierarchial in nature, a crosstabulation of the pretest data was done. The crosstabulation produced the results in Table 6. Five observations are missing due to incomplete data on the students' test results. Level scores were computed by allotting one point for each correct response to the items within each level produced by the cluster analysis. A criterion of correctly answering at least two-thirds of the items within each level was chosen as suggested by the CSMS project. Criteria for level attainment then is as follows:

If Level 1 score is less than 3, classify as Level 0.

If Level 1 score is greater than 2, and Level 2 score is less than 4, classify as Level 1.

If Level 1 score is greater than 2, Level 2 score is greater than 3, and Level 3 score is less than four, classify as Level 2.

If Level 1 score is greater than 2, Level 2 score is greater than 3, Level 3 score is greater than three, and Level 4 score is less than 3, classify as Level 3.
Table 6  Crosstabulation of graphing levels by teacher
Pretest random sample
N = 125

<table>
<thead>
<tr>
<th>LEVEL</th>
<th>COUNT</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>ROW TOTAL</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>ROW PCT</td>
<td>14.3</td>
<td>14.3</td>
<td>28.6</td>
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<td>3.6</td>
<td>3.6</td>
<td>3.6</td>
<td>3.6</td>
</tr>
</tbody>
</table>

COLUMN TOTAL  | 23   | 26   | 28   | 9    | 26   | 8    | 120 |
TOTAL          | 19.2 | 21.7 | 23.3 | 7.5  | 21.7 | 6.7  | 100 |

Number of missing observations = 5
If Level 1 score is greater than 2, Level 2 score is greater than 3, Level 3 score is greater than three, and Level 4 score is greater than 2, classify as Level 4.

The results of the crosstabulation show the majority of students (77.5 per cent) operating at Levels 0, 1, or 2 at the beginning of their precalculus year.

4.5 Pre- and posttest comparisons

Figure 2 compares pretest clusters with posttest item facilities taken from the random sample of 125 C²PC students. Level structure is fairly well maintained with facilities in Levels 2, 3, and 4 increasing by approximately one level. (See Appendix C for posttest linkages and inter-item correlation matrices.) Items 11 and 13 have dramatic changes in facility which could be explained in part by the instructional methods used in the C²PC course.

Figure 3 compares pretest clusters with posttest item facilities taken from the control sample of 32 students (Teacher 3). Again, level structure is maintained with students showing little level growth, especially within Level 4. Note here items 11 and 13 do not exhibit the dramatic growth as with the C²PC group. The facility for item 11 has improved somewhat while the facility of item 13 has remained within the same level. These items were maintained in their respective levels since they exhibit "normal behavior" in the control group.

Using the same criteria for level attainment as for the pretest, a crosstabulation of both group's test results was performed (see Tables 7 and 8). The majority of the control group (68.7 per cent) remained at Levels 0, 1, or 2 while 73.1 per cent of the C²PC students were operating at the higher levels of 3 and 4.
Figure 2  Pretest clusters compared with posttest random sample facilities
Figure 3 Pretest clusters compared with posttest control facilities
Table 7  Crosstabulation of graphing levels by teacher  
Posttest random sample  
N = 125

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<th>4</th>
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<td></td>
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<td>5</td>
<td>14</td>
<td>6</td>
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<td>30</td>
<td>19</td>
<td>25</td>
<td>11</td>
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Number of missing observations = 6
Table 8  Crosstabulation of graphing levels by teacher
Posttest control sample
N = 32

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<td>7</td>
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<td>4</td>
<td>3</td>
<td>9.4</td>
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</table>

COLUMN 32
TOTAL 100

4.6 Hypotheses and results

Research hypotheses addressed in this research were:

1) \( H_0: \) Students enrolled in \( C^2PC \) project will have same group posttest means as students enrolled in standard precalculus course vs.
Hₐ: Students enrolled in C²PC project will have significantly greater group posttest means as compared to students in standard precalculus course.

2) H₀: The number of "high level" items answered correctly is independent of the number of low level items answered correctly vs.

Hₐ: Students will not be able to answer an average number of "high level" items correctly without being able to answer more than an average number of "low level" items, implying a hierarchy.

The first null hypothesis was rejected in favor of the alternative based on the results of Duncan's multiple range test (see Section 3.3). Students enrolled in the C²PC project had significantly higher group posttest means (p < 0.05) than the control sample, with mean differences ranging from 2.5072 to 8.75. Discussion of these differences follow in Chapter Five.

The second hypothesis was also rejected in favor of the alternative based on the results of the Guttman scalogram analysis. Analysis of the pretest random data provided a coefficient of reproducibility of 0.9975, indicating a valid hierarchy exists in the levels.

4.7 Summary

The facility ordered complete linkage produced four levels. A factor analysis using an oblique rotation between the underlying factors did not confirm the complete linkage model and contributed no valuable information to the study.

Pretest clusters (levels) were then compared with posttest results from both the C²PC and control groups. Level structure appeared to be preserved for both groups, suggesting a valid construct.
The Guttman scalogram analysis furthered the validity of the hierarchy. A coefficient of reproducibility of 0.9975 was determined from the pretest random sample.

Crosstabulation of the pre- and posttest data show more growth in understanding in the C²PC group than in the control strongly suggesting the importance of using graphing technology within the classroom.

The results from the analyses suggest that the response to the chapter title be a resounding "Yes."
CHAPTER 5

GRAPHING LEVELS OF UNDERSTANDING

5.1 Introduction

A graphing levels instrument was designed and administered to over 200 precalculus students as both a pre- and posttest and was found to be reliable and valid. Using the students' test results (an indication of what the students appear to understand) on the individual items, the facility ordered complete linkage grouped the questions into four clusters or levels. Characterizations of the levels were based on the grouped items and student problem solving strategies revealed in the posttest interviews. Pre- and posttest comparisons of the level structure were made using both the C^2PC and control groups' test results. The results of the study indicate that an understanding of functions and their graphs for precalculus students can be characterized by levels.

5.2 Forming the Clusters

The clusters were formed by utilizing the students' pretest responses to the Graphing Levels Test that was developed as part of the research process. Facility ordered complete linkage, a modified cluster analysis developed by the CSMS project, was used to group the items. Twenty one of the twenty five items were grouped into four clusters or levels. The number of levels was based on the "facility band splits" (see Figure 2). A criteria span of not more than 20 percent was chosen, based on the CSMS project suggestions.

To validate the formation of four levels further, a factor analysis was also performed using the pretest results. Factor analysis was chosen as a means for verification of the cluster analysis based on the CSMS project research process and on its use for separating variables into groups based on the covariant relationships among the variables. However, factor analysis is not a reliable means for the grouping of
dichotomous (categorical) variables versus continuous variables. The CSMS project also noted "very easy and very difficult items cannot have high correlations so in general would not load highly on one factor...", regardless of how similar the items were in content. This was the project group's argument for not using factor analysis for determining levels. Yet this was not the case in the present study. Notice in the first subscale, item 9 with facility 79.2 percent loaded along with item 13 having a facility of 15.2 percent and that subscale two consists of two difficult and two easy items (see Table 5). This is an indication of just how tenuous the observed factor structure is in this case.

After examining the items within each subscale, no reasoning for the groupings was readily apparent. Factor analysis of the posttest data proved worse with the oblique rotation failing to converge. For the above mentioned reasons and difficulties encountered, factor analysis was not considered a reliable means for determining the levels of graphical understanding.

5.3 Characterizing the Clusters

After the items were grouped by the facility ordered complete linkage (see Section 4.3), the following criteria were also satisfied before the items were considered to constitute a level of understanding, as suggested by the CSMS project:

i) the items had to be of approximately the same level of difficulty; they were limited to span a band of facility of not more than twenty per cent;

ii) there would be some measure of mathematical coherence to the clustered items as determined by the author and other personnel of the C^2PC project;

iii) the clusters of items would be scalable in the sense that a student's success (measured by answering at least 2/3 of the clustered items correctly) on a high level entailed success on all lower levels.
Level 1 consisted of items 1, 2, 5, 9, and 21 (see Appendix C for clusters). These items had the highest facilities and characterized a base of understanding for graphs of functions. Item 2 (Figure 4) is given as an example from this level. After examining the items and reviewing the interview tapes, the following item characteristics were suggested:

i) recognition of the graph of a parabola placed in different positions, i.e. rotated 90 or 180 degrees;

ii) simple interpretation of information from a graph;

iii) knowledge of effects of a positive or negative $x^2$ coefficient on a graph of a quadratic function;

iv) basic intuition of continuity;

v) initial vocabulary developed.

2. If we rotate the above graph so it looks like the graph below, then we have the graph of

A. a point.

B. a line.

C. a parabola.

D. an ellipse.

E. a circle.

![Figure 4 Item 2 from the Graphing Levels Test](image)

These characteristics are very similar to those suggested by van Hiele's Level 1 and the CSMS Level 1 of understanding. Van Hiele Level 1 capabilities include recognition of information labeled on a figure (similar to ii above), association of the correct name with
a given figure (similar to i), and development of initial vocabulary (skill v). CSMS Level
1 capabilities include interpretation of a scattergram and of block graphs, (similar to ii),
and recognition that a straight line represents a constant rate (similar to iii).

Level 2 consisted of items 6, 8, 14, 15, 18 and 20. Item 20 (Figure 5) is provided
as an example from this level. Characteristics of these items and the level itself are:

i) recognition of increasing or decreasing behavior of functions;
ii) translates given verbal information into a simple sketch of a graph;
iii) use of initial vocabulary learned in Level 1;
iv) recognition of basic properties and uses of graphs of functions;
v) awareness of the effect of changing the scale;
vi) simple interpolation from a graph.

Note the abilities associated with this level are a "step up" from those listed in Level
1. Thus, the graphing levels exhibit the property of adjacency, what was intrinsic in the
preceding level becomes extrinsic in the current level, a property defined by Usiskin in
describing the van Hiele levels.

A Level 2 understanding of graphs is similar to van Hiele's Level 2 "analytic"
characteristics in that students begin to focus on components and properties of the figure.

Level 3 also has six items; 3, 7, 10, 11, 12, and 19. Item 12 (Figure 6) is
provided to reflect Level 3 capabilities. Level characteristics and related abilities based on
the clustered items are:

i) use of properties of graphs of functions to determine functions from non-
functions;

ii) recognition of connection between a graph and its algebraic representation;
20. The graph below is of the function \( f(x) = x^3 - 4x^2 - 37x + 40 \). The solutions of the equation \( x^3 - 4x^2 - 37x + 40 = 0 \) are:

A. \( x = -120, 50, 80 \)
B. \( x = -5, 1, 8 \)
C. \( x = -5, 1, 8, 50 \)
D. \( x = 0 \)
E. \( x = 40 \)

Figure 5 Item 20 from the Graphing Levels Test

iii) use of properties of functions to construct graphs;

iv) ordering of properties to determine information;

v) understands and uses definitions of functions and their graphs.

Note, the mathematical connections necessary for students to understand the use and importance of graphs is not made until students are operating at Level 3. Table 6 shows only 19.2 percent of the pretest random sample operating at Level 3 at the beginning of the school year with 77.5 percent functioning at Level 2 and lower. Posttest results show over 70 percent of the C^2PC sample operating at Levels 3 or 4 while only 31.3 percent of the control (10 out of 32) attained those levels. These results again suggest benefits from the use of graphing technology within the mathematics classroom.
12. For which of the following values of \( m \) is the graph of \( x^2 - 2x + my^2 + (2m +1)y = 0 \) a parabola?

A. \( m = -\frac{1}{2} \)
B. \( m = 0 \)
C. all values of \( m \) except \(-\frac{1}{2}\)
D. all values of \( m \) except \( 0 \)
E. all values of \( m \) except \(-\frac{1}{2} \) and \( 0 \)

Figure 6 Item 12 from the Graphing Levels Test

Level 4 contains the four most difficult items, 13, 16, 24, and 25 as determined by the students' test results. Item 13 (Figure 7) is provided as an example from this level.

Characteristics of this level as suggested by the clustered items are:

i) use of given information to construct a graph;
ii) use of information from a graph to deduce more information;
iii) can deduce consequences from given information (e.g. understands geometrical transformations);
iv) recognizes from given information what is required to find.

13. This graph is increasing on which intervals?

A. \((-\infty, -10] \) and \([10, \infty)\)
B. \([-10, -3] \) and \([3, 10]\)
C. \((-\infty, 0] \) and \([3, \infty)\)
D. \([-3, -1.7] \) and \([1.7, 3]\)
E. \((-\infty, -1.7] \) and \([1.7, \infty)\)

Figure 7 Item 13 from the Graphing Levels Test
The characteristics of the four levels are very similar if not identical to ones suggested by the CSMS project and research related to van Hiele levels of geometrical understanding. Other similarities between the graphing levels and results of the CSMS project are found by comparing the levels with the four stages of "mathematical" understanding. These four stages were determined by relating all topics investigated in the CSMS project. Stage 1 includes the easiest items and so does Level 1 (by design). Generally only one step is needed to complete a Stage 1 item as is true in Level 1 items. For example, items 1 and 2 are merely "one-step" identification questions. CSMS states Stage 1 items are concerned with understanding the meaning of new conventions, while Stage 2 involves the application of these conventions. This characteristic is also exhibited in the items from Level 2. See item 15 for an example. Stage 3 items involves an aspect of abstraction; questions were not always connected with a diagram. Strategies are required to solve problems, not simple interpretation. Level 3 items also reflect these characteristics (see Figure 6, item 12). Stage 4 items involve aspects of abstract reasoning. "... mathematics as an abstract system, and not merely a way of quantifying real world phenomena appears" (Hart, 1980). Perhaps this can be said of Level 4 items 24 and 25.

As one advances from Level 1 to Level 4, note the items

i) become more complex (compare items 1 and 25);

ii) require more interpretation (compare items 6 and 12);

iii) require relating more ideas (compare items 14 and 13);

iv) depend on more knowledge (compare items 5 and 24);

v) require more complex problem solving strategies (compare items 15 and 16).

These characteristics also attest to the levels forming a hierarchy.
The abilities associated with each level also appear to move from interpreting figural information (IFI) to visual processing (VP) as one progresses through the levels (Bishop, 1983). IFI involves understanding a visual display and its associated spatial vocabulary. IFI involves being able to read, understand, and interpret information from the display, such as a graph. "It is an ability of content and of context..."

VP involves "visualization and the translation of abstract relationships and nonfigural information into visual terms." VP allows one to manipulate and transform visual representations and visual imagery. "It is an ability of process..."

Items from Levels 1 and 2 (1, 2, 5, 9, 14, 18, et cetera) and some from Level 3 (items 3, 7, 11) involve the IFI abilities to read, understand, and interpret information from a graph. However, items 12 and 19 from Level 3 and items 16 and 25 from Level 4 require the VP skills of transforming visual representations and translating nonfigural information into visual terms.

Bishop (1983) notes a dichotomy between IFI and VP related to training and teaching studies. "Intuitively, IFI ability seems much easier to train and develop than VP, if only because of the public, communicable nature of IFI in comparison with the private, personal nature of VP" (Bishop, 1983). Perhaps the incorporation of graphing technology into the training and teaching studies, and also the curriculum, would reduce the dichotomous nature of the two abilities. The results from the C²PC group suggest this to be true.

Transcripts from the twenty seven student posttest interviews provided more insight into level characteristics based on pretest results (see Appendix D). Level 1 items involved use of facts that students are generally required to memorize more so than other levels. For example, it was very difficult for most of the students to say why the graph
in item one was a parabola; they just knew it was. Or why did a negative coefficient on an \( x^2 \) term of a quadratic function flip the graph; it just did.

It is interesting to note van Hiele's comments regarding memorization. He argues memorization of rules, facts, et cetera, should occur during the fifth phase of the learning process, that of integration.

"...it is clear why many teachers switch too quickly to this last phase: they satisfy the wishes of their pupils and they take a shortcut to the goal. However, such a shortened learning process never can lead to real understanding of a concept; the pupils will get the use of a verbal network of relations not based on an experience of action. Such a network of relations not rooted in the personality of the pupil is in danger of quick dissolution. Moreover, because of the absence of connections with other activities, it cannot have any educational value" (van Hiele, 1986).

Level 2 items involved more discourse on the part of the student; item response explanation did not simply consist of the word "Because." Yet when pressed as to why they did things in a particular order, some students had no explanation. For example, when asked which graph was increasing between \( x=1 \) and \( x=3 \) (item 14), students could explain why they chose as they did. Yet when asked what response they would give if someone viewed a graph of a positive sloped line from right to left and said the graph was decreasing, the students tended to lack for words.

Level 3 and 4 items involved use and knowledge of properties of functions and their graphs, hence the low facilities for these items at the pretest stage. Students lack of knowledge of the vocabulary of precalculus terms (i.e. trigonometric function, complex number, increasing function, etc.) contributed to the low facility levels. However, the
vocabulary indicated concepts needed to understand and operate at the advanced levels and hence was symptomatic of the increased complexity of the item.

5.4 Characterizing the Students

Twenty seven students were interviewed after the posttest in April (see Appendix D for interview transcripts). The students interviewed were chosen from each of the six classes participating in the study. The interview consisted of the student answering preselected questions from the Graphing Levels Test and providing their reasoning or strategies behind each response. The selection of items, performed by the author, was done prior to the interview. The questions were selected in somewhat of a random manner, either odds or evens or some other combination of items. Analysis of data had not taken place at the time of the interviews so items were not clustered into levels.

Similarities and differences between the students operating at different levels, and students operating at the same levels, were found. These similarities and differences existed in the students' problem solving strategies, their vocabulary, their graph sketching ability, their ability to verbalize their cognitive processes, et cetera.

All four levels of understanding were represented by the interviewed students. Some were in violation of the hierarchy, i.e. they skipped a level. The interviews provided further insight in the area of graphical understanding and student learning.

Two of the twenty seven students interviewed attained only a Level 1 understanding of functions and their graphs based on their posttest scores. They were students 4-2 and 3-4 (see Appendix E for level scores). The first identification number for each student represents the teacher number and the second number indicates the order of appearance in the interview in relation to the other students having the same teacher.
Neither of these students felt comfortable in determining where a function was increasing or decreasing. Both incorrectly determined the coordinates of an isolated point (item 3, Level 3) due to disregarding the scaling on the x-axis. In determining the maximum of a function based on an equation alone (item 9, Level 3) neither tried to draw a sketch of the graph (a Level 2 skill). Both of them failed to use translations correctly. Student 3-4 did not know what a cubic function was and both forgot the definition of complex numbers (see van Hiele's comment on memorization). Both exhibited a lack of understanding of symmetry within a parabola (item 15, Level 2). Comments from these students included "I'm not really sure how you get graphs from equations like that," "I'm not too good with the graph," and "I've heard all of this before and seen this before but not together," (referring to determining where a graph was increasing; see Figure 7). Student 3-4 guessed correctly on item 20 (Level 2) but stated she did not really know the answer or why she chose as she did other than the fact that those points were intercepts. This type of response indicates the "power" of the graph to draw one's attention to important features, as noted by Vonder Embse (1987).

Eight of the twenty seven were categorized as Level 2 students. Similarities and differences can be found between these eight based on their interviews. These students have the ability to use the graph correctly to solve an item; they have an ability to sketch graphs; they note scale differences; they have difficulty in solving items that involve performing several tasks at one time, as in item 13. They are still building vocabulary and learning to read mathematics. Student 4-3 read h(x + 3) as "h times (x + 3)" and g(x) as "x of g" then later as "g to the x."

Where the Level 2 students truly differed was on their ability to answer higher level questions. Some knew the relationship between real and complex numbers (item 24).
Others could perform translations correctly (item 25). Some could solve a word problem, item 16.

Based on the interviews, even though the score determined the level of understanding, one may predict a lower level of understanding for some of these students. This is possible since students can perform at higher levels without true understanding of what they are doing. Van Hiele terms this "level reduction." This is why the author believes that communication of teacher-to-student and student-to-student be required in the mathematics classroom. Communication of ideas and concepts is necessary to progress through the levels and for one to have a true understanding of the topic, not just a performance capability. Note this communication of ideas is the explication phase within van Hiele's phases of learning. Students need to pass through each of five phases (see Section 2.3) before attempting the next level of understanding. If phases are omitted from the student's learning process, the understanding will be only superficial, as was evident in some of the interviews. The importance of the teacher in the student's learning process should not be underestimated or overlooked. They are responsible for providing students the opportunities and activities for each of the five phases, as suggested by van Hiele. Teachers are not separate from the curriculum; they are an integral part of it.

Level 3 students could verbalize their responses and actions better than Level 1 or 2 students. Differences between C^2PC Level 3 students and a control Level 3 were detected from comments like "The graph made it alot easier," "I'm gonna look at the graph and see where the zeros are and then that would tell me what the solutions are," versus "I could use the quadratic equation (instead of formula)," "Well, I wish I knew what this looked like," and "The graph means very little to me." The author believes Student 3-2 can operate at Level 3 but has only a Level 1 or 2 graphical understanding,
based on her interview responses. Level 3 students could draw sketches more readily than students from Level 2. Some connections between a graph and its equation were made but the students were not necessarily consistent throughout the test.

Level 4 students had the most brief interviews as compared to the other students. Their responses were generally quick and to the point. If incorrect strategies were chosen, they were quickly detected and remedied. It was not until interviews with Level 4 students that an "error" was detected on item 24 (see Figure 8). The Level 4 students pointed out that since real numbers are part of the complex number system, the problem has three complex solutions. This was not the author's intent for the item. Here again is an argument for the necessity of communication. Item analysis does not account for student responses in scoring an exam. How would one know why a Level 4 student "erred" on item 24 versus a Level 1 or 2 student without asking the students to provide their reasoning?

24. Consider the graph of \( y = g(x) \) shown below. Let \( y \) be a cubic function. How many complex number solutions does the equation \( y = g(x) \) have?

\[
\begin{align*}
A. & \quad 0 \\
B. & \quad 1 \\
C. & \quad 2 \\
D. & \quad 3 \\
E. & \quad 4
\end{align*}
\]

Figure 8 Item 24 from the Graphing Levels Test
A C^2PC Level 4 student noted "It's easier with a graphing calculator," (referring to solving some problems) but still solved the problems quickly and correctly without the use of the calculator.

The control Level 4 student, 3-3, performed as well as the other Level 4 students except when it came to using geometric transformations in item 25. Since graphing is not extensively incorporated into the precalculus curriculum, fairly simple transformations, such as vertical and horizontal translation, are generally not discussed when referring to graphs of higher order polynomials.

In summary, the level characteristics based on the clustered items identified abilities of students. The abilities of the students based on their interview responses were in general agreement with those determined by the clustered items. It was possible, however, for students to have scores placing them at a higher level of understanding than was evident from their abilities during the interview. The author believes this is due to the curriculum, in that teachers have a tendency to get to the "formula" first instead of students "discovering" the ideas themselves. Memorization of facts take place too early in the students' learning process and understanding of concepts is then limited. This reduction of level (van Hiele, 1986) occurred in both the C^2PC and control groups. Hence, characterizing students according to levels of understanding cannot be done solely by examining test scores; they are a reference point. Communication between teacher and student is necessary to fully determine the students understanding of the topic.

5.5 Summary

Using the facility ordered complete linkage analysis and other level criteria as suggested by the CSMS project, four clusters, or levels, were determined. Level characteristics were determined based on the clustered items and results from the student interviews.
Many similarities were found between the graphing levels determined in this study and the CSMS graphing levels, the four stages of mathematical understanding based on results from the CSMS project, and van Hiele's levels of understanding.

Responses of the students during their interviews further support the level structure as determined by the cluster analysis. Some students' responses indicated they had a lower level of understanding than indicated by their test score. The author believes it is possible for a student to perform at a level of understanding that is beyond their true understanding of the concept. The differences in understanding the topic and performing superficial skills is evident only when communication between the student and teacher takes place. Thus, the author believes verbal, and pictorial, communication of ideas should be a part of the mathematics curriculum.

This study provides further evidence to support a learning levels theory of understanding, specifically that of graphing levels of understanding.
CHAPTER SIX
DISCUSSION AND RESULTS

6.1 Introduction

In order to investigate students' understanding of functions and graphs within a precalculus curriculum, a graphing levels instrument was designed and administered to over two hundred precalculus students. Analysis of a random pretest sample clustered the items into four levels. Posttest analyses, along with student interviews, indicated that the level structure was invariant across both the C^2PC and control groups and was hierarchial in nature.

The C^2PC groups significantly outperformed the control on the posttest. Similarities and differences between the groups along with implications for changes within the mathematics curriculum are discussed.

6.2 Group similarities and differences

Comparison of pre- and posttest results for both groups show both similarities and differences. Level 1 items remained in essentially the same facility band for both groups. Item 2 was the only "outlier" for the C^2PC group with its facility decreasing to 74.4 percent. The author believes this was due in part to the students disregarding the arrows on the graph of the function and focusing on the idea of the viewing rectangle not being large enough to capture the whole graph of an ellipse.

Once at Level 2, item facility bands change dramatically between the two groups. Level 2 items for the C^2PC group all grouped in the pretest Level 1 facility band, ranging in facility from 83.2 to 94.4 percent. Inter-item correlations greater than 0.2 still existed...
between the Level 2 items for the posttest data, along with Level 2 items correlating with Level 1 items, e.g., item 14 linking with item 9, item 5 linking with item 20, etc. This "jump" in levels was expected by the end of one year; it demonstrates a growth in graphical understanding.

Unfortunately, the students in the control group did not demonstrate such a growth. Four of the six items in Level 2 (items 6, 15, 18, and 20) remained in the same band of facility with only one item, 14, demonstrating increased understanding and item 8, showing a decreased understanding. Item 14 asks the students to select from five graphs one that is increasing between x=1 and x=3. Based on the facility level of 87.5 percent, it appears the control students knew to select a positive-sloped linear function. Yet the posttest interviews of control students indicate a lack of a true understanding as to what increasing over particular intervals means. Item 13 (see Figure 7) also addresses this concept but in a different fashion. For the control group, this item remained difficult with a posttest facility level of 21.9 percent. The interviewed students focused on the arrows "pointing up" and had a difficult time of selecting appropriate intervals. Response B proved to be the greatest distractor for these students (28 percent chose this response) indicating a lack of understanding in how to read a graph, or of how a graph increases over a particular interval, or even in how to focus on an interval of a graph, graphing skills assumed basic when beginning the study of calculus.

Item 8 required an understanding of what a maximum y-value was and an ability to use graphs to determine if a function attained a maximum y-value. Response A was the greatest distractor for the control group (38 percent chose this response) indicating a misconception between the ideas of "infinitely large" and "maximum value."

Another item of interest was 20 (see Figure 5). For the control, the facility for this question essentially remained the same; 60 percent for the pretest and 59.4 percent for the
posttest. One student's comments for this question were "The graph means very little to me." To find the zeros, the student tried to substitute in the response choices in the given equation and then determining if they were points on the graph. Out of frustration, the student pleaded "Tell me there's an easier way to do this."

Again the C2PC students demonstrated a marked increase of understanding at Level 3 with most items moving up at least one level in facility. The author believes this is due to the increased emphasis on graphing using technology. Inter-item correlations greater than 0.2 spanned across Level 1, Level 2 and Level 3 items, also probably due to the graphing emphasis. For example, item 11 increased in facility from 52 to 93.6 percent. Based on C2PC student interviews, the reason for this increase was largely based on knowledge of the vertical line test for functions. Some students, in explaining why the test was used, referred to the definition for (well-defined) functions, indicating an understanding for using the vertical line test versus the other response of "I just know it works."

The control group students' understanding of Level 3 items barely improved. Item 7 was the only item to jump two levels, increasing in facility from 49.6 to 96.9 percent. This item requires distinguishing a trigonometric function from a quadratic, cubic, and an exponential function. Control group posttest interviews indicate a familiarity with the sine and cosine functions. However, when asked why they chose their response, the control students had difficulty in naming the other functions. They knew sine and cosine were terms used in trigonometry and, in most of the responses, based their answer on this limited knowledge. A greater percentage of the C2PC students interviewed could name the three other functions in addition to recognizing the oscillating behavior of the sine function.
Further indications of the little improved understanding are reflected in the facility levels of the other Level 3 items. Items 11 and 3 moved to the Level 2 facility span. Item 3 indicates a student's ability to simultaneously read a point's coordinates and focus on the scale on the axes. Items 19 and 10 remained at the Level 3 facility span while item 12 moved down in facility from 36.8 to 28.1 percent.

Students in the control group reflected very little growth in Level 4 items (see Figure 3). Four students were interviewed from this group and demonstrated their lack of knowledge of the use of graphs to answer questions related to functions. Similar results have been reported by the national and international studies. Students generally cannot make the connection between an algebraic and a geometric representation. On the other hand, students in the C^2PC group showed a marked increase in facility for Level 4 items (see Figure 2). The author believes this increase was due to the extensive use of computers/calculators within the classroom as related to the teaching of functions and their graphs.

Except for a few items that indicated dramatic growth in understanding for the C^2PC students, the level structure appears invariant to a marked extent for both groups. The "outlying" items are believed to be results of the C^2PC curriculum.

6.3 Van Hiele and Instruction

Along with van Hiele levels of understanding, van Hiele suggested five phases of learning within each level of understanding (see Section 2.3). The phases provide a guide for instruction for the teacher and promote the understanding within a level and movement onto the next higher level. The author believes the C^2PC project includes the five phases implicitly within their curriculum design.
The first phase, information, allows the students time to become acquainted with the new topic. The course was designed to provide more than one or two days per topic, allowing the necessary time for the students to become familiar with the topic.

Guided orientation is provided in the second phase of learning, as suggested by van Hiele. The C²PC curriculum provided ample "materials" for this phase with the use of the computer package and/or the graphics calculator. The homework exercises included needed short tasks for exploration and experimentation by the students, building on their information gained in the first phase.

The author believes the third phase to be one of the most important of the five, that of explication or discussion. Unfortunately, in most classrooms, this phase is generally passed over based on student test results. For example, if the class as a whole performs well on a test, the teacher assumes understanding of the topic has taken place. But this need not be the case, as was shown by the student interviews. In order for the teacher to confirm his/her belief that true understanding has occurred, time must be allotted to the students for discussing what they think they know. The author observed this during classroom visits to the C²PC pilot sites. While the students were at the computers working, they were discussing what should or would happen when they graphed a function or were trying to solve a problem using the grapher. This time will also allow for development of a "spatial vocabulary" associated with graphs involved in interpreting figural information (IFI) (Bishop, 1983).

The fourth phase is that of free orientation where students are given a variety of tasks which could be completed in several ways and are open-ended. Again, the C²PC curriculum allows for this phase at the computer or by using the calculator. Students are provided time to experiment and in many cases went well beyond what was requested of them in the task, simply because they were in control of the technology and were curious
to see what would happen if they tried something different. The immediacy of results provided by a computer allows students to create many examples within a short period of time and make generalizations. The computer provides the experiences necessary to make mathematical connections.

Working with a computer or graphing calculator may also increase the students' visual processing (VP). The technology allows students to manipulate and transform graphs. Later, these manipulations and transformations could be done mentally without use of the "concrete manipulative" due to the great number of previous examples, examples generated by the students themselves.

Students review and summarize what they have learned in the fifth phase of learning; integration. Most curricula provide for this phase but it is generally directed by the teachers and is a one-way operation, i.e. the teacher tells the student what will be on the test. If students have worked their way through the first four phases, with time allowed for discussion between themselves, then it should be possible that the review session could be handled by the students with only little guidance from the teacher. This would be the ideal situation. Van Hiele believes this is where memorization of facts or rules should take place. Unfortunately, teachers have a tendency to present rules and formulas too soon and students have a tendency to memorize them before a true understanding has occurred.

6.4 Implications for the "Graphing Curriculum"

This research was designed to determine and characterize levels of understanding for functions and their graphs within the precalculus curriculum. It could be argued that the age span was too narrow to provide valid results but the author disagrees. The CSMS project found children at every level in each of the secondary age groups investigated (Hart, 1981). This will probably be true for most groups of students as
educational experiences provided them vary early on and will be diverse throughout their schooling. Age alone does not imply a particular level of understanding. The CSMS project states "...when selecting mathematics material for presentation to children it is unwise to base this primarily on age" (Hart, 1980). Experience along with age contribute to a child’s knowledge and understanding. The teacher then is an integral part of a child’s education since they dictate the experiences provided the children.

Four levels of understanding were determined by the pretest analysis and were also evident in the posttest analyses of both the C²PC and the control group test results. The characteristics of these levels were also consistent with past research results in describing levels of understanding. Yet marked differences occurred between the groups, implying the benefits of the use of technology within the classroom.

The C²PC group had higher group posttest means than the control implying their increased experience with graphs improved their understanding of and ability to use graphs. Posttest results indicated the majority of C²PC attained Levels 3 or 4 while less than one third of the control group attained these high levels. Again, the author believes the increased experience with technology along with the C²PC curriculum accounts for these differences in level attainment.

Perhaps just as important as the levels of understanding are what van Hiele terms the phases of the learning process. In order to progress through the levels, the students must pass through the five phases described in Sections 2.3 and 6.3. The necessity of the explication or discussion phase in the learning process was made evident in the interviews. Students, particularly those from Levels 1 and 2, had difficulty in explaining their strategies and were lacking in "mathematics" vocabulary. Some Level 3 students only knew the needed skills for solving problems but had no idea why they were doing what they doing. For example, in item 9 students are asked what is the maximum y-
value of the function $y = x^2 + 4x - 9$. Many students' first strategy was to try and factor the equation. They had no idea why; it was a stimulus-response item for them. See a quadratic equation; first thing to do is factor the equation. "Violation" of the order of the phases produces results such as this. Memorization without understanding "has no educational value" (van Hiele, 1986). Educators must be careful not to violate the order in the learning process. They should provide more experiences for the students and allow time for the students to discuss their findings.

Results from this study indicate changes for the precalculus curriculum. Technology needs to be incorporated into the curriculum, specifically that related to graphing. If the precalculus curriculum is to prepare one for calculus, groundwork for calculus concepts and the corresponding skills must be laid. Part of this important groundwork is related to an understanding of functions and their graphs. As noted by the NCTM Commission on Standards (1988), among the most important of the mathematical connections to be made in the curriculum are those between algebra and geometry. These connections can and should be developed in the precalculus curriculum. Graphing technology, as demonstrated by this study, is a definite aid to the student in developing the mathematical connections. Technology provided C²PC students with an increased example base and a greater number of graphing experiences than the control, furthering the C²PC students' understanding of the connection between an function and its graph.

The hierarchial nature of the graphing levels suggests an order in the presentation of the topic (see Section 5.3). The greatest task for the educator is to allow students opportunities for making their own connections and not to provide shortcuts early on. For example, instead of telling the students what the graph of a quadratic equation looks like, allow them to experiment, a feat now possible with the aid of a graphing calculator. Not
only does this allow the student to make their own generalization of what the graph of a quadratic looks like, it provides them exercises in creating quadratic equations.

In summary, results of this study show learning of functions and their graphs occurs in levels. The hierarchical nature of the levels implies an order to the presentation of topics related to graphing.

Results also imply the use of technology in the classroom improves student understanding of functions and their graphs by providing an increased example base. The interaction between student and computer also provided for making mathematical connections, connections necessary to understand and use graphs.

The student interviews also made evident the importance of the explication phase in the process of learning. Students need an opportunity to make conjectures and discuss their findings. Technology provides the means to investigation and immediate feedback.

6.5 Future Research

At the present time, the Graphing Levels Test is very broad in scope. Future studies should be directed toward refining the graphing instrument. Examining subtopics such as scaling, problem solving, geometric transformations, et cetera may suggest further refinements for the level characterizations. Other instruments could have more items for each graphical concept considered, rather than just one or two as in the present research.

There needs to be further investigation into the importance of the five phases of the learning process, as described by van Hiele, particularly that of the explication phase. Studies comparing the retention of students' knowledge and understanding acquired by differing learning processes, i.e. those "violating" the learning process and those not, need to be performed.

A graphing instrument needs to be administered to other age groups to determine if the level structure suggested by this research is maintained. Groups of students using and
not using technology should be tested to determine if the benefits of using the graphing technology are consistent across other ages.
GRAPHING LEVELS TEST  Aug, 1987

Turn the answer sheet so that "The Ohio Early College Mathematics Placement Testing Program" is on the top. Please do the following before taking the test. [Use only #2 pencil. Please ask for one if you do not have any].

1. Code in your name.
2. Below the section provided for your name, code in your high school grade.
3. In the section provided for your high school ID, code in the appropriate number:
   0001 Upper Arlington    0002 Centennial
   0003 Walnut Ridge       0004 Franklin Heights
4. In the upper right corner of the answer sheet write the test date.

DIRECTIONS

This test consists of 25 items. You will have at most 35 minutes to work on the test. Please give this test your best effort.

In each item, you will be asked a question, to complete a statement, or to solve an equation or inequality. Each item has only one correct answer.

Example: 1. The equation $3 - 5x + 4x^2 = 1$ is of degree:
   A. 1  B. 2  C. 3  D. 4  E. 5

In this example, the correct answer is "B." You would then mark the answer sheet by darkening in the appropriate circle as shown below. Please use a #2 pencil and press hard.

(A) (B) (C) (D) (E)

Please try to answer all items. There is no penalty for guessing. If you have difficulty with a particular item, move on and come back to it later. Do any scratch work on the test paper, not the answer sheet.

Thank you for your cooperation.

PLEASE DO NOT TURN THIS PAGE UNTIL YOU ARE ASKED TO DO SO.
GRAPHING LEVELS TEST

1. The graph below is a
   A. point.
   B. line.
   C. parabola.
   D. ellipse.
   E. circle.

2. If we rotate the above graph so it looks like the graph below, then we have the graph of
   A. a point.
   B. a line.
   C. a parabola.
   D. an ellipse.
   E. a circle.

3. What are the coordinates of point P?
   A. (-4, 2)
   B. (2, -4)
   C. (-8, 2)
   D. (4, -4)
   E. (-4, 4)
4. Which is the graph of an exponential function?

I. \[ \text{Graph A} \]
II. \[ \text{Graph B} \]
III. \[ \text{Graph C} \]
IV. \[ \text{Graph D} \]

A. I
B. II
C. III
D. IV
E. None of the above

5. Consider the graphs of equation A: \( y = ax^2 + bx + c \) and of equation B: \( y = -(ax^2 + bx + c) \).

A. The graph of B is wider than the graph of A.
B. The graph of B is not as wide as the graph of A.
C. The graph of B is an upside down version of the graph of A.
D. The graph of B is rotated 90° in relation to the graph of A.
E. The graph of B is identical to the graph of A.

6. Which of the following points is symmetric to \((3, -5)\) with respect to the origin?

A. \((-5, 3)\).
B. \((5, -3)\).
C. \((3, -5)\).
D. \((-3, 5)\).
E. \((-3, -5)\).
7. Which is the graph of a trigonometric function?

I. 

II. 

III. 

IV. 

A. I
B. II
C. III
D. IV
E. None of the above

8. Which graph has a maximum y value?

I. 

II. 

III. 

IV. 

A. I
B. II
C. III
D. IV
E. None of the above
9. What is the maximum value of \( y \) if \( y = x^2 + 4x - 9 \).

A. \( y = 9 \)
B. \( y = 13 \)
C. \( y = -9 \)
D. \( y = -13 \)
E. There is no maximum value of \( y \).

10. If the vertex of a parabola is (2, 5) and the coefficient of the \( x^2 \) term in the quadratic function is negative, which of the following points could not be on the curve?

A. (-5, -44)
B. (4, -11)
C. (0, 1)
D. (-1, -4)
E. (-3, 30)

11. Which of the following are graphs of functions, \( y = f(x) \)?

A. I and II only
B. II and III only
C. III and IV only
D. I and III only
E. I and IV only
12. For which of the following values of $m$ is the graph of $x^2 - 2x + my^2 + (2m + 1)y = 0$ a parabola?

A. $m = -1/2$
B. $m = 0$
C. all values of $m$ except $-1/2$
D. all values of $m$ except $0$
E. all values of $m$ except $-1/2$ and $0$

13. This graph is increasing on which intervals?

A. $(-∞, -10]$ and $[10, ∞)$
B. $[-10, -3]$ and $[3, 10]$
C. $(-∞, 0]$ and $[3, ∞)$
D. $[-3, -1.7]$ and $[1.7, 3]$
E. $(-∞, -1.7]$ and $[1.7, ∞)$

14. Which graph is increasing between $x = 1$ and $x = 3$?

A. I
B. II
C. III
D. IV
E. V
15. The point (3,0) is the vertex of a parabola symmetric with respect to a vertical line. If (5, 4) is on the parabola, determine another point on the parabola.

A. (-5, 4)
B. (5, -4)
C. (1, 4)
D. (-1, 4)
E. Not enough information is given.

16. A rectangular pen is built so that one side is against the wall of a barn. The other sides of the fence must total 50 feet in length. If y represents the area of the pen and x the length of the side of the fence perpendicular to the barn, which of the following graphs describes the relationship between y and x?

A. I
B. II
C. III
D. IV
E. None of the above
17. Let \( y_1 = x^2 \) and \( y_2 = (x - a)^2 \). How is the graph of \( y_2 \) obtained from the graph of \( y_1 \)? By

A. moving the \( y_1 \) curve up \( a \) units.
B. moving the \( y_1 \) curve down \( a \) units.
C. moving the \( y_1 \) curve left \( a \) units.
D. moving the \( y_1 \) curve right \( a \) units.
E. stretching the \( y_1 \) coordinates \( a \) units.

18. Which are graphs of the same set of ordered pairs?

I. 

\[ \begin{array}{c}
\text{III.} \\
\text{IV.}
\end{array} \]

A. I and II.
B. II and III.
C. III and IV.
D. I and IV.
E. II and IV.
19. Consider the graph below. Which interval on the x-axis describes the solution to $x^2 < 4$?

A. 

B. 

C. 

D. 

E. None of the above

20. The graph below is of the function $f(x) = x^3 - 4x^2 - 37x + 40$. The solutions of the equation $x^3 - 4x^2 - 37x + 40 = 0$ are:

A. $x = -120, 50, 80$
B. $x = -5, 1, 8$
C. $x = -5, 1, 8, 50$
D. $x = 0$
E. $x = 40$

21. What is the total number of points on the line shown below?

A. Two.
B. Two hundred.
C. Ten thousand.
D. One million.
E. There is an infinite number.
22. Which equation is represented by the graph below?

A. $r = \ln(5t)$  
B. $r^2 + t^2 = 5$  
C. $r = \sin(5t)$  
D. $r = 5\sin(t)$  
E. $r = e^{5t}$

23. Consider the graph of $y = f(x)$ shown below. How many real number solutions does the equation $f(x) - 3 = 0$ have?

A. 0  
B. 1  
C. 2  
D. 3  
E. 4

24. Consider the graph of $y = g(x)$ shown below. How many complex number solutions does the equation $y = g(x)$ have?

A. 0  
B. 1  
C. 2  
D. 3  
E. 4
25. Consider the graph of $y = h(x)$ shown to the right. Which of the graphs below is a representation of $y = -1 + h(x + 3)$?

A. I  
B. II  
C. III  
D. IV  
E. V

This is the end of the test. You may go back to previous problems if you wish.
GRAPHING LEVELS TEST  Aug./April, 1987-88

Turn the answer sheet so that "The Ohio Early College Mathematics Placement Testing Program" is on the top. Please do the following before taking the test. [Use only #2 pencil. Please ask for one if you do not have any].

1. Code in your name.
2. Below the section provided for your name, code in your high school grade.
3. In the section provided for your high school ID, code in the appropriate number: 0001 Upper Arlington 0002 Centennial 0003 Walnut Ridge 0004 Franklin Heights
4. In the upper right corner of the answer sheet write the test date.

DIRECTIONS

This test consists of 25 items. You will have at most 35 minutes to work on the test. Please give this test your best effort.

In each item, you will be asked a question, to complete a statement, or to solve an equation or inequality. Each item has only one correct answer.

Example: 1. The equation $3 - 5x + 4x^2 = 1$ is of degree:
   A. 1   B. 2   C. 3   D. 4   E. 5

In this example, the correct answer is "B." You would then mark the answer sheet by darkening in the appropriate circle as shown below. Please use a #2 pencil and press hard.

A B C D E

Please try to answer all items. There is no penalty for guessing. If you have difficulty with a particular item, move on and come back to it later. Do any scratch work on the test paper, not the answer sheet.

Thank you for your cooperation.

PLEASE DO NOT TURN THIS PAGE UNTIL YOU ARE ASKED TO DO SO.
1. The graph below is a

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B. line.
C. parabola.
D. ellipse.
E. circle.

2. If we rotate the above graph so it looks like the graph below, then we have the graph of

A. a point.
B. a line.
C. a parabola.
D. an ellipse.
E. a circle.

3. What are the coordinates of point P?

A. (-4, 2)
B. (2, -4)
C. (-8, 2)
D. (4, -4)
E. (-4, 4)
4. Which is the graph of an exponential function?

A. I
B. II
C. III
D. IV
E. None of the above

5. Consider the graphs of equation A: \( y = ax^2 + bx + c \) and of equation B: \( y = -(ax^2 + bx + c) \).

A. The graph of B is wider than the graph of A.
B. The graph of B is not as wide as the graph of A.
C. The graph of B is an upside down version of the graph of A.
D. The graph of B is rotated 90° in relation to the graph of A.
E. The graph of B is identical to the graph of A.

6. Which of the following points is symmetric to \((3, -5)\) with respect to the origin?

A. \((-5, 3)\).
B. \((5, -3)\).
C. \((3, -5)\).
D. \((-3, 5)\).
E. \((-3, -5)\).
7. Which is the graph of a trigonometric function?

A. I
B. II
C. III
D. IV
E. None of the above

8. Which graph has a maximum y value?

A. I
B. II
C. III
D. IV
E. None of the above
9. What is the maximum value of \( y \) if \( y = x^2 + 4x - 9 \).

A. \( y = 9 \)
B. \( y = 13 \)
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D. \( y = -13 \)
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A. \((-5, -44)\)
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C. \((0, 1)\)
D. \((-1, -4)\)
E. \((-3, 30)\)

11. Which of the following are graphs of functions, \( y = f(x) \)?

A. I and II only
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C. III and IV only
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E. I and IV only
12. For which of the following values of m is the graph of $x^2 - 2x + my^2 + (2m + 1)y = 0$ a parabola?

A. $m = -\frac{1}{2}$  
B. $m = 0$  
C. all values of m except $-\frac{1}{2}$  
D. all values of m except 0  
E. all values of m except $-\frac{1}{2}$ and 0

13. This graph is increasing on which intervals?

A. $(-\infty, -10]$ and $[10, \infty)$  
B. $[-10, -3]$ and $[3, 10]$  
C. $(-\infty, 0]$ and $[3, \infty)$  
D. $[-3, -1.7]$ and $[1.7, 3]$  
E. $(-\infty, -1.7]$ and $[1.7, \infty)$

14. Which graph is increasing between $x = 1$ and $x = 3$?

A. I  
B. II  
C. III  
D. IV  
E. V
15. The point (3,0) is the vertex of a parabola symmetric with respect to a vertical line. If (5, 4) is on the parabola, determine another point on the parabola.

A. (-5,4)  
B. (5,-4)  
C. (1, 4)  
D. (-1, 4)  
E. Not enough information is given.

16. A rectangular pen is built so that one side is against the wall of a barn. The other sides of the fence must total 50 feet in length. If y represents the area of the pen and x the length of the side of the fence perpendicular to the barn, which of the following graphs describes the relationship between y and x?

I.  
II.  
III.  
IV.  
A. I  
B. II  
C. III  
D. IV  
E. None of the above
17. Let $y_1 = x^2$ and $y_2 = (x - a)^2$. How is the graph of $y_2$ obtained from the graph of $y_1$? By

A. moving the $y_1$ curve up $a$ units.
B. moving the $y_1$ curve down $a$ units.
C. moving the $y_1$ curve left $a$ units.
D. moving the $y_1$ curve right $a$ units.
E. stretching the $y_1$ coordinates $a$ units.

18. Which are graphs of the same set of ordered pairs?

A. I and II.
B. II and III.
C. III and IV.
D. I and IV.
E. II and IV.
19. Consider the graph below. Which interval on the x-axis describes the solution to $x^2 < 4$?

A. $-2 \rightarrow 2$
B. $-2 \rightarrow 2$
C. $-2 \rightarrow 2$
D. $-2 \rightarrow 2$
E. None of the above

20. The graph below is of the function $f(x) = x^3 - 4x^2 - 37x + 40$. The solutions of the equation $x^3 - 4x^2 - 37x + 40 = 0$ are:

A. $x = -120, 50, 80$
B. $x = -5, 1, 8$
C. $x = -5, 1, 8, 50$
D. $x = 0$
E. $x = 40$

21. What is the total number of points on the line segment shown below?

A. Two.
B. Two hundred.
C. Ten thousand.
D. One million.
E. There is an infinite number.
22. The length of a shadow is directly proportional to the height of the object. Which graph below shows the length of a shadow \((L)\) as a function of the height \((h)\)?

A. I  
B. II  
C. III  
D. IV  
E. None of the above

23. Consider the graph of \(y = f(x)\) shown below. How many real number solutions does the equation \(f(x) - 3 = 0\) have?

A. 0  
B. 1  
C. 2  
D. 3  
E. 4
24. Consider the graph of \( y = g(x) \) shown below. Let \( y \) be a cubic function. How many complex number solutions does the equation \( y = g(x) \) have?

\[ \begin{array}{c}
\text{A. 0} \\
\text{B. 1} \\
\text{C. 2} \\
\text{D. 3} \\
\text{E. 4}
\end{array} \]

25. Consider the graph of \( y = h(x) \) shown to the right. Which of the graphs below is a representation of \( y = -1 + h(x + 3) \)?

\[ \begin{array}{c}
\text{I.} \\
\text{II.} \\
\text{III.} \\
\text{IV.} \\
\text{V.}
\end{array} \]

\[ \begin{array}{c}
\text{A. I} \\
\text{B. II} \\
\text{C. III} \\
\text{D. IV} \\
\text{E. V}
\end{array} \]

This is the end of the test.
You may go back to previous problems if you wish.
Turn the answer sheet so that “The Ohio Early College Mathematics Placement Testing Program” is on the top. Please do the following before taking the test. [Use only #2 pencil. Please ask for one if you do not have any].

1. Put your name in the section provided in the upper left corner of the page. Place one letter per block, skipping one block between your last and first name, and first name and middle initial. Darken the circles containing the appropriate letters for your name.

2. Below the section provided for your name, is a section for your high school grade. Darken the appropriate circle. (Freshman Leave BLANK.)

3. In the section provided for high school ID, darken completely the appropriate circles: 0001 Centennial, 0002 Upper Arlington.

4. In the upper right corner of the answer sheet write the test date.

5. In the section entitled “Background Questions”, code in the following appropriate information on line A: 1 Algebra 1, 2 Algebra 2, 3 Geometry, 4 Transitions, 5 Precalculus, 6 Calculus.

DIRECTIONS

This test consists of 20 items. You will have at most 35 minutes to work on the test. Please give this test your best effort. In each item, you will be asked a question, to complete a statement, or to solve an equation or inequality. Each item has only one correct answer.

Example: 1. The following equation has degree of:

\[3 - 5x + 4x^2 = 1\]

A. 1  B. 2  C. 3  D. 4  E. 5

In this example, the correct answer is "B". You would then mark the answer sheet by darkening in the appropriate circle as shown below. Please use a #2 pencil and press hard.

A  B  C  D  E

Please try to answer all items. If you have difficulty with a particular item, go on to the next item and come back to it later. Do not use calculators, computers, books, or notes. Do any scratch work on the test paper, not the answer sheet.

Thank you for your cooperation.

PLEASE DO NOT TURN THIS PAGE UNTIL YOU ARE ASKED TO DO SO.
VAN HIELE TEST

1. The graph below is

A. a point.
B. a line.
C. a parabola.
D. an ellipse.
E. a circle.

2. If we turn the above graph sideways so it looks like the graph below, we then have the graph of

A. a point.
B. a line.
C. a parabola.
D. an ellipse.
E. a circle.

3. Which of the following is a quadratic function?

A. \( f(x) = x + 7 \)
B. \( f(x) = 2x + 5 \)
C. \( f(x) = 2x^2 + 3x \)
D. \( f(x) = x^3 + x^2 \)
E. \( f(x) = x^4 + 3x^2 - 3 \)

Over Please
4. The graphs of quadratic functions are
   A. sometimes parabolas, sometimes circles.
   B. always parabolas.
   C. always circles.
   D. always ellipses.
   E. sometimes circles, sometimes ellipses.

5. If the coefficient of the $x^2$ term in the equation $y = ax^2 + bx + c$ is positive the graph of the equation opens
   A. upwards.
   B. downwards.
   C. to the right.
   D. to the left.
   E. in a direction which cannot be determined from the given information.

6. What is the vertex of the graph below?
   
   ![Graph with vertex at (-0.25, 0)]
   A. (-0.25, 0)
   B. (0, 1)
   C. (4.25, 0)
   D. (2, 5)
   E. The graph has no vertex

Over Please
7. Which of the following graphs have a maximum y value?

A. I  
B. II 
C. III  
D. IV  
E. None of the above

8. What is the maximum value of y if \( y = -(x - 2)^2 - 5 \)?

A. \( y = 2 \).  
B. \( y = -2 \).  
C. \( y = 5 \).  
D. \( y = -5 \).  
E. There is no maximum value of y.

Over Please
9. Let \( y = -(x - 2)^2 - 5 \). What value of \( x \) corresponds to the value \( 2 \) for \( y \)?

A. \( x = 2 \).
B. \( x > 2 \).
C. \( x = -5 \).
D. \( x < -5 \).
E. No such value for \( x \) exists

10. Let \( y = -(x - 2)^2 - 5 \). For \( x \) greater than 2 and increasing, the corresponding values for \( y \)

A. increase.
B. decrease.
C. increase then decrease.
D. decrease then increase.
E. are all equal to -5.

11. Which of the following are graphs of functions, \( y = f(x) \)?

A. I and II only
B. II and III only
C. III and IV only
D. I and III only
E. I and IV only

Over Please
12. Which of the following is a correct and complete definition for function?

A. A function is a set of ordered pairs.
B. A function is a relation.
C. A function is a set of ordered pairs in which the second entry is uniquely determined by the first entry.
D. A function is a graph which can be drawn without lifting your pencil.
E. A function is a polynomial of degree higher than one.

13. For which of the following values of \( m \) is the graph of
\[ x^2 - 2x + my^2 + (2m +1)y = 0 \]
a parabola?

A. \( m = -1/2 \)
B. \( m = 0 \)
C. all values of \( m \) except \(-1/2\)
D. all values of \( m \) except \( 0 \)
E. all values of \( m \) except \(-1/2\) and \( 0 \)

14. If \( y = x^2 - 4 \), which of the following statements is true?

A. When \( x \) is greater than 2, \( y \) is negative.
B. When \( x \) is greater than 2, \( y \) is greater than 0.
C. When \( x \) is less than 2, \( y \) is always negative.
D. When \( x \) is less than 2, \( y \) is always positive.
E. When \( x \) is greater than 2, \( y \) is always less than 5.

Over Please
15. A rectangular fence is to be built so that the fourth side is against the wall of a barn. The three sides of the fence must total 50 feet in length. Let y represent the area of the enclosure and x the length of the side of the fence perpendicular to the barn. Which of the following graphs describes the relationship between y and x?

I. \[ y = x \]

II. \[ y = -x + 100 \]

III. \[ y = x^2 \]

IV. \[ y = (x - 2)^2 \]

A. I
B. II
C. III
D. IV
E. None of the above

16. Let \( y_1 = x^2 \) and \( y_2 = (x - 2)^2 \). How is the graph of \( y_2 \) obtained from the graph of \( y_1 \)? By applying

A. a vertical shift upwards of 2.
B. a vertical shift downwards of 2.
C. a horizontal shift of 2 to the left.
D. a horizontal shift of 2 to the right.
E. a vertical stretch of 2.

Over Please
17. If the point (2, -3) is the vertex of a parabola, \( y = ax^2 + bx + c \), what is the equation of the line of symmetry of the parabola?

A. \( x = 2 \).
B. \( x = -3 \).
C. \( y = 2 \).
D. \( y = -3 \).
E. Not enough information is given to determine the line of symmetry.

18. Which of the following graphs describe the same relation?

- I. 
- II. 
- III. 
- IV. 

A. I and II.
B. II and III.
C. III and IV.
D. I and IV.
E. II and IV.
19. The point (3,0) is the vertex of a parabola symmetric with respect to a vertical line. If the point (5, 4) is on the parabola, determine another point on the parabola.

A. (-5,4)  
B. (5,-4)  
C. (1, 4)  
D. (-1, 4)  
E. Not enough information is given.

20. Consider the graph below. Which interval on the x-axis describes the solution to $x^2 < 4$?

A. $\left(-2, 2\right)$  
B. $\left(-2, 2\right)$  
C. $\left(-2, 2\right)$  
D. $\left(-1, 1\right)$  
E. None of the above

This is the end of the test
You may go back to previous problems if you wish
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<th>Item #</th>
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<th>Factor 3 (KR 20: -0.02)</th>
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Note: only loadings of .4 or greater are included.
Table 10

Reliability Coefficients for van Hiele Levels for Quadratic Functions and their Graphs

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### Table 11

**Statistics for Individual Courses**

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<th>Course Title</th>
<th>Number of Subjects</th>
<th>Mean Score</th>
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### Table 12

**Duncan Multiple Range Test on Course Means**

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Note the change in order of the courses. They are now listed in increasing order by overall mean score.
Table 13

Level Attainment by Course

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Table 14

Statistics for Precalculus Sections

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Sections 3 and 4 used the C²PC curriculum for graphing.
Appendix C
Instructions for interpreting the "Pretest Clusters" figure.

The vertical axis designates the facility level for the questions. The numbers next to the points represent the item number. A vertical line connecting points indicates an inter-item correlation of 0.2 or greater. The Facility Ordered Complete Linkage figure which follows provides the correlations between the linked items.

A horizontal line indicates that particular item is linked with another item from the Graphing Levels Test. All inter-item correlations are provided in matrix form following the complete linkage correlations.

Example:

Item 21 is linked with item 9
Item 9 is linked with item 8
Item 1 is linked with item 14
Item 5 is linked with item 2 and item 19 (not shown)
Item 14 is linked with item 19
PRETEST CLUSTERS
Random Sample
N = 125
## Facility Ordered Complete Linkage

**Pretest Random Sample**

\[ N = 125 \]

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**ITEM CORRELATION MATRIX**
Cluster One

21. What is the total number of points on the line segment shown below?

A. Two.
B. Two hundred.
C. Ten thousand.
D. One million.
E. There is an infinite number.

![Line Segment Diagram]

1. The graph below is a

A. point.
B. line.
C. parabola.
D. ellipse.
E. circle.

![Parabola Diagram]

5. Consider the graphs of equation A: \( y = ax^2 + bx + c \) and of equation B: \( y = -(ax^2 + bx + c) \).

A. The graph of B is wider than the graph of A.
B. The graph of B is not as wide as the graph of A.
C. The graph of B is an upside down version of the graph of A.
D. The graph of B is rotated 90° in relation to the graph of A.
E. The graph of B is identical to the graph of A.
9. What is the maximum value of $y$ if $y = x^2 + 4x - 9$.

A. $y = 9$
B. $y = 13$
C. $y = -9$
D. $y = -13$
E. There is no maximum value of $y$.

2. If we rotate the above graph so it looks like the graph below, then we have the graph of

A. a point.
B. a line.
C. a parabola.
D. an ellipse.
E. a circle.
14. Which graph is increasing between $x = 1$ and $x = 3$?

A. I
B. II
C. III
D. IV
E. V

15. The point $(3,0)$ is the vertex of a parabola symmetric with respect to a vertical line. If $(5, 4)$ is on the parabola, determine another point on the parabola.

A. $(-5,4)$
B. $(5,-4)$
C. $(1, 4)$
D. $(-1, 4)$
E. Not enough information is given.
6. Which of the following points is symmetric to (3, -5) with respect to the origin?

A. (-5, 3).
B. (5, -3).
C. (3, -5).
D. (-3, 5).
E. (-3, -5).

8. Which graph has a maximum y value?

I.  

II.  

III.  

IV.  

A. I
B. II
C. III
D. IV
E. None of the above
18. Which are graphs of the same set of ordered pairs?

A. I and II.  
B. II and III.  
C. III and IV.  
D. I and IV.  
E. II and IV.

20. The graph below is of the function \( f(x) = x^3 - 4x^2 - 37x + 40 \).  
The solutions of the equation \( x^3 - 4x^2 - 37x + 40 = 0 \) are:

A. \( x = -120, 50, 80 \)  
B. \( x = -5, 1, 8 \)  
C. \( x = -5, 1, 8, 50 \)  
D. \( x = 0 \)  
E. \( x = 40 \)
11. Which of the following are graphs of functions, $y = f(x)$?

A. I and II only
B. II and III only
C. III and IV only
D. I and III only
E. I and IV only
7. Which is the graph of a trigonometric function?

A. I  
B. II  
C. III  
D. IV  
E. None of the above
19. Consider the graph below. Which interval on the x-axis describes the solution to $x^2 < 4$?

A.  

B.  

C.  

D.  

E. None of the above

10. If the vertex of a parabola is (2, 5) and the coefficient of the $x^2$ term in the quadratic function is negative, which of the following points could not be on the curve?

A. (-5, -44)
B. (4, -11)
C. (0, 1)
D. (-1, -4)
E. (-3, 30)
3. What are the coordinates of point P?

- A. (-4, 2)
- B. (2, -4)
- C. (-8, 2)
- D. (4, -4)
- E. (-4, 4)

12. For which of the following values of $m$ is the graph of $x^2 - 2x + my^2 + (2m + 1)y = 0$ a parabola?

- A. $m = -1/2$
- B. $m = 0$
- C. all values of $m$ except $-1/2$
- D. all values of $m$ except 0
- E. all values of $m$ except $-1/2$ and 0
16. A rectangular pen is built so that one side is against the wall of a barn. The other sides of the fence must total 50 feet in length. If $y$ represents the area of the pen and $x$ the length of the side of the fence perpendicular to the barn, which of the following graphs describes the relationship between $y$ and $x$?

A. I  
B. II  
C. III  
D. IV  
E. None of the above
24. Consider the graph of \( y = g(x) \) shown below. Let \( y \) be a cubic function. How many complex number solutions does the equation \( y = g(x) \) have?

A. 0  
B. 1  
C. 2  
D. 3  
E. 4

13. This graph is increasing on which intervals?

A. \(( -\infty, -10] \) and \([10, \infty)\)  
B. \([-10, -3] \) and \([3, 10]\)  
C. \((-\infty, 0] \) and \([3, \infty)\)  
D. \([-3, -1.7] \) and \([1.7, 3]\)  
E. \((-\infty, -1.7] \) and \([1.7, \infty)\)
25. Consider the graph of $y = h(x)$ shown to the right. Which of the graphs below is a representation of $y = -1 + h(x + 3)$?

I. 

II. 

III. 

IV. 

V. 

A. I  B. II  C. III  D. IV  E. V

This is the end of the test. You may go back to previous problems if you wish.
Facility Ordered Complete Linkage
Posttest Random Sample
N = 125

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Facility Ordered Complete Linkage  
Posttest Data  
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(STANDARD PRECALCULUS CLASS; N=33)

| Q1  | Q2  | Q3  | Q4  | Q5  | Q6  | Q7  | Q8  | Q9  | Q10 | Q11 | Q12 | Q13 | Q14 | Q15 | Q16 | Q17 | Q18 | Q19 | Q20 | Q21 | Q22 | Q23 | Q24 | Q25 |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| Q1  | .48 | -.15 | -.21 | -.08 | -.18 | .52 | .17 | .17 | .10 | .07 | .23 | .11 | .09 | .15 | .19 | .21 | .04 | .12 | .12 | .11 | .13 | .19 | .12 |
| Q2  | .07 | -.29 | -.16 | .15 | .07 | .10 | .00 | .17 | .12 | .18 | .03 | .43 | .10 | -.18 | .18 | .07 | .21 | .25 | .10 | .13 | .29 | .17 |
| Q3  | .00 | .17 | .01 | .22 | .33 | -.04 | .01 | .01 | .02 | .02 | .24 | .26 | .02 | -.33 | .42 | .09 | .12 | .17 | .31 | .32 | .21 | .17 |
| Q4  | -.01 | .22 | .15 | .11 | .16 | .10 | .03 | .20 | .19 | -.10 | .07 | .22 | -.03 | -.11 | .09 | .21 | .09 | .21 | -.13 | .01 | .13 | .19 |
| Q5  | .08 | .42 | -.20 | .13 | .56 | .05 | .20 | .22 | .18 | .16 | .16 | .11 | .33 | .01 | .11 | .17 | .11 | .01 | -.11 | .19 | .19 |
| Q6  | .12 | .34 | -.05 | .26 | .06 | .12 | .19 | .13 | .03 | .22 | .12 | .19 | .15 | .03 | .12 | .19 | .13 | .03 | .12 | .19 | .13 |
| Q7  | -.23 | -.19 | .17 | .25 | .11 | .10 | .07 | .14 | .11 | .37 | .13 | .17 | .22 | .05 | .23 | .05 | .20 | .13 | .23 | .04 | .04 |
| Q8  | -.15 | .11 | .02 | .05 | .21 | .20 | .09 | .20 | .10 | .04 | .25 | .25 | .05 | .07 | .20 | .20 | .13 | .23 | .04 | .04 | .04 |
| Q9  | .40 | .11 | .11 | .04 | .22 | .50 | -.12 | .09 | .11 | .47 | .44 | .44 | .45 | .45 | .08 | .30 | .21 | .34 | .18 | .18 | .40 |
| Q10 | .01 | .01 | .22 | .31 | .03 | .13 | -.33 | -.06 | .24 | .07 | .36 | .07 | .08 | .30 | .16 | .11 | .12 | .18 | .10 | .13 | .28 |
| Q11 | .34 | .18 | .20 | .18 | .30 | .03 | .09 | .16 | .09 | .14 | .13 | .14 | .28 | .18 | .03 | .09 | .11 | .12 | .18 | .10 | .17 |
| Q12 | .10 | .10 | .01 | .23 | .13 | .13 | .14 | .10 | .23 | .12 | .23 | .23 | .17 | .10 | .16 | .13 | .12 | .18 | .10 | .17 |
| Q13 | .10 | -.21 | .18 | .25 | -.07 | .07 | .29 | .26 | .10 | .10 | -.19 | -.19 | .07 | .03 | .23 | .31 | .33 | .15 | .01 | .13 |
| Q14 | .20 | .21 | .02 | .09 | .12 | .23 | .02 | .20 | .01 | .01 | .37 | .07 | .01 | -.03 | .11 | .07 | .11 | .12 | .23 |
Pretest and Posttest Facilities

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ITEM ANALYSES

PRETEST: Random Data (N=125)

KR 20: 0.665

Mean item difficulty: 0.458
Mean item discrimination: 0.346

Mean test score: 12.55  Standard deviation: 3.59
Median: 12  Standard error: 0.029
Mode: 12

POSTTEST: Random Data (N=125)

KR 20: 0.739

Mean item difficulty: 0.239
Mean item discrimination: 0.356

Mean test score: 19.02  Standard deviation: 3.52
Median: 19  Standard error: 0.028
Mode: 21

POSTTEST: Standard Precalculus class (N=32)

KR 20: 0.636

Mean item difficulty: 0.460
Mean item discrimination: 0.317

Mean test score: 13.5  Standard deviation: 3.28
Median: 13  Standard error: 0.103
Mode: 15
Appendix D
Student Interviews

Students were randomly selected from a list of names provided by the teacher. The list consisted of those students who were willing to participate in an interview.

Students were informed of the purpose of the interview at the onset, that is, to help teachers in furthering student understanding of functions and their graphs.

General instructions included the following. Students were asked to write their name at the top of the test. When the interviews were later transcribed, reference was made to the student's test when it was difficult to decipher what was said on the tape. Students were also asked to circle their responses. They were permitted to write on the test itself if they wanted to clarify some point or to draw a sketch as an aid in answering a question. Students were asked to read the question out loud. This provided a reference point when transcribing the tape. After responding to a question, the students were asked to explain how or what they did in order to answer the question. No help was provided by the author, only general encouragement to provide all steps in their strategies. Students did not answer all questions, only ones preselected by the author.

The interviews were performed in a room in each school during the student's study hall so as not to disrupt their school day too much. The interviews lasted 30-40 minutes depending on the student and how willing they were to "share" their problem solving strategies. Students were interviewed one-on-one.

The tape recorder used was a small voice-activated recorder. The recorder would shut off when there was silence; when the student would break the silence, the recorder would only pick up part of the student's first words. This made the tapes somewhat difficult to transcribe.

The students are identified by their teacher number and the order of their appearance in the interview process. Hence Student 4-1 is a student enrolled in Teacher 4's class and was the first student from that class to be interviewed. Below each student's identification number is their attained level based on their score on the Graphing Levels Test.
All the teachers, except Teacher 3, were involved in the C²PC project. Teachers 1, 2, 3, and 6 taught in suburban school districts. Teacher 2 taught an advanced section of the precalculus course.

Teachers 4 and 5 taught in the Columbus city school district.
Q: 1.
A: Now do you want me to pick one out of there?
Q: Yea, you can go ahead and mess them up now. (Laugh.)
A: (nothing)
Q: Now, you say it's a parabola. This is where something might seem easy to you, but I want you to say why you did that. You either know it's a parabola, or there's something about... whatever.
A: I just...
Q: Okay... 3.
A: ... and the y-coordinate would be minus four, so it would be (2, -4).
Q: Okay. 5.
A: Um,...
Q: Which two is that?
A: C and D.
Q: Okay. Is it upside down... how a, um, just a little sketch over here for me of... well, first of all, just draw any old parabola you want. I don't care... Now according to C, what would that do to that picture you have drawn there?
A: It would just...
Q: What would it look like? It says it's an upside down version. What does that communicate to you? Like if this was the original one, then...
A: It would be something like this (points).
Q: Okay. And then D, is it rotating 90°, okay, what would that look like? Again now, if that's you're original one... those are your two choices then.
A: Okay. I would (?) the inverse.
Q: What would it look like? It says it's an upside down version. What does that communicate to you? Like if this was the original one, then...
A: It would be something like this (points).
Q: Okay. And then D, is it rotating 90°, okay, what would that look like? Again now, if that's you're original one... those are your two choices then.
A: Okay. I would (?) the inverse.
Q: Oh, I see. Opposite or inverse.
A: So I'd probably go with D -- or C.
Q: Okay, alright. And this is... but you're not 100% sure, right?
A: No.
Q: Okay, that's fine. 7.
Q: Are you familiar with the term trigonometric or was that... is it still fuzzy?
A: Yea, because we're still going over it, but I think it's... I think it's, I think it'd be this one. Like sine, cosine...
Q: None of these other ones?
A: (Laugh.) Well, I'm not sure about that.
Q: Okay. 9.
A: (I couldn't get it) for this one?
Q: Explain what you're doing.
A: I'm just going to try to graph it...
Q: Do you expect that or... ?
A: Yea.
Q: Okay. Why?
A: Um, because of the $x^2$.
Q: Okay, what are you looking for? Tell me...
A: I'm trying to find the maximum value of $y$.
Q: Okay... Getting it may well say these answers (?). What will you try to do with your picture then if, if you're trying to find the maximum value of $y$. What are you gonna look for?
A: ... do I have to answer these or, just skip over it?
Q: You can, if you want to go ahead and put a question mark next to that one. Some thing, uh... 11. That's fine.
A: Okay, E because... vertical line, and it only hits the graph once.
Q: And that makes it, what?
A: A function.
Q: A function. 13.
Q: How'd you do that?
A: Because it's increasing, then it starts to come back down here, so, and I estimate that's about 1.7.
Q: Okay. And then, is that the only place it's increasing then?
A: And from ... (E circled.)
Q: 15.
Q: What made you draw it like that?
A: Because they told me to, the vertex is um, (3,0), so, and then there's a point on it at (5,4) ... and it's symmetric to a vertical line, so that means this side'll be the same as this side ...
Q: Okay.
A: and ... 
Q: Now ...
A: hm, (laugh), ...
Q: Because ...
A: Um, you can be at three, and the next point's, um 5 ...
Q: Okay.
A: ... and that's like over 2 and up 4 ...
Q: Okay.
A: ... so if you go back 2 and up 4 it'll be ...
Q: 17.
A: Um, let's see. I forget how you do this ... Uhh, I'll have to come back to that one. I don't really understand.
Q: Would it help if we were to, instead of maybe saying a, put in a number? Why don't you go ahead and do that and maybe make up your own number.
A: Okay.
Q: What'd you try?
A: I tried 2 and it moved it over, to the right 2. But they never specified that's a negative or positive number.
Q: O-kay. So that'll make a difference?
A: Um hm. Um ... It does. It moves it the other way.
Q: Okay. Let's just say that a is positive, and go with that.
A: Okay.
Q: What would it be then?
A: It'd be D.
Q: To the right ... Just for the, I'd like you to do 18 too.
A: Okay ...
Q: ... it mean two of the same set or is the question already throwing you?
A: Throwing me.
Q: Okay ...
A: ... these two, these two.
Q: Okay. So it's between I or III and I or IV. Okay, what are you looking for? Are you, you ruled this one out automatically. Why? Tell me why you did that.
A: I'm not sure about that either. (Laugh.)
Q: Oh, okay. (Laugh.) ... What are you gonna look for now to, to maybe help you figure out ...
A: If these have the same coordinate at their ...
Q: Tell me one more time why you picked that answer and then I'm gonna play it back.
A: Okay. I chose I and II because both, they both have the same vertices and they cross the x-axis at the same spot.
Q: 19.
A: ...
Q: You picked that one because ...
A: Um, it says where x^2 is, um, less than 4 ...
Q: Um hm ...
A: ... and that's all this down here ... I don't know how you say it.
Q: Alright.
A: It's from nega ... (-2 to 2, Answer B.)
Q: Okay. 21.
A: Um ... I don't know about that one. (Laugh.)
Q: Okay. What, are you tossed up between a couple of answers or ... what, are you on 20 or 21?
A: 21.
Q: Okay.
A: Um ...
Q: But it's a guess ...
A: Um hm. It's just a guess.
Q: What would be ... maybe another answer in your mind that's kind of ...
A: No. I've never seen this one before.
Q: Yea, put a question mark. Can you tell me — I don't know, can you give me an idea of what's, what's uh, throwing you about the question? These are good points, too, that help us.
A: Um ...
Q: What's the total number of points, so ... Would you rule out, are there any answers you would rule out right away? Let's do it that way maybe.
A: Yea, two.
Q: Alright. You would rule out 200. ... possible? Yea ... alright. So we're just kind of, maybe because what, you can't see them or ...
A: Yea, and plus I don't know, I never ...
Q: Oh, okay. Alrighty, 23.
A: ... (chose E. None of the above.)
Q: Okay, because ...
A: Because it crosses the x-axis once.
Q: Okay, go ahead and try 22 for me also.
A: Okay ...
Q: Okay. Let's try number 22. We have, let's see if we can interpret the first sentence. The length of the shadow is directly proportional to the height of an object. Now does that, if you had to draw a picture for me of, of an object and it's shadow, um, ... the shadow, the length of the shadow is proportional to the height. What, does that communicate anything to you?
A: No. it's ...
Q: Go ahead. You can draw a picture. Like draw a tree.
A: ... a tree.
Q: Okay, now where is it's shadow?
A: ... it's shadow should be right ...
Q: Now, what happens as the tree gets larger then, let's say?
A: ... If I had to find the closest it would probably be that one (?)
Q: Okay. None of the other ones are showing that as, the tree gets bigger, how, okay h is on the horizontal axis. Okay, what is, uh, let's say this graph showing you about height and length? What does that tell you?
A: ...
Q: As the height gets, I don't know, you can either go smaller or larger. What is this graph telling you about the length. Go ahead and pick a direction and then, maybe somehow guess what the thing is saying. Or maybe it's saying nothing at all.
A: ... it's not, at least in the first quadrant ...
Q: Okay.
A: ... because that's not exactly on, like a 45 degree angle.
Q: I don't mean to, (chuckle) confuse you, I ... you know. You can go ahead and circle C as your answer, but I was trying to pull a lot of stuff out of, (chuckle), so whichever. You'd like to say III, but now you're saying well, since it's not a 45 degree angle, do you ... well, what does that mean? Now if it was a 45 degree angle, ...
A: Na, I mean, it's not ...
Q: What would point, if it was a 45 degree angle, what would points look like on the line, like what would their coordinates be?
A: They'd be like, like ...
Q: Now, let's put it back into the problem. That's saying that if the, if the height is 2 ...
A: The length must be 2.
Q: Okay. If that, if you really want to go with that. Golly, I don't wanna, I'm a salesman here. Alrighty. Go, uh, last, let's see what's on this page. Let's go ahead, go ahead and do 24 and 25 for me.
A: I'm not sure, but I'm pretty sure its D.
Q: Okay, complex number solutions does the equation have. Alright, again, why 3?
A: Because it crosses ...
Q: And that, and that relates to, uh ... the number of solutions. Right?
A: Um hm.
Q: Okay. 25.
A: Okay, number IV, or D.
Q: Okay.
A: Because this is your, um, your horizontal shift, ...
Q: Okay.
A: ... and it shifts three, so it'd have to move, this point would move here.
Q: Okay.
A: And then your vertical shift is a negative 1 ...
Q: Um hm.
A: ... so you have to move everything down 1, and that's how ...
Q: And that's it. Okay. That was pretty painless. Thank you.

Student 4-2
1
Q: 1. Okay, how did you know it was a parabola?
A: Um, just by learning it in class. Say it was a parabola.
Q: Okay. Try 2 then.
A: 2 is, in my opinion, an ellipse.
Q: Okay, because ...
A: Uh, the process of elimination because I couldn't remember that.
Q: Okay.
A: Um, ...
Q: But you rule out ... Tell me why you rule out the other ones then.
A: Um, a point is just a little dot ... a line would be going straight ...
Q: Okay.
A: ... the parabola would not, the two points on it, but you can't draw a straight line vertically and touch two points, you can only touch one, and a circle would be going around. So, I ...
A: Um, the coordinates are (2, -4).
Q: Okay.
A: And that's just by, well you go horizontally and vertically.
Q: Okay, you're doing wonderful.
A: That's about as good as I...
Q: Let's try, um, get down to 5.
A: Okay. This one I put C. That's...
Q: Okay.
A: And that's because of the negative sign. It would turn it... It would all be the same except upside
  down, like...
Q: Alright.
A: I don't know if that's good.
Q: Oh no, that's fine. Now, um, ... you know this because of...
A: Learning it in class.
Q: Just learning it? Okay.
A: Just like that...
Q: Okay, good.
A: ... I think.
Q: Those are good. You're doing fine. Okay, flip the paper over. Mostly we'll be doing odds.
A: Okay.
Q: 7.
A: Trigonometric, do you mean like sine and cosine and stuff like that?
Q: Um hm.
A: Then that'd be 1.
Q: Okay. Got that right away.
A: Yea. We just learned that in class about, like graph (I missed it).
Q: Okay. Fill in this part. I'll have ... 9.
A: There is no maximum value.
Q: Why?
A: Because, um, ... x could equal anything...
Q: Um hm.
A: ... and so you never know. And if it equals a hundred, even if you like, work it through...
Q: Um hm.
A: ... it still wouldn't be any of ... I mean, there is no limit to it.
Q: Okay. 11.
A: I put I and IV.
Q: Okay.
A: ... there is one, and ... they kept saying this is, if you can ... same thing as horizon—vertical line
test.
Q: Okay.
A: Do that just once ...
Q: Okay.
A: ... and that's a function.
Q: 13.
A: Um, well I'm not good at it, increasing stuff. Um, ... for this I'd probably put ... that it was
  negative ∞...
Q: Um hm.
A: So I ruled out B and D.
Q: Okay.
A: And it, it looked like it ended with infinity, which all of them say they do ... and then it's increasing
  up to this point ... I'm not too hot with this.
Q: Doing good.
A: Increases with ... ten — wait ... I think put A.
Q: Okay.
A: But I'm not sure, that's not right is it?
Q: Well, it doesn't make any difference.
A: Oh.
Q: Because, I just. I'm interested more in how you're doing it whether you would get it right or wrong.
A: Oh. I didn't, I don't know if that's right.
Q: So, you're telling me you're going up until here, and you pick this, and then it's going down. And then you tell me it starts going up here again and so you picked this interval to correspond ... That's fine.
A: Okay.
Q: Okay.
A: 15.
Q: 15.
A: (5,4) is on it, a parabola, I would ... I would probably say, uh, like this person did. It'd be like, that and there's, uh ... three, four, five ... three, four, five ... here so it'd be (-5,4).
Q: Okay.
A: In my opinion, because, the x-axis has to be the same.
Q: Okay. 17.
A: And then on 17 ... well, this ... in my opinion it just had like ...
Q: Okay.
A: By moving that a or whatever.
Q: Um hm.
A: So if you're moving it to this side, positive side.
Q: Okay.
A: So it'd be right, moving the y curve, or, right. That's what I put.
Q: Okay. Try, do 18 too.
A: Okay, which are graph ... Okay. I remember doing this ... I, I'm pretty sure I put I and II ...
Q: Okay.
A: ... and that's because this is almost up to 9, and this is to 9 ...
Q: Um hm.
A: ... and this, I almost said this one, but this is with 2, 4, 6 so it'd be ... there out so 1, 2, 3, and this is like right between it, more to the 5, so it'd be 3.
Q: Okay.
A: And the same with that side.
Q: 19.
A: I'd probably say, I don't know if this is right, so I'd say x, this ... so it's less than .
Q: Um hm.
A: ... so I'd put all the numbers less than 2.
Q: Okay. And how did you do that?
A: It's ...
Q: How did you get from here to here?
A: I took the square root of it. I don't know, the square root of it. I don't know if that's right, but that's what I did.
Q: Okay, so you used algebra and didn't, the graph didn't help you at all?
A: Uh, I'm not too good with graph it.
Q: Okay. 21.
A: There's and infinite number because you can just have points and points and points and points.
Q: Okay. 23.
A: ... y - 3 = 0, right?
Q: Um hm.
A: It'd be like y = ... y = 3 ... I think one ...
Q: Okay.
A: ... but I'm not sure. That's just guessing.
Q: Okay, and what are you, what's influencing your guess to make it one instead of 2 or 3 or 4 or none?
A: Um, well it's just that y if it equals 0 then I just like, took algebra and put it to y = 3, and I just said if there's only one solution, then
Q: Okay, alright. (Intruder!) She's handling it quite well. 24 and 25 and then we'll be done.
A: Okay ... complex numbers does it show? ... By complex, do you mean like the, like, square root thing, like that ...
Q: Um ...
A: — I mean i, or whatever?
Q: Yea.
A: Oh, I wouldn't know this one.
Q: Okay.
A: Um, . . .
Q: If you want, you can just put a question mark by it, or if you wanna make . . .
A: I know we did this but I just can't remember. It was a while ago.
Q: Okay.
A: So, I don't know. (25.) We did this one, and I don't know if I did it right . . . it's negative . . . first of all, I'd say it has to be upside down because it's -1 . . .
Q: Okay.
A: . . . so, take this one, this one, and this one . . . and then k would stretch it, but it doesn't say how much, plus 3 I'd move to . . . negative 3, this way. That's not right
Q: (Are you ready to) give up?
A: I don't know, I might be . . . Okay, it is . . . I don't know, I figure . . . well maybe the . . . Yea. It is. I was thinking something else.
Q: Okay. Well you can refer back, you answered something like that, where was it . . .
A: Yea.
Q: Here it is. When you said x - a you said . . .
A: Yea, so it's plus, so it'd be going this way.
Q: Okay. Thank you.
Q: 1.
A: The graph below is a ... (C) because of memorization it's a parabola.
Q: Okay, that's fine. You wanna do 2 in this case too.
A: I'm not really sure on this one.
Q: Okay ... Which ones are you tossed up between, or is there some we can eliminate?
A: Um, I know it's not a point, a line, or a circle ...
Q: Okay.
A: ... so it's between a parabola and ellipse ...
Q: Okay.
A: ... and I don't think it's an ellipse, so I'd say it's also a parabola.
Q: Okay. Why, now you ruled out the ellipse. Why? What, what made you ...
A: I think that an ellipse is sort of basically like a circle ...
Q: Uh huh.
A: ... completely connected.
Q: Okay. And those arrows don't mean that. What do the-- these arrows ...
A: They continue out in a straight line.
Q: Okay. Alright, 3.
A: ... were, (4,-4) ...
Q: Okay.
A: Just by memorizing that, x is the vertical axis ...
Q: Um hm.
A: ... and y is the horizontal and you count on where they are, so ...
Q: Okay. Okey-doke. 5.
A: Which would be, um, the graph would be an upside down version of graph A by memorizing that, when you put a negative in front of it it's a reflection through the x-axis.
Q: Okay.
A: I'm pretty sure.
Q: 7.
A: Which is the graph of a trigonomic function? Number II, or wait ... I know that I and III aren't ...
Q: Okay.
A: ... because they fail the horizontal test.
Q: Okay.
A: So does II, so it's just gonna be number IV.
Q: Okay. The horizontal test, what's that? Explain that to me.
A: Um, if you draw a horizontal line through any wide point ...
Q: Uh huh ...
A: ... it can only hit the graph in one place.
Q: Okay. 9.
A: There's gonna be no maximum point. By knowing that, if it's just a square, then it's just gonna go, the maximum's gonna be indefinite.
Q: Okay. 11.
A: Number II by using the horizontal test again.
Q: Okay.
A: ... That's the only ones I can see ...
Q: III, right?
A: Yea ... because if you do the ... horizontal test it'd fail.
Q: Okay.
A: With I, the same thing would happen there. I'm not sure on number 11.
Q: Okay. That's fine. Go ahead and put a question mark out there, and then I'll just remember well, I've got it on tape why it doesn't work. Okay. 13.
A: ... That's gonna increase ... on the, -→ to something and then something to positive ∞.
Q: Okay.
A: And then it up, then you sort of have to figure out which points are the maximum and minimum.
Q: Okay.
A: These points would be -2 as the maximum, and then positive 2 for the minimum. The closest answer is E using 1.7 instead of 2.
Q: Okay. 15.
A: Um, ...
Q: Go ahead and draw the graph.
A: Yea ... um, then if, that the parabola is gonna be the same on both sides, you can draw a vertical line basically through (3,0) ...
Q: Um hm.
A: Then you'll know point (5,4) is on it, so the corresponding point will be on the other side, and just by ... knowing that the x is 5 for the point, it would have to be over 2 from the vertex which would be 1 and then it'd stay up 4. And that's, the answer's C.
Q: Okay. 17.
A: Uh, this is the part that always gets me confused. It's gonna be a shift of either vertical or horizontal.
Q: Okay.
A: ... I can eliminate some of them ... eliminate E ... and it's gonna be ... I remember that if it's inside of the parentheses it's gonna be the opposite of whatever it is and you only move ...
Q: Okay.
A: ... that's sort of messed up thinking, but it's gonna be on the x-axis which would be right to left, since it's negative, it'll go to the right ...
Q: Okay.
A: which is D.
A: ... Um, I just look at the graphs and see which ones closest resemble each other ...
Q: Um hm.
A: ... but then I have to take into consideration the scale.
Q: Um hm.
A: So ... I and II, by looking, so I have to convert, just knowing where the ... scaling is with that being 5, and moving that down.
Q: Okay. 19
A: Here I just know that solving for a square can be negative or positive and ... in that, the answer to it would either be -2 to 2, so it has to be less than that, I would know, and on the number line it would be from -2 through 2.
Q: Okay. 21.
A: There is an infinite number of points because wherever there is, you can name the ordered pair for wherever you choose.
Q: Okay. 23.
A: I know that a real solution is whenever the graph crosses the x-axios -- axis, we see that it only crosses once, so you know that's only one.
Q: Okay. 25.
A: And, using the rules of shifting ... (Note: student reads question h times (x+3) instead of h of (x+3)).
Q: Um hm.
A: ... it would be, -1, so it's gonna be a shift of, the ... horizontal axis ...
Q: Um hm.
A: ... first, so you can eliminate ... it would just be a one so you can eliminate III ...
Q: Um hm.
A: ... and it's gonna have another shift of 3, since it's gonna be the opposite of what that sign is ...
Q: Um hm.
A: ... it'll be to the left which is graph number ... V, which is E.
Q: Alright. Try 24 too.
A: Um, the number of solution equals the number of possible times it can cross the x-axis I think...
(Note: student reads g(x) x of g, then g to the x).
Q: Um hm.
A: . . . and since this crosses three times so it'll be none, but you also have to know that it's gonna be, if you have one complex answer you have to have a second one . . .
Q: Um hm.
A: . . . because if the reverse is gonna always be true . . .
Q: Okay.
A: . . . so it would either be . . . B or C but I think since it crosses, the graph crosses three times, doesn't mean 0 complex numbers.
Q: Okay. What are, the um, if it intersects three times. What are those solutions called if they're not complex then?
A: Real solutions.
Q: Oh, okay. Alright, well, that's it. Thank you very much.

Student 4-4

4

Q: Okay. Number 1. The graph below is a . . .
A: (Circles C. Parabola.)
Q: . . . parabola. Why?
A: Because it opens upwards. That's the way I learned it.
Q: Okay. 2.
A: Well, it's a parabola because, it's like x^2 turned sideways.
Q: Okay, so rotating it does, uh . . .
A: It's still a parabola.
Q: Alright. Down to 3.
A: Uh . . . (4,-4) because it's the way you count the points on a graph.
Q: Okay.
A: The way they're labeled.
Q: 5.
A: It's an upside down version of the graph of A because the -1 just flips it over.
Q: 7. You can skip to 7.
A: I chose 1 because it looks like sine.
Q: Okay.
A: It's a repeating function.
Q: What do the others look like?
A: Um, the others are exponentials, parabolas and cubics.
Q: Okay.
A: Number 9?
Q: Um hm.
A: No maximum value.
Q: Okay. Because . . .
A: Because, uh . . . it would just keep -- it's a parabola going bigger.
Q: Oh, okay.
A: So the range goes to infinity. 11?
Q: Yes.
A: I and IV because all the others have more than one x-coordinate . . . more than one y-coordinate for each x-coordinate.
Q: Okay. That's some kind of test you have for a function, right?
A: Um, well you can draw lines . . .
Q: Okay.
A: ... to tell, that's just the (x,y) value.
Q: Okay. 13.
A: Um, the answer is E. I think, because, uh, as x goes from \(-\infty\) to -1.7, the y-value increases, and as x goes from 1.7 to infinity, the y-value increases.
Q: Okay. 15.
A: Um, (1,4) because, uh... um, how do I explain that... The left side of a parabola is a reflection of the right side.
Q: Okay. Good. 17.
A: Moves the y1 curve right a units.
Q: Alright. ... that?
A: Just ...
Q: Okay (chuckle, chuckle.)
A: ... It's really hard. I could explain it, but it'd take a while.
Q: Let's do 18, too.
A: II and IV because they're the same except for the, um... the... what are these?
Q: The tic marks, ...
A: Yea ...
Q: ... the scales?
A: Yea, the scale, it's just a little bit different with it.
Q: Okay.
A: 19?
Q: Yes.
A: It's B because, if x is between -2 and 2 ...
Q: Um hm.
A: ... then y is less than 4.
Q: Okay. How'd you do that? Did you use the graph to help you decide that?
A: Um, I just , I... you can see this point, ...
Q: Uh huh.
A: ... and if it's larger this way (draws)'...
Q: Um hm.
A: ... or this way (draws) you know that the y is gonna be above this line. I mean that's,...
Q: Okay.
A: ... looking at it (I couldn't get that).
Q: Okay. 21.
A: E because, uh... that's by definition, ...
Q: Okay.
A: ... I guess. 23.
Q: Yes.
A: Wait a second ... I say 3 solutions because, uh, f(x) = 3 in 3 different spots.
Q: Let's go ahead and do 22.
A: Um, it's III because, uh, it's an exponential and II's a... s... exponential, I don't know what you'd call it.
Q: Um hm.
A: Um, and IV ... as x gets larger, y -- or as ... as x gets larger, y gets smaller,...
Q: Okay.
A: ... but the question said as x gets larger, y gets larger. So the only solution is III.
Q: Okay. Or none of the above. Does that make sense? Does III fit?
A: Uh, yea. III fits.
Q: Okay. 24 and 25. Let me see, do 24, too.
A: The answer is A, 0 because it has three real solutions because it crosses the x-axis three times.
Q: Um hm ... and it can't...
A: Yea, and a cubic can only have three solutions.
Q: Okay. One more.
A: Um, the answer is V because move the original graph down one unit and to the left three units.
Q: Okay. And this is, um... have you... using your calculator -- has that helped you at all in terms of figuring out these kind of questions or is it mostly I've, uh, memorized that, you know, a process which -- or is it a combination of both or...?
A: It's probably a combination of both. Um, in the beginning, you know, I used the graph to get a lot of answers...
Q: Um hm.
A: ... even though I didn't really understand it, and the graph helped me understand it faster but I didn't really understand it until I could do it without the graph in it.
Q: Okay. Sounds good. Alright, well thank you very much. That's it.

Student 4-5
2

Q: Yea. Start with -- we'll go ahead and do the whole first page there, 1, 2, and 3.
A: Okay.
Q: So, this is Julie tape recorder.
A: Number 1. The graph below is a... parabola...
Q: Okay.
A: ... because that's what I was taught...
Q: Okay.
A: ... in school.
Q: 2.
A: ... it's still a parabola because it has the same shape.
Q: Okay. Sounds good. 3.
A: ... 2, which is the x-coordinate...
Q: Um hm.
A: ... so I can eliminate all these, and my answer's B.
Q: Okay. Now we jump to 5.
A: The answer is C...
Q: Okay.
A: ... because the negative turns it upside down which is what we were taught.
Q: Okay. Uh, 7. Sorry.
A: That's okay. Okay, for this problem, I remember this one well (laugh). For this one, I put IV...
Q: Okay.
A: ... because I knew that (II) was a parabola...
Q: Um hm.
A: ... and I knew that was a sine (I) function...
Q: Okay.
A: ... and I knew this one was, um, $x^2$ (III)...
Q: Okay.
A: ... and this one I hadn't learned yet really what it was, so I just used elimination.
Q: Okay. It could be none of the above, I mean not to make you change your answer, but...
A: I never pick none of the above. Never.
Q: (Chuckle.) Okay. Your strategy (chuckle), okay.
A: I don't know why, I just never do (chuckle).
Q: Alright. 9.
A: Well, ... there is none, because ... the vertex is on the negative so it's going upward and so there's a minimum and no maximum.
Q: Okay. 11.
A: That's not one because that's $x^2$... x cubed, I won't pick you ... I think I put these two on the test because I didn't know what that one was...
Q: I'm sorry, I missed which one.
A: I put II and III on the test...
Q: But you didn't know which...
A: Because I didn't know what a circle was...
Q: Oh, I see.
A: ... because we haven't done it yet...
Q: Okay.
A: ... and I knew, um, when we plugged in the sine chart I knew we did f(x)...
Q: Um hm.
A: ... and then we put sin x and so I just plugged in sin x and I know that... it was definitely II.
Q: Okay. 13.
A: From a negative, and you go up, increasing of course... it has to start at -> because it's an e... and, then... okay, I put A on my test, but I was confused because I thought this should be a positive...
Q: Okay.
A: ... 10, that's what I put because I knew it was 10...
Q: Okay.
A: ... from what the graph showed me but there wasn't a positive...
Q: Alright.
A: so I just...
Q: I'm sorry. 15.
A: Okay. Well, I knew -5... -1... you didn't know how wide this was, that's the first thing.
Q: Okay
A: So that kind of threw me off... But, um... -5 could be. (5,-4) couldn't be because it was going up.
Q: Okay.
A: ... (1,4) could be, (-1,4) could be... Well, which one did I pick... I'd probably put B and if I didn't that's what I would choose now...
Q: Okay.
A: ... since all three of those work.
Q: And we don't know how -- Okay. 17.
A: Oh geez. I think it's gonna look something like this. (Draws a graph.) Okay, it's gonna be A because negative times negative is positive...
Q: Um hm.
A: ... so you're gonna move it up so many positive a's.
Q: Okay.
A: I think. (Chuckle.)
Q: Alright. 19.
A: What I did on the test is I just took the square root...
Q: Okay.
A: ... of each one.
Q: That's fine.
A: That helped me. And then, I knew that these two were definite points... I knew it couldn't be this one because I knew it would be 2 and... the solution I solved.
Q: Okay.
A: So I was looking at B or C, and because these went outside the parabola and did not include the parabola...
Q: Um hm.
A: ... I chose B...
Q: Okay.
A: ... because that included them.
Q: 21.
A: Well, I put there is an infinite number because on a line there is no...
A: ... set amount of points. I mean these two are definite points, but then there's all these points on here and you don't know if you're gonna go half spaces of spaces ...
Q: Okay.
A: ... so I put infinite (chuckle).
Q: Okay. 23.
A: Okay ... I did this because this is the way I was taught by Dr. Demana ...
Q: Okay.
A: ... was drawing a horizontal line parallel to the x-axis ...
Q: Um hm.
A: ... and however many times the line touches is how many solutions ... 3.
Q: Okay. Go ahead and try 22.
A: Okay ... well ... she went ... since you're starting with L, I would start ... it's either between (?) or II or IV.
Q: Okay.
A: Just, I don't know why, I just would.
Q: Okay.
A: And then ...
Q: If we draw, draw a picture, like of a tree, in a sketch or something, and a shadow.
A: Um hm.
Q: Go ahead and just -- now what is, what is the problem telling you about the relationship between the length of the shadow and the height of the object? What is ...
A: Oh, the shadow's gonna be shorter ...
Q: Okay.
A: ... than the object, so it's gonna bend ...
Q: Okay.
A: ... so I would pick II.
Q: Okay. Now go ahead and do the last two, and then that's it.
Q: Um hm.
A: Um hm ... okay ... it's only gonna have 3, so I'll eliminate all these. Now I have to take a very lucky guess. (Laugh.)
Q: Now what made you say -- you're looking ... You drew a couple lines. What do those lines ... ?
A: Well, the lines tell me that there's gonna be at least 3 solutions whether they're complex or not.
Q: Okay.
A: So, there can't be 4, that's what I meant to do, ...
Q: Okay.
A: ... so it can be any number, and I don't think it can be 0, just 'cause they're always are complex.
Q: Okay.
A: Um, ... and the, from my experience, I've never had three complex answers ...
Q: Okay.
A: ... with a total of three, so I'd probably pick either between 1 or 2.
Q: Okay.
A: And I'd probably go with 2 just because I heard that if you guess C you get 'em right. (Laugh.)
Q: Oh. (Laugh.) Okay.
A: That's what our ACT class taught us.
Q: Alright.
Q: 25.
A: Okay, um, well if it's negative, it's going to ... okay, 3 ... this is going to move 3 places, and I don't remember which way ... okay, I know it's not this one
Q: Julie continued on the last problem.
A: Okay, because it's, um, it has a stretch of one ...
Q: Um hm.
A: ... you move one. I'm gonna ... move this way three, and up or down one -- I haven't decided yet. Down one. I chose IV.
Q: Okay.
A: That looks the best to me.

Student 4-6

2

Q: Question number 1.
A: Okay, um, it's a parabola...
Q: Okay.
A: ... because I memorized it that way. That's how they taught us.
Q: Okay. That's fine. 2.
A: Okay, this is also a parabola.
Q: If you rotate the graph, then nothing happens to it? It's still a parabola?
A: Right.
Q: Okay. 3.
A: It would be (4,-4).
Q: So the, the 2 means something on the x-axis?
A: Right. The 2 means that um, that's how they divided the line. Every point is 2.
Q: Okay. Alright, let's go to 5.
A: Okay. Let's see, B... the graph of B is an upside down version of the graph of A.
Q: Okay.
A: This would change all these in here, all the
Q: Um hm.
A: ... so...
Q: Number 7.
A: IV.
Q: Okay. Because...
A: Um, it's actually the way I remember things of...
Q: For instance, you can also use process of elimination. Like, if you can tell me why doesn't I, II, or III work?
A: Um hm.
Q: Is there a particular reason you said no, they're not trigonometric functions or...
A: ... I couldn't tell you.
Q: Okay.
A: I, I probably remember, but I can't now.
Q: Okay. That's fine.
A: It's not right in my head right now.
Q: Okay. Number 9.
A: I pick E...
Q: Okay.
A: there is no maximum value of y, because, um, x could be infinite...
Q: Okay.
A: ... an infinite number.
Q: Alright. Number 11.
A: It'd be IV.
Q: They're in pairs so we gotta find another one.
A: Oh, okay.
Q: That's alright, it could be what you say IV. Now we have to find another one that goes with it.
A: Okay, um...
Q: What made you rule out or, or take those two?
A: Um, I eliminated that... um, ... I took a guess --
Q: Anything special about those two that...
A: Um, ... I just, uh...
Q: If nothing comes to your mind, that's, we'll take that as it is. Okay. Number 13.
A: ...
Q: Do you want to skip it and go on?
A: Yea, it's ...
Q: That's okay. Put a question mark by that one for me. We'll go on and see what happens. 15.
A: ... Can I write on this?
Q: Yea, sure you can.
A: Okay, it'd be A, (-5,4) ...
Q: Okay.
A: ... because, um, well, you could, for one side, then the parabola would be the same on the other side.
Q: Oh, okay. And 17.
A: ... E.
Q: Okay.
A: Um, because I remember the way a, a certain equation is ...
Q: Um hm.
A: ... and, you learn that, like, each part has a certain way that it moves.
Q: Okay. Right.
A: You know, stretch or shift.
Q: Okay.
A: And --
Q: And so the a there means a (stretch)?
A: Right.
Q: Okay. Alright, I want to do 18, too.
A: Okay, ... I and IV.
Q: Okay, because ...
A: No wait, it's not I and IV. It's III and that's ... II.
Q: Okay.
A: Because, um, the, there's, coordinates are the same. It's on a different scale, but ...
A: ... B ... um ... because it doesn't contain the answer -2 and 2.
Q: Okay ... And, but it's not C, right?
A: ... Um hm. Yea, it's not C.
Q: Because ... Can you tell me why it wouldn't ... ?
A: Well, because C is getting an answer that would be either too big for it, too big to ...
Q: Okay.
A: ... fit.
Q: Are you using the graph or are you doing some algebra?
A: Oh, I'm using both. I'm using, because you just look here ...
Q: Uh huh.
A: ... and, it shows those points and it's inside there. It's not out there.
Q: Okay. 21.
A: Okay ... Uh, I don't know about this one.
Q: Okay.
A: Can we come back to that one?
Q: Yea. 23.
A: ... Um, let's see, ... I real solution, because it crosses the x-axis one time.
Q: Okay. Um, let's go—do 24, too.
A: 24. I'd say there's 3.
Q: Okay. Because ...
A: It crossed the x-axis 3 times.
Q: So it's the same for complex number of solutions as, I forget what was asked on the other one, for real number solutions?
A: Yea.
Q: Okay. 25.
A: Okay... 3 would be a shift to the left.
Q: Okay.
A: ... and... shift 3.
Q: Is that everything? The -1 up here means what now again?
A: ... a horizontal shift 1. I'm not exactly sure if it's...
Q: Okay.
A: ... up or down... There's no shifts that go down.
Q: Well here's the original graph up here.
A: Right here.
Q: Okay. In that one, the peak up here is at where?
A: It's at 2. Okay, so it goes down.
Q: Okay.
A: That's a negative...
Q: Okay.
A: ... and it goes over 3 to the left...
Q: Okay.
A: ... and that's 1, 2, 3.
Q: Okay. Look back at, let's see questions, now... wait a minute... 17 now. Going back to this one again...
A: Um hm.
Q: ... and then comparing that with 25... Now you said the a was a stretch... and then on 25...
A: Um hm.
Q: ... it would be a similar piece of that problem. Or, maybe they're not. I don't know. They're both asking you to move a graph around, so I was just seeing if we're being consistent.
A: Um hm.
Q: And if we are, we are. If we're not, well, let's figure it out.
A: Well, this is kind of like the equation that we learned in class.
Q: Okay.
A: And I just...
Q: Okay.
A: ... this, like in certain...
Q: Okay. So does everything fit then?
A: Um, yea.
Q: Okay. Alright, do you want to try going back to the other one, or...
A: Yea. Let's go back to 13.
Q: Increasing.
A: Um, yea. Let's see...
Q: None of the answers seem to fit, or it's...
A: Yea, it doesn't...
Q: Okay, what would you like to say then. Let's say... Draw it on the picture or give me an interval where you think the graph is increasing and we'll leave it at that if you don't want to pick any of those.
A: Well, um, I guess one of them has to be right.
Q: Maybe we'll find out if... We can see if we can make one fit. Where would you say first of all? Where is the graph increasing? Can you show me on there?
A: Uh, ... increasing from -10 to ∞...
Q: Okay. -10 to ∞. Are you focusing on the y-values or the x-values? Which are...
A: I think these are x or y...
Q: Okay. That's fine. Is there another one?
A: Yea, I think there was. 21.
Q: Okay.
A: Are they asking for, like, the whole thing? Because...
Q: What is the total number of points...
A: I'm thinking there's an infinite number...
Q: Okay, but... Something's bothering you.
A: Because there can be any number of dots on this line or any coordinates, I mean...
Q: Okay.
A: ...we just make this scale smaller...
Q: Um hm.
A: ...so it's, there's an infinite number. That's what I'm thinking.
Q: Go ahead and circle that then. That's fine.
A: Okay.
Q: Um...but if we change the scale or something then that would mean that we have a different
number, a smaller number or...?
A: Yea. That's, like, this is 1 right here...
Q: Okay.
A: ...it'd be .01 I guess, I...
Q: Okay. Well that's fine then. I think that's it, right? It was just the two.
A: Yea.
Q: Alright. Lucky number 13. (Laugh.) Well, thank you.

Student 1-1
3

A: Okay, so, the gra- it just says the graph below is a -- and I would just say parabola.
Q: Okay. You can just go ahead and circle it...
A: Okay.
Q: ...the response.
A: Am I supposed to give an explanation?
Q: Yea, like what, what makes you know that that's a parabola and not a point, a line, a circle...?
A: I don't know. Because I guess we, we've always, I've seen parabolas since, for a couple years so I
mean I just know it by looking at it that it's a parabola.
Q: What are some of the -- if you had to tell somebody who had no idea what a parabola was...?
A: Oh, it's got a, a vertex and the two points, the ends both poi, plah, both point up in the same
direction. Like, if it was a, like, a third root they would, one, they'd be going in opposite directions.
A: I would say parabola because, it's none of the other ones.
Q: Okay. Does rotating it, then according to your answer, rotating it doesn't really change what it is.
A: I guess, I guess not.
Q: Okay. Since it just doesn't fit the other ones.
A: Yea.
Q: Okay. Let's go on to 4.
A: Um...
Q: Are you familiar with this word, exponential, or...?
A: Yea, I guess. I would, I don't know, but I think I'd put IV.
Q: Okay.
A: So I would put IV again.
Q: Go ahead. Yea.
A: I don't know why I...
Q: What if --
A: Because this is a trigonometric function...
Q: Okay.
A: ...that's a parabola, that's a cubic...
Q: Okay.
A: ...and that's part of an, of an... I don't know what those things are called, so I would pick that
one.
Q: Okay. And none of the above, you didn't want to pick that one?
A: No. Am I wrong?
Q: Oh, it makes no difference.
A: Okay.
Q: But I'm interested in are your strategies.
A: Okay. 6?
Q: Yea.
A: So that would be...
Q: You can draw pictures on there if you want to...
A: Oh, okay. So that would be up here, so that's (-3,5). But I knew that because if it's respect to the
origin then it, the 3 is the negative and the 5 is the positive because...
Q: So you didn't need the picture, you just...
A: No.
Q: Oh, okay. I didn't want to con – make you draw one.
A: Okay. That's okay. It's easy to see that way.
Q: Okay.
A: 8?
Q: Yea.
A: This would be IV, because both of these (II and III) don't have a specific point... and this one keeps
going so it... this one has a maximum value instead of more than just...
Q: Let's do 9 in this case, too.
A: 9? Okay... I don't know... This I would...
Q: Are you trying, are you putting numbers in or...
A: No, I was gonna try and factor that but it...
Q: Oh, okay.
A: I looked at these answers and I was thinking I would put an answer in for it.
Q: Um hm.
A: It still wouldn't help. I don't know, I guess I would guess on this one.
Q: Okay. That's fine.
A: I don't know.
Q: You don't know.
A: Yea.
Q: Okay, fine.
A: 10?
Q: Um, yea.
A: It's negative, so it would be like that... then we would... let's see... (-5,44) there... (0,1)
could be there, (-1,-4) could be there, (-3,30) couldn't be there.
Q: Okay, you drew that sketch pretty fast. Um, how did you know that it went down versus up or
sideways?
A: Well, it said that, it told me that the vertex was there and it was a negative quadratic function...
Q: Um hm.
A: ... so I knew that the negative made it flipped.
Q: Okay.
A: So it would be pointing down.
Q: Maybe there's something in this problem that maybe could help you do 9 since we just guessed at that
one. I don't know, I'm just...
A: Um, well, I know that... I know that this picture would look I'm guessing maybe something like
that.
Q: Okay.
A: So...
Q: Does that picture help now or is that...?
A: Well, I don't know, I don't know where the vertex is so...
Q: Okay.
A: ... I guess it could help, but I don't know.
Q: Alright. So we'll still stay with 9.
A: I'll stay with 9. Should I do 11?
Q: Let me look at it. No.
A: 12. I guessed on this one. I remember.
Q: Okay.
A: Um, I have no idea what to do ...
Q: Put, um, well if you had to make up your own equation for a parabola, what would you write?
A: For a parabola?
Q: Yea, any old parabola or any -- is there anything that you would know that you'd have to have?
A: Uh, yea. A squared term.
Q: Okay.
A: The first, you know, the $x^2 + ...$ It would have to be in the form $ax^2 + bx + c ...$
Q: Okay.
A: ... squared, I guess, or something like that.
Q: Alright, does that help at all?
A: Um ... 
Q: Okay. What would you do in this situation?
A: Guess. (Chuckle.)
Q: Okay.
A: Um ... I don't know. Just put down an answer?
Q: Yea, and then, uh, well, if you try some what ... ?
A: Well, let's see ... group those together and group those together. That really wouldn't help much ... . take out a $y$ right here ... that doesn't help either. I think that's what I did before.
Q: If you put in $m = - \frac{1}{2}$, what happens?
A: If I put in -- if I put in - $\frac{1}{2}$ for $m$?
Q: Um hm.
A: Well, I'm gonna get a - $\frac{1}{2} y$ squared
Q: Um hm.
A: ... I guess it's 'cause there's two different -- there's a $x$ and there's a $y$. So I guess that's what's confusing me.
Q: Oh, okay. Well go ahead and what would you guess then and ... ?
A: Um, all the values except for $-\frac{1}{2} ...$. That sounds good.
Q: Okay. We'll go with that. 14.
A: I [one] ... increasing. This one's increasing between 1 and 3.
Q: What is increasing? If you had to explain again to someone who doesn't know anything about this at all.
A: The line.
Q: Okay.
A: Is that what you mean?
Q: Yea, or whatever. I mean, if you had to -- if they say which graph is increasing between $x$ is 1 and $x$ equal 3 and you had to really completely explain ...
A: Um.
Q: ... you know, well, I went here first, and then this is what increasing means to me.
A: Oh, I see. Well, because it's increasing. I don't know ...
Q: Okay.
A: ... how to explain it.
Q: But the rest fail?
A: Yea, well this one's -- well the parabola, uh, this particular ...
A: ... section of the parabola goes down ...
Q: Um hm.
A: ... and the same with this one. Moves down ... This one, I don't even think is gonna go to 3 but it's ...
Q: Um hm.
A: This, the whole thing is decreasing. And so is this one.
Q: What if, what if someone started from 3 to 1? ... you know, this arrow points up and this is always increasing.
A: Well, because you go left to right ... I'm probably not helping much, am I?
Q: No, you're doing fine. This is good. 16/
A: [A rectangular pen is built so that one side is against — ] Oh, I hate these. [Continues with the question.] Okay, 50 ... well, it can't go all the way to 50 because if you had ... wait the other side's — no wait, this side of the fence perpendicular to the barn ... here's the barn, and here's the fence. ... y represents the area of the pen and x the side ... (?) so ... 50 ... it can't go all the way to 50 so it can only go to 25 because then you're gonna have to have ... Well, it can't be either one of those two because it's not a negative number we're dealing with.
Q: Okay.
A: And I do—
Q: That's a good strategy.
A: And I don't think it can be this one because if this side was 50, then you ...
Q: Um hm.
A: ... wouldn't have any, it's two sides.
Q: Okay.
A: So it has to be this one.
Q: Okay.
A: How about that?
Q: Alright. 18.
A: Well, let's see ... between 1 and 5 and this is between 1 ... that's not right. One, two ... that's three. It's those two.
Q: Okay.
A: Do I have to explain what I did?
Q: Yea, go ahead.
A: Um, because the, the way the graphs are shown they're drawn with different, uh, ranges, and the two ranges that match up ... Okay. 20?
Q: Yes.
A: ...
Q: What are you gonna look for, or what are you gonna do?
A: Well, I'm gonna look on the graph and see where the zeros are and then that would tell me what the solutions are.
Q: Okay.
A: So this is 5, so this is (counts) 4, 3, 2, this is 1, (counts) 6, 7, that's 8, one more ... -5. So those are my answers, or my solutions.
Q: Okay. Does that make sense to you or is this something -- well, I've learned that little trick? If I just look on there, that's how it works.
A: Well, 'cause it says where it's equal to 0 and I know that the x-axis is 0 ...
Q: Okay.
A: ... so it would make sense that --. And we've done a lot of work with that stuff.
Q: Okay. Go ahead and do 21, too.
A: Well ... I don't know ... total number of points ...
Q: Are there any answers you would rule out automatically or ... ?
A: Well, I guess I would rule out an infinite number because the -- it's not an ongoing line up there. It stops.
Q: Okay.
A: So I guess it wouldn't be that.
Q: Alright.
A: And I guess it wouldn't say two because on a line there are more than two ... it's just, just a lot ...
Q: Um hm.
A: ... so I wouldn't say two ... I would pick ... I guess it's just a guess then.
Q: Okay.
A: So I'd pick C. Because it's in the middle.
Q: Okay.
A: (Chuckle.)
Q: Okay (chuckle.) 22.
A: It's directly proportional, it would have to be this graph because ... it means when, if the length is, as the shadow, if the -- if the building was larger than the, shadow would be larger so it shows that as one goes up, the other goes up. So I picked that one.
Q: This one doesn't work, I [one]?
A: Well, I su-- , I suppose it does ... Will that one work, too (chuckle)?
Q: What's different about the two then? What made you ...?
A: Because, um ... 
Q: What -- or I could say --
A: Well I don't know. How could it be a negative, of a building? But I suppose we have a negative length over here ... I don't know, I just would, I would have picked this one because that's how we were showed direct proportions like that.
Q: Okay.
A: I would just, um, circle that one.
Q: Okay, that would be good.
A: Should I do this one?
Q: Let's, um, go to the next page and do 24 and 25.
A: Well ... solutions in this one ... those two aren't, like, real exact. But this one is ... -1, but, so I would have to say two because this is a real number.
Q: Okay. And the other ones ...
A: Are complex numbers.
Q: Oh, they're complex because they're not, on the ...
A: Exactly shown on the axis ...
Q: Okay.
A: I guess.
Q: And the last one.
A: I hate this (chuckle).
A: Well, this is x + 3, so plus 3 means it would move ... means that that's a vertical shift, I mean a horizontal shift to the left three, and, -1 means it's moved down 1, so it would be this one over here.
Q: Okay ...
A: Okay.
Q: That's fine. Okay, let's go back and just -- do three also.
A: Two, four ... negative four. [Circles D, (4,-4).]
Q: Okay.
A: Those are, slash marks mean two.
Q: Okay.

Student 1-2

Q: 1.
A: (Circled C, parabola.) ... drilled into our heads.
Q: Yea (laugh).
A: He just, always said that, um...
Q: Do 2.
A: ... (It's not?) a point or a line... I don't know what an ellipse is... parabola, but it's not... it's not a function... I'll just, guess with D because...
Q: Okay, that fact that turning it --. You said that it's not a function, or so...
A: Um hm.
Q: We want a -- now we're not sure if whether it's a parabola or not simply because we've (flipped, turned?) it, right?
A: Right.
A: (Circled B, (2,-4).)
Q: . . .  as if, again, if you were explaining it to someone...
A: x, you just go over two for the x, and then it's -4 because you just go down...
Q: Alright, do 4... Are you familiar with this word exponential, or is it...?
A: It's an exponent, like, x to the third or whatever.
Q: Okay.
A: Okay, this would be ... a squared function... be a cubic, this looks like sine... and I don't remember what this is... I'm not sure if this is what you're thinking, but exponential function, like, which... looking for. Oh, a function. Well, all, all these are a function... I'm not sure. I just guessed on this one, I don't even...
Q: Okay. Then that's fine... Simply because it's --, any strategies to that guess or...
A: (Chuckle.)
Q: This is the good part, yea.
A: Alright, this is squared, ...
Q: Uh huh.
A: ... this is cubic, and I'm not sure what this is, and...
Q: 6.
A: (-3,5).
Q: Why are you...
A: . . . around... respect to the origin if you, like, turn it... It's (-3,5).
Q: Where is the origin?
A: (0,0).
Q: 8.
A: Number IV. That's because -- okay, num --, I just -- I'm gonna eliminate the other ones.
Q: Okay.
A: Number I, it shows it's continuing on y...
Q: Um hm.
A: . . . so that, scratches that one. Number II, there's no maximum.
Q: Um hm.
A: Number III, it's the same way.
Q: Number 9.
A: . . .
Q: Are -- what are you thinking? Are you...
A: . . . . . I mean, oh, I just meant it's gonna be going like...
Q: Okay.
A: . . . um...
Q: Is there enough information in, in your picture... to answer the question?
A: Well, if that's correct...
Q: Um hm.
A: . . . then there would not be a maximum y value, but I...
Q: What if you have to guess? I mean, if you... based on what you, you've done so far.
A: Well, based on what I've done so far, I would have to say this, but...
Q: Okay. But you're hesitant because you don't know if...
A: I'm not sure if that's right.
Q: What, um... If it had to be one of these other choices...
A: What would I guess?
Q: ... your picture to match up with one of those answers, let's say... Okay, that's fine. Yea, you
can always say when you, when you're tired of the question (laugh), rather than -- um, 10.
A: It all depends on...
Q: ... the last one. I missed it. If you... -1, -4.
A: And -3. Thirty I wasn't even thinking, if that's the one... The other ones, I was just stuck because
it would just...
Q: Um hm.
A: ... Like, maybe if I was given another point here...
Q: Um hm.
A: ... then I could draw it to scale, but you don't know...
Q: But this one really doesn't fit?
A: No.
Q: Okay.
A: (-3,30) would be up here.
Q: How did you know it went down? I missed that part.
A: Oh, when it said and the coefficient of the x² term is negative...
Q: Um hm.
A: ... would flip it.
Q: Does that makes sense to you or is that...
A: Yea, that makes sense.
Q: ... something that you've memorized it? Well, if the coefficient's negative it goes..
A: ... it's... I've never...
Q: 12.
A: ... Well, I'm gonna set it up like y = ...
Q: Okay. Is there any way to, to...
A: ... this would be ...
Q: So just, plug it in for x.
A: This for x.
Q: So if you put in 0 for x, what happens?
A: It would not be, like... 1, and x =... 
Q: 13 ... What does increasing mean to you and how could you tell what we... ?
A: Just, like, ...
Q: You said this was increasing backwards. What does...
A: It's going to -∞... value, is increasing, too.
Q: 16.
A: Okay... x represent the length of a side, the total length is 50 feet...
Q: Um hm.
A: ... you got two x's, so you... 50 - 2x.
Q: Um hm. You can keep thinking.
A: Um... and I don't know but that's got inside...
Q: Uh huh.
A: Yea, but this would not... I was just thinking that because this side was... if x is 1
A: ... eliminates II and IV because the... exist.
Q: You can't have negative area.
A: Okay, so it's gotta be between 1 and 3... the most the x can be would be 25... the answer is 1.
Q: Okay. Why is the most 25?
A: Because, for, if you stuck 25 in for x -- wait, let me think. Oh no, that's wrong. If you stuck 25...
25 times 2 is 50... which would be 0...
Q: Um hm.
A: So, that would just... yea.
Q: 18.
A: Was I right?
Q: Doesn't matter...
A: Oh.
Q: ... I want to know how you'd do these problems more than anything. That's what's interesting. You're doing a very good job.
A: Okay. This one's . . . if x is (0,9), this is .1 so that can't be . . . that's to 3 and . . .
Q: Um hm.
A: . . . 'cause that's .1, there's I and II so that one's out so it's I and II . . .
Q: 20.
A: . . . (-)5, 1 and 8. B.
Q: Because . . .
A: To find the solution, you just find where it crosses the x-axis and at that point . . .
Q: . . . the x-axis and maybe not make your points up here or down here.
A: Or for, just because I know that a solution is where x = 0 . . .
Q: Um hm.
A: . . . and I figured, gosh, I figured . . . this . . . um . . . y something or . . . Well, the maximum and the minimum.
Q: Okay.
A: I know it's not that.
Q: Okay . . . if this had said a hundred, you'd still go to where it crossed the x-axis.
A: If 0 . . .?
Q: Yea, if, if that said $x^3 - 4x^2 - 37x$ . . .
A: No, I would have drawn a line at a hundred.
Q: Alright, let's do 21, and we'll be done.
A: . . . (infinite num)ber.
Q: Okay.
A: You just know that from way back when . . .
Q: Okay.
A: . . . you just know there's an infinite number of points on a line . . . five.
Q: Alright. Well, thank you very much. That's it.

Student 1-3
Q: We'll start with 1, 2, and 3. We'll do all those.
A: It's a parabola . . .
Q: You have to tell --
A: . . . because I just learned it, and it's like the other ones that I know of, and then, number 2?
Q: Um hm.
A: I would say an ellipse.
Q: Okay, because . . .
A: It's not a parabola any more and it can't be any of the other options.
Q: Okay, I -- I'll ask if -- like, the rotation now has, you say it's not a parabola any more. Is it because we've turned it? Is that --
A: Right.
Q: Okay.
A: And then, uh, what are the coordinates of the point? [question 3]. Uh, (2, -4).
Q: Okay, because . . . Just explain everything even though it's so basic.
A: Because, um, it's just two over and down four.
A: Woa . . . exponential function . . .
Q: Is that unfamiliar? Exponential?
A: It rings a bell . . .
Q: Okay.
A: I, uh . . . it would be . . . I'd choose IV because it's the . . .
Q: Okay.
A: It looks like . . .
Q: Do you know what the other three are? Is it — have you ruled those out because you know what they are or . . .?
A: Well, that's like $x^2, x^3,$ and $x^4$ . . . It's just, that's the only one that's, like, different.
Q: Okay. You won't choose none of the above because . . .
A: I don't know. I never -- I try not to choose none of the above. (Chuckle.)
Q: Okay (chuckle). Um, 6.
A: (-3,5).
Q: Because . . .
A: Because it crosses the, well, it goes through -- it's opposite. It's like in a mirror.
Q: Um hm.
A: The origin, or (0,0).
Q: Okay. Try 8.
A: Um, IV.
Q: Okay.
A: Or D because there's a maximum point. The other ones don't have a maximum point.
Q: Alright, 9.
A: There is no maximum value.
Q: Well -- how do you know that?
A: Because for every nu-, for, for any number of $x$ . . .
Q: Um hm.
A: . . . you could get an infinite number of y. For -- you know, an infinite number of $x$ you can get an infinite number of y. Unless . . .
Q: I guess 8 was kind of asking which graph has a maximum $y$ value in terms of $a$, of a picture, and this is now an equation. Can you tie the two together or, like, for instance
A: Um.
Q: . . . give me a picture for this one maybe or ?
A: This one has an -- it goes upwards, or it's -- so there's a maximum -- or, there is no maximum value. Unless you --
Q: But how did you know it went up.
A: Because $x^2$. Positive $x^2$.
Q: Oh, okay.
A: If you have a, like, negative $x$, and that's why it just flips and goes down.
Q: 10.
A: Let me see . . . (-3,30). I think that's right.
Q: Um hm.
Q: Okay. 12.
A: . . .
Q: What are you thinking of? Do you know?
A: I'm just trying to put all the numbers in there. Seeing if it works.
Q: Oh, in where? Where are you going?
A: Um, into $m$ . . .
Q: Okay.
A: . . . and seeing if that's work. You know, seeing if I can get a 0 out of it. So if that's $-\frac{1}{2}$ . . .
Q: Um hm.
A: . . . that would be -1, plus 1 . . .
Q: Um hm.
A: . . . $-\frac{1}{2}$ . . . is, uh . . . wait . . . I don't know.
Q: Okay.
A: Should I just... What should I do, just — ?
Q: You can just go ahead and guess if you want, and then kind of give me some idea as to
A: Why?
Q: ... why you chose, you know, why this was my guess.
A: (Laugh.) ... if \( m \) is ... I would say ... I really don't know.
Q: Okay.
A: It'd take time. That's why I...
Q: Okay. You can go ahead and if you want to put a question mark next to it ... Were you, trying to
look for, maybe -- I noticed you were gonna put in ...
A: Trying to ...
Q: ... numbers for \( m \) ...
A: Trying to cancel. To see if, like, you know, like, if I put one-ha -- \(-\frac{1}{2}\) ...
Q: Um hm.
A: ... and then I'm multiplying, then, like, plus 1 ...
Q: Um hm.
A: ... then, try to cancel that out, and then ... that's 0, then \( y \) times that would make that 0, and then
put another number in here.
Q: Um hm.
A: And that's where I got like ...
Q: Do you know what you wanted to cancel in particular? Um ...
A: To make as many, um, zeros as possible ...
Q: Okay.
A: But that wouldn't, I don't know ... it's not right for some reason.
Q: Is the question confusing? Do you know what it's asking for?
A: [Rereads the question.] Oh, a parabola! Um, to make it a parabola ...
Q: Um hm.
A: ... okay ... I would say one -- \(-\frac{1}{2}\) ...
Q: Okay. Because ...
A: Because zero --, if you put 0 in the \( 2m + 1 \) ...
Q: Um hm ...
A: Then, it'll always have one \( y \) ...
Q: Um hm.
A: ... and that wouldn't work, because ... I change ...
Q: Go ahead. That's fine.
A: Yea, 0, because, if I put 0 in there, that'll make one \( y \) ...
Q: Um hm.
A: ... and if I put 0 in, um, in \( y^2 \), that'll make, that'd cancel that out and that would make one \( y \) equal,
or -ly ...
Q: Um hm.
A: \( \ldots = x^2 - 2x \).
Q: The \( y^2 \), you didn't want the \( y^2 \) in there?
A: No. I wanted to get rid of that.
A: Um, just where -- if I'm tryin' to -- you know, like, look in between, like -- \( x =1 \) is here, and then ...

Q: Um hm.
A: Um ... and just look in between those 2 ...
Q: Um hm.
A: ... and the periods are ... um, like 1 is increasing. Just looking at each of 'em.
Q: What particularly is increasing? How do you...
A: The line show that, uh, the line is just go -- rising upwards.
Q: Okay.
A: And... the other ones, they don't have it. They're all going downwards.
Q: Okay.
A: Slope downwards. They're going like... you know, positive, positive. The other ones are like... positive or else...
Q: If I look at, let's say IV, and... III and I, if I look at it this way, it looks like it's... increasing if I go that way. What would you say to someone who...?
A: It's not, the way I look at it is it's increasing both ways -- the x value and the y value. Um, this one's increasing on the y value...
Q: Um hm.
A: ... but it's decreasing on the x value.
Q: Sounds good. 16.
A: I hate these. These are the worst... I didn't even answer this one.
Q: Okay, do you want to try now?
A: I'll give it a shot and see what happens. Okay... the other sides of the fence must total 50 feet in length, okay, so... the, um, wouldn't it be the height...
Q: Uh huh.
A: ... represents the area of the pen... let x be the length of the side of the fence perpendicular to the barn... that's x. Which of the following graphs describes the relationship between x and y?
Q: Is there enough information that, um, any of the pictures you could rule out automatically, or...
A: Well, I'll never choose a negative.
Q: Which one's that?
A: Like, II and IV I'd never choose.
Q: What's negative?
A: The y values are negative...
Q: Oh, okay.
A: ... so... I think, the barn... area. Isn't that it, the area?
Q: Okay.
A: It wouldn't be negative, so...
Q: I, III, or none of the above. But we don't choose none of the above -- or at least we try not to (chuckle).
A: We try not to (chuckle).
Q: Okay, so...
A: Um... I would say III because... it just looks like it has... it's going backwards
Q: Um hm.
A: ... well, because it has 50 place marks...
Q: Um hm.
A: ... and this only --, um, I only has 25.
Q: Okay. 18.
A: Um...
A: Well, if they're the same ordered pairs...
Q: Um hm.
A: ... they'd look somewhat the same...
Q: Um hm.
A: ... except that the, um... the range or whatever...
Q: Um hm.
A: ... the units or whatever would be different.
Q: The scale?
A: The scale, yea.
Q: Um hm.
A: That's what I'm looking for right now, but... two, four... I don't understand, like, the same set of ordered pairs so that... I'd say I [one] and, I [one] and II?... Yea, I and II because... they, like II goes in, um, a scale of 5...
Q: Um hm.
A: Um hm.
A: ... and 10, or, yea, a scale of 5 both ways, and I [one] goes in ... it looks like I [one] is 9 units up and ...
Q: Um hm.
A: ... three units across and it looks the same, um, (?) II also.
Q: Okay. So the response is ...
A: A. I [one] and II.
Q: Okay, um, 20.
A: Okay ... No, it's like, yea. I'm looking for the zeros on the graph ...
Q: Um hm.
A: ... and so --
Q: What are those? What are the zeros?
A: ... 1, and two, three ... 8. So it's B. B is the answer.
Q: So you looked at these ...
A: Where -- yea ...
Q: ... This one is --
A: Where it crosses the x-axis.
Q: This one isn't ...
A: No, that ... I didn't look at because -- it crosses the x-axis for the answers.
Q: Does that make sense that you would?
Tape 2

Student 1-3 (cont)

Q: Try it again. Alright. Asif continued, and...
A: You were saying something like if...
Q: If this was 50, would you do the same thing?
A: I'd, uh, draw a line at the, the y = 50...
A: ... and look for the zeros there.
Q: Okay. You wouldn't look on the x-axis at that time?
A: No. The zeros are...
Q: Okay. 21.
A: There is an infinite number. Uh, hold on. What is the total number of points... Yea, I think there's an infinite number because, it's like, um, I know there's two points on the sides, but...
Q: Um hm.
A: ... I mean, you connect the two points, there's like... between them.
Q: Okay. 22.
A: ... Could you tell me if it was proportional, as x gets bigger, y gets bigger, or would it be... is it, like...?
Q: Um... Well, is there anything in the problem that would help you... to think about -- to answer that question? We're talking about shadows...
A: Um... if the height of the object is more than the, the length of the shadow is greater... I would say, III, because... II and IV makes the shadow be falling. Um... um, as the length decreases or something...
Q: Um hm.
A: ... or as the -- as the... either, yea, as the height decreases, the length increases which... said is IV, and then, um... uh, one, I really don't understand.
Q: Okay. Well that's it.

Student 1-4

2

Q: Alright, go ahead and start with question number 1.
A: C. Parabola. 2. If we --
Q: Uh, we -- well, why? See this is (chuckle).
A: Why? (Chuckle.) Just because I, know it's a parabola. (Chuckle.)
Q: Any, any, like if you had to explain this to somebody who knows nothing about this and asked why, is there anything that you could tell them about the shape that you say, well, this is what makes it a parabola, okay, versus an ellipse or a circle or a line or something like that?
A: It looks like a V (laugh).
Q: Okay.
A: Okay? (Chuckle.)
Q: Yep, this is good. Good.
A: Okay, number 2.
Q: Alright.
A: ... of an ellipse.
Q: Okay.
A: Just because (chuckle) I know it's not a point or a line or a circle so (chuckle).
Q: Okay, so by process of elimination...
A: Process of elimination (chuckle).
Q: That's good.
A: Exactly.
Q: Go ahead and circle that. Um, okay, so rotating it evidently now is gonna change what it is.
A: Um hm.
Q: Okay. Fine. 3.
A: (2,-4) . . .
Q: Um hm.
A: . . . simply because 2 tells you how far to go . . .
Q: Um hm.
A: . . . on the x and . . . -4 is how far on the y.
Q: Okay. Good. 4.
A: Oh.
Q: Now is that word familiar, exponential?
A: Um hm.
Q: Okay.
A: It means it's just, like, x to the - x^2 or something.
Q: Okay.
A: Um, I think it's IV.
Q: Okay.
A: I'm not really sure (giggle).
Q: Okay. It's just a feeling.
A: Um hm. I know that's a cubic (III) . . .
Q: Um hm.
A: . . . that's a parabola (II), so, I'm just guessing. (Laugh.)
Q: Okay. And none of the above, you don't . . . You're gonna pick one. Okay.
A: Yea (giggle).
Q: That's fine. Uh, 6.
A: It'd be (-3,5) because when it's symmetric to the origin, you change both the x and the y.
Q: Okay . . . makes sense, I mean if you had to explain to somebody -- why not just change, uh, the y, or why not just change the x? What makes it, when you change both, symmetric with respect to the origin?
A: Well, you can graph it . . .
Q: Okay.
A: . . . and then figure out, put the point (3,-5) and then you want it symmetric to the origin so it has to be over here and you have to change both. So you can just graph it and figure it out.
Q: Okay. 8.
A: Um . . . [IV?] because the rest of -- um . . . the rest of 'em are going infinite in the -- in the y direction.
Q: Okay.
A: Well, wait a minute (laugh) . . . I'm not sure actually (laugh).
Q: Okay. Is there, is there a tie or, do they all fail your test or . . . ?
A: Actually, tha -- it's none of the above I think . . .
Q: Okay.
A: . . . because they are all, this one keeps going in that direction . . .
Q: Um hm.
A: . . . and so does this one.
Q: Okay. So none have a maximum y.
A: Um hm.
Q: Okay. These are all continuing in some direction . . .
A: Well, this one has a maximum y (laugh).
Q: It does?
A: Yea. (Laugh.)
Q: Okay.
A: It's D.
Q: You're sure?
A: Positive. It takes me a while (laugh).
Q: Oh, no. That's, that's fine. No hurry . . . Why did that one have a maximum y?
A: Because it does have a maxim -- I'm thinking, I was thinking max and min.
A: . . . value.
Q: Okay.
A: That's why . . . Because it -- it stops. It has a maximum.
A: . . . I guess what you would just do is just graph it.
Q: Um hm.
A: . . . But I'm really not sure how to do it.
Q: Okay. Can -- could you draw a sketch? It doesn't have to be accurate, but, I mean how would, maybe how would . . . like, is there anything that, uh . . . would help in the problem?
A: I'm not sure (laugh). I would probably just guess on that problem (laugh)
Q: Okay, go ahead. What would you guess?
A: . . . A.
Q: Okay, so --
A: No, actually I would guess E.
Q: Oh, my.
A: (Laugh).
Q: That's a big jump.
A: Um . . . Because if it's an $x^2$ it would probably be continuing infinitely in the . . . um, positive direction. Just, I'm not sure (laugh).
Q: Okay. 10.
A: Well, I would draw it, try and draw it.
Q: Um hm.
A: . . . the vertex is at (2,5) so you'd go 2 . . . and it's, the coefficient is negative, um, I'd probably guess on this one, too (laugh).
Q: Okay.
A: Um, I would probably just guess C.
Q: Okay, is that, um, s -- phrase the coefficient of the $x^2$ . . .
A: It's supposed to do something, but I forget (laugh).
Q: Okay. Alright. Simply because it's in the middle, or it's, uh . . . what made it stand out to you?
A: . . . I don't know. Just, because it's not very close to that point. It could go around it or to the other way something.
Q: Okay. Um, 12.
A: . . . either (laugh).
Q: Okay . . . Do you understand the question or is that the part that's confusing you or . . . ? . . . but you don't know, how to go from there?
A: Yea (laugh).
Q: Okay . . . forget the question I said and write . . . any equation for a parabola.
A: An equation of a parabola?
Q: Yea.
A: . . . You'll have a squared -- $x^2$ term . . .
Q: Um hm.
A: . . . like . . .
Q: . . . is in your equation that you just gave me that's also in here or, could you somehow compare what you just said to what's here? That would maybe help you answer the question.
A: . . .
Q: Okay. That's fine. Go ahead and put a question mark there or what would you guess or did you guess or . . . ?
A: . . . (laugh).
Q: Okay. Try 14.
A: Let's see . . . increasing then increasing so it must be A.
Q: What are you looking for? You did this quickly.
A: Just, where it's increasing. Going up.
Q: Okay. What about if I looked at this one here and I said, if you look from 3 to 1 the function is going up? Does, does that count?
A: Huh uh . . . just between -- from this, going this way to this way.
Q: Um hm.
A: Because I - you'd read the graph from 1 to 3.
Q: So they're smooth. Alright, 16.
A: ... We've done these so many times and I just (laugh), I really forget how to do them ... Well, I think when I did this problem, the thing that confused me is like ... 
Q: Um hm.
A: ... there's not that big a difference between the graphs and, ...
Q: Um hm.
A: ... I really forget how to do 'em.
Q: Is there any that you can rule out or that you ... ?
A: You know it has to be 50 feet and there's a hundred there already so that's way too big.
Q: Okay.
A: ... Um ... (laugh) putting that -- well, because I mean, if a hundred is like ... I'd put E for that answer.
Q: Okay. 18.
A: ... Um ... I and II, just because I went through and ... figured out where they were on the ... 
Q: Okay.
A: This is at 3 and this is at 9 and this is about 3 and 9 also.
Q: Don't, um ... they don't really look alike like these three do. Is that ...
A: Hm um.
Q: Why is that? ... Change the --
A: Because the scale is different on the x and y axis.
Q: Okay. 20.
A: ... Okay, where they're equal to zero you just figure out where it hits the x-axis, and it'd be these three places.
Q: Um hm.
A: ... 1, -5 and 8. So I'd put B.
Q: Okay. If, um, now this, this process you went through. How would you explain why, why did you look on the x-axis, uh ... ?
A: Well, you want to know where that equation is equal to zero ...
Q: Um hm.
A: ... and the x-axis is where they're equal to zero so wherever they cross the x-axis would be ...
Q: If this had said 50 then would you do the same thing or -- ?
A: No, you'd go to, um, y = 50 so you'd draw a line at 50 and then
Q: Um hm.
A: ... wherever they cross that line.
Q: Okay. Uh, 21.
A: It's an infinite number because, I mean there are so many points on a line that ...
Q: Um hm.
A: ... I mean, it's just an infinite number.
Q: Does, what if this had said 10 and 10 or something. Would, would changing the scale make any difference?
A: No. As long as there's a straight line there's always points.
Q: Okay. 22.
A: Um, I would say number III because, just because the, L and H are even as they go along. They're just -- it's, it's ... they're proportional.
Q: Okay, why did you rule out ... ?
A: Well, because the two curved ones aren't, that starts to go straight up so it wouldn't be directly proportional and this one ... it could be, too, but it's all -- it's going in a negative direction.
Q: Um hm.
A: You can't have a negative, um, length or height.
Q: Okay.
A: This one (II)?
Q: Um hm.
A: It's also a curve, too.
Q: Okay, so we can't...?
A: Well, it wouldn't be directly proportional.
Q: Oh. Okay. 24.
A: I'm not sure on this one (laugh).
Q: Okay.
A: Um... it says how many complex number solutions.
Q: Have you talked about the um, the complex numbers?
A: Yea (laugh).
Q: Well, I don't know. I haven't been following the, um... Let's see. Is there any similarities between this problem and this one that maybe would help, or not?... Are these the complex solutions? This ...
A: That's where it's equal to zero.
Q: Okay.
A: I don't, I don't know if I'd figure the same thing.
Q: Okay.
A: I probably would guess 3 because it has three... I'd just put D then.
Q: Alright. Thank you very much. That's it.
A: Okay.

Student 1-5
3

Q: Start with 1.
A: Parabola.
Q: Okay.
A: Do you want me to...
Q: Yea. Go ahead and circle...
A: It's got the characteristics of a parabola with the, well it's the graph of $x^2$...
Q: Um... 
A: and it...
Q: Oh, go ahead. I'm sorry. I didn't want to stop you.
A: No. It's got a, it's got some vertical shifts.
Q: 2.
A: ... a parabola except it's rotated on its side.
Q: So rotating a graph of this... which is what it is.
A: Right.
Q: And that's important. Okay. 3.
A: ... (4,-4). You just have to look at the -- see, then they put the ... you just have to figure it out according to the scale.
A: Oh, okay. Um, 4... Uh...
Q: Is that term familiar, exponential function?
A: Yea. It's, it's gonna be to a power...
Q: Okay.
A: ... and it could be second power or third power. Um, it would be... the graph right here... No...
Q: That doesn't make sense because this one could be to a power too.
Q: Um...
A: Ss -- that's the sine graph.
Q: Um...
A: ... Maybe you think that it would be this one, but an exponential function is... I think I put IV...
Q: Um...
A: ... on my other one.
Q: Because...
A: Well, jus -- I don't know. After taking so many tests...
Q: (Laugh.)
A: ... you figure that, well, you know that's the sine, and these two are the same and so you know both of 'em have to a power...
Q: Okay.
A: ... and then this is the only one that's different.
Q: Okay. You also have none of the above.
A: Well ... I think I put IV.
Q: Okay.
A: That's ... that's the graph of a ... that's a squ ...
Q: Alright, so now we, we're stuck. You have -- what are you, what are you stuck between? Like IV and none of the above or ... ?
A: Yea. IV and none --
Q: ... between II and III or ... ?
A: Well, they're not gonna give you one that has two possible answers.
Q: Okay.
A: So I'm stuck before, between IV and none of the above ...
Q: Okay.
A: ... and I'm trying to remember what the equation looked for ...
Q: Um hm.
A: ... for this ... It was ... I'd guess the square root sign ...
Q: Um hm.
A: ... 1 plus, the square root of something.
Q: Would that be exponential then?
A: Yea, I think it's IV.
Q: Okay, 1 plus the ...
A: It's got the, if it's got the --
Q: ... square root of something.
A: There would be, I remember there was another one that had a second, and it would be in the opposite quadrant like.
Q: Okay.
A: And there would be another piece...
Q: ... Do you think part of this graph is missing maybe. Like ...
A: I didn't know there were some just like with the one.
Q: Okay. 6.
A: Uh, 5.
Q: No, we'll, we'll go to 6.
A: Oh, okay. ... (-3,5) which is D ... The opposite of both of the ... the opposite quadrant. If you were going to ... across the x ...
Q: Um hm.
A: ... you would switch the y ...
Q: Um hm.
A: ... you would switch the x value if you're going across the ...
Q: respect to the origin ...
A: Right.
Q: ... you would want to be in the opposite quadrant ...
A: Right.
Q: Okay. Number 8.
A: It'd be IV ... these, are bo--, these, II and III are both alike and they're gonna be increasing out to infinity ...
Q: Um hm.
A: ... and this ... increasing to negative infinity.
Q: Um hm. 9.
A: Factor ... I'd factor but I don't think it's factorable.
Q: Okay.
A: So put it in the ...
Q: Why were you trying to factor?
A: I was —
Q: What was that gonna do for you?
A: I was just gonna try and factor so I could get the x values...
Q: Okay.
A: ... get rid of the square sign...
Q: Okay.
A: ... and then you can solve them separately, as separate equations.
Q: Okay.
A: To do that, so you have to put in the quadratic... plus or minus the square root of 52.
Q: Okay.
A: That's about... I'm solving for x...
Q: Okay.
A: ... negative value for x, so I do—, so I can eliminate the x and just have a number.
Q: Okay.
A: But, thinking that there is no maximum y value because I'm thinking of the normal graph of x^2...
Q: Um hm.
A: ... seeing that it goes... the, uh, where you move it the...
Q: Okay.
A: ... vertical shifts and stuff.
Q: Okay.
A: And that's a vertical shift down...
Q: Um hm.
A: ... and that is, um... but anyway, there's nothing that would...
Q: Going through this...
A: Well, that didn't really help me.
Q: It didn't really help. It was the picture that...
A: Well, yea I'm just, I was, kinda util—, I'd look at it and then I couldn't figure out where to start so I'd just...
Q: Okay.
A: ... kinda mess around with it... and that doesn't seem like it's gonna equal... I think, well, let me try and picture the picture...
Q: Okay. 10.
A: Okay.
Q: You drew it down. Wh—, how did you do that or why?
A: Okay, it's the x^2 term, and they say that it's... the negative means that it's gonna be a... function...
Q: Okay.
A: ... and so rather than going to positive infinity it's gonna go to negative infinity.
Q: Okay.
A: ... (0,1), I can cross that one out... it'd have -- it's gotta have a slope around there, it's not gonna be... because, you're going negative...
Q: Um hm.
A: ... 3 and then you're going up 30... and, it's reflecting... maximum y value it can have is 5. There's no way 30's gonna be on it.
Q: Okay. 12.
A: ... This is one of those ones I guessed on.
Q: Okay. What's the question asking? In your own words could you tell me what the confusing part is?
A: Well, they're... asking for the values of m...
Q: Okay.
A: ... and then there's y terms and x terms...
Q: Um hm.
A: ... and it's kinda messy.
Q: Alright... Those values of m's. What --
A: Looking at this \( m \) right here...
Q: Um hm.
A: ... I know that that can't be \(-\frac{1}{2}\)... by the 2, it's gonna be positive \( \frac{1}{2} \) meaning \( m \) has to be \(-\frac{1}{2}\).
Q: Um hm.
A: ... to get that to 0.
Q: Okay.
A: If, if you knew \( y \) was 0, you'd ... it wouldn't matter what \( m \) was.
Q: Okay.
A: ... that leaves ... in \( E \).
Q: Okay.
A: right now 'cause there the only ones -- 'cause it can't equal that, and so ... and that, this \( m \) value right here ... squared it could be anything.
Q: Um hm.
A: It couldn't be 0 because if it was it would cancel that out...
Q: Okay.
A: ... and so it would have to be \( E \).
Q: You, you want the \( y^2 \) in ...
A: Right.
Q: ... and you want the \( y \) in ...
A: Right.
Q: ... and then that equation would be a parabola. Okay. 14.
A: [Circles A. L]
Q: ... looking for? How did you make that decision?
A: It -- I, look along the x-axis ...
Q: Um hm.
A: ... from 1 to 3. And then look at the graph that's in that section and ... well, that and look where the arrows are going.
Q: Okay.
A: If it's going up it's increasing.
Q: What if I, let's just say I picked this one and I ... is that increasing? How would you ...?
A: No, you do it from left to right.
Q: Oh, okay. So we have an order.
A: Yea. You go from ... start at the left.
Q: Um hm.
A: It's just like reading a book I think.
Q: Um hm.
A: ... left to right. But, yea, if you came the other way ... If you're starting you go left to right.
Q: Okay. 16.
A: ... Just from memory ...
Q: Um hm.
A: ... of what we did, it's not gonna be a negative, so I can cross these two out 'cause you can't have a negative distance ... that leaves these two ...
Q: Okay.
A: ... and then looking at the numbers ... the fence must be 50 feet ... Just by looking at the scale ...
Q: Um hm.
A: ... 'cause that goes up to 300.
Q: Um hm.
A: ... Well, that means that the barn is here ...
Q: Um hm.
A: ... but ... \( y \) is the area, it'd be 50. So if this is 50 ... It'd be, the equation would be, uh ... 50 ... plus 2x ... \( y \) which is the area ...
Q: Okay. Where'd the 2 come from?
A: Two sides coming out ...
Q: Okay.
A: ... so that's $2x$ ... Um, even if you ... it can't be greater than ... look at the graph ...
Q: Um hm.
A: ... and look at the information they give me there. This one's one, two hundred, three hundred, four hundred ... 600 plus, and this one's 300 plus. So just looking at the numbers and what they would be ...
Q: Um hm.
A: ... this one's not gonna be more than 300. 'Cause if you, you can't have this $x$ go out far, because this one's gonna be 50 ... I'm trying to find the area, it's only length times width.
Q: Um hm.
A: So that's ... I'm gonna have $2x$ 'cause that's figuring out perimeter. So then it's $50x$ equal to the $y$, and so ... Look at the, the answer could be anywhere from 0 to 50 ...
Q: Um hm.
A: ... $x$, so this one's 50 and this one's only 25 or so ... answer's gotta be in here somewhere.
Q: Okay.
A: ... Um, it's gonna be the side coming out perpendicular to the barn.
Q: Okay. Alright.
A: It could be anywhere up to 50 ...
Q: Um hm.
A: ... because if it ... to be 50 ... I'm not making sense.
Q: (Chuckle.)
A: Well, I know I can cross these two out (II and IV).
Q: Okay.
A: Then ...
Q: None of the above (chuckle). There's ... 
A: ... throw that in.
Q: Yes.
A: I would, I would, I would pick III but ...
Q: Okay.
A: ... just because I've got ... 
Q: But you're not really comfortable with that.
A: Well, ...
Q: Okay. 18.
A: I'm gonna look at the $y$ values to start with.
Q: Um hm.
A: These are po ... is at nine ...
Q: Okay.
A: ... at six, this one's less than five ...
Q: Um hm.
A: ... and this one's at three ... that one's ten on the scale ...
Q: Okay. 20.
A: ... Okay, I'm just looking at the scale.
Q: Okay.
A: That one's at 1. There's only gonna be three solutions.
Q: How'd you know that?
A: It crosses the $x$-axis three times.
Q: Okay.
A: ... at 5. So one's gonna be -5. That makes two and there's only gonna be three of 'em, so I'd go ahead pick B.
Q: Okay ... You looked at where these intersected. Why did you do that? Is ...
A: It's asking where this function equals 0 ...
Q: Uh huh.
A: ... and on the $x$-axis it was where it's gonna be equal to 0 ...
Q: ... it said 50 what would you do?
A: You would go up on the $y$-axis to 50 ...
Q: Um hm.
A: ... you could draw a line ... 
Q: Okay.
A: ... and then where they cross that line is gonna be those.
Q: Does the graph make -- does, to find the solution easier? Could you have done it algebraically or ... ?
A: The graph makes it a lot easier.
Q: Oh, okay.
A: I looked at it, you look at it and just go Yea!
Q: (Chuckle.) Okay. How about 21.
A: ... It would be infinite.
Q: Okay. Because ...
A: It's just asking for ... line segments, and these arrows are going out to infinity, and so there's no real ...
Q: 22.
A: ... Okay, it says that the, the length which is $L$ which is gonna be on the y-axis ...
Q: Um hm.
A: ... is directly proportionally to the height of the object.
Q: Um hm.
A: ... Should have a ... every y unit they go up, they should go up equally ... stay the same, at the same ... it's gonna be either III or IV because these aren't, these aren't gonna be directly proportionally (I and II) ...
Q: Okay.
A: ... because of they're curved. It's gonna be -- it's gonna have to be a straight line.
Q: Okay.
A: And ... but these are moving proportional ...
Q: Um hm.
A: ... but this one's in the negative and this one's in the positive.
Q: Can you rule one out?
A: ... I'm gonna rule out the negative because you can't have a negative height.
Q: Okay.
A: And so I'll put III.
Q: Alright. Makes sense to me. 24.
A: [Reads the question.]
Q: Have you talked about complex numbers?
A: Um, yea, a little bit.
Q: Okay.
A: ... You just look at, you look at the graph ...
Q: Um hm.
A: ... and at the x-axis ...
Q: Okay.
A: ... and it crosses three times ... D.
Q: Okay.
A: There would only be three.
Q: Okay.

Student 6-1
2

Q: Go ahead. You can start with the first one. Okay, let me get my numbers here so I know which ones to tape ... Yea, do 1, 2, and 3.
A: And it's a parabola because, it's what (laugh), this is what parabola graphs look like (laugh).
Q: Okay, and you can go ahead and circle the graph -- circle the answer ...
A: Okay.
Q: ... that you choose on there. Um... If you had to describe why that's a parabola to somebody
though, what...?
A: Um...
Q: It's like why is a square a square. That kind of thing.
A: ... Um, bec--, I don't know. It's because it's just what a parabola looks like. The different ones.
Q: Okay. But you can tell it's not an ellipse or a circle.
A: Yea.
Q: How -- what's the difference then? Let's maybe look at it that way.
A: An ellipse is like a circle it just...
Q: Okay.
A: It looks like a circle, and a circle isn't (laugh)....
Q: Round.
A: Yea.
Q: Okay. So this is missing some of those features that...
A: Just, yea.
Q: Okay. Number 2.
A: It'd have to be a parabola also.
Q: So moving it...
A: Yea, shifting it to different...places. It's still the same thing.
Q: Okay. Go ahead and circle it. Alright, 3.
A: ... (2,-4).
Q: Because...
A: You go first on the x-axis and then down or up the y-axis.
Q: Okay, and you just count the...
A: Yea, the little lines.
Q: Okay. Alright. 5.
A: ... I'd have to say the graph of B is an upside down version of the graph of A because, um,
multiplying by the negative would change every-- change what was positive negative and so it would
be just flip-flopped over.
Q: Okay.
A: Changed around.
Q: Alright. 7.
Q: Is that word familiar to you (trigonometric)?
A: Yea (laugh).
Q: Okay.
A: Um, graph I [one] because it's the graph of, I think it's the sine.
Q: Okay.
A: Or cosine, one or the other. They look about the same to me (laugh).
Q: Okay. Um, and you're certain the other three aren't?
A: Yea, this is, like x cubed, this is a parabola, and this is... I forget but, that's how I remember this
graph...
Q: It's not a trig.
A: Yea.
Q: Okay. Uh, let's try 9.
A: Hm... Probably there is no maximum value for y because it's got various answers. Depending on
what you put in here depends on what this would be.
Q: Okay. So then it, it doesn't reach any...
A: Yea, well--
Q: ... kind of peak value.
A: Yea. Depending on what I guess you'd put in place of x.
Q: Okay. 11.
A: Um... Probably... hm, I think I and IV. I think (laugh). Because it's confusing.
Q: Okay.
A: Hm...
Q: Well what's making you say that?
A: Because this, number II, is the inverse graph of the sine.
Q: Um hm.
A: This is a circle and I don’t think that it would be an \( f(x) \) function. A parabola is and I think that this one is.
A: \(-\infty\) to \(-10\) and, um, 3 to \(\infty\). Well, -10 to \(\infty\) because in between here it’s decreasing. It’s going down. Here it’s going up.
Q: Okay. Is it going down here, though, because the arrow is pointing that way or \(\ldots\)?
A: Um \(\ldots\) I’m trying to think how to explain that (laugh).
Q: Alright, try 14 and then maybe we’ll come back to 13.
A: Okay.
Q: To help answer the question.
A: Um \(\ldots\) Probably graph I [one] because it’s, it’s going upward, so like if you’re coming from the left side \(\ldots\)
Q: Um hm.
A: \(\ldots\) of the graph, like here, it’s going up.
Q: Okay.
A: Then I guess the arrows really don’t have very much to do with it.
Q: Okay. Is there a purpose for the arrows then? Are they\(\ldots\), what do they serve?
A: It’s showing that it keeps going on. That’s the end behavior.
Q: Oh, okay. (Laugh.)
A: That’s the word (laugh).
Q: Alright. And so we’ve ruled out all the other ones in this case.
A: Yea.
Q: Okay. Alright. 15.
A: Probably \((-5,4)\). To a vertical line so \(\ldots\) yea, it’d be \(\ldots\) yea it’d be \((-5,4)\) because if it’s symmetric with a vertical line what’s ever’s on the left side is on the right side or vice versa.
Q: Okay. So did you really need to know where the vertex was in this problem or was the fact that \(\ldots\) ?
A: Mm, yea because \((5,4)\) could have been the vertex and if that was so then \((-5,4)\) couldn’t have been on the graph.
Q: Okay. Uh, 17.
A: Hm \(\ldots\) moving the \(y\) graph to the right \(a\) units.
Q: Because \(\ldots\)
A: Mm, because it’s like a vert\(\ldots\) a horizontal move to get the same graph. It’s the same graph, it’s just moved over.
Q: Okay. How do you know it’s not left versus right or up or \(\ldots\) ?
A: Because the sign, the sign of the \(a\). If it’s a negative \(a\) then you move it to the right. If it’s a positive, you move it to the left. Just the opposite.
Q: Okay.
A: That’s what \(\ldots\)
Q: Um, 18.
A: \# \(\ldots\) 1 [one] and IV.
Q: Okay. And how did you check that?
A: Just going up and down the, the \(x\) and \(y\) axises.
Q: Okay. And, and you were looking at, counting or just \(\ldots\) ?
A: Yea. Checking to see how far they were up and \(\ldots\)
Q: Okay.
A: \(\ldots\) whatever \(\ldots\)
Q: Alright. Uh, 19.
A: \# \(\ldots\) B, between \(\ldots\) Yea, between -2 and 2 because it’s less than.
Q: Okay. And how, how would you explain how you determined that to someone?
A: If it’s for less than, you’d have \(\ldots\), since the line \(y = 4\) is this way \(\ldots\)
Q: Um hm.
A: \(\ldots\) then you’d have to look below if it \(\ldots\) since it’s less than the \(y^2\) graph which is the rest of it up here.
Q: Okay. And then you had to do what after you figured you were gonna look at this piece of the graph?
A: Well, looking on the x-axis to find out where there's, well, then that way you can fix your other thing.
Q: Okay. 21.
A: Mm, two.
Q: Okay.
A: Because there aren't any arrows so it couldn't be there is an infinite number of solutions or numbers.
Q: Okay. So there's only two points on the, on the line segment.
A: ... No, that wouldn't be right then (laugh). Well, you got like all these.
Q: Um hm.
A: I don't know how many's there (laugh) ... Wouldn't be an infinite because the number -- the arrows would mean that they're going, it's going on forever ...
Q: Okay.
A: ... so that'd make it an infinite. I don't know if there's two hundred, ten thousand or one million.
Q: Now it's a toss up ...
A: Yea.
Q: ... between which responses? A, B, C, D? Which ones? You're ruling out E ...
A: Yea.
Q: ... because there's no arrows. Okay ... Any other you can --
A: It wouldn't be A either because ...
Q: Okay.
A: ... these are the end points.
Q: Okay.
A: I guess it'd be sorta between B, C, and D. Yea.
Q: Okay, one of those. Well --
A: Maybe this one.
Q: Which one? And we'll -- just go ahead and pick one.
A: B.
Q: Okay ... You wanta circle it and that way ...
A: Ooh (laugh).
Q: 23.
A: One.
Q: Okay. You did that pretty fast.
A: Yea, 'cause it crosses the x-axis once (laugh).
Q: And then you know that's how many ...?
A: Yea, that means ...
Q: Okay. Does -- and so it doesn't make any diff-- this 3?
A: Huh uh.
Q: Okay. Go ahead and try 22.
A: ... Um ... I would say, graph III ... It's, I don't know. It's the one I guess I gue-- I guess it, I guess (laugh).
Q: No particular ...?
A: No. No particular reason, just I guessed it (laugh).
Q: Okay. I and II. Could you, could you tell me which ones did you rule out? Was it ...
A: You would probably rule out these two on the negative side.
Q: Okay.
A: Because those wouldn't make any sense I guess.
Q: Okay.
A: They're negative. I don't think an object's negative.
Q: Okay. Alright, twenty-uh, five. Wait, let me see ... Yea, do 24 and then 25.
A: ...
Q: Do you know -- is that term familiar, complex?
A: Um hm.
Q: Okay.
A: Probably ... I would say two.
Q: Because...
A: Because, um... because — well, I'm not sure what's because (laugh).
Q: Did you use the graph to help you answer the question or was it well...?
A: Yeah, it was kinda like well, this looks good.
Q: Okay.
A: I guess maybe because these ones that cross the x-axis are real solutions.
Q: Okay.
A: They're zeros. Maybe these ones are imaginary or something.
Q: Okay.
A: Because when you — the -1, you flip it over the x-axis...
Q: Okay.
A: ... and then you move it to the left 3. The whole graph.
Q: Okay. How did where did you pick up moving it to the left 3. Which part of it?
A: The, um, x + 3.

Student 6-2
3
A: Okay, the first question...
Q: Oh, no. Read the question.
A: Okay. It's a parabola. The reason why I know that is 'cause, um... when we were learning parabolas, um, we figured out what the shape would be.
Q: Um hm.
A: And then we learned how to, um, transform them, like, like $h - k$, I think that's what it is.
Q: Um hm.
A: And we, and then from, from the x, the $x^2$, which is this...
Q: Um hm.
A: ... we learned how to transform them down and vertical shift and stuff like that.
Q: Okay. You can go ahead and circle that response.
A: Okay. The next one... it's um, still a parabola. They did a transformation.
Q: Okay. So flipping and rotating it...?
A: Flipping it over the x-- y-axis. The x-axis.
Q: Okay. It doesn't change what it is?
A: No, it's still got the shape of a...
Q: Okay.
A: ... a parabola.
Q: 3.
A: You look to the x-coordinate first...
Q: Um hm.
A: ... which is over, uh, let's see, going by two so it's 4, and the y, the y-axis is going by ones...
Q: Um hm.
A: ... so you're down one, two, three, four which is negative.
Q: Um hm.
A: So -4 and this is positive 4 so the coordinates, this is the x so the first coordinate is 4 and the second is -4, so it's D.
Q: Okay. Good. 5.
A: It'd be, um, you... the negative, it'd be reflected, x so it'd be... it would be, well, this is an upside down version of graph A.
Q: Okay. What made it, be an upside version?
A: The reflection over the...
Q: What told you that it was a reflection?
A: The negative sign.
Q: Okay. 7.
Q: Is that word familiar (trigonometric)?
A: Yea.
Q: Okay.
A: Sines and cosines and tangents. Um, it's um, it's... number I [one] because, um, this has a cycle...  
Q: Um hm.
A: ... and it repeats itself. You don't know if it's pi or 2pi but...
Q: Um hm.
A: ... this doesn't have a cycle and neither does that. That's a parabola and that's an x^3 and that's a square root of 2 I think. Square root of x or, I'm not quite sure what that one is.
Q: Okay. You could rule those out?
A: Yea.
Q: Okay. Number 9.
A: I don't know if I could factor that or not?
Q: What will the factoring do for ya?
A: It, um...
Q: If it would...
A: ... I think it'd, it'd get to... then you set them equal to 0.
Q: Um hm. And then what would you do? You'd have...
A: And then it'll be the zeros, wouldn't it?
Q: Okay.
A: Um... where it hits the x-axis.
Q: Okay.
A: Um...
Q: Where would you go from there. How would that -- let's just pretend it had, okay. How would that have helped you determine what the maximum y value was... Or was that just something you always tried...  
A: I think that there is -- I don't think there is a maximum y value because it has the shape of a parabola...
Q: Um hm.
A: ... and parabolas, I think the y values going to infinity.
Q: Okay. Now, how do you know that graph's pointing up, not right, left, down, or...?
A: Well, um... well it's positive so you know it's taking the shape up. Because if it was negative it would be going down.
Q: Oh, okay. If it was negative and going down would it then have a maximum y value or still because it's, the arrows are pointing down it wouldn't happen?
A: ... You have a, wait a minute. That wouldn't have a minimum either.
Q: Well, would it have a maximum? Let's still ask the same question.
A: If it was negative?
Q: Um hm.
A: I think wherever it -- if I took that shape.
Q: Um hm.
A: Right there would about...
A: ... I don't know this one.
Q: Okay. Any, um, let's see. We're asking which of these are functions. They all could pass...
A: ... the vertical line test, so that'd be...
Q: ... What was that? The vertical line...?
A: The vertical line test.
Q: And what does that tell you? Or how do you use that?
A: Um, if it doesn't pass the vertical line test, then it's not a function.
Q: What is that test? How would you describe that to me?
A: You would take a, like, a ruler, and anywhere along the graph...
Q: Okay.
A: ... you ... pass, if it passes only through one point, then it's a function.
Q: Oh, okay.
A: Take number III.
Q: Um hm.
A: You could take a vertical line test anywhere and it passes through two points. And it has, it even has to be failed once and it's not a function.
Q: Okay. Alright. So in that case we come up with what? I [one] and IV. Okay. Did you just think about ... ?
A: Yea at the very last minute.
Q: Okay (chuckle). Um, 13.
A: It's increasing from \(-\infty\) to this point ...
Q: Um hm.
A: ... which is ... -1.7 and then it's increasing from this point on to positive \(\infty\) ... 
Q: Um hm.
A: ... which is 1.7. It's increasing from \(-\infty\) to -1.7 and positive 1.7 to positive \(\infty\).
Q: You were reading, um, off of the x-axis. Why not the, um, y-axis to say something like \(-\infty\) up to whatever this number is, uh, 10 and then from here up?
A: Ten, um ... just say you're taking the maximum value point ...
Q: Um hm.
A: ... and um, ... 
Q: ... do it. Okay. Try 14 'cause that's a similar question and maybe ...
A: Okay ... Because on the interval of (1,3), on this one, that's decreasing and that's decreasing ...
Q: What if someone came and said well, looking at this graph between 1 and 3 I see this arrow's going up and that's increasing?
A: ... Um, ...
Q: Could you convince them that they're, uh, or that you disagree?
A: Going from the right you mean? It's decreasing.
Q: If you were going from 3 to 1 and looking at it ...
A: Um hm.
Q: Okay, so you were looking at here between x is 1 and 3 and telling me about the graph increasing. Now maybe, knowing what you've done here can you maybe explain why we didn't focus on the 10 and I guess this is a negative ten down here and you said it's going from \(-\infty\) to one point see-- , -1.7 and 1.7 upward.
A: Um ... 
Q: Why do we look at the x versus what was happening to the ...? What was happening 'cause that what you were doing.
A: Um ... Is it because of, um ... because of the domain?
Q: I was just curious. No problem. Um, 15.
A: ... You would have the point (1,4).
Q: Okay.
A: Because um ... his is the line of symmetry, and whatever's on this side is on the opposite side is how I got it. And I got the point, what is it, (1,4).
Q: Okay. How did you know the vertical line went through right there and ...?
A: 'Cause, um, (3,0) is the vertex and ... it's, in this problem, doing parabolas, I just learned how, about line of symmetry.
Q: Okay. 17.
A: ... If'd be C, moving the y1 curve left a units.
Q: And why is that?
A: It's due from the transformations we learned and ... I guess it's like an equation. You know, it's \(h - k\) ...
Q: Okay.
A: ... and, um, if you move negative you'd su--, wait, no. You move to the right. Let me think ... You would think if it's negative you move to the left ...
Q: Um hm.
A: ... but you move the opposite direction, to the right.
Q: Well, that's fine. Okay, nine—, no 18. You can do 18.
A: I [one] and II.
Q: You did that pretty fast. How?
A: I just, um...
Q: What'd you look at? - -
A: The first thing I looked was, I started with this graph...
Q: Um hm.
A: ... and this said it was three and then this, that was between zero and five...
Q: Um hm.
A: ... it was in between, closer to five so I figured that was three. I knew that would be closed because that's point 1...
Q: Um hm.
A: ... and I knew that because that was six and none of those are even by six.
Q: Okay. Great. I [one] and II. Alright. 19.
A: ... is this two and that's four? Isn't it? I mean...
Q: Oh, is this...?
A: The scale.
Q: Um, well let's see.
A: It looks like it.
Q: This is four...
A: Yea.
Q: ... so if the y-coordinate is four...
A: Yea.
Q: ... how could you determine what the corresponding x-coordinate was?
Q: Is there any way of doing it? You were saying it's four, so when x is four, y is four. Is that right?
A: Um... 'Cause two squared equals four.
Q: Okay.
A: ... Um... squared... If you get, it's between -2 and 2.
Q: Okay.
A: Because um, well from the graph... goes from positive 2 to -...
Q: Um hm.
A: ... the x^2 is greater and the same with -2 to the -...
Q: Um hm.
A: ... that's greater. So...
Q: So it makes sense to look below?
A: Yea.
Q: All those numbers, uh, would they fit if I picked some x value between here and here, it would be less than four and I squared it, and it would be less than 4 then? Is that correct?
A: Um hm.
A: Infinite number.
Q: Okay. Could you...
A: ... The number of points like this and this and that and that. I mean...
Q: Too many to count?
A: Yea!
Q: Alright, 23.
A: ... Um... From the graph you know it's a, something, an x to the third graph...
Q: Um hm.
A: ...
Q: Cubic?
A: Right.
Q: Okay...
A: Cubic. Okay. And from doing solutions in math, you know that there's a possibility of three...
A: ... there's a, there's three solutions ... in this graph there's one real and then there's two irrational.
Q: Okay.
A: ... Well, wherever it hits the x-axis it's a zero, and zeros are solutions.
Q: The three makes no difference in the problem, the f(x) - 3?
A: No, it didn't.
Q: Okay.
A: I don't think it determines how many real solutions ...
Q: Okay.
A: ... or determine what that solution is.
Q: I skipped 22. I'd like you to try that one, too.
A: ... I have no idea on this question.
Q: Okay. Any guesses or ... ?
A: I was gonna say III.
Q: Okay.
A: ... It would be a guess (chuckle).
Q: Okay. But there's something about that one versus the other three that stands, do you know what makes it stand out that that would be a good guess or ... ?
A: Um ... uh, there's a ... 
Q: Is it because it's response C or you didn't even notice that? You were just looking at the graphs?
A: I was just looking at the graph.
Q: Okay. 24 and 25 and then we're done.
A: It has a total of three because a cubic equation has three solutions whether rational or irrational, real or nonreal.
Q: Okay. 25.
A: ... do these ... These transformations I can't really do well ... I'm gonna have to say number III.
Q: Okay. Um, okay, alright, what made the graph flip? If I asked you to look back at the equation, what piece of the, uh, equation told you to flip it upside down? Where's that?
A: ... reflect. That's the vertical, shift ... h tells whether it's reflexive or not.
Q: Okay.
A: It tells it's a, um, horizontal shift to the left 3.
Q: Okay ... Does it, does it reflect or not? I, i ... 
A: No.
Q: It doesn't. Okay. So it's not III. But it's got a horizontal shift to the ... 
A: Left.
Q: ... left.
A: Three.
Q: Okay.
A: Vertical shift down ... 
A: It goes up two there.
Q: Well this is, this is the graph. We're saying this is the graph. Now ...
A: Oh, okay.
Q: ... apply this transformation to this. So that's your reference.
A: Alright. So this is not one of my choices ...
Q: Um hm.
A: ... goes down one then it's, that point goes over three ... 
Q: Okay.
A: ... which gives us the point (3,1). That one right there.
Q: That's it. Thank you very much.
A: It's a parabola.
Q: Okay.
A: Um, just because it curves upward.
Q: Okay.
A: It's got an upward...
Q: Okay, what would, let's say, what makes it different from an ellipse? How do you know that it's, or a circle or...?
A: The ellipse would be enclosed on both ends.
Q: Okay.
A: A parabola has one open end.
Q: Okay. So the arrows mean something to you then?
A: Uh huh. Yea, they continue going upward and they won't ever... ever connect.
Q: Okay. Alright. 2.
A: It's a parabola also because you're not doing anything to the graph, you're just turning it.
A: It'd be (2,-4).
Q: Okay.
A: The answer B... It's to the right two and down four.
Q: Okay, two units and then...
A: Two units to the right.
Q: Okay.
A: Okay, I'm sorry.
Q: That's okay (chuckle). That's good. Alright, what do we have next. Um, 5.
A: ...Okay, uh, I think it would be answer C. It says the graph of B is an upside down version of the graph of A because...
Q: Um hm.
A: ... it's negative.
Q: Okay.
A: I'd flip it over into... a vertical flip.
Q: Okay. 9. Did we do ?? I skipped 7. I'm sorry, I thought we went too many pages. There it is.
Q: Is that term familiar, trigonometric?
A: Yea.
Q: Okay.
A: Um, graph number I [one] looks like the graph of a trigonometric function.
Q: Um hm.
A: The sine graph.
Q: Okay... you ruled the rest out?
A: Um... III sort of looks like it might be a sine graph...
Q: Um hm.
A: ... but it, it uh, no because it continues to go downward...
Q: Um hm.
A: ... and it continues to go upward.
Q: Okay.
A: So it wouldn't -- it couldn't be a sine or a cosine... That's why I picked answer I [one].
Q: Okay. 9.
A: ... four... 13... I don't know why (chuckle).
Q: Okay.
A: ... I think that's more like a guess than anything else.
Q: ... it's here, if, if... you think of, uh, any explanation that uh...
A: I can't think of any explanation.
Q: ... do this, do this, do this, do this? Okay, that's fine. B. 13. I mean B, \( y = 13 \). Go ahead and circle it for me. And the next one's 11.
A: ... It's I and II ... because it's symmetric about the y-axis.
Q: Okay, and that one is too?
A: Yea.
A: ... It's increasing ... Okay, I, I believe it's answer, answer D.
Q: Um hm.
A: It increases from -3 to -1.7, which is right here.
Q: Okay.
A: ... and then it also increases from positive 1.7 ... actually it keeps on increasing ... Alright, it's answer E. I didn't look at it too well.
Q: Okay.
A: Increasing from \(-\infty\) all the way up to -1.7
Q: Um hm.
A: ... and from 1.7 to positive \(\infty\).
Q: Okay. What's um, now let's see. We're, you're saying it's going from here to here it's increasing ...
A: Um hm.
Q: ... and I see the \(-\infty\) but why wouldn't you say ten here and then say from negative ten up?
A: Uh, because you want to go by, um, judge it by the do—, its domain.
Q: Okay.
A: From negative —, if you squish it all into the x-axis — or into the y-axis — x-axis ...
Q: Okay.
A: ... you're gonna read your numbers across the x-axis.
A: Although I'd say answer I [one] ...
Q: Um hm.
A: ... although it is increasing from \(-\infty\) to positive \(\infty\) ...
Q: Okay.
A: ... and I just put answer I [one] because one and three — it's increasing between one and three ...
Q: Um hm.
A: ... along with negative to positive \(\infty\).
Q: Okay. What if someone came over here and said, alright, if I look from three to one and I follow, it looks like this graph would be increasing because of this arrow right here. What would you say to them?
A: Um, I've always been taught that you read increasing and decreasing from left to right.
Q: Okay.
A: If the graph's going up from left to right it's increasing. If it's going down from left to right it's decreasing.
Q: Okay. That's good. 15.
A: (1,4). The point (1,4).
Q: Okay. How'd you—
A: Answer C.
Q: How'd you do that?
A: Um, I just put graph, well ...
Q: Go ahead.
A: I, uh, figured out in my head ...
Q: Um hm.
A: ... that if the, and the, if the vertex of the, is at point (3,0) ...
Q: Um hm.
A: ... it's over three and it's on the x-axis.
Q: Okay.
A: If you go over five and up four ...
Q: Um hm.
A: ... and it says that, uh, with respect to a vertical line, so it would have to be a vertical line and symmetric ...
Q: Um hm.
A: ... you just, I would, uh, I moved two over from three ...
Q: Um hm.
A: ... to get five on the x-axis ...
Q: Um hm.
A: ... so I'd bring two down. That'd give me a one and I still move up four because it's symmetric ...
Q: Okay.
A: ... so I got point (1,4).
Q: How would you explain, um, what symmetry means then in this case to someone? If you had to explain to them when it says it's symmetric about a line, how would you explain that to someone?
A: Um, if it's, okay on this one it's symmetric to a vertical line ...
Q: Um hm.
A: ... and, uh, at three ...
Q: Um hm.
A: ... so, I would, uh, basically say that if you went ahead and graphed from point (3,0) to point (5,4) ...
...
Q: Um hm.
A: ... then it's gonna be symmetric. It's gonna be the same, it's gonna be like a mirrored effect to the other si— to the opposite side.
Q: Okay. Alright, how did you know that the vertical line went right through (3,0)? What...
A: Well, 'cause it says the vertex of a parabola is symmetric with a vertical line. So the vertex would be the bottom-most point or the top-most point of a parabola. In this case it is the bottom-most point of the parabola.
A: Um ... it would be, um ... um, the curve right. Moving the y, the y1 curve right a units which would be ...
Q: Okay.
A: Because if the a is inside the parentheses ...
Q: Um hm.
A: ... it's a horizontal shift.
Q: Okay. And how do you know right versus left or ...?
A: Um, it's, I've just always memorized it as ...
Q: Okay.
A: ... as opposite of what's inside the parentheses, so ...
Q: Okay.
A: So I made it positive which would move it right a units.
Q: Alrighty. That's 19 or 17?
A: 17.
Q: Okay, 18 and 19.
A: ... Um, answer would be I [one] and II.
Q: Okay.
A: ... Um, uh I counted up on the, on the y-axis, uh, I counted up to 9 ...
Q: Um hm.
A: ... and I answered II over here. Uh, it was just, it was just below 10 ...
Q: Um hm.
A: ... so I figured that was about the same for the y-axis.
Q: Okay.
A: And on the x-axis, uh, the parabola is stretched, or went to each side 3 units ...
Q: Um hm.
A: ... and on answer number II it was just before the 5 on each one.
Q: Okay.
A: So I figured they were approximately the same thing.
Q: Good. 19.
A: ... Um, it would be, be answer B ...
Q: Um hm.
A: ... when um ... like it'd be open intervals at -2 and 2 and it's any-- x has to be anywhere between -2 and 2 with open intervals at each end.
Q: Okay. How'd you get the...
A: Um.
Q: ... the two. Did you use the graph ...
A: No ...
Q: ... or did you just look at the ...?
A: I didn't use the graph at all.
Q: Okay.
A: I just looked at it and, uh, just trial and error I guess. I don't know.
Q: Okay.
A: ... Once I started, I knew it wouldn't be a real high number so I just ...
Q: Um hm.
A: ... picked 2 as my starting number.
Q: Okay.
A: And it has to be less than -- it can't be equal to four so if the ends start at -2, -2 squared is equal to four ...
Q: Okay.
A: ... so it would have to be somewhere, it would have to be greater than -2 and less than 2.
Q: Is there any way you can maybe see a connection to the picture to, to answer the, the problem?
A: Sure, um, you could draw vertical lines straight down from the point of intercept of the line \(x^2\) ...
Q: Um hm.
A: ... an-- or the parabola \(x^2\) and the line \(y = 4\). And I come down and I'm at 2 and that's where they intersect ...
Q: Um hm.
A: ... and so's that has to be less than and so I'd a been in between somewhere greater than -2 ...
Q: Um hm.
A: ... and somewhere less than 2.
Q: Okay. Um, 2.1.
A: There is an infinite number. Um, it's answer E ...
Q: Um hm.
A: ... there is an infinite number because, um ... if you count all the points, there's, there's an infinite number of points in that line right there.
Q: Alright. 22.
A: ... I think I'll go with answer III.
Q: Okay. Because ...
A: Which is C. Um, I know that the height of the object and the length of the shadow both have to be positive. So they both -- the graph has to be in the first part, it has to be in the first quadrant.
Q: Um hm.
A: Um, and I didn't go along with, uh, I didn't go along with I [one] because the length of the shadow automatically starts at one in that first quadrant so I had, I figured I'd start at (0,0) because ...
Q: Okay.
A: ... if it's zero height there's gonna be zero shadow.
Q: Good point. Alright, uh, 23.
A: ... It has, uh, one real number solution, according to the graph ...
Q: Um hm.
A: ... um, because the graph only crosses the x-axis once.
Q: Okay. So the \(f(x) - 3\), that doesn't change it at all? It'll still just have one?
A: It'll still just have one.
Q: Okay. 24 and 25 and then we're done.
A: ... Okay, according to the graph, uh, being a cubic also ...
Q: Um hm.
A: ... that'd show me that it had three because cubed is three ...
Q: Okay.
A: ... and on the graph it crossed the x-axis three times.
Q: So we can use it for both complex and real? We just look at the ...
A: Um hm.
Q: Alright, good. Uh, twenty-- oh, the last one.
A: ... Um ... Okay, I went with uh, answer number V.
Q: Okay.
A: Um, because it would be a horizontal shift of -3 ...
Q: Um hm.
A: ... and, uh, the only graph with a horizontal shift of -3, which is three to the left ...
Q: Um hm.
A: ... is answers number III and IV (student meant V). And answer number III I automatically ruled out because it showed the graph as being negative ...
Q: Um hm.
A: ... h(x). And that's not how it is.
Q: Alright. What's the minus I do?
A: Okay, it just a vertical shift down ...
Q: Oh, okay.
A: ... one.
Q: Very good. Thank you.

Student 6-4
3

Q: Number 1.
A: Parabola.
Q: Okay. Because. I know it's (laugh).
A: It just is, that's all.
Q: What makes it different from the rest?
A: I know what a point is and it's not a point and it's definitely not a line ...
Q: Okay.
A: ... uh, it's definitely not a circle.
Q: What's different from a circle and, and the parabola. What ...?
A: It's ...
Q: If you had to explain to someone.
A: A circle is just, I mean a parabola doesn't connect.
Q: Aha. Okay. What do the arrows mean?
A: It goes on to infinity.
Q: Oh, okay. Alright, number 2.
A: ... mm ... I'd say still a parabola.
Q: So rotating won't change what it is?
A: No.
Q: Alright. 3.
A: ... (2, -4).
Q: Okay. How'd you do that?
A: It's coordinates are (x, y).
Q: Um hm.
A: Go over to 2 on the x-axis ...
Q: Um hm.
A: ... down to -4.
Q: Okay. 5.
A: ... I'd say the graph of B is upside down version of graph A.
Q: Okay. Because ...
A: Because that's negative and it'd flip it around the x-axis.
Q: Okay. Uh, 7.
A: Yea.
Q: Are you familiar with that word (trigonometric)?
A: Yea.
Q: Okay.
A: . . . I'd say letter, oh, number I [one] . . .
Q: Um hm.
A: . . . because it looks like a sine graph.
Q: And the others don't look like any . . .?
A: No, they're just function graphs.
Q: Okay. Number 9.
A: . . . I don't know. I don't get that one.
Q: Okay. Um . . . what would that -- do you know what that would be the graph of,
   \[ y = x^2 + 4x - 9 \]?
A: Yea, it'd be a parabola.
Q: Okay. Any ideas if I had, say just draw a sketch and I don't care if it's in the right position on the
   axes but . . .? Would that help or . . .?
A: Nope.
Q: . . . What does it's basic shape look like?
A: It's a parabola.
Q: Okay.
A: I mean you can't, it doesn't factor out so . . .
Q: Okay.
A: . . . I don't understand what it means maximum value here.
Q: Okay . . . A: I don't know what I put before but . . .
Q: Okay.
A: . . . I probably guessed.
Q: What if we, let's see . . . if we did 8? . . . Still have the same problem? Is this one easier or more
difficult?
A: I don't think . . .
Q: Okay. Alright. Well, that's fine. So if you had to guess on this one what would you say . . . for a .
   maximum y value?
A: Um . . . I don't know . . .
Q: Okay.
A: Probably a negative number.
Q: A negative number because that would . . .?
A: . . . Well I don't know if a positive number would get a maximum value.
Q: Um hm.
A: . . . I'd probably say B.
Q: Okay. Go ahead and circle that one and we'll go on to 11.
A: . . . I [one] and III -- I [one] and II.
Q: I [one] and II. Okay, because . . . What sets them apart from the other ones?
A: Well, one's a parabola . . .
Q: Um hm.
A: . . . and the other one looks like . . . an inverted sine graph.
Q: Okay. A: A circle's not a function . . .
Q: Okay.
A: . . . and this other thing is not a function either.
A: (Laugh.) It just don't look like one.
A: . . . From \[ \infty \] to -1.7 and 1.7 to \[ \infty \].
Q: Okay. How'd you get that? What did you look for?
A: The maximum's here.
Q: Okay. And then -- but that max is 10. You say it goes to 1.7.
A: Well, you have to look down on the y-axis here.
Q: Okay.
A: 'Cause it goes this way.
Q: Okay.
A: So you go from ∞, it increases 'til you get to here . . .
Q: Um hm.
A: . . . and then it starts decreasing and then increases again after you get past 1.7 here.
A: . . . Number I.
Q: Okay. And how did you decide that?
A: Just looked for 1 and 3 on the x-axis.
Q: What if, uh, someone came and said, alright, from 3, if I look from 3 over one, this graph looks like it's increasing. What would you, what would you tell them?
A: I'd just start from the origin and go over 'til you start, 'til you get to 1, and then I'd just get between this interval and it's going . . . out . . . I don't know how I'd explain that but that's just . . .
Q: Okay.
A: . . . that's what it is.
Q: Okay. 15.
A: . . . (1,4).
Q: Okay. How'd you get that?
A: Well, the vertex is right here at (3,0) . . .
Q: Um hm.
A: . . . and then you plot a point at (5,4) here. Just finish it out. This'd be right there. I mean these are way out. I mean, -1 would be way over here. That wouldn't even look like a parabola right there.
Q: Okay.
A: And, -5 would be down here, and . . . well -4, well wait a minute, -5 would be over here and -4 would be down here and that's not a parabola.
Q: Um, how would you explain symmetry? This says it's symmetric with respect to a vertical line. What if you had to explain what that meant to someone, what . . .?
A: Well, your vertical line would be right here . . .
Q: Um hm.
A: . . . at the origin.
Q: Okay.
A: You just flip it over.
Q: So it's a, this would
A: Yea.
Q: . . . flip over here?
A: This side right here would be the same as that side.
Q: Okay. Uh, 17.
A: . . . I'd say D, moving the y, y₁ curve right a units.
Q: Um hm.
A: Because . . . it'd be a vertical -- or, a horizontal shift right here.
Q: Okay. How'd you --
A: You move it over.
Q: How'd you figure that out? After you you completed the square or from here?
A: After you complete the square. 1 . . .
Q: Okay.
A: . . . couldn't figure it out just by looking at that.
A: . . . I [one] and π.
Q: Okay. And how'd you decide that?
A: Just look for 3 . . .
Q: Uh huh.
A: . . . on here, point 3. That's, this is way off 'cause this is going by 2's . . .
Q: Okay.
A: . . . and these are going by 1's and these are going by .1's so you can't . . .
Q: Where are we? That was 18?
A: Yea.
Q: 19.
A: ... B.
Q: Okay.
A: Between -2 and 2.
Q: How -- did you use the graph or did...
A: Yea.
Q: ... did you just figure it out?
A: I got the points right here.
Q: Okay. How do you know it's in between and it's not, like C?
A: Well, because, I mean you just fill in.
Q: Um hm.
A: If you put, if, yea, these are going, uh, less than 2 and greater than 2...
Q: Um hm.
A: ... so if you put that in it won't make this true.
Q: Okay. That's good. 21.
A: ... That's just common sense.
Q: (Laugh.) Okay. 22.
A: ... I'm gonna have to guess on this problem -- I'm gonna say III.
Q: Okay.
A: Um... the shadow has to be proportional to the height...
Q: Okay.
A: ... so if you got an object this big...
Q: Um hm.
A: ... and it casts a shadow down...
Q: Um hm.
A: ... so that means that this shadow, as far as it goes out, should be like this right here...
Q: Um hm.
A: ... should be the same as that.
Q: Okay.
A: ... I don't know. I... None of these graphs show that. Might not be any of 'em.
Q: Okay. None of these is there. If you think, as you said, the shadow has to be the same as the...
A: As the height.
Q: ... as the height...
A: Yea.
Q: ... and none of these... Okay. That's fine. Alright, uh, 23.
A: ... 3.
Q: Okay. How'd you do that?
A: Well, you have to get f(x) = 3...
Q: Um hm.
A: ... and that'll be a line. It'll be right here... So the three times here, those are real solutions right there.
Q: Solutions because they, they sit on that, that line? Does that --
A: Yea, and --
Q: -- make sense to you or is this, this is something that Mr. Meyers said? This is what we do and...?
A: ... I don't know. That's just the way I been taught I guess.
Q: Okay. Alright, the last two, 24 and 25 and we're done.
A: ... I guess I'd have to say D 'cause...
Q: Okay.
A: ... the only solutions I see is where they hit the y-axis right here -- or the x-axis.
Q: Okay. And the last one.
A: ... Number L
Q: Okay.
A: ... Well, -1, that's a reflection. It's flipped around the y-axis.
Q: Okay.
A: And this is a ... horizontal shift ...
Q: Um hm.
A: ... three units to the right.
Q: Okay, good. That's it. We're done.

Student 6-5
Violation: 1-0-1-1

Q: Okay. Number 1.
A: Parabola.
Q: Okay.
A: Because that's what we learned at the beginning (chuckle) of the year ...
Q: Okay.
A: ... and last year.
Q: You can just go ahead and circle that for me.
A: Just the letter or the ... ?
Q: Uhhuh. But whatever you want to do. Um, what makes it -- but if you have to explain to somebody why this really is a parabola versus -- why isn't it an ellipse or a circle or a point? What makes that stand out as a parabola? What ... ?
A: Because it's the graph of $x^2$ and $x^2$ has the end behavior.
Q: Okay.
A: ... and that's how it goes, if it's like that then they're both out ...
Q: Okay.
A: ... and that's where it is a parabola.
Q: Okay. Alright.
A: Number 2?
Q: Yea.
A: I'm not sure. At least it's not a parabola because it goes sideways ...
Q: Okay.
A: ... because it can't be negative.
Q: Alright.
A: A parabola can't be negative. So, can I guess?
Q: Yea, certainly!
A: I think it'd probably be an eclipse just 'cause you can only take into consideration just half of it 'cause you can't use the negative part.
Q: Okay, so that's not all of the ellipse then? There's more to it?
A: Yea.
Q: Okay. Alright. Um, 3.
A: ... (2,-4).
Q: Okay. How did you --
A: 'Cause you just go over 2 units ...
Q: Um hm.
A: ... and then down 4 and that's where it is.
Q: Okay. Uh, yea. 5.
A: ... Um, C.
Q: Um hm.
A: The graph of B is an upside down version of the graph of A because this is positive and like ...
Q: Um hm.
A: ... the other one would be like that because it's an $x^2$ ...
Q: Um hm.
A: ... and if you put a negative, that just flips it over like that.
Q: Okay. Good, 7.
Student 6-5 (cont)
Violation: 1-0-1-1

A: [Question 7.] ... graph I [one] or, the first one.
Q: Um hm.
A: Because it's the graph of $\sin x$, and it looks like this is $x^2$ and this is $x^3$ so sort of by elimination.
Q: Okay. Good strategy. Um, 9.
A: ... I just don't know. I'm not sure.
Q: Then you can, what we'll do, it's a guess, and that's fine. And then I'd like to know what made you guess that as, you know. Provide as much information as you can about...
A: Okay. E, because, um, this can be, I don't know, you can put any value in there and then y can just go on and on and on.
Q: Okay. That's fine. 11.
A: ... This is what we're doing right now. Polar graphs ...
Q: Um hm.
A: ... and circles and things like that. And this is just the inverse of a sine graph ...
Q: Okay.
A: ... and so E is the one ...
Q: 13.
A: ... E. Because it's going from here, $-\pi$ and then it's going up all the way to right here ...
Q: Um hm.
A: ... I just, think that's close enough (chuckle).
Q: Okay.
A: ... close enough to neg-- between -1 and -2 ...
Q: Okay.
Q: I noticed you read off the x-axis. Why don't you say it goes from $-\pi$ to 10 and then from, uh, what is this, -10 to $\infty$? What makes you use the x-axis?
A: Because that, that's the point where right here ...
Q: Um hm.
A: ... like, down at 1.7 that changes and it starts the other way, and it goes up to that point so it'd be, really it'd be (-1.7,10). Right here would be that point ...
Q: Okay, I see.
A: ... but at 1.7 is where it changes and comes back down.
A: ... A, the first graph. Because from this point to this point the graph is going up. And the rest of them are not.
Q: What if someone came in and said well, if I look at this graph over here, number III, and I go from 3 to 1, I can see that it's going up because this arrow goes up? That's increasing or what would you tell 'em?
A: No, because that's, you're going the wrong way on the axis. If it said negative, you could, you can come over here and you could do that ...
Q: Okay.
A: ... but you can't go backwards. And it's positive going back that way.
Q: Okay. Good. 15.
A: ... A.
Q: Okay.
A: Because, um, with a parabola, it's, being symmetric.
Q: Um hm.
A: Both sides have the same points except for, like this is (5,4) ...
A: ... and then over here it'd be (-5,4).
Q: Um hm.
A: It'd be over on the other side.
Q: Okay. If you had to explain to someone what symmetry meant, how would you explain how a parabola is symmetric or, whatever? What does symmetry mean?
A: If both sides, when you fold it over, they would fall on top of each other. If you fold it over at the line.
Q: Okay. So there’s some line and I can fold it over there and then, what, things, . . . ?
A: Fall on top of each other if they’re symmetric.
Q: Okay. Uh, 17.
A: . . . I’d say C because, um, unless I remember wrong.
Q: Okay.
A: If it’s, you take this . . .
Q: Um hm.
A: . . . the, whatever number that is . . .
Q: Um hm.
A: . . . and if it’s negative then you take it to the left and then if it’s positive you take it to the right.
Q: Okay.
A: Just to move the gra— graph of the $x^2$ graph.
Q: Okay. 18.
A: . . . A, because, um, this graph is just has a larger viewing rectangle than this graph but it’s the same points.
Q: Sounds good. 19.
A: . . . A.
Q: Okay.
A: Because you would just solve for this and then that would be $x < 2$ . . .
Q: Um hm.
A: . . . and then put 2 on the x-axis and anything . .
A: . . . behind what you would use and since that equal to, it's not less than or equal to, it's an open circle.
Q: Okay. Did you use the graph or you just did it algebraically? How . .
A: First I did it algebraically because . .
Q: Um hm.
A: . . . that would be $x < 2$ . . .
Q: Okay.
A: . . . and then I used the graph. To see which graph looked like that.
Q: Okay. Well, this picture here. Does this picture help?
A: . . . No because I didn’t look at it.
Q: Okay.
A: So don’t really know because I didn’t look at it. (Laugh.)
Q: Okay. Well, you can try – I mean, if you hadn’t used the picture to explain your answer . . . It's too confusing or . . . ?
A: . . . No because I could do it because the point where they cross . .
Q: Um hm.
A: . . . is 2, and then it would just be less than because that’s what the thing was anyway. Less than.
Q: Okay. Um, 21.
A: . . . That’s, I think that’s the definition of a line or something.
Q: Um hm.
A: At least, you'd need at least two but there's an infinite number because each one uses a point. It just goes on for as long as you want it.
Q: Okay. 22.
A: . . . Graph II.
Q: Okay.
A: Um, because I know this is the inverse of this . .
Q: Um hm.
A: ... I just picked this because it says the length is, um ... proportional to the height and since it just folds over like that, about right there, if you fold it over then they'd be about the same. And these are just straight so I didn't pick them.

Q: Okay. 23.
A: ... Um, B, because it crosses the x-axis just once.
Q: Okay. The f(x) - 3, the 3 doesn't make any difference then?
A: Yea, you can, by, by using the graph, you can ...
Q: Um hm.
A: ... um, where it crosses you can use that answer, or else you could do it age— algebraically. Because whatever, um, you put in there ...
Q: Um hm.
A: ... that would make that be 0 and 0 = 0, that would be a solution, too.
Q: Okay. 24 and 25. They're on this page ...
A: ... 3, D. I say that because, um, cubic ...
Q: Um hm.
A: ... is x^3 ...
Q: Okay.
A: ... graph, and, um, x^3 graphs have three solutions ...
Q: Um hm.
A: ... plus I'm not sure if this has anything to do with that or not (laugh).
Q: Okay.
A: It crosses the graph three times, so that's why I just put that.
Q: Okay. 25.
A: ... It would be, um, V ...
Q: Okay.
A: ... which is E, um, because, just by using these and moving them around.
Q: Um hm.
A: Start here, the down 1 is, you're here and then, go to the left 3 'cause it's a positive 3 and so you go to the left 3.
Q: Okay.
A: And that makes you be here. And then you just do the same thing with all the other points.
Q: Okay.
A: Just go down and move it over. That's what this is.
Q: Okay. Good. Thank you very much.
A: You're welcome.

Student 5-1

3

Q: Alright. I'd like you to look at question number 1 for me. If you'd rea-- go ahead and read the question and what else -- oh that's right You can draw on this and whatever in terms of your explanation ... The graph below is a what?
A: Uh ...
Q: Rotating won't change it?
A: Huh uh.
Q: Okay ... Go ahead and read the question out loud for me. That helps me on this tape ... Uh, 4 ... Have you talked about exponential this year?
A: Yea.
Q: Okay... Okay, that's fine. Now, why did you rule it out from let's say, I [one]?... What makes it stand out that that's an exponential and not the others? What would...
A: ... Three, ...
Q: Um hm.
A: ... and this one goes to the x-axis, to the nega-, well, it don't have that. But this one's more like a ... parabola. One of them squared, to the squared. This one's just a single, uh, that curves up in one direction.
Q: Okay. Alright, good. Let's see, 6.
A: I think it's (-3,5).
Q: Okay. What does this symmetric to the origin mean to you? You, you put it up here versus putting it in this quadrant or putting it in that quadrant. How would you explain to someone maybe why you put it over there and...?
A: Well, if I put it in the first quadrant, it'd be respect to the ... x-axis...
Q: Um hm.
A: ... and if I put it in this, in the third quadrant, it'd be respect to ... uh, ...
Q: The y.
A: y-axis.
Q: Okay.
A: Yea.
Q: Okay.
A: ... origin because it's ... be both the x and the y-axis.
Q: Okay. What, um ... how would you determine, like, um ... For instance let's say, if it was, you put it here and you said that'd be symmetric to the x-axis. How would you know, how, where to put it in this quadrant. What would you, what would you be doing?
A: Well, in this, you put it in the first quadrant ...
Q: Um hm.
A: ... you have to change the sign of the ...
Q: Okay.
A: ... in the fourth it'd be respect to, or symmetric to y-axis. You have to change the sign of the x.
Q: Um hm.
A: But for it to be symmetric to the origin you have to change the sign of both x and y.
Q: Okay. So, you said like you had to change the sign here of the y coordinate. Um, so that means you leave the sign of the x coordinate the same ...
A: Right.
Q: ... so you'll come out here, um ... 
A: Instead of going down five you go up five.
Q: Okay, so then the distance is the same then ...
A: Right.
Q: ... from the axis. Okay, good. Alright, uh, 8.
A: ... IV has a maximum y value.
Q: How did you know that?
A: Cause the first graph is no maximum, it has a min but not a maximum.
Q: Okay.
A: And, number II is no definite maximum point 'cause it still keeps rising up.
Q: How do you know it still keeps going? What tells you on the graph?
A: The arrow ...
Q: Okay.
A: ... at the end of the parabola.
Q: Okay.
A: Same for number III.
Q: Okay.
A: But number IV has a ...
Q: Good. Uh, 9.
A: ...
Q: Okay, if, if let's just say if you could factor this, what would you do then? Let's just pretend you could factor it. What would have been your next step maybe?
A: Uh, the point of the ...
Q: Okay, so you were gonna try and graph? Is that --
A: Yea.
Q: Okay.
A: ... this point of the graph here.
Q: Okay.
A: Alright, um, you could do it that way or I could just plot points in ...
Q: Um hm.
A: ... x and y.
Q: Okay.
A: ... remember the method 'cause I gotta find the ... this.
Q: Okay ... Let's just --
A: ... to solve it.
Q: Okay. Let's say you, um, had a vertex. Just go ahead and, and say, go ahead and just say that point's the vertex. Alright. What, what else could you, um, ... knowing that then ... will that point always be the maximum value then? If -- once you find that?
A: No 'cause the point, 'cause by the equation, the ...
Q: Um hm.
A: ... the, uh, equation would be going up ...
Q: Okay.
A: ... instead of down.
Q: Okay.
A: So it would be a min instead of a ... value.
Q: ... You're saying that just by looking at, this is positive?
A: Yea.
Q: Is that what you're say--? And then that means this is gonna go up like that?
A: Right.
Q: Okay. So, then if we ask the question what's the maximum value of y ...
A: It'd be E.
A: ...
Q: ... How'd you know it went down this time?
A: Because the x² term is negative.
Q: Okay.
A: Huh, I remember doing this one before ... it'd be E.
Q: Okay, how'd you know that?
A: Because you can go over ...
Q: Uh huh.
A: ... 3 with the x ...
Q: Um hm.
A: ... but you'd be going up 30 which would be ...
Q: Way off.
A: ... off.
Q: Okay. Number 12.
A: [ Reads the question. ]
Q: Whatever, yea. You don't have to read all that. Okay.
A: ... I think this is one of the one's I got wrong.
Q: Oh, (laugh) well don't worry about that, let's. ...
A: Mm  
Q: ... and if you have to guess on something that's fine, too, as long as you can kind of explain why you even made that guess because that's just as interesting to me as, as uh ...
A: Alright.
Q: ... whatever ... Okay, because ... What happens when m is 0?
A: Well if \( x \) was 0 then, getta cancel this one out altogether.
Q: Um hm.
A: This one'd be cancelled out.
Q: Okay... What would you be left with in your equation?
A: \( x^2 - 2x + y \)... If you have...
Q: Um hm.
A: ... then you put the \( x^2 - 2x \) on the other side...
Q: Um hm.
A: ... and you get \( x - 2x \)...
Q: Does that look like an equation for a parabola then?
A: With the \( x^2 \)?
Q: Um hm.
A: Then you substitute \( -x \) out...
Q: Um hm.
A: \( x + ... x - 2 \).
Q: Um hm.
A: ...
A: Hm um.
Q: Was this one, or that wasn't either? The \( x^2 + 2x \).
A: That's not.
Q: Okay. Why wasn't it a parabola? What, what makes it different from an equation for a parabola?
A: Well, it might be but because, I think because since you factored out a \( x \) right here...
Q: Um hm.
A: ... this is just a ... maybe.
Q: Um hm. Oh, okay.
A: But it could turn out to be one. Because of the \( x^2 \).
Q: Okay.
A: Depending what you use.
A: ... Number 14 I put I [one].
Q: Okay. Because...
A: Because between the point on \( x \) 1 and \( x \) 3, it's rising...
Q: Um hm.
A: ... to the right ... choosing number IV because number IV it's going down from 1 to 3.
Q: Um hm.
A: Between that point.
Q: And for II and III and —
A: They also decreasing.
Q: Alright, what if someone came up to you and said, alright, if I go from here to here, if I look at it this direction, I see this graph is going up, so is that increasing? What would you tell them then?
A: ... Well, I say it's decreasing 'cause of the arrow that's right here.
Q: Um hm. Which way are you looking at it? This person was looking at it saying if I start from 3 and I go to 1, this is going up.
A: Well I'm looking from left to right.
Q: Okay. Wou-- and so would you tell them they're wrong or, um, or that oh, well, sometimes you could look at it that way and ... What would you tell them?
A: Well, most of the time you look at it from left to right
Q: Um hm. Okay.
A: ... left to right on a graph.
Q: Okay.
A: So this one's decreasing.
Q: Okay. Uh, let's see. What was that? 14? 16.
A: ... say I [one].
Q: Okay. Because...
A: Um, I changed that.
Q: Okay.
A: III.
Q: Alright. That's fine.
A: Because the graph is representing from 0 to 50...
Q: Um hm.
A: ... for I [one], for the length of it. The length is x.
Q: Okay.
A: Then it shows...
Q: What does that hump represent? All those...
A: Uh...
Q: If you read a y value off of there, what would that represent?
A: The, um... the area.
Q: Okay. Alright. So you go with III? Okay, go ahead and erase I [one] then or I'll get confused later on. 18.
A: ... I would say II and IV.
Q: Okay. Because...
A: Because the scale, even though it's different...
Q: Um hm.
A: ... it still seems like this point and this point is -3 and 3...
Q: Um hm.
A: ... and this is approaching 9...
Q: Okay.
A: ... and this is where... Its I [one] and II.
Q: I [one] and II?
A: Yea.
Q: Okay. What made you change your mind about the, uh, the fourth one?
A: The, uh, scale.
Q: Okay.
A: I just read it right there.
Q: Okay.
A: It's I [one] and II.
Q: Okay, so that changes...
A: Right.
Q: ... this intersection point to what?
A: Six.
Q: To six.
A: Right.
Q: Okay. But then, now, does this one work on both? Does it go up? Uh, you said this was approaching 9.
A: One, two, three, four... yea.
Q: Okay, good. 20.
A: Mm... B.
Q: Okay.
A: For number 20.
Q: How'd you do that so fast?
A: 'Cause the graph. It hits the x-axis at -5...
Q: Um hm.
A: ... positive 1 and positive 8. Them are the solutions, uh...
Q: Okay. What if this had been a 50? Would you still read these three points then if this had said equal to 50? What would you do?
A: Well, then you'd still have to set it to 0.
Q: Uh huh.
A: And then after you set it -- this to 0...
Q: Um hm.
A: ... if this was 50.
Q: Yea.
A: You gotta subtract by 50 on both sides...
Q: Um hm.
A: ... to get a 0...
Q: Okay.
A: ... and then this would be a -10 instead...
Q: Okay.
A: ... and then graph that instead of 40.
Q: Okay. 21.
A: ... Two.
Q: Alright.
A: You only show these two points right here.
Q: Okay... Alrighty. One more. 22.
A: ... III.
Q: Boy, that was quick. (Laugh.) How'd you do that one?
A: Because, as, if it's directly proportional...
Q: Um hm.
A: ... as the, as the height increases the length also increases.
Q: Okay. This one does-- I [one] doesn't work?
A: I [one], it's... it could but most of the graphs when it's directly proportional...
Q: Um hm.
A: ... will be a straight line.
Q: Okay. Alright, that's good. Alright, thank you.

Student 5-2

2

Q: Go ahead and answer question 1 and circle your answer whichever it —
A: Okay. Do you want me to —
Q: Yep.
A: You want me to tell you why I think it's a parabola?
Q: Yea. Why is that a parabola?
A: Okay, I think this is a parabola because in the equation \( y = x^2 \)...
Q: Um hm.
A: ... you are to take a point, such as 1...
Q: Um hm.
A: ... and put it in for the \( x \) so it would, then you would graph it.
Q: Um hm.
A: Go over 1 and up 1 and do that, do the same thing for other points...
Q: Uh huh.
A: ... such as 2. You go over 2 and up 2.
Q: Okay.
A: You know, it's, or 4 or whatever. (Laugh.)
Q: Okay. Fine. So, and then that kind of thing is gonna produce a shape like that.
A: Right.
Q: Is that what you're — Okay. 2.
A: ... It's still a parabola.
Q: Okay. Even if you switch it.
A: Right.
Q: Okay. Or rotate it around...
A: Right.
Q: 3.
A: Um, it's (2,-4).
Q: Okay. How'd you get that?
A: I went to the right 2 . . .
Q: Um hm.
A: . . . and down 4.
Q: And that's how. Go ahead and circle your answer again. Okay. Um, next. 4.
A: Mm, the answer is D.
Q: Okay. Because . . .
A: Because, okay, the equation would be like \( y = x^n \) . . .
Q: Okay.
A: . . . so the \( n \) was standing for the exponent.
Q: Okay.
A: And as you increase the exponent . . .
Q: Um hm.
A: . . . the number gets larger and it goes up.
Q: Okay.
A: It just increases.
Q: Alright, uh, 6.
A: . . . Um, I'd say the answer is -- I'm not sure about this one -- (-3,5).
Q: Okay . . . How'd you . . .
A: Well, or (5,-3).
Q: . . . You're wiggling your pencil. You can go ahead and draw on the paper.
A: Um, I think it's (-3,5).
Q: Okay.
A: All I did was change the signs of the coordinates.
Q: Do you know how -- if you had to explain to somebody or show somebody why is that symmetric to the origin, how would you . . . Okay.
A: Go over 3 and then down 5 . . .
Q: Okay.
A: . . . and -3 and up 5 and it's just, you know, kind of in line with the origin.
Q: Okay.
A: I don't know.
Q: With the origin, so . . .
A: Yea.
Q: . . . if I asked you, if I put a point right here . . .
A: Um hm.
Q: . . . what would that, um, how can I say this, would that be symmetric to anything? We don't have to say symmetric with respect to the origin . . .
A: Um hm.
Q: . . . but is there any other kind of symmetries?
A: None that I see.
Q: Okay. Well, no, I was just saying what if I put -- you have this point here.
A: Uh huh.
Q: And I put one right up here.
A: Like say (3,5).
Q: Yea, what --
A: Well, I'd say that they were symmetric to each other.
Q: Okay.
A: If you drew a line between them.
Q: So I guess what I'm looking for is this key word. This is here and this one's up here. Wha-- how would you explain to somebody --
A: Oh, to the x-axis!
Q: Okay, to the x-axis . . .
A: Okay.
Q: . . . and, um, how did you know it's, you said it was (3,5)? Is that what you said this point --?
A: Um hm.
Q: Why couldn't it be, um, (3,6) or why couldn't it be, um --
A: Well it has --
Q: -- maybe just (2,5) or something?
A: It has to stay the same, the distance from the x-axis, down then when you change it ... 
Q: Um hm.
A: ... you have to keep the same distance.
Q: Okay. (?)
A: Just going in another direction.
Q: Okay, good. That sounds good. Where was I? That's 6, 8.
A: ... That would be, um, the fourth graph, answer D.
Q: Um hm.
A: ... That's because if you were to plot some of these points on each of these graphs -- like say, for instance, in the first one ...
Q: Um hm.
A: ... the, like there are two y values and they're, and they're constantly going up. And this one right here is really low. But the one on IV is at the top and it's decreasing and so this would be like the highest y value.
Q: What about II and III? Do they have a highest one, a highest y value?
A: Um, yea, but there just like going on toward infinity.
Q: Okay.
A: Just keep going, so you can't tell where it's at.
Q: How do you know it keeps going? What's telling you that --
A: The little arrows on the end.
Q: Alright. 9.
A: Um, ...
Q: Keep thinking and talking.
A: Okay. (Laugh.) Can I write?
Q: Um hm.
A: Okay.
Q: Go right ahead .
A: ... I don't know. I don't know what to do ... I would, I would probably graph this (laugh).
Q: Okay.
A: That's what I would do.
Q: What, you're trying to, uh, factor it --
A: Um, factor it or quadratic equation.
Q: Okay, what if, what if it did factor?
A: Okay.
Q: What would you do next then? Let's say it was x - 2 and x + 5. What would you do next?
A: I guess I would say the answer would be 5.
Q: For the maximum ... 
A: Right.
Q: ... value of y?
A: Um hm.
Q: So it does depend what, if it factors, what the maximum y value would be?
A: Right.
Q: Okay ... Well, let me flip back to 8 because 8 was kind of a similar question. This factored into, let's see, we don't have any scale on here, but let's just say it's 1.
A: Um hm.
Q: So this is 1 and this is 2.
A: Right.
Q: So you're saying that a factor of this equation for this graph would have x - 2 in it then, and then that would tell you that, um, y = 2 is the ... 
A: Um hm.
Q: ... maximum value.
A: Okay.
Q: Okay. Well, alright. We're stuck. It doesn't factor.
A: ... I would graph it. That's what I would do. On my calculator (laugh).
Q: Oh, okay. Um ... Well what if w— well, let's see. What if we had, if you had to guess, which is fine.
A: Okay.
Q: Push came to shove and you had to ...
A: ... Um, I would probably ... The answer looks like it would be E., there ...
Q: Okay, because why?
A: ... is no maximum value. I just try to eliminate the answers.
Q: Um hm.
A: Like -9 and -13 don't really make sense, and then 9 and 13, just doesn't — they, they don't look like they fit into the problem anyway.
Q: Okay ... Based on elimination.
A: Um hm.
Q: Alright, E is it. Okay, fine. 10.
A: See, I would draw this out
Q: That's fine.
A: (2,5), go up 5 ... Um, the coefficient of the \( x^2 \) term is negative. It would go down ...
Q: You know it goes down because of what?
A: The coefficient of the \( x^2 \) term is negative so then it goes down.
Q: Okay.
A: ... the following points could not be on the curve?
Q: Okay, so this looks something like this other graph on the other page ...
A: Yea.
Q: ... and that has a maximum y. Okay ... And what's this coefficient?
A: Um ...
Q: Just going back (to question 9).
A: It's just 1.
Q: One?
A: Um hm. It's a positive 1.
Q: So, what, could we maybe use that info— would that information help us to determine a maximum value or not? That's still not good enough? We need to know more?
A: ...
Q: Still don't know? Okay, fine. That's good.
A: Okay.
Q: So it's negative. That's going down.
A: Okay.
Q: Which of the following points is not be on this curve?
A: ... Um ... (-3,30).
Q: Okay.
A: ... So you would go, um, to the left 3 and then up 30. That's quite a ways up there.
Q: Okay. Good. 12.
A: I remember this problem from the test (laugh).
Q: This has been a favorite of mine.
A: Alright. I've seen it so many times and I don't know if I've ever gotten the answer to it. Or if it's ever been explained.
Q: Do you have any guesses again or, you know, if you wanna reason through it. Whichever.
A: Um ... well, I would say ... if it's a parabola ... Okay, um ... I don't think 0 could be ...
Q: Okay.
A: ... I don't think 0 could be a, um ...
Q: What would happen if you put 0 in? What, what, what does it do to the equation?
A: It, um, I don't know how to say this.
Q: Well, you can go ahead and show me. I mean, in this equation ...
A: Um hm.
Q: ... if you put \( m = 0 \) in, what would -- just go ahead --
A: Well, you would --
Q: -- and rewrite the equation.
A: You would have \( x^2 - 2x + \), it would be \( 0 + y \).
Q: Um hm ... and that's not a parabola?
A: Well ... no -- well ... it is now (laugh).
Q: No, don't let me confuse you.
A: Okay ... 'cause if there's, if the \( x \), um, is squared ...
Q: Um hm.
A: ... wouldn't it, it would be a parabola.
Q: Okay.
A: Wouldn't it?
Q: Well, I'm not --
A: OH! (Laugh.)
Q: ... that's a part I won't get to tell you. You get to ... but you can ... to yourself.
A: So, like if you were to solve this for \( y \) ...
Q: Um hm.
A: ... you would have \( y = x^2 - 2x \).
Q: Um hm.
A: ... Um ...
Q: So that might work.
A: Yea.
Q: Alright. What if it was \( -\frac{1}{2} \)? Let's just try that one.
A: \( -\frac{1}{2} \)
Q: Um hm.
A: ... plus the, a fourth \( y \) or whatever.
Q: Um hm.
A: And I just went ahead and did that.
Q: Okay.
A: 2 --
Q: Where'd you get the fourth ...
A: Um ...
Q: ... plus ...
A: Oh, I don't know.
Q: ... Okay. Oh, okay. I sec. You squared it.
A: Yea.
Q: Plus ...
A: ... 2 times \( -\frac{1}{2} \). That would be a negative \( \frac{3}{4} y \).
Q: Would that be a parabola?
A: No.
Q: Okay. You're pretty sure about that. Why not? What makes this one a maybe but this one no?
A: Because there's, um, the \( x \) term and the \( y \) termed are squared ...
Q: Okay.
A: ... in the equation.
Q: Alright. And you -- in parabolas you don't have \( x^2 \)'s and \( y^2 \)'s?
A: Right.
Q: Okay.
A: It would be like \( y = x^2 \), you know, plus whatever.
Q: Okay. Alright, so we can rule out A maybe.
A: Yea.
Q: ... All values of \( m \) except --
A: Well, I think the answer would be C.

Q: C. All values of \( m \) except 1-- Okay.

A: Yea. 'Cause 0 seemed to work.

Q: Okay.

A: It looked like it would.

Q: Okay. Um, where was I? That's 12. 14. You're doing good.

A: ... Okay, it's not graph number II.

Q: It's not? Okay.

A: No, it's not this one... Um... not V because it's going down. The, when you go over to, like, \( x = 1 \)...

Q: Um hm.

A: ... and between \( x = 3 \), you look to see what the \( y \) values are, corresponding...

Q: Okay.

A: ... with that and they're going down. They're also going down over here... and they're going down on this one. They go up in this area right here.

Q: Okay.

A: The \( y \) value's getting larger.

Q: Okay. So I [one] looks like it.

A: Um hm.

Q: What if someone came and said, what if I start at 3 and I -- then I go to 1...

A: Yea.

Q: ... and if I look at it this way, this graph is increasing. What would you say to them, because you ruled it out. You said it's decreasing. What would you tell them?

A: Well, I just looked at it from left to right.

Q: Okay.

A: I look at it from, you know, starting at the 1...

Q: Okay.

A: ... and going over to 3 and see which way it's going.

Q: Okay. Would you tell them --

A: That, that seemed backwards to me.

Q: Okay.

A: To look at it that way.

Q: Say it's backwards, so you would say I don't think you did it right...

A: Yea.

Q: ... I think this way is...?

A: I just always thought you would start at the left of the graph and continue on.

Q: Okay. Alright, let's see. Number 16.

A: I'm thinking about that one now! (Laugh.)

Q: Oh. Well, I don't mean to confuse you again. It's just seeing, you know, I want you to convince me of that.

A: 'Cause if you were doing that, this one would be decreasing.

Q: Um hm. What would happen to the rest of them then?

A: They would all be decreasing. And this one, would be getting larger there. So would that.

Q: How many answers would you have then?

A: Um, four.

Q: Four. Would you feel comfortable then having four right answers?

A: No.

Q: No. Okay.

A: Sorry. Okay, number 16.

Q: Is that what I said? Yes, that's what I said.

A: Gosh, this problem! (Laugh.) [Reads the question.] Okay, first of all, you know it's not graph II or IV because they've got negative values...

Q: Um hm.

A: ... and when you're talking about, like, a length or a distance...

Q: Good point.
A: ... you, can't have negatives. Okay. So, now all I have is, like, I [one] and III to look at.
Q: Or none of the above.
A: Right. Okay.
Q: Everyone seems to avoid that one. (Laugh.)
A: Yea. (Laugh.) Okay ... y represents the area ... really, um ...
Q: What are you thinking about? Can you ...?
A: Um, I was just, I was just looking at like ... you know this is like a parabola ...
Q: Um hm.
A: ... so there'd be an $x^2$. I was thinking, like, you know $10^2$ is 100 ...
Q: Um hm.
A: ... but when you go up here ...
Q: It's not.
A: It's not. And here's a hundred right here ...
Q: Um hm.
A: ... and when you try to follow that 10 up, it's 300, it's close to 300.
Q: Um hm.
A: So that I, I would say that none of these graphs ...
Q: Okay.
A: ... describe ...
Q: Okay.
A: ... the relationship.
Q: 18.
A: Okay, on this one all you have to do is look at some different points.
Q: Okay. Is that an ordered pair? Is a point an ordered pair or ...?
A: Yea.
Q: Okay.
A: ... This, this one has a ...
Q: How far did it go up?
A: It went up 10 ...
Q: Okay.
A: ... and over 3, and this one doesn't quite go up 10 ...
Q: Um hm.
A: ... but it looks like it might go over to 3.
Q: Okay. What about IV?
A: IV goes over to 3 ... It goes up 9.
Q: Okay.
A: So I'm counting, one, then, that's 3, and that's only 1 ...
Q: Okay.
A: ... 'cause that's .1. This problem looks so easy.
Q: So we're looking for a graph to go up what, 10, on the, on the y-axis? Is that what it is?
A: Well, okay. I think it's these two 'cause this one goes over 3. It looks like this could be about 3.
Q: Okay.
A: I said this was 8 or 9 ...
Q: Um hm.
A: ... and then that's almost a 10.
Q: Okay.
A: So, say II and IV have the same set of ordered pairs.
Q: Okay. Now you know this is 3 ...
A: Um hm.
Q: ... because it's ... this line represents ...? What's that little line represent right there?
A: 5.
Q: 5. Okay, so you know then it's, it's less than that and --
A: Right. But it's more than 0.
Q: -- toward 3.
A: Yea.
Q: Okay.
A: It's around there.
Q: Okay. Uh, --
A: OH NO! Wait a minute. Each one of these is 2 so that's 2, 4, 6. Right?
Q: Oops. Sorry.
A: (Laugh.)
Q: I didn't mean to -- Oh.
A: No, you're ... Oh no! (Laugh.)
Q: Okay, now we're, we're back to where we started.
A: Okay, so ...
Q: We know it's not between II and IV now.
A: Right.
Q: So we've got, is it between -- and you've ruled out III because of the .1 on the y ...
A: Right. It's just too small.
Q: ... so it's either I [one] and IV -- could that be a possibility?
A: No.
Q: No. Okay. So that leaves I [one] and II.
A: Um hm.
Q: Well, let's see ...
A: But it doesn't look right.
Q: What doesn't look right? ... What doesn't, let's see.
A: ... Well, this is 9.
Q: Okay.
A: One, two, three, four, five, six ... Yea.
Q: Okay.
A: And this was almost 10.
Q: Okay.
A: And that's 3, and that looks like it's close to 5 so it's I [one] and II.
Q: I [one] and II. Alright.
A: Okay.
Q: I think we've got it. Okay. 20. Let's skip that one and go to the next page. 22.
A: Okay. Directly proportional means that as one gets larger, so does the other one.
Q: Okay.
A: And --
Q: Which one of those pictures show that?
A: Well ... when we did these problems in class they weren't lines.
Q: Okay.
A: They were exponential ...
Q: Okay.
A: ... graphs.
Q: Alright.
A: So it's not III or IV.
Q: Okay.
A: And this one is decreasing ...
Q: Um hm.
A: ... 'cause it's going to the left and so the negative, you know, these are negative numbers over here.
Q: Okay.
A: So I'd say, wait a minute (laugh). It looks like the answer would be I [one]. The first graph ... It's ...
Q: Um hm. You do have none of the above again. Just to remind you.
A: Right. That's what I was just thinking about ... as the height gets larger ... length of the shadow to the height --
Q: What did that graph say? As the height's getting larger, what's happening to the height -- or the length --?
A: The length, the length is also getting larger.
Q: Okay. Does that fit?
A: The length of a shadow is directly proportional... Yea.
Q: Okay. Good. So we go for I [one].
A: (Laugh.)
Q: Alright, uh, 24. And we are done.
Q: Now, have you talked about complex numbers yet? Is that familiar?
A: Um hm.
Q: Okay.
A: ... Okay, first when I, first look at this...
Q: Um hm.
A: ... and it wants to know how many solutions the equation has...
Q: Um hm.
A: ... I look to see how many times it intercepts the x-axis.
Q: Okay. And how many is that?
A: Three.
Q: Okay. So is that it? You're not sounding like that's it but (laugh)...
A: Um, well, it looks like that's the only way to tell. There, you know, there isn't an equation for this function...
Q: Um hm.
A: ... you know, 'cause sometimes there might be little checks or something. I don't know (laugh).
Q: Alright. The only thing we know about it is what it is, it's, um...?
A: It's a cubic.
Q: A cubic. What's that mean? What would it...?
A: Um, it has two turning points...
Q: Okay.
A: ... and it'll intersect the x-axis at three points.
Q: Okay.
A: But how many complex number solutions? Well, this looks like it intersects at just one.
Q: Um hm.
A: Right there, on the number...
Q: Um hm.
A: ... so that's not a complex number.
Q: Okay.
A: But these two look like they're, like, some, somewhere in between...
Q: Um hm.
A: ... some kind of, um, decimal...
Q: Okay.
A: ... you know. So I'd say it has two.
Q: Okay. So having a decimal then a number will make it a complex.
A: Right.
Q: Okay.
A: Well...
Q: No?
A: Well, I'm just saying a complex number. It's not...
Q: It's not something that's gonna hit --
A: It's like π.
Q: Okay.
A: It's just we--...
Q: It's a weird one.
A: Yea.
Q: Okay. That's -- and we, and we're not for sure--
A: It's not a precise, even number like one or two...
Q: Okay.
A: ... where you can just look at it and tell where it's going through.
A: Okay.
Q: Thank you very much.
Q: Alright. He's gonna start with question number 3.
A: ... (2,-4).
Q: Okay. How'd you do -- go ahead and circle your answer now on there, too, and how did you do that?
A: Uh, it's, the first number is x-coordinate ...
Q: Uh huh.
A: ... over two, and the second is y, negative down four.
Q: And you just counted those little tick marks. Okay, uh, the next one. 4 I think I circled. Did I circle that? Yea.
A: Okay. Read it?
Q: Um hm.
Q: Now is that word familiar? Have you had exponential ... ?
A: Yea.
Q: Okay. Fine.
A: Uh ... I guess II and III. I don't know.
Q: ... We have a tie. Uh-oh. And I only --
A: The only thing --
Q: -- have one choice down there (laugh).
A: Okay. IV.
Q: Oh-oh. Okay, how'd you do that? Process of elimination or ... ?
A: Well, exponential growth ...
Q: Um hm.
A: ... and that's just the graph of it.
Q: That you remember.
A: Yea. I was thinking first exponent.
Q: Okay. Go ahead and circle that for me, and we'll go down to 6.
A: Uh ... D, (-3,5).
Q: Okay.
A: I just flipped over on the x and y axis.
Q: So symmetry with respect to the origin you're saying is a double flip? You just sai--
A: Yea.
Q: Okay. How would you explain to someone -- let's do a flip about the x-axis, uh, where would that put it then? Put the, if I just said do a flip. Alright, how would you explain symmetry to the x-axis to somebody? I mean, very, they're eighth graders and you wanna tell them.
A: I just fold it over.
Q: It's a fold, so that what will happen when the point --? I mean, they could fold it over and maybe the point, why wouldn-- couldn't the point be up here? How would you explain to them that if it started down here, and you folded it, they could say, well I could, could I fold the point up here ...
A: You have to fold it on the x-axis.
Q: ... or here? Okay. So what does that, um ... give out. What does that -- I guess I'm looking for distance. What does that mean about the distance from the x-axis?
A: It's the same.
Q: It's the same. Okay, that's what I'm looking for. Um, 8.
A: Uh, IV.
Q: IV. Okay, why?
A: Because all the other ones either, uh, keep, keep going up towards y, all of them.
Q: How do you know they --
A: This has a single maximum point.
Q: How do you know they keep going? What's telling you that?
A: It's a complete graph going up.
Q: Have I gotten to circle any more? Let me just, 9.
A: I would graph that on my calculator.

Q: Okay. Is there any way you could maybe think of a sketch, a bare-bone sketch, let's say, that would give you enough information maybe to answer the question? You're trying to figure out what, if it has a maximum y value.

A: It doesn't.

Q: Uh-oh. It doesn't. How'd you know?

A: Because it's positive, the $x^2$ is positive, the $a$ is positive.

Q: What would a rough sketch look like? I mean, just...

A: [Draws a rough sketch.]

Q: So, and that's, is that all you need then to answer this question? You don't...?

A: Yea.


A: E, because it just doesn't make sense.

Q: It just, --

A: I couldn't reach it.

Q: — where is it?

A: It's in the second quadrant.

Q: Okay, and...

A: It wouldn't reach that.

Q: Okay, fine. Uh, 12.

A: I would have done that on my calculator, too.

Q: (Laugh.) What would you have done? Oh, you would have --

A: I just plugged it in.

Q: Okay. Well, go ahead and start. I don't think it'll take that long.

A: A parabola. It's just 0.

Q: This, oh, --

A: $m = 0$. What happens to the equation when you put $m = 0$ in? What's it look like?

A: Uh, a quadratic equation.

Q: And is that, that's a parabola then, right?

A: Yea.


A: It's I [one].

Q: Okay. All the rest are...

A: Yes. Yea. They're all, the other ones are decreasing.

Q: What if someone said hey, if I start here at 3 and I go to 1, this graph looks like it's increasing? What would you tell them?

A: ... You have to read left to right.

Q: Okay. Fine. Uh, 16.

A: It's easier with a graphing calculator.

Q: Yea.

A: I know it's one of these two (I and II)... One of those two. Okay. How do we figure out...

A: How did I get that?

Q: Or no, that, maybe I should have you explain it but I've been following your, what you've been doing. But how could you figure out now which one of those two would be that? You've got...

A: Uh...

Q: Is there a nice --

A: I could --

Q: — point to pick?

A: — factor it.

Q: Factor it?

A: Yes.

Q: Okay. What would it look like if you factored it? Would that...

A: ... I [one] because the two solutions for $x$ are 0 and, negative, negative 25.
Q: Okay.
A: Something like that.
Q: See, you didn’t need your calculator. You did it. Alright, that — since the bell rang, that’s as far as we’ll get.

Student 5-4
4

Q: Start with question number 3. So just go ahead.
A: You want me to read this?
Q: Um hm.
A: Let’s see, each thing on the xy-axis is a two. Each little ...
Q: Um hm.
A: ... line.
Q: Okay.
A: Okay, so it’s over two of the little lines so that’s a four.
Q: Okay.
A: Okay, and then each on the y is one, so it’s down four. So it’s a (4,4). (4,-4) rather.
Q: Okay.
A: So that’s E.
Q: (?)
A: No. No it’s not. It’s D.
Q: It’s D. Wonderful. Okay. Thank you, that was good. That was 3? Let’s go to 4.
A: I’m looking at the little graphs, okay.
Q: Um hm.
A: The first one is probably sine or cosine. I can’t ever remember which is which.
Q: Okay.
A: Okay, and the second one’s a parabola, and the third one’s a graph of, of something with a cubic ...
Q: Um hm.
A: ... and the fourth is the one with the exponential ...
Q: Okay.
A: ... and that’s growth, exponential growth ...
Q: Okay.
A: ... so the answer’s D.
Q: So you’re sure. And none of the above, you just ...
A: No.
Q: It’s, you’re sure it’s one of those.
A: Yea.
Q: You’re doing this real well. Okay.
A: This is the 87th time I’ve taken this test.
A: I’m looking at all the little answers ...
Q: Um hm.
A: ... and, oh, I hate these. I’m going to draw this out here.
Q: Good.
A: You can look at the little test later ...
Q: Um hm.
A: ... if you want to. Three, one, two, three, four, five. It’s over here. Um, so this would be over five. Wait, it’d be over here. That would be up 5 and over 3, so it would be (-3,5). That’s D ...
Q: Okay.
A: ... I think.
Q: How would you explain symmetry to the origin to someone other than —
A: Well, if you take the paper and you fold it ...
Q: Um hm.
A: ... it, then the point would be exactly where it is over there...
Q: Okay.
A: ... if you fold it there.
Q: Okay.
A: Like that.
Q: Alright. Number 8.
A: ... Well, let's see. It's not number II because it goes left and right...
Q: Um hm.
A: ... and that doesn't work. And it's not number III because it goes left and right the other way...
Q: Um hm.
A: ... and that doesn't work either. Okay, um, but the first one has, it goes up...
Q: Um hm.
A: ... for infinity so it has no maximum unless you want to say infinity and that's not even anything.
Q: Okay.
A: ... conceivable. So it has to be number IV...
Q: Okay.
A: ... because it goes down.
Q: Alright. Good. 9.
A: Um, well you could graph this.
Q: Yea.
A: You could sit there, so I think I will. I'll pretend I am (laugh).
Q: Okay. What will you be looking for on the graph when you graph it to determine this maximum value?
A: Well, I'll be looking to see if it goes upside down which it won't. (Laugh.)
Q: Okay.
A: And then...
Q: How did you know it won't?
A: Because the coefficient of the $x^2$ term isn't negative.
Q: Okay. Alright.
A: So...
Q: Well, if it goes up now, does that make a problem?
A: Yes, because then there will be no maximum value...
Q: Okay.
A: ... unless you wanna say infinity again...
Q: Okay.
A: ... like that's inconceivable, so it's E, there is no maximum value.
Q: Okay. Good. 10.
A: 10. We're doing them right in order here.
Q: Yea. (Laugh.)
A: Okay, um, two, one, two, three, the way I could write off my little x and y axis and put it there...
Okay, (0,1), it might be there.
Q: Um hm.
A: Let me see... It couldn't be E because it's above the vertex and it won't go up, it'll go down.
A: Ooh, we skipped one. Exciting.
Q: Um hm.
A: That's a mouthful.
Q: Yes, isn't it?
A: ... Um... Okay, if it's, if it has two squared terms it'll really mess it up I think...
Q: Okay.
A: ... so if $m = 0$ then that'll just make it easier, so I guess that would be B.
Q: Okay. So a parabola only ha-- you said has one squared term in it? Is that--
A: Well, I like to think of it that way.
Q: Okay.
A: It makes life easier.
Q: Okay. We'll go for that. 14.
A: So you look at the little graphs . . .
Q: Um hm.
A: ... and you find the little x = 1 and you go over to 3 and it's there.
Q: Um hm.
A: So you look and see what's going up. Well, it's obviously not number II because it's going straight down . . .
Q: Um hm.
A: ... and number III is going curvy down . . .
Q: Um hm.
A: ... (laugh) and number IV is going down until it gets to 3 and then it goes up again and ... and number V is decreasing also so it's number I [one]. That's A.
Q: Okay. What would you tell a student who came and said, well if I start from 3 and I go to 1, this is increasing? What would you tell them?
A: Well, I'd tell them that since they put x = 1 first that you... 
Q: Um hm.
A: ... would go from 1 to 3.
Q: Okay.
A: Unless they just wanted to be wrong ...
Q: Okay.
A: ... and then I would just hit them.
Q: Oh, okay (laugh).
A: No, I won't.
Q: Okay (laugh). 16.
A: Just accept it.
Q: Yes. (Laugh).
A: Okay, oh, oh, a word problem. I have to read a word problem. [Reads the question.] Okay, well, it must total 50 feet so you have to go out to 50 on the x-axis I . . .
Q: Um hm.
A: ... think. Yea, so it must be III because that's, well, no wait. IV does it, too. Well screw me up. )
Q: Um hm.
A: ... right now.
Q: Yes.
Q: So between III and IV.
A: Yea. Well, you can't have negative area and since the graph of IV is, shows a negative area . . .
Q: Um hm.
A: ... and y = area then it must be number III . . .
Q: Okay.
A: ... because you can't have negative area . . .
Q: Alright.
A: ... unless you are on some sort of drug.
Q: 18.
A: Okay, look at the scale.
Q: Um hm.
A: La la la. Okay, count up the little, one, two, three ... nine. Okay, this is nine. I'll just write it up there and make it easier for me.
Q: Okay.
A: Okay, I would guess that it's I [one] and II. Why? Well, because, okay, the scale. Okay, for the I [one], it's one and one so it crosses at three and negative three and then it goes up to nine . . .
Q: Um hm.
A: ... on the y-axis, axis. Access, blah.
Q: Um hm.
A: Okay, and then on number II it's five and ten so it, or five and five, so it's like, less than ten but more than five on the y so . . .
Q: Um hm.
A: ... it looks kind of like a nine.
Q: Yea.
A: Okay, and then on the x it's less than -- more than half but not quite as much as four would be so it ...
Q: Okay.
A: ... would probably be three, so it would be A which is I [one] and II.
Q: Okay. 20.
A: Blah ... Um, well, I looked at the graph and it crosses the x-axis at -5, 1, and 8 and that's what B says ... 
Q: Okay.
A: ... so I'll put that.
Q: Alright. Why did you look at the x-axis? What did ...?
A: Because where it crosses the x-axis is where the solutions are. The roots.
Q: Okay.
A: Because that's where y = 0.
Q: If this had said 50 then --
A: The curve equals 0.
Q: Okay.
Q: What would you say then, if this had been a 50 here instead of a 0?
A: Well, then you would look to see where the graph crossed the line 50. y = 50.
Q: Okay.
A: Or x = -- y = 50.
Q: (Laugh.) Okay. 21.
A: ... It's infinite because, um, well there, are like so many points on that line.
Q: Um hm.
A: It's just ridiculous, so it's infinite.
Q: Okay. One million's not ridiculous enough, though, right?
A: No.
Q: Oh, okay.
A: Because, I mean, you're gonna sit there and count a million? You'll say no, no, it's infinite. Let's not bother with it.
Q: Okay (laugh). 22.
A: Um, so that would be ... L(h), I guess. So that would be ... as length increases, height increases, so it would probably be I [one] ...
Q: Um hm.
A: ... because that's what it looks like. I suppose.
Q: Okay.
A: I don't know why, it's just a guess.
Q: Okay.
A: I'm great at guessing.
Q: Is, is II, does II show that? You said as, as, what'd you say? As length increases, height or did you say as height increases?
A: I put length as height.
Q: Okay.
A: As length increases.
Q: Okay, so does II demonstrate that? I mean, as length --
A: No, it shows, number II shows that, um, as length increases, height decreases and ...
Q: Okay.
A: ... that doesn't make any sense as far as that goes.
Q: III?
A: III, it's ... as, well, it's pretty much the same, it's just that ...
Q: Um hm.
A: ... it doesn't look right.
Q: Okay.
A: It doesn't look like the kind of thing I'm used to, and IV it's decreasing also so that doesn't make any sense.
Q: Okay. And none of the above, you, do believe that this would be it?
A: In my heart, yes.
Q: (Laugh) Okay. I always hate to dissuade students but ... Alright, um —
A: Why, am I wrong?
Q: No, no, but sometimes when I point out none of the above they go, "Oh, yea!" you know, and they panic. But, no, I don't mean to do that. I just want to make sure they didn't miss it.
A: It doesn't matter.
Q: Okay. 24.
A: [Reads the question.]
Q: Now, have you talked about complex numbers yet?
A: Yea, but that was a long time ago ... 
Q: Oh, okay.
A: ... so don't expect me to be right. (Laugh.)
Q: (Laugh.) Okay, I won't.
A: Um, well, it doesn't have any because each time it crosses ... the x-axis it's a real solution ...
Q: Um hm.
A: ... and real solutions as far as I know aren't complex ...
Q: Okay.
A: ... so it would be 0.
Q: Okay.
A: And since it's a cubic function it only has three solutions so it's A.
Q: Okay. 25.
A: Okay, so looking at this, la la la ... 
Q: Um hm.
A: Okay, you start in the parentheses so you move it over to the left 3.
Q: Why left?
A: Because it's ... that's the way you do it.
Q: (Laugh.)
A: No ...
Q: That's the easy response, isn't it?
A: Well there was a reason once but ... 
Q: Okay.
A: ... it just became -- well ...
Q: Well, I wondered. It says x + 3 and you said move it to the, what'd you say?
A: Left.
Q: To the left which, I mean, + 3.
A: Well, it's, it's x minus, um, whatever the varia-- or the constant little ...
Q: Oh.
A: ... symbol you want to use for the constant would be.
Q: Okay.
A: And then, like, plus, it would be minus a minus so ...
Q: Oh, okay.
A: ... that's why.
Q: Alright, so we move it to the left 3 --
A: Just in case you were wondering. Okay.
Q: I was. I was.
A: (Laugh.) Or else you wouldn't have asked. You must like me. Okay. Um, okay, so you move it to the left 3 so da, da da, da da. Right, right here. One, two, three, so it's up here. And then you move it down one because you, um, because, um, the -1 is like the little k thingie.
Q: Um hm.
A: So you just move it down one so it'd be there and then it'd be, let's see, one, two, three, four ... so it'd be there and ... there. So it'd be like that, so that looks like ... V. V, E.
Q: Okay, that's it. We're done.
A: Oh, goodie.

Student 3-1
Violation: 0-1-0-0

Q: Number 3.
A: You just want me to explain this to you?
Q: Uh huh. Yea.
A: I think it's (4,-4) because of this 2 and it's going by 2 so . . .
Q: Uh huh.
A: . . . and over 4 on the x-axis and down -4 on the y.
Q: Okay. Go ahead and circle that response for me there. Okay. Uh, 5.
A: . . . Okay, I think it's C.
Q: Okay.
A: The graph of B is an upside down version because it's negative.
Q: Okay. And that, you know, other than the, um, if you had to explain why a negative flips it, is that just from memory or could you think of some way you could explain it to someone why that makes it an upside down version?
A: I can't really think of why. I was just thinking from memory that it would be negative so that would be, like, the opposite.
Q: Okay. Good. 7.
Q: Now, trigonometric, is that familiar?
A: Right.
Q: Okay.
A: Sine, cosine.
Q: Okay.
A: Um, I think it's I [one] . . .
Q: Okay.
A: . . . because it looks like a sine graph, or wave.
Q: Okay. 9.
A: . . . I have no idea on this one.
Q: Okay.
A: I, I would, I would jus- I guess I'd say E because it could be any of the numbers for x's.
Q: Okay.
A: Well, I guess there could be a maximum value. I don't know. I just think E.
Q: Okay, as a guess. Um, 10.
A: . . . C. Because if . . .
Q: Okay.
A: . . . one of the functions is negative then they can't both be positive (referring too the x, y coordinate).
Q: Both of the . . .
A: Yea.
Q: . . . of these things can't be . . .? Okay, so that's what that sentence means?
A: If the coefficient is negative . . .
Q: Uh huh.
A: . . . you mean the x term is negative or the . . .
Q: The, well, . . .
A: . . . the term in front of it?
Q: . . . the x² term.
A: Well then, . . .
Q: Well . . .
A: . . . it would be positive anyway, right?
Q: . . . the coefficient of the x² term. There you go.
A: Okay, I'm just gonna say C.
Q: Okay. Good. 11.
A: Hm... I don't know. I really have no idea what I'm doing.
Q: Okay. A guess? What would you --
A: Probably IV.
Q: Okay, they have to come in pairs.
A: Oh, okay. I [one] and IV.
Q: Okay. Just, alright. Um, uh, try 12.
A: . . . I'd say E 'cause . . .
Q: Okay.
A: . . . if it was a - \( \frac{1}{2} \) and 0 then it would make it 0 in the equation.
Q: What one, what woul—, whon, blah, makes what 0 in the equation?
A: Well, -2.
Q: Okay.
A: If you multiply, if, if \( x = 0 \) then it would make the equation, I don't know. (Laugh.) That's a guess.
Q: Okay.
A: That's one of those. I'm not sure.
Q: Alrighty. What else? Um, let's see. 13.
A: . . . I'll say D, 'cause here it's increasing from -3 . . .
Q: Um hm.
A: . . . It's the only one that says it's increasing from -3 and -- oh, no. Whoa. That's decreasing. Never mind. Wait.
Q: Want to use an eraser?
A: Okay. Let me see.
Q: Now you said it's decreasing why? I missed that point. Because --
A: It's decreasing because the arrow's going down here.
Q: Oh. Okay.
A: Okay? I think it's this one because it's increasing here from 1.7 to \( \infty \) . . .
Q: Um hm.
A: . . . and then increasing from \( -\infty \) to -1.7 there. I don't know --
Q: Okay.
A: -- this one. (Laugh) I guess. I don't know. I just know that this right here is 1.7 to \( \infty \).
Q: Okay. Alright. 15.
A: . . . B.
Q: Okay.
A: (5,-4). 'Cause on the opposite side of the parabola it'd be flipped over because it is symmetric.
Q: Okay. Uh, let's see. 17.
A: . . . I would say that's B. Moving the \( y_1 \) curve down a units because your line's gonna be at about a .
Q: Okay. 19.
A: . . . I don't know. I have no idea.
Q: Is there any way . . . have you, or, if we just gave you the inequality --. I noticed you're referring to the, to the graph. Is there any way if we just gave you this and based on some of these responses that you could solve that or is it gonna be a stab in the dark?
A: It would be a total stab in the dark. I think I got it wrong. I don't know.
Q: Okay. Put a question mark there to the side so I'll know. Okay. 20.
A: . . . I guess it would be B. 'Cause of where it hits the graph, -5, 1 and 8.
Q: Okay. If this had said 50, would you still maybe look at the points on the, on the axis or would your strategy change or -- ?
A: It will change because it wouldn't be 0 on the axis. On the x.
Q: Any ideas what you would do or you just know that it would be different?
A: (Laugh). I know it would be different.
Q: Okay.
A: I'm not sure what I would do.
Q: Okay. 21.
A: Infinite, because the ar— there's arrows which it goes infinitely so it goes on forever (?).
Q: No, but it is a line segment versus a line.
A: Oh, oh...
Q: ... Those are supposed -- just supposed to be points.
A: ... Um, I don't know, 'cause I don't know how you could estimate that.
Q: So it will change --
A: I just still say infinite --
Q: -- or no, you say it won't change --
A: -- If it was a line it would be infinite but I don't know what a line segment does. Because if it ends then it wouldn't be infinite ... Maybe there are just two, on either end. I don't know.
Q: Okay. Put, um, put a question mark, too, or a question mark also I guess (laugh).
Q: And we'll, okay. What do I have? Uh, try 22.
A: ... I'll say III because it's just a straight line so you can tell that it's proportional.
Q: Okay.
A: I don't know.
Q: And this is a straight line, but that one won't do it?
A: No, because, well ... I don't know. Just for some reason ... You got like a negative value over here . . .
Q: Um hm.
A: ... so these are just all positive values.
Q: Okay.
A: So that's why.
Q: Alright. Um, 23.
A: ... I'm just gonna say two because the graph increases up to 2.
A: [Reads the question.]
Q: Complex-- have you talked about complex numbers yet?
A: Huh uh.
Q: Oh, okay. So this would be totally new. Let's try 25 then.
A: [Reads the question.]
Q: What are you looking for? Can you tell me or -- ?
A: I, I thought that it was centered somewhere at the point would be \( x = -3 \),
Q: Okay.
A: -3 would be on the \( x \)-axis.
Q: Okay.
A: So I guess it would be \( V \), that's the only place where I can find -3 on the \( x \)-axis.
Q: Here's one, but it's, but it's upside down.
A: Oh . . .
Q: I realize that.
A: Right.
Q: Does this --
A: But one --
Q: -- -1 do anything?
A: Yea, 'cause \( y = -1 \) so it would be 1 on the \( y \)-axis so I would say \( V \).
Q: Alright. Hey, we're done.
A: (Laugh.)
Student 3-2
3

Q: I'd like you to start with, um, we'll do all 1, 2, 3 questions here on this first page.
A: Okay. Uh, the first one's a parabola.
Q: Okay. Go ahead and circle it. And how do you know that's a parabola?
A: Oh, analyt. We're doing a whole bunch of stuff like this.
Q: Okay.
A: So...
Q: If, if you had to explain to somebody, though, what's --
A: Why it's a parabola?
Q: Yea, versus an ellipse or a circle or a whatever.
A: Okay, well a circle, um, the radius is all the same...
Q: Um hm.
A: ... and an ellipse, um, two of the sides, well, it can be either x or y...
Q: Um hm.
A: ... but the same sides are the same length and the other sides are gonna be different. It's like a A.
Q: Okay.
A: Um, a parabola opens up on one end and can go up and down, left and right.
Q: Okay. That's good.
A: [Starts to read question 2.]
Q: That's good. Read the question out loud and I'll pick that up, too.
A: If you rotate it it won't matter.
Q: Okay.
A: It's still a parabola.
Q: Okay. Go ahead and circle it for me. I'll remember that --
A: Sorry, I keep forgetting that it's a...
Q: (Laugh.)
A: Okay, well your scale's off here...
Q: Okay.
A: ... so it goes 2, 4, 6, 8.
Q: Um hm.
A: Not 2, 3 or something. So the point is at (4,-4).
Q: Okay. Um, 4.
Q: Yea, that's right. Have you talked about exponential functions?
A: Yea...
Q: Okay.
A: Well, I took trig so...
Q: Okay.
A: Um, this is a sine curve (I)...
Q: Um hm.
A: ... that's a parabola (II). This is, I'm not sure. Kind of bizarre (IV).
Q: Um hm.
A: Um, and this is, I forget what it's called. But it's like $x = y^2$ (III) or something.
Q: Okay.
A: So it's number I [one].
Q: Okay. Uh, what do we have? 6.
A: You just draw it.
Q: Okay.
A: (3,-5) is way down here, so it has to be, uh, (-3,-5). It's on the other side of the y-axis.
Q: Okay. So symmetry, uh --
A: It's either can be symmetric with the y-axis or the x-axis.
Q: Okay.
A: So it -- I suppose it could have been up here, too . . .
Q: Okay. So --
A: -- which would have been (3, +5).
Q: What does the origin have to do with it? What is the origin?
A: It -- well, yea. It's symmetric to the origin.
Q: Okay.
A: It can be symmetric to the origin, to the y, or to the x.
Q: Okay.
A: Or too, yea.
Q: Alright. 8.
A: It has to be number I [one].
Q: Okay.
A: Because it goes, it goes positively to an infinity.
Q: Okay.
A: This one goes positively, um, to x and this goes negatively . . .
Q: Um hm.
A: . . . and this goes negatively.
Q: So I [one]. Okay. Uh, what was that? 8? 9.
A: I didn't know how to do this . . .
Q: Okay.
A: . . . to be totally honest.
Q: Okay.
A: Um . . .
Q: Good, good. Honesty.
A: (Laugh.) That I am. Um, you could factor it I suppose.
Q: Um hm. And then what would you do?
A: Um . . . Well, you could do it by trial and error. It would take you forever.
Q: What would your trial and error be? What would you --
A: Well, you could, you'd be, you can use these.
Q: Okay.
A: That's the easiest way to do it. You gotta use 2 or you'd be here forever.
Q: Um hm.
A: But using answers that they give you. Uh . . . I guessed.
Q: Okay.
A: And I know I didn't pick E. I don't know why.
Q: Okay.
A: But, um . . . I think it was thir--, I think I put 13, instead of 9.
Q: Okay.
A: Um . . .
Q: Because it's bigger or, was it -- ?
A: Yea, and there was an, well the problem has a 9 in it . . .
Q: Okay.
A: . . . so I figured that y could be more than 9, for a maximum.
Q: How could it be more than 9?
A: Um . . .
Q: If you had to explain the psychology.
A: Well, the only real reason it couldn't be would be maybe if you had, if x was negative.
Q: Okay.
A: See, I was thinking what x would be and . . .
Q: Um hm.
A: . . . all the answers are y =.
Q: Okay.
A: So, if you put, like . . . See it would take forever. I can't work backwards like this. Because I'm putting in something for y and then . . .
Q: Um hm.
A: ... trying to find out what x is. And I'd rather put in something for x and find out what y is.
Q: Especially since it's solved for y.
A: Yes, it would help! Um, I suppose you'd have to do it, quadratic equation.
Q: Um hm.
A: Which I hate...
Q: Would that provide a maximum value or, or is it gonna give you other information --
A: No, it's gonna give you, I suppose what is equal to y, or what could equal y. So if, I don't think I even did it this way. But, um, it might be able to tell you -- especially it could tell you if you had a negative at the radical.
Q: Um hm.
A: So, finish you get 16, 1, 2, 3 ... And my math.
Q: That's fine.
A: I'm taking forever (laugh).
Q: That's fine.
A: (Laugh...) 3, so that'd be 2 ... that tells me a lot (laugh).
Q: (Laugh...) Are we getting closer to 9 or 13?
A: Um, well, we're probably getting closer to 9 actually. That's what it looks like. Uh, -2 ... I have a mathematical question.
Q: Okay.
A: Can I ju-, can I cancel this out?
Q: Um hm.
A: \( \pm \sqrt{13} \). So I would have to do this on my calculator. \( \sqrt{13} \) would be ... I just learned how to do this. Um ... do it ... You could get positive, you could get positive or negative answers ... \( \ldots \) Wouldn't you?
Q: From this?
A: Yea.
Q: \( \ldots \) Wouldn't you, um ...
A: 'Cause you'd have -2 + \( \sqrt{13} \) ...
Q: Um hm
A: \( \ldots \) and -2 - \( \sqrt{13} \).
Q: Right.
A: So ...
Q: But, if, uh, if you use a quadratic formula, do you, can you recall what you generally use it for? Is it to determine maximum values or is it to --
A: It'd be distance, wouldn't it? ... We're finding what's equal to y ... It equals -2 \( \pm \sqrt{13} \) ... We could, we could put a question mark here, or if you want to go ahead and, and, and circle something based on what you've been doing here, either way would be fine.
A: Well, it looks like it would be 9 then ...
Q: Um hm.
A: \( \ldots \) instead of 13, except I don't know if it's positive or minus ... positive or negative. So maybe there is no maximum value for y ... 'Cause then you'd have to ...
Q: Okay, leave it as a question mark. I've got ... points on here. So this is, well, this is good. You brought up some good concerns. Let's try 10.
A: \( \ldots \) E. Just 'cause I figured the parabola went down.
Q: How did you figure that?
A: 'Cause it, uh, the x term in the quadratic function is negative.
Q: Okay.
A: \( \ldots \) And then you just plot where all the points are and that one's the only one that couldn't be in there. So you wouldn't have to plot all those points.
Q: Is, is there anything in this, you got a rough sketch of this with just a, um --
A: Okay —
Q: — vertex —
A: — if this is negative then, then this has, because x here is positive...
Q: Okay.
A: . . . then this has to be positive. So you'd have -2 + \sqrt{13}.
Q: Um hm.
A: Which I still have no idea what \sqrt{13} actually is. Um...
Q: Does it help or not? I, I don't know. I was just seeing if you could —
A: Well, oh, okay. Then the answer would be positive. I mean, by going, because this is positive...
Q: Um hm.
A: The answer would be positive. So I would probably guess positive 9 then.
Q: You can go ahead and circle it but you can leave your question there because we're still guessing.
Q: Oh, okay.
A: . . . So it can be anything.
Q: . . . So those are functions basically or, or . . . ?
A: Oh, okay . . . That's what I'm doing. That, that's, it's these two then.
Q: Okay.
A: Because these two aren't functions.
Q: How are you determining that? How do you know it's . . .
A: Vertical line test.
Q: Oh, okay. So, we eliminated . . . Okay. 12.
A: Now I feel really dumb.
Q: Why?
A: 'Cause that was easy.
Q: Oh. (Laugh.)
A: Oh no. I did guess on this one.
Q: Okay.
A: Uh . . .
Q: Do you write a, an equation? I mean, just make something up for a parabola, what, what kind of equations, if you graph 'em, look like parabolas?
A: Oh, okay. Um . . .
Q: And then maybe . . .
A: Well, 'cause, okay . . . and add one . . .
Q: Um hm.
A: . . . and so you have to add one over here . . . Oh, okay . . . look close?
Q: What did you, you've —
A: I, I've divided it is what I did.
Q: Okay . . . So you, you're making up another equation. Is that right?
A: Um hm.
Q: Okay.
A: And I only know how to do that 'cause it's what we're doing in analyt now.
Q: Okay does that help?
A: Um, I don't know. (Laugh.)
A: . . . Okay, could you set each one equal to 1 then? Go x - 1 = 1 . . . See, I don't know what to do with my square 'cause otherwise I wouldn't just added one here and went x = 2 . . . and then square both sides?
Q: Will that help you determine what m is?
A: Well, if, if m = \frac{1}{2} it would be within this.
Q: Okay.
A: So that would be alright then... If you do it over here, you get -1 -- you get 0... Well, it, it could
be 0... You're asking what m is, so I guess m could equal 0...
Q: Let me see what's going on... You got this from expanding this? Is that right?
A: Um hm.
Q: \( my^2 - 2my + \ldots \)
A: One y.
Q: ... What's that of?
A: So, you go...
A: \( \ldots m + y^2 \)
Q: ... If you, if you took it back you've got, you'd get \( m^2 \), you'd get a \( y^2 \).
A: Um hm.
Q: ... 'cause you'd have to square your last term?
A: ... Then you ha--
Q: Let's look at what you've written here...
A: Then \( y \) -- or \( m \) couldn't... If this was -- if \( m \) was 0...
Q: Um hm.
A: ... it would cancel this out and it would cancel that out totally.
Q: Wait, I'm sorry. Which would it cancel out now? That --
A: This and this.
Q: Which you've written, right? You're not pointing back to the original equation? You're pointing
here?
A: Right.
Q: Okay.
A: And then you'd get \((x - 1)^2 + y\)...
Q: Um hm.
A: \( \ldots \) equals \( \ldots 1 \).
Q: Now you have to ask yourself, is that a parabola? An equation for a parabola.
A: \( \ldots \) No, it's an equation for a, an ellipse...
Q: Okay... You know that because...
A: Because that's what we're doing now. (Laugh.)
Q: Oh, okay.
A: We're doing ellipses and circles and parabolas.
Q: Okay.
A: And this would be a, if you want to know about your ellipse, it's, uh, centered at \((1,0)\)...
Q: Okay.
A: \( \ldots \) Um ... it couldn't be an ellipse either ... because it wouldn't have any, uh ... there isn't
anything ... I'm totally flustered now and...
Q: Alright. We can come back to it later if there's time. Go ahead and put a question mark again, and
let's try 13.
A: \(-\infty \) to \(-10 \) ... which would be, -10's down here ... Well, the graph is ... negative ... I guessed it
was D.
Q: Okay. Because it's ...
A: Because, well, this would work (response A) but they have their order wrong.
Q: Oh.
A: You can't go \(-\infty \) to \(-10 \). That would be in here...
Q: Um hm.
A: \( \ldots \) and it's going this way and you're trying to go that way.
Q: Um hm.
A: \( \ldots \) And then 10, this, this does increase ...
Q: Um hm.
A: \( \ldots \) between ... gearing up this way, but this half's wrong.
Q: Okay.
A: And then you just do that with every one. And the -10 to -3, which I'm not quite undersur-- quite
understand where that is. It would ... I guess it would be in here.
Q: Um hm.
A: That little space.
Q: Where is the interval you've picked then? It's from -3...
A: -3 to here...
Q: Um hm.
A: ... which is still going the wrong way I guess. From here to there, in that one.
Q: ... Is, this piece here? Is that what you --
A: Um hm.
Q: What makes you think it's going the wrong way? Is...
A: Well, because it went...
Q: This
A: ... -3 to \(-\frac{1}{2}\), which, would go ... Well, this half goes this way and this half goes that way.
Q: Oh, okay.
A: So that's wrong ... So what, it would go ...?
Q: So to determine this increasing you're looking at your, the, the arrows? Is that what ...?
A: Yea.
Q: Okay.
A: ... You know, I don't think any of them work, by this method.
Q: 15.
A: It's symmetric with respect to the vertical. I put the vertical line through the vertex then the other side at this point over here is going to be symmetric to this point because they gave me (5,4). It could very easily be (1,4). The distance between 5 is 2 so the distance between 3 and this is 2, make that one. And that's the same. So it's C.

Q: Symmetry here you were focusing on these two had to be the same number.
A: Yea, the distance between the vertical line.
Q: Okay. Is that how you would explain symmetry to somebody? You'd say ...
A: Well, it's the same distance on both sides, in this case with the vertical line being, uh, x=3, it would the vertex of the parabola. (Pause.) In order for them to be symmetrical, it has to be that way.

Q: 17.
A: Hmm. Well, I wish I knew what this looked like. I know they told us about a thousand times and I never remember. \( y = x^2 \). So if \( y \) goes up 1, then \( x \) will go up whatever it is, so if \( y = 1 \) and \( x = 1 \), but \( x \) is squared so \( x \) is still 1. That's 2 and 4. How is the graph of \( y^2 \) obtained from that? It would be ... Is that going to help me at all?

Q: What you've written down?
A: Uh huh.
Q: Well, let's see. Um, \( y_1 = x^2 \), you're saying this is the graph?
A: Yea.
Q: Okay.
A: I think that's probably the principle in which I tried to solve the problem.
Q: Okay.
A: I think ... well to tell you the truth I don't know if this is an equation of a straight line.

Q: But you're pretty sure \( y_1 \) is?
A: Yea.
Q: Okay. So the \( x - a \) quantity is what's throwing you off from making it a straight line.
A: Uh huh.
Q: Okay.
A: Probably.
A: This point can go in any direction for a line. But if it went \( a \) units down it would still be on the same line as \( a \) units over. It would be on the same line in different places.
A: It means both of them?
Q: Well the \( y_1 \) coordinates.
A: You mean stretching them out, this way?
Q: Uh huh.
A: Gee,... this is depressing. I don't understand if you stretched the y coordinates how the, what that's gonna do. I mean it's gonna ... You're slope will be the same. I guess I would probably think that you would have to move it somewhere. Would it help me to work that out?
Q: I don't want to tell you too much, I mean considering what you're trying. Let's leave it as a question mark an let's try 19.
A: Then this isn't straight any more ... because this is \( y = \ldots \). Describe the (rereads the question). This part of the graph ... This would have to be ... I know this is a point \( x = 0 \) and \( x = 2 \). This would have to be ...
Q: Would you draw now what you said it would have to be?
A: \( x < 4 \) so it's, this is a part of the graph which means it can be between -2 and 2. Which means it has to be B.
Q: How do you know if... What made you take the part of the graph below the line versus above it or...
A: Because an x number that's gonna be squared is always gonna turn out positive.
Q: Okay.
A: So it doesn't matter if it's negative. You can still take all this, except you can't take these because if you take 3 it's gonna be 9 and that's greater than 4.
Q: Okay.
A: So you have to use this line. It's everything down here.
Q: Okay. 20.
Q: You're focusing on...
A: The equation. The graph means very little to me.
Q: Okay.
A: Well, I would probably put in all these points in here and then try to solve it. I might be... There has to be an easier way.
A: If it's 0, you get x = 40. If x is 0, then this is 40. Which looks right because it's on the graph.
Q: Uh huh.
A: If x is 40, then it's gonna be huge.
Q: Okay.
A: If x is way out here then it's gonna be way up here, which could possibly on the graph. (Pause.) If x is 1... (Figures.) Tell me there's an easier way to do this.
Q: I can't do that. That would mess up the whole interview.
A: Well, I don't know what it is.
Q: What did you find when you put in 1?
A: You get 0.
Q: Is that, is that one of the solutions to the equation then or no? What are we looking for?
A: Um, these are the solutions so you have to see if they work.
Q: Work where?
A: Well, if they... you can put them in the equation and what you get for f(x) has to be, f(x) = y so it has to be the y coordinate on the graph because the graph's not wrong.
Q: Okay.
A: So 0 works, 40 works, 1 works. Which means when the answer is -
Q: Oh, I see. You're putting these in and if it fits the graph, you're saying that's a solution.
A: Um hm.
Q: Okay.
A: But 40 was way up here somewhere. It wasn't on this graph. Let me see, 0 is... okay 0 we got 40, which we said was on there. It could be 50.
Q: Did you notice what the solutions of this equation equal to 0 are? Will that help you narrow in...
A: That confuses me.
Q: Oh, okay. Okay.
A: The solutions of the equation... So whatever number we put in here has to cancel out and equal 0? (Pause) 0 wouldn't work because we'd get 40 = 0.
A: 40 can't work, it'd be way too big.
Q: Um hm.
A: And, well, if we put 1 in, 1 = 0. It can't be that one then, so it's one of these two.
Q: Okay.
A: And, if 40 didn't work there's no way 50'll work.
Q: Okay.
A: So it has to be B.
Q: Okay. Fine. Is that it?
A: I guess. It sounds good. (Laugh.)
Q: Alright. Well, okay. 21.
A: Okay.
Q: 21.
A: There is an infinite number.
Q: Okay. How do you know that?
A: Because it goes in both directions.
Q: Okay. 22.
A: (Reads the question) ... as a function of the height (h) which means L = f(h) possibly? maybe?
Which ..
Q: Great.
A: They're directly proportional so that means if we get one, we have to get one for h.
Q: Oh hm.
A: Which looks like ... none of these.
Q: Which is a response by the way. A lot of students miss over that one, but it's there.
A: Well this one's ... it can't be that one, it has to be even. Because it would be a straight line and
these are too bizarre.
Q: Okay.
A: And you really can't have a length if you're measuring, you can't have it negative anyway.
Q: Correct.
A: So it would only be that part of the graph or this part of the graph.
Q: What about the curvy ones up there? Too bizarre?
A: Yeah they're ... it ... Well, wait a minute. (Refers to graph).
Q: Yeah, 2 ...
A: Yeah. They don't ... If it's ... I get so confused. If they're directly proportional then 1 is 1.
Q: That makes sense in terms of the shadow of the height of the object?
A: No. Not at all.
Q: Well, what would you think it would be if you're talking about a shadow of an object?
A: A shadow should be a lot larger than the object.
Q: Directly proportional means if this is this high, then it's shadow is gonna be the same length ...
A: Which the sun must be directly below ... (End of class.)

Student 3-3
4

A: Do you want me to just read this?
Q: Yeah, uh huh.
Q: Number 3.
A: (4,-4).
Q: Okay. You want to go ahead and circle that on there and then if you would, just tell me how you
figured that out.
A: Just because each one of these is 2 and that's the second line over and each one of these is 1 and you
just go down, and it would be (4,-4).
Q: So there's an order you have to go in? You went along the right first and then you went down.
A: Yeah, across. It would be across 4 and down 4.
Q: Okay. Look at the next page, number 5.
A: The graph of B is rotated, no wait ... I'll say C. The graph of B is an upside version of the graph of
A.
Q: Okay. What made you choose that one?
A: It's kind of a guess, but just because it's the opposite (refering to negative sign in front of equation B.)
Q: Okay.
A: This is a parabola, so I thought this would just be the opposite.
Q: 90° doesn't do it? You were first gonna choose that.
A: Well, if it was 180 I would've chosen it, but 90, I didn't ...
Q: Alright, go ahead and circle C. Alright, 7.
Q: Are you familiar with trigonometry?
A: Yeah. A.
Q: Because ...
A: I had trig last semester and I remember one going like that, going through the origin like that.
Q: Okay. What about the other ones?
A: This one's just a parabola (referring to Graph II) and that's not one (Graph III).
Q: Um. hm.
A: This one, looks like a hyp-, four looks like a hyperbola. Whatever it is I know it's not trigonometric, and this one (Graph III) isn't trigonometric. All the ones we did were symmetrical, it was the same all the way through and this one isn't the same (Graph III).
Q: Okay.
A: If it woulda gone like that it could've been one.
Q: Okay. Um, let's try 9.
A: I'd say E. There is no maximum value of y.
Q: Because...
A: Because whatever uh, it doesn't put any limitations on x so you could put a million in for x and y would be real big. You could put 2 million in and y would just be bigger.
Q: Okay. 11.
A: I think it's I and IV. It's E. The vertical line test... it only hits in one point. II and III would have more than one point.
A: Negative infinity to negative... all the way up to -1.7 and then 1.7 to infinity... It increases if it goes left to right if it's going up.
Q: Um hm.
A: It decreases, if it goes down it decreasing.
Q: You gave the numbers -1.7 and 1.7.
A: Top and bottom.
Q: You're leading back down to the x-axis. Okay. 15.
A: Here's 0, it's the vertex. (Student draws a sketch.)
Q: Um hm.
A: (5,4) is over to the right 2 and up 4.
Q: Um hm.
A: So it would have to be the same on the left. Over to the left 2 and up 4.
Q: Okay. Good. 17.
A: Um, I'd say... because $y = x^2$, wherever x is you just go to the right x units and $y_2$ you move to the right x units and back to the left a units. y and go back to the left a units.
Q: What made you choose left versus up, down, right or...
A: Because it says $x - a$ so you're gonna go over right for x and then minus a so you're gonna move you back to the left.
Q: Okay.
A: If it would have been a plus, I would've gone to the right.
Q: Okay. 19.
A: Is there anything greater than 2 or less than negative 2... given 4 so $x^2$ is less than 4.
Q: How did you arrive at that answer? What did you do?
A: Just thought about it, like anything outside of the boundaries of (2,-2) is gonna be given a 4.
Q: How did you get 2?
A: Because 2 is like your zero point.
Q: Um hm. One is not a zero point? I mean...
A: If you put in 2, then it'd be 4 is less than 4.
Q: Um hm.
A: And that's how you explain it. I just set it equal to...
Q: Did you use the graph of it to explain it or not?
A: Um, this would just be... (Action)
Q: Alrighty. Good. 21.
A: An infinite number.
Q: It doesn't make any difference that it's a line segment versus a line or...
A: It could be one billionth, comma or one one-billionth or whatever.
Q: Okay.
A: And they give you choices but none of those choices you couldn't figure out anyways.
Q: 20.
A: B. -5, 1, 8. It would give you your... This is the x-axis and those three points ... are at zero.
Q: Okay. What if perhaps the equation had said this was equal to 50? Would there be a way to use the graph to get that one or would you have to use some other means?
A: If it was 50?
Q: Um hm.
A: I think it would be wherever it crossed up here at 50. Wherever the graph is at.
Q: Okay, good. 22.
A: As the x increases the y increases directly proportional to it?
Q: One doesn't do that?
A: One doesn't do that, two doesn't do that, and four decreases.
Q: 23.
A: If y = 3,...
Q: Um hm.
A: ... it equals 3 up there. That's it, in three points.
Q: Good. 24.
Q: Have you talked about complex numbers yet this year?
A: Huh uh.
Q: I? Imaginary numbers?
A: Yes, we've done that.
Q: Okay, those are the same thing.
A: I'd guess three.
Q: Okay. Guesses are fine. Why would you guess three?
A: Because if I was to make a horizontal line I'd hit the graph in three points.
Q: Okay. 25.
A: x + 3, this one starts at ... x is zero, so x + 3 moves x to the 3 units to the right ...
Q: Um hm.
A: ... move y negative one units so one unit down ...
Q: How did you know to move the -1 would go down versus go back to the left or something?
A: Because, um, y moves up or down.
Q: It doesn't affect the x. It just,... you knew it affected the y to move down?
A: You would just write it y + 1 = f(x).
Q: Good. Thank you very much.

Student 2-1
4

Q: 3.
A: You answer this by counting over across on the x-axis - positive x-axis, two over and then count down four, so the coordinate would be (2, -4) which is choice B.
Q: Okay. 4.
Q: Is that term familiar?
A: Yes. Um, the one that looks most like an exponential function is item IV which is D.
Q: Okay.
A: I is a sine wave, II is a parabola, and III is a cubic.
Q: You do have the choice of -
A: Process of elimination.
Q: Okay. You do have the choice of none of the above there so none of those might be exponentials.
A: I think it's IV.
Q: Okay that's fine. 5.
A: I consider choice C the best, the reason being the negative sign in front of the quantity. It reflects the graph across the x-axis.
Q: Okay. 6.
A: I always do this by making a little graph . . . and I count over . . . even though you switch the signs I
still like to draw a graph . . . -5 and that's in the fourth quadrant so the other point would be in the
second, and it would be positive . . . it would be negative three, positive 5.

Q: Okay. How would you explain symmetry to the origin then to somebody who had no idea what that
meant?
A: Um, you take the point and you reflect it across the x-axis, which would be (3,5) and then reflect that
over again across the y to make it (3,-5).

Q: Okay.
A: Um, choice I looks remarkably like a sine wave.
Q: Okay. And that's a trig function?
A: Yes.
Q: 9.
A: I would use the quadratic formula with this.
Q: Okay.
A: What it would . . . hold on a minute . . . Okay, it's a . . . it's a posi-. this is a . . . quadratic and that
would be a parabola and since it is . . . not a . . . negative function . . . I don't think there would be a
maximum value.
Q: Okay.
A: Maybe not, maybe not.
Q: What do you mean by it's not a negative function, what does that . . . ?
A: That means that there is no maximum vertex. f looks like . . . (draws a graph). The open ends of the
parabola point to the positive y side.
Q: Okay. 10.
A: If the coefficient of the \( x^2 \) term is negative then the function will have the open ends pointing to the
negative y, the third and fourth quadrants. Therefore, any of the points which is above \( x \), I mean \( y^2 \), I
mean \( y = 5 \) will not be on in that . . . I really have no idea. (Student has correct idea but does not see
solution in choices.)
Q: Okay. 12.
A: What I would do is I would go through the answer choices and see if that would work.
Q: What's going to make it look like a parabola? Do you have something in mind or are you just gonna
. . . ?
A: If it looks a parabola then the equation would be . . . the \( my^2 \) term would drop out . . .
Q: Okay.
A: . . . the \( (2m + 1)y \) term would also drop out because . . . I can't dc this one, I mean, I can't . . .
(Student is frustrated that answer does not come immediately to him.)
Q: Go ahead and do it --
A: I can't.
Q: -- and then, and then after you've done it without talking, then maybe come and explain it. That
would be fine too.
A: Okay, those terms would drop out . . . I would just skip through these answers . . .
Q: Okay.
A: Okay. Putting in - \( \frac{1}{2} \) does not drop out the \( my^2 \) term and that won't give a parabola. Um, putting in
0, um, will make the, um, \( my^2 \) term drop out but it won't make the \( 2m + y \) term drop out and that's
needed to transfer to the other side to get a \( y = \) something over there so 0 would work so it's B.
Q: Well that's good. 13.
A: It's increasing where the graph goes from the bottom left to the top right . . .
Q: Okay.
A: . . . and it goes and use the x-axis . . . and the one that, the choice that just looks closest is . . .
choice E.
Q: So you look at it from left to right where it's going up . . .
A: It's increasing on the x-axis.
Q: Okay . . . and then you find that point and then you what, and then you . . .
A: You find that point, okay, if it's starting, if, in the third quadrant it's going all the way down it will start with negative infinity . . .
Q: Okay.
A: . . . and it peaks somewhere and that x coordinate will be -1.7 . . .
Q: Okay, good.
A: . . . and then positive and then . . . in the fourth quadrant 1.7 to infinity because that's where it bottoms out and then it starts to go back up again.
Q: Okay. Good. 16.
A: II and IV are automatically eliminated because they are in the fourth quadrant, and in this situation they would have to be in the first quadrant.
Q: Okay.
A: The area would be 1 times x because 2 of, x being the perpendicular . . . y the total lengths would equal 50 and you could solve for y and replace that back and do the equation for the area.
Q: Um hm.
A: The area equals 50x - 2x^2 . . .(Student decides he has written area incorrectly in terms of x and tries to rewrite equation in terms of y). times 25 - \frac{1}{2} y . . .
Q: What made you switch from your x to your y?
A: I can't do that one. (Student gets very frustrated and wants to abandon problem.)
Q: Okay. 17.
A: Shift a units . . .
Q: Um hm.
A: . . . to the right a units . . .
Q: 19.
A: . . . part is look at the interval of the graph, the x-coordinates, um, which are below the line and you . . . it intersects the line, so it would be from -2 . . .(to positive 2).
Q: 20.
A: It intersects the x-axis . . . -5, 1 and 8. Choice B.
Q: Okay. What if it said 50? Would you change your strategy then in terms of finding a solution.
A: Just subtract 50 from both sides. I believe. If I had my calculator I would graph it.
Q: Would you have to draw a different graph this way?
A: Yea. They're not the same.
Q: Oh, okay. 21.
A: (No comment picked up on tape. E circled)
Q: 22.
A: . . . the height . . . directly proportional . . . and if the shadow increases with the height it would be . . . C which is graph III.
Q: 23.
A: And what I would do is I would shift the graph down 3 units, and I would draw a new graph, or what would be easier is to draw a new x-axis through y = 3.
Q: 24.
A: If it's a cubic function it will have three solutions, and since there is three intersections through the x-axis they are all real and therefore, um, three complex solutions because all real solutions are complex.
Q: 25.
A: You need to draw a whole series of graphs . . . I would shift it to the left 3 . . . and I would move that down 1 . . . and that resembles graph V which would be E.
Q: Okay. And that's it.

Student 2-2
3
Q: 3.
A: Just look on the x-axis, you see the scale is two on each segment . . .
Q: Yes, exactly.
A: ... so it would be 4, and then the scale is one on the y so it would be 4.
Q: Go down and circle it. 5.
A: Okay. It would be upside down because the $x^2$ is changed, and you flip it over.
Q: 7.
Q: Is that term familiar (trigonometric)?
A: Yea. That would be, uh, graph I. I don't remember if it's the sine or cosine, but I know the interval is one. It goes between one and negative one.
Q: And the rest aren't any trig...?
A: No.
Q: Okay. 9.
A: There wouldn't be a maximum value because the $x^2$ is positive.
Q: 11.
A: Uh, I and IV would be functions because if you draw a straight line through 'em, these cross more than once.
Q: Okay. 13.
A: It would be increasing here (points), from -1.7 to $\to$. Um, about, I guess this is 1.7 again to $\to$, so.. it's E.
Q: 15.
A: Um, is the vertex five... I just do this by the picture. If this is three it's gonna be up 2 over 4 so this is symmetric over here so it would be like (1,4).
Q: Okay. How would you explain what symmetry means to somebody?
A: Um, good question. If you draw a line, you can draw a straight line through... you can, if you go over a certain amount this way and go up, you're gonna do this... it's hard to explain. I don't know.
Q: Okay. How about in terms of folding?
A: Yea. You can just fold it across the line of symmetry.
Q: And then what would happen to the two different pieces if you fold, would they...?
A: They'd be right on top of each -- they'd be the same graph.
Q: Okay. 17.
A: Um, you'd move it $a$ units to the right, just, I don't know. We've just been taught that. So like $x - 6$, you'd move it six units to the right, and if it was negative, if it was plus here (points), you'd move it to the left.
Q: Okay. 19.
A: Um, between negative... it would be... you want it less than so it would be everything under that line, $y = 4$, so that would be from -2 to 2.
Q: Alright.
A: Um, and it's not less than or equal to so that's the reason the points are open.
Q: How would you explain to somebody why you focused on the points below this line versus above?
A: Because this would be the quantities where $y$ is less than four.
Q: Okay. 20.
A: Well, you have a graph of it so you just look to see where it crosses the zero for the $y$ and it would be... -5, 1, and 8.
Q: What if the equation had said this was equal to 50? What would your strategy be then?
A: You can either move it down -- subtract the 50 on this side and then you'd get a different graph, or you could draw, you could draw $y = 50$ and see where it crosses there.
Q: Okay, good. 21.
A: It's infinite.
Q: It doesn't make any difference that it's a line segment versus a line?
A: No.
Q: Okay. 22.
A: Um, ... I believe this is a directly proportional graph (Graph I).
Q: Okay, why?
A: Why?
Q: Yes.
A: Because as the height is increasing, the length of the shadow is.
Q: Um um. What made you rule out II, IV and III? Any ideas, or...
A: Um, these are kinda... those would be inversely proportional, wouldn't they?
Q: Okay.
A: These two would. Um, I guess I know from science, too, what a directly proportional graph is, so I thought... I don't know how you would explain it.
Q: Alright. So I fits?
A: Yes.
Q: Okay. 23.
A: Well you'd shift the whole graph down three, because that's what it tells you to do...
Q: What tells you to do that?
A: The negative three takes the whole function and you move it down three...
Q: Okay.
A: ... and then you'd have three because then this would all be dropped down... you could draw a line across and you can see that it would cross three times.
Q: Okay, good. 24.
A: Um, three, because aren't the reals under the complex number set?
Q: Okay.
A: Is that right?
Q: Oh, I can't answer yes or no. (Giggle, giggle.)
A: Oh, okay.
Q: You have to go with your gut feeling. (Giggle, giggle.)
A: That's what I'll go with.
Q: Okay. 25.
A: Um... okay this is the graph here... shifted down one, it would be moved to the left three... um... right here, E.
Q: How did you do that now?
A: Start with the -1 since you're starting with this, you're gonna shift it down one.
Q: Okay.
A: And then x + 3 tells you you move it to the left three.
Q: Okay, and how did you know that this wasn't, like, move it to the left one versus -- you knew it had to go down?
A: Yea, it's a vertical shift. If it's... I cannot explain this (laugh).
Q: (Laugh.)
A: If it doesn't... if it's outside of the equation (referring to $h(x+3)$ versus $h(x)+3$)... this side here would be the horizontal.
Q: Okay.

Student 3-4

1

Q: 3.
A: (2, -4).
Q: Why?
A: Because you just follow the x and...
Q: The x first? There's an order?
A: Yea.
Q: Okay.
A: Whether that's positive or negative.
Q: 5.
A: ... this gets to be that it...
Q: Are you stuck between some? Are there any that you can rule out or...?
A: Well, see I don't really, I'm not really sure how you get graphs from... equations like that, but I know that his negative is squared so it's gonna be the same, but those other two aren't gonna be the same, so... maybe it's identical. I'll try that.
Q: Looking at this equation, you don't know what kind of picture it would make, is that right?
A: Right.
Q: Alright, so what we're gonna have here is a guess, the answer is five?
A: Right, E.
Q: Okay, but you told me why, and that's good. 7.
Q: Is the term familiar, trigonometric?
A: Yea, I... well, let me think... if it's what I'm thinking of... sine, cosine. Okay, number I.
Q: Okay.
A: I'm not real sure, but... I'm not sure which it is but I just remembered that. Cosine or sine curve starts and go up in an up and down motion.
Q: Okay.
A: One starts, um, high up on the y-axis and that one starts at the origin.
Q: You're confident that II, III and IV aren't, or...?
A: Well, I don't know about III because it starts higher up on the y-axis but, it's not... it's not, um, equally distributed... throughout the x.
Q: Okay. 9.
A: Well, I guess I would say that... there is no maximum value of y because... I think that if you'd be able to put in any x number, y could be equal to...
Q: Okay, let's go back and do question 8, too.
A: Number... number 4 because it... goes and it stops, it has a stopping point whereas the other ones could go on.
Q: Good. 11.
A: Do you want me to take a guess at it?
Q: Yea. Go ahead.
A: ... hm...
Q: Okay, just go ahead and put a question mark by that one and we'll just go on. 13.
A: ...
Q: Do you know what the question is asking? Have you talked about increasing and decreasing?
A: Yea. I mean, I know what...
Q: Okay. Could you show me on the picture where it's increasing?
A: just arrows...
Q: Okay.
A: I'd say B. [-10,-3] and [3,10] ... I'm not sure now, but... I've heard all of this and seen this before but not together, so I guessed... if you three on each side, and down and up to ten, that's where that, the end of... where the arrows are, so that's where the arrow's gonna still keep going.
Q: Okay. Try 14.
A: ... from, um, on the x-axis, if you go from one to three, it's going upwards, possibly there...
Q: If someone said, well look at three over here, and if I start from three and I go to one, this graph looks like it's increasing because the arrow is going up. What would you tell them.
A: I would just pick this one first because this is going, it goes, it's on the positive side and it would start heading negative, whereas this is going positive and positive. So I guess I'd still pick that.
Q: Okay. 15.
A: ...
Q: How did you know it went up like that? Instead of sideways, or...
A: Oh, I don't. (Laugh.) I'm just, well, I'm experimenting here.
Q: Okay.
A: Well, because the next, because the next point is at (5,4).
Q: Okay.
A: ... I'd say not enough information because you don't know how wide the parabola is.
Q: Okay. Oh, I was just going to ask you, too, but since you've chosen this, that's fine, because I was going to say is there anything that tells you that the parabola doesn't go like that? (Draws a figure.)
A: I'll go with E.
Q: 17.
A: E. Because... adding a negative $a$... well, since it's negative, I think that you would move down or left. Those are negative... I'm not sure whether, which one... I first started with an upside down because I was thinking of this as a... direct pair, what you call it? But that's not right...

Q: Again, you can go down and put a question mark after this so I don't circle, let's see what was it, B and... Let's see, we're not sure. We know the negative's gonna move it down to the left, but which way, we just don't know.

A: Right.

Q: Okay, fine.

A: I just guessed.

Q: Okay. 19.

A: It would be A.

Q: Okay.

A: graph.

Q: Okay.

A:... it was anything not including 2... one, one squared is one and that's less than four, and as for the rest of these, A. is the one that is true, it would include one.

Q: 21.

A: An infinite number.

Q: It doesn't make any difference if it's a line or a line segment? Does that change or...?

A: I don't think so, because I don't think there's any way of measuring it.

Q: Okay. We skipped 20. I want to do that one, too. 20.

A:... I don't know. I need to get, I need a... numbers.

Q: Does the graph help at all or is it...?

A: No. I need my calculator and then it would. (Laugh.)...

Q: You can go ahead and put a question mark unless you want to try a guess.

A: Let's see... my guess would be B. because where the, where the graph hits is on those points, but I don't know why.

Q: Okay. Would this one count, too? It hits here on this axis.

A: No, because we're just talking about $x$. I just looked at the $x$-axis.

Q: Okay, that's fine. 22.

A:... I don't think any of them do because... I can't figure out how they work.

Q: Okay.

A: I don't understand it.

Q: What does that mean to you, that the length of the shadow is directly proportional to the height of an object?

A: That it can be, it can be, um... like... thing and I can't say it. Um,...

Q: Do you want to draw a picture or something?

A: You can divide equally. Um, like the shadow is, um... that's two and the shadow would be four (I couldn't make it out)... and the shadow could be... and it could be divided into other ones.

Q: Okay. I see what you're saying.

A: I guess it's... maybe it would be I or II, but I...

Q: 23.

A: This is $x$... and three from this point where it touches the line...

Q: Um hm.

A: There isn't one.

Q: C'mon, pick this up.

A: (Laugh.)

Q: Say that one more time. That you drop it down...

A: You drop it down three because it's minus three...

Q: ... and there'd only be one...

A:... left, because it's on two right now.

Q: Okay.

A: There'd only be one left... (I couldn't get it) I think it is.

Q: Okay. 24.

A: I don't know what a cubic function is.
Q: Okay. Have you talked about complex numbers...
A: No.
Q: ...imaginary numbers?
Q: Okay. Go ahead and just, we'll leave that one if you aren't familiar with the terms then there's...
A: (I couldn't get her answer)
Q: Okay, that's fine. 25.
A: ...doo, doo, doo...point negative one...well, it hits over here at negative one, and it goes...
goes over...
Q: We lost it from there, but we got the negative one.
A: Yea...I really don't know.
Q: Okay, that's fine.
A: (I couldn't get it)
Q: Are you doing anything like this in class or...?
A: ...We had sine and cosine graphs...
Q: Um hm. Nothing weirder like this?
A: No.

Student 2-3
Missing data [3]

Q: 3.
A: It would be um,...the answer would be (4,-4). I should just explain it?
Q: Yea.
A: Um,...well, on the coordinate system, we use the Cartesian system, uh...
Q: Um hm...
A: ...you have an ordered pair, uh, where, uh, the first coordinate is the location along the x-axis and the second would be along the y-axis, and this one has a tic mark every two, every... (I couldn't get that) two, and so, um...the dot you would, falls on number four on the graph. And then you count down negative four on the y-axis.
Q: Okay, that's good. Alright, number 5.
A: ...The second equation would be a conversion (inversion?) over the...I guess the x-axis.
Q: Um hm. And why is that?
A: Um,...because the quantity, the whole thing to the negative would just reverse all the values, so, it wouldn't be any wider, um...it's rotated, it wouldn't be identical so it would be C., it's upside down.
Q: 7.
Q: Is that term familiar, trigonometric?
A: Yea, um. This one is a quadratic, number II is a quadratic, um, III is a cubic because of the...shape and the three, three...real solutions...
Q: Um hm.
A: ...um, this is exponential because of the shape, and this is sine.
Q: Okay.
A: Oh no, wait. Is it a sine? ...No, this is the only trigonometric.
Q: Okay.
A: So I guess, A.
Q: Alright. 9.
A: Um, it's a quadratic equation, the graph is a parabola and it's not inverted, so, there would be no maximum...because it opens up.
Q: 11.
A: That would be, uh, I and IV because, um, to be a function...I guess, uh...it would have to—a relation has to have only one pairing of...one y value for every x or something like that, and you check it with the horizontal line test. If the line crosses it at any point then it's not a function, and...
Q: 13.
A: Um, . . . let's see, the graph's increasing when it's . . . rising from left to right...
Q: Um, hm . . .
A: . . . so, it would be . . . behind this mark and this one (points) . . .
Q: Um, hm . . .
A: . . . which would be, uh, -1.7 to -∞, 1.7 and . . .
Q: You read you interval off of which axis then, you were looking at the x or the y?
A: Um, the x-axis.
Q: Okay.
A: Well, actually, yea, okay. You would find the equation . . .
Q: And what is actually increasing in . . .
A: The values of y are increasing . . . domain, the domain that's marked here.
Q: Okay. Let's back up and do 12.
A: I've seen this.
Q: Many times (laugh).
A: I don't want to get it wrong (laugh). Um, \( x^2 - 2x + m^2 + (2m + 1)y = 0 \) a parabola. Um, I never remember how to solve it, but I keep thinking, um, zero would cancel all the \( m \) terms leaving you with, I guess .. . = 0, and then, uh, . . . (figures), so I guess that would be a parabola, . . . and cancel out . . . which of the following values of \( m \) is the graph . . . zero . . . zero . . . one-half for \( m . . . \) this would make, um, . . . cancel out two . . . that would be one . . . two would cancel out the y . . . okay, and then you would only have . . . which is B.
Q: Okay, \( m = 0 \).
A: This?
Q: You're saying if you graph this, this would look like a parabola?
A: It'd be inverted.
Q: Okay, how do you know that's a parabola? What are the characteristics of the equation?
A: Well, I guess it's actually a quadratic . . . equals y, which would, you know, um, the zero is implied, I imagine a plus zero in it.
Q: What makes a quadratic? How do you know that?
A: Um, with the, the um . . . degree of two in there, . . . degree of two.
Q: Okay. 15.
A: (Rereads the question.) . . . so, symmetry you would say, . . . \((5,4)\) is two away from three and so two on the other side of three, would be 1, so \((1,4)\).
Q: Okay. How would you explain symmetry to someone?
A: Um, . . . if something is symmetric about something . . .
Q: Um hm.
A: . . . then, um, either side of the . . . the axis I guess you would say . . .
Q: Of this line . . .
A: . . . on either side of this line, um, . . . this would be equidistant from this, from that line . . . If the, okay, if the, if the graph were inverted across that line it would intersect . . . flip it across.
Q: Okay. good. 17.
A: Um, \( y^2 . . . \) so this is just a translation . . . Um, subtracting \( a \) from, from the squared \( x \) term, the quantity . . . from the translations that we've done, you would move that, uh, the whole graph over to the right \( a \) units, uh, if you added it, it would mean to the left . . .
Q: Um hm .
A: Do you want me to explain translations?
Q: Uh, well, what do they -- how did you know that . . . since there's a minus sign it moves to the right. How did you know that?
A: Um, . . . I would say the definition of a horizontal . . . just a translation-- , you know, just the way it's been taught to us, um, it's . . . I don't really know the logic behind it, but . . .
Q: It works . . .
A: It works. (Laugh.)
Q: Okay, alright. What response then would that be?
A: It's D.
Q: Okay. 19.
A: ... um ... it's similar to the earlier one. You would look at, um ... (I couldn't get it) ... okay, here's your, here's your, your graph of \( y = x^2 \) ...

Q: Um hm.

A: ... um, a parabola, and ... at \( y = 4 \), um, it's on the line, so, what they're asking for is where, what \( y \) values, what are the \( y \) values ...

Q: Um hm.

A: ... or the \( x \) values where, what is the interval of the \( x \) values where the \( y \) values of the parabola is less than the ... 

Q: Um hm.

A: ... the line across, you see ... and these are less so ... um, it would be um, ...

Q: You looked below because you said those values are ... 

A: You just look where the values of the \( x^2 \) drop below the, the \( y \) values drop below the. 

Q: Alright. 20.

A: The solutions, um, they set it equal to zero ...

Q: Um hm.

A: ... so you're looking to find where that equation equals zero, which would mean, uh ... where the lines intersect the \( x \)-axis.

Q: Okay.

A: And just by looking, um, ... (counts the tic marks on the graph) ...

Q: Okay.

A: So you have -5.

Q: Okay. What if it had said that this equation was equal to 50? What strategy would you use then?

A: Um, you could graph the line, um, \( y = 50 \), and just use the same, the same, uh, technique, you know. Just find the new points where it intersects.

Q: Okay. 22.

A: Um, the uh, the \( y \)-axis ... the vertical axis is always the dependent variable, um, and so you'd look to see whether, uh, length or height is proport -- is dependent ...

Q: Um hm.

A: ... um, so the height; would be ... it would be, um, the independent variable, of course I guess that's up or down (?) ...

Q: Okay.

A: It would take the place of the \( x, x \) value ... which graph shows the length of a shadow as a function of the height? ... Okay, I proportion ... oh, this is an exponential which would mean it wasn't with a constant proportion ...

Q: Um hm.

A: ... uh, this one is inverted, it drops which would mean it's not ... then this one is proportional but indirectly because they ... as these increase, they would go opposite of this first one so it would be III.

Q: Okay. 23.

A: How many real number solutions? Um ... okay, again they're saying ... um, the graphs of \( y = f(x) \) and \( x - 3 \) ... drop the graph three units down because \( f(x) \) -- you know, the translations ...

Q: Okay.

A: ... you would put ... um, equals zero again, so real number solutions where, where, um ... the graph equals zero. Normally, in the, in the \( y = f(x) \) graph, there would be one ...

Q: Um hm.

A: ... because these are real because they get on top of (cross ?) the \( x \)-axis ...

Q: Okay.

A: ... um, and it's dropped down ... it would be, uh, three,

Q: Okay.

A: ... so it would be D.

Q: Okay, good. 24.

A: ... Um, complex number solutions ... \( y = g(x) \) ... \( y \) is a cubic function which means it has three solutions ... complex number solutions, complex number, I, I think it's imaginary, and ... (three?) complex solutions because of, of being a cubic function.

Q: Try the last one. 25.
A: Okay, um... this is the, the $y = h(x)$ thing, um... three, it will shift it, shift, uh... (put ?) the $x$ on the origin, would shift the, shift the graph, uh, three units to the right — or to the left, uh... the negative one would drop the whole thing down one, so... those are the translations.

Q: Okay, that's it. Thank you very much.
Appendix E
The following data were taken from the students' response patterns on the Graphing Levels Pretest. A "1" represents attainment of a particular level while a "0" represents failure to pass the criteria for attainment. The criteria for attainment of each level is as follows:

- **Level 1**: correct responses on 3 of the 5 Level 1 items.
- **Level 2**: correct responses on 4 of the 6 Level 2 items.
- **Level 3**: correct responses on 4 of the 6 Level 3 items.
- **Level 4**: correct responses on 3 of the 4 Level 4 items.

Asterisks next to a line of data represent a violation of the hierarchy. Some students did not answer all items hence their line of data is blank. A total of 114 students gave complete data.

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<th>Student #</th>
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