INFORMATION TO USERS

The most advanced technology has been used to photograph and reproduce this manuscript from the microfilm master. UMI films the original text directly from the copy submitted. Thus, some dissertation copies are in typewriter face, while others may be from a computer printer.

In the unlikely event that the author did not send UMI a complete manuscript and there are missing pages, these will be noted. Also, if unauthorized copyrighted material had to be removed, a note will indicate the deletion.

Oversize materials (e.g., maps, drawings, charts) are reproduced by sectioning the original, beginning at the upper left-hand corner and continuing from left to right in equal sections with small overlaps. Each oversize page is available as one exposure on a standard 35 mm slide or as a 17" × 23" black and white photographic print for an additional charge.

Photographs included in the original manuscript have been reproduced xerographically in this copy. 35 mm slides or 6" × 9" black and white photographic prints are available for any photographs or illustrations appearing in this copy for an additional charge. Contact UMI directly to order.

Accessing the World’s Information since 1938

300 North Zeeb Road, Ann Arbor, MI 48106-1346 USA
The Steinberg module and the top cohomology of arithmetic groups

Reeder, Mark Stephen, Ph.D.
The Ohio State University, 1988
PLEASE NOTE:

In all cases this material has been filmed in the best possible way from the available copy. Problems encountered with this document have been identified here with a check mark √.

1. Glossy photographs or pages ______
2. Colored illustrations, paper or print ______
3. Photographs with dark background ______
4. Illustrations are poor copy ______
5. Pages with black marks, not original copy ______
6. Print shows through as there is text on both sides of page ______
7. Indistinct, broken or small print on several pages √
8. Print exceeds margin requirements ______
9. Tightly bound copy with print lost in spine ______
10. Computer printout pages with indistinct print ______
11. Page(s) ______ lacking when material received, and not available from school or author.
12. Page(s) ______ seem to be missing in numbering only as text follows.
13. Two pages numbered ______. Text follows.
14. Curling and wrinkled pages ______
15. Dissertation contains pages with print at a slant, filmed as received √
16. Other ________________________________________________________

_______________________________________

_______________________________

UMI
THE STEINBERG MODULE AND THE
TOP COHOMOLOGY OF ARITHMETIC GROUPS

DISSERTATION

Presented in Partial Fulfillment of
the Requirements for the Degree Doctor of
Philosophy in the Graduate School of
The Ohio State University

By

Mark Stephen Reeder, B.A., M.S.

1988

Dissertation Committee:
Avner Ash
Ruth Charney
Guido Mislin

Approved by
Advisor
Department of Mathematics
To my mother
Rosalia Stamatakos
and the memory of my father
Fritz Adrian Reeder
Acknowledgments

I am very grateful to Professor Avner Ash for suggesting the original topic of this dissertation, for his own work which inspired most of the results presented here, and for his patience with me.

I also wish to recognize here the influence on my work of Professor Donna M. Testerman, who introduced me to Lie theory and whose mathematical aesthetics will always pervade my own.
Vita

October 6, 1959 ....................................................... Born—Los Angeles, California

1982 ........................................................................... B.A., Humboldt State University, Arcata, California

1982–1985 ................................................................. Graduate Teaching Fellow, University of Oregon, Eugene, Oregon

1983 ........................................................................... M.S., University of Oregon, Eugene, Oregon

1985–Present ....................................................... Graduate Teaching Assistant, Ohio State University, Columbus, Ohio

Fields of Study

Algebraic Groups, Representation theory, Automorphic Forms
# Table Of Contents

Dedication ......................................................... ii
Acknowledgments .............................................. iii
Vita ........................................................................ iv
Introduction ...................................................... 1

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>I. Modular Symbols and Restriction to a Cusp</td>
<td>§1. The Steinberg Representation</td>
<td>17</td>
</tr>
<tr>
<td></td>
<td>§2. The Restriction Map</td>
<td>21</td>
</tr>
<tr>
<td>II. Applications to Subgroups of SLₙ(ℤ)</td>
<td>§1. Partial Descriptions of the Top (Co)Homology</td>
<td>32</td>
</tr>
<tr>
<td></td>
<td>§2. Restriction for Γ&lt;SLₙ(ℤ)</td>
<td>35</td>
</tr>
<tr>
<td></td>
<td>§3. The Top Homology of SLₙ(ℤ)</td>
<td>37</td>
</tr>
<tr>
<td></td>
<td>§4. Γ=Γ₀(n,N)</td>
<td>43</td>
</tr>
<tr>
<td></td>
<td>§5. n≤4</td>
<td>50</td>
</tr>
<tr>
<td>III. The Borel-Serre Boundary</td>
<td>§1. The Cover and its Restriction Maps</td>
<td>54</td>
</tr>
<tr>
<td></td>
<td>§2. The Split Groups of Rank 2</td>
<td>58</td>
</tr>
<tr>
<td>IV. Counting Cusps.</td>
<td>§1. Bruhat Cells in Pⁿ⁻¹(ℤ/N)</td>
<td>68</td>
</tr>
<tr>
<td></td>
<td>§2. Symmetry in the Formula</td>
<td>73</td>
</tr>
<tr>
<td></td>
<td>§3. The Tits Building of SL₃ modulo Γ₀(n,N)</td>
<td>76</td>
</tr>
<tr>
<td>V. Old Forms for GLₙ.</td>
<td>§1. Notation and Preliminaries</td>
<td>78</td>
</tr>
<tr>
<td></td>
<td>§2. Polynomial Whittaker Vectors</td>
<td>79</td>
</tr>
<tr>
<td></td>
<td>§3. The Polynomial Mellin Transform</td>
<td>80</td>
</tr>
<tr>
<td></td>
<td>§4. Old Forms</td>
<td>87</td>
</tr>
</tbody>
</table>

Bibliography .................................................. 92
Introduction

Let $G$ be a reductive algebraic group defined over $\mathbb{Q}$, $\Gamma$ an arithmetic subgroup. Most of the results in this dissertation have their motivational roots in the study of the "cuspidal cohomology" of $\Gamma$. This is a generalization of Eichler's parabolic cohomology for subgroups of $SL_2(\mathbb{Z})$. To give the definition, we need an analytic view of $H^*(\Gamma)$. See [9].

Let $K$ be a maximal compact subgroup of $G(\mathbb{R})$, $X$ the symmetric space $G(\mathbb{R})/K$. The Eilenberg–Maclane cohomology of $\Gamma$ with real or complex coefficients may be computed as the cohomology of the complex of $\Gamma$ invariant differential forms on $X$. A version of the deRham theorem then yields

$$H^*(\Gamma) \cong H^*(g,K;C^\infty(\Gamma\backslash G(\mathbb{R})))$$

where the right side is $(g,K)$ cohomology, and $g$ is the Lie algebra of $G(\mathbb{R})$.

Let $L^2_0(\Gamma\backslash G(\mathbb{R})) \subseteq C^\infty(\Gamma\backslash G(\mathbb{R}))$ denote the subspace of "cuspidal functions": those which give the zero function when averaged over left translations by $U(\mathbb{R})\cap \Gamma \backslash U(\mathbb{R})$ where $U$ is the unipotent radical of any proper $\mathbb{Q}$–parabolic subgroup of $G$. It turns out that the natural map
\[ H^\bullet(g,K; L^2_0(\Gamma \backslash G(\mathbb{R}))^{\infty}) \rightarrow H^\bullet(g,K; C^\infty(\Gamma \backslash G(\mathbb{R}))) \]
is injective, and the image in \( H^\bullet(\Gamma) \) is called the \textit{cuspidal cohomology} of \( \Gamma \), denoted \( H^\bullet_{\text{cusp}}(\Gamma) \).

These cohomology classes correspond to cuspidal automorphic forms in the following way [9], [15]. The space \( L^2_0(\Gamma \backslash G(\mathbb{R}))^{\infty} \) is the direct sum of the infinity components of those cuspidal automorphic representations \( \pi = \pi_\infty \otimes \pi_f \) of \( G(A) \) which contain a fixed vector under \( \Gamma \). Here \( A = \mathbb{R} \times A_f \) is the adele ring of \( \mathbb{Q} \). Moreover,

\[ H^\bullet(g,K; L^2_0(\Gamma \backslash G(\mathbb{R}))^{\infty}) \simeq \oplus_{\pi} H^\bullet(g,K; \pi_\infty) \otimes \pi_f^\Gamma, \]

where the sum is over all cuspidal automorphic representations \( \pi \). Thus, the cuspidal classes in \( H^\bullet(\Gamma) \) correspond to \( \Gamma \)-fixed vectors in those cuspidal automorphic representations \( \pi \) with \( \pi_\infty \) having non-trivial \((g,K)\) cohomology. We can similarly define \( H^\bullet_{\text{cusp}}(\Gamma,F) \), where \( F \) is a finite dimensional representation of \( G(\mathbb{R}) \), and thereby detect those \( \pi \) with \( \pi_\infty \otimes F \) having \((g,K)\) cohomology.

For \( G = SL_2 \) and \( F_n \) the \( n \)-dimensional representation of \( SL_2(\mathbb{R}) \), these \( \pi_\infty \) are the discrete series representations with lowest \( K \)-type \( \pm(n+1) \) corresponding to the (anti)holomorphic cusp forms for \( \Gamma \) of weight \( n+1 \), and the isomorphism

\[ H^\bullet_{\text{cusp}}(\Gamma,F_n) \simeq \oplus_{\pi} H^\bullet(g,K; \pi_\infty \otimes F_n) \otimes \pi_f^\Gamma \]

An advantage of the group cohomological viewpoint is that (in principle, and sometimes in practice) it gives a combinatorial way of studying the representation-theoretic space
A disadvantage is that there is, at present, no purely group cohomological characterization of $H^\ast_{\text{cusp}}(\Gamma)$. However, we can assert ([20] 6.2) that cuspidal classes must vanish upon restriction to $\Gamma \cap P$ for every maximal $\mathbb{Q}$-parabolic subgroup $P$ of $G$. Moreover, for small $G$ and certain cohomological dimensions (e.g., $\text{SL}_3$ and $\text{Sp}_4$ in dimension 3), the vanishing on all the $\Gamma \cap P$'s is also a sufficient condition for a class to be cuspidal.

My original goal, suggested by my advisor Avner Ash, was to compute the cuspidal cohomology $H^3_{\text{cusp}}(\Gamma_0(3,N),\mathbb{C})$ where $\Gamma_0(3,N)$ is the subgroup of $\text{SL}_3(\mathbb{Z})$ whose elements have first column congruent to $(*,0,0)$ modulo $N$. This is the largest "Hecke type" congruence subgroup (except for $\text{SL}_3(\mathbb{Z})$ itself, which has no cuspidal cohomology with $\mathbb{C}$ coefficients) and hence is the easiest nontrivial example. In [4] Ash, Grayson and Green had, in some sense, solved this problem for $N=p$, a prime. They found the cuspidal cohomology as a space of $\mathbb{C}$-valued functions on $\Gamma_0(p,N)\backslash \text{SL}_3(\mathbb{Z})$, satisfying some linear equations. Some of the equations express the cocycle conditions, while others characterize the vanishing of restriction to $\Gamma \cap P$ for $P$ running through a set of representatives of $\Gamma_0(3,p)$-conjugacy classes of maximal $\mathbb{Q}$-parabolic subgroups of $\text{SL}_3$. These equations were solved electronically for $p \leq 113$. Moreover, the Hecke algebra acts naturally on the solution space, and the Hecke eigenvalues on $H^3_{\text{cusp}}(\Gamma_0(3,p),\mathbb{C})$ were investigated. However, the latter type of equations in [4] do not apply to composite $N$ and hence relatively
few computations were possible with small level. The difficulty
with non-prime $N$ is that, as we shall see in this thesis, there are
$2 \cdot (\# \text{divisors of } N)$ $\Gamma_0(3,N)$-conjugacy classes of maximal
$\mathbb{Q}$-parabolics in $\text{SL}_3$. This means we need more equations to
determine the kernels of the restriction maps. Now, Ash found the
equations heuristically, using "modular symbols" (defined below),
and using the fact that $N$ was prime, he was able to prove that his
equations were correct. It was difficult to generalize this last
step the composite case (due to nontrivial homology in $\mathbb{J}_G/\Gamma_0(3,N)$,
where $\mathbb{J}_G$ is the Tits building of $\text{SL}_3$ over $\mathbb{Q}$). Thus, it seemed like
a good idea to get a computational understanding of modular
symbols in the first place, and derive the equations for
restriction to $\Gamma \cap \mathcal{P}$. Since the general ideas are valid for, and
have applications to any reductive $\mathbb{Q}$-group, we return to that
setting. The term "modular symbol" originally meant the
homology class for a discrete subgroup of $\text{SL}_2(\mathbb{R})$ given by a
geodesic in the upper half plane $\mathbb{H}$ pushed down to $\Gamma \backslash \mathbb{H}$. As
developed by Ash, Mazur and others [3],[5], "modular symbol" now
refers to an analogous submanifold of any arithmetic locally
symmetric space. More precisely, we have a
Definition: Let $\mathcal{X}$ denote the Borel-Serre bordification [8] of
$X=\text{G}(\mathbb{R})/\mathbb{K}$. Suppose $T$ is a maximal $\mathbb{Q}$-split torus of $G$ and $\Gamma$ is a
torsion free arithmetic subgroup of $G(\mathbb{Q})$ lying in $G(\mathbb{R})^0$. The
closure of the image of
$T(\mathbb{R})^0 \hookrightarrow G(\mathbb{R}) \rightarrow \Gamma \backslash \mathcal{X}$
is a submanifold-with-boundary whose Poincare dual is an element of $H^v(\Gamma \backslash \mathcal{X}, \mathbb{Z}) = H^v(\Gamma, \mathbb{Z})$ where $v = \dim X - \dim T$. We denote this cohomology class by $[T]_\Gamma$ and call it a "(minimal) modular symbol". The assumptions $\Gamma < \mathbb{G}(\mathbb{R})^0$ and torsion-free are made for technical simplifications only, and can be fulfilled by passing to a subgroup of finite index if necessary. The term "minimal" refers to the fact that $T$ is a Levi factor for a minimal $\mathbb{Q}$-parabolic subgroup of $G$. One can mimic the above construction with larger Levi factors, but we shall not do so here, and "modular symbol" will always mean "minimal modular symbol".

The classes $[T]_\Gamma$ are useful because of the following

**Duality Theorem** (Borel-Serre, [8]) There is a $G(\mathbb{Q})$ module $I$ such that for any $\Gamma$ module $A$, we have a natural isomorphism $H^v(\Gamma, A) \cong H_{v-q}(\Gamma, I \otimes A)$.

An immediate corollary is that $v$ is the highest possible nonvanishing dimension of $H^v(\Gamma, A)$. Another consequence is that the modular symbols $[T]_\Gamma$ generate $H^v(\Gamma, \mathbb{Z})$. This is a crucial fact and shortly I will sketch the proof. The dualizing module $I$ is called the "Steinberg representation" of $G(\mathbb{Q})$ because it can be constructed as the top homology of the Tits building $J_G$ (see chapter I for the definition of $J_G$). (Some people refer to the top cohomology of $J_G$ as the Steinberg representation, and the two representations are different here, as opposed to the situation of finite groups of Lie type). If we fix a minimal $\mathbb{Q}$-parabolic subgroup $B$ of $G$, then $I$ has a basis.
\{\tau_T; T \text{ a maximal } \mathbb{Q}\text{-split torus of } B\}.

The \(B(\mathbb{Q})\)-action is given by \(\tau_T \cdot b = \tau_T b\), and \(\tau_T\) affords the sign character of the relative Weyl group of \(T\) in \(G\). Other than this, the \(G(\mathbb{Q})\) action on \(I\) is not well understood in general.

Here is why the modular symbols generate the top cohomology of \(\Gamma\). See [3]. The Tits building \(\mathcal{J}_G\) is a retract of \(\mathcal{J}\), and since \(\mathcal{X}\) is contractible we get an isomorphism \(I \rightarrow H_c(\mathcal{X}, \mathcal{Z})\), where \(r = \dim T\). On the other hand, by Borel-Serre duality with \(A = \mathbb{Z}\), \(q = 0\), we have an isomorphism

\[
\mathcal{I} \otimes_{\mathbb{Z}} \mathbb{Z} \rightarrow H^v(\Gamma, \mathbb{Z}) \cong H_c(\Gamma \backslash \mathcal{X}, \Gamma \backslash \mathcal{Z}).
\]

Moreover, the following diagram commutes,

\[
\begin{array}{c}
I \\[0.5em]
\alpha \downarrow \\
\mathcal{I} \otimes_{\mathbb{Z}} \mathbb{Z} \\
\beta
\end{array}
\]

and \(\tau_T \in I \mapsto [T] \in H_c(\Gamma \backslash \mathcal{X}, \Gamma \backslash \mathcal{Z})\). Here \(\alpha\) and \(\beta\) are the nature maps. Since \(\alpha\) is surjective, so is \(\beta\). It follows that the \([T]_r\)'s generate \(H^v(\Gamma, \mathbb{Z})\). Notice that, thinking topologically, it is not obvious that \(\beta\) is surjective, while this is tautological for the algebraically defined \(\alpha\). Also, all of this works with \(\mathbb{Z}\) replaced by an arbitrary coefficient module \(A\), where the geometric picture is less clear. This led me to make the Steinberg module \(I\) the central object, and to think of the top cohomology of \(\Gamma\) algebraically, as nothing but some quotient of \(I\).

Let me now describe the results in this thesis. The first goal is to understand the restriction map \(H^*(\Gamma, A) \rightarrow H^*(\Gamma \cap \mathcal{P}, A)\).
There is an analogous dualizing module $I_p$ (the Steinberg module for a Levi factor of $P$) for $\Gamma \cap P$ and Borel-Serre duality holds for $\Gamma \cap P$, with the same dualizing dimension $v$. Hence, there must be a commutative diagram

$$
\begin{array}{c}
H^q(\Gamma, A) \to H^v(\Gamma, I \otimes A) \\
\downarrow \text{rst} \downarrow \\
H^q(\Gamma \cap P, A) \to H^v(\Gamma \cap P, I_p \otimes A).
\end{array}
$$

The main result in chapter I is a formula for $r(P,A)$. When $q=v$, $r(P,A)$ is a map

$$
r(P,A): I \otimes \Gamma A \to I_p \otimes \Gamma \cap P A,$$

and the formula involves a sum over certain Weyl group elements. Geometrically, the boundary of the modular symbol $[T]$ may be thought of as a union of smaller-dimensional modular symbols for the proper $\Phi$-parabolics in $G$ which contain $T$. Roughly speaking, the formula for $r(P,A)$ expresses the restriction of a big modular symbol as a sum of smaller modular symbols of the same "type" as $P$, taking into account $\Gamma$-conjugacy and the twisting of $A$.

Before settling down to compute cuspidal cohomology for $\Gamma_0(3,N)$, we explore, in chapter II, some consequences of Borel-Serre duality and our formula for $r(P,A)$, when $G=\text{SL}_n$. The main reason for considering such a special case is the often-used fact ([5], [21]) that $I$ is generated over $\text{SL}_n(\mathbb{Z})$ by a single $\tau_T$. Some of the results, however, are valid for any split $\Phi$-group.
To describe the results in chapter II, we change notation slightly and set $G = \text{SL}_n(\mathbb{Z})$, $S = \text{SO}_n(\mathbb{Z})$, $\Gamma$ a subgroup of finite index in $G$, and $A$ a $\mathbb{C}G$-module. (In most results, we could replace $\mathbb{C}$ by a field of characteristic greater than $n+1$). The top homology version of Borel-Serre duality is

**Thm:** (Borel-Serre) There is a natural isomorphism $H^r(\Gamma, A) \approx \text{Hom}_\mathbb{C}(I, A)$.

Using this result and the fact that $\tau_T$ generates $I$ over $G$, we get

**Corollary:** There is an injection $H^\ast(G, A) \hookrightarrow A_\mathbb{C} = \{a \in A | s \cdot a = \epsilon(s)a \text{ for all } s \in S\}$. Here $S$ is viewed as the normalizer in $G$ of the diagonal torus $T$ of $\text{SL}_n(\mathbb{Q})$, and $\epsilon$ is the sign representation of the Weyl group.

Let $V$ be the standard $n$-dimensional representation of $\text{SL}_n(\mathbb{C})$, viewed as an $\text{SL}_n(\mathbb{Z})$-module. Let $\Lambda^\ast V$, $S^\ast V$ denote the exterior and symmetric algebras of $V$, and let $\text{Ad}$ be the adjoint representation of $\text{SL}_n(\mathbb{Q})$.

**Thm.:** $H^r(G, A) = 0$ for $A = \Lambda^\ast V$, $\text{Ad}$, $S^\ast V$, where $r$ is restricted as follows: $n$ even or $r$ odd $\Rightarrow r \leq n+1$, $n$ odd and $r$ even $\Rightarrow r \leq n(n-1)$.

The proof consists of showing that $\epsilon$ does not occur in $A$, viewed as an $S$-module. This uses Invariant theory, viewing the integral matrices in $O(n, \mathbb{R})$ as the Weyl group of type $C_n$.

We now discuss the inclusion map on homology from a parabolic subgroup of $\text{SL}_n$. Let $P = LU$ be a $\mathbb{Q}$-parabolic of $\text{SL}_n$. Write $P = P(\mathbb{Z})$, etc. So $L$ is a subgroup of index two in a direct
product of $GL_r(\mathbb{Z})$'s for various $r$. Let $v_L$ be the virtual cohomological dimension of $L$ and $i_\kappa$ the composition $H_{v_L}(L, A^U) \approx H_{v}(P, A) \rightarrow H_{v}(G, A)$. The formula for $r(P, A)$ in chapter I can be used to prove the following.

**Thm.:** Define $j: (A^U)_\epsilon \rightarrow A$ by $j(a) = \sum_{w \in W(P)} \epsilon(w) w^{-1} a$, where $W(P)$ is a set of representatives in $S$ for $S/SDL$. Then the following diagram commutes:

\[
\begin{array}{cccc}
H_{v}(G, A) & \hookrightarrow & A_{\epsilon} \\
\uparrow i_\kappa & & \uparrow j \\
H_{v_L}(L, A^U) & \hookrightarrow & (A^U)_\epsilon
\end{array}
\]

Note that $\epsilon|S\cap L$ is the sign character of the Weyl group of $T$ in $L$.

**Corollary:** Let $A_\lambda$ be the rational $SL_n(\mathbb{Q})$ module of highest weight $\lambda$, with respect to the upper triangular matrices. Let $\Delta_L$ be the simple roots which generate $L$. Suppose that the Dynkin diagram of $SL_n$ contains a node which is not connected to $\Delta_L \cup \text{support}(\lambda)$. Then $i_\kappa$ is identically zero.

On the other hand, one can show that $i_\kappa$ is often non-zero. For example, if $P$ is a minimal $\mathbb{Q}$-parabolic and the coefficients of the fundamental weights in $\lambda$ are even and positive, then $i_\kappa$ is nonzero. Similarly, if the derived group of $L$ is a product of $SL_2$'s, we can get nonzero lifts of products of cusp forms for $SL_2$. For these results, one shows explicitly that the image of $H_{v_L}(L, A^U) \hookrightarrow (A^U)_\epsilon \rightarrow A_\epsilon$ is nonzero. If $A$ is an $SL_n(\mathbb{F}_p)$ module,
this technique can often be used to identify A as an \( S_{n}(\mathbb{F}_{p}) \)-composition factor of \( H^{\nu}(\Gamma(p),\mathbb{Z}) \), where \( \Gamma(p) \) is the full congruence subgroup of level \( p \).

We now begin to compute \( \ker \text{rst}: H^{\nu}(\Gamma,A) \to H^{\nu}(\Gamma \cap P,A) \).

In order to apply the formula for \( r(P,A): I \otimes_{T} A \to I_{p} \otimes_{\Gamma \cap P} A \) in chapter I effectively, we need to lift the map \( r(P,A) \) to a more understandable space. We first observe that there is a canonical isomorphism

\[
I_{p} \otimes_{\Gamma \cap P} A \cong I_{p} \otimes_{\Gamma \cap U} A_{\Gamma \cap U}
\]

where \( \Gamma(L) \) is the projection of \( \Gamma \cap P \) into \( L \), and then arrive at the following commutative diagram:

\[
\begin{array}{cccc}
J(\Gamma,A) & \to & J(\Gamma(L),A_{\Gamma \cap U}) \\
\downarrow m_{\Gamma} & & \downarrow m_{\Gamma(L)} \\
r(P,A): I \otimes_{T} A & \to & I_{p} \otimes_{\Gamma \cap U} A_{\Gamma \cap U} \\
\downarrow \cong & & \downarrow \cong \\
\text{rst}: H^{\nu}(\Gamma,A) & \to & H^{\nu}(\Gamma(L),A_{\Gamma \cap U})
\end{array}
\]

where

\[
J(\Gamma,A) = \{ f: G \to A | f(g \gamma) = \gamma^{-1} f(g) \text{ for all } g \in G, \gamma \in \Gamma \},
\]

\[
m_{\Gamma}(f) = \sum_{g \in G} \tau_{T} \cdot g \otimes f(g) \text{ and}
\]

\( T \) is a maximal \( \mathbb{Q} \)-split torus of \( L \). \( J(\Gamma(L),A_{\Gamma \cap U}) \) and \( m_{\Gamma(L)} \) are defined similarly. The map \( p_{L} \) is given by
\[ p_L(f)(\ell) = \ell^{22} \sum_{w \in P} \epsilon(w) w g f(g), \text{ for } \ell \in L. \] We then have
\[ p_L^{-1}(\ker m_{r(L)}) / \ker m_{r} \cong \ker \text{rst}. \]

Now consider the case \( r = r_0(n, N) \). In order to compute \( p_L \), we need to have representatives of the \( r \)-conjugacy classes of the \( \mathbb{Q} \)-parabolics of \( \text{SL}_n \) which are \( \text{SL}_n(\mathbb{Z}) \)-conjugate to \( P \). Moreover, if \( P^\mu \) is one such, and \( x \in P(\mathbb{Z}) \mu r \), say \( x = \mu \rho \gamma \), we need an explicit formula for \( p \) modulo \( \mu \Gamma \rho P \), in terms of matrices. This is done for arbitrary \( n \) and maximal parabolics \( P \).

Skipping ahead to chapter III, I have also found the \( r \)-conjugacy classes of minimal \( \mathbb{Q} \)-parabolic subgroups of \( \text{SL}_n \). This is equivalent to finding the orbits of the upper triangular integer matrices on \( \mathbb{P}^{n-1}(\mathbb{Z}/N) \). The orbits are collected into "cells", as in the Bruhat decomposition when \( N \) is prime. However, there are now many orbits per cell, instead of just one. The symmetric group \( S_n \) permutes the cells, and the number of orbits per cell is given by a function which is invariant under \( S_n \). When \( N = p^r \), \( p \) a prime, this function is, on each cell, a rational function in \( p \). A by-product of this computation is the first Betti number of \( G / \Gamma \) for \( \text{SL}_3 \).

Back to Chapter II. To complete the computation of \( \ker \text{rst} \), we need to know \( \ker m_r \) and \( \ker m_{r(L)} \). Unfortunately, these have only been determined for \( n = 3, 4 \) (by Ash, in another context, using the "well-rounded retract" in [1]). For \( n = 3, 4 \) and \( A = \mathbb{C} \) with trivial \( \Gamma \) action, we end up with an explicit system of linear equations whose solution space is isomorphic to
For nontrivial coefficients, we get, instead of equations, an explicit subquotient of a certain finite dimensional vector space.

From the cohomology of the Borel-Serre boundary for $\text{SL}_3$ (chapter IV), we deduce that

$$\ker \text{rst}: H^0(\Gamma, \mathbb{C}) \to H^0(\Gamma \cap \mathcal{P}^\mu, \mathbb{C}).$$

In chapter IV, we compute $H^\ast(\Omega \text{or} / \Gamma, A)$, where $\Gamma$ is congruence subgroup of $G(\mathbb{Z})$, and $A$ is a finite dimensional rational $G(\mathbb{C})$ representation, for $G=\text{SL}_3, \text{Sp}_4, G_2$. We also compute the $G(\mathbb{Z})/\Gamma$ module structure on this space. It follows from Poincare duality and a result of Kazhdan that restriction to the full boundary is surjective in the top cohomological dimension, at least for $\mathbb{Q}$-coefficients. Roughly speaking, the cohomology of the boundary is orthogonal to the cuspidal cohomology in $H^\ast(\Gamma, A)$. In this thesis, I actually only use the result mentioned in the previous paragraph, but the dimensions of the cohomology of the
boundary presented here would provide a check on the computer computation of cusp forms alluded to above.

The computation of $H^*(\mathfrak{X}/\Gamma,A)$ was already done for $G=\text{Sl}$ and $\text{Sp}_4$ with $A=\mathbb{C}$ by Lee-Schwermer and Schwermer in [20] and [23]. My methods are essentially theirs, adapted to work for all three groups at once, with twisted coefficients.

For $G_2$, we can use vanishing theorems from $(g,K)$ cohomology [31] to get $H^6(\Gamma,\mathbb{C})\cong H^6(\mathfrak{X}/\Gamma,\mathbb{C})$. ($cd\Gamma=6$ in the $G$ case). So we have actually computed $H^6(\Gamma,\mathbb{C})$ for $\Gamma$ a subgroup of $G_2(\mathbb{Z})$ which contains a congruence subgroup. We can also conclude from this computation that $H^2(G_2(\mathbb{Z}),\mathbb{C})\cong H^6(G_2(\mathbb{Z}),\mathbb{C})=0$.

The cohomology classes for $\Gamma$ which restrict nontrivially to the Borel-Serre boundary (which must be noncuspidal) are "lifts" of cusp forms on groups of smaller rank, so are inductively "already known". Cuspidal classes themselves can be "already known" in another sense: they may come from cuspidal cohomology from some larger arithmetic subgroup of the same $\mathbb{Q}$-group $G$. More specifically, there are various ways, besides via the ordinary restriction, that a cusp form for $\Gamma_0(n,d)$ induces a cusp form on $\Gamma_0(n,N)$, when $d|N$. Forms induced from smaller level are called "old". For $n=2$, the classical Atkin-Lehner theorem (see [13]) on cusp forms for $\Gamma_0(2,N)$ characterizes these old cusp forms as obstructions to diagonalizing the Hecke operators at the primes dividing $N$. In terms of group cohomology, the Atkin-Lehner theorem can be phrased as follows. Let $A$ be an
irreducible finite dimensional $\text{SL}_2(\mathbb{C})$-module. Let $m \mid N$, and $d \mid (N/m)$. Define a map

$$T_d : H^1_{\text{cusp}}(\Gamma_0(2,m),A) \to H^1_{\text{cusp}}(\Gamma_0(2,N),A)$$

as follows: If $u : \Gamma_0(2,m) \to A$ is a 1-cocycle for $\Gamma_0(2,m)$, then

$$T_d(u)(g) = (u)_1 g^{-1}(u)$$

is a 1-cocycle on $\Gamma_0(2,N)$. Let $H_{\text{old}}$ be the subspace of $H^1_{\text{cusp}}(\Gamma_0(2,N),A)$ generated by the images of all the $T_d$'s, for $m \mid N$ and $d \mid (N/m)$. Let $H_{\text{new}}$ be the largest subspace of $H^1_{\text{cusp}}(\Gamma_0(2,N),A)$ on which the Hecke operators can be simultaneously diagonalized.

**Theorem (Atkin-Lehner):** $H^1_{\text{cusp}}(\Gamma_0(2,N),A) = H_{\text{new}} \oplus H_{\text{old}}$, and the decomposition is orthogonal with respect to the Petersson inner product.

If we view modular forms as representations of $\text{GL}_2(\mathbb{A})$, where $\mathbb{A}$ is the adele ring of $\mathbb{Q}$, the Atkin-Lehner theorem can be phrased purely in the context of representations of $\text{GL}_2(F)$, where $F = \mathbb{Q}_p$, or any non-archimedean local field. So let $F$ be such, $\mathfrak{o} = \text{ring of integers in } F$, $\pi = \text{a generator of the maximal ideal of } \mathfrak{o}$. Let $K(n)$ be all matrices $\left( \begin{array}{cc} a & c \\ \pi & d \end{array} \right)$ in $\text{GL}_2(\mathfrak{o})$ with $a-1, c \in \pi^n \mathfrak{o}$. Casselman's local version of the Atkin-Lehner theorem is

**Theorem (Casselman [13]):** Let $(\rho, V)$ be an irreducible infinite-dimensional admissible representation of $\text{GL}_2(F)$. Then there is a positive integer $C(\rho)$ such that

a) $V^{K(C(\rho))}$ is one dimensional, spanned, say, by $\varphi_0$. (Here $V^H$ means fixed vectors of $H$).

b) $m < C(\rho) \Rightarrow V^{K(m)} = 0$. 

c) $m \geq C(p) \Rightarrow \{ p \rho \mathfrak{p}, 0 \leq \omega_m - C(p) \}$ is a basis of $V_{K(m)}$.

A cusp form for $\Gamma_0(2,N)$ is **new** in the sense of Atkin-Lehner if it generates an irreducible automorphic representation $\rho$ with

$$C(p) = \prod p^{C(p)} = N.$$ In general, $C(p) | N$, and our cusp form is "induced" from a form on $\Gamma_0(2,C(p))$. The form is **old** if $C(p) < N$.

In order to separate "old classes" from "new classes" in $H^3_{\text{app}}(\Gamma_0(3,N), A)$, we need the analogue of Casselman's theorem for $G L_3$ (and why not $G L_n$?). Now

$$K(m) = \{ g \in G L_n(\mathfrak{o}) | \text{col}_1(g) = 1, 0, \ldots, 0 \mod n^m \mathfrak{o} \}$$

and $V$ is assumed to be generic. This is not a restriction in our case, since the local components of any cuspidal automorphic representation of $G L_n$ are generic.

Parts a) and b) of Thm.2 were already proven for $G L_n$ by Jacquet, Piatetski-Shapiro and Shalika in [17]. Using their ideas, I was able to prove the analogue of c) for $G L_n$, $n \geq 3$. We conclude this introduction with a description of this result.

The basis of $V_{K(m)}$ for $m \geq C(p) = C$ is now given by the translates of $\psi_0$ by a certain natural subspace of the Hecke algebra $\mathcal{H}$ of $G L_{n-1}(F)$ (viewed embedded in $G L_n(F)$ in the lower right corner) as follows.

$\mathcal{H}$ is by definition the space of locally constant functions on $G_{n-1} = G L_{n-1}(F)$ with compact support, which are bi-invariant under $K = G L_{n-1}(\mathfrak{o})$. $\mathcal{H}$ is an algebra under convolution product, and has a basis given by the characteristic functions $\psi(a_1, \ldots, a_{n-1})$ of the
double cosets $K \backsim K$, where $a_i \in \mathbb{Z}$ and $a_1 \leq a_2 \leq \ldots \leq a_{n-1}$. In fact (see [12]) $\mathcal{X} \simeq \mathbb{C}[T_1, \ldots, T_{n-1}]$ where $\psi(1, \ldots, 1, 0, \ldots, 0) \mapsto \text{const} \cdot T_1$. Via this identification, $\mathbb{C}[T_1, \ldots, T_{n-1}]$ acts on the $GL_n(F)$ representation $V$ as follows: For $\psi \in \mathcal{X}$, $v \in V$,

$$
\psi \ast v = \int_{G_{n-1}} \psi(h^{-1}) |\det h|^{-\frac{1}{2}} \rho(h)(v) \, dh.
$$

We can now state

**Thm:** If $P \in \mathbb{C}[T_1, \ldots, T_{n-1}]$ has degree $d$, then $P \ast \psi_0 \in V^{K(c+d)}$, and the map $\mathbb{C}[T_1, \ldots, T_{n-1}]_{\text{deg } d} \to V^{K(c+d)}$ given by $P \mapsto P \ast \psi_0$ induces an isomorphism $\mathbb{C}[T_1, \ldots, T_{n-1}]_{\text{deg } d} \cong V^{K(c+d)}/V^{K(c+d-1)}$. 
Chapter I
Modular Symbols and Restriction to a Cusp

§1. The Steinberg Representation

Let $F$ be an arbitrary field, $G$ an algebraic group defined over $F$, $\ell = \text{F-rank of } G$. Recall the definition of the Tits building $\mathcal{J}_G$ of $G$ ([10],[8] chapter 8). This is an $\ell - 1$ dimensional simplicial complex whose vertices $\sigma_G(P)$ correspond to the maximal $F$-parabolic subgroups $P$ of $G$. If $0 \leq j \leq \ell - 1$, the $\ell - j$ simplices are the $F$-parabolic subgroups $P$ such that the $F$-rank of the derived group of $P/R_uP$ is $j-1$. ($R_uP$ is the unipotent radical of $P$). $\mathcal{J}_G$ has the homotopy type of a bouquet of $\ell - 1$ spheres. $G(F)$ acts on $\mathcal{J}_G$ on the right. We can orient $\mathcal{J}_G$ as follows: Fix a minimal $Q$-parabolic subgroup $B$ of $G$. Choose an order on the set of maximal parabolics containing $B$, say $P_1 < \ldots < P_R$. An arbitrary maximal parabolic is conjugate to a unique $\bar{P} \in \{ P_1, \ldots, P_R \}$. If $P$ and $R$ are two distinct maximal parabolics, we declare $P < R$ iff $\bar{P} < \bar{R}$. Note that $P(F)$ fixes the vertices in the closure of $\sigma_G(P)$, so $P(F) = \text{Stab}_{G(F)} \sigma_G(P)$ preserves orientation on $\sigma_G(P)$.

Define

$$I = I_G = H_{\ell - 1} (\mathcal{J}_G, \mathbb{Z}).$$

(1)
This is a module for $G(F)$, called the "Steinberg representation" of $G(F)$. Note that $I$ is a submodule of $C_{R-1}(J_G, Z) \cong Z \otimes_{B(F)} G(F)$. If $T$ is a maximal $F$-split torus of $G$, then the subcomplex of $\mathcal{J}_G$ consisting of those parabolic subgroups which contain $T$ is homeomorphic to a $k-1$ sphere and is called the "apartment" corresponding to $T$. The fundamental class of this apartment is an element of $I$ which we shall denote by $\tau$. If $T<B$, then $\tau = \sum_{w \in W(G,T)} \epsilon(w) \sigma_G(B^w)$, where $W(G,T) = \text{Norm}_G(T)/\text{Cent}_G(T)$ is the relative Weyl group of $T$ in $G$, and $\epsilon$ is its sign character. If $U$ is the unipotent radical of $B$, then $\tau$ freely generates $I$ as a $U(F)$-module.

Let $P$ be a $F$-parabolic subgroup of $G$, $L$ a Levi subgroup of $P$, $\pi: P \rightarrow L$ the projection, $k$ the $F$-rank of $[L,L]$. The projection $\pi$ gives an isomorphism of simplicial complexes $\mathcal{J}_P \cong \mathcal{J}_L$. Hence we have a canonical isomorphism $I_P \cong I_L$, and we will identify these two spaces. The canonical action of $P(F)$ on $I_P$ is the pullback of the action of $L(F)$. We will define a map

$$s_P: I \rightarrow I_P$$

as follows. Let $Q$ be a minimal $F$-parabolic subgroup of $G$. Set

$$s_P(\sigma_G(Q)) = \sigma_P(Q) \text{ if } Q \subseteq P, \quad s_P(\sigma_G(Q)) = 0 \text{ otherwise. \quad (2')}$$

Note that if $Q \subseteq P$ then $s_P(\sigma_G(Q))$ is a $k-1$ cell in $\mathcal{J}_P$. Thus we have a map

$$s_P: C_{R-1}(J_G, Z) \rightarrow C_{k-1}(J_P, Z).$$

The following result illustrates the functorial nature of $s_P$, and shows in particular that $s_P$ is surjective.
(1.1) Proposition:

i) If $T$ is a maximal $F$-split torus of $P$ and $\tau$ (resp. $\tau_P$) is the corresponding element of $I$ (resp. $I_P$), then $s_P(\tau) = \tau_P$.

ii) $s_P$ is $P(F)$-equivariant.

iii) $s_P$ induces an isomorphism from the coinvariants of $R_{\mathfrak{u}} P(F)$ in $I$ to $I_P$.

Proof: 1) Let $B$ be a minimal $F$-parabolic subgroup containing $T$, so that

$$\tau = \sum_{w \in \mathcal{W}(G,T)} \varepsilon(w) \sigma_G(B^w).$$

(3)

Now $B^w \subseteq P$ if and only if $w \in \mathcal{W}(P,T)$, a natural subgroup of $\mathcal{W}(G,T)$. Hence

$$s_P(\tau) = \sum_{w \in \mathcal{W}(P,T)} \varepsilon(w) r_P[\sigma_G(B^w)] = \sum_{w \in \mathcal{W}(P,T)} \varepsilon(w) \sigma_P(B^w) = \tau_P.$$  (4)

ii) We must show, for $\rho \in P(F)$, that

$$s_P(\sigma_G(Q^\rho)) = s_P(\sigma_G(Q)) \cdot \rho.$$  (5)

This follows from the observation that $Q^\rho \subseteq P \iff Q \subseteq P$.

iii) Since every element of $I$ is of the form $\tau \cdot u$ with $u \in R_{\mathfrak{u}} B(F) < P(F)$, i) and ii) imply that $s_P(I) = I_P$.  (6)

Let $I(N)$ denote the coinvariants of $N = R_{\mathfrak{u}} P$ in $I$. If $x \in I$, let $x$ denote its image in $I(N)$. For $n \in N(F)$, we have $x \cdot n = x$. Since $s_P$ is $P(F)$-equivariant and $N(F)$ acts trivially on $I_P$, $s_P$ descends to $I(N)$. Let $U$ be the unipotent radical of a minimal $F$-parabolic subgroup of $L$, the Levi subgroup of $P$ which contains $T$. Then every element $x$ of $I$ may be written

$$x = \sum c_i \tau \cdot u_i \eta_i$$ with $c_i \in \mathbb{Z}$, $n_i \in \mathcal{Z}(N(F))$, $u_i \in U(F)$, $u_i \neq u_j$ if $i \neq j$.  (7)

Then ii) and the fact that $U(F) < P(F)$ imply

$$s_P(x) = s_P(\sum c_i \tau \cdot u_i) = \sum c_i \tau_P \cdot u_i.$$  (8)
Since \( \tau_p \) freely generates \( I_p \) as a \( \mathbb{Z}U(\mathbb{F}) \)-module, we get that 
\[ s_p : I(N) \rightarrow I_p \] is injective. \( \square \)

(1.2) Remark: Suppose \( Q = P^\mu \), some \( \mu \in G(\mathbb{F}) \). Then we have an isomorphism 
\[ c_\mu : I_p \rightarrow I_Q, \] given on the chain level by \( \sigma_p(B) \rightarrow \sigma_Q(B^\mu) \).
Note that
\[ c_\mu(\tau_p) = \tau_Q, \] and for \( p \in P(\mathbb{F}) \), \( \eta \in I_p \), we have
\[ c_\mu(\eta \cdot p) = (c_\mu \eta) \cdot \mu^{-1} p \mu. \] (9)

The following lemma describes a crucial feature of \( s_p \).

(1.3) Lemma: Let \( \Delta \) be a subgroup of \( G(\mathbb{F}) \), \( \eta \in I \). Then the map 
\( \Delta \rightarrow I_p \) given by \( x \mapsto s_p(\eta \cdot x) \) is supported on finitely many cosets 
\( \delta(\Delta \cap P(\mathbb{F})) \) for \( \delta \in \Delta \).

Proof: Let \( \sigma_G(B_1), \ldots, \sigma_G(B_n) \) be those \( \ell - 1 \) simplices \( \sigma_G(B) \) in the support of \( \eta \) for which \( B \) is \( \Delta \)-conjugate to a subgroup of \( P \). If no such \( B_i \) exist, then \( x \mapsto s_p(\eta \cdot x) \) is identically zero on \( \Delta \). For each \( 1 \leq i \leq n \), choose \( \delta_i \in \Delta \) such that \( \delta_i^{-1} B_i \delta_i \prec P \).

Suppose \( \delta \in \Delta \), and \( s_p(\eta \cdot \delta) \neq 0 \). Then \( \exists \ 1 \leq k \leq n \) such that \( \delta^{-1} B_k \delta \prec P \). Since two conjugate parabolic subgroups containing the same minimal parabolic subgroup must be equal, we have \( \delta P \delta^{-1} = \delta_k P \delta_k^{-1} \). Since \( P \) is its own normalizer, we get \( \delta \delta_k^{-1} \in \delta_k P \delta_k^{-1} \).

Hence \( \delta \in \delta_k(\Delta \cap P(\mathbb{F})) \).
§2. The Restriction Map

In this section, G is a reductive \( \mathbb{Q} \)-group and \( \Gamma \) is a torsion-free arithmetic subgroup of \( G(\mathbb{Q}) \), with \( \Gamma \) contained in the identity component of \( G(\mathbb{R}) \). We can arrange this by passing to a subgroup of finite index if necessary. \( A \) is a left \( \mathbb{Z} \Gamma \) module with added restrictions to be specified along the way. Let \( d \) be the dimension of the symmetric space \( X \) associated to \( G(\mathbb{R}) \), \( l = \text{rank}_{\mathbb{Q}} G \).

Let \( P \) be a \( \mathbb{Q} \)-parabolic of \( G \). According to Borel and Serre [8], there are natural isomorphisms

\[
H^q(\Gamma, A) \rightarrow H_{d-q}(\Gamma, I \otimes A),
\]

\[
H^q(\Gamma \cap P, A) \rightarrow H_{d-q}(\Gamma \cap P, I_P \otimes A).
\]

(This may require some comment. See (2.2) below and the remarks following it). We find the missing link in the commutative diagram

\[
H^q(\Gamma, A) \rightarrow H_{d-q}(\Gamma, I \otimes A) \\
\downarrow \text{res} \quad \quad \quad \quad \quad \quad \downarrow ?
\]

\[
H^q(\Gamma \cap P, A) \rightarrow H_{d-q}(\Gamma \cap P, I_P \otimes A).
\]

We then let \( q = d - l \) and give a more explicit formula for the right hand map. We will use this formula in our examples when \( G = SL_3, SL_4 \).

Lemma (1.3) enables us to define a "transfer" map

\[
H_* (\Gamma, I \otimes A) \rightarrow H_* (\Gamma \cap P, I_P \otimes A)
\]

even though \( [\Gamma : \Gamma \cap P] = \infty \). The precise definition of this map and some of its properties are contained in the following short digression on general group cohomology.

First, we make some conventions: All Hom spaces will be in categories of right modules for various groups. We form the
"tensor product" $A \otimes B$ of two right modules by giving $B$ its canonical left module structure: $g \cdot b = b \cdot g$ for $g$ in the group, $b \in B$.

(2.0) **Definition and Remarks:**

Let $G_1 \leq G_2$ be groups, $M_i$ right $ZG_i$ modules, $p \in \text{Hom}_{G_i}(M_2, M_1)$. We do not assume $[G_2:G_1] < \infty$. Suppose $p$ satisfies the following condition:

(*) Given $m \in M_2$, the map $g \mapsto p(mg): G_2 \to M_1$ has support contained in finitely many $G_2/G_1$ cosets.

Let $\mathbf{C} \to \mathbb{Z}$ be a projective resolution of $\mathbb{Z}$ of right $ZG_2$-modules ($\text{pr}ZG_2$ for short). Define $t(p): \mathbf{C} \otimes_{G_2} M_2 \to \mathbf{C} \otimes_{G_1} M_1$ by

$$
t(p)(x \otimes m) = \sum_{g \in G_2/G_1} x \cdot g \otimes p(mg).
$$

(11)

The sum makes sense by (*), and $t(p)$ is clearly a chain map, so we get

$$
t(p): H_n(G_2, M_2) \to H_n(G_1, M_1).
$$

We remark for later use that if $n=0$ and $H_0(G_i, M_i)$ is identified with $\mathbb{Z} \otimes_{G_i} M_i$, then $t(p): \mathbb{Z} \otimes_{G_2} M_2 \to \mathbb{Z} \otimes_{G_1} M_1$ is given by

$$
t(p)(1 \otimes m) = \sum_{g \in G_2/G_1} 1 \otimes p(mg).
$$

(12)

We recall the definition of cap product. Let $G$ be a group, $M$ and $N$ right $ZG$-modules. Let $\mathbf{C} \to \mathbb{Z}$ be a $\text{pr}ZG$. $G$ acts on $\mathbf{C}$ on the right. Then $\mathbf{C} \otimes \mathbf{C} \to \mathbb{Z}$ is again a $\text{pr}ZG$, with diagonal $G$-action [11]. Let $\sum x_j \otimes y_j \in (C \otimes C)_q$, where $x_j, y_j \in C_j$. Let $m \in M$, $\varphi \in \text{Hom}_G(C_p, N)$ (i.e., $\varphi(x \cdot g) = g^{-1} \cdot \varphi(x)$ for $x \in C_p, g \in G$). Then

$$
n: H_q(G, M) \otimes H_p(G, N) \to H_{q-p}(G, M \otimes N)
$$

is given on the chain level by
Let \( G_1 < G_2, M_i \) and \( N_i \) right \( \mathbb{Z}G_i \) modules, \( \sigma \in \text{Hom}_{G_1}(N_2, N_1) \), \( \rho \in \text{Hom}_{G_1}(M_2, M_1) \). Assume that (*) holds for \( \rho \). Then (*) also holds for \( \rho \otimes \sigma : M_2 \otimes N_2 \rightarrow M_1 \otimes N_1 \) and the following diagram commutes:

\[
\begin{align*}
n: & \text{H}_q(G_2, M_2) \otimes \text{H}(G_2, N_2) \rightarrow \text{H}_q(G_2, M_2 \otimes N_2) \\
\downarrow \text{t}(\rho) & \downarrow \text{res}_\sigma \downarrow \text{t}(\rho \otimes \sigma) \\
n: & \text{H}_q(G_1, M_1) \otimes \text{H}(G_1, N_1) \rightarrow \text{H}_q(G_1, M_1 \otimes N_1)
\end{align*}
\]

Here \( \text{res}_\sigma = \sigma^* \ast \text{res}(G_2 | G_1) \).

**Proof:** This is identical to the proof for the ordinary restriction map. Let \( C. \rightarrow \mathbb{Z} \) be a pr\( \mathbb{Z}G_2 \), \( x_j, y_j \in C_j, m \in M, \varphi \in \text{Hom}_{G_2}(C_p, M) \). We compute

\[
\begin{align*}
t(\rho \otimes \sigma) [\sum x_{q-i} \otimes y_i] & \otimes m \cap \varphi) = (-1)^{pq} t(\rho \otimes \sigma) [x_{q-p} \otimes (m \otimes \varphi(y_p))] \\
= (-1)^{pq} \sum_{g \in G_2 / G_1} [x_{q-p} \cdot g] \otimes [\rho(mg) \otimes \sigma(g^{-1} \varphi(y_p))] \\
= (-1)^{pq} \sum_{g \in G_2 / G_1} [x_{q-p} \cdot g] \otimes [\rho(mg) \otimes \sigma(\varphi(y_p \cdot g))] \\
= \sum g [\sum_i (x_{q-i} \cdot g \otimes y_i \cdot g) \otimes \rho(g^{-1} m)] \cap [\text{res}_\varphi] \\
= t(\rho) [\sum_i (x_{q-i} \otimes y_i) \otimes m] \cap [\text{res}_\varphi]. \quad \square \quad (14)
\end{align*}
\]

We are going to apply this with \( G_2 = \Gamma, G_1 = \Gamma_p, M_2 = I, M_1 = I_p, \rho = sp : I \rightarrow I_p \).

We have seen in (1.3) that (*) holds for \( sp \).

Now \( \Gamma \) and \( \Gamma \cap P \) are duality groups \([8] \) with dualizing modules \( I \) and \( I_p \) respectively. We have \( H_{d-\text{H}}(\Gamma, I) \approx H_{d-\text{H}}(\Gamma \cap P, I_p) \approx \mathbb{Z} \), and the cohomological dimensions of \( \Gamma \) and \( \Gamma \cap P \) (cd \( \Gamma \) for short) are both
The following lemma implies the same statement for $\Gamma \cap P$ and $I_P$. Recall that $N=R_dP$. Let $n=\dim N$, $\Gamma_L=\pi(\Gamma \cap P)$.

(2.2) **Lemma**: Let $t=cd\Gamma_L$, $M$ a $\Gamma \cap P$-module, $M^{\Gamma \cap N}$ the invariants of $\Gamma \cap N$ in $M$. Then $H_d(t;\Gamma \cap P,M)\cong H_d(t;\Gamma_L,M^{\Gamma \cap N})$, and $H_q(\Gamma \cap P,M)=0$ if $q>d-\ell$. Moreover, $t=d-\ell-n$.

**Proof**: Since $N(R)/\Gamma \cap N$ is a $K(\Gamma \cap N,1)$ space which is a compact oriented manifold, $cd(\Gamma \cap N)=n$, and Poincare duality holds for $H_*(\Gamma \cap N,M)$. Consider the Hochschild-Serre spectral sequence associated to the exact sequence $1\rightarrow \Gamma \cap N\rightarrow \Gamma \cap P\rightarrow \Gamma_L\rightarrow 1$.

$$E_2^{pq}=H_p(\Gamma_L,H_q(\Gamma \cap N,M)) \Rightarrow H_{p+q}(\Gamma \cap P,M).$$

Since $d-\ell=cd\Gamma \cap P$, we have $d-\ell\leq t+n$. On the other hand, if $r>t$ or $s>n$ then $E_2^{rs}=0$. Hence, $E_\infty^{tn}=E_2^{tn}$. Note that $M^{\Gamma \cap N}\cong H^0(\Gamma \cap N,M)\cong H_n(\Gamma \cap N,M)$.

Set $M=I_P$. Since $\Gamma \cap N$ acts trivially on $I_P$, we get $E_\infty^{tn}=H_t(\Gamma_L,I_P)\cong Z$, since $I_P$ is the dualizing module for $\Gamma_L$. It follows that $d-\ell=t+n$. □

Choose generators $e\in H_{d-\ell}(\Gamma,I)$, $e_P\in H_{d-\ell}(\Gamma,P,I_P)$.

(2.3) **Lemma**: $t(s_P): H_{d-\ell}(\Gamma,I)\rightarrow H_{d-\ell}(\Gamma \cap P,I_P)$ is an isomorphism.

**Proof**: We have $t(s_P)(e)=ke_P$ for some $k \in Z$. (15)

We apply (2.1) and the remark prior to it to get a commutative diagram

$$
\begin{array}{ccc}
H_{d-\ell}(\Gamma,I) & \otimes & H^{d-\ell}(\Gamma,Z\Gamma) \\
\downarrow t(s_P) & & \downarrow \text{res} \\
H_{d-\ell}(\Gamma \cap P,I_P) & \otimes & H^{d-\ell}(\Gamma \cap P,Z\Gamma)
\end{array}
\rightarrow
\begin{array}{ccc}
I \\
\downarrow r(P,Z\Gamma) \\
I_P \otimes \Gamma \cap P Z\Gamma
\end{array}
$$

where

$$r(P,Z\Gamma)(\eta)=\sum_{\gamma \in \Gamma/\Gamma \cap P} s_P(\eta \gamma) \otimes \gamma^{-1}. \quad (16)$$
For $\varphi \in H^{d-2}(\Gamma, \mathbb{Z})$, we have $r(P, \mathbb{Z})\langle e\varphi \rangle = k[e_P \cap \text{res}(\varphi)]$. Now

$$I_P \otimes_{\Gamma \cap P} \mathbb{Z} \Gamma \simeq (I_P \otimes 1) \oplus \sum_{x \neq \gamma \in \Gamma / \Gamma \cap P} I_P \otimes \gamma^{-1},$$

as $\mathbb{Z}[\Gamma \cap P]$ modules. Let $pr: I_P \otimes_{\Gamma \cap P} \mathbb{Z} \Gamma \rightarrow I_P$ denote the projection onto the first factor. Let $\tau \in I$, $\tau_P \in I_P$ correspond to the maximal $\mathbb{Q}$-split torus $T < P$. Then

$$pr[r(P, \mathbb{Z})\langle \tau \rangle] = pr[\tau_P \otimes 1 + \sum_{x \neq \gamma \in \Gamma / \Gamma \cap P} s_P(\eta \gamma) \otimes \gamma^{-1}] = \tau_P. \quad (17)$$

Now let $\varphi \in H^{d-2}(\Gamma, \mathbb{Z})$ be such that $e\varphi = \tau$. Then

$$\tau_P = pr[r(P, \mathbb{Z})\langle \tau \rangle] = pr[k[e_P \cap \text{res}(\varphi)]] \in kI_P. \quad (18)$$

But $kI_P$ is $P(\mathbb{Q})$-invariant and $\tau_P$ generates $I_P$ over $P(\mathbb{Q})$, so we must have $kI_P = I_P$. Since $I_P$ is free abelian, we see that $k = \pm 1$. The lemma follows. □

(2.4) Proposition: We may choose $e, e_P$ so that for every $\Gamma$-module $A$ there is a commutative diagram

$$
\begin{array}{ccc}
H^q(\Gamma, A) & \xrightarrow{\text{en}} & H_{d-q}^*(\Gamma, I \otimes A) \\
\downarrow \text{res} & & \downarrow \text{t}(s_P \otimes 1_A) \\
H^q(\Gamma \cap P, A) & \xrightarrow{e_P \cap \text{res}} & H_{d-q}^*(\Gamma \cap P, I_P \otimes A).
\end{array}
$$

Proof: By (2.3), we may take

$$e_P = t(s_P)(e). \quad (19)$$

The Borel-Serre duality theorem says that $e\varphi$ and $e_P \cap \text{res}(\varphi)$ are isomorphisms. The proposition now follows from (2.1) with $N_1 = N_2 = A, \sigma = 1_A$. □
In our examples, we will only consider the restriction map on the top cohomology groups. Fix a $Q$-parabolic $P$ and a maximal $Q$-split torus $T$ of a Levi subgroup $L$ of $P$. Let $\tau$ and $\tau_P$ be the corresponding elements of $I$ and $I_P$. We will describe the restriction of $\tau \cdot g \boxtimes a \in I \boxtimes rA$ to a conjugate $Q = P^\mu$ in terms of $\tau_P$, $\mu$, and $P$.

(2.5) Remark: Suppose $Q = P^\mu$, for some $\mu \in G(\alpha)$, and $A$ is in fact a module for the subgroup of $G(Q)$ generated by $\Gamma$ and $\mu$. We have an isomorphism $\mu_P : I_Q \boxtimes r_Q A \to I_P \boxtimes r_P A$ given by $\eta \boxtimes a \mapsto c_{\mu} \eta \boxtimes a$. (Remark (1.2) gives the definition of $c_{\mu}$ and is used to show this is well defined).

We also denote by $\mu_G : I \boxtimes r A \to I \boxtimes r A$ the map $\mu_G (\eta \boxtimes a) = \eta \mu^2 \boxtimes a$. It is easy to verify that

$$c_{\mu} s_p (\eta) = s_p (\eta \mu) \quad \text{for all } \eta \in I, \quad (20)$$

and

$$\mu_P \circ (s_q \boxtimes 1_A) = (s_p \boxtimes 1_A) \circ \mu_G. \quad (21)$$

Notation: We set $\Gamma(P, \mu) := \mu \Gamma \cap P$, $\Gamma(L, \mu) := \pi(\mu \Gamma \cap P)$, $\Gamma(N, \mu) := \mu \Gamma \cap N$.

If $A$ is a module for $\Gamma(N, \mu)$, we abbreviate $A_\mu := A_{\Gamma(N, \mu)}$, the coinvariants of $\Gamma(N, \mu)$ in $A$, and let $a \mapsto a : A \to A_\mu$ be the canonical map. Since $\Gamma(N, \mu)$ acts trivially on $I_P$, there is a canonical identification

$$I_P \boxtimes \Gamma(P, \mu) A = I_P \boxtimes \Gamma(L, \mu) A_\mu. \quad (22)$$

Finally, define a linear map
\[ p_\mu: \mathbb{Z}G(\mathfrak{q}) \to \mathbb{Z}[P(\mathfrak{q})/\Gamma(P,\mu)] \]

by

\[ p_\mu(g) = \text{coset of } p \text{ if } g \in \rho \mu \Gamma \]

\[ = 0 \text{ if } g \notin \rho(\mathfrak{Q})\mu \Gamma. \]  

We further define \( l_{P,\mu}: \mathbb{Z}G(\mathfrak{q}) \to \mathbb{Z}[L(\mathfrak{q})/\Gamma(L,\mu)] \) to be \( \pi \circ p_\mu \).

(2.6) Proposition: Let \( \Delta \) be a torsion-free arithmetic subgroup of \( G(\mathfrak{q}) \), \( P=LN \) (Levi decomposition) a \( \mathfrak{Q} \)-parabolic of \( G \), \( Q=P^\mu \), where \( \mu \in G(\mathfrak{q}) \). Let \( T \) be a maximal \( \mathfrak{Q} \)-split torus of \( L \) with corresponding elements \( \tau, \tau_L \) in \( I, I_p \). Define a map

\[ r(P,\mu,A): I \otimes \mathcal{R} \to I \otimes \mathcal{R}[L(\mathfrak{q})/\Gamma(L,\mu)]A_\mu \]

by

\[ r(P,\mu,A) = \mu \circ t(s_{\mathfrak{q}} \otimes 1_A). \]  

(see (2.0) and (2.5))  

(24)

Assume that \( A \) is the restriction to \( \Delta \) of a module for \( \Delta < G(\mathfrak{Q}) \) such that \( \Delta \) contains \( \Gamma, \mu \) and a complete set of representatives of \( W(G,T) \). Then

i) There is a commutative diagram

\[
\begin{array}{ccc}
H^{d-\ell}(\Gamma,A) & \xrightarrow{\epsilon_{\mathfrak{q}}} & I \otimes \mathcal{R} \\
\downarrow \text{res} & & \downarrow r(P,\mu,A) \\
H^{d-\ell}(\Gamma\cap Q,A) & \Rightarrow & I_p \otimes \mathcal{R}[L(\mathfrak{q})/\Gamma(L,\mu)]A_\mu,
\end{array}
\]

ii) If \( g \in \Delta, \alpha \in A \), then

\[ r(P,\mu,A)[\tau \cdot g \otimes a] = \sum_{w \in \mathcal{R}^P} [\epsilon(w) \tau_L \cdot p_\mu(wg) \otimes p_\mu(wg) \otimes g \otimes g], \]  

where \( \mathcal{R}^P \) is any set of coset representatives for \( W(P,T) \backslash W(G,T) \).

Proof: The bottom isomorphism is the composition

\[ x \mapsto \mu_\rho(e_{\mathfrak{q}} \cap x): H^{d-\ell}(\Gamma\cap Q,A) \to I_q \otimes \mathcal{R}[Q,A] \to I_p \otimes \mathcal{R}[L(\mathfrak{q})/\Gamma(L,\mu)]A_\mu = I_p \otimes \mathcal{R}[L(\mathfrak{q})/\Gamma(L,\mu)]A_\mu. \]
From (2.4), we get the commutative diagram

\[
\begin{align*}
\text{H}^{d-\xi}(\Gamma, A) & \xrightarrow{\text{res}} \text{I} \otimes_{\Gamma} A \\
\downarrow \text{res} & \downarrow \text{t}(e_{Q} \otimes 1_A) \\
\text{H}^{d-\xi}(\Gamma \cap Q, A) & \xrightarrow{\text{I} \otimes \Gamma \cap Q} \text{I} \otimes \Gamma \cap Q \otimes A
\end{align*}
\]

Now \(i\) follows from the definition of \(r(\nu, A)\).

Recall that \(\tau \cdot w = \varepsilon(w) \tau \) for \(w \in W\). (26)

We begin with a \(\text{Claim:} \quad s_{p}(\tau \cdot g) = \begin{cases} 
\varepsilon(w) \tau_{L} \cdot wg & \text{if } wg \in P \langle Q \rangle, \text{some } w \in W \\
0 & \text{if } g \notin WP \langle Q \rangle
\end{cases} \) \(27\)

Note that the right hand side does not depend on the representative of \(w\) since \(\text{Cent}_G T \langle Q \rangle\) fixes \(\tau_L\) and \(T \subset P\).

\(\text{Reason:}\) The faces of the apartment \(\tau \cdot g\) are the parabolics which contain \(T^g\). If \(s_{p}(\tau \cdot g) \neq 0\), then \(P\) is one of these, so \(T \cap gP^{-1}\). Choose minimal parabolics \(B_1\) and \(B_2\) contained in \(P\) and \(gP^{-1}\) respectively, with \(T \subset B_1 \cap B_2\). Since \(W\) is transitive on the minimal parabolics containing \(T\), \(\exists \ w \in W\) such that \(B_2 = w^{-1}B_1w\). Since \(P \langle Q \rangle\) is transitive on its own minimal parabolics, \(\exists \ p \in P \langle Q \rangle\) with \(pB_1p^{-1} = g^{-1}B_2g = g^{-1}w^{-1}B_1w\). It follows that \(wg \in P \langle Q \rangle\). Now assume \(wg \in P \langle Q \rangle\), for some \(w\). By (1.1),

\[
\tau_{L} \cdot wg = s_{p}(\tau) \cdot wg = s_{p}(\tau \cdot wg) = \varepsilon(w) s_{p}(\tau \cdot g).
\]

This proves the \(\text{Claim.}\)
We now compute
\[ \mu_p \circ t(s_0 \otimes 1_A) [\tau \cdot g \otimes a] = \mu_p [\sum_{r \in r_n \cap r_0} r_0(\tau \cdot g \gamma) \otimes \gamma^{-1} a] \text{ (by (2.0))} \]
\[ = \mu_p [\sum_{r \in r_n \cap r_0} c_{t_n} s_p(\tau \cdot g \gamma \mu^{-1}) \otimes \mu^{-1} a] \text{ (by (2.5))} \]
\[ = \sum_{r \in r_n \cap r_0} s_p(\tau \cdot g \gamma \mu^{-1}) \otimes \mu^{-1} a \]
\[ = \sum_{\epsilon \in \mu \gamma \mu^{-1}} s_p(\tau \cdot g \mu^{-1} \gamma) \otimes \gamma^{-1} a \]
\[ = \sum \epsilon(w) \tau_L \cdot w \mu^{-1} \gamma \otimes \gamma^{-1} a \text{ , (29)} \]
where the sum is over \( \{(\gamma, w) \in \mu \gamma / \Gamma(P, \mu) \times W_p | w \mu^{-1} \gamma \in P(\Omega)\} \), by the Claim. It is easy to see that projection onto the second factor gives a bijection from this set onto \( \{w \in W_p | w \gamma \in P(\Omega) \mu \gamma \} \).

If \( x \in P(\Omega) \mu \gamma \), write \( x = p_\mu(x) \mu \gamma(x) \) where \( \gamma(x) \in \Gamma \). Two choices of \( p_\mu(x) \) or \( \gamma(x) \) differ by an element of \( \Gamma(P, \mu) \). If \( w \mu^{-1} \gamma \in P(\Omega) \), with \( \gamma \in \mu \gamma \), then \( w \gamma \in P(\Omega) \mu \gamma \) and we may take \( p_\mu(w) = w \mu^{-1} \gamma \), \( \gamma(w) = \mu^{-1} \gamma^{-1} \mu \). Hence
\[ \mu_p \circ t(r_0 \otimes 1_A) [\tau \cdot g \otimes a] = \sum \epsilon(w) \tau_L \cdot p_\mu(w) \otimes p_\mu(w)^{-1} w \gamma a \text{ , (30)} \]
where the sum is over \( w \in W_p \) such that \( w \gamma \in P(\Omega) \mu \gamma \). Since \( p_\mu \) is defined to be zero off \( P(\Omega) \mu \gamma \), we can just sum over all \( W_p \), and this is the statement of ii). \( \square \)

(2.7) Remarks: i) \( \Gamma(N, \mu) \) acts trivially on \( I_p \) so \( \tau_L \cdot p_\mu(w) = \tau_L \cdot 1_p \mu(w) \).
If \( A \) is a rational \( \mathbb{C}G \) module then \( A_{\Gamma(N,\mu)} = A_{\mathbb{N}(\mathbb{Q})} \) since \( \Gamma(N,\mu) \) is Zariski dense in \( N(\mathbb{C}) \). Hence we may replace \( p_{\mu}(wg) \) by \( \iota_{\mu}(wg) \) in (2.6).11).

11) If \( A \) is a \( \mathbb{G}(\mathbb{Q}) \)-module, this formula completely describes \( r(p,\mu,\Lambda) \), in principle, since \( \tau \) generates \( I \) over \( \mathbb{G}(\mathbb{Q}) \). If \( G = \text{SL}_n \), then in fact \( \tau \) generates \( I \) over \( \text{SL}_n(\mathbb{Z}) \) [AR]. This is why we can weaken the assumptions on \( A \) in this case and still claim that this formula gives the values of \( r(p,\mu,\Lambda) \) on all of \( I \otimes_\mathbb{R} A \).

We will also have use for the homology version of (2.6). Let \( A \) be as in that proposition. For simplicity, we let \( \mu = 1 \) and omit it in our notation. The top homology version of Borel-Serre duality is \( H_{d-g}(\Gamma, A) \approx \text{Hom}_{\mathbb{T}}(I, A) \) and similarly for \( H_{d-p}(\Gamma(L), A^{(n)}) \) [11]. Define \( j : \text{Hom}_{\mathbb{R}(\mathbb{L})}(I_{\mathbb{P}}, A^{(n)}) \to \text{Hom}_{\mathbb{T}}(I, A) \) by

\[
 j(\psi)(x) = \sum_{\gamma \in \Gamma \cap P} \gamma \psi(\varepsilon_p(x \cdot \gamma)) .
\]

Let \( i_* \) be the composition \( H_{d-p}(\Gamma(L), A^{(n)}) \to H_{d-g}(\Gamma(P), A) \to H_{d-g}(\Gamma, A) \).

(2.8) Proposition: 1) The maps \( i_* \) and \( j \) correspond under Borel-Serre duality.

2) For \( g \in \mathbb{G}(\mathbb{Z}) \), \( \psi \in \text{Hom}_{\mathbb{R}_{\mathbb{L}}}(I_{\mathbb{P}}, A^{(n)}) \), we have

\[
 j(\psi)(\tau \cdot g) = \sum_{w \in \mathbb{W}_P} \varepsilon(w)(wg)^{-1} p(wg) \psi[\tau \cdot p(wg)].
\]

Proof: Let \( A' = \text{Hom}_{\mathbb{Z}}(A, \mathbb{Z}) \) and consider the natural pairings \( (,): \)

\[
 [I \otimes_{\mathbb{R}} A'] \otimes \text{Hom}_{\mathbb{T}}(I, A) \to \mathbb{Z}
\]

\[
 (,)_L : [I_{\mathbb{P}} \otimes_{\mathbb{L}} A^{(n)}] \otimes \text{Hom}_{\mathbb{R}(\mathbb{L})}(I_{\mathbb{P}}, A^{(n)}) \to \mathbb{Z}
\]

Let \( x \in I \), \( \lambda \in A' \), \( \psi \in \text{Hom}_{\mathbb{R}_{\mathbb{L}}}(I_{\mathbb{P}}, A^{(n)}) \). Then
\[
\langle \sum_{\Gamma/\Gamma(P)} \gamma^{-1} \psi(x \gamma), \lambda \rangle = \langle \sum_{\Gamma/\Gamma(P)} \gamma \psi(x \gamma), \lambda \rangle = \langle j(\psi)(x), \lambda \rangle = (x \otimes \lambda, j(\psi)).
\]

This, together with (2.6) proves 1).

Using the Claim in the proof of (2.6), we get
\[
\sum_{\Gamma/\Gamma(P)} \gamma \psi[s_p(\tau \cdot g \gamma)] = \sum_{w \in \omega} \varepsilon(w)(wg)^{-1}p(wg) \psi[\tau_l \cdot p(wg)],
\]
which gives 2). □

(2.9) Remark: If char(k) is zero or sufficiently large, then Borel-Serre duality holds when \( \Gamma = G(\mathbb{Z}) \) (see Chapter II). In this case, we have \( p(wg) = 1 \) and
\[
\langle \sum_{\Gamma/\Gamma(P)} \gamma \psi(\tau \cdot g \gamma) \rangle = \sum_{w \in \omega} \varepsilon(w)(wg)^{-1}\psi(\tau_l).
\]
Also, we have maps
\[
\text{Hom}_{\tau}(I, A) \to A, \quad \text{Hom}_{\tau}(I_{\Gamma}, A^{\Gamma(N)}) \to A^{\Gamma(N)}
\]
given by evaluation at \( \tau \) and \( \tau_{l} \), respectively. We thus have a commutative diagram
\[
\begin{array}{ccc}
\text{Hom}_{\tau}(I, A) & \to & A \\
J \uparrow & & j \uparrow \\
\text{Hom}_{\tau}(I_{\Gamma}, A^{\Gamma(N)}) & \to & A^{\Gamma(N)}
\end{array}
\]
where \( j(a) = \sum_{w \in \omega} \varepsilon(w)w^{-1}a \). (36)
Chapter II
Applications to Subgroups of $\text{SL}_n(\mathbb{Z})$

§1. Partial Descriptions of the Top (Co)Homology

Let $G = \text{SL}_n(\mathbb{Z})$, $\Gamma$ a subgroup of finite index in $G$, $v = \frac{1}{2}n(n-1) = \text{vcd}(\Gamma)$. In this section, we define certain spaces of functions on $G$ (special cases of which have appeared in the work of Ash ([1],[2]) and Soule ([26]) which approximate the top cohomology of $G$. We rely heavily on the fact ((1.1)) that $I$ is a cyclic $\mathbb{Z}G$-module. This will enable us to get more explicit information about the restriction maps, as well as some applications to the top homology of $\text{SL}_n(\mathbb{Z})$.

Let $T$ be the diagonal torus of $\text{SL}_n$ with respect to the basis $e_1,...,e_n$ of $\mathbb{Q}^n$. We may take representatives of the Weyl group $W(G,T)$ in $S = \text{SO}_n(\mathbb{Z}) < G$. Define an element $h \in G$ by

$$h(e_1) = -e_2, \ h(e_2) = -e_1 - e_2, \ h(e_i) = e_i \text{ for } i \geq 3.$$  \hspace{1cm} (37)

Let $\tau \in I$ correspond to $T$.

(1.1)\textbf{Theorem} ([5],[21]): $I = \tau \cdot \mathbb{Z}G$. \hspace{1cm} (38)

Note that $G$ acts transitively on the $\text{SL}_n(\mathbb{Q})$-conjugacy classes of $\mathbb{Q}$-parabolic subgroups of $\text{SL}_n$, so $G$ satisfies the hypotheses on $\Delta$ in chap.I(2.6), for any $\mathbb{Z}G$-module $A$. However, throughout this chapter, we shall assume that $A$ is a finite dimensional left
kG-module, where k is a field of characteristic p and either p=0 or p>n+1. The following lemma assures that Borel-Serre duality holds for G and A.

(1.2) Lemma: If p>n+1, there exists a prime \( \ell > 2 \) such that \( p \nmid |\text{SL}_n(F_\ell)| \).

Proof: We must show that \( p>n+1 \Rightarrow \exists \ell > 2 \) such that \( p \nmid (\ell^n-1)(\ell^{n-1}-1)...(\ell^2-1) \). Choose \( m \in \mathbb{Z} \) such that \( m(\text{mod } p) \) generates \( (\mathbb{Z}/p\mathbb{Z})^\times \). Let \( \ell \) be an odd prime in the arithmetic progression \( \{m+k|k \in \mathbb{N}\} \). Then \( \ell(\text{mod } p) \) generates \( (\mathbb{Z}/p\mathbb{Z})^\times \). Hence \( \ell^{r-1}(\text{mod } p) \Rightarrow p-1 \leq r \Rightarrow n+1 < p \leq r+1 \Rightarrow r > n \). □

Remarks: i) The converse is also true since \( p=n+1-a \), \( a>0 \Rightarrow p|(|\text{SL}_n(F_\ell)|).

ii) The same proof shows, for any split \( \Phi \)-group \( G \) with Coxeter number \( H \), that \( p>H+1 \Rightarrow \exists \ell \) with \( p \nmid |G(F_\ell)| \). This is because \( |G(\ell)| \) is of the form \( \ell^m(\ell^H-1) \prod (\ell^{d_i}-1) \) with \( M \in \mathbb{N} \) and \( H>d_i \) for all \( i \). The converse is true for \( G \) classical, but false for \( G_2 \). Here \( H=6 \), and \( |G_2(F_\ell)| = \ell^6(\ell^6-1)(\ell^2-1) \). Note that \( 5 \leq 6+1 \) but \( 5 \nmid |G_2(F_\ell)| \) for \( \ell=3,7,13,... \).

Define

\[ J(\Gamma,A) = \{ f:G \to A \mid f(g\gamma) = \gamma^{-1}f(g) \text{ for all } \gamma \in \Gamma, g \in G \} . \]

We will show that \( H^\nu(\Gamma,A) \) (resp. \( H_v(\Gamma,A) \)) is a quotient (resp. subspace) of \( J(\Gamma,A) \). Let \( \lambda \) denote the left action of \( G \) on \( J(\Gamma,A) \) given by \( [\lambda(g)f](x) = f(g^{-1}x) \). Set \( J(\Gamma,A)_e = \langle (e(s)-\lambda(s))f \mid s \in S, f \in J(\Gamma,A) \rangle \), and \( J(\Gamma,A)_h = \{ f \in J(\Gamma,A) \mid \lambda(h)f = f \} \). Finally, we define
K(\Gamma,A) = J(\Gamma,A)/J(\Gamma,A)_e + J(\Gamma,A)^h

and \ W(\Gamma,A) = \{ f \in J(\Gamma,A) \mid \lambda(s)f = \varepsilon(s)f \text{ for all } s \in S, (1+\lambda(h)+\lambda(h^2)f=0) \}. 

If \ \Gamma \text{ is normal in } G, \text{ then } J(\Gamma,A) \text{ is a left } G/\Gamma\text{-module via } (g\cdot f)(x) = g[f(xg)]. \text{ This action lifts to } K(\Gamma,A) \text{ and preserves } W(\Gamma,A). \text{ Also } G/\Gamma \text{ acts on } I \otimes rA \text{ by } g \cdot (\eta \otimes a) = \eta g^{-1} \otimes ga. \text{ Recall that we have defined } \text{Hom}_r(I,A) \text{ to be those linear maps } \Psi:I \rightarrow A \text{ such that } \Psi(\eta \gamma) = \gamma^{-1} \Psi(\eta g). \text{ Finally, } G/\Gamma \text{ acts on Hom}_r(I,A) \text{ by } (g \cdot \Psi)(\eta) = g\Psi(\eta g).

(1.3) \text{Proposition:} \text{ There is a surjection } m_r:K(\Gamma,A) \twoheadrightarrow H^r(\Gamma,A), \text{ and an injection } \Psi: H^r(\Gamma,A) \hookrightarrow W(\Gamma,A). \text{ These are } G\text{-equivariant if } \Gamma \subseteq G. 

(\text{This was proved for trivial coefficients in } [2], \text{ using the} \text{ "well-rounded retract"})

\text{Proof: } \text{Define } m_r:J(\Gamma,A) \rightarrow I \otimes rA \text{ by } m_r(f) = \sum_{g \in G/\Gamma} \tau g \otimes f(g), \text{ where, as usual, } \tau \in I \text{ corresponds to } T. \text{ From } ([5], (2.2)) \text{ one computes that } 1+h+h^2 \text{ annihilates } \tau, \text{ and it follows that } m_r \text{ is defined on } K(\Gamma,A). \text{ Define elements of } J(\Gamma,A) \text{ as follows: for } a \in A, g \in G, \text{ set } f_{ga}(x) = \gamma^{-1} a \text{ if } x = g \gamma, \text{ } f_{ga}(x) = 0 \text{ if } x \notin g \Gamma. \text{ Note that }

\sum_{G/\Gamma} \tau x \otimes f_{ga}(x) = \tau g \otimes a. \quad (39)

By (1.1),

I = \sum_{G/\Gamma} \tau g \mathbb{Z} \Gamma. \quad (40)

It follows that the \( \tau g \otimes a \) for \( g \in G/\Gamma, a \in A \) generate \( I \otimes rA \). \text{ This shows surjectivity. The } G/\Gamma\text{-equivariance is clear.}

Next, we have a natural isomorphism \( H_\mu(\Gamma,A) \cong \text{Hom}_r(I,A) \). \text{ This is the top homology version of Borel Serre duality. (See } [11] \text{ pg. 204). For } F \in \text{Hom}_r(I,A), \text{ define a function } \Phi(F): G \rightarrow A \text{ by}
\[ \Psi(F)(g) = F(\tau g). \] It is again easy to check that \( \Psi(F) \in W(\Gamma, A) \). Using (1.1) again, we see that \( \Psi \) is injective. \( \square \)

**Remark:** Let \( \langle , \rangle \) and \( ( , ) \) be the natural pairings on \( A \otimes A^* \) and \( (I \otimes_I A) \otimes \text{Hom}_I(I, A^*) \), respectively. Define \( ( , ) : K(\Gamma, A) \otimes W(\Gamma, A) \to k \) by \( (f, \psi)_I = \sum_{g \in G/\Gamma} (f(g), \psi(g)) \). One checks that \( (m_{\Gamma}(f), F) = (f, \Psi(F))_I \), and that
\[
(f_{\beta, a}, \psi)_I = \langle a, \psi(g) \rangle.
\]
Hence the pairing between \( H^\nu(\Gamma, A) \) and \( H_p(\Gamma, A^*) \) lifts to a perfect pairing between \( K(\Gamma, A) \) and \( W(\Gamma, A^*) \). Computationally, it appears easier to work with \( W(\Gamma, A) \), rather than \( K(\Gamma, A) \). The following lemma, which is a consequence of the above remark, says \( W(\Gamma, A) \to H^\nu(\Gamma, A) \), when, for example, \( A \) is trivial.

(1.4) **Lemma:** Suppose \( A \) admits a positive definite \( \Gamma \)-invariant bilinear form \( \langle , , \rangle \). Then the natural composition \( W(\Gamma, A) \to J(\Gamma, A) \to K(\Gamma, A) \) is an isomorphism.

§2. **Restriction for \( \Gamma < \text{SL}_n(\mathbb{Z}) \)**

We now study the restriction map \( r_{\mathcal{P}, \mu} \) from chapter I, showing that it lifts to a map on \( J(\Gamma, A) \). We have arranged our formula for \( r_{\mathcal{P}, \mu} \) so that we may always restrict to a standard parabolic subgroup. So let \( \mathcal{P} \) be a parabolic containing the upper triangular matrices, \( L \) the standard Levi factor of \( \mathcal{P} \) (so \( T \triangleleft L \)). We have \( L < L_1 \times \ldots \times L_r \), where \( L_i \cong \text{GL}_{n_i} \) and \( n_1 + \ldots + n_r = n \). It is easy to see that the Tits building \( \mathcal{J}_L \) is the Cartesian product of the \( \mathcal{J}_{L_i} \)'s. Hence
I_L = I_{1_L} \otimes \cdots \otimes I_{r_L}. \quad \text{Let } \tau_i \in I_{1_L} \text{ correspond to the maximal } \mathbb{Q} \text{-split torus } T \cap L_i \text{ of } [L_i, L_i]. \quad \text{Then } \tau_L = \tau_1 \otimes \cdots \otimes \tau_r. \quad \text{If } r_i \geq 2, \text{ let } h_i \in L_i \text{ be analogous to } h. \quad \text{I.e.,}

\begin{align*}
    h_i \cdot e_{q_1^+ + \cdots + q_{i-1}^+ + 1} &= e_{q_1^+ + \cdots + q_{i-1}^+ + 2}, \\
    h_i \cdot e_{q_1^+ + \cdots + q_{i-1}^+ + 2} &= -e_{q_1^+ + \cdots + q_{i-1}^+ + 1} - e_{q_1^+ + \cdots + q_{i-1}^+ + 2}, \\
    h_i \cdot e_j &= e_j \text{ for all other } j.
\end{align*}

If } r_i = 1, \text{ set } h_i = 1 \text{ and put } h_L = \prod h_i. \quad (42)

If } B \text{ is a } k_L\text{-module and } \Gamma_L \text{ has finite index in } L, \text{ then as above, we define}

\begin{align*}
    J(\Gamma_L, B) &= \{f : L \to B| f(\lambda \gamma) = \gamma^{-1} f(\lambda) \text{ for all } \lambda \in L, \gamma \in \Gamma\}, \\
    J(\Gamma_L, B)_e &= \{f(s) - \lambda(s) f| s \in S \cap L, f \in J(\Gamma_L, B)\}, \\
    J(\Gamma_L, B)^{h_L} &= \{f \in J(\Gamma_L, B)| \lambda(h_L) f = f\}, \\
    K(\Gamma_L, B) &= J(\Gamma_L, B) / J(\Gamma_L, B)_e^+ J(\Gamma_L, B)^{h_L}.
\end{align*}

Then (1.3) and (1.4) are valid for these spaces as well, with } \Gamma_L \text{ replacing } \Gamma. \text{ Let}

\begin{align*}
    m_r : J(\Gamma, A) &\to K(\Gamma, A) \to I \otimes r_A = H^r(\Gamma, A) \quad \text{and} \\
    m_{\Pi L, w} : J(\Gamma(L, \mu), A_{\Pi L, w}) &\to K(\Gamma(L, \mu), A_{\Pi \Pi L, w}) \to I_p \otimes \Pi_{L, w}A_{\Pi \Pi L, w} = H^p(\Gamma(L, \mu), A_{\Pi \Pi L, w}) \text{ be as in (1.3). See chapter I §2 for notation.}
\end{align*}

Here

\begin{equation}
    v_L = \sum_{1 \leq i \leq r} \frac{1}{2} \eta_i(n_i-1) = vcd[\Gamma(L, \mu)]. \quad (43)
\end{equation}

(2.1) Proposition: There is a map } \rho_{P, \mu} : J(\Gamma, A) \to J(\Gamma(L, \mu), A_{\Pi \Pi L, w}) \text{ such that } r(p, \mu, A) \circ m_r = m_{\Pi L, w} \circ \rho_{P, \mu}. \text{ It is given by}

\begin{equation}
    \rho_{P, \mu}(f)(\lambda) = \lambda^{-1} \sum \varepsilon(w) w g(f) \quad (44)
\end{equation}

where the sum is over } \mathcal{B}(\lambda) = \{((w, g) \in W_{\Pi L, w} \cap \Gamma \cap \Pi_{L, w}| P(\lambda g) = \lambda\}. \quad (45)
Proof: We compute
\[ r(p,\mu,A) \circ m(f) = r(p,\mu,A) \left[ \sum_{g \in G/\Gamma} \tau g \otimes f(g) \right] \]
\[ = \sum_{w \in W} \epsilon(w) \sum_{g \in G/\Gamma} \tau_{L} p_{\mu}(wg) \otimes p_{\mu}(wg)^{-1} wgf(g) \]
\[ = \sum_{w \in W} \epsilon(w) \tau_{L} \otimes \ell^{-1} wgf(g) \]
\[ = \sum_{w \in W} \tau_{L} \otimes p_{\mu}(f)(l) = m_{\mu}(\pi) \circ p_{\mu}(f). \tag{45} \]

(2.2) Remarks: i) By Chapter I (2.6) we conclude that
\[ \ker(\ker: H^{\nu}(\Gamma,A) \to H^{\nu}(\Gamma \cap P_{\mu},A)) \simeq \rho_{P_{\mu}}^{-1}(\ker m_{\mu})/\ker m_{\pi}. \]
ii) Suppose \( A = k \), with trivial \( \Gamma \)-action. By (1.4), \( K(\Gamma,k) \) can be identified with \( W(\Gamma,k) \). The map \( p_{\mu}: W(\Gamma,k) \to W(\Gamma(L),k) \) is
\[ p_{\mu}(f)(l) = |W| \sum f(g), \tag{46} \]
sum over all \( g \in G/\Gamma \) such that \( l_{P_{\mu}}(g) = l \).

§3. The Top Homology of \( SL_{n}(\mathbb{Z}) \).

We pause here to give some applications and examples of the Borel–Serre theorem and our results up to this point. The first result is a version of (1.3) in the case \( \Gamma = G = SL_{n}(\mathbb{Z}) \).

(3.1) Proposition: There is an injection
\[ H_{\nu}(G,A) \hookrightarrow \{ a \in A | sa = \epsilon(s)a \text{ for all } s \in S, \text{ and } (1+h+h^{2})a = 0 \}. \]

Proof: By (1.1), we can choose a torsion-free normal subgroup \( \Gamma_{1} \) of \( G \) such that \( \text{char}(k) \nmid [G:G_{1}] \). Then \( H_{\nu}(G,A) \simeq H_{\nu}(\Gamma_{1},A)^{G/G_{1}} \hookrightarrow W(\Gamma_{1},A) \). One checks that the map \( f \mapsto f(1) \) gives an isomorphism of this last space onto \( \{ a \in A | sa = \epsilon(s)a \text{ for all } s \in S, \text{ and } (1+h+h^{2})a = 0 \} \). \( \square \)

We next give a simple formula for the homology inclusion map from a parabolic subgroup. Let \( P \) be a \( \Phi \)-parabolic subgroup of \( SL_{n} \), with Levi factor \( L \) and unipotent radical \( U \). We write \( P, U, L \) for
P(\mathbb{Z}),\ldots\text{etc.} \ L \text{ is a subgroup of index two in a direct product of } GL_r(\mathbb{Z})'s, \text{ for various } r. \ \text{Set } v_L = vcd(L) \text{ and let } i_* \text{ be the composition } H_{v_L}(L,A^U) \rightarrow H_{v}(G,A). \ \text{By Chapter I (2.8), we have a map } j: (A^U)_* \rightarrow A_*, \text{ whose restriction to } H_{v_L}(L,A^U) \text{ is } i_*, \text{ and }

\[ j(a) = \sum_{w} e(w) w^{\lambda} a. \]  

(47)

Let \( \bar{k} \) be an algebraic closure of \( k \), and \( L' \) the derived group of \( L \).

(3.2)\textbf{Proposition:} \ Let \( V \) be the finite dimensional irreducible rational \( SL_n(\bar{k}) \) representation of highest weight \( \lambda \) (with respect to the maximal torus \( T \) corresponding to \( \tau \) and a Borel subgroup of \( P \) containing \( T \)). \ Suppose \( V \) remains irreducible for \( SL_n(k) \). \ Let \( \Delta_P \) be the simple roots generating \( L' \). \ If the Dynkin diagram of \( SL_n \) contains a node not connected to any node of \( \Delta_P \cup \text{support}(\lambda) \), \ then the map \( H_{v}(P,V) \rightarrow H_{v}(G,V) \) induced by the inclusion \( P \subset G \) is identically zero.

\textbf{Proof:} \ Let \( \alpha \) be a simple root not contained in \( \Delta_P \cup \text{support}(\lambda) \). \ To \( \alpha \) there corresponds a subgroup \( M \cong SL_2 \) of \( G \). \ The reflection \( s_\alpha \in W \) corresponding to \( \alpha \) has a representative \( s \) in \( M(\mathbb{Z}) \). \ Hence \( s_\alpha \lambda = \lambda \) and \( s \in \text{Cent}_G(L(\bar{k}')) \). \ The fixed point space \( V^s \) is generated over \( L(k)' \) by a highest weight vector which is fixed by \( s \). \ It follows that \( s \) acts trivially on \( V^s \). \ However, by the formula for \( j \), if \( v \in V^s \) and \( w \in W \) are such that \( e(w) = -1 \) and \( \bar{w}v = v \) for some representative \( \bar{w} \) of \( w \), \ we must have \( j(v) = 0 \). \ Setting \( w = s_\alpha \), we see that \( j \) vanishes on \( (V^s)_* \), so \( i_* \) must be zero. \ \Box

In the other direction, the following observation about \( j \) is obvious.
**Lemma:** Let $V$ be a finite dimensional rational $\text{SL}_n(\mathbb{R})$-module. Suppose the weights appearing in $v \in V^U$ are $\mu_1, \ldots, \mu_s$ and that $w \in W(G, T)$, $w \mu_m = \mu_1 \Rightarrow w \in W(L, T)$. Then $j(v) \neq 0$.

**Example:** Let $n = 3$, $P = \text{Stab}(e_1, e_2)$. Then $L \cong \text{GL}_2(\mathbb{Z})$. Let $V = S^{10}V_3$, the tenth symmetric power of $V_3$, where $V_n$ is the natural $n$-dimensional representation of $\text{SL}_n(\mathbb{C})$. It is well known (cf. [1]) that

$$f(e_1, e_2, e_3) = e_1^8 e_2^2 - 3e_1^6 e_2^4 + 3e_1^4 e_2^6 - e_1^2 e_2^8$$

belongs to $H_1(L, V^U)$, viewed as a subspace of $V^U$, as in (2.1). (This corresponds to the $+1$ eigenspace of on the cusp forms of weight $12$ for $\text{SL}_2(\mathbb{Z})$.) We have

$$j(f) = f(e_1, e_2, e_3) - f(e_1, e_3, e_2) + f(e_2, e_3, e_1),$$

and one checks that this gives a nonzero element of $H_3(\text{SL}_3(\mathbb{Z}), S^{10}V_3)$. We remark that our computation of the cohomology of the Borel–Serre boundary in Chapter III shows that $\dim H_3(\text{SL}_3(\mathbb{Z}), S^{10}V_3) = 1$. (50)

Hence $i_\pi$ is surjective in this case.

Now suppose $G = \text{SL}_4(\mathbb{Z})$. Let $P_3 = \text{Stab}_G(e_1, e_2, e_3)$, $P_2 = \text{Stab}_G(e_1, e_2)$. Write $P_r = L_r U_r$ with standard Levi factors $L_r$. Let $V = S^{10}V_4$, and let $i_{r, \pi}$ be the corresponding homology inclusion maps. Let $i_{1, \pi}$ be the homology inclusion from the previous paragraph, induced by the inclusion $L_2(L_3 \cap U_2) < L_3$. We have $i_{2, \pi} = i_{3, \pi} \circ i_{1, \pi}$. By (3.2), $i_{2, \pi}$ is zero. Since $i_{1, \pi}$ is surjective, we get $i_{3, \pi} = 0$. Hence the lifting of $f$ stops at $n = 3$. This phenomenon is not surprising in view of (3.6) below.
On the other hand, \( j(f) \neq 0 \) when \( n = 4 \),
\[ P = LU = \text{Stab}_G(0 \left< e_1 \right>< e_1, e_2, e_3 ) \] and \( V \) has highest weight \( \lambda: \)
\[ \text{diag}(x_1, x_2, x_3, x_4) \mapsto (x_1 x_2)^{10} \]. Here we assume \( k = \mathbb{C} \). Then the module \( V \)
can be realized as the submodule of \( S^{10}(\wedge^2 V_4) \) generated by\( (e_1 \wedge e_2)^{10} \). We have an \( L \)-module homomorphism
\[ \varphi: (e_2, e_3) \rightarrow \Lambda^2 V_4 \text{ via} \]
\[ \varphi( ae_2 + be_3 ) = ae_1 \wedge e_2 + be_1 \wedge e_3. \] (51)
The image of
\[ \varphi^{10}: S^{10}(e_2, e_3) \rightarrow S^{10}(\Lambda^2 V_4) \]
is \( V^U \). Now
\[ \varphi^{10}(f) = (e_1 \wedge e_2)^8 (e_1 \wedge e_3)^2 - (e_1 \wedge e_2)^6 (e_1 \wedge e_3)^4 + (e_1 \wedge e_2)^4 (e_1 \wedge e_3)^6 \]
\[ - (e_1 \wedge e_2)^2 (e_1 \wedge e_3)^8. \] (52)
Write \( \varepsilon_i \) for the weight of \( e_1 \) and \( \mu_1, ..., \mu_4 \) for the weights (in order)
appearing in \( \varphi^{10}(f) \). Set \( W = \text{W}(G, T) \) and \( \langle s_{a_2} \rangle = \text{W}(L, T) \). Note that
\[ s_{a_2} \mu_1 = \mu_4, s_{a_2} \mu_2 = \mu_3 \text{ and } \text{W} \cdot \{ \mu_1, \mu_4 \} \cap \text{W} \cdot \{ \mu_2, \mu_3 \} = \emptyset \] since the weights have different lengths under the standard \( W \)-invariant
inner-product on \( \langle \varepsilon_1, ..., \varepsilon_4 \rangle \). Also, \( \text{Stab}_W \mu_i = \{1\} \) for each \( i \). Hence the
\( \mu_i ' s \) satisfy the conditions of (3.3), so \( j( \varphi^{10}(f)) \neq 0 \). Since
\[ \varphi^{10}(f) \in H_1(L, V^U), \] we have \( 0 \neq j( \varphi^{10}(f)) \in H_6(\text{SL}_4(\mathbb{Z}), V) \).

The following result describes a family of similar liftings.

(3.5) Proposition: Let \( \{ \alpha_1, ..., \alpha_n \} \) be the simple roots of \( G \) with
respect to \( T \) and \( B \). Let \( V \) be the irreducible rational \( \text{SL}_n(\overline{k}) \)-module
with highest weight \( \lambda \). Suppose \( \langle \lambda, \alpha_i \rangle \) is even and positive for all
\( i \). Then \( H_\nu(\text{SL}_n(\mathbb{Z}), V ) \neq 0 \).

Proof: We will produce a nonzero class by lifting from \( B(\mathbb{Z}) \), the
integral upper-triangular matrices. We have
Let $v_+$ be a highest weight vector for $B(k)$. Since the $\langle \lambda, \alpha \rangle$'s are all even, $v_+ \in V^B(Z)$. Now $j : V^B(Z) \to V$ is simply

$$j(v) = \sum_w \epsilon(w) w \cdot v.$$  \hfill (53)

Since $\lambda$ is regular, the weights are distinct and $j(v_+) \neq 0$. The result follows. □

Remarks: 1) This argument will yield a similar result for other $Q$-groups. 2) Consider the case $k = \mathbb{F}_p$, $p > n+1$. Then $\Gamma(p)$, the principal congruence subgroup of level $p$, acts trivially on $V$. Hence $\text{Hom}_{\text{SL}_n(\mathbb{Z})}(I, V) \cong \text{Hom}_{\text{SL}_n(\mathbb{F}_p)}(I \otimes \Gamma(p) \mathbb{Z}, V) \cong \text{Hom}_{\text{SL}_n(\mathbb{F}_p)}(H^*(\Gamma(p), \mathbb{Z}), V)$. Thus, the conditions of (3.5) imply that $V$ is a composition factor of $H^*(\Gamma(p), \mathbb{Z})$, viewed as an $\text{SL}_n(\mathbb{F}_p)$-module.

We now give some rough vanishing theorems for $H_p(\text{SL}_n(\mathbb{Z}), A)$ for certain familiar representations $A$ over $\mathbb{C}$. As above, $V_n$ is the natural module for $\text{SL}_n(\mathbb{C})$ and $S^rV_n$ is the symmetric algebra. Let $\Lambda^r V_n$ be the exterior algebra of $V_n$, and $A^\text{ad}$ the adjoint representation of $\text{SL}_n(\mathbb{C})$. These are to be viewed as $\text{SL}_n(\mathbb{Z})$-modules.

(3.6) Theorem: Let $A$ be one of $\Lambda^r V_n$, $A^\text{ad}$, or $S^r V_n$, where $r$ is restricted as follows: $n$ even or $r$ odd $\Rightarrow r \leq n+1$, $n$ odd and $r$ even $\Rightarrow r \leq n(n-1)$. Then $H_p(\text{SL}_n(\mathbb{Z}), A)$ is zero.

Note: This bound on $r$ is not sharp

Proof: The proof consists of showing that $\epsilon$ does not occur in $A$, viewed as an $S$-module. For this we use invariant theory.
Let \( C \) be the Weyl group of type \( C_n \), acting on \( V_n \) by permutations and changes of sign of the \( e_i \)'s. Let \( \sigma \) be the sign character of \( C \). Then \( S = \ker \sigma \), and \([C:S]=2\). Let \( y \in C \setminus S \). There are exactly four one-dimensional characters of \( C \): \( 1, \sigma, \psi, \sigma \psi \). Here \( 1 \) is the trivial character of \( C \) and \( \psi \) is defined by \( \psi|S = \varepsilon \) and \( \psi(y) = 1 \). Now \( \Lambda^r V_n \) is irreducible for \( C \), for all \( 0 \leq r \leq n \) [10]. Since \( \Lambda^0 V_n = 1, \) \( \Lambda^n V_n = \sigma \) as \( C \) representations, and \( \sigma|S \neq \varepsilon \), we get \( H_v(SL_n(\mathbb{Z}), \Lambda^r V_n) = 0 \).

If \( H \) is a subgroup of \( C \), \( \chi \) a character of \( H \), let \( R(H, \chi) \) be the \( \chi \)-isotypic subspace of the symmetric algebra \( S^* V \). For \( 1 \leq i \leq n \), let \( T_i = s_i(e_1^2, ..., e_n^2) \in S^{2i} V \) where \( s_i(x_1, ..., x_n) \) is the \( i \)th elementary symmetric polynomial. Also set \( f_\psi = \prod_{i<j}(e_i^2-e_j^2) \) and \( f_\sigma \phi = \prod_i e_i \). Then \( R[C,1] = \mathbb{C}[T_1, ..., T_n] \), \( R[C, \sigma] = R[C,1] \cdot f_\sigma \phi \) (cf. [29] p.410 ex. 58). By ([27] 4.7), \( R[C, \psi] \) and \( R[C, \sigma \psi] \) are also free \( R[C,1] \)-modules of rank one.

**Claim:** \( R[C, \psi] = R[C,1] f_\psi \), \( R[C, \sigma \psi] = R[C,1] f_\sigma \phi \).

**Reason:** Suppose \( R[C, \psi] = R[C,1] g_\psi \), \( R[C, \sigma \psi] = R[C,1] g_\sigma \phi \). We have \( f_\psi \in R[C, \psi] \), \( f_\sigma \phi \in R[C, \sigma \psi] \), so \( \deg f_\psi \geq \deg g_\psi \) and \( \deg f_\sigma \phi \geq \deg g_\sigma \phi \). On the other hand, \( g_\psi \cdot g_\sigma \phi \) affords \( \sigma \) and \( R[C, \sigma] = R[C,1] f_\psi \cdot f_\sigma \phi \). Hence \( \deg g_\psi + \deg g_\sigma \phi \geq \deg f_\psi + \deg f_\sigma \phi \). Therefore, all the inequalities are equalities, and we must have \( f_\psi = \text{const} \cdot g_\psi \), \( f_\sigma \phi = \text{const} \cdot g_\sigma \phi \) and the claim follows.

It is easy to see that \( \text{Ind}(S,C, \varepsilon) = \psi \Theta \sigma \psi \). By Frobenius reciprocity, we get \( R[S, \varepsilon] = R[C,1] \cdot f_\psi \Theta R[C,1] \cdot f_\sigma \phi \). The asserted vanishing of \( H_v(SL_n(\mathbb{Z}), S^r V) \) follows by counting degrees.
For the adjoint representation, recall that $\mathbb{C} \oplus \text{Ad} \cong V_n \otimes V_n^*$. It suffices to show $H_p(\text{SL}_n(\mathbb{Z}), V_n \otimes V_n^*) = 0$. Since $S$ is represented on $V_n$ by real matrices, $V_n \cong V_n^*$ as $\mathbb{C} S$-modules. Hence $V_n \otimes V_n^* \cong V_n \otimes S^2 V_n \oplus \Lambda^2 V_n$. We have seen that $S^2 V_n$ and $\Lambda^2 V_n$ have zero $\epsilon$-component. This concludes the proof of (3.6). □

§4. $\Gamma = \Gamma_0(n,N)$

In this section, we record some matrix computations which are motivated by our formula for the restriction map. For a positive integer $N$, let $\Gamma'_0(n,N)\pm$ be the set of all matrices in $\text{GL}_n(\mathbb{Z})$ whose first column is congruent to $t(u,0,\ldots,0)$ mod $N$, where $u$ is any element of $(\mathbb{Z}/N)^\times$. Let $\Gamma_1(n,N)$ denote those elements of $\Gamma_0(n,N)\pm$ for which $u=1$. Fix $\Gamma = \Gamma_0(n,N)$ to be the determinant one matrices in $\Gamma_0(n,N)\pm$. As before, $G = \text{SL}_n(\mathbb{Z})$. We will determine the $\Gamma$-conjugacy classes of maximal $\mathfrak{Q}$-parabolic subgroups of $\text{SL}_n$, and explicitly compute the maps $I_{P,\mu}$.

For $1 \leq k \leq n$, let $P_k = \text{Stab}_G(e_1,\ldots,e_k)$, $L_k = P_k \cap P_k$. For typographical reasons, we write elements of $L_k$ as follows: If $A$ and $B$ are matrices of size $k \times k$ and $(n-k) \times (n-k)$ respectively, let $[A,B]$ denote the $n \times n$ matrix with $A$ in the upper left corner, $B$ in the lower right corner and zeros elsewhere. For $m \mid N$, we set

$$\Delta(m,k) = \{[(a_{ij}),(b_{ij})] \in L_k : m \mid a_{ii} \text{ for } 2 \leq i \leq k, \text{Nm}^2 \mid b_{kk+1} \text{ for } k+1 \leq i \leq n, \text{ and } (m,\text{Nm}^2) \mid a_{11} - b_{k+1,k+1}\}.$$

So we have containments of finite index:

$$G \cap [\Gamma_1(k,m) \times \Gamma_1(n-k,\text{Nm}^2)] < \Delta(m,k) < G \cap [\Gamma_0(k,m)^\pm \times \Gamma_0(n-k,\text{Nm}^2)^\pm].$$
If \( v=\sum c_i e_i \), \( c_i \in \mathbb{Z} \), let \( (v)=\gcd\{c_i : c_i \neq 0\} \), and for \( c \in \mathbb{Z} \setminus \{0\} \), let \( (v,c)=\gcd((v),c) \). Finally, \( \pi \) is the projection \( P_k \rightarrow L_k \).

(4.1) Proposition:  

1) A system of representatives for the double cosets \( P_k \setminus G / \Gamma \) is given by \( \{\mu=I+m\epsilon_{k+1}, l \leq m \text{ and } m \mid N\} \).

2) Let \( g \in G \), \( g\epsilon_1 = g_1 + g_2 \), where \( g_1 \in \langle e_1, \ldots, e_k \rangle \), \( g_2 \in \langle e_{k+1}, \ldots, e_n \rangle \). Then \( g \in P_k \mu \Gamma \) if and only if \( (g_2,N) = m \).

3) \( \pi(\mu \Gamma \cap P_k) = \Delta(m,k) \). There is a natural \( L_k \)-equivariant surjection \( (\mathbb{Z}/m)^k \times (\mathbb{Z}/N)^{n-k} \rightarrow L_k / \pi(\mu \Gamma \cap P_k) \).

Proof: Note that 2) implies 1). Let \( \mu = I+m\epsilon_{k+1} \). We begin by showing \( g \in P_k \mu \Gamma \Rightarrow (g_2,N) = m \). For any matrix \( M \), write \( M \epsilon_1 = M_1 + M_2 \) as we did for \( g \). Let \( p \in P_k \) be such that \( \mu^\dagger pg \in \Gamma \). Write \( p = \cdots \).

Then

\[ (\mu^\dagger pg)_1 = A \cdot g_1 + U \cdot g_2, \quad \text{and} \quad (\mu^\dagger pg)_2 = B \cdot g_2 - mc \cdot \epsilon_{k+1}, \]

where \( c \) is the coefficient of \( \epsilon_1 \) in \( A \cdot g_1 + U \cdot g_2 \). Let \( b \) be the coefficient of \( \epsilon_{k+1} \) in \( B \cdot g_2 \). Then

\[ \mu^\dagger pg \in \Gamma \Rightarrow Nb - mc \quad \text{and} \quad (c,N) = 1. \]

Hence

\[ (b,N) = m. \]

Also, \( N \) divides the other coefficients of \( B \cdot g_2 \), so

\[ (g_2,N) = (B \cdot g_2,N) = (b,N) = m. \]

We next show \( (g_2,N) = m \Rightarrow g \in P_k \mu \Gamma \). It is easy to see that we can find \( x \in SL_k(\mathbb{Z}) \), \( y \in SL_{n-k}(\mathbb{Z}) \) such that

\[ (a) \quad x \cdot e_1 = (g_1)^{-1} g_1, \quad (b) \quad y \cdot e_1 = (g_2)^{-1} g_2, \quad \text{and} \quad (c) \quad [x^2, y^2] g \cdot e_1 = (g_1) e_1 + (g_2) e_2. \]

(58) Also, there exists \( \gamma \in \Gamma \) such that

\[ [x^2, y^2] g \gamma \cdot e_1 = m' e_1 + me_{k+1} \text{ where } m = (g_2,N) \text{ and } (m,m') = 1. \]
We now show \( g \in \mathcal{P}_k \mu \Gamma \) where \( \mu = I + mE_{k+1} \).

**Case 1:** \( k = 1 \). By Dirichlet's theorem, \( \exists u \in \mathbb{Z} \) such that
\[
(m' + um, N) = 1. \tag{60}
\]
Choose \( a, b \in \mathbb{Z} \) such that
\[
b(m' + um) - aN = 1. \tag{61}
\]
Define \( p \in \mathcal{P}_k \) by
\[
\begin{align*}
p \cdot e_2 &= u e_1 + (m' + um) e_2 + Ne_3, \\
p \cdot e_3 &= a e_2 + b e_3, \\
p \cdot e_i &= e_i \text{ for } i \neq 2, 3.
\end{align*}
\]
One checks that
\[
\mu^{-1} p(x^1, y^2) g \cdot e_1 = (m' + um) e_1 + mNe_3, \tag{63}
\]
so \( \mu^{-1} p(x^1, y^2) g \in \Gamma \).

**Case 2:** \( k > 1 \). Choose \( a, c \in \mathbb{Z} \) such that
\[
am' + cm = 1. \tag{64}
\]
Since, for \( l \in \mathbb{Z} \), \( a_l = a + lm \), \( c_l = c - lm' \) also satisfy this equation, and \( (a, m) = 1 \), we may again use Dirichlet's theorem to assume \( (a, N) = 1 \).
Choose \( b, d \in \mathbb{Z} \) so that
\[
ad - bN = 1. \tag{65}
\]
Define \( p' \in \mathcal{P}_k \) by
\[
\begin{align*}
p' \cdot e_1 &= ae_1 + Ne_2, \\
p' \cdot e_2 &= be_1 + de_2, \\
p' \cdot e_{k+1} &= ce_1 - m^2 m' e_2 + e_{k+1}, \\
p' \cdot e_i &= e_i \text{ for } i \neq 1, 2, k+1.
\end{align*}
\]
Again, one checks that
\[
\mu^{-1} p'(x^1, y^2) g \cdot e_1 = (am' + cm) e_1 + m' Ne_2, \text{ so } \mu^{-1} p'(x^1, y^2) g \in \Gamma. \tag{67}
\]
This proves ii).
Proof of iii): Let $X=(x_{ij})$ be an $n \times n$ matrix. Then $\mu X \mu^{-1}$ differs from $X$ only in the first column and $k+1^{th}$ row. We have

$$\mu X \mu^{-1} \cdot e_i = \sum_{1 \leq i \leq n} (x_{i1} - m \cdot x_{i,k+1}) e_i$$

$$+ [x_{k+1} \cdot m - m(x_{k+1,k+1} + mx_{1,k+1})]e_{k+1}.$$ 

$$e_{k+1} \cdot \mu X \mu^{-1} = [x_{k+1} \cdot m - m(x_{k+1,k+1} + mx_{1,k+1})]e_1 + \sum_{2 \leq j \leq n} (x_{k+1,j} + mx_{1,j}) e_j. \quad (68)$$

Let $[A, B]$ be in $L$, with $A=(a_{ij})_{1 \leq i \leq k, j \leq k}$, $B=(b_{ij})_{k+1 \leq i \leq k, j \leq n}$. Then $[A, B] \in \pi(\mu \cap P_k)$ if and only if we can find integers $x_{ij}, 1 \leq i, j \leq n$ satisfying

i) $\mu(x_{ij}) \mu^{-1} \in P_k(Q)$ (note that $\det(\mu(x_{ij}) \mu^{-1}) = \det([A, B] = 1)$

ii) For all $1 \leq i \leq k$, $a_{ij} = x_{ij} - mx_{1,i}$, and $a_{ij} = x_{ij}$ if $2 \leq j \leq k$

iii) For all $k+1 \leq j \leq n$, $b_{k+1,j} = x_{k+1,j} + mx_{1,j}$, and $b_{ij} = x_{ij}$ if $k+2 \leq i \leq n$, and

iv) For all $2 \leq i \leq n$, $x_{ii} \equiv 0 \pmod{N}\). 

We first show $\pi(\mu \cap P_k) \subseteq \Delta(m,k)$. Let $[A, B] = \pi(\mu(x_{ij}) \mu^{-1}) \in L$, where $x_{ij}$ are integers satisfying i)—iv). We show $[A, B] \in \Delta(m,k)$. By ii) and iv), we have $m|a_{ii}$ for $2 \leq i \leq k$, so $A \in \Gamma_0(m,k)$. Also, i) and (1) imply

$$x_{ii} \equiv mx_{1,i} \equiv 0 \pmod{k+2 \leq i \leq n}. \quad (69)$$

By iv), $N^{-1}|x_{(k+1)}$ and by iii) we get $N^{-1}|b_{(k+1)}$ for $k+2 \leq i \leq n$, so $B \in \Gamma_0(N/m,k)$. It remains to show that $a_{ii} \equiv b_{i,k+1} \equiv 0 \pmod{m,N^{-1}}$. From ii) we get $x_{ii} \equiv x_{1,i} \pmod{m}$. By i) and (1) again, we have

$$0 = m^2 x_{k+1,i} + x_{1,i} - (x_{k+1,k+1} + mx_{1,k+1}) \quad (by \ i) \ and \ (1))$$

$$= m^2 x_{k+1,i} + x_{1,i} - b_{k+1,k+1} \quad (by \ iii)). \ Hence \ x_{1,i} \equiv b_{k+1,k+1} \equiv 0 \pmod{(N^{-1}).} \quad (70)$$

By the Chinese Remainder Theorem, such an $x_{1,i}$ exists if and only if $a_{1,i} \equiv b_{k+1,k+1} \pmod{(m,N^{-1})}$.
We now show $\Delta(m,k) \subseteq \pi(n^m \cap P_k)$. Suppose $[A,B] \in \Delta(m,k)$. We construct an $n \times n$ integral matrix $(x_{ij})$ satisfying i)–iv). As remarked above, we can solve the congruences

$$x_{11} \equiv b_{k+1,k+1}(\mod (Nm^{-1})), \quad x_{11} \equiv a_{11}(\mod m)$$

for $x_{11}$, by the definition of $\Delta(m,k)$. Set

$$x_{k+1} = m(b_{k+1,k+1} x_{11}), \quad x_{1l} = N, \quad x_{1k+1} = m^2(N-a_{1l}), \quad \text{for } 2 \leq i, k \leq n. \quad (71)$$

Then $x_{k+1} \equiv 0(\mod N)$, and $a_{1l} = x_{1l} - mx_{k+1}$. Set

$$x_{ij} = a_{ij} \text{ if } 2 \leq j \leq k, = 0 \text{ if } k+1 \leq j \leq n. \quad (72)$$

Then set

$$x_{k+1,j} = -mx_{ij} \text{ if } 2 \leq j \leq k = b_{k+1,j} \text{ if } k+1 \leq j \leq n. \quad (73)$$

Next, set

$$x_{ik+1} = b_{ik+1} \text{ for } k+2 \leq i \leq n. \quad (74)$$

It follows that $x_{11} \equiv 0(\mod N)$ for $k+2 \leq i \leq n$. Thus iv) is now satisfied. The first and $k+1$st rows and columns of $(x_{ij})$ have now been completed. The choice of the remaining $x_{ij}$'s is now clear: If $(c_{ij}) = [A,B]$, set $x_{ij} = c_{ij}$ for $i, j \neq 1, k+1$. Now i), ii) and iii) are fulfilled, so $\mu(x_{ij}) \mu^{-1} \in \mu(n^m \cap P_k)$ and $[A,B] = \pi(\mu(x_{ij}) \mu^{-1})$. This completes the proof of the proposition. \(\square\)

We are now able to compute $I_{P,m}: G/\Gamma \to \mathbb{Z}[L/\Gamma(L,\mu)]$. Identify $\mathbb{Z}^n = \mathbb{Z}\langle e_1, \ldots, e_n \rangle$, $\mathbb{Z}^k = \mathbb{Z}\langle e_1, \ldots, e_k \rangle$, $\mathbb{Z}^{n-k} = \mathbb{Z}\langle e_{k+1}, \ldots, e_n \rangle$, thought of as column vectors. If $v \in \mathbb{Z}^n$ is primitive (i.e., $(v) = 1$), let $[v]_n$ be the coset it determines in $G/\Gamma = SL_n(\mathbb{Z})/\Gamma_0(n, N)$. If $x_1 \in \mathbb{Z}^k$ and $x_2 \in \mathbb{Z}^{n-k}$ are primitive, let $[x_1]_m \times [x_2]_{Nm^{-1}}$ be the coset they determine in $L/\Gamma(L,\mu)$ (see (4.1) iii)). If $k$ or $n-k = 1$, we omit the corresponding
term. If \( x_1 \) and \( x_2 \) are not necessarily primitive, but instead \( (x_1,x_2)=1 \), we write \([x_1,x_2]_N\) for \([t(x_1,x_2)]_N\). We have \([x_1,x_2]_N \in P_k \mu \Gamma / \Gamma\) if and only if \( (x_2,N)=m \), by (4.1) ii).

(4.2) Proposition: The map \( l_{P \mu} : G / \Gamma \to \mathbb{Z}[L / \Gamma(L,\mu)] \) is zero off \( P_k \mu \Gamma / \Gamma \) and for \([x_1,x_2]_N \in P_k \mu \Gamma / \Gamma\), we have

\[
[l_{P \mu}([x_1,x_2]_N)]_N = \begin{cases} 
[x_1]_m x_2 m^{-1} & \text{if } 2 \leq k \leq n-2 \\
[u x_2 m^{-1}]_N m^{-1} & \text{if } k=1, \text{ where } \gcd(u,N)=1 \text{ and } u x_1 = 1 (\text{mod } m) \\
[u x_1]_m & \text{if } k=n-1, \text{ where } \gcd(u,N)=1 \text{ and } u x_2 = m (\text{mod } N)
\end{cases}
\]

Proof: The first assertion holds by definition. Suppose \( d_1,\ldots,d_r \in \mathbb{Z}, \gcd(d_1,\ldots,d_r,N)=1, \text{ and } r \geq 2 \). Then it is easy to see that there exists \( c \in \mathbb{N} \) such that \( \gcd(d_1+cN,d_2,\ldots,d_r)=1 \). Using this fact, we get the following:

1) Suppose \( k,n-k \geq 2, x_1 \in \mathbb{Z}^k, x_2 \in \mathbb{Z}^{n-k}, (x_1,x_2)=1, (x_2,N)=m \). Then \( \exists a \in \mathbb{Z}^k, b \in \mathbb{Z}^{n-k} \) such that \( (a)=(b)=1, x_1 = a (\text{mod } m) \) and \( x_2 = b m (\text{mod } N) \).

1i) Notation as in i). Suppose \( k=1 \). Then \( \exists u \in \mathbb{Z}, b \in \mathbb{Z}^{n-1} \), such that \( (u,N)=(b)=1, u x_1 = a (\text{mod } m) \), and \( u x_2 = b (\text{mod } N) \).

1ii) Suppose \( k=n-1 \). Then \( \exists u \in \mathbb{Z}, a \in \mathbb{Z}^{n-1} \), such that \( (u,N)=(a)=1, u x_1 = a (\text{mod } m) \), and \( u x_2 = b (\text{mod } N) \).

We need to compute the map \( P_k \mu \Gamma / \Gamma \to P / \mu \Gamma \cap P \), \( p \mu \Gamma \to p(\mu \Gamma \cap P_k) \) in terms of matrices. Given \([x_1,x_2]_N \in P_k \mu \Gamma / \Gamma\), it suffices to find \( p \in P_k \) such that \( p \mu \Gamma = [x_1,x_2]_N \). If \( \text{col}_1 p = t(c,0), \text{col}_k p = t(y,d) \), then \( \text{col}_1 p \mu = t(c+my,md) \). Let \( a,b \) and \( u \) be as in i)—iii). Write
\( x_i = a + my, \ ux_i = 1 + my, \ ux_i = a + my \)  

(75)

In cases i), ii) and iii) respectively.

Let \( p = \begin{cases} \ast y & \text{if } 2 \leq k \leq n-2 \\ 0 & \text{if } k = 1 \\ b \ast & \text{if } k = n-1 \\ 1 \end{cases} \)

(76)

Here "\( \ast \)" means any completion to an integral matrix of determinant one.

Then \( \text{col}_j p \mu = \left\{ \begin{array}{ll} t(x_1, x_2) & \text{if } 2 \leq k \leq n-2 \\ t(ux_1, ux_2) & \text{if } k = 1, n-1 \end{array} \right\} \pmod{N} \)

Therefore,

\[ p \mu \Gamma = [x_1, x_2]_N, \]  

(76)

so we get

\[ 1_{p \mu}(x_1, x_2) = \pi(p) \pmod{\Gamma(L, \mu)} = \left\{ \begin{array}{ll} [a]_m[b]_{nm^{-1}} & \text{if } 2 \leq k \leq n-2 \\ [b]_{nm^{-1}} & \text{if } k = 1 \\ [a]_m & \text{if } k = n-1 \end{array} \right\} \]

\[ = \left\{ \begin{array}{ll} [ux_1]_{nm^{-1}} & \text{if } 2 \leq k \leq n-2 \\ [ux_2]_{nm^{-1}} & \text{if } k = 1 \\ [ux_1]_m & \text{if } k = n \end{array} \right\} \]  

(77)
Using this result, one can compute the set $l_{P,\mu}^{-1}(\ell)$ for any $\ell \in \mathbb{L}/\mathbb{L}(L,\mu)$. We give the answer for $SL_3$.

(4.3) Corollary: \[ l_{P,\mu}^{-1}[a]_{Nm^2} = \{ [x]_N | x_1 \equiv \pm 1 \pmod{Nm^2,m}, x_2 \equiv ma \pmod{N} \} \]

Corollary:\[ l_{P,\mu}^{-1}[a]_m = \{ [x]_N | (x_2,N)=m, m^2x_2 \equiv \pm 1 \pmod{Nm^2,m}, x_1 \equiv a \pmod{m} \}. \]

§5. $n \leq 4$

In this section, we assume $n \leq 4$, and $A$ is a finite-dimensional $kG$-module, and $\text{char } k$ is zero or $> n+1$. We can appeal to results of Ash and Soule which determine the top cohomology of $\mathbb{L}$ precisely.

(5.1) Proposition: For $n \leq 4$ the injection $H_1(\mathbb{L},A) \hookrightarrow W(\mathbb{L},A)$ is also surjective.

Proof: (sketch) In [1], Ash constructs an isomorphism $\Phi: W(\mathbb{L},\mathbb{C}) \rightarrow H_1(\mathbb{L},\mathbb{C})$. As we shall see, his proof applies to the case of nontrivial coefficients, and the assertion of the proposition follows by counting dimensions.

First, recall the definition of the "Well-Rounded Retract" (see [1]). This is the subset $Y$ of $SO_n(\mathbb{R}) \setminus SL_n(\mathbb{R})$ consisting of those quadratic forms $Q$ such that $\min\{Q(x)|x \in \mathbb{Z}^n \setminus 0\}=1$ and the vectors attaining this minimum span $\mathbb{R}^n$. For $E \subseteq \mathbb{Z}^n$, let $\sigma(E)=\{Q \in Y|Q(x)=1 \Leftrightarrow x \in E\}$. The $\sigma(E)$'s give a $G$-cellulation of $Y$. We have $\dim Y = \frac{1}{2}n(n-1) = v$, and for $n \leq 4$ there is a unique $G$-orbit of top dimensional cells, one of which is $\sigma_0 = \sigma(\pm e_1, \ldots, \pm e_n)$. Also, $S=SO_n(\mathbb{Z})=\text{Stab}_G\sigma_0$ and the orientation character of $S$ on $\sigma_0$ is given by the sign character $\varepsilon$. For $2 \leq r \leq n$, let $\sigma_r = \sigma(\pm e_1, \ldots, \pm e_n, \pm(e_1+\ldots+e_r))$. 
Then $C_{n-1}Y = \bigoplus_{2 \leq r \leq n} \sigma_r \cdot \mathbb{Z}G$. Let $p_2: C_{n-1}Y \otimes \mathbb{R}A \to \sigma_2 \cdot \mathbb{Z}G \otimes \mathbb{R}A$ be the projection onto the $r=2$ summand.

If $f \in J(\Gamma,A)$ and $f(sg) = \varepsilon(s)f(g)$ for all $s \in S, g \in G$, set

$$\Phi f = \sum_{s \in S} \sigma_0 g \otimes f(g) \in C_{n}Y \otimes \mathbb{R}A. \quad (78)$$

The key point is that $p_2 \Phi f = 0 \iff \Phi f = 0$. This follows from the fact ([1] Prop. 4.6) that if $a \in \mathbb{C}_{N}Y$ and support$(\partial a) \cap \sigma_2 \cdot G = \emptyset$, then $\partial a = 0$. It remains to determine $\ker p_2 \Phi$. We have

$$\Phi f = \partial \sum_{s \in S} \sigma_0 g \otimes f(g) = \sum_{s \in S} \sigma_0 \otimes (\sigma_0 g \otimes f(g)). \quad (79)$$

Now $\sigma_2$ appears with coefficient one in $\sigma_0, \sigma_0 h, \text{ and } \sigma_0 h^2$, with coefficient zero in all other translates of $\sigma_0$. Thus, the coefficient of $\sigma_2$ in $\Phi f$ is $f(1) + f(h) + f(h^2)$, and the coefficient of $\sigma_2 \cdot g$ in $\Phi f$ is $f(g) + f(hg) + f(h^2g)$. It follows that $\ker p_2 \Phi W(\Gamma,A)$. Finally, it is easy to see that $\Phi: W(\Gamma,A) \to \ker \partial = H_v(\Gamma,A)$ is surjective. \[\square\]

(5.2) \textbf{Corollary:} The surjection $\eta_r: K(\Gamma,A) \to H^r(\Gamma,A)$ is also injective for $n \leq 4$.

If we restrict to a cusp with $[L,L]$ simple, there is an element $h_L \in L$ which plays the same role in the definition of $W(\Gamma'(L,\mu),A_{\mu})$ as $h$ does in $W(\Gamma',A)$ (see §2).

(5.3) \textbf{Corollary:} The map $p_{P_{\mu}}$ of (2.1) induces a map

$$p_{P_{\mu}}: K(\Gamma,A) \to K(\Gamma'(L,\mu),A_{\mu}) \text{ with } \ker p_{P_{\mu}} = \ker r(p_{\mu,A}).$$

If $\mu = k$ with trivial $\Gamma$-action, then by the identification in (1.4), we have

$$p_{P_{\mu}}: W(\Gamma,k) \to W(\Gamma'(L,\mu),k)$$
given by
\[ p_{\mu}(f)(l) = \sum_{g \in L_\mu^{-1}(l)} f(g), \]  

sum over \( L_\mu^{-1}(l) \), and \( p_{\mu} \approx \ker r_{\mu} \).

**Proof:** This follows from (2.1), (2.2) and (5.2).

We can now prove the main result in this chapter.

\((5.4)\) **Theorem:** \( H^3_{\text{ker}}(\Gamma_0(3,N), \mathbb{C}) \) is isomorphic to the vector space of all functions \( f: \mathbb{P}^2(\mathbb{Z}/N) \rightarrow \mathbb{C} \) satisfying the following conditions:

i) \( f(sx) = \varepsilon(s)f(x) \) for all \( s \in S_0(\mathbb{Z}), x \in \mathbb{P}^2(\mathbb{Z}/N) \).

ii) \( f(x) + f(hx) + f(h^2x) = 0 \) for all \( x \in \mathbb{P}^2(\mathbb{Z}/N) \).

iii) For each divisor \( m \) of \( N \), each \( (c,d) \in (\mathbb{Z}/m)^2 \) and each \( (a,b) \in (\mathbb{Z}/Nm^{-1})^2 \), we have

\[ \sum_{x \in S(1,m,a,b)} f(x) = \sum_{x \in S(2,m,c,d)} f(x) = 0, \]  

where

\[ S(1,m,a,b) = \{(x_1,x_2,x_3) \in \mathbb{P}^2(\mathbb{Z}/N) \mid (x_2,x_3,N) = m, x_1 = \pm 1 \mod (Nm^{-1},m), x_2m^{-1} = a \mod Nm^{-1}, x_3m^{-1} = b \mod Nm^{-1}\}, \]

\[ S(2,m,c,d) = \{(x_1,x_2,x_3) \in \mathbb{P}^2(\mathbb{Z}/N) \mid (x_3,N) = m, x_3m^{-1} = \pm 1 \mod (Nm^{-1},m), x_1 = c \mod m, x_2 = d \mod m\}. \]

**Remark:** We are viewing elements of \( \mathbb{P}^2(\mathbb{Z}/N) \) as represented by column vectors. Also, if \( g \in SL_3(\mathbb{Z}), x \in \mathbb{P}^2(\mathbb{Z}/N) \), then \( gx \) denotes the natural action of \( SL_3(\mathbb{Z}) \) on \( \mathbb{P}^2(\mathbb{Z}/N) \).

**Proof:** A \( \Gamma_0(3,N) \)-invariant function on \( SL_3(\mathbb{Z}) \) is a function on \( \mathbb{P}^2(\mathbb{Z}/N) \). Conditions i) and ii) say that \( f \in W(\Gamma_0(3,N), \mathbb{C}) \). By (4.3) and (5.3), condition iii) says the restriction of \( f \) to each of the \( \Gamma_0(3,N) \)-inequivalent maximal parabolics is zero. Arguing as in [20], we get that
$H^3_{\text{cusp}}(\Gamma_0(3,N), \mathbb{C}) = \bigcap Q \ker \text{rst}: H^3(\Gamma_0(3,N), \mathbb{C}) \to H^3(\Gamma_0(3,N) \cap Q, \mathbb{C})$

where $Q$ runs over the $\Gamma$-conjugacy classes of maximal $Q$-parabolic subgroups of $\text{SL}_3$. The result now follows. □
§1. The Cover and Its Restriction Maps

We begin with some general remarks, most of which are well-known. See [8], [20], [21], [23]. Let $H$ be a connected $\mathcal{Q}$-group (not necessarily reductive), $\Delta$ a torsion-free arithmetic subgroup of $H$, $K$ a maximal compact subgroup of $H(\mathbb{R})$. Let $e_H$ be the Borel-Serre compactification of the $K(\Delta,1)$ space $K\backslash^0 H(\mathbb{R})/\Delta$, where $^0H:=\cap\ker(\chi^2)$, $\chi$ running over the $\mathcal{Q}$-characters of $H$. This diverges from the customary notation $\text{cl}[e'(H)]$ for the sake of visual convenience.

Let $\mathcal{P}(H,\Delta)$ be a set of representatives of $\Delta$-conjugacy classes of $\mathcal{Q}$-parabolic subgroups of $H$. The boundary of $e_H$ equals $\bigcup_{P\in \mathcal{P}(H,\Delta)} e_P$. For any $\Delta$ module $A$, viewed as a coefficient system on $e_H$, there is a spectral sequence [16] abutting to $H^*(\mathfrak{e}_H, A)$ with $E_1$ term

$$E_1^{pq}=\oplus H^q(e_p, A)$$

where the sum is over $\{P\in \mathcal{P}(H)\mid \text{rank}_\mathcal{Q}[P/R_P]=\text{rank}_\mathcal{Q}[H/R_H]-p\}$. The differential $d_i^{pq}: E_i^{pq} \to E_i^{p+1,q}$ is given on each summand by...
\[ \oplus \oplus r_{PQ} : H^q(e_P, A) \to \oplus H^q(e_Q, A) \]

where both direct sums are over the maximal elements in \( \mathcal{P}(P, \Delta \cap P) \) and \( r_{PQ} \) is the restriction map. If \( E^1_{PQ} = 0 \) for \( p > 1 \) then the spectral sequence degenerates and we get an exact sequence

\[ 0 \to \ker d_1^{1n} \to H^n(\mathfrak{e}e_H, A) \to \coker d_1^{1n-1} \to 0. \]

This happens, for example, if rank \( Q H \leq 2 \) or if \( \Delta \) has torsion and \( A \) is chosen appropriately.

We shall look more closely at the \( r_{PQ} \)'s. Let \( U \) be the unipotent radical of \( H \) and \( L \) a Levi subgroup with \( \pi : H \to L \) the canonical projection. Then \( \pi(\Delta) \) is an arithmetic subgroup of \( L \) and \( \pi \) induces a locally trivial fibration \( e_H \to e_L \) with typical fiber \( e_U \). There is another spectral sequence \([22]\) with \( E^2 \) term

\[ E^2_{PQ} = H^P(e_L, H^Q(e_U, A)) \Rightarrow H^{P+Q}(e_P, A). \tag{83} \]

We shall always assume that \( A \) is a finite dimensional complex \( H(\mathbb{C}) \)-module. The Lie algebra \( \mathfrak{u} \) of \( U(\mathbb{R}) \) induces vector fields on \( e_U \) (a manifold with empty boundary), and a standard averaging procedure gives an isomorphism \( H^*(e_U, A) \cong H^*(\mathfrak{u}, A) \) between de Rham and Lie algebra cohomologies ([9] Chap. VII).

Let \( Z(\ell) \) be the center of the universal enveloping algebra of \( L(\mathbb{C}) \). The diagonalizable action of \( Z(\ell) \) on \( H^1(\mathfrak{u}, A) \) induces an action on \( E^2_{PQ} \) which commutes with the differentials. By Kostant's theorem, extended by Casselman and Osborne ([19], [14]), the space \( H^1(\mathfrak{u}, A) \) affords a single eigencharacter for \( Z(\ell) \) and

\[ \text{Hom}_{Z(\ell)}( H^1(\mathfrak{u}, A), H^1(\mathfrak{u}, A) ) = 0 \text{ if } i \neq j. \tag{84} \]
We get the following result, which seems to be first noticed by Harder:

**(1.1) Lemma:** The spectral sequence $E_{pq}^*$ degenerates at the $E_2$ term. Moreover, there exist unique subspaces $V_{q_i}(P) \subseteq H^q(e_P, A), 0 \leq i \leq \dim u$ with the properties

i) $V_{q_i}(P) \cong H^i(e_L, H^{q-1}(u, A))$

ii) $H^q(e_P, A) = \bigoplus V_{q_i}(P)$.

Indeed, $V_{q_i}$ is the direct sum of those eigenspaces for $Z(\ell)$ in $H^q(e_P, A)$ with eigencharacter equal to a $Z(\ell)$ eigencharacter on $H^{q-1}(u, A)$.

Now let $Q \supseteq P = LU$ be a containment of $Q$-parabolic subgroups of a reductive $Q$-group $G$, and let $\Gamma$ be an arithmetic subgroup of $G(Q)$, so $\Gamma \cap P$ and $\Gamma \cap Q$ are arithmetic subgroups of $P$ and $Q$ respectively. We have fibrations and a commutative diagram

$$
\begin{array}{ccc}
eq U & \rightarrow & e_P \\
\downarrow & & \downarrow \\
eq U & \rightarrow & e_Q
\end{array}
$$

Hence there is yet another spectral sequence

$$F_{2, pq}^* = H^p(e_{\pi(Q)}, H^q(u, A)) \Rightarrow H^{p+q}(e_Q, A). \quad (85)$$

The algebra $Z(\ell)$ also acts on $H^p(e_{\pi(Q)}, H^q(u, A))$, and commutes with $r_{pq}$. As above, this proves

**(1.2) Lemma:** The spectral sequence $F_{pq}^*$ degenerates at the $F_2$ term and there exist unique subspaces $W_{q_i}(P, Q) \subseteq H^q(e_Q, A), 0 \leq i \leq \dim u,$
such that
i) $W_q(P,Q) \simeq H^1(e_{\pi Q}, H^{-1}(\mathfrak{u}, A))$

ii) $H^q(e_Q, A) = \oplus W_q(P, Q)$

iii) $r_{PQ}: H^q(e_P, A) \to H^q(e_Q, A)$ maps $V_q(P)$ to $W_q(P,Q)$ and makes the following diagram commute:

$$
\begin{array}{ccc}
V_q(P) & \to & H^1(e_{L}, H^{-1}(\mathfrak{u}, A)) \\
\downarrow & & \downarrow \\
W_q(P,Q) & \to & H^1(e_{\pi L}, H^{-1}(\mathfrak{u}, A))
\end{array}
$$

From now on, we assume that $\Delta$ is a torsion-free normal subgroup of $H(\mathbb{Z})$. We also suppose that $H$ is split over $\mathbb{Q}$. Then $H(\mathbb{Z})$ is transitive on the $H(\mathbb{Q})$ conjugacy classes of $\mathbb{Q}$-parabolics. These correspond to the subsets $J$ of $S_H$, a base of the $\mathbb{Q}$-root system of $H/R_J H$. Denote this correspondence by $J \mapsto P(J)$. We can write $\mathfrak{e}_H$ as the union of submanifolds-with-boundary $Y_J$, $J \subseteq S_H$, as follows. Set $\mathfrak{R} = \Delta \setminus H(\mathbb{Z})$, and note that $\mathfrak{R}$ acts on $\mathfrak{e}_H$ on the right. Put, for $J \subseteq S_H$,

$$
Y_J = e_{P(J)} \times_{P(J)} \mathfrak{R}.
$$

Then $Y_J \cap Y_I = Y_{J \cap I}$ and $\mathfrak{e} Y_J = U_{I \subseteq J} Y_I$. Moreover, $\mathfrak{e}_H$ is obtained as a gluing of the $Y_J$'s along their common boundaries. Thus $\mathfrak{e}_I^{pq}$ may be written

$$
\mathfrak{e}_I^{pq} = \bigoplus_{|S_H - J| = p+1} H_q(Y_J, A).
$$

We have

$$
H^*(Y_J, A) \simeq \text{ind}_{P(J)} \mathfrak{R} [H^*(e_{P(J)}, A)] \text{ as } \mathfrak{R} \text{ modules.}
$$
If $I \subseteq J$, then the restriction map $r_{JI}: H^\ast(Y_J,A) \rightarrow H^\ast(Y_I,A)$ is interpreted as a map on induced modules as follows: By Frobenius reciprocity the restriction

$$r_{[J,I]}: H^\ast(e_{[J,I]},A) \rightarrow H^\ast(e_{[I]},A)$$

induces a map

$$r_{[J,I]}: H^\ast(e_{[J,I]},A) \rightarrow \text{ind}_{[I]} r_{[J,I]} H^\ast(e_{[I]},A).$$

Now apply the functor $\text{ind}_{[J]} H$, to get $r_{JI} = \text{ind}_{[J]} H [r_{[J,I],P(I)}].$

§2. The Split Groups of Rank 2

Let $G$ be one of $S_{3}, Sp_{4}, G_{2}$, given its $\mathbb{Q}$-structure as a Chevalley group. Let $\Gamma$ be the full congruence subgroup of level $N \geq 3$. Fix a maximal $\mathbb{Q}$-split torus $T$ and simple roots $\{\alpha_{1}, \alpha_{2}\}$ ($\alpha_{1}$ is short for $Sp_{4}$ and $G_{2}$). Let $\{\alpha_{1}^{\vee}, \alpha_{2}^{\vee}\}$ and $\{\lambda_{1}, \lambda_{2}\}$ be the corresponding coroots and fundamental dominant weights, respectively. Let $P_{0}$ denote the corresponding minimal $\mathbb{Q}$-parabolic subgroup. For $r=1,2$ ($r \in \{1,2\}$ is always to be read mod 2), let $s_{r}$ be the corresponding Weyl group reflection, and let $P_{r} = P_{0} \cup P_{0}s_{r}P_{0} = L_{r}U_{r}$ be the standard maximal $\mathbb{Q}$-parabolic subgroups, with $L_{r} \supset T$. Let $\pi_{r}: P_{r} \rightarrow L_{r}$ be the projection. Throughout, it is understood that the spaces $e_{G}, e_{P_{r}}, e_{L_{r}}, e_{U_{r}},$ are meant with respect to the arithmetic groups $\Gamma, \Gamma \cap P_{r}, \pi_{r}(\Gamma \cap P_{r}), \Gamma \cap U_{r},$ respectively. We set
\[ \Gamma(L_r) := \pi_r(\Gamma \cap P_r) \cong \Gamma(2,N), \]
the full congruence subgroup of degree two and level \( N \).

For a \( \mathbb{Q} \)-group \( H \), let \( H^0 \) be the identity component of \( H \). Now \( T(Z)/T(Z) \cap 0L_r 0(Z) \cong L_r(Z)/0L_r 0(Z) \cong \mathbb{Z}/2 \) and a nontrivial representative is given by \( \alpha_{r+1}^\vee(-1) \). Thus any rational character \( \lambda \) of \( T \) with \( \langle \lambda, \alpha_r^\vee \rangle \in 2\mathbb{Z} \) is trivial on \( T(Z)/0L_r 0(Z) \) so may be viewed as a character of \( L_r(Z)/0L_r 0(Z) \) and hence as a character of \( \tilde{P}_r \), since \( \pi_r(\Gamma) \subseteq 0L_r 0(Z) \). For \( r=1,2 \), set

\[ \epsilon_r = \lambda_{r+1} \text{ viewed as a character of } \tilde{P}_r, \]
\[ \delta_{r+1} = \lambda_{r+1} \text{ viewed as a character of } \tilde{P}_0. \]

We also need to observe that \( L_r(Z) \cong SL_2(Z) \times (\pm 1) \) iff \( \alpha_{r+1}^\vee(\pm 1) \)
centralizes \( L_r \) iff \( \langle \alpha_r, \alpha_{r+1}^\vee \rangle \) is even. Otherwise \( L_r(Z) \cong GL_2(Z) \).

Hence \( L_r(Z) \cong GL_2(Z) \) in all cases except \( G=Sp_4 \) with \( r=2 \), for then \( \langle \alpha_2, \alpha_1^\vee \rangle = 2 \). Also \( 0L_r(R) \cong SL_2^\pm(R) \) in all cases except this last, when \( 0L_r(R) \cong SL_2(R) \times (\pm 1) \). Note that for each \( k \in \mathbb{N} \), there are exactly two distinct \( k+1 \) dimensional complex representations \( \mathcal{O}_{k+1} \) of \( 0L_r(R) \), distinguished by \( \langle \lambda, \alpha_r^\vee(-1) \rangle = \pm 1 \). Let \( A \) be the finite dimensional irreducible \( G(C) \) module with highest weight \( \lambda = c_1 \lambda_1 + c_2 \lambda_2 \).

The goal of this section is to compute the \( G \) module structure of \( H^*(G, A) \). We state the result.

(2.1) **Theorem:** Let \( m= \dim \ De_G \). Then for \( 0 \leq q \leq [m/2] \) we have

\[ H^m(\mathcal{O} G, A) \cong H^0(\mathcal{O} G, A) \cong \mathbb{C} \text{ if } c_1 = c_2 = 0, =0 \]

otherwise,

\[ H^{m-q}(\mathcal{O} G, A^*) \cong H^q(\mathcal{O} G, A) \cong \bigoplus_{r=1,2} \text{ind}_{\mathbb{P}_r^\ast} [H^1_{\text{cusp}}(\Gamma(2,N), \mathcal{O}_{k,r})] \otimes D(q) \]
Here $k(q,r)$ is a pair $(k,\pm)$. $D(q)$ and $k(q,r)$ are given as follows.

\[
D(1) = \begin{cases} \text{coker} \left( \text{ind}_{Fr} \delta_c \rightarrow \text{ind}_{F_1} \delta_c \right) & \text{if } c_r = 0 \neq c_{r+1} \\ H^1(I_0, \mathbb{C}) & \text{if } c_1 = 0 = c_2 \end{cases}
\]

(Recall that $I_0$ is the Tits building of $\Phi$-parabolic subgroups of $G$).

\[
D(2) = \begin{cases} \text{ind}_{P_2} \delta_c \epsilon_c \theta^c & (Sp_4, c_2 = 0 \neq c_1) \\ \text{ind}_{P_1} \delta_c \epsilon_c \oplus \text{ind}_{P_2} \delta_c [1_{Fr}] & (Sp_4, c_1 = 0 = c_2) \\ \oplus_{c_r} \text{ind}_{P_r} \delta_c [1_{Fr}] & \text{all other cases} \end{cases}
\]

$D(3) = 0$ (for $G_2$ only)

\[
\begin{array}{ccc}
k & \pm \\
c_r & c_{r+1} & (all) \quad q = 1, r = 1, 2 \\
c_1 + c_2 + 1 & c_{r+1} & (Sl_3) \quad q = 2, r = 1 \\
c_1 + 2c_2 + 2 & c_2 & (Sp_4) \quad q = 2, r = 1 \\
k(q,r)= & c_1 + 3c_2 + 3 & c_2 \quad (G_2) \quad q = 2, r = 1 \quad \text{(cont.)} \\
c_1 + c_2 + 1 & c_1 & (all) \quad q = 2, r = 2 \\
2c_1 + 3c_2 + 4 & c_1 + c_2 + 1 & (G_2) \quad q = 3, r = 1 \\
c_1 + 2c_2 + 2 & c_1 + c_2 + 1 & (G_2) \quad q = 3, r = 2 \\
\end{array}
\]

The sign $\pm$ is determined by the parity of the integer which appears ($even \Rightarrow +$).
In the notation of §1, we set $V_{q_i}(r) := V_{q_i}(P_r)$, and $W_{q_i}(r) := W_{q_i}(P_r, P_0)$ for $r=1,2$. Also set $\mu = \dim U_r = 2,3,5$ for $G = SL_3, Sp_4, G_2$ (independent of $r$).

(2.2) Lemma: For $r=1,2$, we have $W_{q_0}(r) = W_{q_1}(r+1)$ for $1 \leq q \leq \mu$. We emphasize that this is equality, not just isomorphism.

Proof: We study the $T$-weights occurring in $H^q(e_{P_0}, \mathcal{A})$. First, set $W^r = \{ w \in W | w^1 \alpha_r > 0 \}$. These are given as follows:

$SL_3$: $W^1 = \{1, s_2, s_2s_1\}$, $Sp_4$: $W^1 = \{1, s_2, s_2s_1s_2\}$,
$G_2$: $W^1 = \{1, s_2, s_2s_1, s_2s_1s_2\}$. The $W^2$'s are obtained in all cases by interchanging $s_1$ and $s_2$.

Let $w_q(r)$ be the unique element of $W^r$ with length $q$. Observe that

$$s_r w_{q_1}(r) = w_q(r+1).$$

By Kostant's theorem [19], the $T$-weights in $H^q(e_{P_0}, \mathcal{A})$ are $w_q(1) \cdot \lambda$ and $w_q(2) \cdot \lambda$, where $w \cdot \lambda = w(\lambda + \rho) - \rho$ and $\rho = \lambda_1 + \lambda_2$. Both weights have multiplicity one. On the other hand, by §1 and Kostant's theorem again, the $Z(L_r)$-weights in $H^q(e_{P_0}, \mathcal{A})$ are $w_q(r) \cdot \lambda$ and $s_r w_{q_1}(r) \cdot \lambda = w_q(r+1) \cdot \lambda$. These weights also have multiplicity one, and the weight spaces are $W_{q_0}(r)$ and $W_{q_1}(r)$, respectively. Hence $W_{q_0}(r)$ is also the $T$-weight space $w_q(r) \cdot \lambda$ and $W_{q_1}(r)$ is the $T$-weight space $w_q(r+1) \cdot \lambda$. The lemma follows. □

Set $Y_r = Y_{\{\alpha_r\}}$ for $r=1,2$, $Y_0 = Y_{\{\}}$. Since the cohomological dimension of $\Gamma(L_r)$ is one, we have

$$H^q(Y_r, \mathcal{A}) = \bigoplus_{i=0,1} \text{ind}_{P_r}^G V_{q_i}(r)$$

and
Let $f(q,1,r): \text{ind}_P \delta V_q(r) \to \text{ind}_P \delta W_q(r)$ be the restriction of the map $H^q(Y, A) \to H^q(Y_0, A)$ (see §1).

The lemmas (1.1), (1.2) and (2.2) combine to give the following

(2.3) Proposition: For $1 \leq q \leq \mu$, the differential $d_1^{0q} \mathbb{E}_1^{0q} \to \mathbb{E}_1^{1q}$ is described by the following diagram:

\[
\begin{array}{ccc}
\text{ind}_P \delta V_q(1) & \xrightarrow{f(q,1,1)} & \text{ind}_P \delta W_q(1) \\
H^q(Y_1, A) = & \oplus & \oplus \\
\text{ind}_P \delta V_q(1) & \xrightarrow{f(q,0,1)} & \text{ind}_P \delta W_q(1) \\
& = H^q(Y_0, A) \\
\end{array}
\]

In particular, $(\text{co})\ker d_1^{0q} = (\text{co})\ker [f(q,0,1) + f(q,1,2)] \oplus (\text{co})\ker [f(q,1,1) + f(q,0,2)]$

For $q=0$, the map $d_1^{00}: \mathbb{E}_1^{00} \to \mathbb{E}_1^{10}$ is given by
\[ H^0(Y_1, A) = \text{ind}_{g_1}^{\ast} V_{00}(1) \xrightarrow{f(0,0,1)} \text{ind}_{g_0}^{\ast} W_{00}(1) \]

\[ H^0(Y_2, A) = \text{ind}_{g_2}^{\ast} V_{00}(2) \xrightarrow{f(0,0,2)} \text{ind}_{g_0}^{\ast} W_{00}(2) \]

Each map \( f(q,i,r) \) is induced from a map \( r(q,i,r): V_{q,i}(r) \to \text{ind}_{g_0}^{\pi} \) \( W_{q,i}(r) \) which may be identified with the restriction map

\[ H^1(\omega_{1}, H^q(-1(u_r,A)) \to H^1(\omega_{2}, H^q(-1(u_r,A))). \]

(2.4) Lemma: The following is a description of the \( \pi_r \)-module structures of the relevant (co)ker \( r(q,i,r) \)'s.

\[ \ker r(q,0,r) = 0 \]
\[ \ker r(q,1,r) \cong H^1_{\text{ad}}(\omega_{1}, H^q(-1(u_r,A))) \]
\[ \text{coker } r(0,0,r) \cong \begin{cases} \text{ind}_{g_0}^{\pi} \lambda & c_1 \neq 0 \neq c_2 \\ \text{coker}(\varepsilon_r c_{\text{tr}} \to \text{ind}_{g_0}^{\pi} [\delta_r c_{\text{tr}}]) & c_r = 0 \neq c_{r+1} \end{cases} \]
\[ \text{coker } r(q,1,r) \cong \begin{cases} 0 & q=0, \text{ or } 1 \leq q \leq \mu, \text{ or } q=1 \text{ with } c_r \neq 0 \\ \varepsilon_r c_{\text{tr}} & q=1, G=Sp_4, r=2, c_2=0 \\ \varepsilon_r c_{\text{tr}} + 1 & q=1 \text{ and } c_r = 0 \text{ in the remaining cases} \end{cases} \]

Proof: The assertion about the kernels is clear. We compute \( \text{coker } r(0,0,r) \). This is \( \text{coker} [ H^0(\omega_{1}, H^0(u_r,A)) \to H^0(\omega_{2}, H^0(u_r,A)) ] \). A \( \pi_r \)-modules,

\[ H^0(\omega_{1}, H^0(u_r,A)) \cong \lambda \text{ if } c_r = 0, \text{ and is zero otherwise, and} \]
\[ H^0(\omega_{2}, H^0(u_r,A)) \cong \text{ind}_{g_0}^{\pi} \lambda. \] We next determine \( \text{coker } r(q,1,r) \).
This is
\[ \text{coker } [ \pi^1(\ell_r, \pi^{q-1}(\ell_r, \mathbf{A})) \to \pi^1(\ell_r, \pi^{q-1}(\ell_r, \mathbf{A}))]. \]
By Poincare duality (or an elementary argument) this coker is zero if \( \pi^{q-1}(\ell_r, \mathbf{A}) \) is not the trivial \( \Gamma(L_r) \) module. We are thus reduced to the case \( q=1, c_r=0 \) (we are ignoring \( q=\mu+1 \)), and we have
\[ \text{coker } r(1,1,r) = \lambda \otimes \text{coker } [ \pi^1(\ell_r, \mathbf{C}) \to \pi^1(\ell_r, \mathbf{C})]. \]

It is easy to see that when \( L_r(\mathbb{Z}) \cong \text{GL}_2(\mathbb{Z}) \), the element \( [ \ ] \) acts by \(-1\) on this latter cokernel. When \( L_r(\mathbb{Z}) \cong \text{SL}_2(\mathbb{Z}) \times \langle \pm 1 \rangle \), the element \(-1\) acts trivially on the latter cokernel. Hence (see the remarks prior to (2.1))
\[ \text{coker } r(1,1,r) = \lambda \otimes \epsilon_r, \text{ except for } G=\text{Sp}_4 \text{ and } r=2 \text{ when } \text{coker } r(1,1,2) = \lambda. \] This finishes the computation. \( \square \)

\textbf{(2.5) Lemma:} For \( 0 \leq q \leq \mu \), \((\text{co})\text{ker } d_{ij0}^{0q}\) is given as follows

\begin{enumerate}
    \item \( \text{ind}_{P_0}^{\mathbb{B}} \lambda \) if \( c_i \neq 0 = c_2 \)
    \item \( \text{coker } d_{ij0}^{00} \approx \text{coker } (\text{ind}_{P_r}^{\mathbb{B}} \epsilon_r c_i = \rightarrow \text{ind}_{P_0}^{\mathbb{B}} [\delta_r c_{i+1}]) \) if \( c_r = 0 \neq c_{r+1} \)
    \item \( H^1(\mathbb{Z}/\mathbb{Z}, \mathbb{C}) \) if \( c_1 = 0 = c_2 \)
    \item \( \ker d_{ij0}^{00} = \mathbb{C} \) if \( c_1 = c_2 = 0 \), zero otherwise.
    \item \((\text{co})\text{ker } d_{ij0}^{0q} \approx \bigoplus_{r=1}^{q-1} \text{ind}_{P_r}^{\mathbb{B}} (\text{co})\text{ker } r(q,1,r) \) if \( 1 \leq q < \mu \).
    \item \( \text{co} \text{ker } d_{ij0}^{0\mu} = 0 \)
    \item \( \ker d_{ij0}^{0\mu} \approx \bigoplus_{r=1}^{\mu} \text{ind}_{P_r}^{\mathbb{B}} \ker r(\mu,1,r) \oplus \bigoplus_{r=2}^{\mu} \text{ind}_{P_r}^{\mathbb{B}} V_{i0}(r), \) and \( V_{i0}(r) = \epsilon_r c_i \) if \( G=\text{SL}_3 \text{ and } c_{i+1} = 0 \),
        \approx \epsilon_r c_i \) if \( G=\text{Sp}_4, G_2 \text{ and } c_r = 0 \), and is zero otherwise.
\end{enumerate}
Proof: The first two isomorphisms in i) follow from (2.3), (2.4) and the fact that if \( c_{r+1} \neq 0 \) then \( V_{00}(r+1) = 0 \).

Assume \( c_1 = c_2 = 0 \). Then \( H^0(u_r, A) \) is the trivial \( T(\mathbb{Z}) \) module. Hence \( d_{i0} \) may be identified with the natural map
\[
\bigoplus_{r=1,2} \text{ind}_{P_r} \overline{b} [1_{P_r}] \to \text{ind}_{P_2} \overline{b} [1_{P_2}].
\]

On the other hand, \( G(\mathbb{Z}) \) is transitive on the \( G(\mathbb{Q}) \)-conjugacy classes of \( \mathbb{Q} \)-parabolic subgroups, so the simplicial cochain complex which computes \( H^2(\mathbb{G}/\Gamma, \mathbb{C}) \) may be identified with the sequence
\[
0 \to 1_5 \to \bigoplus_{r=1,2} \text{ind}_{P_r} \overline{b} [1_{P_r}] \to \text{ind}_{P_2} \overline{b} [1_{P_2}].
\]
I claim that this sequence is exact. View \( \text{ind}_{P_r} \overline{b} [1_{P_r}] \) as the space of \( \overline{P}_r \)-invariant functions on \( G \). If
\[
f \oplus g \in \ker \left( \bigoplus_{r=1,2} \text{ind}_{P_r} \overline{b} [1_{P_r}] \to \text{ind}_{P_2} \overline{b} [1_{P_2}] \right)
\]
then \( f = g \) and is invariant under \( P_1 \) and \( P_2 \). By [28], \( G(\mathbb{Z}/N) \) is generated by root groups corresponding to \( T \). Since each root group is contained in \( \overline{P}_1 \cup \overline{P}_2 \), we have that \( f \) is constant. The claim follows, and we're done with i).

Recall that \( V_{q_1}(r) \cong H^1(\ell_{r^1}, H^{q_1}(u_r, A)) \). Hence for \( 1 \leq q < \mu, r = 1, 2 \), we have \( V_{q_0}(r) = V_{01}(r) = 0 \), so ii) and iii) follow from (2.3).

For \( q = \mu \), we have coker \( f(\mu,1,r) \cong \text{ind}_{P_r} \overline{b} \ker r(\mu,1,r) = 0 \) by (2.4). It follows that coker \( d_{i0} \) is zero, so iv) holds.

Since the group algebra \( \mathbb{C}G \) is semisimple, if \( f_i: A_1 \to B, i = 1,2 \), are \( \mathbb{C}G \) module maps with \( f_1 \) surjective, we have
\[
\ker[f_1-f_2; A_1 \oplus A_2 \to B] \cong \ker f_1 \oplus A_2. \]
This implies
\[
\ker d_{i0} \cong \bigoplus_{r=1,2} \text{ind}_{P_r} \overline{b} \ker r(\mu,1,r) \oplus [\bigoplus_{r=1,2} \text{ind}_{P_r} \overline{b} V_{\mu 0}(r)].
\]
Now
\[
V_{\mu 0}(r) \cong H^0(\ell_{r^1}, H^\mu(u_r, A)) \cong H^0(\ell_{r^1}, H^0(u_r, A^\ast)), \]
and
$H^0(u_r,A^*)$ has weight $c_2\lambda_1+c_1\lambda_2$ for $Sl_3$, $c_1\lambda_1+c_2\lambda_2$ for $Sp_4, G_2$. This gives the last statement of ν).

**Proof of theorem (2.1):** The space $\mathfrak{a}e_G$ is homeomorphic to compact orientable manifold, hence satisfies Poincare duality. Moreover, the $\bar{G}$ module $H^q(\mathfrak{a}e_G,A)$ is realizable over $\mathbb{R}$ since $A$ is. It follows that $H^q(\mathfrak{a}e_G,A) \cong H^{m-q}(\mathfrak{a}e_G,A^*)$ as $\bar{G}$ modules. It is only necessary therefore, to compute $H^q(\mathfrak{a}e_G,A)$ for $0 \leq q \leq \lfloor m/2 \rfloor$. We have

$$H^q(\mathfrak{a}e_G,A) \cong \ker d^q \oplus \mathrm{coker} d^{q-1}.$$ 

These kernels and cokernels are computed in (2.4) and (2.5). We need only remark that $H^{q-1}(u_r,A)$ is the $0_{L_r,0}(C)$ representation with high weight $\langle w_{q-1}(r), \lambda, \alpha_r \rangle \lambda_r$. Setting $k(q,r) = \langle w_{q-1}(r), \alpha_r \rangle$, and letting $\pm$ be the parity of $\langle w_{q-1}(r), \lambda, \alpha_{r+1} \rangle$, we get the table in (2.1). □

(2.6) **Corollary:** Let $\Gamma$ be the full congruence subgroup of $G_2(\mathbb{Z})$ of level $N$. Then $H^6(\Gamma,\mathbb{C}) \cong \bigoplus_{r=1,2} \mathrm{ind}_{\Gamma}^{\bar{G}}[H^1_{\alpha\beta}(\Gamma(2,N),\mathbb{C})] \oplus H^1(G/\Gamma,\mathbb{C})$.

**Proof:** By Kazhdans theorem on $H^1(\Gamma,\mathbb{Z})$ [18] and Poincare duality, the restriction map $H^6(e_G,\mathbb{C}) \rightarrow H^6(\mathfrak{a}e_G,\mathbb{C})$ is surjective. On the other hand, the vanishing theorems of Kumaresan-Vogan-Zuckerman [31] tell us that the discrete spectrum of $L^2(\Gamma\backslash G(\mathbb{R}))$ has zero $(g,K)$ cohomology in dimension 6 for $G_2$. It follows that $H^6(\Gamma\backslash G(\mathbb{R})/K)$ is
zero so that $H^6(\mathfrak{e}_G, \mathbb{C}) \to H^6(\mathfrak{a}_G, \mathbb{C})$ is injective. The result now follows from our computation of $H^6(\mathfrak{a}_G, \mathbb{C}) \cong H^1(\mathfrak{a}_G, \mathbb{C})$. □

The following result is required in chapter II.

(2.7) Corollary: For $\Gamma$ an arithmetic subgroup of $G = \text{SL}_3, \text{Sp}_4, \text{G}_2,$ and $A$ any finite dimensional $G(\mathbb{C})$ representation, the direct sum of restriction maps $H^q(\mathfrak{a}_G, A) \to \bigoplus H^q(e_P, A)$ where $P$ runs over the $\Gamma$-conjugacy classes of maximal $Q$-parabolic subgroups of $G$, is injective for $3 \leq q \leq \dim \mu + 1$.

Proof: By (2.4) and (2.5),

$$\text{coker } d_{1}^{0q-1} \cong \bigoplus_{r=1,2} \text{ind}_{P_r}^{G} \text{coker } r(q-1,1,r) = 0 \text{ for } 2 \leq q-1 \leq \mu.$$ Viewing the spectral sequence $E^{pq}$ as the Meyer-Vietoris sequence, we have an exact sequence

$$\bigoplus H^{q-1}(e_P) \to \bigoplus H^{q-1}(e_Q) \to \bigoplus H^q(e_Q) \to \bigoplus H^q(e_P)$$

where the $P$'s (resp. $Q$'s) run over the $\Gamma$-conjugacy classes of maximal (resp. minimal) $Q$-parabolic subgroups of $G$. This first map is $d_{1}^{0q-1}$, so is surjective. This implies the result. □
§1. Bruhat cells in $\mathbb{P}^{n-1}(\mathbb{Z}/N)$

Let $U$ be the group of unipotent upper triangular matrices with integral entries. We will determine the orbits of $U$ on $\mathbb{P}^{n-1}(\mathbb{Z}/N)$. As a consequence of this calculation we get the number of top dimensional cells in $\mathcal{G}\mathcal{G}/\Gamma_0(n,N)$. We also find, but cannot "explain", a symmetry among the Bruhat cells which is undetected when $N$ is prime. The case $n=2$ can be found in [24].

Throughout, $N$ and $n$ are fixed. Objects whose definition depends on $N$ or $n$ will not have these letters in their notation.

Definition:  
1) $\mathcal{B} = \{ \sigma = (\sigma_0, \ldots, \sigma_n) \in \mathbb{N}^{n+1} \mid 1=\sigma_n|\sigma_{n-1}|\ldots|\sigma_1|N=\sigma_0 \}$

2) for $\sigma \in \mathcal{B}$, $k=0,1,\ldots,n-1$, set
   
   $a_k(\sigma) = \sigma_k/\sigma_{k+1}$

   $c_k(\sigma) = \gcd\{a_0a_1\cdots a_i \cdot \cdots a_k \mid i=0,\ldots,k \} \text{ if } k>0$, $c_0(\sigma) = 1$

   $d_k(\sigma) = \gcd\{a_0\cdots a_{k-1}/c_{k-1}(\sigma),a_k \}$

3) $Y_k(\sigma) = Y(\sigma_1,\ldots,\sigma_{k+1})$ is a set of representatives in $\mathbb{Z}$ for the kernel of $\mod d_k(\sigma) : \mathbb{Z}^*/a_k(\sigma) \to \mathbb{Z}^*/d_k(\sigma)$.

Note that these definitions depend only on the first $k+1$ terms of $\sigma$.  

68
(1.1) **Proposition:** A set of representatives of the orbits of \( U \) on \( \mathbb{P}^{n-1}(\mathbb{Z}/N) \) is given by
\[
\mathcal{C} = \bigcup_{\sigma \in \mathbb{A}} \{ [x_1, x_2, \ldots, x_{n-1}]_N \mid y_k \in \gamma_k(\sigma) \}.
\]
Here \([\ldots]_N\) denotes the image in \( \mathbb{P}^{n-1}(\mathbb{Z}/N) \) of a primitive (column) vector in \( \mathbb{Z}^n \).

The proof will consist of a series of lemmas. For \( m \mid N \), set
\[
S(m) := \{ x \in \mathbb{Z}/N^\ast \mid x \equiv 1 (Nm^{-1}) \}.
\]

(1.2) **Lemma:** For any integers \( 0 \neq c \neq d \mid mN \), the sequence
\[
S(m) \to \mathbb{Z}/d^\ast \to \mathbb{Z}/s^\ast \to 1
\]
is exact, where \( s = (Nm^{-1}, dc^{-1}) \) and the maps are the natural ones.

**Proof:** Let \( \pi : \mathbb{Z}/N \to \mathbb{Z}/d \) be "mod d". Thus
\[
\pi S(m) = \{ x \in \mathbb{Z}/d^\ast \mid x \equiv 1 (Nm^{-1}) \text{ and } x \text{ has a lift in } \mathbb{Z} \text{ prime to } N \}.
\]
We first find \( y \in \mathbb{Z} \) satisfying \( y \equiv x(d), y \equiv 1(Nm^{-1}), (y,N)=1 \).

Let \( p = \Pi \{ \text{primes dividing } d \}, q = \Pi \{ \text{primes dividing } m \text{ but not } d \text{ or } Nm^{-1} \}, r = \{ \text{primes dividing } Nm^{-1} \} \).

Then if \( s \) is a prime dividing \( N \), \( s \) divides exactly one of \( p, q, r \). Choose \( e \in \mathbb{N} \) large enough so that \( dp^e, Nm^{-1} \mid re \). By the Chinese Remainder theorem, \( \exists y \in \mathbb{Z} \) such that \( y \equiv x(p^e), y \equiv 1(q^e), y \equiv 1(r^e) \). Recalling that \( (x,p^e)=1 \), we see that \( y \) has the required properties. This shows that
\[
\pi S(m) = \{ z \in \mathbb{Z}/d^\ast \mid x \equiv 1(Nm^{-1}) \}.
\]
In \( \mathbb{Z}/d \), \( (Nm^{-1}) = (s) \). Hence
\[
\pi S(m) = \{ z \in \mathbb{Z}/d^\ast \mid x \equiv 1(s) \}.
\]
Now, given \( x \in \mathbb{Z}/s^\ast \), the above argument with \( N, Nm^{-1}, d \) replaced by \( d, 1, s \) respectively, shows that \( \exists y \in \mathbb{Z} \) such that \( y \equiv x(s) \) and \( (y,d)=1 \). Hence \((\text{mod } s) : \mathbb{Z}/d^\ast \to \mathbb{Z}/s^\ast \) is surjective. □.
Let \( X(c,d) = \{ y \in \mathbb{Z} / d \mid (y,d) = c \} \). Then \( S(m) \) acts by multiplication on \( X(c,d) \) and \( \pi S(m) \) acts on \( X(1,dc^{-1}) = \mathbb{Z} / dc^{-1} \).

\[(1.3) \textbf{Lemma:} \] The composition \( (\mod \delta) \circ (t \mapsto tc^{-1}) : X(c,d) \to \mathbb{Z} / dc^{-1} \to \mathbb{Z} / \delta \) induces bijections \( S(m) \backslash X(c,d) \to \pi S(m) \backslash \mathbb{Z} / dc^{-1} \to \mathbb{Z} / \delta \).

\text{Proof:} \ This \ is \ immediate \ from \ (1.2). \ \Box

\[(1.4) \textbf{Lemma:} \] For any \( x \in X(c,d) \), \( \text{Stab}_{S(m)} y = S((cNd',m)). \)

\text{Proof:} \ Write \( y= cu \), \( u \in \mathbb{Z} / dc^{-1} \). Then

\begin{align*}
\text{Stab}_{S(m)} y &= \{ x \in S(m) \mid xcu \equiv cu \ (d) \} \\
&= \{ x \in S(m) \mid xu \equiv u \ (dc^{-1}) \}
\end{align*}

= \{ x \in S(m) \mid x \equiv 1 \ (dc^{-1}) \} \quad \text{(since \( (u,dc^{-1})=1 \))}

= \{ x \in \mathbb{Z} / N^* \mid x \equiv 1 \ \text{mod} \ Nm^2 dc^{-1} (Nm^2,dc^{-1}) \}

= S(cmd^2 (Nm^2,dc^{-1})) = S((cNd',m)) \quad \Box

(89)

Matrix multiplication shows that double coset representatives for \( U \backslash S_1 n(\mathbb{Z}) / \Gamma_0 (n,N) \) are given by

\[ \left\{ [x_1,...,x_n] \mid x_1 \in \mathbb{Z} / N, x_2 \in \mathbb{Z} / x_1, x_3 \in \mathbb{Z} / (x_1,x_2),...,x_{n-1} \in \mathbb{Z} / (x_1,...,x_{n-2}), x_n \in \mathbb{Z} / (x_1,...,x_{n-1})^* \right\} \]

modulo the diagonal action of \( \mathbb{Z} / N^* \). Setting \( m = \sigma k c_{-1}(\sigma), \ d = \sigma k, \ c = \sigma k+1, \) we get

\( \delta = (Nm^2,dc^{-1}) = (a_0(\sigma) \cdots a_{k-1}(\sigma)c_{k-1}(\sigma)^{-1}, a_k(\sigma)) = d_k(\sigma). \) Now (1.3) says that \( Y(\sigma_1,...,\sigma_{k+1}) \) is a cross section for the orbits of \( S(\sigma_k c_{k-1}(\sigma)) \) on \( \mathbb{Z} / a_k(\sigma)^* \).
Lemma: Let $k \in \{1, \ldots, n-1\}$, $i \in \{1, \ldots, k+1\}$, $\sigma \in \mathfrak{B}$. Then for all $x_i \in X(\sigma_i, \sigma_{i+1})$ we have $\text{Stab}_{S(\sigma_k c_{k-1}(\sigma))} x_{k+1} = S(\sigma_{k+1} c_k(\sigma)) = \bigcap_{1 \leq i \leq k+1} \text{Stab}_{Z/N^*} x_i$.

Proof: For the left equality, it suffices by (1.4) to see that $(\sigma_{k+1} N c_k^{-1}, \sigma_k c_{k-1}(\sigma)) = \sigma_{k+1} c_k(\sigma)$, and this is easy. For the other equality, use the definition of $S(m)$ and the first equality to see the following picture:

\[
\text{Stab}_{Z/N^*} x_1 \supseteq \text{Stab}_{S(\sigma_1)} x_2 \supseteq \text{Stab}_{S(\sigma_2 c_1(\sigma))} x_3 \supseteq \cdots \supseteq \text{Stab}_{S(\sigma_k c_{k-1}(\sigma))} x_{k+1}
\]

S($\sigma_1$) $\supseteq$ S($\sigma_2 c_1(\sigma)$) $\supseteq$ S($\sigma_3 c_2(\sigma)$) $\supseteq$ $\cdots$ $\supseteq$ S($\sigma_k c_{k-1}(\sigma)$).

Lemma: Let $x_i \in \mathbb{Z}/N$, $x_k \in \mathbb{Z}/(x_1, \ldots, x_{k-1})$ for $k=2, \ldots, n-1$, $x_k \in \mathbb{Z}/(x_1, \ldots, x_{n-1})^*$. Then $(x_1, \ldots, x_n)$ is conjugate under $\mathbb{Z}/N^*$ to an element of our set of alleged representatives, $\mathfrak{C}$. I.e., $\exists u \in \mathbb{Z}/N^*$, $\sigma \in \mathfrak{B}$, $y_i \in Y(\sigma)$ such that $ux_k \equiv \sigma_k y_{k-1} \mod(x_1, \ldots, x_{k-1})$.

Proof: We first construct $u, \sigma, y_i$, with $ux_k \equiv \sigma_k y_{k-1} \mod(x_1, \ldots, x_{k-1})$. Choose $u_1 \in \mathbb{Z}/N^*$ such that $u_1 x_1 \equiv \sigma_1(N)$, where $\sigma_1 \mid N$. Thus $(\mathbb{Z}/N)/x_1 = \mathbb{Z}/\sigma_1$. Recall that $S(\sigma_1) = \text{Stab}_{\mathbb{Z}/N^*}(u_1 x_1)$ and that $u_1 x_2 \in \mathbb{Z}/\sigma_1 = \bigcup_{d \mid \sigma_1} d[\mathbb{Z}/\sigma_1 d^2]$. Choose $\sigma_2 \mid \sigma_1$ such that $u_1 x_2 \in \sigma_2 \mathbb{Z}/\sigma_1(N)$. Now $Y(\sigma_1, \sigma_2)$ is a cross-section of the orbits of $S(\sigma_1)$ on $\mathbb{Z}/\sigma_1(N)$ so $\exists u_2 \in S(\sigma_1)$ and $y_1 \in Y(\sigma_1, \sigma_2)$ with $u_2 y_1 x_2 \equiv \sigma_2 y_1(\sigma_1)$. Note that $u_2 u_1 x_1 \equiv u_1 x_1(N)$. We continue in this way: Assume that $\sigma_1, u_i, y_{i-1}$ $i=2, \ldots, k$ have been chosen so that $\sigma_k \mid \sigma_{k-1} \cdots \mid \sigma_1$, $u_i \in S(\sigma_{i+1} c_i(\sigma))$, $y_{i-1} \in Y(\sigma_1, \ldots, \sigma_i)$
and \( u_1 u_2 \cdots u_k x_i \equiv \sigma_i y_{k+1} (\sigma) \). Note that \( \sigma \) has no been completely defined yet, but \( c_{i-2}(\sigma) \) is defined. Now \( u_k \cdots u_1 x_{k+1} \in \sigma_{k+1} \mathbb{Z}/a_k(\sigma)^* \) for some \( \sigma_{k+1} | \sigma_k \), and \( \gamma(\sigma_1, ..., \sigma_{k+1}) \) is a cross section for the orbits of 
\[
S(\sigma_k c_{k-1}(\sigma)) = \bigcap_{1 \leq i \leq k} \text{Stab}_{\mathbb{Z}/a_i(\sigma)^*} u_1 u_2 \cdots u_i x_i
\]
on \( \mathbb{Z}/a_k(\sigma)^* \), by (1.5). Hence \( \exists u_{k+1} \in S(\sigma_{k+1}(\sigma)) \) and \( y_k \in \gamma(\sigma_1, ..., \sigma_{k+1}) \) such that \( u_{k+1} \cdots u_1 x_{k+1} \equiv \sigma_{k+1} y_k (\sigma_k) \). In this way we choose \( u_i, \sigma_i y_{i-1} \) for \( i=1, ..., n \) (put \( y_0 = 1 \)).

Put \( u = u_1 \cdots u_n \) (the product is in \( \mathbb{Z}/N^* \)). Then for \( k=1, ..., n \),
\[
u x_k = u_1 \cdots u_k x_k,
\]
since \( u_j \in \text{Stab}_{\mathbb{Z}/a_j(\sigma)^*} u_1 \cdots u_k x_k \) for \( j>k \). Hence \( u x_k = \sigma_k y_{k-1} (\sigma_{k-1}) \).

Finally, we must show that \( (x_1, ..., x_k) = \sigma_k \forall k=1, ..., n \). This is seen by using the fact \( (\sigma, \sigma_{i+1} y_i) = \sigma_{i+1} (a_i(\sigma), y_i) = \delta_{i+1} \) \( (y_i \in \mathbb{Z}/a_i(\sigma)^* \) and starting with \( (x_1, x_2) = (\sigma_1, \sigma_2 y_1) = \sigma_2 \).

(1.7) Lemma: The elements of \( G \) represent each orbit of \( U \) on \( \mathbb{P}^{n-1}(\mathbb{Z}/N) \) at most once. I.e., if \( \sigma, \sigma' \in G \), \( y_i \in \gamma_i(\sigma) \), \( y_i' \in \gamma_i(\sigma') \), and if \( \exists u \in \mathbb{Z}/N^* \) with \( u \sigma_{i+1} y_i = \sigma_{i+1} y_i' (N) \), then \( \sigma = \sigma' \) and \( y_i = y_i' \) for all \( i=0, ..., n-1 \).

Proof: First of all, \( u \sigma_1 = \sigma_1' (N) \) and \( \sigma_1, \sigma_1' \) are both divisors of \( N \). Hence \( \sigma_1 = \sigma_1' \). Now let \( k \in \{0, ..., n-1\} \) be such that \( \forall i<k \), we have \( \sigma_{i+1} = \sigma_{i+1}' \) and \( y_i = y_i' \). This means
\[
u \in \bigcap_{0 \leq i \leq k-1} \text{Stab}_{\mathbb{Z}/a_i(\sigma)^*} [\sigma_{i+1} y_i] = S(\sigma_k c_{k-1}(\sigma))
\]
by (1.5).
For all \( d|\sigma = \sigma \), \( X(d, \sigma_k) \) is \( u \)-invariant and \( \mathbb{Z}/\sigma_k = U_{d|\sigma_k} X(d, \sigma_k) \). Now \( \sigma_{k+1} y_k * \in X(\sigma_{k+1}, \sigma_k) \) and are conjugate by \( u \). Therefore \( \sigma_{k+1} = \sigma_{k+1} \) and \( y_k, y_k \) lie in the same orbit of \( S(\sigma_k, \sigma_k) \) on \( \mathbb{Z}/\mathbb{Z}/a_k(\sigma)^* \), hence by (1.3), in the same coset of \( \ker(\mathbb{Z}/a_k(\sigma)^* \rightarrow \mathbb{Z}/d_k(\sigma)^*) \).

Since \( \psi(\sigma_1, ..., \sigma_{k+1}) = \psi(\sigma_1, ..., \sigma_{k+1}) \) is a set of coset representatives for this kernel, we have \( y_k = y_k \). □

This completes the proof of (1.1).

§2. Symmetry in the Formula

It follows from (1.1) that we may write

\[
|U \setminus S_n(\mathbb{Z})/\Gamma_0(n,N)| = \sum_{\sigma \in \mathcal{B}} \prod_{1 \leq k \leq n-1} \psi(d_k(\sigma)),
\]

where \( \psi \) is the Euler \( \psi \)-function. Now \( \mathcal{B} \) is in one to one correspondence with the set of ordered "multiplicative partitions" of \( N \) having length \( n \): \( \{(a_0, ..., a_{n-1}) \in \mathbb{N}^n | a_0 \cdots a_{n-1} = N \} \). The symmetric group on \( n \) letters acts on this set by permuting coordinates. The goal of the next few lemmas is to prove that \( \prod_{1 \leq k \leq n-1} \psi(d_k(\sigma)) \) is invariant under this action, by giving a different expression which is evidently invariant. However, I am unable to give a conceptual "reason" for this symmetry.

The symmetry is trivial in the \( n=2 \) case: if \( d|N \) then \( \psi(d, N d^{-1}) \) is invariant under \( d \rightarrow N d^{-1} \). In general, each \( \psi(d_k(\sigma)) \) is not invariant, but the product is. When \( N=p^r \), the multiplicative partitions of \( N \) correspond to the ordinary partitions of \( r \), and the
formula for \( |U \setminus \mathbb{S}|_{n}(\mathbb{Z})/\Gamma_0(n,N)| \) simplifies to a rational function in one indeterminate evaluated at \( p \).

From now on, \( \sigma \) will be fixed, so we write \( c_k \) instead of \( c_k(\sigma) \), etc. Using the facts that \( c_{k-1}d_k = c_k \) and \( \varphi(x)\varphi(y) = \varphi(xy)\varphi(x,y)(x,y)^{-1} \), we get

\[
\prod_{1 \leq k \leq n-1} \varphi(d_k) = \varphi(c_{n-1}) \prod_{1 \leq k \leq n-1} \varphi(c_{k-1},d_k) (c_{k-1},d_k)^{-1}. \tag{91}
\]

Since \( \varphi(c_{n-1}) \) is clearly invariant under permutations of the \( a_i \)'s, we must show that \( \prod_{1 \leq k \leq n-1} \varphi(c_{k-1},d_k) (c_{k-1},d_k)^{-1} \) is invariant. We need more

**Notation:** Put \( T = \{1, \ldots, n-2\} \), \( \bar{T} = \{0, \ldots, n-1\} \). For \( i \in T \), set \( b_i = (c_i, d_i) \), and for \( A \subseteq T \), put \( b(A) = \gcd\{b_i | i \in A\} \). Similarly, if \( i \in \bar{T} \), recall that \( a_i = \sigma_i(\sigma_i)^{-1} \), and put, for \( B \subseteq T \), \( a(B) = \gcd\{a_j | j \in B\} \). Finally, if \( J \) is a set of indices, let \( J_k \) = the collection of \( k \)-element subsets of \( J \).

For \( x \in \mathbb{N}^+ \), set \( \psi(x) = \psi(x)x^2 = \prod_{p \mid x}(1 - p^{x^2}) \). The following lemma is an elementary generalization of the fact that \( \psi(x)\psi(y) = \psi(xy)\psi(x,y) \), and we omit the proof.

\[(2.1) \text{Lemma: } \text{Let } x_1, \ldots, x_m \in \mathbb{N}^+, \ J \subseteq \{1, \ldots, m\}. \text{ Then} \]

\[
\prod_{1 \leq k \leq m} \psi(x_i) = \prod_{1 \leq k \leq m} \psi[\prod_{A \in J_k} \gcd(x_i, i \in A)].
\]

We come now to the main result.

\[(2.2) \text{Lemma: } \prod_{1 \leq k \leq n-1} \varphi(d_k) = \varphi(c_{n-1}) \prod_{1 \leq m \leq n-2} \psi[\prod_{B \in J_{m+2}} a(B)]. \]
Proof: By (2.1), we must show
\[
\psi[\prod_{B \in \bar{T}_{m+2}} a(B)] = \psi[\prod_{A \in T_m} b(A)].
\]
This is equivalent to showing
\[
P \mid \prod_{B \in \bar{T}_{m+2}} a(B) \iff P \mid \prod_{A \in T_m} b(A).
\]
Now it follows readily from the definitions that
\[
P \mid d_{k+1} \iff P \mid (a_0 \cdots a_k a_{k+1}) \quad \text{and} \quad P \mid c_k \iff P \mid \text{two of } a_0, \ldots, a_k.
\]
Hence \(P \mid b_k \iff P \mid a_{k+1} \) and two of \(a_0, \ldots, a_k\). Let \(A = \{i_1, \ldots, i_m\} \subseteq T_m\).
Then \(P \mid b(A) \iff P \mid (a_{i_1+1}, \ldots, a_{i_m+1}) \) and two of \(a_0, a_1, \ldots, a_m\)
\[
\iff P \mid a(B), \text{ some } B \in \bar{T}_{m+2} \text{ with }
\{i_1+1, \ldots, i_m+1\} \subseteq B \subseteq \{0, \ldots, i_1, i_1+1, \ldots, i_m+1\}.
\]
The lemma follows from this. □

(2.3) Proposition: \(\prod_{1 \leq k \leq n-1} \varphi(d_k)\) is invariant under permutation of the \(a_i\)'s.

Proof: By (2.2), it suffices to see that \(\prod_{B \in \bar{T}_{m+2}} a(B)\) is invariant. But this is clear since the product is over all possible subsets of \(\{0, \ldots, n-1\}\). □

Now assume that \(N = p^r\), some \(r > 0\). By (2.3), we may assume \(a_{n-1} | a_{n-2} | \cdots | a_0\). Then \(c_k = a_1 \cdots a_k\) and \(d_k = a_k\). Let \(a_i = p^{r_i}, r_0 \geq \cdots \geq r_{n-1} \geq 0\). Then
\[
\prod_{1 \leq k \leq n-1} \varphi(d_k) = \prod \varphi(p^{r_k}) = p^{(r-r_0)(1-p^{-1})^{(r)}},
\]
where \(r_i\) is the smallest \(i\) such that \(r_i \geq a_0\) (resp \(r_i > 1\)) if \(p \neq 2\) (resp \(p = 2\)). Let \(K_0 = \text{Stab}_S \sigma\). We have shown
Corollary: \( |\mathbb{U}\setminus \mathbb{P}^{n-1}(\mathbb{Z}/p^r)| = n! \sum |K_0^r| p^{(r-r_0)(1-p^{-1})}\). 

where the sum is over \( \{\sigma=(p^{r_0}, p^{r_{1-2}}, \ldots) \mid r_0 \geq \cdots \geq r_{n-1} \geq 0 \text{ and } \sum r_i = r \} \).

If \( r \ll n \), the \( \sigma \)'s will have many zeros so \( |K_\sigma| \) will be large. In particular, if \( r=1 \), \( |K_\sigma|=(n-1)! \) for all \( \sigma \) and the sum above has only one term with \( \epsilon(\sigma)=0 \), \( r=r_0 \). I.e., we have one \( U \) orbit for each coset of \( S_n/\text{Stab}(1,0,\ldots,0) \). Hence \( S_n \) is playing a more complicated version of its role in the finite field case.

§3. The Tits building of \( SL_3 \) modulo \( \Gamma_0(3,N) \).

When \( n=3 \), the set of representatives for \( U\setminus \mathbb{P}^2(\mathbb{Z}/N) \) given in (1.1) becomes

\[
C=\{[\sigma_1, u\sigma_2, v] : \sigma_2|\sigma_1|N, u \in \mathbb{Z}/(a_0, a_1)^*, v \in \mathbb{Z}/(a_2, a_0 a_1(a_0, a_1)^{-1})^* \},
\]

where as usual, \( a_i=\sigma_i^\sigma_i^{-1} \).

Since \( SL_3(\mathbb{Z}) \) is transitive on \( SL_3(\mathbb{Q}) \)-classes of \( \mathbb{Q} \)-parabolics, the \( \Gamma_0(3,N) \) conjugacy classes of minimal \( \mathbb{Q} \)-parabolic subgroups correspond to the orbits in \( B\setminus \mathbb{P}^2(\mathbb{Z}/N) \) where \( B \) is the group of integral upper triangular matrices having determinant one. We see that the action of the integral diagonal matrices \( T \) on \( [x_1, x_2, x_3] \in C \) changes one or two of the signs of the coordinates. Hence representatives of \( B\setminus \mathbb{P}^2(\mathbb{Z}/N) \) are given by

\[
C'=\{[\sigma_1, u\sigma_2, v] : \sigma_2|\sigma_1|N, u \in \pm 1\mathbb{Z}/(a_0, a_1)^*, v \in \pm 1\mathbb{Z}/(a_2, a_0 a_1(a_0, a_1)^{-1})^* \}.
\]

Define, for \( n \in \mathbb{N}^+ \), \( \Psi(n)=\frac{1}{2} \Psi(n) \) if \( n \neq 1,2 \), \( \Psi(n)=1 \) for \( n=1,2 \). Then the formula for \( |B\setminus \mathbb{P}^2(\mathbb{Z}/N)| \) is obtained from that for \( |U\setminus \mathbb{P}^2(\mathbb{Z}/N)| \).
upon replacing \( \psi \) by \( \psi' \) and we get
\[
|B \setminus P^2(\mathbb{Z}/N)| = \sum_a \psi'(a_0a_1, a_0a_2, a_1a_2) \psi'(a_0,a_1,a_2)(a_0,a_1,a_2)^{-1}. \tag{94}
\]

(3.1) **Examples:** In the following table we have worked out the cases \( N=p, p^2, p^3, p^4, \) for \( p \) an odd prime. The columns list, in order, the level \( N \), a representative of each \( S_3 \) orbit on \( \{\sigma=(\sigma_0,\sigma_1,\sigma_2,\sigma_3): \sigma_3|\sigma_2|\sigma_1|\sigma_0=0\} \), the corresponding \( a(\sigma)=(a_0,a_1,a_2) \), the size of the \( S_3 \) orbit through \( \sigma \), the number of \( B \) orbits of type \( \sigma \), and the first Betti number of \( \mathcal{G}/\Gamma_0(3,N) \). Recall that this simplicial complex is a graph with \( 2(\# \text{divisors of } N) \) vertices (see Chapter II) and \( |B \setminus P^2(\mathbb{Z}/N)| \) edges.

| \( N \) | \( \sigma \) | \( a(\sigma) \) | \( |S_3\cdot \sigma| \) | \#B orbits | \( \dim H_1(\mathcal{G}/\Gamma_0(3,N)) \) |
|------|------|------|------|---------|------------------|
| \( p \) | \( p,1,1,1 \) | \( p,1,1 \) | 3    | 1        | 0                |
| \( p^2 \) | \( p^2,1,1,1 \) | \( p^2,1,1 \) | 3    | 1        | \( \frac{1}{2}(3p-7) \) |
|         | \( p^2,p,1,1 \) | \( p,p,1 \) | 3    | \( \frac{1}{2}(p-1) \) |                  |
| \( p^3 \) | \( p^3,1,1,1 \) | \( p^3,1,1 \) | 3    | 1        | \( \frac{1}{4}(p^2+10p-27) \) |
|         | \( p^3,p,1,1 \) | \( p^2,p,1 \) | 6    | \( \frac{3}{2}(p-1) \) |                  |
|         | \( p^3,p^2,p,1 \) | \( p,p,p \) | 1    | \( \frac{3}{4}(p-1)^2 \) |                  |
| \( p^4 \) | \( p^4,1,1,1 \) | \( p^4,1,1 \) | 3    | 1        | \( \frac{1}{4}(3p^2-11) \) |
|         | \( p^4,p,1,1 \) | \( p^3,p,1 \) | 6    | \( \frac{3}{2}(p-1) \) |                  |
|         | \( p^4,p^2,p,1 \) | \( p^2,p,p \) | 3    | \( \frac{3}{4}(p-1)^2 \) |                  |
|         | \( p^4,p^2,1,1 \) | \( p^2,p^2,1 \) | 3    | \( \frac{3}{4}p(p-1) \) |                  |
§1. Notation and Preliminaries

Let $k$ be a finite extension of the $p$-adic numbers $Q_p$, $v$ the valuation of $k$, $O$ the ring of integers in $k$, $\omega$ a generator of the maximal ideal of $O$, $q$ the cardinality of residue field, $G_r=GL_r(k)$, $K_r=GL_r(O)$, $N_r$ upper triangular matrices in $G_r$, $A_r$ diagonal matrices in $G_r$, $B_r=N_rA_r$. We view $G_{r-1}<G_r$ and $K_{r-1}<K_r$ embedded in the upper left corner. The Haar measure on $G_r$ is chosen so that $\text{vol}(K_r)=1$.

Let $\alpha_1,\ldots,\alpha_{r-1}$ be the simple roots for $A_r$ with respect to $B_r$. We have $\alpha_i[\text{diag}(a_1,\ldots,a_r)]=a_i/a_{i+1}$. The modulus of $B_r$ is $|2p|$ where $p$ is one half the sum of the positive roots. If $g=nak$, $n\in N_r$, $a\in A_r$, $k\in K_r$, set $|\alpha_i(g)|=|\alpha_i(a)|$. Let $\tau$ be a character of the additive group of $k$ with $\tau$ trivial on $O$, $\tau(\pi^i)\neq 1$. Let $\theta$ be the resulting generic character of $N_r$. That is, for $x=(x_{ij})\in N_r$,

$$\theta(x)=\prod_{1\leq i<j\leq r-1} \tau(x_{i,j+1}).$$

If $\lambda$ is a character of $B_r$, the induced representation

$$\text{Ind}(B_r,G_r,\lambda)$$

consists of those locally constant $C$-valued functions $f$ on $G_r$ such that $f(bg)=|\rho(b)|\lambda(b)f(g)$ for $b\in B_r$, $g\in G_r$. 

78
The induced representation $\text{Ind}(N_r,G_r,\theta)$ is defined similarly except the modulus is trivial.

Let $\mathcal{H}_r$ be the convolution algebra of $K_r$ bi-invariant functions on $G_r$ with compact support. We have the Satake isomorphism (see [12])
\[ \mathcal{H}_r \cong \mathbb{S}_r \]
where $\mathbb{S}_r = \mathbb{C}[T_1, \ldots, T_r, T_r^{-1}]$ and $T_i$ is the $i^{th}$ elementary symmetric polynomial in indeterminates $X_1, \ldots, X_r$. If $x = (x_1, \ldots, x_r) \in (\mathbb{C}^r)^r$, then evaluation at $x$ determines an algebra homomorphism $\lambda_x : \mathcal{H}_r \to \mathbb{C}$.

The point $x$ also determines an unramified character of $B_r$ by $\text{diag}(a_1, \ldots, a_r) \mapsto \prod x_i^{\chi_x}$, extended to $B_r$. Then the $K_r$ fixed vectors in $\text{Ind}(B_r, G_r, \lambda_x)$ are one dimensional and afford the $\mathcal{H}_r$ eigencharacter $\lambda_x$.

Let $\Delta(r) = \{ f = (f_1, \ldots, f_r) \in \mathbb{Z}^r \mid f_1 \geq \ldots \geq f_r \}$, $\Delta(r,n) = \{ f \in \Delta(r) \mid \sum f_i = n \}$. For $f \in \mathbb{Z}^r$, let $\omega^f = \text{diag}(\omega^{f_1}, \ldots, \omega^{f_r}) \in A_r$.

§2. Polynomial Whittaker Vectors

Let $x \in (\mathbb{C}^r)^r$. Let $W(g) = W(g,x_1, \ldots, x_r, \tau)$ be the unique $K_r$-fixed vector in $\text{Ind}(N_r, G_r, \theta)$ such that $W(e) = 1$. Shintani [25] has given the following formula for $W(g)$. Let $n \in N_r$, $k \in K_r$, $f \in \mathbb{Z}^r$. Then
\[ W(n \omega^f k) = \theta(n) |\rho(\omega^f)| \chi_f(x), \]
where $\chi_f(x) = 0$ unless $f \in \Delta(r)$, in which case $\chi_f(x)$ is the character of the rational $GL_r(\mathbb{C})$ representation with highest weight (with respect to $B_r$) $\text{diag}(a_1, \ldots, a_r) \mapsto \prod a_i^{\chi_f}$. Here is the explicit formula for $\chi_f$, $f \in \Delta(r)$:
We can replace the $x_i$'s by indeterminants $X_1,...,X_r$ and make the following

\textbf{(2.1) Observations:}

i) $X_f(X_1,...,X_r) = T_r^r X_{f,0}$

where $X_{f,0} \in \mathbb{C}[T_1,...,T_{r-1}]$. (Factor out the bottom row of the determinant.)

ii) \{ $X_f(X_1,...,X_r)$ \mid $f \in \Delta(r)$ \} is a linearly independant set in $S_r$. (This is because distinct $f$'s give distinct representations.)

Thus, Shintani's formula defines a map $G_r \to S_r$, $g \mapsto W(g, X_1,...,X_r, \tau^-)$ = $W(g, X_1,...,X_r)$. By (2.1), we can write

$$W(\omega f, X_1,...,X_r) = T_r^r W_0(\omega f, X_1,...,X_r)$$

where

$$W_0(\omega f, X_1,...,X_r) \in \mathbb{C}[T_1,...,T_{r-1}]$$

§3. The Polynomial Mellin Transform

Let $\Psi$ be a $\mathbb{C}$-valued function on $G_r$ satisfying

\textbf{(3.1) }

i) $\Psi(ng) = \theta(n) \Psi(g)$ for all $g \in G_r, n \in \mathbb{N}_r$, 

ii) $\Psi$ is right invariant under a compact open subgroup of $G_r$. 


iii) For each integer \( n \), \( \{ g \in G_r \mid \nu(\text{det}g) = n, \nu(g) \neq 0 \} = N_r \mathbb{Z}_n \) where \( \mathbb{Z}_n \subset G_r \) is compact.

As in [17], we define

\[
\int \varphi(g) W(g, X_1, \ldots, X_r) |\text{det}g|^{s} \, dg \text{ to be } \sum a_n(\varphi) \mathbb{Z}_n,\text{ a formal Laurent series with coefficients in } S_r,\text{ where } \vartheta = q^s \text{ and }
\]

\[
a_n(\varphi) = \int_{N_r \backslash G_r} \varphi(g) W(g, X_1, \ldots, X_r) \, dg \in S_r,
\]

\( \nu(\text{det}g) = n \)

The hypotheses (3.1) imply that the integrand has compact support on the region of integration.

(3.2) **Lemma**: Suppose \( \varphi \) satisfies (3.1) and is also right \( K_r \)-invariant. Then

\[
\int_{N_r \backslash G_r} \varphi(g) W(g, X_1, \ldots, X_r) |\text{det}g|^{s} \, dg = 0 \text{ implies } \varphi = 0.
\]

**Proof**: This (3.5) of [17] but the proof given here seems simpler. Since \( \varphi \) satisfies (3.1)i) and is right \( K_r \)-invariant, \( \varphi \) is determined by \( \varphi(\omega^f), f \in \mathbb{Z}^r \). Moreover, \( \varphi(\omega^f) = 0 \) unless \( f \in \Delta(r) \) [17(3.2)]. Thus,
\[ a_n(\psi) = \sum_{f \in \Delta(r,n)} \psi(\omega^f) W(\omega^f, X_1, \ldots, X_r) \] (99)

and the sum is finite. Since the Laurent series is zero, we have \( a_n(\psi) = 0 \) for all \( n \). By (2.1 ii), the \( W(\omega^f, X_1, \ldots, X_r) \)'s are linearly independant elements of \( S^r \), so \( \psi(\omega^f) = 0 \) for all \( f \in \Delta(r,n) \), for every \( n \). Hence \( \psi = 0 \). \( \square \)

Let \( \pi \) be an irreducible generic representation of \( G_r \), with respect to \( \tau \). Recall [17] this means we have a nonzero \( G_r \) homomorphism \( \pi \rightarrow \text{Ind}(N_r, G_r, \theta) \). Let \( \mathcal{W}(\pi, \tau) = \mathcal{W}(\pi) \) be the image of \( \pi \) under this homomorphism. This is the "Whittaker model of \( \pi \) with respect to \( \tau \)." Let \( \psi \in \mathcal{W}(\pi) \). Then [17 (3.2)] there exists a constant \( C \) such that \( \psi(\gamma) \neq 0 \) implies

\[ |\alpha_i(\gamma)| \leq C \text{ for all } \gamma \in G_r, \ 1 \leq i \leq r-1. \]

It follows that the function

\[ \psi_1(\gamma) = \psi \begin{pmatrix} 0 & g \\ 1 & 0 \end{pmatrix} |\det g|^{-1/2} \text{ on } G_{r-1} \text{ satisfies (3.1) with } r-1 \text{ instead of } r. \]

For \( \psi \in \mathcal{W}(\pi) \), we define

\[ \Psi(\psi, Y) := \psi(\psi, Y, X_1, \ldots, X_{r-1}, \tau) \]

\[ := \int_{N_{r-1} \backslash G_{r-1}} \psi_1(\gamma) W(\gamma, X_1, \ldots, X_{r-1}, \tau^-) |\det g|^s \, dg. \]

If \( \psi \in \mathcal{W}(\pi)^{K_{r-1}} \) then from (3.2) and the fact that restriction of functions in \( \mathcal{W}(\pi) \) to \( G_{r-1} \) is injective, we get

(3.3) \[ \Psi(\psi, Y) = 0 \Rightarrow \psi = 0. \] (100)

In ([17] §4], it is shown that for each \( \psi \in \mathcal{W}(\pi) \), there exists

\( \Theta(\psi) = \Theta(\psi, X_1, \ldots, X_{r-1}) \in S_{r-1} \) such that
Here $P_\pi$ is the polynomial in one variable over $\mathbb{C}$ such that $\mathcal{O}_{\pi}(Y)^{-1}$ is the $L$-factor attached to $\pi$. Viewing $\sum a_n(\psi)$ as an element of $\mathbb{C}[[T_1, \ldots, T_{r-1}, T]^{-1}]$, we can write

$$\Theta(\psi, YX_1, \ldots, YX_{r-1}) = \Psi(\psi, Y, X_1, \ldots, X_{r-1}, T^{-1}) \prod_{i=1}^{r-1} \mathcal{O}_{\pi}(YX_i). \quad (101)$$

(3.4) \quad \Theta(\psi) = \sum a_n(\psi) \prod \mathcal{O}_{\pi}(X_i). \quad (102)

As in [17], we note that $\Psi$ is in the space of Laurent series in $Y$ over $S_{r-1}$, having only a finite number of terms. This space is a torsion-free $S_{r-1}[T]$-module. From (3.3), we get

(3.5) The map $\psi \mapsto \Theta(\psi)$: $\mathcal{O}(\pi)^{K-1} \rightarrow S_{r-1}$ is injective.

(3.6) Lemma: $\Theta(\psi) \in \mathbb{C}[T_1, \ldots, T_{r-1}]$ if and only if $a_n(\psi) \in \mathbb{C}[T_1, \ldots, T_{r-1}]$ and $a_n(\psi) = 0$ for $n < 0$.

Proof: Observe that

$$\Theta(\psi, X_1, \ldots, X_{r-1}) \in \mathbb{C}[T_1, \ldots, T_{r-1}] \iff \left[ \sum a_n(\psi) Y^n \right] \prod \mathcal{O}_{\pi}(YX_i) \in \Theta(\psi, YX_1, \ldots, YX_{r-1})$$

belongs to $\mathbb{C}[T_1, \ldots, T_{r-1}][Y]$.

The "if" direction is now clear. Recall that $\mathcal{O}_{\pi}(0) = 1$. Suppose $a_n(\psi) = 0$ for $n < N$, $a_N(\psi) \neq 0$. The coefficient of $Y^N$ in $\Theta(\psi, YX_1, \ldots, YX_{r-1})$ is $a_N(\psi)$. Therefore $N > 0$ and $a_N(\psi) \in \mathbb{C}[T_1, \ldots, T_{r-1}]$.

Let $n > N$. The coefficient of $Y^n$ in $\Theta(\psi, YX_1, \ldots, YX_{r-1})$ is $a_n(\psi) + \text{terms involving } a_m(\psi), m < N$. Thus $a_n(\psi) \in \mathbb{C}[T_1, \ldots, T_{r-1}]$ by induction. \(\square\)

For $\psi \in \mathbb{H}_r$, we let $P_\psi \in S_{r-1}$ be the corresponding element under the
The Satake isomorphism. For $1 \leq i \leq r$, let $f(i) = (1, ..., 1, 0, ..., 0) \in \mathbb{Z}^r$ with exactly $i$ entries equal to 1. Let $\psi_i \in \mathcal{H}_r$ be $q^{(r-1)/2}$ times the characteristic function of $K_r \omega^{f(i)} K_r$.

Lemma: $P_{\psi_i} = T_i$. (This explains the normalization of $\psi_i$).

Proof: This is surely well-known, but for lack of a reference, I include the computation.

The space $\text{Ind}(B_r, G_r, \lambda_x)$ contains the function $F(g)$ on $G_r$ defined by

$$F(nak) = |\rho(a)| \prod x_{i}^{\nu(a)},$$

for $n \in N_r$, $a \in A_r$, $k \in K_r$. For any $\psi \in \mathcal{H}_r$,

$$\psi \star F = \lambda_x(\psi) F = P_{\psi}(x) F,$$

so we must show $\lambda_x(\psi_i) = T_i(x)$.

Now

$$q^{(r-1)/2} \lambda_x(\psi_i) = q^{(r-1)/2} \psi_i \star F(e) = \int_{K_r \omega^{f(i)} K_r} F(h) dh.$$  (104)

We now write this double coset as a union of right cosets. Let $I_i \subset \mathbb{Z}^r$ be the set of $r$-tuples of zeros and ones with exactly $i$ entries equal to one.

For $\epsilon \in I_i$, let $N(\epsilon) = N_r \cap K_r \cap \omega^\epsilon K_r \omega^{-\epsilon}$. Set $N_0 = N_r \cap K_r$. Then [25]

$$K_r \omega^{f(i)} K_r = \bigcup_{\epsilon \in I_i} \bigcup_{x \in N_0 / N(\epsilon)} x \omega^\epsilon K_r$$

(105)

Hence

$$\int_{K_r \omega^{f(i)} K_r} F(h) dh = \sum_{\epsilon \in I_i} \sum_{x \in N_0 / N(\epsilon)} F(x \omega^\epsilon) = \sum_{\epsilon \in I_i} [N_0 : N(\epsilon)] |\rho(\omega^\epsilon)| \prod x_{j}^{\nu(\epsilon)}.$$  (106)
Let $W$ be the Weyl group of $A_r$ in $G_r$. Let $e(i) = (0,...,0,1,...,1) \in I_r$. For $\epsilon \in I_r$, there is a unique element $w_\epsilon \in W$ such that i) $w_\epsilon(\omega^{e(i)})w_\epsilon^{-1} = \omega^\epsilon$ and

ii) $w_\epsilon$ has strictly smaller length $l(w_\epsilon)$ (with respect to the simple reflections determined by $B_r$) than any other element of $W$ satisfying i). It is standard that if $\alpha$ is a positive root not involving $\alpha_{r-1}$, then $w_\epsilon\alpha$ is also positive.

Claim 1: $[N_G : N(\epsilon)] = q^{l(w_\epsilon)}$. (107)

Proof: Let $\alpha$ be a positive root. Then $\alpha[\text{diag}(a_1,...,a_r)] = a_k/a_j$ with $k<j$. For $t \in k$, let $x_\alpha(t) = I + tE_{k,j} \in G_r$. It is not hard to see that

$$N(\epsilon) = \prod_{\alpha > 0} x_\alpha(0) \prod_{\alpha > 0} x_\alpha(\omega \Theta).$$ (108)

Since $l(w_\epsilon) = \# \{\alpha > 0 | w_\epsilon \alpha < 0\}$, Claim 1 is proven.

Claim 2: $|\rho(w_\epsilon)| = q^{-l(w_\epsilon)}|\rho(\omega^{e(i)})|$. (109)

Proof: The left hand side is $|w_\epsilon \rho(\omega^{e(i)})|$. If $\alpha$ is a simple root, then $|\alpha(\omega^{e(i)})| = q$ if $\alpha = \alpha_{r-1}$, 1 if not. Hence it suffices to show that the coefficient of $\alpha_{r-1}$ in $w_\epsilon \rho - \rho$ is $-l(w_\epsilon)$. Since $w_\epsilon \rho - \rho = \sum_{\alpha > 0} w_\epsilon \alpha$, it suffices to show that $\alpha > 0, w_\epsilon \alpha < 0 \Rightarrow$ the coefficient of $\alpha_{r-1}$ in $w_\epsilon \alpha$ is $-1$. Since we are in the $GL_r$ root system, it is enough to show this coefficient is nonzero. But if the coefficient is zero, we have $-w_\epsilon \alpha > 0$, $w_\epsilon^{-1}(-w_\epsilon \alpha) = -\alpha < 0$, contradicting a feature of $w_\epsilon$. Thus Claim 2 holds.
We now see that \([N_\Theta:N_\Theta(e)]|\rho(\omega^e)| = |\rho(\omega_e^{(1)})|\). In particular, this is independent of \(e\). It follows that
\[
q^{(r-1)/2} \lambda_x(\psi) = |\rho(\omega_e^{(1)})| \sum_{\epsilon \in I_1} T_{x} \xi_{\epsilon} = |\rho(\omega_e^{(1)})| T_{x}(x) = |\rho(\omega_e^{(1)})| T_{x}(x). 
\]
(110)

Finally, \(|\rho(\omega_e^{(1)})| = q^{(r-1)/2}\), by an easy induction. This finishes the Lemma.

\[\square\]

We now define an action of \(K_{r-1}\) on the \(G_r\) representation \(\mathcal{W}(\pi)\). Define, for \(\psi \in \mathcal{W}(\pi)\), \(\psi \in K_{r-1}\), and \(g \in G_r\),
\[
\psi \ast \psi(g) = \int_{K_{r-1}} \psi(gh) \psi(h^{-1}) |\det h|^{-1/2} dh.
\]
(Recall our embedding \(K_{r-1} < G_r\)). In this way \(K_{r-1}\) acts on \(\mathcal{W}(\pi)\) and preserves the fixed vectors of \(K_{r-1}\).

(3.8) Lemma: For \(\psi \in K_{r-1}\) and \(\varphi \in \mathcal{W}(\pi)\), we have \(\Theta(\psi \ast \psi) = P_\psi \Theta(\varphi)\).

Proof: By (3.4), it suffices to show \(a_n(\psi \ast \psi) = P_\psi a_n(\varphi)\) for all \(n \in \mathbb{Z}\). Since \(\psi_1\) has compact support modulo \(N_{r-1}\) on \(\{g \in G_{r-1} | v(\det g) = n\}\) and \(\psi\) has compact support on \(G_{r-1}\), we can use Fubini's theorem to compute
\[ a_n(\psi \ast \varphi) = \int_{N_{r-1} \backslash G_{r-1}} \int_{G_{r-1}} \varphi(g, h) \psi(g) W(g, X_1, \ldots, X_{r-1}) \, dg \, dh \]

\[ = \int_{N_{r-1} \backslash G_{r-1}} \int_{G_{r-1}} \varphi(g, h) \psi(h^{-1}) W(g, X_1, \ldots, X_{r-1}) \, dg \, dh \]

\[ = \int_{N_{r-1} \backslash G_{r-1}} \int_{G_{r-1}} \varphi(g) \psi(h) W(gh, X_1, \ldots, X_{r-1}) \, dh \, dg \]

\[ = \int_{N_{r-1} \backslash G_{r-1}} \psi(g) \int_{G_{r-1}} \varphi(h) W(g, X_1, \ldots, X_{r-1}) \, dh \, dg \]

\[ = \psi_0(\psi_1 \ast \varphi) \cdot \Theta(\varphi). \quad \Box \quad (111) \]

§4. Old Forms

Let \( \varphi_0 \) be the new vector in \( \mathcal{W}(\pi) \). The new vector is defined to be the unique vector in \( \mathcal{W}(\pi)_T \) such that \( \Theta(\varphi) = 1 \) ([17], (4.1)).

(4.1) Proposition: The map \( \mathcal{F} : \mathcal{K}_{r-1} \rightarrow \mathcal{W}(\pi)_{K-1} \) given by \( \mathcal{F}(\psi) = \psi \ast \varphi_0 \) is a linear isomorphism which intertwines the regular representation of \( \mathcal{K}_{r-1} \) on itself with the convolution action of \( \mathcal{K}_{r-1} \) on \( \mathcal{W}(\pi)_{K-1} \).

Proof: By (3.8) we have \( \Theta(\psi \ast \varphi_0) = P_\varphi \) for any \( \psi \in \mathcal{K}_{r-1} \). Hence \( \varphi \) is injective. Let \( \psi \in \mathcal{W}(\pi)_{K-1} \). By the Satake isomorphism there exists \( \psi_\varphi \in \mathcal{K}_{r-1} \) such that \( P_{\psi_\varphi} = \Theta(\varphi) \). Then (3.8) gives \( \Theta(\psi_\varphi \ast \varphi_0) = \Theta(\varphi) \). By (3.5), we get \( \psi_\varphi \ast \varphi_0 = \varphi \). Hence \( \mathcal{F} \) is surjective. The equivariance is clear. \( \Box \)
(4.2) Lemma: If $\varphi \in \mathcal{W}(\pi)^{K_{n-1}}$ and $\Theta(\varphi, \psi_1, ..., \psi_{r-1}) \in K_{r-1}$ is obtained by replacing $T_i$ by $\psi_i$ in $\Theta(\varphi, T_1, ..., T_{r-1})$ then $\varphi = \Theta(\varphi, \psi_1, ..., \psi_{r-1}) \ast \varphi_0$.

Proof: Let $\psi = \Theta(\varphi, \psi_1, ..., \psi_{r-1})$. Applying $\Theta$ to the right hand side of the conclusion gives $P_\varphi$. By (3.7), $P_\varphi = \Theta(\varphi)$. Applying $\Theta$ to the left hand side also gives $\Theta(\varphi)$. By (3.5), the two sides must be equal. □

For $m \in \mathbb{N}$, let

$$K(m) = \{ (a_r) \in K_r | a_{j} \in \omega^m \Theta \text{ for } 1 \leq j \leq r-1, \quad a_r \equiv 1 \mod \omega^m \} ,$$

$$K(\infty) = \bigcap_{m \geq 1} K(m).$$

Since $\rho$ is admissible,

$$\mathcal{W}(\pi)^{K(\infty)} = \bigcup_{m \geq 1} \mathcal{W}(\pi)^{K(m)}.$$ To describe $\mathcal{W}(\pi)^{K(m)}$ in terms of translates of $\varphi_0$, we will do the same for $\mathcal{W}(\pi)^{K(\infty)}$, and keep track of the level.

(4.3) Proposition: Assume $\varphi \in \mathcal{W}(\pi)^{K_{n-1}}$. Then $\varphi \in \mathcal{W}(\pi)^{K(\infty)}$ if and only if $\Theta(\varphi) \in \mathbb{C}[T_1, ..., T_{r-1}]$.

Proof: By (3.6), it suffices to show that $\varphi \in \mathcal{W}(\pi)^{K(\infty)}$ if and only if

$$a_n(\varphi) \in \mathbb{C}[T_1, ..., T_{r-1}]$$

and $n < 0 \Rightarrow a_n(\varphi) = 0$ both hold.

For $f \in \mathbb{Z}^{r-1}$, set $\varphi_f = \varphi^f$. Since $\varphi \in \mathcal{W}(\pi)^{K_{n-1}}$, the relation $\varphi_f \neq 0$ implies $f \in \Delta_{r-1}$. It is easy to see [J-PS-S §3] that $\varphi \in \mathcal{W}(\pi)^{K(\infty)}$ if and only if the relation $\varphi_f \neq 0$ implies $f \in \Delta(r-1)$ and $f_{r-1} \geq 0$. Now

$$a_n(\varphi) = \int_{N_{r-1} \setminus G_{n-1}} \psi_f W(g, X_1, ..., X_{r-1}) dg = q^{n/2} \sum_{f \in \Delta(r-1,n)} \psi_f W(\omega^f, X_1, ..., X_{r-1}). \tag{112}$$

By (98), $W(\omega^f) = T_{r-1} W_0(\omega^f)$, with $W_0(\omega^f) \in \mathbb{C}[T_1, ..., T_{r-2}]$. Thus $W(\omega^f)$ is a polynomial if and only if $f_{r-1} \geq 0$. The "only if" part is now
clear. For the other direction, set
\[ a_n^{-}(\varphi) = \sum_{f \in \Delta(r-1,n)} \varphi_f W(\omega^f). \]
Since \( a_n(\varphi) \) is a polynomial, so is \( a_n^{-}(\varphi) \). By (2.4) we may write
\[ a_n^{-}(\varphi) = \sum_{i=1}^{M} p_i T_{r-1}^{-1} \] where \( p_i \in \mathbb{C}[T_1, \ldots, T_{r-2}] \). But \( a_n^{-}(\varphi) \in \mathbb{C}[T_1, \ldots, T_{r-1}] \) implies \( p_i = 0 \) for all \( i \), so in fact \( a_n^{-}(\varphi) = 0 \). By (96) and (2.1) ii), \[ \{W(\omega^f) \mid f \in \Delta(r-1)\} \] is linearly independant. Hence each \( \varphi_f \) appearing in \( a_n^{-}(\varphi) \) is zero, whence the result. □

Let \( \omega^N \) be the conductor of \( \pi \). Thus \( \varphi_0 \) is fixed by \( K(N) \) but not \( K(N-1) \). Moreover, \( \varphi_0 \) spans \( W(\pi)^{K(N)} \). For \( \varphi \in W(\pi)^{K(\infty)} \), set
\[ \text{Level } \varphi = d, \]
where \( \varphi \) is fixed by \( K(N+d) \) but not fixed by \( K(N+d-1) \).

(4.4) Proposition: Level \( \varphi = \deg \Theta(\varphi) \).

Note: \( \deg \Theta(\varphi) \) is the degree of the highest degree monomial in the \( T_i \)'s which appears in \( \Theta(\varphi) \) with nonzero coefficient.
Proof: It is clearly true if \( \varphi = \varphi_0 \). By (3.7) and (3.8), for any \( \varphi \in W(\pi)^{K(\infty)} \) we have
\[ \deg \Theta(\varphi) = 1 + \deg \Theta(\varphi). \]
Claim: Level \( \varphi = 1 + \text{Level } \varphi \).

Admit this for the moment. By (4.2) we may write
\[ \varphi = \Theta(\varphi, \varphi_1, \ldots, \varphi_{r-1}) \cdot \varphi_0. \] Using the Claim and induction on \( \deg \Theta(\varphi) \) we are done. It remains to prove the Claim.
For any integer \( m \), let \( U(m) = \{(a_{ij}) \in G_r \mid a_{ii} = 1 \text{ for all } i, a_{rj} \in \omega^m \text{ for } 1 \leq j < r, \text{ all other entries zero}\} \). It is an easy matrix multiplication to see that \( U(m) \) and \( K(\infty) \) generate \( K(m) \). Suppose \( \Psi = d \). We will show that \( \psi_i \Psi \) is invariant under \( U(N + d + 1) \).

We keep the notation used in the proof of (3.7) except we now set \( N(\epsilon) = N_0 \cap \omega^{-\epsilon} K_{r-1} \omega^{\epsilon} \). Recall that \( N_0 = N_{r-1} \cap K_{r-1} \).

Then
\[
K_{r-1} \omega^{f(1)} K_{r-1} = U_{\epsilon \in I_1} \bigcup_k K_{r-1} \omega^{\epsilon} \chi_k \quad (114)
\]

Note that \((\omega^\epsilon)U(N + d + 1)(\omega^{-\epsilon}) \subset U(N + d)\), and \( N_0 \) normalizes \( U(N + d + 1) \), since \( K_{r-1} \) does. Hence if \( u \in U(N + d + 1) \), \( x \in N_0 \), \( \epsilon \in I_0 \), then \((\omega^\epsilon)uxu^{-1}(\omega^{-\epsilon}) \in U(N + d)\). Since \( \Psi \) is invariant under \( U(N + d + 1) \), we get
\[
\psi_i \Psi(gu) = q^{(1-\epsilon-1)/2} \sum_{\epsilon, x} \Psi(gux^2 \omega^{-\epsilon}) = q^{(1-r-1)/2} \sum_{\epsilon, x} \Psi(g\epsilon^2 \omega^{-\epsilon}) = \psi_i \Psi(g). \quad (115)
\]

It follows that \( \psi_i \Psi \) is invariant under \( K(N + d + 1) \).

To finish the Claim and the Proposition, it remains to show that \( \psi_i \Psi \) is not invariant under \( K(N + d) \). Suppose the contrary. We have \( d > 0 \) by ([17] §5). If \( d = 0 \) then by the uniqueness of the new vector, there is a constant \( C \) such that \( (\psi_i) \Psi = C \Psi \). This contradicts (4.1). Assume \( d > 0 \). Let \( f = (-d, \ldots, -d) \in \mathbb{Z}^{r-1} \). Let \( \Psi \in K_{r-1} \) be the characteristic function of \( K_{r-1} \omega^f K_{r-1} \). Using the fact that \( \omega^f U(N) \omega^{-f} = U(N + d) \), one easily computes that \( \psi \Psi \) and \( \Psi \psi_i \Psi \) are both invariant under \( U(N) \). Hence they are both invariant under \( K(N) \). By uniqueness again, \( \psi \Psi \psi_i \Psi \) is a constant multiple of \( \psi_i \Psi \). Thus \( \Theta(\psi \Psi \psi_i \Psi) = P_{\Psi} \Theta(\psi_i \Psi) \) is a constant multiple of \( \Theta(\psi_i \Psi) \). But
$P\psi = T_{r-1}^{-d}$, by (3.7), and $\Theta(\psi_1 \psi) \neq 0$ by (3.5). This is contradiction, and we are done. □

(4.5) **Corollary:** Let $(\pi, V)$ be an irreducible generic representation of $G_r$ with conductor $\omega^N$. Let $\mathbb{C}[T_1, \ldots, T_{r-1}]_d$ be the degree $d$ homogeneous polynomials. Then for all $d > 0$, the map $F: \mathbb{C}[T_1, \ldots, T_{r-1}]_d \to V^{KN+d}/V^{KN+d-1}$ such that $F(P) = \text{the coset of } P(\psi_1, \ldots, \psi_{r-1}) \psi_0$, is an isomorphism.

(4.6) **Remark:** 1) Let $\sigma$ be the graph automorphism of $G_r$. It is defined by $\sigma(g_{ij})^2 = (g_{r-i, r-j})$. If we replace $K(m)$ by $\sigma K(m)$ and embed $G_{r-1}$ in the lower right hand corner, then (4.5) remains true.

2) Let $K_0(m) = \{g \in K_r | \text{col}_1 g \equiv (\ast, 0, \ldots, 0) \mod \omega^m\}$. Suppose the central character of $\pi$ is trivial on $\Theta^\ast \cdot \text{Id}$. Since $\sigma K_{r-1}$ commutes with the operators $\pi(\text{diag}(a, 1, \ldots, 1))$, (4.5) holds for such $\pi$ with the groups $K(m)$ replaced by $K_0(m)$. 
Bibliography

5. A. Ash, L. Rudolph: The modular symbol and continued fractions in higher dimensions, Inv. Math. 55, (1979) 241-250
10. N. Bourbaki: Groupes et algebre de Lie, Ch. 4,5 et 6, Elements de Mathematique, XXXIV, Paris: Hermann (1968)


22. J. McCleary: Users guide to spectral sequences, Publish or Perish (1985)


28. R. Steinberg: Lectures on Chevalley groups, Lecture notes, Yale Univ. (1967)
