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Resonated holograms for efficient optical interconnects: Theoretical analysis and experimental verification

Griffith, Paul Combs, Ph.D.
The Ohio State University, 1988

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Resonated Holograms for Efficient Optical Interconnects:
Theoretical Analysis and Experimental Verification

A Dissertation
Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the
Graduate School of the Ohio State University

by

Paul Combs Griffith, M.S.

1988

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I would like to thank my wife, Tauna, for all of her encouragement and support, without which this would not have been possible.
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# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>ACKNOWLEDGEMENTS</th>
<th>ii</th>
</tr>
</thead>
<tbody>
<tr>
<td>VITA</td>
<td>iii</td>
</tr>
<tr>
<td>LIST OF TABLES</td>
<td>vii</td>
</tr>
<tr>
<td>LIST OF FIGURES</td>
<td>viii</td>
</tr>
</tbody>
</table>

## I. INTRODUCTION

1.1 Background ......................................................... 2
1.2 Problem Statement ................................................ 4
1.3 Literature Review .................................................. 4
1.4 Contents of the Dissertation ............................... 8

## II. THEORY

2.1 Introduction ....................................................... 10
2.2 General Four Mirror Resonated Hologram Model .......... 11
2.3 Ideal Transmission Hologram .................................. 19
2.4 Optimization: Qualitative Discussion ..................... 24
2.5 Optimization: Two, Three, and Four Mirror Resonated Holograms 28
2.6 Ideal Reflection Hologram ...................................... 42
2.7 Frequency, Angle, and Temporal Response of the Resonated Hologram 47

iv
# LIST OF TABLES

<table>
<thead>
<tr>
<th></th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Components of the Resonated Hologram Demonstration Apparatus</td>
<td>79</td>
</tr>
<tr>
<td>2</td>
<td>Hologram characteristics</td>
<td>96</td>
</tr>
<tr>
<td>3</td>
<td>Semitransparent mirror intensity characteristics</td>
<td>96</td>
</tr>
<tr>
<td>4</td>
<td>Three mirror resonated hologram results — Hologram 1</td>
<td>97</td>
</tr>
<tr>
<td>5</td>
<td>Four mirror resonated hologram results — Hologram 1</td>
<td>98</td>
</tr>
<tr>
<td>6</td>
<td>Three mirror resonated hologram results — Hologram 2</td>
<td>99</td>
</tr>
<tr>
<td>7</td>
<td>Four mirror resonated hologram results — Hologram 2</td>
<td>100</td>
</tr>
<tr>
<td>8</td>
<td>Experimental data on angular response of hologram.</td>
<td>159</td>
</tr>
<tr>
<td>9</td>
<td>Angles corresponding to peaks and nulls in diffracted power</td>
<td>161</td>
</tr>
<tr>
<td>10</td>
<td>Hologram parameters computed using experimental data.</td>
<td>162</td>
</tr>
<tr>
<td>11</td>
<td>DCG hologram sensitizing procedure.</td>
<td>185</td>
</tr>
<tr>
<td>12</td>
<td>Exposure procedure for DCG hologram.</td>
<td>186</td>
</tr>
<tr>
<td>13</td>
<td>Developing procedure for DCG hologram.</td>
<td>188</td>
</tr>
</tbody>
</table>
# LIST OF FIGURES

1. Schematic drawing of a resonated hologram. ........................................ 5
2. Four mirror resonated transmission hologram model. ........................ 14
3. Fields for the four mirror resonated hologram. ............................... 14
4. Diffraction and transmission coefficients for a transmission hologram 15
5. Paths for determining phases. ...................................................... 27
6. Two mirror resonated hologram. .................................................. 29
7. Three mirror resonated hologram #1. ............................................ 30
8. Three mirror resonated hologram #2. ............................................ 30
9. Four mirror resonated hologram .................................................. 31
10. Comparison of $P_{\text{out}}^{\text{max}}$ for 0, 2, 3, and 4 mirror resonated holograms. $A = 0.$ ................................................................. 40
11. Comparison of $P_{\text{out}}^{\text{max}}$ for 0, 2, 3, and 4 mirror resonated holograms. $A = 0.1.$ ................................................................. 40
12. Comparison of $P_{\text{out}}^{\text{max}}$ for 0, 2, 3, and 4 mirror resonated holograms. $A = 0.2.$ ................................................................. 41
13. $P_{\text{out}}'$ for a three mirror resonated hologram as a function of mirror reflectivity. $S = 0.05$ .............................................................. 42
14. $P_{\text{out}}'$ for a four mirror resonated hologram as a function of mirror reflectivity. $S = 0.05$ .............................................................. 43
15 $\mathcal{P}'_{\text{out}}$ for a three mirror resonated hologram as a function of mirror reflectivity. $S = 0.2$ ........................................ 43
16 $\mathcal{P}'_{\text{out}}$ for a four mirror resonated hologram as a function of mirror reflectivity. $S = 0.2$ ........................................ 44
17 Four mirror resonated reflection hologram model. ................. 45
18 Diffraction and transmission coefficients for a reflection hologram . 46
19 Frequency response of three and four mirror resonated holograms. 57
20 Bandwidth of three and four mirror resonated holograms. .......... 58
21 Round trip path for light incident at non-Bragg angle. ............... 61
22 Angular response of three and four mirror resonated holograms. .. 64
23 Half-power angular halfwidth ............................................. 65
24 Time constant for resonated hologram. ................................. 67
25 Dual frequency response curves for the resonated hologram. ..... 70
26 Bounds upon the maximum output of three mirror resonated holo- 
   gram with dual frequency input. ........................................ 72
27 Bounds upon the maximum output of four mirror resonated holo- 
   gram with dual frequency input. ........................................ 73
28 Resonated Hologram Demonstration Apparatus schematic............. 76
29 Photograph of the Resonated Hologram Demonstration Apparatus. 77
30 Photograph of the Resonated Hologram ................................ 78
31 Fringes from nonparallel mirror resonator due to two laser modes . 81
32 Control circuitry for one piezo-electric mirror mount. ............... 83
33 Schematic drawing of detector amplifiers. ............................ 84
34 Photograph of hologram used in experimental system. .............. 86
<table>
<thead>
<tr>
<th>Page</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>35</td>
<td>Output of three mirror resonated hologram as a function of mirror reflectivity for Hologram 1: Experimental points plotted between theoretical bounds for single and dual frequency.</td>
<td>97</td>
</tr>
<tr>
<td>36</td>
<td>Output of four mirror resonated hologram as a function of mirror reflectivity for Hologram 1: Experimental points plotted between theoretical bounds for single and dual frequency.</td>
<td>98</td>
</tr>
<tr>
<td>37</td>
<td>Output of three mirror resonated hologram as a function of mirror reflectivity for Hologram 2: Experimental points plotted between theoretical bounds for single and dual frequency.</td>
<td>99</td>
</tr>
<tr>
<td>38</td>
<td>Output of four mirror resonated hologram as a function of mirror reflectivity for Hologram 2: Experimental points plotted between theoretical bounds for single and dual frequency.</td>
<td>100</td>
</tr>
<tr>
<td>39</td>
<td>Hologram model</td>
<td>145</td>
</tr>
<tr>
<td>40</td>
<td>Comparison between the theoretical and experimental angular response of a hologram.</td>
<td>163</td>
</tr>
<tr>
<td>41</td>
<td>Transmission hologram diffraction and transmission coefficients</td>
<td>164</td>
</tr>
<tr>
<td>42</td>
<td>Fields for determining diffraction and transmission coefficients for a transmission hologram.</td>
<td>165</td>
</tr>
<tr>
<td>43</td>
<td>Reflection hologram diffraction and transmission coefficients</td>
<td>170</td>
</tr>
<tr>
<td>44</td>
<td>Photograph of the hologram creation apparatus</td>
<td>187</td>
</tr>
</tbody>
</table>
CHAPTER I
INTRODUCTION

The design and implementation of a resonated hologram are presented in this dissertation. A theoretical model is developed for the response of a plane-wave hologram located in a four mirror resonant structure. An experimental system is used to verify this theory for two transmission holograms. Using this resonant structure, the signal output by a hologram with low diffraction efficiency can be greatly increased, even for lossy holograms.

The motivation for investigating this topic comes from the use of holograms as optical interconnecting devices in the areas of optical and electronic computing [1]. The number of interconnections possible with a single emulsion is limited by the minimum required diffraction efficiency of each connection [2]. A low diffraction efficiency results in an inefficient system in terms of power. Placing a hologram which has a low diffraction efficiency in a resonator cavity can result in a significant increase in the amount of diffracted light. This will significantly increase the number of holograms possible within the emulsion because the lower limit on diffraction efficiency will be reduced.

The resonated hologram presented in this dissertation has the potential to perform a large number of optical interconnections without sacrificing efficiency. The light beam directing properties of the hologram are utilized along with the high efficiency of a resonant cavity to achieve an efficient optical interconnection.
device.

1.1 Background

Resonators have been used for many years in electrical and optical areas. In electrical engineering metallic cavities are used as electromagnetic resonators in devices such as the klystron and bandpass filters in waveguides. In the area of optics the best known example of an optical resonator is found in the laser. In both cases, the resonant modes are those waves where the fields add up in phase from multiple bounces in order to greatly increase the output of the cavity.

Holograms are well known for their intriguing three dimensional images. Replicated holograms have appeared on the covers of at least two issues of National Geographic Magazine as well as many other magazines. They are used for security purposes on many credit cards and are even being investigated for use on our paper money.

In a more general application holograms have the ability to generate an output beam with a desired wavefront and direction in response to a given input beam. They can bend, focus, and redirect light beams. In this context they are extremely valuable for many aspects of optical computing, including the area of optical interconnecting devices.

In the area of computing, whether it is performed electronically or optically, there is a growing need for efficient interconnecting devices. In electronics wires of one form or another have been used in the past but they are rapidly reaching their limit. As very large scale integration (VLSI) technology continues to shrink the size of an integrated circuit (IC) and increase the number of circuits on one chip, it becomes more difficult to connect all of the circuits to each other as well as to the external world. Furthermore, the shrinking size is causing data rate limits
due to capacitive and inductive effects. This limits the performance of electronic computers. Optical computers are also limited in performance by the large number of interconnections required.

Holograms are one answer to this problem. In optical computing it is natural to use a hologram to redirect and reshape the light beams in order to make efficient interconnections. One example of their use is in optical associative memories using neural networks [3]. In electronic computing optics can also play a role. Using light emitters at each output and detectors at each input eliminates the need for getting wires between these locations. Connections can be made optically utilizing the space above the chip. This has the advantage of high speed and minimizes the density problem since light beams can pass through each other without interference. Again, holograms can be used to make the actual optical connections.

While the beam directing properties of holograms make them a natural candidate for optical interconnecting devices, they have the drawback that the number of efficient interconnections using current materials is limited. Each connection requires a separate hologram superimposed in the same emulsion. With conventional materials there is a rapid trade-off between the number of holograms and the diffraction efficiency of each. A low diffraction efficiency means that a large part of the light is not diffracted. This translates into wasted power and a potentially high background noise level. In many applications power is at a premium and this waste cannot be tolerated.

The concept of a resonated hologram was developed to address this problem. It enables the wasted power to be reduced without sacrificing the number of interconnections. The hologram continues to provide the interconnection mechanism while the resonant cavity increases the efficiency of the process.
A schematic drawing of this system is shown in Figure 1. The input light enters the system through semitransparent mirror $M_1$. It is partially diffracted and partially transmitted by the hologram. The diffracted light exits the system through semitransparent mirror $M_4$. The undiffracted light is collected by mirror $M_3$ and sent back through the hologram. The resulting diffracted light is collected by mirror $M_2$ and returned to the hologram where the transmitted beam will now exit the system through mirror $M_4$. By adjusting the mirror separation it is possible to cancel the wave backscattered from $M_1$ and direct all of the light out of the cavity via mirror $M_4$. This means a significant increase in the output signal over that produced by the hologram alone. For a lossless perfect hologram and perfect mirrors it is possible to achieve an effective diffraction efficiency of 100% regardless of how low the diffraction efficiency of the hologram is. In a practical system the necessary minimum diffraction efficiency will be determined by the noise level.

1.2 Problem Statement

The purpose of this work is to present a theoretical analysis and experimental verification of the resonated hologram performance.

1.3 Literature Review

The idea of a resonated hologram for use as an efficient optical interconnect is quite recent [4]. But the concepts it draws upon (optical resonators, holograms, and optical interconnects) are well known and are the subject of much current as well as past research. An overview of this work will be given here.

Optical resonators have existed since late in the nineteenth century. The Fabry-Perot interferometer [5] is one of the earliest and best known examples.
Figure 1: Schematic drawing of a resonated hologram.

This parallel plane mirror device is still in use today both in its original form and in multiple mirror configurations [6–8]. The analysis of the simple Fabry-Perot interferometer is well known [5] but there is still work going on today dealing with variations on the simple resonator involving loss [9], spherical mirrors [10], and misalignment properties [11]. Fabry-Perot type resonators are even being created with optical fibers [12–14]. Work on curved mirror resonators was originally done for microwaves by Fox and Li [15] and for millimeter waves through optical wavelengths by Boyd, Kogelnik, and Gordan [16,17]. The results of this work are used in lasers, the most widespread application of optical resonators.

Holograms have not been around as long as optical resonators but they are also an area of much current research. The idea of a hologram was first proposed by Gabor for use in electron microscopy [18,19]. His idea was improved by Leith and Upatnieks with the use of an off-axis reference beam [20–22]. Since then there have
been many analyses to predict the behaviour of holograms. These have ranged from the coupled-wave theory by Kogelnik [23] to the rigorous derivations by Gaylord, Moharam, and Glytsis [24,25] and include work by Solymar, Jordan, and Parry [26–28].

The creation of low loss, low noise, high efficiency holograms is an area of much activity. Many materials and processes are being explored. A lot of work is being done with silver-halide holograms concentrating on the processing [29–31], especially in the areas of developers [32] and bleaches [33,34]. The other emulsion-based hologram getting a lot of attention is dichromated gelatin (DCG). From the beginning work by Shankoff, Curran, and others [35–40] to very recent work by Georgekutty [41] and many people in between [42,43] the processing has been refined, the reliability increased, and the loss decreased. The result is that very low loss, low noise holograms can be made with the DCG process.

The resonated hologram concept merges the areas of holograms with that of resonators. But this is not the first such merger. Pole and Wieder introduced holograms in laser cavities to improve reconstruction and for laser output coupling [44,45]. Lee, Bartholomew, and Cederquist created a system for optical information processing utilizing a Fabry-Perot interferometer with a hologram between the mirrors [46]. In more recent work, holograms have been coupled with resonating phase-conjugate mirrors to form holographic associative memories [47–49]. Here the goal was not efficiency (since phase conjugate mirrors are not efficient devices) but instead the ability to reconstruct images stored within the hologram given only part of the reconstruction beam (i.e. access stored data given only part of its address).

Very recently work was reported by Steier, et. al., [50] on the use of a resonant cavity to enhance optooptical light deflection. This work is very similar to the work
in this dissertation. A real-time hologram was written in a LiNbO$_3$:Fe crystal and placed in a two mirror cavity and also in a ring cavity. The theory was restricted to lossless transmission holograms and only one mirror was assumed to be lossy. Steier reports demonstrating an improvement by a factor of 5.6.

The area of optical interconnects has always been of interest to the field of optical computing but recently has gained interest in the field of electronic computing as well [1]. In the latter field the interest is generated by the inability of conventional wires to perform the number of desired interconnections at the required speeds given the density involved. These applications include internal connections within a chip [51], connections between chips [52–54], and the distribution of clock signals [55].

Various schemes for implementing optical interconnects in both optical and electronic computers have been proposed. These include classical optics (lenses, prisms, and mirrors) [56–58], optical fiber [4,59,60], lenslet arrays [61], and holograms [4,62–64]. Each of these has some advantages and disadvantages.

In this dissertation an analysis of the general four mirror resonated hologram will be pursued. This analysis is valid for a lossy, dielectric, possibly nonreciprocal plane-wave phase hologram resonated with two, three, or four mirrors. It is conducted in the same manner as Collins [63], which is similar in approach to the analyses of Steier [50], Urquhart [14], and Brierley [13], but with fewer initial assumptions so as to have much broader applications than any previous analyses. The analysis for the output power of the resonated hologram procedes in a similar manner to that for the simple Fabry-Perot interferometer [5].

An analysis for the frequency and angular response of the resonated hologram is also conducted and a brief consideration of the temporal response is done. Only Collins [63] has previously considered the angular response of the resonated holo-
gram. The consideration of frequency and temporal response is done here for the first time.

In addition to a theoretical analysis, this dissertation contains experimental results verifying key parts of the theory. These experimental results demonstrate increases in the power output for the resonated hologram by up to a factor of seven times the power output of the unresonated hologram. This is larger than any results which have been previously published.

1.4 Contents of the Dissertation

The remainder of this dissertation contains the theoretical analysis and experimental verification of the resonated hologram concept. Chapter II contains the relevant theory concerning the analysis of the general four mirror resonated hologram for both transmission and reflection holograms. The conditions for maximum output of a two, three, and four mirror resonated hologram will be given and these outputs compared. Finally, the response to small angle and frequency changes and the temporal response will be examined. In Chapter III the experimental verification of key parts of the theory from Chapter II will be presented. The apparatus and procedures used will be presented and the data verifying the theory for two, three, and four mirror resonated transmission holograms will be given. Chapter IV summarizes the results and conclusions of this dissertation and presents some ideas for future work.

The appendices contain information which is important but would distract from the main line of thought if included in the body of the dissertation. Appendix A contains all of the rigorous mathematical proofs, mostly dealing with the optimization of the resonated hologram, that are mentioned elsewhere in the dissertation. In Appendix B is a theoretical analysis of the response of hologram.
It consists of an analysis of a plane wave hologram using the coupled-wave theory in much the same manner as Kogelnik [23]. The results of this analysis are used to derive diffraction and transmission coefficients by which the hologram can be modeled and it is determined when an assumption of symmetric coefficients, as desired for this analysis, can be made. Finally, Appendix C details the creation of the DCG holograms used in Chapter III. The procedures used and the reasons for using them are explained.
CHAPTER II

THEORY

2.1 Introduction

In this chapter the theoretical model for the resonated hologram will be developed. This model will be analyzed for both a transmission and a reflection lossy dielectric hologram. Equations for both output and backscatter will be derived and examined, showing the increase in effective diffraction efficiency which can be obtained by resonating.

The analysis will be conducted first for an extremely general case which has applications to more than just holograms; then various assumptions will be made in order to examine specific cases of interest. The results of this general analysis will be used to analyze a resonated transmission hologram. The outputs for two, three, and four mirror resonated holograms at Bragg incidence will be derived and compared with the unresonated output and with each other. Then it will be shown that all of these results are equally applicable to a resonated reflection hologram. Finally, the response of the resonated hologram to frequency and angle changes will be examined, as well as a brief consideration of temporal response. These will be used to predict the response of the resonated hologram to an input consisting of two distinct frequencies (e.g. laser with two longitudinal modes).
2.2 General Four Mirror Resonated Hologram Model

In this section the model for the four mirror resonated hologram will be presented and a general analysis conducted. The initial assumptions will be given, parameters defined, and the mathematical form of the system presented. This form will be analyzed and the pertinent equations derived. Although a transmission hologram will be assumed in this analysis, it is equally valid for a reflection hologram, as will be shown in Section 2.6.

Initial Assumptions

In any analysis there are some initial assumptions and restrictions necessary to set up a mathematical model. These will be enumerated here.

1. Light is polarized perpendicularly to the plane of the analysis (i.e. TE mode).
   This is assumed to simplify the analysis, but the results are equally valid for parallel-polarized light. The only difference lies in the hologram interaction and this is covered in Appendix B, Section 6.

2. There exist only four propagation directions for the fields. (A hologram at Bragg incidence and mirrors at normal incidence is a sufficient condition for this assumption and will initially be assumed.)

3. Operation is steady-state.

4. System is analyzed for infinite plane waves of the form $e^{-j(k \cdot \vec{x} - \omega t)}$. This simplifies the analysis and is valid if the angular spread of the light is sufficiently small.
These assumptions are sufficient, but not all are necessary, for this analysis to be valid. It is thought that many of them (especially the plane wave assumption) can be loosened although no attempt to pursue that is carried out here.

**Model**

Figure 2 shows a diagram of the four mirror resonated hologram model. It consists of a plane wave hologram of thickness $d$ and two pairs of parallel mirrors, each normal to one of the Bragg angles ($\theta_0$ and $\gamma_0$) of the hologram. Although the fringes are shown orthogonal to the surface, the analysis is not restricted to this case. The hologram is assumed to be a thick dielectric phase hologram although it can be lossy. The mirrors are semitransparent, potentially lossy mirrors. The input beam enters the system through mirror $M_1$. Two baselines, parallel to the two Bragg angles and intersecting in the center of the hologram, will be used in this analysis. They provide a self-consistent set of optical axes. Without loss of generality it will be assumed that the the average index of refraction of the emulsion is the same as that of the surrounding medium. This basic model will be used throughout the analysis although some minor changes (e.g. near Bragg rather than on Bragg incidence) will be made.

The fields which will be used to analyze this system are shown in Figure 3 with arrows indicating their directions of propagation. Inside the resonator the fields are separated into waves traveling toward the hologram ($E_{i-}$) and waves traveling away from the hologram ($E_{i+}$). These fields are evaluated at the point where the baselines intersect the surface of the emulsion. Outside the resonator we have the input wave ($E_{in}$), the transmitted waves ($E_t$), and the backscattered wave ($E_b$), all of which are evaluated at the surface of the mirrors. The backscattered wave is the sum of the fraction of the input wave which is reflected by $M_1$ and the
transmitted wave \( E_1 \) (not shown).

The variables used are defined below:

\[ l_i \equiv \text{Optical distance from mirror } M_i \text{ to the edge of the emulsion along the baseline.} \]

\[ E_{\text{in}} \equiv \text{Input field at incident surface of mirror } M_1. \]

\[ E_i \equiv \text{Transmitted field at exit surface of mirror } M_i \text{ (due to } E_{i+}). \]

\[ E_{\text{b}} \equiv \text{Field of the backscattered light at the exit surface of mirror } M_1. \]

\[ E_{i+} \equiv \text{Field at the surface of the hologram heading toward mirror } M_i. \]

\[ E_{i-} \equiv \text{Field at the surface of the hologram heading away from mirror } M_i. \]

\[ S_i \equiv \text{Field diffraction coefficient for } E_{i-}. \]

\[ T_i \equiv \text{Field transmission coefficient for } E_{i-}. \]

\[ j r_i \equiv \text{Field reflectance coefficient for mirror } M_i \ (r_i \text{ is real}). \]

\[ t_i \equiv \text{Field transmission coefficient for mirror } M_i \ (t_i \text{ is real}). \]

**General Analysis**

The first step in the analysis is to write a series of linear equations relating \( E_{i+}, E_{i-}, E_{\text{in}}, E_i, \) and \( E_{\text{b}} \) based upon the diffraction and transmission of the hologram and the relationships imposed by the mirrors. This analysis will start in the same manner as in the literature [63] but will not make as many assumptions so as to have more general applications. The hologram is modeled by transmission and diffraction coefficients \( T_i \) and \( S_i \) as shown in Figure 4. These coefficients represent the amplitude and phase changes imposed by the hologram. They give the complex value of the fields at the exit surface of the hologram relative to the value of the reference field at the input surface of the hologram evaluated on the baselines. At this point the transmission and diffraction coefficients are taken to
Figure 2: Four mirror resonated transmission hologram model.

Figure 3: Fields for the four mirror resonated hologram.
Figure 4: Diffraction and transmission coefficients for a transmission hologram be different for each of the four directions. Later, an assumption of symmetry will be made in order to obtain further results. These coefficients are discussed more in the next section and an explicit derivation of them is located in Appendix B, Section 4.

The equations modeling the light-hologram interactions are given first.

\begin{align*}
E_{1+} &= T_3 E_3^- + S_4 E_4^- \quad (2.1) \\
E_{2+} &= S_3 E_3^- + T_4 E_4^- \quad (2.2) \\
E_{3+} &= T_1 E_1^- + S_2 E_2^- \quad (2.3) \\
E_{4+} &= S_1 E_1^- + T_2 E_2^- \quad (2.4)
\end{align*}

The relationships between \( E_i^- \) and \( E_i^+ \) imposed by the mirrors are then written.

\begin{align*}
E_{1-} &= j r_1 e^{-j k l_1} E_{1+} + t_1 e^{-j k l_1} E_{\text{in}} \quad (2.5) \\
E_{2-} &= j r_2 e^{-j k l_2} E_{2+} \quad (2.6)
\end{align*}
\[ E_3^- = j r_3 e^{-j k l_3} E_3^+ \] (2.7)
\[ E_4^- = j r_4 e^{-j k l_4} E_4^+ \] (2.8)

The transmitted and backscattered fields are the outputs. They are defined as

\[ E_i = t_i e^{-j k l_i} E_i^+ \] (2.9)
\[ E_b = j r_1 E_{in} + E_1 \] (2.10)

We eliminate \( E_i^- \) by substituting equations (2.5)-(2.8) into (2.1)-(2.4). Then we eliminate \( E_i^+ \) by substituting the resulting equations into equation (2.9) for \( i=1-4 \). This reduces the analysis to a set of four linearly independent equations for the \( E_i \) in terms of the input \( E_{in} \).

\[ 0 = \frac{1}{t_1} e^{j k l_1} E_1 - j \frac{r_3}{t_3} e^{-j k l_3} E_3 - j \frac{r_4}{t_4} e^{-j k l_4} S_4 E_4 \] (2.11)
\[ 0 = \frac{1}{t_2} e^{j k l_2} E_2 - j \frac{r_3}{t_3} e^{-j k l_3} S_3 E_3 - j \frac{r_4}{t_4} e^{-j k l_4} T_4 E_4 \] (2.12)
\[ t_1 e^{-j k l_1} T_1 E_{in} = -j \frac{r_1}{t_1} e^{-j k l_1} T_1 E_1 - j \frac{r_2}{t_2} e^{-j k l_2} S_2 E_2 + \frac{1}{t_3} e^{j k l_3} E_3 \] (2.13)
\[ t_1 e^{-j k l_1} S_1 E_{in} = -j \frac{r_1}{t_1} e^{-j k l_1} S_1 E_1 - j \frac{r_2}{t_2} e^{-j k l_2} T_2 E_2 + \frac{1}{t_4} e^{j k l_4} E_4 \] (2.14)

The next step is to solve for each \( E_i \) in terms of \( E_{in} \) and the parameters of the resonated hologram system. In order to simplify this analysis the parameters will be grouped according to the following definitions.

\[ g_i \equiv -j \frac{r_i}{t_1 t_i} e^{-j k (l_i - l_1)} \frac{E_i}{E_{in}} \] (2.15)
\[ \rho_i \equiv \frac{j}{r_i} e^{j k l_i} \] (2.16)

The \( g_i \) are convenient reduced variables proportional to the relative output at mirror \( M_i \) and the \( \rho_i \) are proportional to the change in the field going from the
hologram to mirror $M_i$ and back. Substituting these definitions into equations (2.11)-(2.14) and placing the resulting system of equations into matrix form will yield

$$
\begin{pmatrix}
0 \\
0 \\
T_1 \\
S_1
\end{pmatrix}
= 
\begin{pmatrix}
\rho_1 & 0 & T_3 & S_4 \\
0 & \rho_2 & S_3 & T_4 \\
T_1 & S_2 & \rho_3 & 0 \\
S_1 & T_2 & 0 & \rho_4
\end{pmatrix}
\begin{pmatrix}
g_1 \\
g_2 \\
g_3 \\
g_4
\end{pmatrix}
\tag{2.17}
$$

This equation can be solved by multiplying both sides by the inverse of the matrix. (The matrix inverse was found using Gaussian elimination.) This yields the solution:

$$
g_i = \frac{b_i}{d} \tag{2.18}
$$

where

$$
b_1 = \frac{(T_1 T_2 - S_1 S_2)(T_3 T_4 - S_3 S_4)}{\rho_1 \rho_2 \rho_3 \rho_4} - \frac{T_1 T_3}{\rho_1 \rho_3} - \frac{S_1 S_4}{\rho_1 \rho_4} \tag{2.19}
$$

$$
b_2 = -\left(\frac{S_1 T_4}{\rho_2 \rho_4} + \frac{T_1 S_3}{\rho_2 \rho_3}\right) \tag{2.20}
$$

$$
b_3 = \frac{1}{\rho_3} \left( T_1 - \frac{T_4(T_1 T_2 - S_1 S_2)}{\rho_2 \rho_4} \right) \tag{2.21}
$$

$$
b_4 = \frac{1}{\rho_4} \left( S_1 + \frac{S_3(T_1 T_2 - S_1 S_2)}{\rho_2 \rho_3}\right) \tag{2.22}
$$

$$
d = 1 - \frac{T_1 T_3}{\rho_1 \rho_3} - \frac{T_2 T_4}{\rho_2 \rho_4} - \frac{S_1 S_4}{\rho_1 \rho_4} - \frac{S_2 S_3}{\rho_2 \rho_3} + \frac{(T_1 T_2 - S_1 S_2)(T_3 T_4 - S_3 S_4)}{\rho_1 \rho_2 \rho_3 \rho_4} \tag{2.23}
$$

Combining equations (2.15), (2.16), and (2.18) gives the relative output fields

$$
\frac{E_i}{E_{in}} = t_1 t_2 e^{-jk(l_1 + l_2)\rho_1 \frac{b_i}{d}}. \tag{2.24}
$$
Substituting equations (2.16) and (2.19)-(2.23) into (2.24) gives us explicit expressions for the \( \frac{E_i}{E_{in}} \).

\[
\frac{E_1}{E_{in}} = \frac{j t_1^2}{D} [(T_1 T_2 - S_1 S_2)(T_3 T_4 - S_3 S_4) r_2 r_3 r_4 e^{-j k 2 (l_1 + l_2 + l_3 + l_4)} \\
+ T_1 T_3 r_3 e^{-j k 2 (l_1 + l_3)} + S_1 S_4 r_4 e^{-j k 2 (l_1 + l_4)}] \quad (2.25)
\]

\[
\frac{E_2}{E_{in}} = \frac{j t_1 t_2}{D} e^{-j k (l_1 - l_2)} [S_1 T_4 r_4 e^{-j k 2 (l_2 + l_4)} + T_1 S_3 r_3 e^{-j k 2 (l_2 + l_3)}] \quad (2.26)
\]

\[
\frac{E_3}{E_{in}} = \frac{t_1 t_3}{D} e^{-j k (l_1 + l_3)} [T_1 + T_4 (T_1 T_2 - S_1 S_2) r_2 r_4 e^{-j k 2 (l_2 + l_4)}] \quad (2.27)
\]

\[
\frac{E_4}{E_{in}} = \frac{t_1 t_4}{D} e^{-j k (l_1 + l_4)} [T_1 - S_1 (T_1 T_2 - S_1 S_2) r_2 r_3 e^{-j k 2 (l_2 + l_3)}] \quad (2.28)
\]

where

\[
D = 1 + T_1 T_3 r_1 r_3 e^{-j k 2 (l_1 + l_3)} + T_2 T_4 r_2 r_4 e^{-j k 2 (l_2 + l_4)} \\
+ S_1 S_4 r_1 r_4 e^{-j k 2 (l_1 + l_4)} + S_2 S_3 r_2 r_3 e^{-j k 2 (l_2 + l_3)} \\
+ (T_1 T_2 - S_1 S_2) (T_3 T_4 - S_3 S_4) r_1 r_2 r_3 r_4 e^{-j k 2 (l_1 + l_2 + l_3 + l_4)} \quad (2.29)
\]

Substituting equations (2.25) and (2.29) for \( \frac{E_i}{E_{in}} \) and \( D \) into the equation for \( E_b \) given by (2.10) yields

\[
\frac{E_b}{E_{in}} = \frac{j}{D} [r_1 + T_1 T_3 (r_1^2 + t_1^2) r_3 e^{-j k 2 (l_1 + l_3)} + T_2 T_4 r_1 r_2 r_4 e^{-j k 2 (l_2 + l_4)} \\
+ S_1 S_4 (r_1^2 + t_1^2) r_4 e^{-j k 2 (l_1 + l_4)} + S_2 S_3 r_1 r_2 r_3 e^{-j k 2 (l_2 + l_3)} \\
+ (T_1 T_2 - S_1 S_2) (T_3 T_4 - S_3 S_4) (r_1^2 + t_1^2) r_2 r_3 r_4 e^{-j k 2 (l_1 + l_2 + l_3 + l_4)}]
\]

\[
(2.30)
\]

Equations (2.25)-(2.30) are the most general results for the four mirror resonated hologram.
2.3 Ideal Transmission Hologram

In order to progress further in the analysis, it will now be assumed that the hologram is a nearly lossless, dielectric transmission hologram at Bragg incidence. An explicit derivation of the diffraction and transmission coefficients for this hologram is given in Appendix B, Section 4 following Kogelnik’s coupled-wave analysis due to a small sinusoidal index of refraction modulation [23]. It is shown in Appendix B, Section 5 that for low loss holograms we can approximate the transmission and diffraction coefficients by

\[ S_1 = S_3 = -j\sqrt{c}S e^{-j(\phi_T + \phi_S)/2} \]  
\[ S_2 = S_4 = -\frac{j}{\sqrt{c}}S e^{-j(\phi_T + \phi_S)/2} \]  
\[ T_1 = T_3 = T e^{-j\phi_T} \]  
\[ T_2 = T_4 = T e^{-j\phi_S} \]  

where

\[ \phi_T \equiv kn/d/\cos \theta_0 \]  
\[ \phi_S \equiv kn/d/|\cos \gamma_0| \]  
\[ c \equiv \sqrt{\cos \theta_0/|\cos \gamma_0|} \]  

\( \theta_0 \) is the angle of incidence of \( E_{1-} \) relative to the hologram normal

\( \gamma_0 \) is the angle of incidence of \( E_{2-} \) relative to the hologram normal

\( n \) is the average index of refraction of the emulsion

\( d \) is the thickness of the emulsion

\( S, T \) are real

\( S \) is a reduced field amplitude diffraction coefficient and \( T \) is a field amplitude transmission coefficient. \( \phi_T \) and \( \phi_S \) are the phase shifts due to translation through
the hologram at the Bragg angles. The absolute value signs are introduced so that these definitions can be used for a reflection hologram as well.

The phases contained in equations (2.25)–(2.30) are grouped into convenient combinations as follows:

\[
\begin{align*}
\phi_0 & \equiv 2k(l_1 + l_3) + 2\phi_T \\
\phi_1 & \equiv 2k(l_2 + l_4) + 2\phi_S \\
\phi_2 & \equiv 2k(l_2 + l_3) + \phi_T + \phi_S \\
\phi_3 & \equiv \phi_0 + \phi_1 - \phi_2 = 2k(l_1 + l_4) + \phi_S + \phi_T \\
\phi_4 & \equiv \phi_1 - \phi_2 = 2k(l_4 - l_3) + \phi_S - \phi_T
\end{align*}
\]  

The first three phases are linearly independent and will be used extensively. However, occasionally it will be useful to use the last two so they are also given here. \(\phi_0, \phi_1, \phi_2, \) and \(\phi_3\) are phases corresponding to the roundtrip pathlength between mirror pairs \(M_1M_3, M_2M_4, M_2M_3,\) and \(M_1M_4\) respectively. \(\phi_4\) is a difference between phases.

These definitions are used to simplify the form of the equations for the transmitted and backscattered fields. Substituting equations (2.31)–(2.34) into the field expressions given by equations (2.26)–(2.30) and utilizing the phase definitions contained in equations (2.38)–(2.40) gives

\[
\begin{align*}
\frac{E_b}{E_{in}} &= \frac{j}{D} \left\{ r_1 + T^2[(r_1^2 + t_1^2)r_3e^{-j\phi_0} + r_1r_2r_4e^{-j\phi_1}] \\
&\quad - S^2[r_1r_2r_3e^{-j\phi_2} + (r_1^2 + t_1^2)r_4e^{-j(\phi_0 + \phi_1 - \phi_2)}] \\
&\quad + (T^2 + S^2)^2(r_1^2 + t_1^2)r_2r_3r_4e^{-j(\phi_0 + \phi_1)} \right\} \\
\frac{E_2}{E_{in}} &= \frac{t_1t_2e^{-j(\phi_0 - \phi_2)/2}}{D} \sqrt{cST}[r_4e^{-j\phi_1} + r_3e^{-j\phi_2}] \\
\frac{E_3}{E_{in}} &= \frac{t_1t_3e^{-j\phi_0/2}}{D} T[1 + (T^2 + S^2)r_2r_4e^{-j\phi_1}]
\end{align*}
\]
\[
\frac{E_4}{E_{\text{in}}} = \frac{j t_1 t_4 e^{-j(\phi_0 + \phi_1 - \phi_2)}}{D}\sqrt{cS[1 - (T^2 + S^2)r_2 r_3 e^{-j\phi_2}]} (2.46)
\]
\[
D = 1 + T^2[r_1 r_3 e^{-j\phi_0} + r_2 r_4 e^{-j\phi_1} - S^2[r_2 r_3 e^{-j\phi_2} + r_1 r_4 e^{-j(\phi_0 + \phi_1 - \phi_2)}] + (T^2 + S^2)r_1 r_2 r_3 r_4 e^{-j(\phi_0 + \phi_1)} (2.47)
\]

It will next be assumed that all of the mirrors are lossless, i.e.
\[
t_i^2 = 1 - r_i^2, (2.48)
\]

and we shall use the intensity diffraction and transmission coefficients for the hologram defined by
\[
S = ||S_1 S_2|| = ||S_3 S_4|| = S^2 (2.49)
\]
\[
T = ||T_1 T_2|| = ||T_3 T_4|| = T^2 (2.50)
\]
\[
A = 1 - T - S (2.51)
\]

where
\[
0 < S, T
\]
\[
0 \leq A.
\]

\(A\) is a quantity characterizing the loss of the hologram due to scattering, absorption, etc.

Thus, equations (2.43)-(2.47) become
\[
\frac{E_\text{ph}}{E_{\text{in}}} = \frac{j}{D}\left\{r_1 + T[r_3 e^{-j\phi_0} + r_1 r_2 r_4 e^{-j\phi_1}]
- S[r_1 r_2 r_3 e^{-j\phi_2} + r_4 e^{-j(\phi_0 + \phi_1 - \phi_2)}]
+ (S + T)^2 r_2 r_3 r_4 e^{-j(\phi_0 + \phi_1)}\right\} (2.52)
\]
\[
= \frac{j}{D}\left\{[r_1 + (1 - A)r_3 e^{-j\phi_0}][1 + (1 - A)r_2 r_4 e^{-j\phi_1}]
\right\}
\]
- $S[r_3 + r_4 e^{-j(\phi_1 - \phi_2)}][e^{-j\phi_0} + r_1 r_2 e^{-j\phi_2}]$ \hfill (2.53)

\[
\frac{1}{\sqrt{c}} \frac{E_2}{E_{in}} = \frac{\sqrt{(1 - r_1^2)(1 - r_2^2)}}{D} e^{-j(\phi_0 - \phi_2)/2} \sqrt{ST} [r_4 e^{-j\phi_1} + r_3 e^{-j\phi_2}]
\] \hfill (2.54)

\[
\frac{E_3}{E_{in}} = \frac{\sqrt{(1 - r_1^2)(1 - r_3^2)}}{D} e^{-j\phi_0/2} \sqrt{T} [1 + (1 - A)r_2 r_4 e^{-j\phi_1}]
\] \hfill (2.55)

\[
\frac{1}{\sqrt{c}} \frac{E_4}{E_{in}} = -j \frac{\sqrt{(1 - r_1^2)(1 - r_4^2)}}{D} e^{-j(\phi_0 + \phi_1 - \phi_2)/2} \sqrt{S} [1 - (1 - A)r_2 r_3 e^{-j\phi_2}]
\] \hfill (2.56)

\[
D = 1 + T[r_1 r_3 e^{-j\phi_0} + r_2 r_4 e^{-j\phi_1}]
- S[r_2 r_3 e^{-j\phi_2} + r_1 r_4 e^{-j(\phi_0 + \phi_1 - \phi_2)}]
+ (T + S)^2 r_1 r_2 r_3 r_4 e^{-j(\phi_0 + \phi_1)}
\] \hfill (2.57)

\[
= [1 + (1 - A)r_1 r_3 e^{-j\phi_0}][1 + (1 - A)r_2 r_4 e^{-j\phi_1}]
- S e^{-j\phi_2}[r_2 + r_1 e^{-j(\phi_0 - \phi_2)}][r_3 + r_4 e^{-j(\phi_1 - \phi_2)}]
\] \hfill (2.58)

These equations represent the output fields of the general resonated hologram, and in fact are valid for any four port optical resonator which can be modeled in this manner.

This is as far as we will take the general analysis. In order to proceed further we will designate just one of the four outputs as being of interest and proceed to maximize it.

**Specific Model of Interest**

At this point the desired output shall be designated as $E_4$, the output of mirror $M_4$. Up till this point in the analysis, any transmitted field could be taken as the desired output. The hologram was designed to make $E_4$ the output field so
it is natural to let the resonated hologram's output be $E_4$. However, there is no reason why $E_2$ could not be chosen as the output. This is an interesting possibility where a resonated transmission hologram could act like a reflection hologram.

The quantities in which we are interested are the relative output power ($P_{\text{out}}$) and the relative power backscattered ($P_b$). These will now be explicitly defined and used throughout the rest of this dissertation. The relative output power represents the efficiency of our system so we will attempt to maximize it. The relative power backscattered represents light sent back to the source and we would like this to be small. No attempt to minimize $P_b$ will be made, we will merely evaluate it when $P_{\text{out}}$ is maximized. The relative output power is evaluated using equation (2.56).

It is given by

$$P_{\text{out}} = \left| \frac{E_4}{E_{\text{in}}} \right|^2$$

$$= \left| \frac{\mathcal{N}}{\mathcal{D}} \right|^2$$

where

$$\mathcal{N} = \sqrt{(1 - r_1^2)(1 - r_4^2)}\sqrt{S[1 - (1 - A)r_2r_3e^{-j\phi_2}]}$$

The relative power backscattered ($P_b$) is evaluated using equation (2.53). It is given by

$$P_b = \left| \frac{E_b}{E_{\text{in}}} \right|^2$$

$$= \left| \frac{\mathcal{N}_b}{\mathcal{D}} \right|^2$$

where

$$\mathcal{N}_b = [r_1 + (1 - A)r_3e^{-j\phi_0}][1 + (1 - A)r_2r_4e^{-j\phi_1}]$$

$$- S[r_3 + r_4e^{-j(\phi_1-\phi_2)}][e^{-j\phi_0} + r_1r_2e^{-j\phi_2}]$$
We are interested in having all of our light exit the system through mirror \( M_4 \). Therefore, we will set \( r_2 = r_3 = 1 \) (i.e. let mirrors \( M_2 \) and \( M_3 \) be completely reflective). Any light exiting through mirrors \( M_2 \) or \( M_3 \) represents lost power. Requiring these outputs \( (E_2 \) and \( E_3 \)) to be zero means that \( r_2 = r_3 = 1 \), by the following argument. Examining equation (2.55) it can be seen that for \( E_3 = 0 \), \( r_3 \) must be unity (because \( r_1, r_4 < 1 \) if there is to be any output). Likewise, examining equation (2.54) it can be seen that for \( E_2 = 0 \) when \( r_3 = 1 \), \( r_2 \) must be unity. Substituting \( r_2 = r_3 = 1 \) into equations (2.58), (2.61), and (2.64) yields

\[
N = \sqrt{(1 - r_1^2)(1 - r_4^2)}\sqrt{S[1 - (1 - \Delta)e^{-j\phi_2}]}
\]

\[
N_b = [r_1 + (1 - \Delta)e^{-j\phi_0}][1 + (1 - \Delta)r_4e^{-j\phi_1}]
- S[1 + r_4e^{-j(\phi_1 - \phi_2)}][e^{-j\phi_0} + r_1e^{-j\phi_2}]
\]

\[
D = [1 + (1 - \Delta)r_1e^{-j\phi_0}][1 + (1 - \Delta)r_4e^{-j\phi_1}]
- S[e^{-j\phi_2}][1 + r_1e^{-j(\phi_0 - \phi_2)}][1 + r_4e^{-j(\phi_1 - \phi_2)}]
\]

\[ (2.65) \]

\[ (2.66) \]

\[ (2.67) \]

2.4 Optimization: Qualitative Discussion

Having set up the model for the resonated hologram we now proceed to optimize the phases and mirror reflectivities in order to maximize the relative output power, \( P_{\text{out}} \). In this section a qualitative analysis will be conducted showing that the value of the phases which maximize \( P_{\text{out}} \), henceforth designated the optimum phases, are not independent of the other system parameters (mirror reflectivity, diffraction efficiency, and hologram loss). In the next section quantitative results will be given.

The conditions for maximizing the output power are more complex for a four mirror resonated hologram than for a simpler resonator (e.g. Fabry-Perot resonator), because a four mirror resonated hologram is in essence three parallel
plane mirror Fabry-Perot resonators coupled by the hologram. For a single two
mirror resonator the optimum phases are independent of the rest of the system
parameters. This is not the case for the resonated hologram configuration, as will
be shown here for the first time. This analysis will lend credence to the quanti-
tative results for the optimum phases cited in the next section. The associated
rigorous mathematical derivation of the optimum phases (and optimum mirror
reflectivities) is conducted in Appendix A, Section 2.

In this analysis we will consider four phases: $\phi_0, \phi_1, \phi_2$, and $\phi_3$. Any three of
these form a linearly independent set from which the fourth (or any other phase
of interest) can be determined. By considering all four, however, we shall see that
a phase analysis which does not take any other parameters into consideration will
lead to a contradiction in the optimum values determined for these phases.

A tuned resonator is commonly said to resonate when the fields from successive
bounces are in phase. In other words, waves at the input of the resonator should
be in phase and waves at the output of the resonator should be in phase. Taking
this as defining the optimum phases we will proceed to determine the optimum
values of $\phi_0, \phi_1, \phi_2,$ and $\phi_3$. We will find that the results we get are contradictory,
as predicted.

In order to determine the phases of these waves we shall examine five paths
within the resonator. These are given below along with their associated phases
and also shown in figure 5. The notation $M_i \rightarrow M_j$ is used to indicate the direct
path from mirror $M_i$ to $M_j$.

\[ M_1 \rightarrow M_4 \ : \ \Phi_{14} \]
\[ M_1 \rightarrow M_3 \rightarrow M_2 \rightarrow M_4 \ : \ \Phi'_{14} \]
\[ M_1 \rightarrow M_3 \rightarrow M_1 \ : \ \Phi_{13} \]
$M_4 \rightarrow M_2 \rightarrow M_4 : \Phi_{42}$

$M_4 \rightarrow M_1 \rightarrow M_4 : \Phi_{41}$

The phases for diffraction and transmission by the hologram are given by equations (2.31)-(2.34). Those for path lengths between mirror pairs are given by equations (2.38)-(2.40). In addition we need the following phase terms:

$$kl_i = \text{phase due to traveling from } M_i \text{ to the emulsion.}$$

$$\pi/2 = \text{phase due to reflection from a mirror.}$$

By adding the phases due to path length, hologram, and reflections we get

$$\Phi_{14} = k(l_1 + l_4) + \left( \frac{\phi_s + \phi_r}{2} + \pi/2 \right)$$

$$\Phi_{14}' = k(l_1 + l_4) + \left( \frac{\phi_s + \phi_r}{2} + \frac{\pi}{2} \right) + \phi_r + \phi_s + 2(\pi/2) + 2k(l_2 + l_3)$$

$$= \phi_2 + \pi + k(l_1 + l_4) + \left( \frac{\phi_s + \phi_r}{2} + \frac{\pi}{2} \right)$$

$$\Phi_{13} = 2k(l_1 + l_3) + 2\phi_r + 2(\pi/2)$$

$$= \phi_0 + \pi$$

$$\Phi_{42} = 2k(l_2 + l_4) + 2\phi_s + 2(\pi/2)$$

$$= \phi_1 + \pi$$

$$\Phi_{41} = 2k(l_1 + l_4) + \left( \frac{\phi_s + \phi_r}{2} + \pi/2 \right) + \left( \frac{\phi_r + \phi_s}{2} + \pi/2 \right) + 2(\pi/2)$$

$$= \phi_3 + 2\pi$$

If the waves at the input of the resonator are to be in phase and waves at the output are to be in phase then

$$\Phi_{14}' - \Phi_{14} = 0 \pmod{2\pi} \quad (2.73)$$

$$\iff \phi_2 = \pi \pmod{2\pi} \quad (2.74)$$
Figure 5: Paths for determining phases.

(i) $\Phi_{14}$, (ii) $\Phi_{14}'$, (iii) $\Phi_{13}$, (iv) $\Phi_{42}$, (v) $\Phi_{41}$
\[ \Phi_{13} = 0 \pmod{2\pi} \]  
\[ \Leftrightarrow \phi_0 = \pi \pmod{2\pi} \]  
\[ \Phi_{42} = 0 \pmod{2\pi} \]  
\[ \Leftrightarrow \phi_1 = \pi \pmod{2\pi} \]  
\[ \Phi_{41} = 0 \pmod{2\pi} \]  
\[ \Leftrightarrow \phi_3 = 0 \pmod{2\pi} \]  
\[ \Phi_{52} = 0 \pmod{2\pi} \]  
\[ \Leftrightarrow \phi_2 = \pi \pmod{2\pi} \]  
\[ \Phi_{51} = 0 \pmod{2\pi} \]  
\[ \Leftrightarrow \phi_4 = 0 \pmod{2\pi} \]  

These results, however, lead to a contradiction, because

\[ \phi_0 + \phi_1 = \phi_2 + \phi_3 \]

\[ \pi + \pi \neq \pi + 0 \pmod{2\pi} \]

This contradiction leads to the conclusion that the approach involving only the phases is inadequate. The optimum values of the phases are not independent of the rest of the parameters of the system. As is cited in the next section and derived in Appendix A, Section 2, the optimum phases depend upon the the mirror reflectivities and the hologram's diffraction efficiency and loss.

To proceed further with this analysis it is necessary to determine the configuration of the resonated hologram: two, three, or four mirrors. This will be considered in the next section where optimum reflectivities will be considered as well as optimum phases.

2.5 Optimization: Two, Three, and Four Mirror Resonated Holograms

We will now divide the general four mirror resonated hologram into three distinct subsets: two, three and four mirror resonated holograms. These are separated by the total number of mirrors used to resonate the hologram. A two mirror resonated hologram is formed by omitting the input and output mirrors \( r_1 = r_4 = 0 \),
as shown in Figure 6. There are two types of three mirror resonated holograms. Type #1 has no output mirror \( (r_4 = 0) \) and type #2 has no input mirror \( (r_1 = 0) \). These are shown in figures 7 and 8. The four mirror resonated hologram has both input and output mirrors. It is shown in Figure 9. With each additional mirror the complexity of the system grows but so does the output. Once these three cases have been analyzed, the results will be compared and the tradeoffs of complexity versus output can be seen.

The goal of these analyses will be to determine \( \mathcal{P}_{\text{out}}' \) and \( \mathcal{P}_{\text{out}}|_{\text{max}}' \). \( \mathcal{P}_{\text{out}}' \) is the maximum value of \( \mathcal{P}_{\text{out}} \) when the reflectivities are assumed to be fixed and only the phases can be adjusted for maximum output. This will be used in Chapter III to predict the output of the experimental apparatus. \( \mathcal{P}_{\text{out}}|_{\text{max}}' \) is the maximum value of \( \mathcal{P}_{\text{out}} \) when both the reflectivities and the phases have been adjusted to maximize the output. We will evaluate the backscattered power, \( \mathcal{P}_b \),
Figure 7: Three mirror resonated hologram #1.

Figure 8: Three mirror resonated hologram #2.
when the output has been maximized.

Two Mirror Resonated Hologram

A two mirror resonated hologram (2MRH) is formed by letting $r_1 = r_4 = 0$ ($r_2 = r_3 = 1$ as before), as shown in Figure 6. Substituting these values into equations (2.65)-(2.67) yields the relevant equations for the two mirror resonated hologram.

$$N = \sqrt{S}[1 - (1 - \mathcal{A})e^{-j\phi_2}]$$ (2.81)

$$N_b = e^{-j\phi_0}(1 - \mathcal{A} - \mathcal{S})$$ (2.82)

$$\mathcal{D} = 1 - \mathcal{S}e^{-j\phi_2}$$ (2.83)
Using these to evaluate the relative output power, $P_{\text{out}}$, and the relative power backscattered, $P_b$, yields

$$P_{\text{out}} = \frac{S[1 + (1 - A)^2 - 2(1 - A) \cos \phi_2]}{1 + S^2 - 2S \cos \phi_2} \quad (2.84)$$

$$P_b = \frac{(1 - A - S)^2}{1 + S^2 - 2S \cos \phi_2} \quad (2.85)$$

$P_{\text{out}}$ is a function of the hologram parameters $S$ and $A$ and phase $\phi_2$. We will maximize it with respect to $\phi_2$ by taking the derivative with respect to $\phi_2$ of equation (2.84), setting it equal to zero, and solving for $\phi'_2$, the value of $\phi_2$ which gives a global maximum for $P_{\text{out}}$. This procedure (shown in detail in Appendix A, Section 2) gives the result:

$$\phi'_2 = \pi \text{ (mod } 2\pi) \quad (2.86)$$

$$\Rightarrow P_{\text{out}}|_{\text{max}} = S \left(\frac{2 - A}{1 + S}\right)^2 \quad (2.87)$$

The prime on $\phi_2$ is a convention which shall be used for all phases and reflectivities to designate the optimum value, that which maximizes the relative output power.

Evaluating $P_b$ at this point by substituting $\phi_2 = \pi$ into equation (2.85) yields

$$P_b = \left(\frac{1 - A - S}{1 + S}\right)^2 \quad (2.88)$$

If the hologram is truly lossless then $A = 0$. In this case these results reduce to:

$$P_{\text{out}}|_{\text{max}} = \frac{4S}{(1 + S)^2} \quad (2.89)$$

$$P_b = \left(\frac{1 - S}{1 + S}\right)^2 \quad (2.90)$$

Equations (2.87)–(2.90) give a measure of the optimum performance that can be expected from the two mirror resonated hologram. They will be used to compare
the performance of the two mirror resonated hologram with that of the three and four mirror resonated holograms.

**Three Mirror Resonated Hologram**

A three mirror resonated hologram (3MRH) can be formed in one of two ways: set $r_1 = 0$ or $r_4 = 0$. Both cases will be examined. It will be found that the maximum output is the same for both although the backscatter will differ. In our analysis we will first optimize $P_{\text{out}}$ in terms of phases and mirror reflectivity and then consider the case where mirror reflectivity $r$ is fixed and only the phases can be optimized.

The three mirror resonated hologram #1 has no output mirror ($r_4 = 0$), as shown in Figure 7. Substituting $r_4 = 0$ and $r_1 = r$ into equations (2.65)-(2.67) yields

\[
N = \sqrt{1-r^2}\sqrt{S[1-(1-A)e^{-j\phi_2}]}
\]  
\[N_b = r + (1-A-S)e^{-j\phi_0} - Se^{-j\phi_2}
\]
\[D = 1 + (1-A-S)re^{-j\phi_0} - Se^{-j\phi_2}
\]

$P_{\text{out}}$ can be maximized in terms of $\phi_0, \phi_2$, and $r$ by setting the derivative with respect to these variables equal to zero, solving for the values $\phi'_0, \phi'_2$, and $r'$, and substituting these values back into the expression for $P_{\text{out}}$ to obtain $P_{\text{out}}^\text{max}$. This is done explicitly in Appendix A, Section 2 and yields

\[
\phi'_0 = \pi \quad \text{(mod } 2\pi) \tag{2.94}
\]
\[
\phi'_2 = \pi \quad \text{(mod } 2\pi) \tag{2.95}
\]
\[
r' = \frac{1-A-S}{1+S} \tag{2.96}
\]
\[ \Rightarrow P_{\text{out}} \big|_{\text{max}} = \frac{S(2-A)}{2S+A} \quad (2.97) \]
\[ P_b = 0 \quad (2.98) \]

Notice that for this optimum case all of the power exiting the resonator does so through the output mirror.

We will now consider \( r \) to have a predetermined fixed value rather than be a variable. \( P_{\text{out}} \) can be maximized for \( \phi_0 \) and \( \phi_2 \) using the same procedure as before. It is shown in Appendix A, Section 2 that the optimum value of these phases is dependent upon the value of \( r \) (as was indicated by our phase analysis in the previous section). There exists a boundary value for \( r \)

\[ r_b = \frac{1-S(1-A)}{(1+S)(1-A)} \quad (2.99) \]

at which the character of the solution changes. If \( r \leq r_b \), case \((i)\), then the phases are independent of \( r \) and have the same values as before.

\[ \phi'_0 = \pi \quad (\text{mod } 2\pi) \quad (2.100) \]
\[ \phi'_2 = \pi \quad (\text{mod } 2\pi) \quad (2.101) \]
\[ \Rightarrow P'_{\text{out},i} = \frac{S(1-r^2)(2-A)^2}{[1+S-(1-A+S)r]^2} \quad (2.102) \]

where the subscript \( i \) denotes the solution for case \((i)\). But if \( r \geq r_b \), case \((ii)\), then

\[ \phi'_2 = \cos^{-1} \left\{ \frac{1}{2S} \left[ 1 + S^2 - \left( \frac{1-S(1-A)}{(1-A)r} \right)^2 \right] \right\} \quad (2.103) \]
\[ \phi'_0 = \sin^{-1} \left[ \frac{S \sin \phi'_2}{\sqrt{1+S^2-2S \cos \phi'_2}} \right] \quad (2.104) \]
\[ = \cos^{-1} \left[ \frac{S \cos \phi'_2 - 1}{\sqrt{1+S^2-2S \cos \phi'_2}} \right] \quad (2.105) \]
\[ p'_{\text{out},ii} = \frac{1 - S(1 - \mathcal{A})}{1 - \mathcal{A} - S + \frac{\mathcal{A}(2 - \mathcal{A})}{(1 - \mathcal{A})(1 - r^2)}} \]  

(2.106)

Notice that when \( r = 0 \) these results reduce to the two mirror resonated hologram case.

It is interesting to examine the performance of this lossless three mirror resonated hologram. When \( \mathcal{A} = 0 \) we have

\[ r' = r_b = \frac{1 - S}{1 + S} \]  

(2.107)

\[ p'_{\text{out}}|_{\text{max}} = p'_{\text{out},ii}(r) = 1 \]  

(2.108)

This means that we can achieve 100% output for a range of reflectivities \( r_b \leq r < 1 \) regardless of how small the diffraction efficiency is. One can compensate for any value of \( r \) in this range by changing the phases according to equations (2.103)–(2.105). This is an unexpected but highly desirable result.

Now the three mirror resonated hologram #2 can be considered. It has no input mirror \( (r_1 = 0) \) as shown in Figure 8. Substituting \( r_1 = 0 \) and \( r_4 = r \) into equations (2.65)–(2.67) gives

\[ N = \sqrt{1 - r^2}\sqrt{S[1 - (1 - \mathcal{A})e^{-j\phi_2}]} \]  

(2.109)

\[ N_b = e^{-j\phi_0}[1 - \mathcal{A} - S + (1 - \mathcal{A})^2re^{-j\phi_1} - Se^{-j(\phi_1 - \phi_2)}] \]  

(2.110)

\[ D = 1 + (1 - \mathcal{A} - S)re^{-j\phi_1} - Se^{-j\phi_2} \]  

(2.111)

Notice that the expressions for \( N \) and \( D \) (equations (2.109) and (2.111)) are the same as for the three mirror resonated hologram #1 (equations (2.91) and (2.93)) if \( \phi_1 \) is replaced by \( \phi_0 \), as one might expect. Consequently, we can use the same solution for optimizing with respect to \( \phi_1, \phi_2 \), and \( r \):

\[ \phi'_1 = \pi \pmod{2\pi} \]  

(2.112)
\[ \phi_2' = \pi \pmod{2\pi} \quad (2.113) \]

\[ r' = \frac{1 - A - S}{1 + S} \quad (2.114) \]

\[ \Rightarrow P_{\text{out}}|_{\text{max}} = \frac{S(2 - A)}{2S + A} \quad (2.115) \]

However, the expression for the power backscattered (equation (2.110)) is different from that for the three mirror resonated hologram \#1 (equation (2.92)). Thus, we get

\[ P_b = \left( \frac{A(1 - A - S)}{2S + A} \right)^2 \quad (2.116) \]

which is not necessarily zero.

For the lossless case these results reduce to

\[ r' = \frac{1 - S}{1 + S} \quad (2.117) \]

\[ P_b = 0 \quad (2.118) \]

\[ P_{\text{out}}|_{\text{max}} = 1 \quad (2.119) \]

as expected.

For the case where the value of \( r \) has a predetermined, fixed value the results for \( P'_{\text{out}} \) are the same as for the three mirror resonated hologram \#1. They are given by equations (2.100)-(2.106) where \( \phi_0 \) is replaced by \( \phi_1 \).

It is easy to see that the maximum output of the three mirror resonated hologram is always greater than that of the two mirror resonated hologram. Equation (2.97) for \( P_{\text{out}}|_{\text{max}} \) for the three mirror resonated hologram can be rewritten in the form

\[ P_{\text{out}}|_{\text{max}} = \frac{S(2 - A)^2}{(1 + S)^2 - (1 - A - S)^2} \quad (2.120) \]

which is obviously greater than \( P_{\text{out}}|_{\text{max}} \) for the two mirror resonated hologram which is given by equation (2.87).
Four Mirror Resonated Hologram

The four mirror resonated hologram (4MRH) is the most general form of the resonated hologram. It is the most complex but yields the greatest output if the hologram is not lossless.

The derivation for determining $P_{\text{out}}|_{\text{max}}$ is similar to that employed for the two and three mirror resonated holograms. It is not simple, because of the presence of two independent reflectivities ($r_1$ and $r_4$) and three independent phases ($\phi_0, \phi_1$, and $\phi_2$). The details are given in Appendix A, Section 2; merely the results are given here.

\[
\begin{align*}
\phi_0 &= \phi'_1 = \phi'_2 = \pi \pmod{2\pi} \quad (2.121) \\
\quad r'_1 &= r'_4 = r' = \frac{2}{H} \left(1 - \sqrt{1 - H}\right) - 1 \quad (2.122)
\end{align*}
\]

where

\[
H = \frac{4(1 - A - S)}{(2 - A)^2} \quad (2.123)
\]

\[
\Rightarrow P_{\text{out}}|_{\text{max}} = \frac{S(2 - A)^2(1 - r'^2)^2}{\left[1 - (1 - A)r'^2 + S(1 + r'^2)^2\right]^2} \quad (2.124)
\]

\[
P_{\text{b}}|_{\text{min}} = 0 \quad (2.125)
\]

Substituting the expression for $r'$ into the equation for $P_{\text{out}}|_{\text{max}}$ does not yield a simple expression and hence is not done. The parameter $H$ has no physical significance but it is used merely to simplify the equations.

$P_{\text{out}}'$ is determined as a function of $r$ for the case where $r_1 = r_4 = r$. This restriction enables the functions to be solved analytically over part of the range of reflectivities and a reasonable approximation determined for the rest. This restriction is reasonable since equal reflectivities are required for achieving maximum output for a resonated lossy hologram. Even with this restriction the analysis is
quite complex. It is conducted in Appendix A, Section 2; merely the results will be given here.

We find that there is a critical value of reflectivity at which the character of the solution for $\mathcal{P}_{\text{out}}'$ changes, just as with the three mirror resonated hologram. This critical value is designated $r_c$. For $r \leq r_c$, case (i), we solve explicitly for $\mathcal{P}_{\text{out}}'$, which we will designate $\mathcal{P}_{\text{out},i}'$. For $r \geq r_c$, case (ii), due to the nonlinearity of the equations we obtain a reasonable approximation for $\mathcal{P}_{\text{out}}'$, which we will designate $\mathcal{P}_{\text{out},ii}'$. This approximation is good for low loss holograms and is exact when $\mathcal{A} = 0$.

$$r_c \equiv \frac{1 + \mathcal{S} - (2 - \mathcal{A})\sqrt{\mathcal{S} + \mathcal{S}^2}}{1 - \mathcal{A} - \mathcal{S}}$$  (2.126)

If $r \leq r_c$:

$$\phi_0' = \phi_1' = \phi_2' = \pi \pmod{2\pi}$$  (2.127)

$$\Rightarrow \mathcal{P}_{\text{out},i}' = \frac{\mathcal{S}(2 - \mathcal{A})^2(1 - r^2)^2}{\left[1 - (1 - \mathcal{A})r^2 + \mathcal{S}(1 + r^2)^2\right]^2}$$  (2.128)

If $r \geq r_c$:

$$\phi_2' \simeq \pi \pmod{2\pi}$$  (2.129)

$$\phi_0' = \phi_1' \simeq \cos^{-1}\left\{\frac{(1 - \mathcal{A} - \mathcal{S})(1 + r^2)}{[1 + 2\mathcal{S} + (1 - \mathcal{A})^2]r}\right\}$$  (2.130)

$$\mathcal{P}_{\text{out},ii}' \simeq \frac{(1 + \mathcal{S})[\mathcal{S} + (1 - \mathcal{A})^2][1 - r^2]^2}{\left[1 + \mathcal{S} - [\mathcal{S} + (1 - \mathcal{A})^2]r^2\right]^2}$$  (2.131)

Once again we will examine the response of a lossless hologram by letting $\mathcal{A} = 0$. Substituting this into equations (2.122) and (2.124) gives

$$r' = \frac{1 - \sqrt{\mathcal{S}}}{1 + \sqrt{\mathcal{S}}}$$  (2.132)

$$\mathcal{P}_{\text{out}}|_{\text{max}} = 1$$  (2.133)
These results are identical to those which have been published [63]. However, substituting $\mathcal{A} = 0$ into equations (2.126) and (2.131) yields

$$r_c = r'$$  \hspace{1cm} (2.134)

$$P'_{\text{out},ii}(r) = 1$$  \hspace{1cm} (2.135)

for all values of $r$, $r_c \leq r < 1$. These results are seen here for the first time. They mean that there is a range of reflectivities for which 100% output is obtainable. As with the three mirror resonated hologram one can compensate for any reflectivity in this range by adjusting the phases according to equations (2.129)–(2.130). These results are not unexpected, considering the results for the three mirror resonated hologram, but it is certainly worth noting. Comparing this range with that for the three mirror resonator ($r_b = \frac{1-S}{1+S}$) one finds that this range of reflectivities is much larger.

$P_{\text{out}}|_{\text{max}}$ for the four mirror resonated hologram is greater than or equal to $P_{\text{out}}|_{\text{max}}$ for the three mirror resonated hologram if $0 \leq \mathcal{A} \leq .2$, the range of interest. Equality holds only for the lossless case. This can be seen for in figures 10–12, where $P_{\text{out}}|_{\text{max}}$ is plotted for both the three and four mirror resonated holograms for $\mathcal{A} = 0,.1,$ and $.2$.

**Comparison of Resonated Holograms**

To easily see the tradeoff between the output and complexity of the two, three, and four mirror resonated holograms the results which have been derived shall be plotted for various parameters. $P_{\text{out}}|_{\text{max}}$ is plotted as a function of the hologram diffraction efficiency, $S$, for the two, three, and four mirror resonated holograms as well as for an unresonated hologram (0 mirrors). This is done for $\mathcal{A} = 0,.1,$ and $.2$ in figures 10–12 respectively. It can be seen that the resonated
Figure 10: Comparison of $P_{\text{out}}|_{\text{max}}$ for 0, 2, 3, and 4 mirror resonated holograms. $A = 0$.

Figure 11: Comparison of $P_{\text{out}}|_{\text{max}}$ for 0, 2, 3, and 4 mirror resonated holograms. $A = 0.1$. 
holograms will have a much higher output than the unresonated hologram and with each additional mirror the output increases. With lower diffraction efficiencies the percentage increase is greater. Although the four mirror resonated hologram has a larger output than the three mirror resonated hologram in the presence of loss, this increase is not very large unless the diffraction efficiency is very small. Comparing these graphs it can be seen that as the loss increases the outputs decrease, especially for low diffraction efficiencies. But they are still significantly higher than the unresonated hologram.

We will also look at the dependence of the output upon mirror reflectivity by plotting $P_{out}^I$. Figures 13–16 show $P_{out}^I$ as a function of the power reflectivity, $r^2$, for the three and four mirror holograms for several values of $S$ and $A$. The two mirror resonated hologram is implicitly included in this set by including the point where $r^2 = 0$. It can be seen that the optimum reflectivity for the three mirror

Figure 12: Comparison of $P_{out}|_{\text{max}}$ for 0, 2, 3, and 4 mirror resonated holograms. $A = 0.2$. 

$A = 0$.
resonated hologram is much higher than for the four mirror resonated hologram for the same diffraction efficiency. As the loss increases the optimum reflectivity decreases slightly for the four mirror resonated hologram, but quite a bit for the three mirror resonated hologram.

It can be seen that there are some definite tradeoffs between the number of mirrors used in the resonated hologram, the sensitivity of the resonated hologram to loss, and the expected output. Hence, the optimum resonated hologram will depend upon the situation for which it is to be used.

2.6 Ideal Reflection Hologram

In this section it will be shown that a resonated reflection hologram has the same response as the resonated transmission hologram which has been considered. The functional dependence of a reflection hologram upon the hologram parameters
Figure 14: $P_{\text{out}}'$ for a four mirror resonated hologram as a function of mirror reflectivity. $S = 0.05$

Figure 15: $P_{\text{out}}'$ for a three mirror resonated hologram as a function of mirror reflectivity. $S = 0.2$
Figure 16: $P_{\text{out}}'$ for a four mirror resonated hologram as a function of mirror reflectivity. $S = 0.2$

(thickness, index of refraction change, etc.) is different than that for a transmission hologram. When the expression for the relative output power of a resonated reflection hologram is expressed just in terms of the hologram's external parameters ($S$ and $A$) and the resonator's parameters we will find that it is identical to the expression for the relative output power of a resonated transmission hologram. It will be shown that a resonated reflection hologram generates the same set of field equations as a resonated transmission hologram and hence all of the results of the previous sections are equally true for a resonated reflection hologram.

Figure 17 shows a resonated reflection hologram. Just as in Figure 2, the input is mirror $M_1$, the output is mirror $M_4$, and the baselines are parallel to the Bragg angles and intersect in the center of the emulsion. The holographic plate has been rotated $90^\circ$ but since no coordinate system was used (just path lengths) in the original analysis this will not make a difference. The definitions of $\theta_o$ and
\( \gamma_0 \) remain the same: they are measured from the surface normal to the baselines of \( M_3 \) and \( M_4 \) respectively. Because we are dealing with a reflection hologram, \( \cos \theta_0 \) will now be negative but this will not affect our equations since they only depend upon \( |\cos \gamma_0| \).

The fields are still as defined in Figure 3 for the resonated transmission hologram. The geometries for the diffraction and transmission coefficients are shown in Figure 18. They are defined in the same manner as for the transmission hologram.

The series of linear equations relating \( E_{i+}, E_{i-}, E_i \), and \( E_b \) to \( E_{in} \) which are generated by this model are identical to equations (2.1)-(2.10). This means that the solution for the transmitted and backscattered waves given by equations (2.26)-(2.30) apply to a resonated reflection hologram.

Furthermore, for a nearly lossless dielectric reflection hologram, the transmis-
Figure 18: Diffraction and transmission coefficients for a reflection hologram
sion and diffraction coefficients can be expressed in the form

\[
S_1 = S_3 = -j \sqrt{c} S e^{-j(\phi_r + \phi_s)/2} \tag{2.136}
\]

\[
S_2 = S_4 = -\frac{j}{\sqrt{c}} S e^{-j(\phi_r + \phi_s)/2} \tag{2.137}
\]

\[
T_1 = T_3 = T e^{-j\phi_r} \tag{2.138}
\]

\[
T_2 = T_4 = T e^{-j\phi_s} \tag{2.139}
\]

as verified in Appendix B, Section 4. Notice that these are identical to equations (2.31)–(2.34). Of course, the functional dependence of \( S \) and \( T \) upon the hologram parameters (e.g. emulsion thickness, maximum index of refraction change, etc.) is different for a transmission hologram than for a reflection hologram but both have the same external form.

Consequently, all of the results derived in this chapter are equally applicable to both transmission and reflection holograms. Henceforth, the type of hologram used shall be identified only if it is relevant.

2.7 Frequency, Angle, and Temporal Response of the Resonated Hologram

The analysis conducted so far has assumed a single frequency of light, Bragg angle incidence at the hologram, and normal incidence at the mirrors. Now the effect of small changes in the frequency or angle upon the resonated hologram output will be examined and the results used for a brief consideration of the temporal response.

Both a frequency shift and angle shift, if small, result merely in a simultaneous shift of all three phases, \( \phi_0, \phi_1, \) and \( \phi_2 \), although not necessarily all by the same amount. Consequently, the same general equations for relative output power are valid for both and are dealt with in this section. First we will derive the equations
for the response of the resonated hologram to shifts in all three phases from their optimum values. This analysis will be restricted to the range of reflectivities for which the optimum phase values are $\pi$ (the usual case). This will then be evaluated for the special case (implemented experimentally) where the phase shifts are equal for all three phases. For this case we will define a half-power phase halfwidth and evaluate it for a three and four mirror resonated hologram. Next we will look specifically at the phase shifts due to a change in frequency. For the special case where all three phase shifts are equal the relative output power will be graphed as a function of the change in frequency. In addition, an expression for the half-power bandwidth of the resonated hologram will be derived. Then we will examine the phase shifts due to small changes in the input angle. For the special case where all three phase shifts are equal the relative output power will be graphed as a function of the change in input angle and an expression for the half-power angular halfwidth will be derived. Finally, we will define a time constant for the resonated hologram based upon the bandwidth. Throughout this section we will restrict our analysis to frequency and angle shifts which are sufficiently small that the hologram characteristics are unchanged.

Effects of Phase Shifts on $P_{\text{out}}$

In Section 2.3, equations (2.58), (2.60), and (2.61), the output of the resonated hologram was determined as a function of $\phi_0$, $\phi_1$, and $\phi_2$. We shall use these expressions to determine the response of the resonated hologram to shifts in these phases from their optimum value. In order to keep this analysis as general as possible, we shall not assume that the reflectivities have been optimized, merely that they are small enough that the optimum phases are $\phi_0 = \phi_1 = \phi_2 = \pi \pmod{2\pi}$. 

48
Deviations from the optimum phases can be expressed as

\[
\phi_0 = \pi + \Delta \phi_0 \pmod{2\pi} \quad (2.140)
\]
\[
\phi_1 = \pi + \Delta \phi_1 \pmod{2\pi} \quad (2.141)
\]
\[
\phi_2 = \pi + \Delta \phi_2 \pmod{2\pi} \quad (2.142)
\]

Substituting these equations into equations (2.58), (2.60), and (2.61) for \( P_{\text{out}}, N, \) and \( D \) yields

\[
P_{\text{out}} = \frac{S(1 - r_1^2)(1 - r_4^2)[1 + (1 - A)^2 + 2(1 - A)\cos \Delta \phi_2]}{\|D\|^2} \quad (2.143)
\]

where

\[
\|D\|^2 = 1 + S^2 + (1 - A - S)^2(r_1^2 + r_4^2) + [S^2 + (1 - A)^4]r_1^2r_4^2
\]
\[
- 2(1 - A - S)r_1[1 + (1 - A)^2r_4^2] \cos \Delta \phi_0
\]
\[
- 2(1 - A - S)r_4[1 + (1 - A)^2r_1^2] \cos \Delta \phi_1
\]
\[
+ 2S[1 + (1 - A)^2r_1^2r_4^2] \cos \Delta \phi_2
\]
\[
+ 2(1 - A)^2r_1r_4 \cos(\Delta \phi_0 + \Delta \phi_1)
\]
\[
+ 2(1 - A - S)^2r_1r_4 \cos(\Delta \phi_0 - \Delta \phi_1)
\]
\[
- 2S(1 - A - S)r_1(1 + r_4^2) \cos(\Delta \phi_0 - \Delta \phi_2)
\]
\[
- 2S(1 - A - S)r_4(1 + r_1^2) \cos(\Delta \phi_1 - \Delta \phi_2)
\]
\[
+ 2S[1 + (1 - A)^2]r_1r_4 \cos(\Delta \phi_0 + \Delta \phi_1 - \Delta \phi_2)
\]
\[
+ 2S^2r_1r_4 \cos(\Delta \phi_0 + \Delta \phi_1 - 2\Delta \phi_2) \quad (2.144)
\]

We will now restrict ourselves to the case when the phase shifts are all equal. The conditions which generate this will be covered when we deal with the frequency and angle shifts. Equations (2.143) and (2.144) will be evaluated for both the three and four mirror resonated holograms. For a three mirror resonated hologram with
equal phase shifts \( r_1 = r, r_4 = 0 \), and \( \Delta \phi_0 = \Delta \phi_2 = \Delta \phi \) or else \( r_1 = 0, r_4 = r \), and \( \Delta \phi_1 = \Delta \phi_2 = \Delta \phi \). Substituting either of these conditions into equations (2.143) and (2.144) yields

\[
P_{\text{out},3}(\Delta \phi) = \frac{S(1 - r^2)[1 + (1 - A)^2 + 2(1 - A) \cos \Delta \phi]}{1 + [S - (1 - A - S)r]^2 + 2[S - (1 - A - S)r] \cos \Delta \phi}
\]

(2.145)

For a four mirror resonated hologram with equal mirror reflectivities \( r_1 = r_4 = r \) and equal phase shifts \( \Delta \phi_0 = \Delta \phi_1 = \Delta \phi_2 = \Delta \phi \) we have

\[
P_{\text{out},4}(\Delta \phi) = \frac{S(1 - r^2)[1 + (1 - A)^2 + 2(1 - A) \cos \Delta \phi]}{||D(\Delta \phi)||^2}
\]

(2.146)

where

\[
||D(\Delta \phi)||^2 = [1 - (1 - A)^2 r^2]^2 + [S(1 + r)^2 - 2(1 - A)r]^2 \\
+ 2[1 + (1 - A)^2 r^2][S(1 + r)^2 - 2(1 - A)r] \cos \Delta \phi \\
+ 2(1 - A)^2 r^2[\cos(2\Delta \phi) + 1]
\]

(2.147)

\[
= [1 - (1 - A)^2 r^2]^2 + [S(1 + r)^2 - 2(1 - A)r]^2 \\
+ 2[1 + (1 - A)^2 r^2][S(1 + r)^2 - 2(1 - A)r] \cos \Delta \phi \\
+ 4(1 - A)^2 r^2 \cos^2 \Delta \phi
\]

(2.148)

These equations will be used in the next two subsections.

We will now use these results to define a half-power phase halfwidth, designated \( \Delta \phi_{1/2} \), for the resonated hologram. The half-power phase halfwidth is defined as the value of \( \Delta \phi \) at which the output power equals half the maximum. In other words,

\[
P_{\text{out}}(\Delta \phi_{1/2}) = \frac{1}{2}P_{\text{out}}(0).
\]

(2.149)

We shall evaluate this half-power phase halfwidth for both the three and four mirror resonated holograms.
First we will derive the half-power phase halfwidth for the three mirror resonated hologram.

\[ P_{\text{out},3}(\Delta \phi_{1/2}) = \frac{1}{2} P_{\text{out},3}(0) \]
\[ S(1 - r^2)[1 + (1 - A)^2 + 2(1 - A)\cos \Delta \phi_{1/2}] \]
\[ \Rightarrow \frac{S(1 - r^2)[1 + (1 - A)^2 + 2(1 - A)\cos \Delta \phi_{1/2}]}{1 + [S - (1 - A - S)r]^2 + 2[S - (1 - A - S)r] \cos \Delta \phi_{1/2}} \]
\[ \Rightarrow \frac{(2 - A)^2 \left[ 1 + [S - (1 - A - S)r]^2 + 2[S - (1 - A - S)r] \cos \Delta \phi_{1/2} \right]}{2[1 + S - (1 - A - S)r]^2} \]
\[ \Rightarrow \cos \Delta \phi_{1/2} = \frac{(2 - A)^2 \left[ 1 + [S - (1 - A - S)r]^2 - 2[1 + S - (1 - A - S)r]^2[1 + (1 - A)^2] \right]}{4(1 - A)[1 + S - (1 - A - S)r]^2 - 2(2 - A)^2[S - (1 - A - S)r]} \]

The optimum mirror reflectivity for the three mirror resonated hologram is given by equation (2.96), \( r' = \frac{1 - A - S}{1 + S} \). Substituting this into equation (2.153) and manipulating slightly yields

\[ \cos \Delta \phi_{1/2} = \frac{2(1 + S)^2 - 2(1 + S)(2S + A)(2 - A) - (2S + A)^2 A^2}{2(1 + S)^2 - 2(1 + S)(2S + A)(2 - A) + 4(1 - A)(2S + A)^2} \]

\[ \Rightarrow \Delta \phi_{1/2} = \cos^{-1} \left\{ \frac{(2S + A)^2 A^2}{2(2S + A)(1 - A)(1 + S)^2} \right\} \]

For the lossless case \( (A = 0) \) this reduces to

\[ \Delta \phi_{1/2} = \cos^{-1} \left\{ \frac{1}{1 + \frac{8S^2}{(1 + S)(1 - 3S)}} \right\} \]

These equations will be used in the subsequent subsections.
We will now repeat this procedure to determine the half-power phase halfwidth for the four mirror resonated hologram.

\[ P_{\text{out},4}(\Delta \phi_{\frac{1}{2}}) = \frac{1}{2} P_{\text{out},4}(0) \]

\[ S(1 - r^2)^2 \left[ 1 + (1 - A)^2 + 2(1 - A) \cos \Delta \phi_{\frac{1}{2}} \right] \]

\[ \|D(\Delta \phi_{\frac{1}{2}})\|^2 \]

\[ = \frac{S(1 - r^2)^2(2 - A)^2}{2 \left[ [1 - (1 - A)r]^2 + S(1 + r)^2 \right]^2} \]

\[ \|D(\Delta \phi_{\frac{1}{2}})\|^2 \]

\[ = 2 \left[ [1 - (1 - A)r]^2 + S(1 + r)^2 \right]^2 \left[ 1 + (1 - A)^2 + 2(1 - A) \cos \Delta \phi_{\frac{1}{2}} \right] \]

(2.159)

where \( \|D(\Delta \phi_{\frac{1}{2}})\|^2 \) is given by equation (2.148). Substituting equation (2.148) into (2.159) and rearranging yields

\[ a \cos^2 \Delta \phi_{\frac{1}{2}} + 2b \cos \Delta \phi_{\frac{1}{2}} + c = 0 \]

(2.160)

where

\[ a = 4(2 - A)^2(1 - A)^2r^2 \]

(2.161)

\[ b = (2 - A)^2[1 + (1 - A)^2r^2][S(1 + r)^2 - 2(1 - A)r] \]

(2.162)

\[ c = (2 - A)^2 \left[ [1 - (1 - A)r]^2 + S(1 + r)^2 - 2(1 - A)r]^2 \right] - 2 \left[ [1 - (1 - A)r]^2 + S(1 + r)^2 \right]^2 \left[ 1 + (1 - A)^2 \right]. \]

(2.163)

Utilizing the well known solution for a quadratic equation yields

\[ \cos \Delta \phi_{\frac{1}{2}} = \frac{-b \pm \sqrt{b^2 - ac}}{a} \]

(2.164)

\[ \Delta \phi_{\frac{1}{2}} = \cos^{-1} \left\{ \frac{-b \pm \sqrt{b^2 - ac}}{a} \right\} \]

(2.165)

52
We will evaluate this for the lossless case when the mirror reflectivity has been set to its optimum value (given by equation (2.132)).

\[ r = \frac{1 - \sqrt{S}}{1 + \sqrt{S}} \quad (2.166) \]
\[ a = 16r^2 \quad (2.167) \]
\[ b = 4(1 + r^2)[S(1 + r)^2 - 2r] - 2[(1 - r)^2 + S(1 + r)^2]^2 \quad (2.168) \]
\[ c = 4(1 - r^2)^2 + 4[S(1 + r)^2 - 2r]^2 - 4[(1 - r)^2 + S(1 + r)^2]^2 \quad (2.169) \]

Substituting equation (2.166) into the equations for \( a, b, \) and \( c \) yields

\[ a = \frac{16(1 - \sqrt{S})^2}{(1 + \sqrt{S})^2} \quad (2.170) \]
\[ a = \frac{16(1 - S)^2}{(1 + \sqrt{S})^4} \quad (2.171) \]
\[ b = \frac{16(1 + S)(3S - 1) - 128S^2}{(1 + \sqrt{S})^4} \quad (2.172) \]
\[ b = \frac{16}{(1 + \sqrt{S})^4}(1 - 2S + 5S^2) \quad (2.173) \]
\[ c = \frac{4}{(1 + \sqrt{S})^4}[16S + 4(3S - 1)^2 - 64S^2] \quad (2.174) \]
\[ c = \frac{16}{(1 + \sqrt{S})^4}(1 - 2S - 7S^2) \quad (2.175) \]
\[ \Rightarrow b^2 - ac = \frac{256}{(1 + \sqrt{S})^8}[(1 - 2S + 5S^2)^2 - (1 - 2S + S^2)(1 - 2S - 7S^2)] \quad (2.176) \]
\[ \Rightarrow b^2 - ac = \left[ \frac{64S}{(1 + \sqrt{S})^4} \right]^2 [(1 - S)^2 + S^2] \quad (2.177) \]

Substituting equations (2.171), (2.173), (2.175) and (2.177) into equation (2.165) yields

\[ \Delta \phi_1 = \cos^{-1} \left\{ \frac{1 - 2S + 5S^2 \pm 4S\sqrt{(1 - S)^2 + S^2}}{(1 - S)^2} \right\} \quad (2.178) \]
where the negative root was chosen because the argument for arccosine must be less than or equal to one.

The expressions for the half-power phase halfwidth will be used in the next three sections to determine the half-power bandwidth, angular half-power halfwidth, and time constant for the resonated hologram.

Frequency Response

We will determine the frequency response of the resonated hologram by deriving the phase dependence upon the frequency. With this information we will be able to use the equations for the response of the resonated hologram to phase shifts in order to determine the frequency response of the resonated hologram.

To determine the phase shift due to a change in frequency we will rewrite the equations for phases \( \phi_0, \phi_1, \) and \( \phi_2 \) (equations (2.38)-(2.40)) to show their explicit dependence upon the frequency, \( f \).

\[
\phi_0 = \frac{4\pi}{c} \left( l_1 + l_3 + \frac{nd}{\cos \theta_o} \right) f
\]

\[
= \frac{4\pi}{c} (L_1 + L_3)f
\]

\[
\phi_1 = \frac{4\pi}{c} \left( l_2 + l_4 + \frac{nd}{\cos \gamma_o} \right) f
\]

\[
= \frac{4\pi}{c} (L_2 + L_4)f
\]

\[
\phi_2 = \frac{4\pi}{c} \left( l_2 + l_3 + \frac{nd}{2} \left( \frac{1}{\cos \theta_o} + \frac{1}{\cos \gamma_o} \right) \right) f
\]

\[
= \frac{4\pi}{c} (L_2 + L_3)f
\]

where

\[
c \equiv \text{speed of light in free space}
\]
$L_1 \equiv l_1 + \frac{nd}{2 \cos \theta_0}$  \hspace{1cm} (2.186)

$L_2 \equiv l_2 + \frac{nd}{2 \cos \gamma_0}$  \hspace{1cm} (2.187)

$L_3 \equiv l_3 + \frac{nd}{2 \cos \theta_0}$  \hspace{1cm} (2.188)

$L_4 \equiv l_4 + \frac{nd}{2 \cos \gamma_0}$  \hspace{1cm} (2.189)

To obtain the dependence of the phases upon changes in frequency a perturbational analysis will be made. Let

\[ f = f_0 + \Delta f \]  \hspace{1cm} (2.190)

where $f_0$ is the frequency for which the resonated hologram has been optimized. Then the phases can be expressed as the sum of two terms: a constant term and a linear perturbational term.

\[ \phi_0 = \phi_{0,o} + \Delta \phi_0 \]  \hspace{1cm} (2.191)

\[ \phi_1 = \phi_{1,o} + \Delta \phi_1 \]  \hspace{1cm} (2.192)

\[ \phi_2 = \phi_{2,o} + \Delta \phi_2 \]  \hspace{1cm} (2.193)

where

\[ \phi_{0,o} = \frac{4\pi}{c} (L_1 + L_3) f_0 = \pi \pmod{2\pi} \]  \hspace{1cm} (2.194)

\[ \phi_{1,o} = \frac{4\pi}{c} (L_2 + L_4) f_0 = \pi \pmod{2\pi} \]  \hspace{1cm} (2.195)

\[ \phi_{2,o} = \frac{4\pi}{c} (L_2 + L_3) f_0 = \pi \pmod{2\pi} \]  \hspace{1cm} (2.196)

\[ \Delta \phi_0 = \frac{4\pi}{c} (L_1 + L_3) \Delta f \]  \hspace{1cm} (2.197)

\[ \Delta \phi_1 = \frac{4\pi}{c} (L_2 + L_4) \Delta f \]  \hspace{1cm} (2.198)

\[ \Delta \phi_2 = \frac{4\pi}{c} (L_2 + L_3) \Delta f \]  \hspace{1cm} (2.199)
We will now examine the special case used experimentally where \( L_1 = L_2 \) and \( L_3 = L_4 \). We can define a length, \( \mathcal{L} \), by

\[
\mathcal{L} \equiv L_1 + L_3 = L_2 + L_4 = L_2 + L_3
\]  

(2.200)

The phases are now given by

\[
\phi_0 = \phi_1 = \phi_2 = \pi + \Delta \phi
\]  

(2.201)

where

\[
\Delta \phi = \frac{4 \pi}{c} \mathcal{L} \Delta f
\]  

(2.202)

The frequency response is obtained by replacing \( \Delta \phi \) in the expressions for \( P_{\text{out}} \) derived in the previous subsection (equations (2.145)–(2.148)), by equation (2.202). This response is plotted in Figure 19 for a lossless hologram with diffraction efficiency of 0.2 and optimized mirror reflectivities. In this plot the change in frequency, \( \Delta f \), is normalized to

\[
\Delta f_0 = \frac{c}{2 \mathcal{L}}
\]  

(2.203)

the spectral free range for this system. We will use these results in Section 7 to consider the response of the resonated hologram to an input with two frequencies.

We determine the half-power bandwidth of the resonated hologram using the half-power phase halfwidth. Designate the half-power bandwidth by \( \Delta f_{1/2} \). We define the half-power bandwidth using the relationship between frequency and phase given by equation (2.202).

\[
\Delta f_{1/2} = 2 \left( \frac{c}{4 \pi \mathcal{L}} \Delta \phi_{1/2} \right)
\]  

(2.204)

\[
= \frac{1}{\pi} \Delta \phi_{1/2} \Delta f_0
\]  

(2.205)
Figure 19: Frequency response of three and four mirror resonated lossless holograms ($S = 0.2$) plotted as a function of $\frac{\Delta f}{\Delta f_0}$ where $\Delta f_0 = \frac{c}{2L}$.

where $\Delta \phi_2$ is given by equation (2.155) for the three mirror resonated hologram and equation (2.165) for the four mirror resonated hologram.

The half-power bandwidth, normalized by the spectral free range, is plotted in Figure 20 for both the three and four mirror lossless resonated holograms with optimum mirror reflectivities. Notice that the three mirror resonated hologram has a narrower frequency bandwidth than the four mirror resonated hologram. This is largely due to the fact that the optimum reflectivity for the mirror of a three mirror resonated hologram is much higher than the optimum reflectivity for the mirrors of the four mirror resonated hologram.

It should be noted that the magnitude of the frequency deviation $\Delta f$ is constrained by the implicit assumption that the hologram performance is unchanged. In other words, the frequency deviation must be small relative to the frequency
Figure 20: Normalized half-power bandwidth plotted as a function of diffraction efficiency for three and four mirror lossless resonated holograms with optimum mirror reflectivities.

bandwidth of the hologram measured at the Bragg angle. This assumption is valid for the laser source used in the experimental verification of the resonated hologram performance presented in Chapter III. For that laser $\Delta f \approx 10^{-6}$ and the half power bandwidth of the holograms is $\Delta f / f_0 \approx 0.02$. This is computed using the measured value of the hologram half-power angular halfwidth ($\approx 15$ mrad) and equation (B.21) (Appendix B, Section 3) relating the hologram response due to a frequency shift to the response due to an angle shift.

Angular Response

In this subsection we will determine the analytical expression for the angular response of the resonated hologram. As with the frequency response, we will first express the phases as functions of changes in the incident angle and then use the equations which we have already derived for the phase dependence of the resonated
hologram in order to determine the angular dependence. In a similar manner, we will derive the half-power angular halfwidth of the resonated hologram.

We will now consider the dependence of the phases upon a change in incident angle. As in the rest of this section we shall assume that the optimum phases are all equal to $\pi$ but beyond that we will not constrain the reflectivities.

The first step in this analysis is to verify that the resonated hologram analysis still applies for small changes in the incident angle. Our initial analysis was based upon the assumption that there were only four propagation directions for the fields. We shall now verify that this assumption still holds for small changes in the incident angle.

We shall define the incident angle, $\theta$, in terms of the Bragg angle, $\theta_o$, and the deviation from the Bragg angle, $\Delta \theta_o$.

$$\theta = \theta_o + \Delta \theta \quad (2.206)$$

The diffracted angle will be handled the same way.

$$\gamma = \gamma_o + \Delta \gamma \quad (2.207)$$

To derive the relationship between $\Delta \gamma$ and $\Delta \theta$ we will use the well known grating coupling equation

$$\vec{\sigma} = \vec{\rho} - \vec{K} \quad (2.208)$$

where $\vec{\rho}$ and $\vec{\sigma}$ are the propagation vectors of the incident and diffracted waves respectively and $\vec{K}$ is the grating vector. Taking just the component parallel to the surface of the emulsion this becomes

$$\sin \gamma = \sin \theta - \frac{K}{k} \sin \phi_g \quad (2.209)$$
where $\phi_g$ is the angle between the grating vector and the emulsion normal. Expanding this equation in a first order Taylor series about the Bragg angles $\theta_o$ and $\gamma_o$ yields

$$\Delta \gamma \approx \Delta \theta \frac{\cos \theta_o}{\cos \gamma_o}$$

which is valid if $|\Delta \theta|, |\Delta \gamma| \ll \pi$.

With the relationship between $\Delta \theta$ and $\Delta \gamma$ determined we can proceed to verify that only four propagation directions still exist. Look at a round trip from mirror $M_1$ to $M_4$ and back, as shown in Figure 21. Light is incident at angle $\theta_o + \Delta \theta$, diffracts at angle $\gamma_o + \Delta \gamma$, reflects off of mirror $M_4$ at angle $\gamma_o + \Delta \gamma'$, diffracts at angle $\theta_o + \Delta \theta'$, and finally reflects off of mirror $M_1$ at angle $\theta_o + \Delta \theta''$.

$$\Delta \gamma' = -\Delta \gamma$$

$$= -\Delta \theta \frac{\cos \theta_o}{\cos \gamma_o}$$

$$\Rightarrow \Delta \theta' = \Delta \gamma' \frac{\cos \gamma_o}{\cos \theta_o}$$

$$= -\Delta \theta$$

$$\Rightarrow \Delta \theta'' = \Delta \theta$$

Therefore, after a round trip the resulting light beam is parallel to the incident beam. An analysis of the rest of the system will show similar results. There is going to be sidestep but we will still have only four directions of propagation for the fields and hence can use the resonated hologram analysis in this situation. Referring to the fields shown in Figure 3, the associated angles of the propagation vectors are:

$$E_{1-}, E_{3+} : \theta_o + \Delta \theta$$

$$E_{2-}, E_{4+} : \gamma_o + \Delta \gamma$$
Figure 21: Round trip path for light incident at non-Bragg angle.

\[ E_{3-}, E_{1+} : \theta_o - \Delta \theta \]  
\[ E_{4-}, E_{2+} : \gamma_o - \Delta \gamma \]  

Now that we know our equations for the system are still valid, we can use them to determine the phase shift associated with the change in angle. To do this without resorting to a complex mathematical analysis we shall utilize the results of the well known analysis for a Fabry-Perot parallel plane mirror resonator at non-normal incidence for small angles [5]. We can do this because we are concerned just with the phases \( \phi_0, \phi_1, \) and \( \phi_2, \) each of which is associated with just two mirrors.

Each pair of mirrors in the resonated hologram act like a Fabry-Perot parallel plane mirror resonator. This is obvious for mirror pairs \( M_1M_3 \) and \( M_2M_4 \) since they are physically parallel. The presence of the hologram possessing diffraction and loss is equivalent to having a lossy cavity. The cavity loss is equivalent to the sum of the diffraction and loss of the hologram. For mirror pairs \( M_2M_3 \) and \( M_1M_4 \)
it may not be as obvious. But if one considers the fact that the axis of the system
is normal to both mirrors one can consider them parallel as well, for all practical
purposes. The hologram serves to connect these two axes as a single path. For
these pairs the hologram transmission and loss combine as the equivalent of loss
in a lossy resonator cavity.

In a parallel plane mirror Fabry-Perot resonator a slight deviation from normal
incidence will manifest itself as a slight decrease in the phase on axis. For an angle
of $\Delta \theta$ this decrease in phase is given by

$$\Delta \phi \approx -\frac{2\pi}{\lambda} L(\Delta \theta)^2$$

(2.220)

for a cavity of length $L$ at wavelength $\lambda$.

For mirror pair $M_1M_3$ the axis is defined by the Bragg angle $\theta_0$ and the change
in angle is $\Delta \theta$. Therefore, the phase shift in $\phi_0$ is given by

$$\Delta \phi_0 = -\frac{2\pi}{\lambda} (L_1 + L_3)(\Delta \theta)^2$$

(2.221)

For mirror pair $M_2M_4$ the axis is defined by the Bragg angle $\gamma_0$ and the change
in angle is $\Delta \gamma$. Therefore, the phase shift in $\phi_1$ is given by

$$\Delta \phi_1 = -\frac{2\pi}{\lambda} (L_2 + L_4)(\Delta \gamma)^2$$

(2.222)

For mirror pair $M_2M_3$ it is a little less straight forward. From $M_2$ to the
center of the hologram the axis is defined by the Bragg angle $\gamma_0$ and the deviation
angle is $\Delta \gamma$. From the center of the hologram to $M_3$ the axis is defined by the
Bragg angle $\theta_0$ and the deviation angle is $\Delta \theta$. This gives a phase shift in $\phi_2$ of

$$\Delta \phi_2 = -\frac{2\pi}{\lambda} [L_2(\Delta \gamma)^2 + L_3(\Delta \theta)^2]$$

(2.223)
We can express all of these phase shifts in terms of $\Delta \theta$ by using equation (2.210) to substitute for $\Delta \gamma$. This yields

\[
\Delta \phi_0 = -\frac{2\pi}{\lambda} (L_1 + L_3)(\Delta \theta)^2 \quad (2.224)
\]

\[
\Delta \phi_1 = -\frac{2\pi}{\lambda} \left[ L_2 \frac{\cos^2 \theta_0}{\cos^2 \gamma_0} + L_4 \frac{\cos^2 \theta_0}{\cos^2 \gamma_0} \right] (\Delta \theta)^2 \quad (2.225)
\]

\[
\Delta \phi_2 = -\frac{2\pi}{\lambda} \left[ L_2 \frac{\cos^2 \theta_0}{\cos^2 \gamma_0} + L_3 \right] (\Delta \theta)^2 \quad (2.226)
\]

We will now consider the special case where $L_1 \cos^2 \gamma_0 = L_2 \cos^2 \theta_0$ and $L_3 \cos^2 \gamma_0 = L_4 \cos^2 \theta_0$. We will define a length, $L_0$, by

\[
L_0 = L_1 + L_3 = (L_2 + L_4) \frac{\cos^2 \theta_0}{\cos^2 \gamma_0} = L_2 \frac{\cos^2 \theta_0}{\cos^2 \gamma_0} + L_3 \quad (2.227)
\]

and a phase, $\Delta \phi$, by

\[
\Delta \phi = -\frac{2\pi}{\lambda} L_0 (\Delta \theta)^2 \quad (2.228)
\]

We have assumed that the phases equal $\pi$ when $\Delta \theta = 0$, so, for the special case being considered, we can write the phases as

\[
\phi_0 = \phi_1 = \phi_2 = \pi + \Delta \phi \quad (\text{mod } 2\pi). \quad (2.229)
\]

The angular response of the resonated hologram is obtained by replacing $\Delta \phi$ in the expressions for the phase dependence, equations (2.145)–(2.148). This response is plotted in Figure 22 as a function of the change in input angle for both the three and four mirror lossless resonated holograms with optimum mirror reflectivities when $S = .2$. We have normalized $\Delta \theta$ to the quantity

\[
\Delta \theta_0 = \sqrt{\lambda/(2L_0)}, \quad (2.230)
\]

the half angle width to the first null. Notice that the response is fairly flat for small values of $\Delta \theta$ and then drops off rapidly.
Figure 22: Angular response of three and four mirror resonated lossless holograms ($S = .2$) plotted as a function of $\frac{\Delta \theta}{\Delta \theta_0}$ where $\Delta \theta_0 = \left(\frac{\lambda}{2L_0}\right)^\frac{1}{2}$

We also define a half-power angular halfwidth. Designate this as $\Delta \theta_1$. We define the half-power angular halfwidth using the relationship between the phase and the change in angle, given by equation (2.228).

$$\Delta \theta_1 = \sqrt{\frac{\lambda}{2\pi L_0} \Delta \phi_1}$$

$$= \sqrt{\frac{1}{\pi} \frac{1}{\Delta \phi_1} \Delta \theta_o}$$

where $\Delta \phi_1$ is given by equation (2.155) for the three mirror resonated hologram and equation (2.165) for the four mirror resonated hologram.

The normalized half-power angular halfwidth is plotted in Figure 23 for both three and four mirror lossless resonated holograms with optimum mirror reflectivity as a function of the diffraction efficiency. Notice that the half-power angular halfwidth is narrower for the three mirror resonated hologram than for the four
mirror resonated hologram because the optimum mirror reflectivity is much higher.

This analysis is limited to small angles where small was defined as $|\Delta \theta|, |\Delta \gamma| \ll \pi$. But actually, there is a much tighter tolerance caused by the implicit assumption that the hologram characteristics do not change. This means that $|\Delta \theta| \ll \Delta \theta_{1/2}$ where $\Delta \theta_{1/2}$ is the half angle width of the hologram. For measured values of the transmission holograms used in Chapter III, $\Delta \theta_{1/2} \simeq 15$ mrad. Thus, this is a very restrictive assumption. However, for the resonated hologram used in Chapter III, $\Delta \theta_o \simeq 1$ mrad, so the graphs in Figure 22 are reasonably accurate over the range shown.

**Temporal Response**

In this subsection we will define a time constant for the resonated hologram using our knowledge of its bandwidth. This will serve as an estimate of the response
time of the resonated hologram, a quantity of much interest because of the ultimate use of this device in the communication part of a computing system.

A common estimate of the time constant of a system is given by the inverse of its bandwidth. We have defined the bandwidth of the resonated hologram, $\Delta f_1$, previously in this section (equation (2.204)). Taking the inverse of this will give us a reasonable estimate of the time constant for the resonated hologram.

$$\tau \equiv (\Delta f_1)^{-1}$$ (2.233)

$$= \left(\frac{c}{2L}\Delta \phi_1\right)^{-1}$$ (2.234)

$$= (\Delta \phi_1)^{-1}\tau_0$$ (2.235)

where

$$\tau_0 = \frac{2L}{c}$$ (2.236)

and $\Delta \phi_1$ is given by equation (2.155) for the three mirror resonated hologram and equation (2.165) for the four mirror resonated hologram. $\tau_0$ is the roundtrip time between mirror pairs $M_1M_3$, $M_2M_4$, or $M_2M_3$.

The normalized time constant is plotted in Figure 24 as a function of the diffraction efficiency for both three and four mirror lossless resonated holograms with optimum mirror reflectivities. For the resonated holograms used in Chapter III the time constants are $\approx 1$ ns.

### 2.8 Dual Frequency Response

This section considers the response of the resonated hologram to an input with two frequencies. The laser used to gather the experimental data contained in Chapter III has two longitudinal modes which means two distinct frequencies of light. Thus, the results of this section will be used to predict the expected output
of our experimental system. Because the relative power distribution between the two modes of the laser is unknown, we shall derive an expression for the least upper bound and greatest lower bound upon the maximum output of the resonated hologram for the range of possible power distributions. The results will be applicable to both the three and four mirror resonated holograms.

In order to model this situation we define the two frequencies and the input power distribution between them. Let the average of the two frequencies be \( f_1 \) and their separation be \( \Delta f_1 \). Then the two frequencies are \( f_1 \pm \Delta f_1/2 \). Define \( \rho \) as the ratio of the input power at frequency \( f_1 + \Delta f_1/2 \) to the total input power.

In this analysis there will be two sets of phases, one for each of the frequencies. They will be represented by the same symbols (\( \phi_0, \phi_1 \), and \( \phi_2 \)) but will be explicitly distinguished when necessary. We will again consider the special case of equal path
lengths between mirror pairs, discussed in the previous section. We can use the results of the previous section to state that for each frequency

\[ \phi_0 = \phi_1 = \phi_2 = \phi. \]  

(2.237)

The path lengths will be allowed to vary slightly in order to adjust the phases for maximum output of the resonated hologram in the presence of two frequencies. The path length is defined by \( L = L_0 + \Delta L \) where \( L_0 \) is the length which sets the phases equal to \( \pi \) (mod \( 2\pi \)) at frequency \( f_1 \).

The phases for the two frequencies are given by

\[
\phi = \frac{4\pi}{c} (L_0 + \Delta L)(f_1 \pm \Delta f_1/2) \\
= \phi_o \pm \phi_f + \Delta \phi_L \pm \Delta \phi_{Lf}
\]  

(2.238)

(2.239)

where

\[
\phi_o = \frac{4\pi}{c} L_0 f_1 = \pi \quad (\text{mod } 2\pi) \]  

(2.240)

\[
\phi_f = \frac{2\pi}{c} L_0 \Delta f_1 \quad (\text{constant}) \]  

(2.241)

\[
\Delta \phi_L = \frac{4\pi}{c} \Delta L f_1 \quad (\text{variable}) \]  

(2.242)

\[
\Delta \phi_{Lf} = \frac{2\pi}{c} \Delta L \Delta f_1 \quad (\text{cross term}) \]  

(2.243)

The cross term can be neglected because we have assumed that \( |\Delta f_1| \ll f_1 \) (to keep the hologram characteristics constant) which means that \( |\Delta \phi_{Lf}| \ll |\Delta \phi_L| \), and we restrict ourselves to the case where \( |\Delta \phi_L| < \pi \), since the path length is changed just to tune the resonator phases. This means that \( \Delta \phi_{Lf} \ll \pi \) and can be neglected. So the phases are given by

\[
\phi = \pi + \Delta \phi_L \pm \phi_f.
\]  

(2.244)
In deriving an expression for the output power we shall assume that the two frequencies are far enough apart that any interaction between them varies fast enough to average to zero and hence can be neglected. This lack of a cross term means that we can add the output powers at each frequency in order to determine the total output power with a dual frequency input. In other words,

$$P_{\text{out,2f}}(\rho, \Delta \phi_L) = (1 - \rho)P_{\text{out}}(\Delta \phi_L - \phi_f) + \rho P_{\text{out}}(\Delta \phi_L + \phi_f)$$  \hspace{1cm} (2.245)

where $P_{\text{out,2f}}$ is the output of the resonated hologram in response to a two frequency input and $P_{\text{out}}$ is the output of the resonated hologram in response to a single frequency input. As was derived in the last section, $P_{\text{out}}$ is given by equation (2.145) for a three mirror resonated hologram and equations (2.146) and (2.148) for a four mirror resonated hologram.

A graph of this situation is located in Figure 25. The relative output power at each of the two frequencies, given by $(1 - \rho)P_{\text{out}}(\Delta \phi - \phi_f)$ and $\rho P_{\text{out}}(\Delta \phi + \phi_f)$, are shown as well as their sum, $P_{\text{out,2f}}(\rho, \Delta \phi)$, plotted as a function of $\Delta \phi$ for a specific value of power distribution $\rho$. Notice that $P_{\text{out,2f}}(\rho, \Delta \phi)$ has a maximum at the point $\Delta \phi_L(\rho)$. The value at this point is designated $P'_{\text{out,2f}}(\rho)$.

The maximum output power, $P'_{\text{out,2f}}(\rho)$, is obtained by tuning the phases of the resonated hologram (it is assumed that the mirror reflectivities have fixed, predetermined values). Because the power distribution factor, $\rho$, between the two frequencies is unknown, we are unable to predict a precise value for $P'_{\text{out,2f}}$. But we are able to determine bounds upon this value.

In Appendix A, Section 4 a rigorous derivation of the least upper bound and greatest lower bound on $P'_{\text{out,2f}}(\rho)$ is given, based upon the fact that the expression for $P_{\text{out}}(\phi)$ is symmetric about $\phi = 0$. It is shown there that the least upper bound on $P'_{\text{out,2f}}(\rho)$ is given by $P'_{\text{out,2f}}(0)$, the case where all of the input
Figure 25: Dual frequency response curves for the resonated hologram.
power is at a single frequency. The greatest lower bound is given by \( P'_{\text{out},2f}(0.5) \), the case where the input power is evenly divided between the two frequencies. Both of these can be seen intuitively from the graphs in Figure 25.

The next step is to evaluate these bounds in terms of \( P_{\text{out}} \). From our definition of \( P'_{\text{out},2f} \) it is apparent that

\[
P'_{\text{out},2f}(0) = P_{\text{out}}(0)
\]

(2.246)

The value of \( P'_{\text{out},2f}(0.5) \) is not as obvious. When the power is split equally between the two modes (\( \rho = 0.5 \)) we have

\[
P_{\text{out},2f} = \frac{P_{\text{out}}(\Delta \phi_L - \phi_f) + P_{\text{out}}(\Delta \phi_L + \phi_f)}{2}
\]

(2.247)

To maximize this for the general case requires substituting the exact expressions for \( P_{\text{out}} \) and solving for the optimum value of \( \Delta \phi_L \) in terms of \( \phi_f, S, A, \) and \( r \). But since we are only really interested in the case when \( \phi_f = 0.83 \text{ rad} \) (equation (2.241) evaluated when \( L_0 = 58 \text{ mm}, \Delta f_1 = 685 \text{ MHz} \)), the value for our experimental system in Chapter III, we can simplify this process. For the range of interest \( P_{\text{out}} \) is a concave downwards function (i.e. negative second derivative) and symmetric about zero. Therefore, when \( \phi_f = 0.83 \text{ rad} \), \( P_{\text{out},2f} \) is maximized when \( \Delta \phi_L = 0 \). This means

\[
P'_{\text{out},2f}(0.5) = P_{\text{out}}(\phi_f)
\]

(2.248)

The upper and lower bounds, \( P_{\text{out}}(0) \) and \( P_{\text{out}}(\phi_f) \), are plotted in figures 26 and 27 for the three and four mirror resonated holograms, respectively, where \( P_{\text{out}} \) is given by equation (2.145) for the three mirror resonated hologram and equations (2.146) and (2.148) for the four mirror resonated hologram. These curves are plotted as a function of mirror reflectivity for a hologram with diffraction efficiency
Figure 26: Upper and lower bounds upon the maximum output power of a lossy three mirror resonated hologram when the input consists of two frequencies. $S = .2$, $A = .1$, $\phi_f = .83$

$S = .2$ and loss $A = .1$. Such curves will be used to predict the bounds upon the experimental data obtained in Chapter III.

2.9 Summary

This chapter has dealt with the theory for a general four mirror resonated hologram. The theory was derived and evaluated for both a transmission and reflection hologram and it was shown that the same equations apply to both types of resonated holograms.

The expressions for the relative output power were obtained for two, three, and four mirror resonated holograms. The results for these were compared to each other and to the unresonated hologram. It was found that the maximum output of the resonated hologram is significantly greater than that of the unresonated holo-
Figure 27: Upper and lower bounds upon the maximum output power of a lossy four mirror resonated hologram when the input consists of two frequencies. 
\[ S = .2, A = .1, \phi_f = .83 \]

gram and it increases with each additional mirror, although the difference between the three and four mirror resonated holograms is significant only for holograms with very low diffraction efficiencies. The percentage increase is greatest for low diffraction efficiencies. The resonated hologram is most sensitive to loss if the diffraction efficiency is small or the mirror reflectivities are large. The optimum reflectivity for the three mirror resonated hologram is much larger than for the four mirror resonated hologram.

Then the effects of small frequency and angle shifts were examined. The frequency and angular response of the resonated hologram were derived. The half-power bandwidth and angular half-width were determined and a time constant for the resonated hologram was defined. Finally, the frequency response was used
to predict the performance of the experimental system implemented in Chapter III when a laser with two longitudinal modes is used as a light source.
CHAPTER III
EXPERIMENT

3.1 Introduction

The resonated hologram apparatus and experimental data are now presented, verifying experimentally key elements of the theory presented in Chapter II. First the system will be described, then the experimental procedure will be explained, and finally the results will be given and compared to those predicted by the theory.

Two transmission holograms will be tested for both the three and four mirror resonators. Four data points, obtained with various reflectivities for the semi-transparent mirrors, will be plotted for each case. This data will show that the experimental output agrees with that predicted by the theory of Chapter II.

3.2 Equipment

In this section the apparatus will be presented and discussed. A diagram and two photos of the system will be given. Each component will be identified and its function explained. Diagrams of the supporting electronic circuitry will also be included. The design constraints affecting component choices and mounting techniques will then be considered.

The resonated hologram demonstration apparatus is shown in Figures 28–30. Figure 28 is a schematic drawing of the system, showing the components and light paths. The area enclosed by the dashed lines is the resonated hologram. The rest
Figure 28: Resonated Hologram Demonstration Apparatus schematic.
Figure 29: Photograph of the Resonated Hologram Demonstration Apparatus.
of the system is support and data taking apparatus. Figure 29 is a photograph of this system. The photograph in Figure 30 is a close-up of the resonated hologram subsystem. These will now be described in more detail.

System Description

This system consists of an input beam, the resonator cavity, the hologram, an output beam, and associated components to align the system and take the data, all of which are located on a granite table. Their function is most easily understood by referring to the schematic in Figure 28 but of course they can also be found in the photographs of Figures 29 and 30. Specifications of the components used can be found in Table 1.

The input beam is generated using a 2 mW helium-neon laser located in the upper left of Figure 28. This beam is spatially filtered by lens $L_1$ and pinhole $P_1$ and then collimated by lens $L_2$. It is directed onto the hologram by mirrors $M_5$
Table 1: Components of the Resonated Hologram Demonstration Apparatus

<table>
<thead>
<tr>
<th>Component</th>
<th>Description</th>
</tr>
</thead>
</table>
| Laser     | Helium-neon laser, Hughes, model 1222H-PC, 2 mW  
  $\lambda = .6328$ nm, $\Delta f = 685$ MHz |
| $L_1$     | 20x microscope objective |
| $P_1$     | 25 $\mu$m pinhole |
| $L_2$     | Planoconvex lens— $\phi$25.4 mm, f.l.=175 mm |
| $I_1, I_2$| Irises |
| $H$       | DCG hologram — made with two plane waves |
| $M_1, M_4$| Semitransparent planar mirrors, $\phi$25.4 mm, t=5.7 mm, quartz,  
  S.F.=$\frac{\lambda}{10}$, AR on back, various reflectivities |
| $M_2, M_3$| Planar dielectric mirrors — $\phi$25.4 mm, t=9.5 mm, Zeodur  
  S.F.=$\frac{\lambda}{10}$, $r^2 = 100\%$ |
| $M_5, \ldots, M_8$ | Aluminum planar mirrors |
| $M_9, M_{10}$ | Semitransparent planar mirrors, $r^2 \simeq .6$ |
| $F$       | Optical flat — $\phi$50 mm, t=12 mm, S.F.=$\frac{\lambda}{10}$ |
| $P_2$     | 200 $\mu$m pinhole |
| $D_r$     | Quantrad detector — 100-PV-BNC |
| $D_o$     | Detector — active area = .75 mm x .75 mm |
and $M_6$, and then stopped down to the size of the hologram by iris $I_1$. Iris $I_2$, found next to the collimating lens $L_2$, is only used when aligning the resonator mirrors and hence is not present when the system is operating.

The optical flat (F), located in the center of Figure 28, is utilized in the monitoring of the backscatter from the resonated hologram as well as in the monitoring of the input beam. The backscattered light from the resonator reflected by the optical flat is observed on screen #1. This light is merely observed visually. The input beam reflection is used to monitor the power of the input beam and the power distribution between the two longitudinal modes of the laser. The optical flat is 12.7 mm thick and the beam diameter is 6 mm, hence the front and back surface reflections from the optical flat are spatially separated.

The front surface reflection of the input beam from the optical flat is used to monitor the input power. This beam is incident upon detector $D_r$, a large area detector located in the center right of Figure 28. The output of the system is referenced to this detector in order to factor out any effects caused by changing laser power.

The back surface reflection of the input beam from the optical flat is used to qualitatively monitor the power distribution between the two longitudinal modes of the laser. These modes have a frequency separation of 685 MHz [65]. The power distribution and frequency of these modes can change significantly over a period of minutes. These changes are sufficient to affect the performance of the resonated hologram. Hence the need to monitor these modes even if only on a qualitative basis. The light beam is directed, using mirrors $M_7$ and $M_8$, onto a Fabry-Perot interferometer with slightly nonparallel plane mirrors, formed by mirrors $M_9$ and $M_{10}$ which are adjusted to be at an angle of approximately 70 $\mu$rad to each other (computed from the fringe spacing). This interferometer can be seen in the lower
Figure 31: Fringes from nonparallel mirror resonator due to two laser modes right of Figure 28. These mirrors are separated by 73 mm and are adjusted so that a parallel fringe pattern whose location is a function of the frequency of the light can be observed on screen #2. When the laser is producing two frequencies, two sets of fringes are present. A photograph of these fringes is shown in Figure 31. It can be seen in this photograph that there are two pairs of fringes separated by a distance approximately twice their widths. The two fringes which comprise a pair are formed by the two frequencies of the laser. The repetition of these pairs is due to the interferometer. By visually comparing the intensities of the two fringes within a pair, a qualitative measure of the power distribution between the two longitudinal modes is possible.

The resonated hologram and output detector are located within the dashed lines in Figure 28 and are shown in the photograph in Figure 30. The four mirrors and the hologram can be seen in the center of this photo. The grid structure
around them are support bars between the mirror mounts in order to increase the stability of the system. The input beam enters the resonator cavity through semitransparent mirror \( M_1 \). The resonator cavity is formed by mirrors \( M_1-M_4 \) with the hologram located in the center. The hologram is mounted on a rotational stage so that it can be adjusted for Bragg incidence. All four mirrors are mounted in Burleigh mirror mounts so that they can be aligned orthogonal to the incident light. Mirrors \( M_1, M_2, \) and \( M_3 \) are further mounted on piezo-electric mounts in order to adjust the phase of each path in the resonator. The distances between the mirrors and the hologram are 29 mm.

The output beam exits through semitransparent mirror \( M_4 \). Its intensity is measured by the output detector \( D_0 \). This detector has a 200 \( \mu \text{m} \) pinhole in front of it in order to measure a small area of the output beam. This ensures uniformity of the measured beam.

Some electronics were used in conjunction with the piezo-electric mirror mounts and the detectors. The electronics can be seen in the bottom of the photograph in Figure 29. The control circuitry for the piezo-electric mirror mounts is located in the lower right and the detector circuitry is located in the bottom center. The digital voltmeter can be seen on the table at the left of the photograph.

The diagram for the circuitry to control one of the piezo-electric mirror mounts is shown in Figure 32. The circuitry for the other two are identical. It consists of a potentiometer to simultaneously vary the voltage across all three piezo-electric rods within the mirror mount, thus effecting a translation of the mirror along the optical axis. There is one potentiometer for each mirror. There is no feedback property to this circuitry, consequently it exhibits overshoot when adjusted and is not stable for more than a few minutes but it was adequate for gathering the necessary data.
Figure 32: Control circuitry for one piezo-electric mirror mount.

The schematic diagram for the detector amplifiers is shown in Figure 33. The current put out by the detectors is converted to voltage by the first op-amp circuit and then amplified by the second op amp circuit. The potentiometer at the input of the second op-amp is used to set the output of the system to zero in the absence of light. The voltage from signal detector $D_o$ is divided by the voltage from reference detector $D_r$ using an analog divider chip (AD535J). The output voltage is read using a digital voltmeter. The initial resistor converting current to voltage was chosen such that the voltages are in the correct range for the divider chip. The detectors and the divider chip were tested and found to be linear within an accuracy of 1%.

Design Considerations

There were several design considerations taken into account when choosing components and their mounts. These involved the choice of hologram, mirrors,
Figure 33: Schematic drawing of detector amplifiers.
and detector system, as well as component placement.

The hologram was made with a dichromated gelatin process rather than a bleached silver-halide process because this gave significantly less loss and hence a much better performance by the resonator. The holograms were created using the blue line of an argon laser ($\lambda = 488$ nm) because of the increased sensitivity of the dichromate to this wavelength. This procedure is discussed in Appendix C. They were reconstructed with the red line of a helium-neon laser ($\lambda = 632.8$ nm) because of its convenience and the preponderance of inexpensive coatings for this wavelength. Since a plane-wave hologram was involved, this shift in wavelength merely corresponds to an increase in the Bragg angle. It does not affect the validity of the data.

The hologram had antireflection-coated optical flats on both sides, optically connected with liquid gates. This eliminated Fresnel reflections at the air/plate interface, making the hologram more efficient, and eliminated wavefront distortion which results from the nonflatness of the hologram plates. Mineral oil was used as the liquid and the edges were sealed with silicone sealant. A photograph of one of the holograms is shown in Figure 34. The plate is located in a mount for stability. It is possible to see the antireflection flat on the front. The hologram is the bright spot in the center of the plate. It can be easily seen because the plate was illuminated from behind at the Bragg angle and the camera was positioned in the path of the diffracted beam.

Four reflectivities were chosen for both semitransparent mirrors $M_1$ and $M_4$ to test the resonated hologram. There were three pairs of mirrors used for $M_1$ and $M_4$ in addition to no mirrors at all. The first pair were created using uncoated quartz, utilizing the Fresnel reflection due to the change in index of refraction.
This gave a very low reflection coefficient. The second pair were dielectric mirrors, chosen because they had very low loss. The third pair of semitransparent mirrors were made by vapor deposition of aluminum using the Solid State Group's facilities on campus. The loss of these mirrors was taken into account. All of the mirrors had an anti-reflection coating on the back surface to minimize loss and potentially troubling reflections.

The output detector, $D_o$, consisted of a small area silicon detector mounted behind a pinhole, which was located in a spatial filter holder. The detector/pinhole combination was mounted on a horizontal/vertical translational stage which was located on a kinematic mount. The small size of the pinhole ensured a region of uniform intensity and uniform diffraction efficiency. It provided the means of easily scanning the light beam simply by adjusting the position of the detector/pinhole combination. The kinematic mount enabled the detector to be easily moved from
the hologram transmitted beam to the diffracted beam and back so that both could be measured. This detector was tested for uniformity by scanning the pinhole and comparing the results to a scan with the detector/pinhole combination. It was found that the detector was quite uniform at this resolution. (The large area detector had no such need of uniformity since it was of a comparable size to the incident reference beam.)

Ideally one would like the resonating mirrors as close to the hologram as possible, not only because this minimizes the size of the system but because it increases the angular alignment tolerances on the mirrors as well as decreases the sensitivity of the resonated hologram to changes in angle and frequency. Consequently, the mirrors were placed as close as possible to the hologram given the size of the mirror mounts and the rotation stage involved.

Mechanical stability was a very important consideration when mounting all of the components. All component mounts were adhered with melted wax directly to the granite table. The wax is a combination of beeswax and rosin. In addition, support bars were used to join opposite pairs of mirror mounts.

3.3 Experimental Procedure

This section covers the procedures for calibrating the detectors, aligning the mirrors, and taking data on the hologram (both resonated and nonresonated). Throughout this section we will assume that iris \( I_1 \), located near the entrance of the resonator, has been adjusted to a diameter of 6 mm. These procedures, along with their relevant equations, are given below.
Callibration

It is necessary to calibrate the detectors and measure the hologram characteristics so that the data which is taken can be reliably compared with the predicted value. This subsection covers the procedures used to accomplish this. First we will cover the procedure for centering the output detector. (The reference detector was slightly larger than the beam which it measures and hence need not be centered precisely.) Then we will discuss the procedure for callibrating the detectors. Finally, we will present the method for measuring the hologram characteristics.

In order to get reliable data it was necessary that $D_0$ always be centered on the same portion of the beam. With any real beam there is some spatial variation in the intensity. To minimize the effects of this it was necessary to have a reliable method of measuring the same position in both the transmitted beam and the diffracted beam. First we centered the pinhole on the detector. We made use of the fact that the active area of the detector is smaller than the incident beam but much larger than the pinhole. Then we centered the detector/pinhole combination upon the beam which it was measuring. The procedure used to accomplish this is given below:

1. Move pinhole $P_2$ both horizontally and vertically to determine the peak output of the detector. This provides a reference value for the next step.

2. Roughly center pinhole $P_2$. Then adjust it horizontally to determine the two points where the the output of the detector drops to 25% of the peak value. This fairly accurately locates the vertical edges of the active area of the detector. Although the intensity of the incident beam is not completely uniform over this range, the drop-off in response of the detector at its edge is sufficiently rapid that this method will locate it reliably within 6 $\mu$m. Since
the pinhole is 200 μm wide and the detector is 750 μm wide this is precise enough. Center the pinhole midway between these points, thus centering the pinhole horizontally on the detector.

3. Adjust the pinhole vertically to determine the two points where the output drops to 25% of the peak value. Center it midway between these values. This will center the pinhole vertically on the detector.

4. Repeat these steps for the detector/pinhole combination. This will center the detector on the beam.

We are interested in knowing the relative output power of the resonated hologram (i.e. the ratio of the output power to the input power). As was done in Chapter II, this quantity shall be designated \( P_{\text{out}} \). \( P_{\text{out}} \) is obtained from the ratio of the output and reference detector voltages, \( V_{\text{out}} \), the output voltage of the analog divider chip in Figure 33, properly calibrated.

\[
P_{\text{out}} = V_{\text{out}} \frac{c_d}{c_a}
\]  

(3.1)

The callibration constant consists of two factors: \( c_d \) and \( c_a \). The detector callibration factor, \( c_d \), takes into account the differences in the responses of the two detector circuits comprising the inputs of the analog divider chip to equivalent signals. The angle callibration factor, \( c_a \), takes into account the angle induced change in the power density of the light due to different exit angles.

The procedure for determining the detector callibration factor, \( c_d \), is straightforward. Place the output detector in the transmitted beam of the optical flat. (This necessitates removing all of the mirrors and the hologram from the system.) Center the detector in the beam. It is now measuring the input light. Measure \( V_{\text{out}} \). We wish to calibrate our system so that this value is one. Thus,
\[ c_i = \gamma_{\text{out}}^{-1} \]  

(3.2)

The angle calibration factor is strictly a function of the incident and transmitted angles of the beam relative to the hologram normal.

\[ c_a = \frac{\cos \theta}{\cos \gamma} \]  

(3.3)

where \( \theta \) is the angle of incidence of the beam at the hologram and \( \gamma \) is the angle of the transmitted beam, both measured relative to the surface normal. (Evaluated at the Bragg angles this is Kogelnik’s slant factor.)

**Experimental Determination of Hologram Parameters**

There are two sets of parameters of the hologram which are needed in order to compare the experimental data with the theoretical prediction. The first set is the Bragg angles, \( \theta_0 \) and \( \gamma_0 \), and the second set is the diffraction efficiency, \( S \), and the loss, \( \mathcal{A} \).

To measure the Bragg angles we mounted the hologram on a calibrated rotation stage (not the rotation stage used in the resonator) and illuminated it with a HeNe laser beam. By front surface reflection the surface normal of the hologram was found and its angle on the scale of the rotation stage recorded. The Bragg angles for the object and reference beams (\( \gamma_0 \) and \( \theta_0 \) respectively) relative to the hologram normal were determined by rotating the hologram until the maximum amount of light was diffracted for each case.

The diffraction efficiency (\( S \)) and loss (\( \mathcal{A} \)) of the hologram is measured using much the same procedure as for determining the detector calibration factor.

1. Insert the hologram in its holder within the resonator but do not add any mirrors. Rotate it until the input beam is at Bragg incidence for the source beam (maximum diffracted output).
2. Place the output detector in the transmitted beam. Center it. Measure $V_{out}$. Since we have determined the calibration factors we can compute the fraction of the incident power being transmitted by the hologram, $T$. This transmitted light is propagating parallel to the incident light ($c_a = 1$) so

$$T = V_{out}c_d \tag{3.4}$$

3. Move the output detector to the diffracted beam. Center it. Once again we know the calibration factors so we can compute the fraction of the incident power being diffracted, $S$. Since this light is diffracted at the Bragg angle we have

$$S = V_{out} \frac{c_d}{c_a} \tag{3.5}$$

$$= V_{out}c_d \frac{\cos \gamma_o}{\cos \theta_o} \tag{3.6}$$


$$A = 1 - S - T \tag{3.7}$$

Alignment

Once the detector has been calibrated and the hologram characteristics measured we are ready to add the mirrors to the system and take the data. First the mirrors will be aligned orthogonal to the incident beams using the controls of the Burleigh mirror mounts. Then the axial positions will be adjusted using the piezo-electric mirror mounts in order to get the proper phases.

The procedure for orthogonalizing the mirrors is given below.

1. Mount all four mirrors in the system. Roughly align them orthogonal to the incident beams.
2. Insert iris $I_2$ (next to the laser). Adjust it to a diameter of 2 mm.

3. Adjust mirror $M_1$ until the backscatter on $I_2$ is centered. This is quite easy because the backscatter will have a bright center and concentric rings due to the diffraction pattern of the 2 mm iris. The bright center will just fill the iris aperture.

4. Remove iris $I_2$. This leaves us with a beam which is 6 mm in diameter.

5. Examine the light backscattered from the resonated hologram apparatus displayed on screen #1 and adjust mirror $M_3$ until the interference fringes in the backscatter are eliminated. $M_3$ and $M_1$ form a Fabry-Perot interferometer and these fringes are the result. When the fringes are eliminated, $M_3$ is optically parallel to $M_1$ and hence is also orthogonal to the input light.

6. Examine the output of the resonated hologram upon a screen placed in front of pinhole #2 (near the output mirror $M_4$). Adjust mirror $M_2$ until the interference fringes in the output are as broad as possible. Mirror $M_2$ and $M_3$ form a Michelson interferometer with the hologram acting as the beamsplitter. Because the mirrors and hologram are not perfect we wind up with circular fringes when they are aligned. (The center fringe is about 2 mm in diameter, hence the use of our 200 $\mu$m pinhole in front of the detector.) When the fringes form concentric circles, $M_2$ is orthogonal to the beam.

7. Block $M_3$, examine the output, and adjust mirror $M_4$ until the output fringes are eliminated. $M_4$ and $M_2$ form a Fabry-Perot interferometer. Hence, when the fringes have been eliminated they are parallel. This means that $M_4$ is orthogonal to the incident light.
If $M_1$ is not present (two or three mirror resonators) then align $M_3$ using iris $I_2$ and align the rest of the mirrors as before.

The procedures for adjusting the phases of the system for the two, three, and four mirror resonated holograms were developed so that just a single phase was adjusted at a time. This makes the procedure quick and clearcut. The procedure is slightly different for each resonator configuration. Recall that $M_4$ has no piezoelectric position control. Hence, it serves as a reference for determining the phases.

This phase adjustment procedure is valid for a single frequency input beam. A mathematical analysis verifying the results of this phase adjustment procedure for a single frequency input beam is given in Appendix A, Section 3. When the input beam has two frequencies, as is the case for the laser used, the phases for each frequency are different and the optimum mirror position is a compromise between the optimum positions for each frequency individually. This can be seen from the analysis of the resonated hologram response to a dual frequency input conducted in Chapter II, Section 8. Consequently, The individual steps of this procedure will not adjust the mirror positions to their optimum values but will get them close. The final step is to make adjustments in all of the mirrors. This will result in the optimum mirror positions.

For the two mirror resonated hologram the phase adjustment procedure is very simple. It has just one step.

1. Adjust the axial position of $M_2$ until the output of the resonated hologram is maximized. This will set $\phi_2 = \pi \pmod{2\pi}$.

For the three mirror resonated hologram the procedure contains three steps and is different depending upon whether $M_1$ or $M_4$ is absent. For the three mirror resonated hologram #1 ($M_4$ is absent) the procedure is:
1. Block $M_2$. Adjust the axial position of $M_1$ until the output is maximized. This will set $\phi_0 = \pi \pmod{2\pi}$.

2. Unblock $M_2$. Adjust the axial position of $M_2$ until the output is maximized. This will set $\phi_2 = \pi \pmod{2\pi}$.

3. Make final adjustments in the axial positions of $M_1$ and $M_2$ if necessary.

For the three mirror resonated hologram #2 ($M_1$ is absent) the procedure is:

1. Block $M_3$. Adjust the axial position of $M_2$ until the output is maximized. This will set $\phi_1 = \pi \pmod{2\pi}$.

2. Unblock $M_3$. Adjust the axial position of $M_3$ until the output is maximized. This will set $\phi_2 = \pi \pmod{2\pi}$.

3. Make final adjustments in the axial positions of $M_2$ and $M_3$ if necessary.

The phase adjustment procedure for the four mirror resonated hologram consists of four steps.

1. Block mirrors $M_2$ and $M_3$. Adjust the axial position of $M_1$ until the output is minimized. This will set $\phi_3 = \pi \pmod{2\pi}$.

2. Unblock mirror $M_3$. Adjust the axial position of $M_3$ until the output is maximized. This will set $\phi_4 = 0 \pmod{2\pi}$. ($\Rightarrow \phi_0 = \pi \pmod{2\pi}$).

3. Unblock mirror $M_2$. Adjust the axial position of $M_2$ until the output is maximized. This will set $\phi_1 = \pi \pmod{2\pi}$. ($\Rightarrow \phi_2 = \pi \pmod{2\pi}$).

4. Make final adjustments in the axial positions of $M_1$, $M_2$, and $M_3$ to maximize the output if necessary.
Once the system is completely aligned the data is taken. This is covered in the next section.

3.4 Results

In this section the experimental results will be presented and compared with the theoretical predictions. This will be done for two transmission holograms with different diffraction efficiencies, both for the three and the four mirror cases. The two mirror case is merely a special subset of the three and four mirror cases when \( r = 0 \) and hence will be implicitly considered.

The pertinent data on the holograms and the mirrors which were used is listed in Tables 2 and 3. This provides the necessary data to substitute into the theoretical equations in order to make a prediction of expected output. The Bragg angles of the holograms, listed in Table 2, were chosen to be different so that any Fresnel reflections which did occur at the hologram would not be in the same direction as a Bragg angle. The diffraction efficiency of the holograms, \( S \), was deliberately chosen to be small because this system was conceived to work with low diffraction efficiencies. The loss, \( A \), was made as small as possible. The mirror reflectivities, listed in Table 3, were chosen to bracket the optimum reflectivity for the four mirror resonated hologram with hologram #1. The quartz and dielectric mirrors were very low loss. The aluminum mirrors had significant loss. Mirrors \( M_2 \) and \( M_3 \) are fully reflective dielectric mirrors modeled by \( r^2 = 1 \).

The experimental data on the performance of the resonated holograms is listed in Tables 4–7 and plotted on the theoretical curves in Figures 35–38 as a function of mirror power reflectivity, \( r^2 \). The performance of Hologram 1 (\( S = .22, A = .14 \)) is given in Tables 4 and 5 and graphed in Figures 35 and 36 for both the three and four mirror resonated hologram configurations, respectively. The performance of
Table 2: Hologram characteristics

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Hologram 1</th>
<th>Hologram 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \theta_0 )</td>
<td>(-36.6^\circ)</td>
<td>(-36.4^\circ)</td>
</tr>
<tr>
<td>( \gamma_0 )</td>
<td>(32.0^\circ)</td>
<td>(32.2^\circ)</td>
</tr>
<tr>
<td>( S )</td>
<td>.22</td>
<td>.054</td>
</tr>
<tr>
<td>( A )</td>
<td>.14</td>
<td>.08</td>
</tr>
</tbody>
</table>

Table 3: Semitransparent mirror intensity characteristics

<table>
<thead>
<tr>
<th>Mirrors ( M_1, M_4 )</th>
<th>( r^2 )</th>
<th>( t^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A none</td>
<td>0</td>
<td>1.00</td>
</tr>
<tr>
<td>B quartz</td>
<td>.04</td>
<td>.95</td>
</tr>
<tr>
<td>C dielectric</td>
<td>.12</td>
<td>.87</td>
</tr>
<tr>
<td>D aluminum</td>
<td>.21</td>
<td>.47, .50</td>
</tr>
</tbody>
</table>
Table 4: Three mirror resonated hologram results — Hologram 1

<table>
<thead>
<tr>
<th>Mirror</th>
<th>Location</th>
<th>$P_{out}$</th>
<th>$P_{out, \text{corrected}}$</th>
<th>Theoretical Bounds</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>N/A</td>
<td>.50</td>
<td>.50</td>
<td>.49, .53</td>
</tr>
<tr>
<td>B</td>
<td>$M_1$</td>
<td>.54</td>
<td>.55</td>
<td>.55, .63</td>
</tr>
<tr>
<td>B</td>
<td>$M_4$</td>
<td>.59</td>
<td>.60</td>
<td>.55, .63</td>
</tr>
<tr>
<td>C</td>
<td>$M_1$</td>
<td>.64</td>
<td>.65</td>
<td>.57, .68</td>
</tr>
<tr>
<td>C</td>
<td>$M_4$</td>
<td>.63</td>
<td>.64</td>
<td>.57, .68</td>
</tr>
<tr>
<td>D</td>
<td>$M_1$</td>
<td>.40</td>
<td>.67</td>
<td>.56, .71</td>
</tr>
<tr>
<td>D</td>
<td>$M_4$</td>
<td>.43</td>
<td>.68</td>
<td>.56, .71</td>
</tr>
</tbody>
</table>

Figure 35: Output of three mirror resonated hologram as a function of mirror reflectivity for Hologram 1: Experimental points plotted between theoretical bounds for single and dual frequency.
Table 5: Four mirror resonated hologram results — Hologram 1

<table>
<thead>
<tr>
<th>Mirror</th>
<th>$P_{out}$</th>
<th>$P_{out}$, corrected</th>
<th>Theoretical Bounds</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>.50</td>
<td>.50</td>
<td>.49, .53</td>
</tr>
<tr>
<td>B</td>
<td>.61</td>
<td>.62</td>
<td>.62, .71</td>
</tr>
<tr>
<td>C</td>
<td>.65</td>
<td>.66</td>
<td>.67, .71</td>
</tr>
<tr>
<td>D</td>
<td>.25</td>
<td>.66</td>
<td>.67, .70</td>
</tr>
</tbody>
</table>

Figure 36: Output of four mirror resonated hologram as a function of mirror reflectivity for Hologram 1: Experimental points plotted between theoretical bounds for single and dual frequency.
Table 6: Three mirror resonated hologram results — Hologram 2

<table>
<thead>
<tr>
<th>Mirror</th>
<th>Location</th>
<th>$P_{out}$</th>
<th>$P_{out}$, corrected</th>
<th>Theoretical Bounds</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>N/A</td>
<td>.15</td>
<td>.15</td>
<td>.15, .18</td>
</tr>
<tr>
<td>B</td>
<td>$M_1$</td>
<td>.17, .21</td>
<td>.17, .21</td>
<td>.18, .25</td>
</tr>
<tr>
<td>B</td>
<td>$M_4$</td>
<td>.19</td>
<td>.19</td>
<td>.18, .25</td>
</tr>
<tr>
<td>C</td>
<td>$M_1$</td>
<td>.19</td>
<td>.20</td>
<td>.20, .31</td>
</tr>
<tr>
<td>C</td>
<td>$M_4$</td>
<td>.24</td>
<td>.24</td>
<td>.20, .31</td>
</tr>
<tr>
<td>D</td>
<td>$M_1$</td>
<td>.11</td>
<td>.19</td>
<td>.19, .36</td>
</tr>
<tr>
<td>D</td>
<td>$M_4$</td>
<td>.19</td>
<td>.30</td>
<td>.19, .36</td>
</tr>
</tbody>
</table>

Figure 37: Output of three mirror resonated hologram as a function of mirror reflectivity for Hologram 2: Experimental points plotted between theoretical bounds for single and dual frequency.
Table 7: Four mirror resonated hologram results — Hologram 2

<table>
<thead>
<tr>
<th>Mirror</th>
<th>$P_{out}$</th>
<th>$P_{out, corrected}$</th>
<th>Theoretical Bounds</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>.15</td>
<td>.15</td>
<td>.15, .18</td>
</tr>
<tr>
<td>B</td>
<td>.21, .27</td>
<td>.22, .27</td>
<td>.22, .33</td>
</tr>
<tr>
<td>C</td>
<td>.27, .33, .35</td>
<td>.28, .34, .36</td>
<td>.26, .49</td>
</tr>
<tr>
<td>D</td>
<td>.13, .17, .19</td>
<td>.35, .45, .49</td>
<td>.27, .62</td>
</tr>
</tbody>
</table>

Figure 38: Output of four mirror resonated hologram as a function of mirror reflectivity for Hologram 2: Experimental points plotted between theoretical bounds for single and dual frequency.
Hologram 2 \((S = .054, A = .08)\) is given in Tables 6 and 7 and graphed in Figures 37 and 38 for both the three and four mirror resonator configurations, respectively. All data is accurate to ±.01. The data for the three mirror resonated holograms was taken for both configurations (no output mirror and no input mirror) for each type of mirror. Because the power distribution between the two modes of the laser is known only qualitatively, the theoretical curves are given for the lower and upper bounds for the maximum output with a dual frequency input, \(P'_{\text{out},2f}\), as derived in Chapter II, equations (2.46) and (2.48). They are plotted for the case of lossless mirrors. The data, corrected for the loss of mirrors \(M_1\) and \(M_4\), is plotted on these curves as well.

In the tables the actual data is given and then the value corrected for the loss of the semitransparent mirrors. By applying this correction it is possible to plot all of the data points on one curve, the curve for lossless semitransparent mirrors. For the three mirror resonated hologram this correction takes the form:

\[
P_{\text{out}, \text{corrected}} = \frac{1 - r^2}{t^2} P_{\text{out}}
\]

and for the four mirror resonated hologram it takes the form:

\[
P_{\text{out}, \text{corrected}} = \frac{(1 - r_1^2)(1 - r_4^2)}{t_1^2 t_4^2} P_{\text{out}}.
\]

Such a correction is possible because the transmission of these mirrors can be factored out of the expression for the resonator output (see equation (2.47)).

A qualitative measure of the power distribution between the modes was made at the time each data point was taken, based upon the relative intensity of the fringes in the output of the slightly nonparallel Fabry-Perot interferometer. When the fringes appeared to be equal in intensity (indicating equal power distribution between the modes) the data points were close to the lower bound. When one
fringe of a pair was obviously brighter than the other (indicating one mode was dominant) the data was nearer to the upper bound.

This data demonstrates a significant increase in effective diffraction efficiency. From the data in Table 7 for the four mirror resonated hologram for hologram #2, it can be seen that for the dielectric mirrors (C) the resonated output increased by a factor of 7 over the unresonated output.

It can be seen from these curves that the resonated hologram is indeed behaving like the theory predicted it would. Due to the changing frequency distribution between the modes several data points were taken. It can be seen that these lie between the predicted bounds. Thus, within experimental error, this verifies that the theories derived in the previous chapter are applicable to this system.

3.5 Summary

In this chapter an experimental system was presented to verify some of the theories derived in the previous chapter. The operation of the system was explained. Then the experimental procedure was described. Finally the experimental results were reported and shown to agree with the theory within experimental error. An increase in effective diffraction efficiency was demonstrated, up to a factor of 7.
CHAPTER IV
SUMMARY & CONCLUSIONS

4.1 Introduction

This chapter summarizes the contents of this dissertation and presents some conclusions about this work. First the contents of each chapter is summarized. Then a list of conclusions about the resonated hologram is enumerated. Finally, some directions for future work are given.

4.2 Summary

The purpose of this work was to present a theoretical analysis and experimental verification of the resonated hologram performance. This has now been accomplished. The motivation for this work came from the need for efficient optical interconnects in both electronic and optical computing. Holograms can provide this if a passive method is found to increase the diffraction efficiency of a hologram. This increase is needed because current materials for holograms can support many holograms with low diffraction efficiencies but only a few holograms with high diffraction efficiencies.

In Chapter I this problem was introduced and the resonated hologram proposed as a solution. Previous work in the field was summarized and compared with the work in this dissertation.
The resonated hologram was modeled and analyzed in Chapter II. It was shown that both transmission and reflection holograms obey the same general equations when resonated and hence all of the results derived for the resonated hologram are equally applicable to both types of holograms. First a set of equations for the transmitted fields of a general four mirror resonated hologram were derived. Then the output and backscattered power were evaluated for the special case of lossless mirrors and symmetric hologram transmission and diffraction coefficients. The maximum output of the two, three, and four mirror resonated holograms were derived as a function of the hologram diffraction efficiency and loss. The results were compared between each other and with the unresonated hologram. The significant increase in output power which could be obtained was shown (100% output for a lossless hologram, regardless of its diffraction efficiency). Then the angular and frequency response of the resonated hologram was derived and used to define a time constant for the resonated hologram. Finally, the response of the resonated hologram to an input with two frequencies was determined and upper and lower bounds on the maximum output power were computed when the power distribution between these two frequencies was free to vary.

The experimental system and the data generated with it were presented in Chapter III. First the operation of the system and the method of taking data were explained. Then the data for two transmission holograms, each used to create both a three and four mirror resonated hologram, was given and compared to the theoretically predicted bounds. It was found that the data fell within the predicted bounds, thus verifying the theories of Chapter II within experimental error.

There are three appendices. Appendix A consists of mathematical proofs of results used in chapters II and III. A coupled wave analysis of the hologram was conducted in Appendix B. This analysis was used to determine the diffraction and
transmission coefficients with which the hologram is modeled. Conditions upon
the loss of the hologram were determined which allow an assumption of symmetric
coefficients to be made. In Appendix C the method used to create the dichromated
gelatin holograms used in Chapter III was detailed. The procedures used and the
reasons for using them were explained.

4.3 Conclusions

A theoretical analysis and experimental verification of the resonated hologram
performance has been done for the two, three, and four mirror resonated hologram.
Some of the conclusions which have resulted from this work are enumerated below.

1. The theories derived in this dissertation are borne out by the experimental
data. They were used to accurately predict the response of an experimentally
implemented resonated hologram system for two, three, and four mirrors.

2. For a lossless hologram it was determined theoretically that an output of
100% was obtainable for any hologram diffraction efficiency over a range of
mirror reflectivities for both the three and four mirror resonated holograms.
The range of reflectivities is much greater for the four mirror resonated holo-
gram.

3. A theoretical comparison between the maximum output power for the two,
three, and four mirror resonated holograms and the unresonated hologram
leads to the following observations.

(a) The maximum output power of a resonated hologram is significantly
greater than that of an unresonated hologram, especially for the three
and four mirror resonated holograms. The percentage increase due to
resonating is greatest for holograms with low diffraction efficiencies.
(b) For a lossy hologram the maximum output increases with each additional mirror, although the difference between the three and four mirror resonated holograms is significant only for holograms with very low diffraction efficiencies.

(c) The maximum output of the resonated hologram is a strong function of the loss of the hologram. The resonated hologram is most sensitive to the loss of the hologram for holograms with low diffraction efficiencies and for high mirror reflectivities.

(d) The optimum reflectivity of the three mirror resonated hologram is much higher than for the four mirror resonated hologram.

(e) The angular and frequency width of the resonated hologram is less for the optimized three mirror resonated hologram than for the optimized four mirror resonated hologram, primarily due to the difference in their mirror reflectivities. Consequently, the time constant of the temporal response is longer for the optimized three mirror resonated hologram. For the resonated hologram implemented experimentally (both three and four mirrors), the bandwidth was on the order of 1 GHz and hence the time constant was on the order of 1 ns.

4. It was demonstrated that operation is possible with a multimode laser.

5. The experimental resonated hologram demonstrated an increase in output power by up to a factor of 7. Even higher outputs are expected with optimized lossless mirrors.

6. The significant increases predicted and observed make this a likely candidate for an efficient optical interconnect.
4.4 Future Work

There are many areas of this work which can be extended. These fall into two categories. First is further analysis and experimental verification of the model presented in this dissertation. Second is the extension of this model to a system with multiple holograms and nonplanar waves.

There are still some areas left to be explored with the model used in this dissertation. These are enumerated below.

1. A sensitivity analysis can be done on the maximum output power of the resonated hologram to determine manufacturing tolerances. The initial work for frequency, angle, and mirror reflectivity has been done, but the sensitivity to mirror alignment and hologram alignment have yet to be done.

2. An analysis of the case when the output mirror is chosen as mirror $M_2$ rather than $M_4$ might prove interesting. This case would cause a resonated transmission hologram to operate like a reflection hologram. A transmission hologram generally has a narrower angular width and is easier to manufacture so this might prove to be better than resonating a reflection hologram.

3. A transient analysis of this system would prove enlightening. This information is necessary if the usefulness of the system is to be completely understood. An estimate of the temporal response has been made based upon the bandwidth of the resonated hologram. Knowledge of the transient response of the system to a pulse, however, will provide much more information.

4. An experimental verification of a resonated reflection hologram would add further credence to the results derived here.
5. Repeating this experiment with a single frequency laser would give much better confirmation of the theoretical equations.

The initial assumptions of plane waves and plane mirrors normal to the Bragg angles were necessary to analyze this system. For a real optical interconnect the emulsion will contain many holograms and many Bragg angles, so such a configuration will not be practical. The next step, therefore, is to extend this analysis to a configuration which allows many holograms to be simultaneously resonated without crosstalk. Two such configurations have been proposed in the literature [63]. The final step is to extend this analysis to nonplanar waves.
APPENDIX A

MATHEMATICAL PROOFS

In this appendix the mathematical proofs referred to in the text are presented. They are grouped into four sections. The first section consists of an analysis of mathematical forms which occur frequently. These analyses are performed once in a general form so that they can be referred to when needed. The second section consists of proofs dealing with the maximization of $P_{\text{out}}$, the relative output power of the resonated hologram. The third section consists of proofs dealing with the alignment procedures. The last section contains a derivation of the upper and lower bounds upon $P'_{\text{out},2f}(\rho)$, the maximum output power of the resonated hologram with a two frequency input.

Throughout this appendix we will use the following notational conventions. A prime attached to a parameter (phases $\phi_i$ and reflectivities $r_i$) indicate its value maximizes or minimizes the function of interest. These will be designated the optimum values. $P'_{\text{out}}$ designates the value of $P_{\text{out}}$ evaluated at the optimum phases. $P_{\text{out}}|_{\text{max}}$ designates the value of $P_{\text{out}}$ evaluated at the optimum phases and reflectivities.

A.1 General Forms

In this section three general mathematical forms will be considered. The results of this section will be utilized in subsequent sections.
Form #1

The first form concerns the minimization of a function of the form

\[ f(\phi) = A + B \sin \phi + C \cos \phi \]  

(A.1)

To determine the minimum value of \( f \), take the derivative of \( f \) with respect to \( \phi \), set it equal to zero, and solve for \( \phi' \), the value of \( \phi \) which minimizes \( f \).

\[ \frac{df}{d\phi} = B \cos \phi - C \sin \phi \]  

(A.2)

\[ \frac{df}{d\phi} = 0 \Leftrightarrow \tan \phi' = \frac{B}{C} \]  

(A.3)

\[ \Rightarrow \phi' = \tan^{-1} \left( \frac{B}{C} \right) \]  

(A.4)

Equation (A.4) has two solutions which differ by \( \pi \). It is easy to see that the solution which produces a minimum value for \( f \) is

\[ \sin \phi' = -\frac{B}{\sqrt{B^2 + C^2}} \]  

(A.5)

\[ \cos \phi' = -\frac{C}{\sqrt{B^2 + C^2}} \]  

(A.6)

\[ \Rightarrow f(\phi') = A - \sqrt{B^2 + C^2} \]  

(A.7)

Form #2

The second form which will be analyzed concerns the maximization of a function of the form

\[ f(\phi) = \frac{1 + A^2 - 2A \cos \phi}{1 + B^2 - 2B \cos \phi} \]  

(A.8)

where

\[ -1 < B < A \leq 1, 0 < A. \]  

(A.9)

To determine \( \phi' \), the value of \( \phi \) which maximizes \( f \), we will take the derivative of \( f \) with respect to \( \phi \), set it equal to zero, and solve for \( \phi' \).
\[
\frac{df}{d\phi} = \frac{(1 + B^2 - 2B \cos \phi)(2A \sin \phi) - (1 + A^2 - 2A \cos \phi)(2B \sin \phi)}{(1 + B^2 - 2B \cos \phi)^2}
\] (A.10)

\[
\frac{df}{d\phi} = 0 \iff \sin \phi'(A - B)(1 - AB) = 0
\] (A.11)

\[\Rightarrow \phi' = n\pi
\] (A.12)

The next step is to determine which of the following two values for \(f\) is the greater.

\[
f(0) = \left(\frac{1 - A}{1 - B}\right)^2
\] (A.13)

\[
f(\pi) = \left(\frac{1 + A}{1 + B}\right)^2
\] (A.14)

\[-1 < B < A \leq 1 \Rightarrow \begin{cases}
1 - A & < 1 - B \\
1 + A & > 1 + B
\end{cases}
\] (A.15)

\[\Rightarrow \left(\frac{1 - A}{1 - B}\right)^2 < 1 < \left(\frac{1 + A}{1 + B}\right)^2
\] (A.16)

\[\Rightarrow f(0) < f(\pi)
\] (A.17)

\[\Rightarrow \phi' = \pi \pmod{2\pi}
\] (A.18)

Form #3

The last form to be analyzed consists of finding the extrema (minimum or maximum) of the function

\[f(\phi) = \frac{A(\phi)}{B(\phi)}.
\] (A.19)

The first and second derivatives are given by

\[f' = \frac{BA' - AB'}{B^2}
\] (A.20)

\[f'' = \frac{BA'' - AB''}{B^2} + \frac{(BA' - AB')(B - 2B')}{B^3}
\] (A.21)

\[= \frac{BA'' - AB''}{B^2} + f' \frac{B - 2B'}{B}
\] (A.22)
At a maximum or a minimum

\[ f' = 0 \Leftrightarrow BA' = AB' \]
\[ f'' = \frac{BA'' - AB''}{B^2} \]

where

\[ f'' \begin{cases} < 0 & \text{if maximum} \\ > 0 & \text{if minimum} \end{cases} \]
\[ \Leftrightarrow BA'' - AB'' \begin{cases} < 0 & \text{if maximum} \\ > 0 & \text{if minimum} \end{cases} \]

We shall use equations (A.23) and (A.26) to determine maximums and minimums in the next two sections.

A.2 Maximizing \( P_{\text{out}} \)

This section contains rigorous proofs relating to maximizing \( P_{\text{out}} \). These are formal mathematical proofs of the results stated in Chapter II, Section 5. They are grouped in subsections according to the resonated hologram (two, three, or four mirrors) to which they apply.

For each of the two, three, and four mirror resonated hologram configurations we will derive expressions for the quantity \( P_{\text{out}} \mid_{\text{max}} \), the maximum value for \( P_{\text{out}} \) which occurs when both the phases \((\phi_0, \phi_1, \phi_2)\) and the reflectivities \((r_1, r_4)\) have been set to their optimum values \((\phi_0', \phi_1', \phi_2', r_1', \text{and} \ r_4')\), and the quantity \( P'_{\text{out}} \), the maximum value for \( P_{\text{out}} \) when just the phases have been set to their optimum values. For the two and three mirror resonated holograms a complete analysis covering all possible cases will be performed. For the four mirror resonated hologram we will restrict the analysis to just the cases that are of importance to
this dissertation. We will calculate $P_{\text{out}}|_{\text{max}}$ exactly, but we will only obtain a close approximation for $P'_{\text{out}}$.

A.2.1 Two Mirror Resonated Hologram

For the two mirror resonated hologram $P_{\text{out}}$ is given by equation (2.84) which is reproduced here for convenience.

$$P_{\text{out}} = S \left[ \frac{1 + (1 - A)^2 - 2(1 - A) \cos \phi_2}{1 + S^2 - 2S \cos \phi_2} \right]$$  \hspace{1cm} (A.27)

This equation can be recognized as being in the form of equation (A.8) (which we have designated form #2) where

$$A = 1 - A$$  \hspace{1cm} (A.28)

$$B = S$$  \hspace{1cm} (A.29)

Consequently, equation (A.18) gives us the solution

$$\phi'_2 = \pi \pmod{2\pi}$$  \hspace{1cm} (A.30)

$$\Rightarrow P_{\text{out}}|_{\text{max}} = \frac{S(2 - A)^2}{(1 + S)^2}$$  \hspace{1cm} (A.31)

which are quoted as equations (2.86) and (2.87).

A.2.2 Three Mirror Resonated Hologram

For the three mirror resonated hologram, $P_{\text{out}}$ is given by equations (2.91) and (2.93). They are rewritten in the form

$$P_{\text{out}} = \frac{S(1 - r^2)[1 + (1 - A)^2 - 2(1 - A) \cos \phi_2]}{\|D\|^2}$$  \hspace{1cm} (A.32)

where

$$\|D\|^2 = 1 + (1 - A - S)^2 r^2 + S^2 - 2S \cos \phi_2$$

$$+ \ 2(1 - A - S) r \left[ (1 - S \cos \phi_2) \cos \phi_0 + (-S \sin \phi_2) \sin \phi_0 \right]$$  \hspace{1cm} (A.33)
Notice that the \( \phi_0 \) dependence of \( P_{\text{out}} \) is entirely due to \( \mathcal{D} \). Thus the first step in maximizing \( P_{\text{out}} \) is to minimize \( ||\mathcal{D}||^2 \) with respect to \( \phi_0 \). Recognizing that equation (A.1) (which we have designated form #1) is imbedded in this equation leads to the immediate solution

\[
\sin \phi'_0 = \frac{S \sin \phi_2}{\sqrt{1 + S^2 - 2S \cos \phi_2}} \tag{A.34}
\]

\[
\cos \phi'_0 = \frac{S \cos \phi_2 - 1}{\sqrt{1 + S^2 - 2S \cos \phi_2}} \tag{A.35}
\]

which, when substituted in the equation for \( P_{\text{out}} \), gives

\[
P_{\text{out}}(\phi'_0) = \frac{S(1 - r^2)[1 + (1 - A)^2 - 2(1 - A) \cos \phi_2]}{\left[\sqrt{1 + S^2 - 2S \cos \phi_2 - (1 - A - S)r}\right]^2} \tag{A.36}
\]

This maximizes \( P_{\text{out}} \) in terms of \( \phi_0 \) for all values of \( \phi_2 \) and \( r \).

The next step is to maximize \( P_{\text{out}}(\phi'_0) \) with respect to \( \phi_2 \). Taking the derivative of equation (A.36) with respect to \( \phi_2 \) and setting it equal to zero gives

\[
\sin \phi_2 \left[ (1 - A)(1 + S^2) - S[1 + (1 - A)^2] \
- (1 - A)(1 - A - S)r\sqrt{1 + S^2 - 2S \cos \phi_2} \right] = 0 \tag{A.37}
\]

which has two solutions:

\((i)\) \( \phi_2 = n\pi \) \tag{A.38}\]

and

\((ii)\) \( (1 - A)(1 + S^2) - S[1 + (1 - A)^2] = (1 - A)(1 - A - S)r\sqrt{1 + S^2 - 2S \cos \phi_2} \tag{A.39}\]

We shall proceed to examine each of these solutions in turn to determine which correspond to maxima and which to minima for \( P_{\text{out}}(\phi'_0) \). We will find
that there is a value of reflectivity, \( r_b \), at which the character of \( P_{\text{out}}(\phi)' \) changes. When \( r \leq r_b \), \( P_{\text{out}} \) has one maximum (\( \phi_2 = \pi \)) and one minimum (\( \phi_2 = 0 \)). But when \( r > r_b \), \( P_{\text{out}} \) has two minima (\( \phi = 0, \pi \)) and two maxima (given by case (ii) above).

The first solution, given by equation (A.38) and henceforth designated case (i), is the one we expected to find from our analysis in Chapter II, Section 4. It will be examined first. The first step in this examination will be to determine which of the two possible values for \( \phi_2 \) (0 or \( \pi \)) results in the larger value for \( P_{\text{out}}(\phi)' \). Then we will determine the conditions under which this value corresponds to a maximum for \( P_{\text{out}}(\phi)' \).

To determine which value of \( \phi_2 \) results in the larger value for \( P_{\text{out}}(\phi)' \), we will evaluate \( P_{\text{out}}(\phi)' \) for each, take their ratio, and determine if it is greater or less than one.

\[
P_{\text{out}}(\phi)'|_{\phi_2=0} = \frac{S(1-r^2)A^2}{[1-S-(1-A-S)r]^2}\quad (A.40)
\]

\[
P_{\text{out}}(\phi)'|_{\phi_2=\pi} = \frac{S(1-r^2)(2-A)^2}{[1+S-(1-A-S)r]^2}\quad (A.41)
\]

\[
\frac{P_{\text{out}}(\phi)'|_{\phi_2=\pi}}{P_{\text{out}}(\phi)'|_{\phi_2=0}} = \left(\frac{2-A}{A}\right)^2 \left[1-S-(1-A-S)r\right] \quad (A.42)
\]

This quantity is strictly greater than one for all values of \( r, 0 \leq r < 1 \). This is shown as follows.

\[
1 - S - \frac{\frac{1-A-S}{1-A}}{1-A} = \frac{AS}{1-A} \geq 0 \quad (A.43)
\]

\[
r < 1 \leq \frac{1}{1-A} \quad (A.44)
\]

\[
\Rightarrow \frac{1-S-(1-A-S)r}{1+S-(1-A-S)r} > \frac{1-S-\frac{1-A-S}{1-A}}{1+S-\frac{1-A-S}{1-A}} \quad (A.45)
\]

\[
= \frac{A}{2-A} \quad (A.46)
\]
\[ \left( \frac{2 - A}{A} \right)^2 \left[ \frac{1 - S - (1 - A - S)r}{1 + S - (1 - A - S)r} \right]^2 > \left( \frac{2 - A}{A} \right)^2 \left( \frac{A}{2 - A} \right)^2 \]
\[ = 1 \]
\[ \Rightarrow P_{\text{out}}(\phi_0')|_{\phi_2 = \pi} > P_{\text{out}}(\phi_0')|_{\phi_2 = 0} \]

so, of the two possible values for \( \phi_2 \), the one which we are interested in is

\[ \phi_2 = \pi \pmod{2\pi} \]  
\[ (A.50) \]
\[ \Rightarrow \phi_0' = \pi \pmod{2\pi} \]  
\[ (A.51) \]
\[ P'_{\text{out},i} = \frac{S(1 - r^2)(2 - A)^2}{[1 + S - (1 - A - S)r]^2} \]  
\[ (A.52) \]

We have verified that \( \phi_2 = \pi \) yields a larger value for \( P_{\text{out}}(\phi_0') \) than \( \phi_2 = 0 \), but this does not preclude it from still being a local minimum. We shall now determine when \( \phi_2 = \pi \) maximizes \( P_{\text{out}}(\phi_0') \) by using equation (A.26) derived for form #3 and the expression for \( P_{\text{out}}(\phi_0') \) given in equation (A.36). At a maximum we have

\[ \left. \frac{d^2 P_{\text{out}}(\phi_0')}{d\phi_2^2} \right|_{\phi_2 = \pi} < 0 \]  
\[ (A.53) \]
\[ \Leftrightarrow [1 + S - (1 - A - S)r]^2(-2S)(1 - r^2)(1 - A) \]
\[ < -2S^2(1 - r^2)(2 - A)^2 \left[ 1 - \frac{(1 - A - S)r}{1 + S} \right] \]  
\[ (A.54) \]
\[ \Leftrightarrow S(2 - A)^2 < (1 + S)(1 - A)[1 + S - (1 - A - S)r] \]  
\[ (A.55) \]
\[ \Leftrightarrow r < \frac{(1 + S)(1 - A) - S(2 - A)^2}{(1 + S)(1 - A)(1 - A - S)} \]  
\[ (A.56) \]
\[ \Leftrightarrow r < \frac{1 - S(1 - A)}{(1 + S)(1 - A)} \]  
\[ \equiv r_b \]  
\[ (A.57) \]
Thus, if \( r < r_b \) then \( \phi_2 = \pi \) maximizes \( \mathcal{P}_{\text{out}}(\phi'_0) \). But if \( r > r_b \) then \( \phi_2 = \pi \) results in a local minimum for \( \mathcal{P}_{\text{out}}(\phi'_0) \), so we shall have to look to case (ii) to find the value of \( \phi_2 \) which yields a maximum for \( \mathcal{P}_{\text{out}}(\phi'_0) \).

Having concluded our examination of case (i) we will now examine case (ii), given by equation (A.39). The first step is to determine when equation (A.39) has a solution for \( \phi'^2 \) and then determine when that solution maximizes \( \mathcal{P}_{\text{out}}(\phi'_0) \). This equation can be written in the form

\[
1 - S(1 - A) = (1 - A)r\sqrt{1 + S^2 - 2S \cos \phi_2} \quad \text{(A.59)}
\]

\[
\Leftrightarrow 1 + S^2 - 2S \cos \phi_2 = \left[ \frac{1 - S(1 - A)}{(1 - A)r} \right]^2 \quad \text{(A.60)}
\]

\[
\Leftrightarrow \cos \phi_2 = \frac{1}{2S} \left[ 1 + S^2 - \left( \frac{1 - S(1 - A)}{(1 - A)r} \right)^2 \right] \quad \text{(A.61)}
\]

Because \( |\cos \phi_2| \leq 1 \) such a solution can exist if and only if (iff)

\[
\left| \frac{1}{2S} \left[ 1 + S^2 - \left( \frac{1 - S(1 - A)}{(1 - A)r} \right)^2 \right] \right| \leq 1 \quad \text{(A.62)}
\]

\[
\Leftrightarrow \frac{1}{2S} \left[ 1 + S^2 - \left( \frac{1 - S(1 - A)}{(1 - A)r} \right)^2 \right] \geq -1 \quad \text{(A.63)}
\]

because \( 0 \leq r < 1 \), due to the physical system we are modeling.

\[
\Leftrightarrow (1 + S)^2 \geq \left[ \frac{1 - S(1 - A)}{(1 - A)r} \right]^2 \quad \text{(A.64)}
\]

\[
\Leftrightarrow r \geq \frac{1 - S(1 - A)}{(1 + S)(1 - A)} \quad \text{(A.65)}
\]

\[
\Leftrightarrow r \geq r_b \quad \text{(A.66)}
\]

From this result we can conclude that the solution for \( \phi'_2 \) given by case (ii) does not exist if \( r < r_b \).

When \( r \geq r_b \), \( \phi'_2 \) is given by equation (A.61), which is rewritten here.

\[
\phi'_2 = \cos^{-1} \left\{ \frac{1}{2S} \left[ 1 + S^2 - \left( \frac{1 - S(1 - A)}{(1 - A)r} \right)^2 \right] \right\} \quad \text{(A.67)}
\]
This has two solutions, but since the dependence of $P_{\text{out}}(\phi_0')$ is only upon $\cos \phi_2$ both solutions will yield equal values for $P'_{\text{out}}$. Evaluating $P_{\text{out}}(\phi_0')$ given by equation (A.36) at this solution yields

$$P'_{\text{out},\text{ii}} = \frac{1 - S(1 - A)}{1 - A - S + \frac{A(2 - A)}{(1 - A)(1 - r^2)}} \quad \text{(A.68)}$$

Now that we have solved and evaluated both cases we can determine the values for $\phi_2$ which maximize $P_{\text{out}}(\phi_0')$. If $r < r_b$ we determined that only $\phi_2 = \pi$ maximized $P_{\text{out}}(\phi_0')$. Hence, $P'_{\text{out}}$ is given by $P'_{\text{out},\text{i}}$. If $r > r_b$ we determined that only $\phi_2$ given by equation (A.67) can possibly result in a maximum. Since $P_{\text{out}}(\phi_0')$ must have a maximum, we can conclude without further work that this maximum is given by the value of $\phi_2$ defined by equation (A.67). Hence, $P'_{\text{out}}$ is given by $P'_{\text{out},\text{ii}}$. If $r = r_b$ then both solutions (i) and (ii) give the same results.

In order to determine $P_{\text{out}}|_{\text{max}}$ we shall solve for $r'$, the value of $r$ which maximizes $P'_{\text{out}}$. To do this we shall determine the reflectivities which maximize $P'_{\text{out},\text{i}}$ and $P'_{\text{out},\text{ii}}$. That reflectivity which results in the larger of these two quantities is $r'$.

Taking the derivative of $P'_{\text{out},\text{i}}$ (defined by equation (A.52)) with respect to $r$, setting it equal to zero, and solving for $r'$ yields

$$r' = \frac{1 - A - S}{1 + S}$$

$$(A.69)$$

$$= \frac{1 - S(1 - A) - A(2 - A)}{(1 + S)(1 - A)}$$

$$(A.70)$$

$$\leq r_b$$

$$(A.71)$$

$P'_{\text{out},\text{ii}}$, given by Equation (A.68), can readily be seen to be monotonically decreasing in $r$ if $A > 0$ and hence is maximized when $r = r_b$. Thus, we can write

$$P'_{\text{out},\text{i}}(r') \geq P'_{\text{out},\text{ii}}(r_b) \quad \text{(A.72)}$$
where equality holds iff \( A = 0 \).

Therefore, the global maximum is given by

\[
\phi_0 = \phi'_2 = \pi \pmod{2\pi} \quad \text{(A.74)}
\]

\[
r' = \frac{1 - A - S}{1 + S} \quad \text{(A.75)}
\]

\[
P'_{\text{out, \text{\it ii}}} \big|_{\text{max}} = \frac{S(2 - A)}{2S + A} \quad \text{(A.76)}
\]

It is interesting to note that for the lossless case \( A = 0 \), \( P'_{\text{out, \text{\it ii}}} = 1 \) for all values of \( r \), \( r_b \leq r < 1 \). Thus, we do not have just a single reflectance at which 100% output is obtained, but rather a range of reflectances.

The results of this analysis are quoted in Chapter II as equations (2.94)-(2.97) and (2.99)-(2.106).

**A.2.3 Four Mirror Resonated Hologram**

This analysis of the four mirror resonated hologram will be conducted for all of the important cases of reflectivities and phases, although not all possible cases will be considered. For the two and three mirror resonated holograms we were able to solve for the phases which gave \( P'_{\text{out}} \) and \( P'_{\text{out, \text{\it max}}} \) for all values of \( r \). But for the four mirror resonated hologram we now have two independent reflectivities, \( r_1 \) and \( r_4 \), and three independent phases, \( \phi_0, \phi_1 \), and \( \phi_2 \), which greatly complicates the analysis. Therefore, we shall proceed in two stages. The first stage will allow \( r_1 \) and \( r_4 \) to be independent but we shall consider only the solutions where the phases are multiples of \( \pi \). The solutions for \( P'_{\text{out, \text{\it max}}} \) for both the two and three mirror resonated holograms were of this form so this is not a highly restrictive assumption. It will be found in this analysis that \( r_1 = r_4 \) yields the maximum output for a resonated lossy hologram. Thus, the second stage of our analysis will
assume this constraint upon the reflectivities and solve for cases when the phases are not multiples of $\pi$. An exact solution for $\mathcal{P}_{\text{out}}$ will not be obtained for this case but a reasonable approximation will be determined.

The equations for the four mirror resonated hologram are given by equations (2.65) and (2.67).

$$
\mathcal{P}_{\text{out}} = \left| \frac{N}{D} \right|^2
$$

(A.77)

where

\[
\begin{align*}
||N||^2 &= S(1 - r_2^2)(1 - r_4^2)[1 + (1 - A)^2 - 2(1 - A)\cos \phi_2] \\
D &= [1 + (1 - A)r_1e^{-j\phi_0}] [1 + (1 - A)r_4e^{-j\phi_1}] \\
&- Se^{-j\phi_2} [1 + r_1e^{-j(\phi_0 - \phi_2)}][1 + r_4e^{-j(\phi_1 - \phi_2)}] \\
\Rightarrow ||D||^2 &= 1 + S^2 + (1 - A - S)^2(r_1^2 + r_4^2) + [S^2 + (1 - A)^4]r_1^2r_4^2 \\
&+ 2(1 - A - S)r_1[1 + (1 - A)^2r_4^2] \cos \phi_0 \\
&+ 2(1 - A - S)r_4[1 + (1 - A)^2r_1^2] \cos \phi_1 \\
&- 2S[1 + (1 - A)^2]r_1^2r_4^2 \cos \phi_2 \\
&+ 2(1 - A)^2r_1r_4 \cos(\phi_0 + \phi_1) \\
&+ 2(1 - A - S)^2r_1r_4 \cos(\phi_0 - \phi_1) \\
&- 2S(1 - A - S)r_1(1 + r_4^2) \cos(\phi_0 - \phi_2) \\
&- 2S(1 - A - S)r_4(1 + r_1^2) \cos(\phi_1 - \phi_2) \\
&- 2S[1 + (1 - A)^2]r_1r_4 \cos(\phi_0 + \phi_1 - 2\phi_2) \\
&+ 2S^2r_1r_4 \cos(\phi_0 + \phi_1 - 2\phi_2)
\end{align*}
\]

(A.80)

Notice once again that the dependence of $\mathcal{P}_{\text{out}}$ upon $\phi_0$ and $\phi_1$ is entirely due to $||D||^2$. Thus, we can minimize $||D||^2$ in order to maximize $\mathcal{P}_{\text{out}}$ with respect to these phases. Further notice that if we interchange $(\phi_0, r_1)$ with $(\phi_1, r_4)$ in these
equations we will still have the original equation. This symmetry will be used to simplify the mathematics in our analysis.

To determine if an operating point is a maximum or a minimum, it is necessary to check that the first derivative is zero and whether the second derivative is negative or positive, respectively. The relevant first and second derivatives are given below.

\[
\frac{d\|D\|^2}{d\phi_0} = -2r_1 \left\{ (1 - A - S)[1 + (1 - A)^2 r_4^2] \sin \phi_0 
+ (1 - A)^2 r_4 \sin(\phi_0 + \phi_1) + (1 - A - S)^2 r_4 \sin(\phi_0 - \phi_1) 
- S(1 - A - S)(1 + r_4^2) \sin(\phi_0 - \phi_2) 
- S[1 + (1 - A)^2] r_4 \sin(\phi_0 + \phi_1 - \phi_2) 
+ S^2 r_4 \sin(\phi_0 + \phi_1 - 2\phi_2) \right\}
\]  
\hspace{1cm} (A.81)

\[
\frac{d\|D\|^2}{d\phi_1} = -2r_4 \left\{ (1 - A - S)[1 + (1 - A)^2 r_4^2] \sin \phi_1 
+ (1 - A)^2 r_1 \sin(\phi_0 + \phi_1) - (1 - A - S)^2 r_1 \sin(\phi_0 - \phi_1) 
- S(1 - A - S)(1 + r_1^2) \sin(\phi_1 - \phi_2) 
- S[1 + (1 - A)^2] r_1 \sin(\phi_0 + \phi_1 - \phi_2) 
+ S^2 r_1 \sin(\phi_0 + \phi_1 - 2\phi_2) \right\}
\]  
\hspace{1cm} (A.82)

\[
\frac{d\|\mathcal{N}\|^2}{d\phi_2} = 2S \left\{ [1 + (1 - A)^2] r_4^2 r_4^2 \sin \phi_2 
- (1 - A - S)r_1(1 + r_4^2) \sin(\phi_0 - \phi_2) 
- (1 - A - S)r_4(1 + r_1^2) \sin(\phi_1 - \phi_2) 
- [1 + (1 - A)^2] r_1 r_4 \sin(\phi_0 + \phi_1 - \phi_2) 
+ 2Sr_1 r_4 \sin(\phi_0 + \phi_1 - 2\phi_2) \right\}
\]  
\hspace{1cm} (A.83)

\[
\frac{d\mathcal{P}_{\text{out}}}{d\phi_2} = 2S(1 - A)(1 - r_1^2)(1 - r_4^2) \sin \phi_2
\]  
\hspace{1cm} (A.84)

\[
\frac{d\mathcal{P}_{\text{out}}}{d\phi_2} = \frac{1}{\|D\|^4} \left\{ \|D\|^2 \frac{d\|\mathcal{N}\|^2}{d\phi_2} - \|\mathcal{N}\|^2 \frac{d\|D\|^2}{d\phi_2} \right\}
\]  
\hspace{1cm} (A.85)
\[
\frac{d^2\|P\|^2}{d\phi_0^2} = -2r_1 \left\{ (1 - A - S)[1 + (1 - A)^2 r_4^2] \cos \phi_0 \\
+ (1 - A)^2 r_4 \cos(\phi_0 + \phi_1) + (1 - A - S)^2 r_4 \cos(\phi_0 - \phi_1) \\
- S(1 - A - S)(1 + r_4^2) \cos(\phi_0 - \phi_2) \\
- S[1 + (1 - A)^2] r_4 \cos(\phi_0 + \phi_1 - \phi_2) \\
+ S^2 r_4 \cos(\phi_0 + \phi_1 - 2\phi_2) \right\} 
\]
(A.86)

\[
\frac{d^2\|P\|^2}{d\phi_1^2} = -2r_4 \left\{ (1 - A - S)[1 + (1 - A)^2 r_4^2] \cos \phi_1 \\
+ (1 - A)^2 r_1 \cos(\phi_0 + \phi_1) + (1 - A - S)^2 r_1 \cos(\phi_0 - \phi_1) \\
- S(1 - A - S)(1 + r_4^2) \cos(\phi_1 - \phi_2) \\
- S[1 + (1 - A)^2] r_1 \cos(\phi_0 + \phi_1 - \phi_2) \\
+ S^2 r_1 \cos(\phi_0 + \phi_1 - 2\phi_2) \right\} \]
(A.87)

It is obvious by inspection that all of the first derivatives equal zero when

\[
\phi_0 = n\pi \quad (A.88)
\]

\[
\phi_1 = m\pi \quad (A.89)
\]

\[
\phi_2 = p\pi \quad (A.90)
\]

which yields a set of eight possible solutions \((m, n, p)\) either odd or even) for maximizing \(P_{\text{out}}\). We shall proceed to examine these in detail. Later, we will examine other solutions.

If \(\phi_2 = 0 \pmod{2\pi}\) then \(\lim_{A \to 0} P_{\text{out}} = 0\). This is obviously not the solution we want. Therefore, set

\[
\phi_2 = \pi \pmod{2\pi}. \quad (A.91)
\]

This reduces our possible solutions to four: \(\phi_0, \phi_1 = 0 \text{ or } \pi, \phi_2 = \pi \pmod{2\pi}\). We will determine the conditions under which these solutions maximize \(P_{\text{out}}\) by
examining second derivatives. Substituting $\phi_2 = \pi$ into equations (A.86) and (A.87), the second derivatives of $||\mathcal{D}||^2$ with respect to $\phi_0$ and $\phi_1$, yields

$$
\frac{d^2||\mathcal{D}||^2}{d\phi_0^2} \bigg|_{\phi_2=\pi} = -2r_1 \left\{ (1 - A - S) \left[ 1 + S + [S + (1 - A)^2] r_4^2 \right] \cos \phi_0 \\
+ (1 + S)[S + (1 - A)^2] r_4 \cos(\phi_0 + \phi_1) \\
+ (1 - A - S)^2 r_4 \cos(\phi_0 - \phi_1) \right\} 
$$
(A.92)

$$
\frac{d^2||\mathcal{D}||^2}{d\phi_1^2} \bigg|_{\phi_2=\pi} = -2r_4 \left\{ (1 - A - S) \left[ 1 + S + [S + (1 - A)^2] r_4^2 \right] \cos \phi_1 \\
+ (1 + S)[S + (1 - A)^2] r_4 \cos(\phi_0 + \phi_1) \\
+ (1 - A - S)^2 r_4 \cos(\phi_0 - \phi_1) \right\} 
$$
(A.93)

These second derivatives are obviously both negative when $\phi_0 = \phi_1 = 0$, meaning that $||\mathcal{D}||^2$ is maximized at this operating point and hence $P_{\text{out}}$ is minimized. Therefore if $||\mathcal{D}||^2$ is to be minimized, we need only consider the cases where $\phi_0$ or $\phi_1$ equals $\pi$.

Substitute $\phi_0 = \pi$ into equations (A.92) and (A.93) and determine in more detail the conditions on $\phi_1, r_1$, and $r_4$ under which these second derivatives are positive.

$$
\frac{d^2||\mathcal{D}||^2}{d\phi_0^2} \bigg|_{\phi_0=\phi_2=\pi} = 2r_1 \left\{ [(1 - A - S)^2 + (1 + S)[S + (1 - A)^2]] r_4 \cos \phi_1 \\
+ (1 - A - S) \left[ 1 + S + [S + (1 - A)^2] r_4^2 \right] \right\} 
$$
(A.94)

$$
\frac{d^2||\mathcal{D}||^2}{d\phi_1^2} \bigg|_{\phi_0=\phi_2=\pi} = -2r_4 \cos \phi_1 \left\{ (1 - A - S)(1 + S) \\
- [(1 - A - S)^2 + (1 + S)[S + (1 - A)^2]] r_1 \\
+ (1 - A - S)[S + (1 - A)^2] r_4^2 \right\} 
$$
(A.95)

Examining equation (A.94) it is apparent that this second derivative is always

123
positive when $\phi_1 = 0$, but when $\phi_1 = \pi$ it is positive iff

$$(1 + S)(1 - A - S) - [(1 - A - S)^2 + (1 + S)[S + (1 - A)^2]] r_4$$

$$+ (1 - A - S)[S + (1 - A)^2] r_4^2 > 0$$

(A.96)

This quadratic has two solutions for $r_4$, one of which is rejected because we have restricted $r_4 < 1$. The other solution is

$$r_4 < r'_b$$

(A.97)

where the boundary value is given by

$$r'_b = \frac{1 - A - S}{S + (1 - A)^2}.$$  

(A.98)

Recall that for the three mirror resonated hologram we also had a boundary value for $r$, given by equation (A.58), at which the character of the solution changed. These boundary values have different expressions but are equal when $A = 0$.

Examining equation (A.95) to determine when the second derivative with respect to $\phi_1$ is positive we find the same boundary appearing except that the constraint is now on $r_1$. This second derivative is positive iff $\phi_1 = 0$ when $r_1 > r'_b$ and $\phi_1 = \pi$ when $r_1 < r'_b$.

We can summarize these results for $\phi_0 = \pi$ (mod $2\pi$) as follows: $||D||^2|_{\phi_0=\phi_2=\pi}$ is a maximum iff $r_1, r_4 < r'_b$ when $\phi_1 = \pi$ and $r_1 > r'_b$ when $\phi_1 = 0$.

This same procedure is repeated for equations (A.92) and (A.93) when $\phi_1 = \pi$.

Because of the symmetry of the equations we will get the same results except with $(\phi_0, r_1)$ interchanged with $(\phi_1, r_4)$.

The net result of this analysis is to reduce our original eight possible solutions (generated by equations (A.88)-(A.90) for maximizing $P_{\text{out}}$ with respect to the phases $\phi_0, \phi_1$, and $\phi_2$ to only three solutions, each valid for a specific range of reflectivities. These three solutions are
(i) \( \phi_0 = \phi_1 = \phi_2 = \pi \pmod{2\pi} \) if \( r_1, r_4 < r_b' \).

(ii) \( \phi_0 = 0, \phi_1 = \phi_2 = \pi \pmod{2\pi} \) if \( r_4 > r_b' \).

(iii) \( \phi_0 = \phi_2 = \pi, \phi_1 = 0 \pmod{2\pi} \) if \( r_1 > r_b' \).

Thus, if \( r_1, r_4 > r_b' \) then both solutions (ii) and (iii) are valid local maximums.

Now that the phases have been defined, the next step is to maximize \( P_{\text{out}} \) in terms of the reflectivities \( r_1 \) and \( r_4 \). This shall be done for each of the three solutions above. The procedure will be to first evaluate \( P_{\text{out}} \), given by equation (A.77), for the phases being considered to determine \( P_{\text{out}}' \); then set the first derivatives of \( P_{\text{out}}' \) with respect to \( r_1 \) and \( r_4 \) equal to zero; and finally solve for the values of \( r_1 \) and \( r_4 \) which maximize \( P_{\text{out}}' \).

Solution (i) \([\phi_0 = \phi_1 = \phi_2 = \pi \pmod{2\pi}]\) yields

\[
P'_{\text{out},i} = \frac{S(2 - A)^2(1 - r_1^2)(1 - r_4^2)}{[1 - (1 - A)r_1][1 - (1 - A)r_4] + S(1 + r_1)(1 + r_4)}^2
\]

(A.99)

\[
\frac{dP'_{\text{out},i}}{dr_1} = 0 \iff (1 - A - S)(1 + r_1 r_4) = [S + (1 - A)^2]r_4 + (1 + S)r_1
\]

(A.100)

\[
\frac{dP'_{\text{out},i}}{dr_4} = 0 \iff (1 - A - S)(1 + r_1 r_4) = [S + (1 - A)^2]r_1 + (1 + S)r_4
\]

(A.101)

Subtracting equation (A.101) from equation (A.100) yields

\[
A(2 - A)(r_1 - r_4) = 0
\]

(A.102)

\[
\Rightarrow A = 0 \text{ or } r_1 = r_4
\]

(A.103)

We shall first consider the case where \( A = 0 \) and then the case where \( r_1 = r_4 \).

Substituting \( A = 0 \) into either equation (A.100) or (A.101) yields
\[
\frac{(1-r_1)(1-r_4)}{(1+r_1)(1+r_4)} = S. \tag{A.104}
\]

Substituting this solution into \( P'_{\text{out},i} \), given by (A.99), and evaluating when \( A = 0 \) yields

\[
P_{\text{out}}\bigg|_{\text{max}} = 1. \tag{A.105}
\]

So for a lossless resonated hologram it is sufficient but not necessary that \( r_1 = r_4 \), as assumed in the literature [63], in order to achieve \( P_{\text{out}} = 1 \).

The other solution is \( r_1 = r_4 = r \). Substitute this into \( P'_{\text{out},i} \), given by equation (A.99), and maximize \( P'_{\text{out},i} \) with respect to \( r \).

\[
P'_{\text{out},i} = \frac{S(2-A)^2(1-r^2)^2}{[1-(1-A)r]^2 + S(1+r)^2]^2} \tag{A.106}
\]

\[
\frac{dP'_{\text{out},i}}{dr} = 0 \iff r^2 - \frac{1+2S+(1-A)^2}{1-A-S}r + 1 = 0 \tag{A.107}
\]

This quadratic equation has two roots. The larger one is rejected because we require \( r < 1 \), leaving the solution \( r = r' \) given by

\[
r' = \frac{1+2S+(1-A)^2}{2(1-A-S)} - \sqrt{\left[\frac{1+2S+(1-A)^2}{2(1-A-S)}\right]^2 - 1} \tag{A.108}
\]

\[
= 1 - \frac{(2-A)\sqrt{4S+A^2} - (4S+A^2)}{2(1-A-S)} \tag{A.109}
\]

\[
= \frac{2}{H} \left(1 - \sqrt{1-H}\right) - 1 \tag{A.110}
\]

where

\[
H = \frac{4(1-A-S)}{(2-A)^2} \tag{A.111}
\]

This factor of \( H \) appears to have no physical interpretation but is useful for notational purposes.
There does not exist a simple expression for $P_{\text{out}}|_{\text{max}}$ when the expression for the optimum reflectivity, $r'$, given by equation (A.110) is substituted into the expression for $P'_{\text{out},i}$ given by equation (A.106). But it can be seen in the graphs shown in figures 10–12 in Chapter II that the value for the four mirror resonated hologram is greater than or equal to the value for the three mirror resonated hologram, where equality holds only if $A = 0$.

We now examine solution (ii) [$\phi_0 = 0, \phi_1 = \phi_2 = \pi, r_4 > r'_6$]. We shall proceed in much the same manner. Substitute these conditions into $P_{\text{out}}$ given by equation (A.77). This yields

\[
P'_{\text{out},ii} = \frac{S(2 - A)^2(1 - r_1^2)(1 - r_4^2)}{[1 + (1 - A)r_1][1 - (1 - A)r_4] + S(1 - r_1)(1 + r_4)^2} \tag{A.112}
\]

\[
\frac{dP'_{\text{out},ii}}{dr_1} = 0 \iff (1 - A - S)(1 - r_1r_4) = [S + (1 - A)^2]r_4 - (1 + S)r_1 \tag{A.113}
\]

\[
\frac{dP'_{\text{out},ii}}{dr_4} = 0 \iff (1 - A - S)(1 - r_1r_4) = -[S + (1 - A)^2]r_1 + (1 + S)r_4 \tag{A.114}
\]

Subtracting equation (A.114) from (A.113) yields

\[
A(A - 2)(r_1 + r_4) = 0 \tag{A.115}
\]

\[
\Rightarrow A = 0 \text{ or } r_1 = -r_4 \tag{A.116}
\]

Substituting $A = 0$ into either equation (A.113) or (A.114) yields

\[
\frac{(1 + r_1)(1 - r_4)}{(1 - r_1)(1 + r_4)} = S. \tag{A.117}
\]

Substituting this equation into (A.112) when $A = 0$ yields $P_{\text{out}}|_{\text{max}} = 1$. This gives yet another solution for 100% output.
If \( A \neq 0 \) then we are left with the solution \( r_1 = -r_4 \) which falls outside of the range of allowed values of reflectivity, since \( r_1 \) and \( r_4 \) were defined to be nonegative. Therefore, the maximum under these conditions must lie at the extreme

\[
r_1 = 0
\]

Substituting this into equation (A.114) yields

\[
r_4 = \frac{1 - A - S}{1 + S} < r'_b
\]

But solution (\( ii \)) is valid only for the range \( r_4 > r'_b \). Because the optimum solution lies in the range of solution (\( i \)), we conclude that solution (\( ii \)) does not yield a global maximum for \( P_{\text{out}} \).

Solution (\( iii \)) is merely a dual of solution (\( ii \)), \((\phi_0, r_1) \leftrightarrow (\phi_1, r_4)\), and hence will have similar results. Therefore, we can conclude that \( P_{\text{out}} \)\(_{\text{max}} \) is given by solution (\( i \)):

\[
\phi_0 = \phi_1 = \phi_2 = \pi \quad (\text{mod } 2\pi)
\]

\[
r_1 = r_4 = r' \quad (A.122)
\]

where \( r' \) is given by equation (A.110).

These results are quoted in Chapter II as equations (2.121)-(2.124).

We shall now look at solutions when the reflectivities have fixed, predetermined values and only the phases are allowed to vary. We will restrict this analysis to the case when \( r_1 = r_4 = r \). For a lossy hologram the optimum reflectivities were found to be equal, so it is reasonable to examine the effect upon the optimum phases when the reflectivities are equal but not optimized.

We shall proceed in much the same manner as before. We will set the first derivatives with respect to the phases equal to zero and solve for the phases of
interest. As with the three mirror resonated hologram we will find that there exists a boundary value for \( r \) at which the optimum phases cease to be equal to \( \pi \).

This is the case that will be pursued here.

Substituting \( r_1 = r_4 = r \) into equations (A.81)-(A.84), the equations for the first derivatives with respect to the phases, yields

\[
\frac{d\|D\|^2}{d\phi_0} = -2r \left\{(1-A-S)[1+(1-A)^2r^2] \sin \phi_0 + (1-A)^2r \sin(\phi_0 + \phi_1) + (1-A-S)^2r \sin(\phi_0 - \phi_1) - S(1-A-S)(1+r^2) \sin(\phi_0 - \phi_2) - S[1+(1-A)^2]r \sin(\phi_0 + \phi_1 - \phi_2) + S^2r \sin(\phi_0 + \phi_1 - 2\phi_2)\right\} \tag{A.123}
\]

\[
\frac{d\|D\|^2}{d\phi_1} = -2r \left\{(1-A-S)[1+(1-A)^2r^2] \sin \phi_1 + (1-A)^2r \sin(\phi_0 + \phi_1) - (1-A-S)^2r \sin(\phi_0 - \phi_1) - S(1-S-A)(1+r^2) \sin(\phi_1 - \phi_2) - S[1+(1-A)^2]r \sin(\phi_0 + \phi_1 - \phi_2) + S^2r \sin(\phi_0 + \phi_1 - 2\phi_2)\right\} \tag{A.124}
\]

\[
\frac{d\|D\|^2}{d\phi_2} = 2S \left\{[1+(1-A)^2r^4] \sin \phi_2 - (1-A-S)r(1+r^2)[\sin(\phi_0 - \phi_2) + \sin(\phi_1 - \phi_2)] - [1+(1-A)^2]r^2 \sin(\phi_0 + \phi_1 - \phi_2) + 2Sr^2 \sin(\phi_0 + \phi_1 - 2\phi_2)\right\} \tag{A.125}
\]

\[
\frac{d\|N\|^2}{d\phi_2} = 2S(1-A)(1-r^2)^2 \sin \phi_2 \tag{A.126}
\]

When \( P_{\text{out}} \) is maximized, \( \frac{d\|D\|^2}{d\phi_j} = 0 \). Setting the first derivative of \( \|D\|^2 \) with respect to \( \phi_0 \) and \( \phi_1 \), equations (A.123) and (A.124), equal to zero and then
subtracting the latter from the former yields

\[ [1 + (1 - A)^2 r^2] (\sin \phi_0 - \sin \phi_1) + 2(1 - A - S) r \sin(\phi_0 - \phi_1) \]
\[ - S(1 + r^2) (\sin(\phi_0 - \phi_2) - \sin(\phi_1 - \phi_2)) = 0 \]  
(A.127)

An obvious solution to this equation is

\[ \phi_0 = \phi_1 \equiv \phi \]  
(A.128)

While this may not be the only solution, it is a useful one and it will be considered here. Substituting this solution along with \(r_1 = r_4 = r\) into the expression for \(\|D\|^2\) given by (A.80) yields

\[ \|D\|^2 = 1 + S^2 + 4(1 - A - S)^2 r^2 + [S^2 + (1 - A)^4] r^4 \]
\[ + 4(1 - A - S)r[1 + (1 - A)^2 r^2] \cos \phi \]
\[ - 2S[1 + (1 - A)^2 r^4] \cos \phi_2 \]
\[ + 2(1 - A)^2 r^2 \cos(2\phi) \]
\[ - 4S(1 - A - S)r(1 + r^2) \cos(\phi - \phi_2) \]
\[ - 2S[1 + (1 - A)^2] r^2 \cos(2\phi - \phi_2) \]
\[ + 2S^2 r^2 \cos(2\phi - 2\phi_2) \]  
(A.129)

We will now maximize \(P_{\text{out}}\) in terms of \(\phi\) and \(\phi_2\) for constant \(r\), considering the cases when \(\phi\) and \(\phi_2\) do not equal \(\pi\), since that case has already been considered. Due to the nonlinear nature of these equations it does not appear to be possible to solve the completely general case. However, it can be observed that \(P_{\text{out}}\) is very sensitive to the value of \(\phi_2\), due to the dependence of \(\|N\|^2\) upon \(\phi_2\) (equation (A.78)). To get an approximate solution we shall continue to let \(\phi_2 = \pi\). We shall see that this is not always the optimum phase, but it does give a good approximation for the value of \(P'_{\text{out}}\).
Substituting $\phi_2 = \pi$ into equation (A.129) yields

$$\|D\|^2|_{\phi_2=\pi} = (1+S)^2 + 4(1-A-S)^2 r^2 + [S + (1-A)^2]^2 r^4$$

$$+ 4(1-A-S) [1+S + [S + (1-A)^2]^2] r \cos \phi$$

$$+ 2(1+S)[S + (1-A)^2] r^2 \cos(2\phi)$$

$$(A.130)$$

$$= [1 + S - [S + (1-A)^2]^2 r^2] + 4(1-A-S)^2 r^2$$

$$+ 4(1-A-S) [1+S + [S + (1-A)^2]^2] r \cos \phi$$

$$+ 4(1+S)[S + (1-A)^2] r^2 \cos^2 \phi$$

$$(A.131)$$

Because this equation is quadratic in $\cos \phi$, we can immediately write that it is minimized (and hence $P_{\text{out}}$ is maximized) when

$$\cos \phi = -\frac{4(1-A-S) [1+S + [S + (1-A)^2]^2] r}{8(1+S)[S + (1-A)^2] r^2}$$

$$(A.132)$$

$$= -\frac{(1-A-S) [1+S + [S + (1-A)^2]^2] r}{2(1+S)[S + (1-A)^2] r}$$

$$(A.133)$$

if this exists. Otherwise the minimum occurs at the extreme $\cos \phi = -1$ ($\phi = \pi$ (mod 2\pi)) as previously determined. This solution is worth pursuing because it indicates the possibility of a situation where $\phi = \pi$ does not maximize $P_{\text{out}}$.

If equation (A.133) is to have a solution then

$$\frac{(1-A-S) [1+S + [S + (1-A)^2]^2] r}{2(1+S)[S + (1-A)^2] r} \leq 1$$

$$(A.134)$$

$$r^2 - \frac{2(1+S)}{1-A-S} r + \frac{1+S}{S + (1-A)^2} \leq 0$$

$$(A.135)$$

$$\Leftrightarrow r_c \leq r \leq r_d$$

$$(A.136)$$

where

$$r_d = \frac{1+S + (2-A) \sqrt{\frac{S+S^2}{S+(1-A)^2}}}{1-A-S}$$

$$(A.137)$$
\[ r_c = \frac{1 + S - (2 - A)\sqrt{\frac{S + S^2}{S + (1 - A)^2}}}{1 - A - S} \]  
\[ \text{(A.138)} \]

We have defined \( r_c \) as the critical value of the reflectance at which the character of the solution changes. The upper bound \( r_d \) is greater than one, hence our prior constraint of \( r < 1 \) remains unchanged.

We will now test our assumption that \( \phi_2 = \pi \) is still the phase that maximizes \( P_{\text{out}} \). To do this we shall evaluate the derivative of \( P_{\text{out}} \) with respect to \( \phi_2 \) when \( \phi_2 = \pi \) to see if it is zero. The derivative of \( P_{\text{out}} \) with respect to \( \phi_2 \) is given by equation (A.85), where \( \frac{d\|x\|^2}{d\phi_2} \) is given by equation (A.84) and \( \frac{d\|d\|^2}{d\phi_2} \) is given by equation (A.83). Substitute into these equations the conditions \( \phi_0 = \phi_1 = \phi \), \( r_1 = r_4 = r \), and \( \phi_2 = \pi \). This yields

\[
\left. \frac{dP_{\text{out}}}{d\phi_2} \right|_{\phi_2=\pi} = -\frac{8S^2(2 - A)^2r(1 - r^2)^2}{\|D\|^4} A^2 \sin \phi \left\{ (1 - A - S)(1 + r^2) + [1 + 2S + (1 - A)^2]r \cos \phi \right\} 
\]
\[ \text{(A.139)} \]

This derivative equals zero under three conditions:

1. \( \phi = n\pi \)  
2. \( A = 0 \)  
3. \( \cos \phi = \frac{(1 - A - S)(1 + r^2)}{[1 + 2S + (1 - A)^2]r} \)  
\[ \text{(A.140)} \]  
\[ \text{(A.141)} \]  
\[ \text{(A.142)} \]

The first condition has already been considered. The second condition will be considered as a special subset of the more general lossy hologram analysis. We would like the third condition to yield the same value for \( \phi \) as we derived previously in equation (A.133) when considering \( \frac{dP_{\text{out}}}{d\phi} = 0 \), but unfortunately this is not the case. This means that our solution \( \phi_2 = \pi \) with \( \phi \) defined by equation (A.133) does not yield the maximum value of \( P_{\text{out}} \) for a lossy hologram.
This does, however, give a good approximation for $P_{\text{out}}'$ for low loss holograms. When $A = 0$ (lossless hologram) equations (A.133) and (A.142) are equivalent and hence we have an exact solution for $P_{\text{out}}'$. Equations refa103 and (A.142) are both slowly and smoothly varying functions of $A$, hence evaluating $P_{\text{out}}$ at $\phi_2 = \pi$ and $\phi$ given by equation (A.133) yields a good approximation for $P_{\text{out}}'$ for low loss holograms.

To determine this approximate value of $P_{\text{out}}'$, we substitute the value of $\phi$ given by equation (A.133) into the expression for $\|D\|^2$ given by equation (A.131) and $\phi_2 = \pi$ into the expression for $\|N\|^2$ given by equation (A.78) and use these to compute our approximate $P_{\text{out}}'$. This yields

$$P_{\text{out}}' \approx \frac{(1 + S)(S + (1 - A)^2)(1 - r^2)^2}{[1 + S - (S + (1 - A)^2)r^2]^2}$$ \hspace{1cm} (A.143)

It is interesting to note that when $A = 0$, we have $P_{\text{out}}' = 1$. Thus, as with the three mirror resonated hologram, we find that for a lossless hologram there is not just a single value of $r$ which yields 100% output but rather a whole range of values, $r_c \leq r < 1$, a range that is very large even for small values of $S$.

To recap, we have determined expressions for $P_{\text{out}}'$ when the reflectivities are equal. For $r < r_c$ we have an exact expression for $P_{\text{out}}'$. But when $r > r_c$ we have only an approximate expression which, at the least, is a lower bound for $P_{\text{out}}'$. These results are quoted in Chapter II as equations (2.126)-(2.131).

### A.3 Phase Adjustments

This section is concerned with the phase adjustment procedures used in Chapter III, Section 3. In this section mathematical proofs of the validity of these phase adjustment procedures for a single frequency will be presented. They are given just for the three and four mirror resonated holograms because the two mirror res-
onated hologram is a trivial case. Throughout this section equations (2.58), (2.60) and (2.61) will be used. For convenience they are reproduced below.

\[ P_{out} = \left| \frac{N}{D} \right|^2 \] (A.144)

where

\[ N = \sqrt{S} \sqrt{(1 - r_1^2)(1 - r_4^2)} \left[ 1 - (1 - \mathcal{A})r_2r_3e^{-j\phi_2} \right] \] (A.145)

\[ D = \left[ 1 + (1 - \mathcal{A})r_1r_3e^{-j\phi_0} \right] \left[ 1 + (1 - \mathcal{A})r_2r_4e^{-j\phi_1} \right] - Se^{-j\phi_2} [r_2 + r_1e^{-j(\phi_0 - \phi_2)}] [r_3 + r_4e^{-j(\phi_1 - \phi_2)}] \] (A.146)

We will also make use of equations (2.42) and (2.43).

\[ \phi_3 = \phi_0 + \phi_1 - \phi_2 \] (A.147)

\[ \phi_4 = \phi_1 - \phi_2 \] (A.148)

A.3.1 Three Mirror Resonated Hologram #1

The phase adjustment procedure for three mirror resonated hologram #1 is as follows: Block mirror \( M_2 \) and adjust the axial position of \( M_1 \) until the output is maximized; unblock \( M_2 \) and adjust the axial position of \( M_2 \) until the output is maximized. This is done to set phases \( \phi_0 = \phi_2 = \pi \) (mod 2\( \pi \)). We will now show that this procedure does set these phases properly.

The first step in the phase adjustment procedure for the three mirror resonated hologram #1 \( (r_4 = 0) \) is to block \( M_2 \). The operating parameters are

\[ r_2 = r_4 = 0, r_3 = 1 \] (A.149)

\[ \Rightarrow N = \sqrt{S} \sqrt{(1 - r_1^2)} \] (A.150)

\[ D = 1 + (1 - \mathcal{A} - S)r_1e^{-j\phi_0} \] (A.151)
By inspection, $||\mathcal{D}||$ is minimized when $\phi_0 = \pi \pmod{2\pi}$, which maximizes $P_{\text{out}}$.

When $M_2$ is unblocked we have

$$r_2 = r_3 = 1, r_4 = 0, \phi_0 = \pi \pmod{2\pi} \quad (A.152)$$

$$\Rightarrow \mathcal{N} = \sqrt{5} \sqrt{1 - r_1^2} [1 - (1 - A)e^{-j\phi_2}] \quad (A.153)$$

$$\mathcal{D} = 1 - (1 - A - S)r_1 - Se^{-j\phi_2} \quad (A.154)$$

$$\Rightarrow P_{\text{out}} = \frac{S[1 - r_1^2][1 + (1 - A)^2 - 2(1 - A)\cos \phi_2]}{[1 - (1 - A - S)r_1]^2[1 + B^2 - 2B\cos \phi_2]} \quad (A.155)$$

where

$$B = \frac{S}{1 - (1 - A - S)r_1} \quad (A.156)$$

This is in the same form as the general case in equation (A.8) (form #2), where $A = 1 - A$. In order to use the solution $\phi_2 = \pi \pmod{2\pi}$ maximizes this form, we must prove that the requisite conditions $0 < B < A \leq 1$ are satisfied.

Obviously

$$A \leq 1 \quad (A.157)$$

$$B > 0 \quad (A.158)$$

Furthermore

$$(1 - A - S)[1 - (1 - A)r_1] > 0 \quad (A.159)$$

$$\Leftrightarrow (1 - A)[1 - (1 - A)r_1] + S(1 - A)r_1 > S \quad (A.160)$$

$$\Leftrightarrow 1 - A > \frac{S}{1 - (1 - A - S)r_1} \quad (A.161)$$

$$\Leftrightarrow A > B \quad (A.162)$$

Therefore, we can use the result given in equation (A.18).

$$\phi'_2 = \pi \pmod{2\pi} \quad (A.163)$$
A.3.2 Three Mirror Resonated Hologram #2

The phase adjustment procedure for three mirror resonator #2 is as follows: Block mirror $M_3$ and adjust the axial position of mirror $M_1$ until the output is maximized; unblock $M_3$ and adjust the axial position of $M_3$ until the output is maximized. The aim of this procedure is to set phases $\phi_1 = \phi_2 = \pi \pmod{2\pi}$. It will now be shown that this procedure does that.

The first step for three mirror resonated hologram #2 ($r_1 = 0$) is to block $M_3$ which gives

$$r_1 = r_3 = 0, r_2 = 1 \quad (A.164)$$

$$\Rightarrow \mathcal{N} = \sqrt{S}\sqrt{1-r_2^2} \quad (A.165)$$

$$\mathcal{D} = 1 + (1 - A - S)r_4 e^{-j\phi_1} \quad (A.166)$$

By inspection $||\mathcal{D}||$ is minimized when $\phi_1 = \pi \pmod{2\pi}$, which maximizes $P_{out}$.

When $M_3$ is unblocked we have

$$r_1 = 0, r_2 = r_3 = 1, \phi_1 = \pi \pmod{2\pi} \quad (A.167)$$

$$\Rightarrow \mathcal{N} = \sqrt{S}\sqrt{1-r_2^2}[1 - (1 - A)e^{-j\phi_2}] \quad (A.168)$$

$$\mathcal{D} = 1 - (1 - A - S)r_4 - Se^{-j\phi_2} \quad (A.169)$$

$$\Rightarrow P_{out} = \frac{S(1-r_4^2)[1 + (1 - A)^2 - 2(1 - A)\cos \phi_2]}{[1 - (1 - A - S)r_4]^2[1 + B^2 - 2B\cos \phi_2]} \quad (A.170)$$

where

$$B = \frac{S}{1 - (1 - A - S)r_4} \quad (A.171)$$

which is the same as equation (A.155) (three mirror resonated hologram #1) if $r_1 \leftrightarrow r_4$. Therefore, the same solution results. $P_{out}$ is maximized when

$$\phi_2' = \pi \pmod{2\pi} \quad (A.172)$$
A.3.3 Four Mirror Resonated Hologram

The phase adjustment procedure for the four mirror resonated hologram is as follows: Block mirrors $M_2$ and $M_3$ and adjust the axial position of mirror $M_1$ until the output is minimized; unblock mirror $M_3$ and adjust the axial position of $M_3$ until the output is maximized; unblock $M_2$ and adjust its axial position until the output is maximized. The goal of this procedure is to set $\phi_0 = \phi_1 = \phi_2 = \pi \pmod{2\pi}$. It will now be shown that this procedure does set the proper phases.

In step 1 of the phase adjustment procedure for the four mirror resonated hologram, mirrors $M_2$ and $M_3$ are blocked. The system is operating effectively as if

$$r_2 = r_3 = 0.$$  \(\text{(A.173)}\)

Substituting equation $r_2 = r_3 = 0$ into (A.145) and (A.146) yields

$$\mathcal{N} = \sqrt{S} \sqrt{(1 - r_1^2)(1 - r_4^2)}$$ \(\text{(A.174)}\)

$$\mathcal{D} = 1 - S r_1 r_4 e^{-i\phi_3}$$ \(\text{(A.175)}\)

It can be seen by inspection that $||\mathcal{D}||$ is maximized when $\phi_3 = \pi \pmod{2\pi}$, thus minimizing $P_{\text{out}}$.

In step 2 mirror $M_3$ is unblocked. The operating parameters are now

$$r_2 = 0, r_3 = 1, \phi_3 = \pi$$ \(\text{(A.176)}\)

$$\Rightarrow \mathcal{N} = \sqrt{S} \sqrt{(1 - r_1^2)(1 - r_4^2)}$$ \(\text{(A.177)}\)

$$\mathcal{D} = 1 + S r_1 r_4 - (1 - A - S) r_1 e^{i\phi_4}$$ \(\text{(A.178)}\)

By inspection of equation (A.178) it is seen that $||\mathcal{D}||$ is minimized when $\phi_4 = 0 \pmod{2\pi}$, which maximizes $P_{\text{out}}$. 

137
In step 3 mirror $M_2$ is also unblocked. The operating parameters are now

$$ r_2 = r_3 = 1, \phi_3 = \pi, \phi_4 = 0 \pmod{2\pi} \quad (A.179) $$

$$ \Rightarrow N = \sqrt{S}\sqrt{(1-r_1^2)(1-r_4^2)[1-(1-A)e^{-j\phi_2}]} \quad (A.180) $$

$$ D = 1 - (1-A-S)r_1 + Sr_1 r_4 $$

$$ \Rightarrow P_{\text{out}} = \frac{S(1-r_1^2)(1-r_4^2)[1+(1-A)^2 - 2(1-A)\cos\phi_2]}{[1-(1-A-S)r_1 + Sr_1 r_4]^2[1+B^2 - 2B\cos\phi_2]} \quad (A.182) $$

where

$$ B = \frac{S+(1-A)^2r_1 r_4 - (1-A-S)r_4}{1-(1-A-S)r_1 + Sr_1 r_4} \quad (A.183) $$

It is seen that this is in the form of equation (A.8) (form #2), where $A = 1 - A$. In order to use the result of that analysis, we must prove that $-1 < B < A \leq 1$.

This is not difficult but does require several steps. We know

$$ A = 1 - A \quad (A.184) $$

$$ \Rightarrow A \leq 1 \quad (A.185) $$

We also know

$$ [1-(1-A)r_1][1-(1-A)r_4] + S(1+r_1)(1+r_4) > 0 \quad (A.186) $$

$$ \Leftrightarrow [1-(1-A-S)r_1 + Sr_1 r_4] + S + (1-A)^2r_1 r_4 $$

$$ - (1-A-S)r_4 > 0 \quad (A.187) $$

$$ \Leftrightarrow \frac{S+(1-A)^2r_1 r_4 - (1-A-S)r_4}{1-(1-A-S)r_1 + Sr_1 r_4} > -1 \quad (A.188) $$

$$ \Leftrightarrow B > -1 \quad (A.189) $$

Finally,

$$ 1 + r_4 > (1-A)r_1(1 + r_4) \quad (A.190) $$
\[ 1 > (1 - A)r_1 + (1 - A)r_1r_4 - r_4 \quad (A.191) \]
\[ (1 - A - S) > (1 - A - S)(1 - A)r_1 + (1 - A - S)(1 - A)r_1r_4 - (1 - A - S)r_4 \quad (A.192) \]
\[ (1 - A) - (1 - A - S)(1 - A)r_1 + S(1 - A)r_1r_4 > S + (1 - A)^2r_1r_4 - (1 - A - S)r_4 \quad (A.193) \]
\[ 1 - A > \frac{S + (1 - A)^2r_1r_4 - (1 - A - S)r_4}{1 - (1 - A - S)r_1 + Sr_1r_4} \quad (A.194) \]
\[ A > B \quad (A.195) \]

We have proven that

\[-1 < B < A \leq 1 \quad (A.196)\]

so we can use the solution of equation (A.18)

\[ \phi_2' = \pi \pmod{2\pi}. \quad (A.197) \]

Three phases have now been fixed, giving

\[ \phi_0 = \phi_3 - \phi_4 = \pi \pmod{2\pi} \quad (A.198) \]
\[ \phi_1 = \phi_4 + \phi_2 = \pi \pmod{2\pi} \quad (A.199) \]
\[ \phi_2 = \pi \pmod{2\pi} \quad (A.200) \]

which will maximize \( P_{\text{out}} \) as desired.

**A.4 Dual Frequency: Bounds on Maximum Output Power**

We will now determine the least upper bound and greatest lower bound upon the relative output power of the resonated hologram when the input power is distributed between two frequencies. First we will define the terms and expressions used in this section. Then we will show that \( P_{\text{out},2f}(\rho) \) is symmetric about \( \rho = .5 \).
This will allow us to restrict our analysis to the range \(0 \leq \rho \leq .5\). Finally, we will determine general expressions for the upper and lower bounds of \(P'_{\text{out},2f}(\rho)\).

The relative output power with two frequencies, \(P_{\text{out},2f}(\rho, \Delta \phi_L)\), is a function of the frequency separation, \(\Delta f_1\), which is proportional to the phase term \(\phi_f\) and assumed to be fixed in this analysis; the fraction of the input power at frequency \(f_1 + \Delta f_1/2\), which is designated \(\rho\); and the phases of the resonated hologram, represented by \(\Delta \phi_L\), which are adjusted by changing the path length between the mirrors. It is expressed in terms of these parameters by

\[
P_{\text{out},2f}(\rho, \Delta \phi_L) = (1 - \rho)P_{\text{out}}(\Delta \phi_L - \phi_f) + \rho P_{\text{out}}(\Delta \phi_L + \phi_f) \quad (A.201)
\]

where \(P_{\text{out}}\) is the relative output power of the resonated hologram at just a single frequency. \(P_{\text{out}}\) is given by equation (2.145) for the three mirror resonated hologram and equations (2.146) and (2.148) for the four mirror resonated hologram. \(P_{\text{out}}(\phi)\) is a symmetric function about \(\phi = 0\). This fact will be used in our derivation of the bounds.

The maximum output power, \(P'_{\text{out},2f}(\rho)\), is obtained by tuning the phases of the resonated hologram (i.e. by adjusting \(\Delta \phi_L\)). Because the power distribution factor, \(\rho\), between the two frequencies is unknown, we are unable to predict a precise value for \(P'_{\text{out},2f}\), so we will determine the greatest lower bound and least upper bound on this quantity.

We will now show that \(P'_{\text{out},2f}(\rho)\) is symmetric about \(\rho = .5\). Let \(\Delta \phi'_L(\rho)\) be the value of \(\Delta \phi_L\) which maximizes \(P_{\text{out},2f}(\rho, \Delta \phi_L)\). In other words,

\[
P'_{\text{out},2f}(\rho) = P_{\text{out},2f}(\rho, \Delta \phi'_L(\rho))
\]

\[
= (1 - \rho)P_{\text{out}}(\Delta \phi'_L - \phi_f) + \rho P_{\text{out}}(\Delta \phi'_L + \phi_f) \quad (A.203)
\]
We will compare this with $P'_{\text{out},2f}(1-\rho)$ and show that they are equal. By definition,

$$P'_{\text{out},2f}(1-\rho, \Delta\phi_L) = \rho P'_{\text{out}}(\Delta\phi_L - \phi_f) + (1-\rho)P'_{\text{out}}(\Delta\phi_L + \phi_f)$$

(A.204)

Using the symmetry of $P'_{\text{out}}$ yields

$$P'_{\text{out},2f}(1-\rho, \Delta\phi_L) = \rho P'_{\text{out}}(-\Delta\phi_L + \phi_f) + (1-\rho)P'_{\text{out}}(-\Delta\phi_L - \phi_f)$$

(A.205)

Interchanging the last two terms yields

$$P'_{\text{out},2f}(1-\rho, \Delta\phi_L) = (1-\rho)P'_{\text{out}}(-\Delta\phi_L - \phi_f) + \rho P'_{\text{out}}(-\Delta\phi_L + \phi_f)$$

(A.206)

$$= P'_{\text{out}}(\rho, -\Delta\phi_L)$$

(A.207)

We know that $P'_{\text{out},2f}(\rho, \Delta\phi_L)$ is maximized when $\Delta\phi_L = \Delta\phi'_L(\rho)$. Therefore, $P'_{\text{out},2f}(\rho, -\Delta\phi_L)$ is maximized when $\Delta\phi_L = -\Delta\phi'_L(\rho)$, which means

$$P'_{\text{out},2f}(1-\rho) = P'_{\text{out},2f}(1-\rho, -\Delta\phi'_L(\rho))$$

(A.208)

$$= P'_{\text{out}}(\rho, \Delta\phi'_L(\rho))$$

(A.209)

$$= P'_{\text{out},2f}(\rho)$$

(A.210)

as stated. Because of this symmetry, we need only consider the range $0 \leq \rho \leq .5$.

Henceforth, we will restrict $\rho$ to the range $0 \leq \rho \leq .5$. The will simplify our derivation of the upper and lower bounds on $P'_{\text{out},2f}(\rho)$.

We next show that the least upper bound on $P'_{\text{out},2f}(\rho)$ is given by $P'_{\text{out},2f}(0)$ and the greatest lower bound is given by $P'_{\text{out},2f}(.5)$. The proof is as follows:

$$P'_{\text{out},2f}(\rho) = P'_{\text{out},2f}(\rho, \Delta\phi_L(\rho))$$

(A.211)
\begin{align}
\Delta \phi_L'(\rho) & \geq \mathcal{P}_{\text{out},2f}(\rho,-\Delta \phi_L'(\rho)) \\
\Leftrightarrow (1 - \rho)\mathcal{P}_{\text{out}}(\Delta \phi_L'(\rho) - \phi_f) + \rho \mathcal{P}_{\text{out}}(\Delta \phi_L'(\rho) + \phi_f) \\
& \geq (1 - \rho)\mathcal{P}_{\text{out}}(-\Delta \phi_L'(\rho) - \phi_f) + \rho \mathcal{P}_{\text{out}}(-\Delta \phi_L'(\rho) + \phi_f)
\end{align}

Utilizing the symmetry of $\mathcal{P}_{\text{out}}$ on the righthand side of this equation yields
\begin{align}
(1 - \rho)\mathcal{P}_{\text{out}}(\Delta \phi_L'(\rho) - \phi_f) + \rho \mathcal{P}_{\text{out}}(\Delta \phi_L'(\rho) + \phi_f) \\
& \geq (1 - \rho)\mathcal{P}_{\text{out}}(\Delta \phi_L'(\rho) + \phi_f) + \rho \mathcal{P}_{\text{out}}(\Delta \phi_L'(\rho) - \phi_f) \\
\Leftrightarrow (1 - 2\rho)\mathcal{P}_{\text{out}}(\Delta \phi_L'(\rho) - \phi_f) \geq (1 - 2\rho)\mathcal{P}_{\text{out}}(\Delta \phi_L'(\rho) + \phi_f) \\
\Leftrightarrow \mathcal{P}_{\text{out}}(\Delta \phi_L'(\rho) - \phi_f) \geq \mathcal{P}_{\text{out}}(\Delta \phi_L'(\rho) + \phi_f)
\end{align}

Now we will look at the effect of a change in $\rho$ upon $\mathcal{P}_{\text{out},2f}'(\rho)$.
\begin{align}
\mathcal{P}_{\text{out},2f}'(\rho - \Delta \rho, \Delta \phi_L'(\rho)) &= (1 - \rho + \Delta \rho)\mathcal{P}_{\text{out}}(\Delta \phi_L'(\rho) - \phi_f) \\
& \quad + (\rho - \Delta \rho)\mathcal{P}_{\text{out}}(\Delta \phi_L'(\rho) + \phi_f) \\
& = (1 - \rho)\mathcal{P}_{\text{out}}(\Delta \phi_L'(\rho) - \phi_f) + \rho \mathcal{P}_{\text{out}}(\Delta \phi_L'(\rho) + \phi_f) \\
& \quad + \Delta \rho [\mathcal{P}_{\text{out}}(\Delta \phi_L'(\rho) - \phi_f) - \mathcal{P}_{\text{out}}(\Delta \phi_L'(\rho) + \phi_f)] \\
& = \mathcal{P}_{\text{out},2f}'(\rho) + \Delta \rho [\mathcal{P}_{\text{out}}(\Delta \phi_L'(\rho) - \phi_f) \\
& \quad - \mathcal{P}_{\text{out}}(\Delta \phi_L'(\rho) + \phi_f)]
\end{align}

In equation (A.216) we have the relationship $\mathcal{P}_{\text{out}}(\Delta \phi_L'(\rho) - \phi_f) \geq \mathcal{P}_{\text{out}}(\Delta \phi_L'(\rho) + \phi_f)$, therefore
\begin{align}
\mathcal{P}_{\text{out},2f}'(\rho - \Delta \rho, \Delta \phi_L'(\rho)) \begin{cases}
\geq \mathcal{P}_{\text{out},2f}'(\rho) & \text{if } \Delta \rho \geq 0 \\
\leq \mathcal{P}_{\text{out},2f}'(\rho) & \text{if } \Delta \rho \leq 0
\end{cases}
\end{align}
Restricting $\Delta \rho \geq 0$ and utilizing the definition of $P'_{\text{out},2f}$ gives us the result

$$P'_{\text{out},2f}(\rho - \Delta \rho) \geq P_{\text{out},2f}(\rho - \Delta \rho, \Delta \phi_L(\rho))$$  \hspace{1cm} (A.221)

$$\geq P'_{\text{out},2f}(\rho)$$  \hspace{1cm} (A.222)

This implies that $P'_{\text{out},2f}(0)$ is the least upper bound on $P'_{\text{out},2f}(\rho)$ and $P'_{\text{out},2f}(0.5)$ is the greatest lower bound.

A.5 Summary

In this appendix the mathematical proofs of the results used in the main body of the dissertation were given. The phases and reflectivities which maximized $P_{\text{out}}$ were determined for the two, three, and four mirror resonated holograms. For the three and four mirror resonated holograms the phases which maximized $P_{\text{out}}$ when the reflectivities were fixed at nonoptimal values were also determined. It was found that there were two types of solutions for these phases, depending upon whether the mirror reflectivity was above or below a critical value. For low reflectivities the optimum phases were found to be independent of the value of reflectivity, but for high reflectivities the phases were dependent upon the value of reflectivity. The phase adjustment procedure given in Chapter III was expressed in mathematical terms and shown to set the phases to their desired values. Finally, the upper and lower bounds upon the maximum output power with two frequencies were derived.
APPENDIX B
HOLOGRAM THEORY

In this appendix a theoretical model for a plane-wave hologram will be developed. It will be shown that the results of this model are consistent with experimentally obtained results. The results of this theory will be used to determine the transmission and diffraction coefficients for the hologram. The assumption that such coefficients are symmetric will be considered and a determination of when such an assumption is valid will be made.

B.1 Coupled-wave Theory

In this section the coupled-wave theory will be developed for both transmission and reflection holograms. This development will follow along the lines of Kogelnik's coupled-wave theory [23]. The only difference between these results and those derived by Kogelnik are that the origin was chosen to lie in the center of the hologram. This does not affect the results. It is done to be consistent with the other work in this dissertation.

The model which is used in this analysis is shown in Figure 39. The $z$-axis is chosen orthogonal to the boundaries of the emulsion. The $x$-axis is parallel to the hologram boundaries and halfway between them. The grating vector, $\vec{K}$, is normal to the fringe planes, located in the $x$-$z$ plane, and makes an angle $\phi$ with respect to the $z$-axis. The propagation vector of the incident wave, $\vec{p}$, makes an
angle of $\theta$ with the $z$-axis ($\theta = \theta_0$ at Bragg incidence). The propagation vector of the diffracted wave, $\mathbf{\sigma}$, makes an angle of $\gamma$ with the $z$-axis ($\gamma = \gamma_0$ at Bragg incidence). The sign convention used for all angles is that measuring counterclockwise from the $z$-axis yields a positive angle. Thus, in this figure, $\theta$ and $\phi$ are negative and $\gamma$ is positive. The incident wave is assumed to have a positive $z$ component because this makes the analysis straightforward. An incident wave with a negative $z$ component will be considered in Section 4. It is assumed for convenience that the average index of refraction of the surrounding medium is the same as that of the emulsion. The light is assumed to be monochromatic and polarized perpendicular to the plane of incidence (i.e. TE mode). The case for parallel-polarized light will be considered at the end of this appendix.

The coupled-wave theory assumes that the angle of incidence is at or near the Bragg angle. Furthermore, only two waves are assumed to be present in the
hologram: the incident (reference) wave, R, and the diffracted (signal) wave, S.

In this analysis it will be assumed that the hologram is a dielectric phase hologram. The hologram can be lossy but it is assumed that there is no amplitude modulation.

The following symbols will be used in this analysis.

\[ E(x, z) \] Complex amplitude of the y-component of the electric field

\[ k(x, z) \] propagation constant

\[ \vec{x} \] position vector = \((x, y, z)\)^T

\[ \vec{K} \] grating vector = \(\frac{2\pi}{\Lambda}(\sin \phi, 0, \cos \phi)^T\)

\[ \Lambda \] fringe spacing

\[ c \] velocity of light in free space

\[ \omega \] angular frequency of light

\[ \lambda \] free space wavelength of light

\[ \epsilon(x, z) \] relative dielectric constant of emulsion

\[ \mu \] permeability of emulsion (assume free space)

\[ \sigma \] conductivity of emulsion

\[ \beta \] average propagation constant

\[ \alpha \] absorption constant

\[ \kappa \] coupling constant

Wave propagation in the grating satisfies the scalar wave equation

\[ \nabla^2 E + k^2 E = 0 \] \hspace{1cm} (B.1)

where

\[ k^2 = \frac{\omega^2}{c^2} \epsilon - j \omega \mu \sigma \] \hspace{1cm} (B.2)
The index of refraction is given by

\[ n = n_0 + n_1 \cos(K \cdot x) \]  

(B.3)

If the wave decays little in the distance of one wavelength and the index of refraction modulation is small, i.e.

\[ \frac{2\pi n_0}{\lambda} \gg \alpha \quad \text{and} \quad n_0 \gg n_1, \]  

(B.4)

then we can express the permittivity as

\[ \epsilon = n^2 \]

\[ \simeq n_0^2 + 2n_0n_1 \cos(K \cdot x) \]  

(B.5)

Substituting this equation into the expression for \( k^2 \) given by equation (B.2) yields

\[ k^2 = \beta^2 - 2j\alpha\beta + 2\kappa\beta(e^{jK \cdot x} + e^{-jK \cdot x}) \]  

(B.6)

where

\[ \alpha = \mu \sigma \frac{\mu}{2n_0} \]  

(B.7)

\[ \beta = 2\pi n_0 / \lambda \]  

(B.8)

\[ \kappa = \pi n_1 / \lambda \]  

(B.9)

The total electric field in the emulsion is the sum of the incident and diffracted waves.

\[ E = R(z)e^{-j\rho \cdot x} + S(z)e^{-j\sigma \cdot x} \]  

(B.10)

where \( \rho \) is the propagation vector of the incident wave and \( \sigma \) is the propagation vector of the diffracted wave.

\[ \rho = \beta(\sin \theta, 0, \cos \theta)^T \]  

(B.11)

\[ \sigma = \rho - K \]  

(B.12)
The Bragg condition occurs when

\[ \| \vec{\sigma} \|^2 = \beta^2 \]  

which, by using equation (B.13), can be expressed as

\[ \cos(\phi - \theta_0) = K/(2\beta_0) \]  

This is used to define the Bragg angle \( \theta_0 \) at wavelength \( \lambda_0 \) \( (\beta_0 = \frac{2\pi}{\lambda_0}) \). Evaluating equation (B.13) at Bragg angle \( \theta_0 \) allows us to express \( \vec{\sigma} \) at Bragg incidence in terms of the second Bragg angle \( \gamma_0 \):  

\[ \vec{\sigma} = \beta(\sin \gamma_0, 0, \cos \gamma_0)^T \]  

We can also determine the relationship between the three angles \( \theta_0, \gamma_0, \) and \( \phi \):

\[ \phi = \frac{\theta_0 + \gamma_0 - \pi}{2} \]  

Define the dephasing parameter

\[ \vartheta \equiv (\beta^2 - \| \vec{\sigma} \|^2)/(2\beta) \]  

\[ = K \cos(\phi - \theta) - K^2\lambda/(4\pi n_0) \]  

when equation (B.13) is used to compute \( \| \vec{\sigma} \|^2 \). Obviously, \( \vartheta = 0 \) at Bragg incidence. At non-Bragg incidence we can approximate this dephasing parameter by a first order Taylor series expansion of equation (B.19) about the Bragg angle and wavelength in terms of \( \Delta \theta = \theta - \theta_0 \) and \( \Delta \lambda = \lambda - \lambda_0 \).

\[ \vartheta = \Delta \theta K \sin(\phi - \theta_0) - K^2\Delta \lambda/(4\pi n_0) \]  

\[ = -\Delta \theta K \cos(\theta_0 - \gamma_0)/2 - \Delta \lambda K^2/4\pi n_0 \]
when equation (B.17) is substituted for $\phi$. This expression will be used in Section 3.

In order to solve for the fields within the hologram we shall first substitute equations (B.6) and (B.10) for $k^2$ and $E$ into the scalar wave equation, (B.1).

Use equation (B.12) for $\sigma$ in order to group the terms of this result which have equal exponentials. It is assumed that we are dealing with waves in the region of the Bragg angles so that the waves in the $\rho + K$ and $\sigma - K$ directions can be neglected along with all higher diffracted orders. This leaves us with two terms representing waves in the $\rho$ and $\sigma$ directions. These terms are linearly independent and their sum is always zero. Hence, the coefficients of each of them must be zero.

Setting these coefficients equal to zero yields

$$ R'' - 2jR'p_z - 2j\alpha\beta R + 2\kappa\beta S = 0 \quad (B.22) $$

$$ S'' - 2jS'\sigma_z - 2j\alpha\beta S + 2\beta\delta S + 2\kappa\beta R = 0 \quad (B.23) $$

where primes indicate differentiation with respect to $z$. In addition, it is assumed that the energy interchange between $R$ and $S$ is slow and hence the second derivatives can be neglected as well. Thus equations (B.22) and (B.23) become

$$ c_r R' + \alpha R = -j\kappa S \quad (B.24) $$

$$ c_s S' + (\alpha + j\delta)S = -j\kappa R \quad (B.25) $$

where

$$ c_r \equiv \rho_z/\beta = \cos \theta \quad (B.26) $$

$$ c_s \equiv \sigma_z/\beta = \cos \theta - \frac{K}{\beta} \cos \phi \quad (B.27) $$

$$ \simeq \cos \gamma \quad (B.28) $$

The solution of the coupled wave equation has the form

$$ R(z) = r_1 e^{\gamma_1 z} + r_2 e^{\gamma_2 z} \quad (B.29) $$
\[ S(z) = s_1 e^{\gamma_1 z} + s_2 e^{\gamma_2 z} \] (B.30)

where \( r_i \) and \( s_i \) are constants depending upon the boundary conditions and \( \gamma_1 \) and \( \gamma_2 \) are constants independent of the boundary conditions which will be determined. Inserting the solutions given by equations (B.29) and (B.30) into the coupled equations (B.24) and (B.25) yields

\[ (c_r \gamma_i + \alpha) r_i = -j \kappa s_i \] (B.31)
\[ (c_s \gamma_i + \alpha + j \vartheta) s_i = -j \kappa r_i \] (B.32)

Multiplying these equations together leads to a quadratic equation for \( \gamma_i \) with the solution

\[ \gamma_{1,2} = \frac{-1}{2} \left( \frac{\alpha}{c_r} + \frac{\alpha}{c_s} + j \frac{\vartheta}{c_s} \right) \pm \frac{1}{2} \sqrt{\left( \frac{\alpha}{c_r} - \frac{\alpha}{c_s} - \frac{\vartheta}{c_s} \right)^2 - 4 \frac{\kappa^2}{c_r c_s}} \] (B.33)

**Transmission Hologram**

We shall now solve for \( r_i \) and \( s_i \) using the boundary conditions for a transmission hologram. (A transmission hologram is characterized by the fact that the \( z \) component of the diffracted wave is positive, which is expressed mathematically as \( c_s > 0 \).) For simplicity, the reference wave will be normalized at the incident boundary. At this boundary the signal wave is zero. Thus the boundary conditions are:

\[ R(-d/2) = 1 \] (B.34)
\[ S(-d/2) = 0 \] (B.35)

Substituting boundary conditions (B.34) and (B.35) into solutions (B.29) and (B.30) yields

\[ r_1 e^{-\gamma_1 d/2} + r_2 e^{-\gamma_2 d/2} = 1 \] (B.36)
\[ s_1 e^{-\gamma_1 d/2} + s_2 e^{-\gamma_2 d/2} = 0 \]  \hspace{1cm} (B.37)

We can solve for \( r_1, r_2, s_1, \) and \( s_2 \) by combining equations (B.36) and (B.37) with (B.31) and (B.32).

\[
(c_r \gamma_1 + \alpha) r_1 e^{-\gamma_1 d/2} = -j \kappa s_1 e^{-\gamma_1 d/2} \tag{B.38}
\]
\[
= j \kappa s_2 e^{-\gamma_2 d/2} \tag{B.39}
\]
\[
= -(c_r \gamma_2 + \alpha) r_2 e^{-\gamma_2 d/2} \tag{B.40}
\]
\[
= -(c_r \gamma_2 + \alpha)(1 - r_1 e^{-\gamma_1 d/2}) \tag{B.41}
\]

\[
\Rightarrow r_1 = -\frac{c_r \gamma_2 + \alpha}{c_r(\gamma_1 - \gamma_2)} e^{\gamma_1 d/2} \tag{B.42}
\]
\[
r_2 = \frac{c_r \gamma_1 + \alpha}{c_r(\gamma_1 - \gamma_2)} e^{\gamma_2 d/2} \tag{B.43}
\]

Evaluating equation (B.32) for \( i = 1, 2 \) and adding yields

\[
c_s(\gamma_1 s_1 e^{-\gamma_1 d/2} + \gamma_2 s_2 e^{-\gamma_2 d/2}) + (\alpha + j \theta)(s_1 e^{-\gamma_1 d/2} + s_2 e^{-\gamma_2 d/2}) \]
\[
= -j \kappa (r_1 e^{-\gamma_1 d/2} + r_2 e^{-\gamma_2 d/2}) \tag{B.44}
\]

Substituting equations (B.36) and (B.37) into (B.44) yields

\[
c_s(\gamma_1 - \gamma_2) s_1 e^{-\gamma_2 d/2} = -j \kappa \tag{B.45}
\]
\[
\Rightarrow s_1 = \frac{-j \kappa}{c_s(\gamma_1 - \gamma_2)} e^{\gamma_1 d/2} \tag{B.46}
\]
\[
s_2 = \frac{j \kappa}{c_s(\gamma_1 - \gamma_2)} e^{\gamma_2 d/2} \tag{B.47}
\]

Substituting the solutions given for \( r_1, r_2, s_1, \) and \( s_2 \) given by equations (B.42), (B.43), (B.46), and (B.47) into the solutions for the coupled wave equations given by (B.29) and (B.30) yields

\[
R(z) = \frac{\gamma_1 + \alpha}{\gamma_1 - \gamma_2} e^{\gamma_2(z+d/2)} - \frac{\gamma_2 + \alpha}{\gamma_1 - \gamma_2} e^{\gamma_1(z+d/2)} \tag{B.48}
\]
\[
S(z) = \frac{j \kappa}{c_s(\gamma_1 - \gamma_2)} \left[ e^{\gamma_2(z+d/2)} - e^{\gamma_1(z+d/2)} \right] \tag{B.49}
\]
the coefficients for the fields within the emulsion. Evaluating this at the exit boundary \((z=d/2)\) yields

\[
R(d/2) = \frac{\gamma_1 + \frac{\alpha}{c_r} e^{\gamma_2 d}}{\gamma_1 - \gamma_2} - \frac{\gamma_2 + \frac{\alpha}{c_r} e^{\gamma_1 d}}{\gamma_1 - \gamma_2} 
\]

\[
S(d/2) = \frac{j\kappa}{c_s(\gamma_1 - \gamma_2)} \left( e^{\gamma_2 d} - e^{\gamma_1 d} \right) 
\]

For convenience, define the following parameters.

\[
\xi \equiv \frac{d}{2} \left( \frac{\alpha}{c_r} - \frac{\alpha}{c_s} - \frac{j\vartheta}{c_r} \right) 
\]

\[
\nu \equiv \frac{\kappa d}{\sqrt{c_r |c_s|}} 
\]

\[
c \equiv \frac{c_r}{|c_s|} 
\]

These are a dephasing parameter, a coupling parameter, and a slant factor, respectively. The absolute value sign in the last two equations is introduced so that these definitions are equally valid for the reflection hologram. Substituting these definitions into the equation for \(\gamma_{1,2}\) given by (B.33) yields

\[
\gamma_{1,2} = \frac{1}{d} \left( \xi - \frac{\alpha d}{c_r} \pm j \sqrt{\nu^2 - \xi^2} \right) 
\]

Substituting these same definitions and equation (B.55) for \(\gamma_{1,2}\) into equations (B.50) and (B.51) for the field coefficients at the exit boundary yields

\[
R(d/2) = e^{\xi - \frac{\alpha d}{c_r}} \left( \cos \sqrt{\nu^2 - \xi^2} - \frac{\xi}{\sqrt{\nu^2 - \xi^2}} \sin \sqrt{\nu^2 - \xi^2} \right) 
\]

\[
S(d/2) = -j\sqrt{c} e^{\xi - \frac{\alpha d}{c_r}} \frac{\nu}{\sqrt{\nu^2 - \xi^2}} \sin \sqrt{\nu^2 - \xi^2} 
\]

These expressions will be used later to define the transmission and diffraction coefficients for a transmission hologram.
Reflection Hologram

We can repeat this procedure to solve for \( r_i \) and \( s_i \) using the boundary conditions for a reflection hologram. (A reflection hologram is characterized by the fact that the \( z \) component of the diffracted wave is negative, which is expressed mathematically as \( c_s < 0 \).) For simplicity, the incident wave will again be normalized at the incident boundary. At the other boundary the signal wave is zero. Thus, the boundary conditions are:

\[
R(-d/2) = 1 \quad \text{(B.58)}
\]
\[
S(d/2) = 0 \quad \text{(B.59)}
\]

Substituting these boundary conditions into the general solutions given by (B.29) and (B.30) yields

\[
r_1 e^{-\gamma_1 d/2} + r_2 e^{-\gamma_2 d/2} = 1 \quad \text{(B.60)}
\]
\[
s_1 e^{\gamma_1 d/2} + s_2 e^{\gamma_2 d/2} = 0 \quad \text{(B.61)}
\]

Combining these equations with the coupled equations of (B.31) and (B.32) gives

\[
\begin{align*}
r_1 &= \frac{-\left(\gamma_2 + \frac{\alpha}{c_r}\right)e^{\gamma_2 d}e^{\gamma_1 d/2}}{\left(\gamma_1 + \frac{\alpha}{c_r}\right)e^{\gamma_1 d} - \left(\gamma_2 + \frac{\alpha}{c_r}\right)e^{\gamma_2 d}} \quad \text{(B.62)} \\
r_2 &= \frac{\left(\gamma_1 + \frac{\alpha}{c_r}\right)e^{\gamma_1 d}e^{\gamma_2 d/2}}{\left(\gamma_1 + \frac{\alpha}{c_r}\right)e^{\gamma_1 d} - \left(\gamma_2 + \frac{\alpha}{c_r}\right)e^{\gamma_2 d}} \quad \text{(B.63)} \\
s_1 &= \frac{-i\bar{\kappa}e^{-\gamma_1 d/2}}{\left(\gamma_1 + \frac{\alpha}{c_s} + \frac{j\bar{\zeta}}{c_s}\right)e^{-\gamma_1 d} - \left(\gamma_2 + \frac{\alpha}{c_s} + \frac{j\bar{\zeta}}{c_s}\right)e^{-\gamma_2 d}} \quad \text{(B.64)} \\
s_2 &= \frac{i\bar{\kappa}e^{-\gamma_2 d/2}}{\left(\gamma_1 + \frac{\alpha}{c_s} + \frac{j\bar{\zeta}}{c_s}\right)e^{-\gamma_1 d} - \left(\gamma_2 + \frac{\alpha}{c_s} + \frac{j\bar{\zeta}}{c_s}\right)e^{-\gamma_2 d}} \quad \text{(B.65)}
\end{align*}
\]

Substituting these expressions for the coefficients into the general solutions given...
by (B.29) and (B.30) yields

\[
R(z) = \frac{(\gamma_1 + \frac{\alpha}{c_r})e^{\gamma_1 d}e^{\gamma_2(z+d/2)} - (\gamma_2 + \frac{\alpha}{c_r})e^{\gamma_2 d}e^{\gamma_1(z+d/2)}}{(\gamma_1 + \frac{\alpha}{c_r})e^{\gamma_1 d} - (\gamma_2 + \frac{\alpha}{c_r})e^{\gamma_2 d}} \tag{B.66}
\]

\[
S(z) = \frac{\frac{i k}{c_s} (e^{\gamma_2(z-d/2)} - e^{\gamma_1(z-d/2)})}{(\gamma_1 + \frac{\alpha}{c_s} + \frac{i \omega}{c_s})e^{-\gamma_1 d} - (\gamma_2 + \frac{\alpha}{c_s} + \frac{i \omega}{c_s})e^{-\gamma_2 d}} \tag{B.67}
\]

Evaluating these equations at the boundaries yields

\[
R(d/2) = \frac{\gamma_1 - \gamma_2) e^{(\gamma_1+\gamma_2)d}}{(\gamma_1 + \alpha/c_r)e^{\gamma_1 d} - (\gamma_2 + \alpha/c_r)e^{\gamma_2 d}} \tag{B.68}
\]

\[
S(-d/2) = \frac{\frac{i k}{c_s} (e^{-\gamma_2 d} - e^{-\gamma_1 d})}{(\gamma_1 + \alpha/c_s + \frac{i \omega}{c_s})e^{-\gamma_1 d} - (\gamma_2 + \frac{\alpha}{c_s} + \frac{i \omega}{c_s})e^{-\gamma_2 d}} \tag{B.69}
\]

Substituting the definitions given by equations (B.52)-(B.54) into (B.33) for \(\gamma_1, 2\) yields

\[
\gamma_1, 2 = \frac{1}{d} \left( \xi - \frac{\alpha d}{c_r} \pm \sqrt{\nu^2 + \xi^2} \right) \tag{B.70}
\]

Notice that we have \(\nu^2 + \xi^2\) for the reflection hologram whereas we had \(\nu^2 - \xi^2\) for the transmission hologram (equation (B.55)). Substituting the same definitions along with equation (B.70) into (B.66) and (B.67) yields

\[
R(d/2) = e^{\xi - \frac{\alpha d}{c_r}} \left( \cosh \sqrt{\nu^2 + \xi^2} + \frac{\xi}{\sqrt{\nu^2 + \xi^2}} \sinh \sqrt{\nu^2 + \xi^2} \right)^{-1} \tag{B.71}
\]

\[
S(-d/2) = \frac{-j \sqrt{\nu} \sinh \sqrt{\nu^2 + \xi^2}}{\xi \sinh \sqrt{\nu^2 + \xi^2} + \sqrt{\nu^2 + \xi^2} \cosh \sqrt{\nu^2 + \xi^2}} \tag{B.72}
\]

These equations will be used later to define the transmission and diffraction coefficients for a reflection hologram.

**B.2 Bragg Incidence**

We shall now consider the special case (used extensively in Chapter 2) where the reference wave is incident at the Bragg angle \(\theta_0\). At Bragg incidence the
The dephasing parameter is zero ($\vartheta = 0$) which, when substituted into equation (B.52) defining $\xi$, yields

$$\xi = \frac{\alpha \delta d}{2} \left( \frac{1}{c_r} - \frac{1}{c_s} \right)$$

(B.73)

which is a real quantity. We also can define a reduced loss parameter by

$$\alpha_0 \equiv \frac{\alpha \delta d}{c_r} - \xi$$

(B.74)

$$= \frac{\alpha \delta d}{2} \left( \frac{1}{c_r} + \frac{1}{c_s} \right)$$

(B.75)

For the transmission hologram, substituting this definition into the field coefficients given by equations (B.56) and (B.57) yields

$$R(d/2) = e^{-\alpha_0} \left( \cos \sqrt{\nu^2 - \xi^2} - \frac{\xi}{\sqrt{\nu^2 - \xi^2}} \sin \sqrt{\nu^2 - \xi^2} \right)$$

(B.76)

$$S(d/2) = -j\sqrt{\nu} e^{-\alpha_0} \frac{\nu}{\sqrt{\nu^2 - \xi^2}} \sin \sqrt{\nu^2 - \xi^2}$$

(B.77)

For the reflection hologram, substituting this definition into equations (B.71) and (B.72) yields

$$R(d/2) = e^{-\alpha_0} \left( \cosh \sqrt{\nu^2 + \xi^2} + \frac{\xi}{\sqrt{\nu^2 + \xi^2}} \sinh \sqrt{\nu^2 + \xi^2} \right)^{-1}$$

(B.78)

$$S(-d/2) = \frac{-j\sqrt{\nu} \sinh \sqrt{\nu^2 + \xi^2}}{\xi \sinh \sqrt{\nu^2 + \xi^2} + \sqrt{\nu^2 + \xi^2} \cosh \sqrt{\nu^2 + \xi^2}}$$

(B.79)

These will be used in Section 4, along with the fact that $\xi$ is a real quantity, in order to model the hologram at Bragg incidence by transmission and diffraction coefficients.

**B.3 Lossless Dielectric Hologram**

The other case of interest occurs when the angle of incidence need not be exactly at Bragg, which is easiest to analyze when the hologram is lossless ($\alpha = 0$). Substituting $\alpha = 0$ this into equation (B.52) for $\xi$ yields
\[ \xi = -jd\theta/(2cs) \]  
(B.80)

which is an imaginary quantity. For convenience, a new parameter, \( \zeta \), will be defined which is a real quantity.

\[ \zeta \equiv -j\xi \]  
(B.81)

\[ = -d\theta/(2cs) \]  
(B.82)

We can substitute equation (B.21) for \( \vartheta \) into this equation for \( \zeta \) in order to see the explicit dependence upon a change in angle, \( \Delta \theta \), or a change in wavelength, \( \Delta \lambda \). This yields

\[ \zeta = \frac{d}{2cs} \left[ \Delta \theta K \cos \left( \frac{\theta_o - \gamma_o}{2} \right) + \Delta \lambda \frac{K^2}{4\pi n_0} \right] \]  
(B.83)

For the transmission hologram, substituting equation (B.81) for \( \xi \) into equations (B.56) and (B.57) gives the coefficients of the fields at the boundaries in terms of \( \zeta \).

\[ R(d/2) = e^{j\xi} \left( \cos \sqrt{\nu^2 + \zeta^2} - j\frac{\zeta}{\sqrt{\nu^2 + \zeta^2}} \sin \sqrt{\nu^2 + \zeta^2} \right) \]  
(B.84)

\[ S(d/2) = -j\sqrt{\nu} e^{j\xi} \frac{\nu}{\sqrt{\nu^2 + \zeta^2}} \sin \sqrt{\nu^2 + \zeta^2} \]  
(B.85)

For the reflection hologram, substituting equation (B.81) into equations (B.71) and (B.72) yields

\[ R(d/2) = e^{j\xi} \left( \cosh \sqrt{\nu^2 - \zeta^2} + j\frac{\zeta}{\sqrt{\nu^2 - \zeta^2}} \sinh \sqrt{\nu^2 - \zeta^2} \right)^{-1} \]  
(B.86)

\[ S(-d/2) = \frac{-j\sqrt{\nu} \sinh \sqrt{\nu^2 - \zeta^2}}{j\zeta \sinh \sqrt{\nu^2 - \zeta^2} + \sqrt{\nu^2 - \zeta^2} \cosh \sqrt{\nu^2 - \zeta^2}} \]  
(B.87)
B.4 Comparison of Model with Experimental Results

In this section we will take the experimental results for the angular response of a hologram, derive the parameters $\zeta$ and $\nu$ based upon the equations which have been derived in the prior sections, and compare the theoretical angular response curve with the experimentally obtained data points. It will be found that they agree quite closely, indicating that the results of our theoretical model are useful.

The hologram which was measured was a dichromated gelatin plane wave hologram, created by the procedure given in Appendix C. It had antireflection coated microscope slides attached to each side and gated with mineral oil to reduce the Fresnel reflections. Consequently, it fits the theoretical model quite accurately. The index of refraction immediately surrounding the emulsion is approximately equal to the average index of refraction of the emulsion and the hologram is a volume phase hologram, any surface relief effects being eliminated by the mineral liquid gate.

The data on the angular response of the hologram was taken by mounting the hologram on a calibrated rotational stage and measuring the relative power diffracted at many angles of incidence. This data is given in Table 8. The raw data (total incident power, total diffracted power, and angle of incidence as measured by the scale of the rotation stage) is given first. The numbers of interest (the relative diffracted power normalized to the diffracted power at Bragg incidence ($\theta_o$) and the angle of incidence relative to the hologram normal) are given next. The Bragg angle is defined as that angle at which the maximum power is diffracted. The Bragg angle, $\theta_o$, and diffraction efficiency, $\eta_o$, are found by averaging the values of data points 1 and 16. Finally, the angles of incidence at the emulsion surface relative to the Bragg angle, designated $\Delta \theta$, are given. These were calculated using
Snell's Law, assuming that \( n_0 = 1.5 \), and will be used to compare the angular response with that predicted theoretically.

For convenience, we will assume that the loss of the hologram is negligible and compare this data with the response predicted for a lossless hologram by equation (B.85). Designate the relative diffracted power of the hologram by \( \eta \). Using equation (B.85) this gives us

\[
\eta = \frac{1}{c} \| S(d/2) \|^2 \tag{B.88}
\]

\[
= \nu^2 \sin^2 \sqrt{\nu^2 + \zeta^2} \tag{B.89}
\]

\[
= \nu^2 \text{sinc}^2 \sqrt{\nu^2 + \zeta^2} \tag{B.90}
\]

where

\[
\nu = \frac{\kappa d}{\sqrt{c_r c_s}} \tag{B.91}
\]

\[
= \frac{\pi n_1 d}{\lambda \sqrt{c_r c_s}} \tag{B.92}
\]

\[
\zeta = \Delta \theta \frac{Kd}{2c_s} \cos[(\theta_0 - \gamma_0)/2] \tag{B.93}
\]

\[
= \chi \Delta \theta \tag{B.94}
\]

We have defined the quantity \( \chi \) because it will be useful notationally in the following analysis. Designate the relative diffracted power at Bragg incidence (\( \Delta \theta = 0 \)) by \( \eta_0 \) which is given by

\[
\eta_0 = \sin^2 \nu_0. \tag{B.95}
\]

We are interested in the quantity \( \frac{\eta}{\eta_0}(\Delta \theta) \), the normalized diffracted power as a function of incident angle. Using the previous equations this becomes

\[
\frac{\eta}{\eta_0}(\Delta \theta) = \frac{\nu^2 \text{sinc}^2 \sqrt{\nu^2 + \chi^2(\Delta \theta)^2}}{\sin^2 \nu_0} \tag{B.96}
\]
Table 8: Experimental data on angular response of hologram.

<table>
<thead>
<tr>
<th>Data</th>
<th>Power</th>
<th>Angle (abs.)</th>
<th>$\eta/\eta_0$</th>
<th>Angle (air)</th>
<th>$\Delta\theta$</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>#</td>
<td>Diff.</td>
<td>Inc.</td>
<td>(deg.)</td>
<td>(norm.)</td>
<td>(deg.)</td>
<td>(mrad.)</td>
</tr>
<tr>
<td>1</td>
<td>1.294</td>
<td>8.743</td>
<td>258° 20.0'</td>
<td>.99</td>
<td>35° 33.0'</td>
<td>0.2</td>
</tr>
<tr>
<td>2</td>
<td>1.187</td>
<td>8.815</td>
<td>257° 47.0'</td>
<td>.90</td>
<td>35° 0.0'</td>
<td>-5.5</td>
</tr>
<tr>
<td>3</td>
<td>1.054</td>
<td>8.880</td>
<td>257° 34.0'</td>
<td>.80</td>
<td>34° 47.0'</td>
<td>-7.7</td>
</tr>
<tr>
<td>4</td>
<td>.935</td>
<td>8.904</td>
<td>257° 23.0'</td>
<td>.70</td>
<td>34° 36.0'</td>
<td>-9.6</td>
</tr>
<tr>
<td>5</td>
<td>.804</td>
<td>8.918</td>
<td>257° 13.5'</td>
<td>.60</td>
<td>34° 26.5'</td>
<td>-11.2</td>
</tr>
<tr>
<td>6</td>
<td>.669</td>
<td>8.941</td>
<td>257° 1.0'</td>
<td>.50</td>
<td>34° 14.0'</td>
<td>-13.4</td>
</tr>
<tr>
<td>7</td>
<td>.538</td>
<td>9.018</td>
<td>256° 51.0'</td>
<td>.40</td>
<td>34° 4.0'</td>
<td>-15.2</td>
</tr>
<tr>
<td>8</td>
<td>.400</td>
<td>8.957</td>
<td>256° 41.0'</td>
<td>.30</td>
<td>33° 54.0'</td>
<td>-16.9</td>
</tr>
<tr>
<td>9</td>
<td>.239</td>
<td>9.006</td>
<td>256° 25.5'</td>
<td>.18</td>
<td>33° 38.5'</td>
<td>-19.5</td>
</tr>
<tr>
<td>10</td>
<td>.119</td>
<td>9.021</td>
<td>256° 10.5'</td>
<td>.09</td>
<td>33° 23.5'</td>
<td>-22.1</td>
</tr>
<tr>
<td>11</td>
<td>.010</td>
<td>9.042</td>
<td>255° 30.0'</td>
<td>.01</td>
<td>32° 43.0'</td>
<td>-29.2</td>
</tr>
<tr>
<td>12</td>
<td>.040</td>
<td>9.068</td>
<td>255° 0.0'</td>
<td>.03</td>
<td>32° 13.0'</td>
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</tr>
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<td>13</td>
<td>.074</td>
<td>9.084</td>
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<td>31° 30.0'</td>
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</tr>
<tr>
<td>14</td>
<td>.049</td>
<td>9.070</td>
<td>253° 43.5'</td>
<td>.04</td>
<td>30° 56.5'</td>
<td>-37.3</td>
</tr>
<tr>
<td>15</td>
<td>.004</td>
<td>9.062</td>
<td>252° 42.5'</td>
<td>.00</td>
<td>29° 55.5'</td>
<td>-58.8</td>
</tr>
<tr>
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<td>1.364</td>
<td>9.062</td>
<td>258° 18.0'</td>
<td>1.01</td>
<td>35° 31.0'</td>
<td>-2</td>
</tr>
<tr>
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<td>1.205</td>
<td>9.082</td>
<td>258° 49.5'</td>
<td>.89</td>
<td>36° 2.5'</td>
<td>5.3</td>
</tr>
<tr>
<td>18</td>
<td>1.000</td>
<td>9.120</td>
<td>259° 10.0'</td>
<td>.73</td>
<td>36° 23.0'</td>
<td>8.7</td>
</tr>
<tr>
<td>19</td>
<td>.704</td>
<td>9.085</td>
<td>259° 33.0'</td>
<td>.52</td>
<td>36° 46.0'</td>
<td>12.6</td>
</tr>
<tr>
<td>20</td>
<td>.299</td>
<td>9.127</td>
<td>260° 7.5'</td>
<td>.22</td>
<td>37° 20.5'</td>
<td>18.5</td>
</tr>
<tr>
<td>21</td>
<td>.102</td>
<td>9.144</td>
<td>260° 35.0'</td>
<td>.07</td>
<td>37° 48.0'</td>
<td>23.1</td>
</tr>
<tr>
<td>22</td>
<td>.030</td>
<td>9.162</td>
<td>261° 7.5'</td>
<td>.02</td>
<td>38° 20.5'</td>
<td>28.6</td>
</tr>
<tr>
<td>23</td>
<td>.081</td>
<td>9.170</td>
<td>262° 27.0'</td>
<td>.06</td>
<td>39° 40.0'</td>
<td>41.7</td>
</tr>
<tr>
<td>24</td>
<td>.010</td>
<td>9.138</td>
<td>264° 13.0'</td>
<td>.01</td>
<td>41° 26.0'</td>
<td>59.0</td>
</tr>
<tr>
<td>25</td>
<td>N/A</td>
<td>N/A</td>
<td>222° 47.0'</td>
<td>N/A</td>
<td>0°</td>
<td>N/A</td>
</tr>
</tbody>
</table>

159
For small values of $\Delta \theta$

\[
\nu(\Delta \theta) \approx \nu_0
\]

\[
\Rightarrow \frac{n}{\eta_0}(\Delta \theta) \approx \frac{\text{sinc}^2 \sqrt{\nu_0^2 + \chi^2(\Delta \theta)^2}}{\text{sinc}^2 \nu_0}
\]  \hspace{1cm} (B.98)

We now have an expression for $\frac{n}{\eta_0}$ in terms of the hologram parameters $\nu_0$ and $\chi$ and the change in angle $\Delta \theta$. In order to compare our experimental data with this theoretical formula, we must determine $\nu_0$ and $\chi$ for this hologram. We will do this by utilizing the experimentally measured angles at which the output peaks and nulls occur.

Examining equation (B.98) we can easily determine expressions for the angles at which the peaks and nulls occur. The nulls occur when

\[
\text{sinc}^2 \sqrt{\nu_0^2 + \chi^2(\Delta \theta)^2} = 0
\]

\[
\Leftrightarrow \nu_0^2 + \chi^2(\Delta \theta)^2 = (m\pi)^2
\]  \hspace{1cm} (B.100)

The location of the peaks is not as simple. The first peak of the function $\text{sinc}^2 x$ occurs when $x = 4.4934$. Therefore, at the angle at which the first peak occurs we have

\[
\nu_0^2 + \chi^2(\Delta \theta)^2 = (4.4934)^2.
\]  \hspace{1cm} (B.101)

Designate the angle at which the $i$th null occurs as $\Delta \theta_{ni}$ and the angle at which the first peak occurs as $\Delta \theta_p$. Substituting these quantities into equations (B.100) and (B.101) yields

\[
\nu_0^2 + \chi^2(\Delta \theta_{n1})^2 = \pi^2
\]  \hspace{1cm} (B.102)

\[
\nu_0^2 + \chi^2(\Delta \theta_p)^2 = (4.4934)^2
\]  \hspace{1cm} (B.103)

\[
\nu_0^2 + \chi^2(\Delta \theta_{n2})^2 = (2\pi)^2
\]  \hspace{1cm} (B.104)
Table 9: Angles corresponding to peaks and nulls in diffracted power.

<table>
<thead>
<tr>
<th>Angle</th>
<th>mrad</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta \theta_{n1}$</td>
<td>28.9</td>
</tr>
<tr>
<td>$\Delta \theta_{p1}$</td>
<td>41.9</td>
</tr>
<tr>
<td>$\Delta \theta_{n2}$</td>
<td>58.9</td>
</tr>
</tbody>
</table>

Taking equations (B.102)–(B.104) in pairs and looking at the difference between them will give us three separate expressions for $x^2$ just in terms of the peak and null angles:

$$
\chi_1^2 = \frac{(4.4934)^2 - \pi^2}{(\Delta \theta_p)^2 - (\Delta \theta_{n1})^2} \tag{B.105}
$$

$$
\chi_2^2 = \frac{(2\pi)^2 - (4.4934)^2}{(\Delta \theta_{n2})^2 - (\Delta \theta_p)^2} \tag{B.106}
$$

$$
\chi_3^2 = \frac{3\pi^2}{(\Delta \theta_{n2})^2 - (\Delta \theta_{n1})^2} \tag{B.107}
$$

all of which are equal. Using equation (B.95) we can get an expression for $\nu_0$ just in terms of our data:

$$
\nu_0 = \sin^{-1} \sqrt{\eta_0} \tag{B.108}
$$

This result is less sensitive to data errors than using equations (B.102)–(B.104).

The angles at which the nulls and peaks occur should be symmetric about the Bragg angle. We will use this fact to minimize our experimental error by averaging the angles for corresponding nulls and peaks in order to find our $\Delta \theta_{ni}$ and $\Delta \theta_p$. Using the data in Table 8, entries 11, 13, 15, and 22–24 gives us values for these angles. They are listed in Table 9.
We will now compute the parameters of the hologram. Substituting the values in Table 9 into equations (B.105)-(B.107) will give us three values for $\chi^2$. These are listed in Table 10 along with their average, $\overline{\chi^2}$. Taking the ratio of the diffracted power to the incident power at the Bragg angle (average entries 1 and 16 in Table 8) will give us $\eta_0 = 0.149$. Substituting this into equation (B.108) yields a value for $\nu_0$ which is also listed in Table 10.

We now substitute the values of $\overline{\chi^2}$ and $\nu_0$ into equation (B.98), the equation for $\frac{g}{\eta_0}(\Delta \theta)$. This function is plotted in Figure 40 along with the experimental data points given in Table 8. Notice how close the experimental data points are to the theoretical curve. This indicates that the results of the coupled-wave theory are useful for modeling a real hologram.

### B.5 Diffraction and Transmission Coefficients

We shall model the hologram for the resonated hologram analysis by transmission and diffraction coefficients. These coefficients represent the amplitude and phase changes imposed by the hologram. They are calculated at the boundaries of
Figure 40: Comparison between the theoretical and experimental angular response of a hologram.
Figure 41: Transmission hologram diffraction and transmission coefficients

the hologram on the baselines which are parallel to the Bragg angles and intersect in the center of the hologram. An example of these coefficients for a transmission hologram are shown in Figure 41. These coefficients will be calculated first for a transmission hologram and then for a reflection hologram.

Transmission Hologram

We shall first analyze the standard case shown in Figure 41-i to determine the transmission coefficient $T_1$ and the diffraction coefficient $S_1$. The results of this analysis will be used to determine the remainder of the transmission and diffraction coefficients shown in Figures 41-ii, 41-iii, and 41-iv.

For convenience, this first case has been recopied in Figure 42. $E_i$ is a plane wave incident beam with propagation vector $\vec{\rho}$ at the Bragg angle $\theta_0$. $E_t$ is the transmitted wave, again with propagation vector $\vec{\rho}$. $E_s$ is the diffracted (signal) wave with propagation vector $\vec{\sigma}$ at the Bragg angle $\gamma_0$.
The transmission and diffraction coefficients will be defined by evaluating the output fields \((E_t, E_s)\) relative to the input field \((E_i)\) at the boundary of the hologram on the baselines. We shall define these locations by \(\vec{x}_i, \vec{x}_t, \) and \(\vec{x}_s\) where

\[
\vec{x}_i = \frac{d}{2}(\tan \theta_o, 0, -1)^T \\
\vec{x}_t = -\vec{x}_i = -\frac{d}{2}(\tan \theta_o, 0, -1)^T \\
\vec{x}_s = \frac{d}{2}(\tan \gamma_o, 0, 1)^T
\]  

Then the transmission and diffraction coefficients are given by

\[
T_1 \equiv \frac{E_t(\vec{x}_t)}{E_i(\vec{x}_i)} \\
S_1 \equiv \frac{E_s(\vec{x}_s)}{E_i(\vec{x}_i)}
\]  

In the coupled wave analysis the electric field was defined by equation (B.10) as
\[ E(x) = R(z)e^{-j\vec{p} \cdot \vec{x}} + S(z)e^{-j\vec{\sigma} \cdot \vec{x}} \]  

(B.114)

Recall that for a transmission hologram \( S(-d/2) = 0 \) so that the field at the incident boundary is given by

\[ E_i(x) = R(-d/2)e^{-j\vec{p} \cdot \vec{x}} \]  

(B.115)

At the output boundary we can split the field into two parts: transmitted and diffracted. These are given by

\[ E_t(x) = R(d/2)e^{-j\vec{p} \cdot \vec{x}} \]  

(B.116)

\[ E_s(x) = S(d/2)e^{-j\vec{\sigma} \cdot \vec{x}} \]  

(B.117)

Evaluating these fields on the baselines yields

\[ E_i(x_i) = R(-d/2)e^{j\phi_t/2} \]  

(B.118)

\[ E_t(x_t) = R(d/2)e^{-j\phi_t/2} \]  

(B.119)

\[ E_s(x_s) = S(d/2)e^{-j\phi_s/2} \]  

(B.120)

where

\[ \phi_t \equiv -2 \vec{p} \cdot \vec{x}_i = 2 \vec{p} \cdot \vec{x}_t \]  

(B.121)

\[ = \beta d/\cos \theta_o \]  

(B.122)

\[ \phi_s \equiv 2 \vec{\sigma} \cdot \vec{x}_s \]  

(B.123)

\[ = \beta d/\| \cos \gamma_o \| \]  

(B.124)

(B.125)

The phase terms \( \phi_t \) and \( \phi_s \) correspond to phase shifts from translation through the hologram along vectors \( \vec{p} \) and \( \vec{\sigma} \) respectively. We have included absolute value signs in the definition of \( \phi_s \) so that it is equally valid for a reflection hologram.
We can substitute these expressions into the equations for the transmission and diffraction coefficients to get

\[
T_1 = \frac{R(d/2)}{R(-d/2)} e^{-j\phi} \quad \text{(B.126)}
\]

\[
S_1 = \frac{S(d/2)}{R(-d/2)} e^{-j(\phi + \phi_0)/2} \quad \text{(B.127)}
\]

Expressions for \(R(-d/2), R(d/2),\) and \(S(d/2)\) are given by equations (B.34), (B.76), and (B.77). Substituting these into (B.126) and (B.127) yields

\[
T_1 = T(1 - \epsilon) e^{-j\phi} \quad \text{(B.128)}
\]

\[
S_1 = -j\sqrt{\epsilon} S e^{-j(\phi + \phi_0)/2} \quad \text{(B.129)}
\]

where

\[
T = e^{-\alpha_0} \cos \sqrt{\nu^2 - \xi^2} \quad \text{(B.130)}
\]

\[
\epsilon = \frac{-\xi}{\sqrt{\nu^2 - \xi^2}} \tan \sqrt{\nu^2 - \xi^2} \quad \text{(B.131)}
\]

\[
S = e^{-\alpha_0} \frac{\nu}{\sqrt{\nu^2 - \xi^2}} \sin \sqrt{\nu^2 - \xi^2} \quad \text{(B.132)}
\]

\[
\nu = \frac{\pi n_1 d}{\lambda \sqrt{c_r |c_s|}} \quad \text{(B.133)}
\]

\[
\xi = \frac{ad}{2} \left( \frac{1}{c_r} - \frac{1}{c_s} \right) \quad \text{(B.134)}
\]

\[
\alpha_0 = \frac{ad}{2} \left( \frac{1}{c_r} + \frac{1}{c_s} \right) \quad \text{(B.135)}
\]

\[
c_r = \cos \theta_0 \quad \text{(B.136)}
\]

\[
c_s \simeq |\cos \gamma_0| \quad \text{(B.137)}
\]

\[
c = \frac{c_r}{|c_s|} \quad \text{(B.138)}
\]

We will now determine \(T_2\) and \(S_2\), the diffraction and transmission coefficients shown in Figure 41–ii. This is the same hologram but now \(\vec{\sigma}\) is the propagation vector of the incident wave and \(\vec{\rho}\) is the propagation vector of the diffracted wave.
It can be easily seen that the previous analysis is still valid and can be used merely by interchanging the angles $\theta_0$ and $\gamma_0$. Since the results depend only upon the cosines of these angles this is equivalent to interchanging $c_r$ and $c_s$.

\[
\begin{align*}
c_r \leftrightarrow c_s & \Rightarrow \xi \rightarrow -\xi \\
\nu \rightarrow \nu \\
\alpha_0 \rightarrow \alpha_0 \\
\Rightarrow \epsilon \rightarrow -\epsilon \\
T \rightarrow T \\
S \rightarrow S \\
c \rightarrow 1/c \\
\phi_T \leftrightarrow \phi_S
\end{align*}
\]

\[
\begin{align*}
T_2 &= T(1 + \epsilon)e^{-j\phi_S} \\
S_2 &= -j\frac{1}{\sqrt{c}}Se^{-j(\phi_T + \phi_S)/2}
\end{align*}
\]

Comparing these with the values for $T_1$ and $S_1$ it can be seen that the transmission coefficient has a factor of $1 + \epsilon$ instead of $1 - \epsilon$ and the diffraction coefficient has a factor of $\frac{1}{\sqrt{c}}$ instead of a factor of $\sqrt{c}$. In addition, the phase of $T_2$ is $\phi_S$ whereas the phase of $T_1$ is $\phi_T$. But the phases of $S_1$ and $S_2$ are the same. For a lossless hologram $\epsilon = 0$ so the magnitude of the transmission coefficients are the same. But the magnitude of the diffraction coefficients are the same only when $\cos \theta_0 = \cos \gamma_0$.

To determine the values of the other two sets of coefficients we will use an argument of symmetry. We cannot proceed in the same manner as was used to find $T_2$ and $S_2$ because the incident wave is traveling in the negative $z$ direction and solution for the transmission hologram done in Section 1 assumed that it was
traveling in the positive $z$ direction. Rather than redo that analysis, we can simply argue from symmetry.

If we define a new set of axes by setting $x \rightarrow -x$, $y \rightarrow -y$, and $z \rightarrow -z$, the equation for the hologram will not change. The equation for the index of refraction change, equation (B.3), is invariant under this change of coordinates because it is dependent upon $\cos(\vec{K} \cdot \vec{x})$ which equals $\cos(\vec{K} \cdot (-\vec{x}))$. But under this new coordinate system our incident wave has a positive $z$ component and is incident at Bragg angle $\theta_0$ for Figure 41-iii and at Bragg angle $\gamma_0$ for Figure 41-iv. Thus, the amplitude and phases will be the same as we have already derived, including the phase shifts due to translation since they are forward traveling waves. Consequently,

$$T_3 = T_1, \quad S_3 = S_1$$  \hspace{1cm} (B.149)  
$$T_4 = T_2, \quad S_4 = S_2$$  \hspace{1cm} (B.150)

This gives us all four of the diffraction and transmission coefficients for a transmission hologram.

**Reflection Hologram**

The diffraction and transmission coefficients for a reflection hologram are derived in much the same manner. These are shown in Figure 43. For the case pictured in Figure 43-i, the results are

$$T_1 = \frac{R(d/2)}{R(-d/2)} e^{-j\phi_T}$$  \hspace{1cm} (B.151)  
$$S_1 = \frac{S(-d/2)}{R(-d/2)} e^{-j(\phi_T + \phi_S)/2}$$  \hspace{1cm} (B.152)

The expressions for $R(-d/2), R(d/2)$, and $S(-d/2)$ for a reflection hologram are given by equations (B.34), (B.78), and (B.79). Substituting these into (B.151)
Figure 43: Reflection hologram diffraction and transmission coefficients
and (B.152) yields

\[ T_1 = Te^{-\alpha_0}e^{-j\phi_T} \]  (B.153)
\[ S_1 = -j\sqrt{c}Se^{-j(\phi_T+\phi_S)/2} \]  (B.154)

where

\[ T = \left( \cosh \sqrt{\nu^2 + \xi^2} + \frac{\xi}{\sqrt{\nu^2 + \xi^2}} \sinh \sqrt{\nu^2 + \xi^2} \right)^{-1} \]  (B.155)
\[ S = \frac{\nu \sinh \sqrt{\nu^2 + \xi^2}}{\xi \sinh \sqrt{\nu^2 + \xi^2} + \sqrt{\nu^2 + \xi^2} \cosh \sqrt{\nu^2 + \xi^2}} \]  (B.156)

and the rest of the parameters are defined as for the transmission hologram by equations (B.133)-(B.138).

We can use these results to determine \( T_2 \) and \( S_2 \), shown in Figure 43-iv. The propagation vector of the incident wave is now \(-\sigma\) and the propagation vector of the diffracted wave is \(-\rho\), but notice that the \( z \) component of the incident wave is still positive and hence our prior analysis can still be used. Mathematically, this change in propagation vectors is equivalent to interchanging \( \theta_0 \) and \( \gamma_0 + \pi \). Therefore, we can use the previous results by interchanging \( c_r \) and \(-c_s\).

\[ c_r \leftrightarrow -c_s \Rightarrow \xi \rightarrow \xi \]  (B.157)
\[ \nu \rightarrow \nu \]  (B.158)
\[ \alpha_0 \rightarrow -\alpha_0 \]  (B.159)
\[ \Rightarrow T \rightarrow T \]  (B.160)
\[ S \rightarrow S \]  (B.161)
\[ \phi_T \leftrightarrow \phi_S \]  (B.162)
\[ \Rightarrow T_2 = Te^{\alpha_0}e^{-j\phi_S} \]  (B.163)
\[ S_2 = -j\frac{1}{\sqrt{c}}Se^{-j(\phi_T+\phi_S)/2} \]  (B.164)
Finally, in order to determine the last two sets of diffraction and transmission coefficients we can utilize the same symmetry argument as for the transmission hologram \(((x, y, z) \to (-x, -y, -z))\). This yields

\[ T_3 = T_1, \quad S_3 = S_1 \]  
\[ T_4 = T_2, \quad S_4 = S_2 \]

which gives us all four diffraction and transmission coefficients for a reflection hologram.

B.6 Conditions for Symmetric Coefficients in the Resonator

In the analysis for the resonated hologram in Chapter 2 it was assumed that the amplitude of the transmission coefficients for both Bragg incidence directions were the same. However, if the hologram is not lossless this can be seen not to be the case. In this section it will be determined how this asymmetry affects the performance of the resonator and when an assumption of symmetry is a reasonable approximation.

This analysis will look at the expressions for \(N\) and \(D\), from which \(P_{\text{out}}\) is computed, in the presence of loss in order to determine the effect this asymmetry has on the equations for the output of the resonated hologram. Then we will derive a condition for which we can reasonable neglect this asymmetry. This analysis will be conducted for both transmission and reflection holograms, considering both the three and four mirror resonated holograms for each case.

Transmission Hologram

For the transmission hologram we will derive a sufficient condition on the loss, \(A\), for which we can approximate the transmission and diffraction coefficients as being symmetric. This will be done by first determining a constraint on \(\epsilon\), the
parameter which causes the asymmetry in the transmission coefficients, relating this to a constraint on $\xi$, one of the hologram parameters, which in turn will be expressed as a constraint upon $\alpha_0$, the reduced loss parameter, which will lead to a constraint upon $A$.

To determine $N$ and $D$ we must first know the transmission and diffraction coefficients of the hologram. The transmission and diffraction coefficients for a lossy transmission hologram are derived following the procedure given in the previous section. They are as follows:

$$T_1 = T_3 = T(1 - e)e^{-j\phi_r}$$  \hspace{1cm} (B.167)

$$T_2 = T_4 = T(1 + e)e^{-j\phi_s}$$  \hspace{1cm} (B.168)

$$S_1 = S_3 = -j\sqrt{c}S e^{-j(\phi_r+\phi_s)/2}$$  \hspace{1cm} (B.169)

$$S_2 = S_4 = -j\frac{1}{\sqrt{c}}S e^{-j(\phi_r+\phi_s)/2}$$  \hspace{1cm} (B.170)

where the parameters that we shall be dealing with are defined as

$$S = e^{-\alpha_0}\frac{\nu}{\sqrt{\nu^2 - \xi^2}}\sin\sqrt{\nu^2 - \xi^2}$$  \hspace{1cm} (B.171)

$$T = e^{-\alpha_0}\cos\sqrt{\nu^2 - \xi^2}$$  \hspace{1cm} (B.172)

$$\varepsilon = \frac{-\xi}{\sqrt{\nu^2 - \xi^2}}\tan\sqrt{\nu^2 - \xi^2}$$  \hspace{1cm} (B.173)

$$\xi = \frac{ad}{2}\left(\frac{1}{c_r} - \frac{1}{c_s}\right)$$  \hspace{1cm} (B.174)

$$\alpha_0 = \frac{ad}{2}\left(\frac{1}{c_r} + \frac{1}{c_s}\right)$$  \hspace{1cm} (B.175)

It can be seen that the diffraction coefficients remain symmetric in the presence of loss (the factor of $\sqrt{c}$ not being loss related) but the transmission coefficients are not symmetric due to the factors of $1 \pm \varepsilon$.

Since we are interested in $P_{out}$ we need to determine how the presence of a nonzero $\varepsilon$ affects $P_{out}$ and when $\varepsilon$ is small enough to be neglected. To deter-
mine the equations for $N$ and $D$ in the presence of loss we will substitute equations (B.167)-(B.170), defining the diffraction and transmission coefficients, into equation (2.28), giving the output of the general four mirror resonated hologram. For the four mirror resonated hologram we will assume that $r_1 = r_4 = r$ and $\phi_0 = \phi_1 = \phi$, which are conditions for obtaining $P_{\text{out}}|_{\text{max}}$. This gives us

\[
N = S(1 - r^2)\left[1 - [T^2(1 - \epsilon^2) + S^2]e^{-j\phi_2}\right] \\
D = 1 + 2T^2r e^{-j\phi}(1 + \epsilon^2) - S^2(e^{-j\phi_2} + r^2e^{-j(2\phi - \phi_2)}) \\
+ [T^2(1 - \epsilon^2) + S^2]^2r^2e^{-j2\phi}
\]

We can neglect $\epsilon$ (i.e. assume symmetric coefficients) if

\[
\lesssim 1 \\
\Rightarrow \frac{\xi^2}{\cos^2 \sqrt{\nu^2 - \xi^2}} \frac{\sin^2 \sqrt{\nu^2 - \xi^2}}{(\nu^2 - \xi^2)} \lesssim 1.
\]

Since

\[
\frac{\sin^2 x}{x^2} \leq 1
\]

for all values of $x$, a sufficient condition on $\xi$ is

\[
\xi^2 \ll \cos^2 \sqrt{\nu^2 - \xi^2}.
\]

To proceed further we must make an assumption to get a lower bound for $\cos^2 \sqrt{\nu^2 - \xi^2}$. We can assume for a low loss, low diffraction efficiency hologram that the transmission will be at least 50%. We know that

\[
T^2 = e^{-2\alpha_0} \cos^2 \sqrt{\nu^2 - \xi^2}.
\]

Therefore, because $e^{-2\alpha_0} \leq 1$,

\[
\cos^2 \sqrt{\nu^2 - \xi^2} > 0.5
\]
Combining this inequality with equation (B.181) yields a sufficient upper bound on $\xi$ given by

$$\xi^2 \leqslant .5.$$  \hfill (B.184)

We shall make a standard engineering approximation and assume that this is satisfied if

$$\xi^2 < .05$$  \hfill (B.185)

$$\Leftrightarrow |\xi| < .2.$$  \hfill (B.186)

From the definitions for $\xi$ and $\alpha_0$ we can write

$$\xi = \left(\frac{c_s - c_r}{c_s + c_r}\right) \alpha_0$$  \hfill (B.187)

$$\Rightarrow \alpha_0 < \frac{.2}{\left|\frac{c_s + c_r}{c_s - c_r}\right|}.$$  \hfill (B.188)

To get a condition on $A$ we need to express $A$ in terms of $\alpha_0$.

$$A = 1 - T - S$$  \hfill (B.189)

where $T$ is the intensity transmission coefficient for the hologram and $S$ is the intensity diffraction coefficient for the hologram. Due to the loss, $T$ will be different for the two Bragg angles so we will take the geometric average of these two. $S$ is also defined as a geometric average to eliminate the angle effects.

$$T = |T_1T_2^*| = |T_3T_4^*|$$  \hfill (B.190)

$$= T^2(1 - \epsilon^2)$$  \hfill (B.191)

$$= e^{-2\alpha_0} \left(\cos^2 \sqrt{\nu^2 - \xi^2} - \frac{\xi^2}{\nu^2 - \xi^2} \sin^2 \sqrt{\nu^2 - \xi^2}\right)$$  \hfill (B.192)

$$S = |S_1S_2^*| = |S_3S_4^*|$$  \hfill (B.193)

$$= S^2$$  \hfill (B.194)
\[
\exp(-2\alpha_0) \frac{\nu^2}{\nu^2 - \xi^2} \sin\sqrt{\nu^2 - \xi^2} = 1 - e^{-2\alpha_0} \left[ \cos^2 \sqrt{\nu^2 - \xi^2} + \frac{\nu^2 - \xi^2}{\nu^2 - \xi^2} \sin^2 \sqrt{\nu^2 - \xi^2} \right]
\]

(B.195)

\[
\Rightarrow A = 1 - e^{-2\alpha_0} \left[ \cos^2 \sqrt{\nu^2 - \xi^2} + \frac{\nu^2 - \xi^2}{\nu^2 - \xi^2} \sin^2 \sqrt{\nu^2 - \xi^2} \right]
\]

(B.196)

\[
A = 1 - e^{-2\alpha_0}
\]

(B.197)

We now can get a sufficient condition on \( A \) by substituting equation (B.188) into (B.197), but it will still be angle dependent. To obtain a more general constraint, we will look at the case when the angles are not bounded and then look at an example when they are restricted by an upper bound.

If all angles are allowed then our only constraint is that \( c_r \) and \( c_s \) are positive, because we are dealing with a transmission hologram. This means that

\[
\left| c_s + c_r \right| > 1 \quad \text{(B.198)}
\]

Combining this with equation (B.188) gives us a sufficient condition on \( \alpha_0 \) expressed by

\[
\alpha_0 < 0.2
\]

(B.199)

This can be translated into a sufficient condition upon \( A \) by using equation (B.197).

\[
A < 0.3
\]

(B.200)

Now let us examine how this constraint changes if we bound our angles by \(|\theta_0|, |\gamma_0| < 0.73\) rad. (This is the angle limit imposed by Snell's law if the emulsion has an index of refraction of 1.5 and it is surrounded by air.) This is equivalent to the bound \( 0.75 < c_r, c_s \leq 1 \) which yields

\[
\left| \frac{c_s + c_r}{c_s - c_r} \right| > 7
\]

(B.201)

Combining this bound with equation (B.188) yields a sufficient condition on \( \alpha_0 \) given by
\[ \alpha_0 < 1.4 \]  \hspace{1cm} (B.202)

which, using equation (B.197), gives us a sufficient condition on \( \mathcal{A} \).

\[ \mathcal{A} < .9 \]  \hspace{1cm} (B.203)

We can obviously assume symmetric coefficients for the second case for any useful hologram and even the first case is not very restrictive. The holograms used in Chapter 3 easily meet these constraints.

For the three mirror resonated hologram we will find that our tolerances are much tighter. The analysis proceeds in much the same manner. \( \mathcal{N} \) and \( \mathcal{D} \) are given by

\[
\mathcal{N} = S \sqrt{1 - r^2} \left[ 1 - \left[ T^2 (1 - \epsilon^2) + S^2 \right] e^{-j\phi_2} \right] \quad \text{(B.204)}
\]

\[
\mathcal{D} = 1 + T^2 r (1 \pm \epsilon)^2 e^{-j\phi} - S^2 e^{-j\phi_2} \quad \text{(B.205)}
\]

The '±' in the expression for \( \mathcal{D} \) is due the fact that we can have two types of three mirror resonated holograms (\( r_1 = 0 \) or \( r_4 = 0 \)). With either type we need

\[ |\epsilon| \ll 1 \]  \hspace{1cm} (B.206)

in order to assume symmetric coefficients. Comparing this with the constraint on the four mirror resonated hologram (\( \epsilon^2 \ll 1 \)) it can be seen that this is a much tighter tolerance.

Following the same procedure as before gives us the sufficient condition upon \( \alpha_0 \):

\[ \alpha_0 < .07 \left| \frac{c_s + c_r}{c_s - c_r} \right| \]  \hspace{1cm} (B.207)

If all angles are allowed this translates into the sufficient condition

\[ \mathcal{A} < .07 \]  \hspace{1cm} (B.208)
If we limit the angles by $|\theta_o|, |\gamma_o| < .73$ rad then a sufficient condition upon $A$ is

$$A < .4$$  \hspace{1cm} (B.209)

The first condition has now become quite tight but the second condition is still quite reasonable and it is met by the holograms used in Chapter 3.

It can be seen that a bound upon the allowed angles can greatly increase the usable range of this assumption of symmetry. Determining what that bound should be is a design choice based upon the system in which the resonated hologram is used.

**Reflection Hologram**

For a reflection hologram we will derive a constraint upon $\alpha d$ which is sufficient for approximating the diffraction and transmission coefficients as being symmetric. Unlike for the transmission hologram, we are unable to express this as a constraint upon $A$, but $\alpha d$ is also a physically measurable quantity, so it will suffice.

We shall proceed in this analysis in much the same manner as for the transmission hologram. Once again we will find that it is the transmission coefficients which become asymmetric due to the loss. However, now the asymmetry is due to $\alpha_0$.

The transmission and diffraction coefficients for a reflection hologram are derived following the procedure in the previous section. This yields

$$T_1 = T_3 = Te^{-\alpha_0 e^{-j\phi_t}}$$  \hspace{1cm} (B.210)

$$T_2 = T_4 = Te^{\alpha_0 e^{-j\phi_s}}$$  \hspace{1cm} (B.211)

$$S_1 = S_3 = -j\sqrt{e}Se^{-j(\phi_t+\phi_s)/2}$$  \hspace{1cm} (B.212)

$$S_2 = S_4 = -j\frac{1}{\sqrt{e}}Se^{-j(\phi_t+\phi_s)/2}$$  \hspace{1cm} (B.213)
where
\[
\alpha_0 = \frac{\alpha d}{2} \left( \frac{1}{c_r} + \frac{1}{c_s} \right)
\]  
(B.214)
and \(S\) and \(T\) are given by equations (B.155), (B.156), and (B.133)-(B.138). The expressions for \(S\) and \(T\) are not used in this analysis and hence are not recopied here.

For the resonated reflection hologram we will substitute these diffraction and transmission coefficients into the same equations from Chapter 2 and make the same assumptions as for the transmission hologram. For the four mirror resonated hologram these assumptions are \(r_1 = r_4 = r\) and \(\phi_0 = \phi_1 = \phi\). They yield

\[
N = S(1 - r^2)[1 - (T^2 + S^2)e^{-j\phi_2}]
\]  
(B.215)
\[
D = 1 + 2T^2re^{-j\phi} \cosh(2\alpha_0) - S^2 \left(e^{-j\phi_2} + r^2e^{-j(2\phi - \phi_2)}\right)
\]  
(B.216)
Thus, we can make the approximation of symmetric transmission coefficients if

\[
cosh(2\alpha_0) \simeq 1
\]  
(B.217)
\[
\Leftrightarrow \alpha_0^2 \ll 1
\]  
(B.218)
\[
\Rightarrow |\alpha_0| < .3
\]  
(B.219)
\[
\Rightarrow \alpha d < \frac{.6}{\left|\frac{1}{c_r} + \frac{1}{c_s}\right|^{-1}}
\]  
(B.220)
This gives us our constraint upon \(\alpha d\) for the four mirror resonated hologram.

For the three mirror resonated hologram, we have

\[
N = S\sqrt{1 - r^2}[1 - (T^2 + S^2)e^{-j\phi_2}]
\]  
(B.221)
\[
D = 1 + T^2re^{-j\phi}e^{\pm j\alpha_0} - S^2e^{-j\phi_2}
\]  
(B.222)
where once again we have \( '\pm ' \) due to the two possible three mirror resonated hologram configurations. For the three mirror resonated hologram we can make the approximation of symmetric coefficients if

\[
e^{\pm 2\alpha_0} \approx 1 \quad (B.223)
\]

\[
\Rightarrow |\alpha_0| \ll .5 \quad (B.224)
\]

\[
\Leftrightarrow |\alpha_0| < .05 \quad (B.225)
\]

\[
\Rightarrow \alpha d < .1 \left| \frac{1}{e_r} + \frac{1}{e_s} \right|^{-1} \quad (B.226)
\]

This gives us our constraint upon \( \alpha d \) for the three mirror resonated hologram.

Comparing the results for the three and four mirror resonated holograms we find that the constraint is much tighter for the three mirror resonated hologram than for the four mirror resonated hologram, which is just what was found with the transmission hologram.

**B.7 Parallel Polarized Light**

In this section the case for light polarized parallel to the plane of incidence (parallel polarized light) will be considered. A detailed analysis will not be presented here, merely an outline of what was shown by Kogelnik [23] and the implications of the results.

It is shown by Kogelnik in his appendix that the only difference between the analysis for perpendicular-polarized light and parallel-polarized light is in the coupling parameter \( \kappa \). For parallel polarized light the following change is made.

\[
\kappa \rightarrow \vec{r} \cdot \vec{s} \kappa \quad (B.227)
\]

where

\[
\vec{r} = \text{unit direction vector of the incident electric field}
\]
\[ \vec{s} = \text{unit direction vector of the diffracted electric field} \]

In physical terms one can express the incident electric field as the sum of two orthogonal components, one parallel to the diffracted electric field and one orthogonal to it. The coupling will occur just with the component of the incident electric field which is parallel to the diffracted electric field hence the reduction in the coupling constant. Notice that the coupling constant can actually be equal to zero if the two direction vectors are orthogonal.

This expression for the coupling coefficient is actually the most general expression. If we use it for perpendicularly polarized light, \( \vec{r} \cdot \vec{s} = 1 \) so we have the same equations as before.

Because this only affects the coupling constant, all of the analysis conducted in this appendix is equally valid for both perpendicularly polarized and parallel polarized light, including the modeling of the hologram by transmission and diffraction coefficients and the form of these coefficients. This is important because it means that these results can be applied to a resonated hologram with parallel polarized incident light.

**B.8 Summary**

In this appendix a theoretical model for a plane-wave hologram was developed and analyzed. The special cases of a lossy hologram at Bragg incidence and a lossless hologram at non-Bragg incidence were looked at. The diffraction and transmission coefficients for this hologram were defined and explicitly given in terms of the hologram parameters. The conditions under which the approximation that these coefficients are symmetric is valid for the three and four mirror resonated hologram were determined. Finally, it was shown that this analysis was
applicable for light polarized parallel to the plane of incidence as well as polarized perpendicular to it.
APPENDIX C

DICROMATED GELATIN HOLOGRAMS

This appendix contains the technical information on the creation of the holograms used in Chapter III for the experimental demonstration of the resonated hologram. The methods for sensitizing, exposing, and developing these dichromated gelatin (DCG) holograms will be given as well as the method for attaching the antireflection (AR) plates.

Much work has been done in recent years to perfect a reliable method of producing sensitive, low loss holograms using dichromated gelatin. This work has yielded several different techniques, no one obviously superior to the others for all applications. Many of these methods require specialized procedures and environmental conditions (e.g. controlled heating and cooling of solutions, a constant humidity environment, etc.). The method used in this dissertation is a simplified technique developed by combining parts of several different procedures [40,66] with a few original modifications.

The formation of the hologram is done in four stages: sensitization, exposure, development, and attaching antireflection coated optical flats. These stages are described in detail in the following sections.
C.1 Sensitization

This section will present the sensitizing procedure to create DCG plates for making the hologram. Since DCG plates are not commercially available, it is necessary to create them. A list of the steps involved is given in Table 11 and are discussed in the remainder of this section.

Kodak 649-F spectroscopic plates are used to form the DCG holograms. These are chosen merely as a reliable source of thick, uniform gelatin on a glass substrate. The plates are fixed, washed, and then cleaned with methyl alcohol to remove all of the silver-halide grains leaving just the clean gelatin.

This gelatin is then impregnated with chromium ions by soaking in a solution of \((\text{NH}_4)_2\text{Cr}_2\text{O}_7\). This solution must be filtered before use in order to minimize the loss of the resulting hologram. The solution has a shelf life of one week.

Once the plates have been soaked the edges are cleaned with a damp sponge, the emulsion is squeegeed using a soft rubber squeegee (e.g. a squeegee for photographic prints), and then the back is cleaned, again with a damp sponge. This results in a more uniform plate and reduces noise due to crystalized chromium on the surface. (Any scatterers present when the plate is exposed will be recorded in the hologram and appear as noise when reconstructed.)

The plate is then placed in a light-tight box at a 10° angle from the horizontal to dry overnight. The plates should dry at least 16 hours before use. They are good up till 40 hours from the time of sensitization. This gives a 24 hour window in which the plates can be used before they must be discarded.
Table 11: DCG hologram sensitizing procedure.

<table>
<thead>
<tr>
<th>Step</th>
<th>Description</th>
<th>Time</th>
<th>Lights</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Fix plates using Kodak rapid fixer.</td>
<td>5 min</td>
<td>off</td>
</tr>
<tr>
<td>2</td>
<td>Wash in running tap water.</td>
<td>10 min</td>
<td>on</td>
</tr>
<tr>
<td>3</td>
<td>Soak in methyl alcohol.</td>
<td>5 min</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Agitate until pink color has dissolved.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>Repeat with clean methyl alcohol.</td>
<td>5 min</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>Dry at room temperature in a vertical position.</td>
<td>15 min</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>Sensitize in a 5% by weight solution of $(\text{NH}_4)_2\text{Cr}_2\text{O}_7$ (filtered using qualitative 1 filter paper).</td>
<td>5 min</td>
<td>off</td>
</tr>
<tr>
<td>7</td>
<td>Clean edges of plate with damp sponge</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>Squeegee emulsion twice</td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>Clean back of plate with damp sponge</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>Place in light-tight box at $10^\circ$ angle and allow to dry.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>Use between 16 and 40 hours after sensitizing.</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table 12: Exposure procedure for DCG hologram.

<table>
<thead>
<tr>
<th>Step</th>
<th>Description</th>
<th>Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Clean back of plate with methyl alcohol.</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>Mount with emulsion toward the light source.</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>Wait for any stresses to stabilize</td>
<td>2 min</td>
</tr>
<tr>
<td>4</td>
<td>Expose</td>
<td>2-3 sec</td>
</tr>
</tbody>
</table>

C.2 Exposure

The exposure procedure is given in Table 12. It is no different from the procedure for exposing silver-halide holograms except that the plate is cleaned before exposure to minimize noise. Lights are off throughout the exposure procedure.

The exposure was made using the blue line of an argon laser ($\lambda = 488.0$) because of the increased sensitivity of the DCG plate at this wavelength. The laser used was a 40 mW argon laser, model 60C/60B made by the American Laser Corporation. The laser beam is split into two beams, an object beam and a reference beam, using a variable beamsplitter. The object and reference beams are each spatially filtered using a 20x microscope objective and a 10 $\mu$m pinhole, collimated with a planoconvex lens to a diameter of 25 mm, and then stopped down to a diameter of 8 mm to create plane waves of uniform intensity. The beam power ratio was 1:1. The power in each beam is 0.4 mW. The reference beam angle was $28^\circ 25'$ and the object beam angle is $23^\circ 05'$. A photograph of this system is shown in Figure 44. The laser is located in the upper right with the beam emitted from its left side. The variable beamsplitter is located in the upper center of the photo. The reference and object beams are
Figure 44: Photograph of the hologram creation apparatus located on the left and right respectively. The film holder is located in the bottom center of the photo.

C.3 Development

The developing procedure is itemized in Table 13. It is quite simple, requiring only running tap water, distilled water, reagent grade isopropyl alcohol, and a hair dryer.

The plate is washed in running tap water to remove any unreacted chromium. It is then gradually dehydrated using solutions of isopropyl alcohol mixed with distilled water in the ratios 50/50, 90/10, and 100/0 by volume. The reason for several stages is to prevent the hologram from tearing away from the glass substrate due to too rapid dehydration. In addition it ensures that the final 100% solution will not be diluted by water from the emulsion. Finally, the plate is dried as it is removed from the alcohol by directing a hot stream of air (in this case from a
Table 13: Developing procedure for DCG hologram.

<table>
<thead>
<tr>
<th>Step</th>
<th>Description</th>
<th>Time</th>
<th>Lights</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Wash in running tap water</td>
<td>10 min</td>
<td>off</td>
</tr>
<tr>
<td>2</td>
<td>Soak in solution of 50% isopropyl alcohol, 50% distilled water. Agitate.</td>
<td>2 min</td>
<td>on</td>
</tr>
<tr>
<td>3</td>
<td>Soak in 90/10 solution. Agitate.</td>
<td>2 min</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>Soak in 100/0 solution. Agitate.</td>
<td>10 min</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>Remove slowly (1cm/min) while drying with stream of hot air.</td>
<td>12 min</td>
<td></td>
</tr>
</tbody>
</table>

hair dryer) onto the emulsion. The goal in drying is to create a drying wedge that moves uniformly across the emulsion. This is supposed to ensure uniform results.

C.4 AR Coating

The final step in producing the DCG holograms is to apply anti-reflection (AR) coated optical flats to both sides of the hologram in order to reduce the Fresnel reflections from the air-emulsion and glass-air interfaces and eliminates any beam distortion from the nonflat substrate. This reduces the reflections to < 0.5% per surface.

The AR plates are mirror blanks with surface flatness of $\lambda/10$ which have been coated with an antireflection coating for the HeNe wavelength on one side. These plates are attached using silicone sealant around the edges and optically gated with mineral oil. First a separator, consisting of four layers of transparent tape and one layer of double-sided tape, is cut into 1mm strips and placed just inside the edge.
of the AR flat. This tape serves to hold the flat in position while the edges are sealed and provides enough of a separation (~ 300μm) that the mineral oil will fill the space without air bubbles. Then the edges are sealed around three quarters of the perimeter. Next, the space between is filled with mineral oil. Finally, the top edge is sealed. A hypodermic needle and syringe are used to apply the mineral oil. The needle is dipped in a container of mineral oil and then placed at the edge of the AR flat. The drop of mineral oil that has adhered to the side of the needle then diffuses between the AR flat and the hologram. (The syringe is merely a convenient handle by which to manipulate the needle. It is not used to draw up fluid.) By proceeding one drop at a time in this manner it is possible to keep the mineral oil off of the AR coated surface. In a similar manner the silicone sealant is also applied with a (different) needle.

In addition to eliminating reflections, this procedure seals the holographic emulsion so that it cannot rehydrate. It is reported in the literature that over time unsealed DCG holograms may lose their diffractive properties due to rehydration, although this was not observed during the course of this dissertation. This possibility appears to be a function of the processing of the hologram and apparently is not a problem with the procedure described in this appendix.

C.5 Summary

The method used for producing the DCG holograms used in this experiment was given in detail. The steps and the reasons for performing them were given. The information in this appendix is referenced in Chapter III, Section 2.
BIBLIOGRAPHY


