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The behavior of foreign-currency prices and option values

Lieu, Der-Ming, Ph.D.
The Ohio State University, 1988

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THE BEHAVIOR OF FOREIGN-CURRENCY PRICES AND OPTION VALUES

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of the Ohio State University

By

Der-Ming Lieu, B.A., M.A., M.S.

* * * * *

The Ohio State University

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DEDICATION

This dissertation is dedicated with heart-felt appreciation to my brother Der-Lu, who helped to finance my college education and take care of our parents in Taiwan. Without his continuing love and support, I would not have completed my Ph. D. degree.
ACKNOWLEDGEMENTS

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CHAPTER I
INTRODUCTION

1.1 Introduction

Path-breaking research on option pricing has been performed by Black and Scholes (1973) and extended importantly by Merton (1973a). More than 200 papers about option pricing, almost every one evolved from the Black and Scholes model, have appeared in professional finance journals¹. Reactions from practitioners are enthusiastic.

Despite its preeminence, the Black-Scholes formula has a few well-known deficiencies. Empirically, model prices appear to differ systematically from market prices (see, e.g., Black [1975], Merton [1976], Galai [1983], Rubinstein [1985]). These biases are usually ascribed to the strong assumptions of the Black-Scholes model:

(1) There are no penalties for short sales.
(2) Transactions costs and taxes are zero.
(3) The market operates continuously.
(4) The risk-free interest rate is constant.
(5) The asset price follows a stationary geometric Brownian motion (or equivalently, the asset price is lognormal with constant variance).
(6) The asset pays no dividends.

¹. The reader is referred to Chapter 4 and 6 of this study for a detailed review.
The option can only be exercised at the terminal date of the contract.

It is no secret that none of these assumptions in reality holds for an American call option. The problem is how relaxations in the basic assumptions would affect the option price. Thorpe (1973) examines the effects of restrictions against the use of the proceeds of short sales. Ingersoll (1976) takes explicit considerations of the effect of differential taxes on capital gains and ordinary income. Gould and Galai (1974) analyze the effect of transaction cost of the Black-Scholes hedged portfolio. Rubinstein (1976) derives the same Black-Scholes formula even though a perfect hedge is prohibited. Brennan and Schwartz (1977) develops an approximate Black-Scholes formula with discrete time trading. Merton (1973b) derives essentially the same Black-Scholes formula with stochastic interest rate. Cox and Rubinstein (1985) study the impact of transaction costs (commission, bid-ask spread, etc.) and taxes on options price. They find that these factors should not change the basic Black-Scholes' formula. It appears that the Black-Scholes model is robust with respect to the first four assumptions.

Merton (1973b) proves that without dividend payments, it is not optimal for the call holder to exercise prior to the expiration date. Merton (1973b) and Thorpe (1973) modify the model to account for dividend payments on the underlying stock, if early exercise is not possible. Roll (1977) develops a valuation formula for unprotected American call options with a known single-dividend distribution prior
to expiration of the option contract. But Blomeyer and Klemkosky (1983) show that the systematic pricing bias observed in the Black-Scholes model is not a dividend bias. Models such as Geske and Johnson (1984) have evolved to handle the early exercise problem. It is still inconclusive whether Roll's American call option formula can explain Black-Scholes' empirical biases. Therefore, the focus is on the assumption (5): the underlying asset price is lognormal with constant variance.

Black and Scholes derive their option-pricing formula by showing that it is possible to form a riskless hedge portfolio that contains a call option, a common stock and a riskless bond. If the underlying distribution is not lognormal, then the Black-Scholes' hedged portfolio would no longer be riskless and the Black-Scholes formula would not be an exact solution. Therefore, whether the underlying distribution satisfies assumption (5) becomes critical if the Black-Scholes arbitrage-free reasoning is to apply.

The idea of modelling the underlying stochastic process as a lognormal process can be traced back to Bachelier (1900). Bachelier's original rationale for using the normal distribution was that the change in security's price over time is the sum of more or less independent but identically distributed (iid) changes that occur day by day, hour by hour. In 1900's version of the Central Limit Theorem, only the normal distribution could plausibly result from an infinite sum of iid random variables. But several problems flow from this assumption:
First, even if the information flow over time can be approximated by an iid random variable, the observed security price could be autocorrelated due to such trading costs as bid-ask spreads (Roll [1984]). Second, by 1925, Paul Levy had demonstrated that the infinite sum of iid random variables does not necessarily lead to a normal distribution, rather, it leads to a much broader class. This broader class is known as the stable distribution, of which the normal is only a special case. Therefore, even if we agree that the information flow can be approximated by iid random variable and trading costs can be ignored, the underlying distribution does not necessarily have to be lognormal. A natural choice would be the log-stable process. But Clark (1973) argues that if the number of random variable is stochastic, then the sum of iid random variables does not necessarily converge to a stable distribution. He proposes a subordinate stochastic process as an alternative. Therefore, whether the log-normal, log-stable or other process is the better choice for modelling security price should be a question that can only be answered in an empirical context.

The majority of empirical evidence for the behavior of security prices (e.g., Fama [1965], Rosenberg [1972], Oldfield, Rogalski and Jarrow [1977]) so far suggests that the lognormal assumption does not hold, at least in stock markets and foreign exchange markets. The consensus of the empirical evidence in these markets indicates that the empirical distributions are more peaked in the center and have fatter tails than a lognormal process entails. These findings lead financial economists to search for a distribution which more
accurately describes the behavior of the asset price. Because a log-stable process\(^2\), with its characteristic exponent less than 2, could generate observations which are more peaked in the center and have fatter tails than a lognormal process, the next empirical question is to check how well a log-stable process can explain the empirical data.

Unfortunately, empirical results do not uniformly support that the underlying distribution for asset prices is log-stable. Among proponents are Fama(1965), Roll(1970), Dusak(1973), Robert and Paulson(1975), Hilliard and Leitch(1976), Westerfield(1977), Cornew and Town(1984). However, Officer (1972) and Hsu, Miller and Wichern (1974) find the stable-distribution hypothesis to be inconsistent, in several respects, with the empirically observed behavior of security price changes. Press(1967, 1974) suspect that we do not have enough tools to distinguish a stable process from other long-tail processes such as a mixed diffusion-Poisson process. Perhaps a more damaging news is that the second moment for a stable process is ordinarily infinite. This implies that the expected value of a call option with a stable process is ordinarily infinite. This led Samuelson (1973, p.15) conjecture that using a stable process to model asset price changes would produce an inconsistent option model\(^3\).

\(^2\) The reader is referred to Appendix A.1 for an introduction to a stable process.

\(^3\) The reader is referred to Appendix A.2 for the proof that the expected value of a call option under a stable process is ordinarily infinite and a discussion of Samuelson's conjecture.
Some alternatives other than a stable process to model asset price changes have been proposed over the years. Blattberg and Gonedes (1974), and Rogalski and Vinso (1978) propose a Student $t$ process. Clark (1973) finds that a subordinate stochastic process model with a finite variance fits cotton price data better than a stable process. Press (1967, 1974) proposes a mixed diffusion-Poisson process as a description of stock price changes. However, none of these has gained general acceptance. Nevertheless, the search for a new probability process other than a lognormal process to model asset price change has created a new direction of option-pricing studies.

Several theoretical option-valuation models that utilize a different stochastic process to model asset price changes have appeared in the literature. Cox and Ross (1976) develops a model for a pure Poisson jump process. Merton (1976a) develops a model that assumes that stock price changes follow a mixed diffusion-Poisson process. Cox (1975) derives an option-pricing formula that is based on a diffusion process that has a constant elasticity of variance (CEV). Geske (1979) derives an option model that allows the variance of stock prices to change according to a firm’s capital structure. Rubinstein (1983) presents an option-pricing model with a displaced diffusion process. Unfortunately, because the parameters estimation for these new models are difficult, empirical tests for these models are very
limited. It is still not conclusive that any of these models performs better than the Black-Scholes model.\(^4\)

From this brief review, the current empirical findings are: (1) no single probability process gains wide acceptance as the describing distribution of the underlying asset price changes. (2) no existing option-pricing model can systematically predict option prices better than the Black-Scholes formula, yet the Black-Scholes formula exhibits a systematic bias. (3) the empirical tests for other new option-pricing models are still too limited to offer conclusive evidence against the Black-Scholes' model. Galai (1983, p.69) in a survey for the empirical studies of option-pricing models concludes that 'The major problem faced by the B-S model or any other model suggested so far, is the nonstationarity of the risk estimator of the underlying stock. The nature of the nonstationarities is not clear'.

This study adds to the literature on testing option-pricing models. We examine two probability models that describe the observed underlying market price changes: a lognormal process and a mixed diffusion-Poisson process. Based on the results of this examination, we test two option-pricing models that make these two distributional assumptions: the Black-Scholes model and Merton's (1976a) model. We

---

4. We present a detailed review for these model in Chapter 4. Regarding the preliminary empirical tests for these models, the reader is referred to Merton (1976b), Becker (1981), Ball and Torous (1983,1985), for Merton's (1976a) mixed diffusion-Poisson model; to MacBeth (1983) for Cox's (1975) CEV model; to Rubinstein (1985) for Rubinstein's (1983) displaced diffusion option-pricing model; to Gulai (1983) for a survey of empirical studies for option-pricing models.
examine how the performance of each competing option-pricing model corresponds to the empirical predictions of the underlying distribution. Furthermore, we propose three ad-hoc models to estimate empirical pricing equations for options. The performance of the empirical pricing equation is used to establish a benchmark criterion for evaluating an arbitrage-free option-pricing model to which the Black-Scholes and the Merton’s option-pricing models belong.

The real-world options market studied here is the foreign-currency options market. There are good reasons for conducting such a study in foreign-currency options market. Foreign-currency options are the latest innovation in the financial markets. Although empirical studies on options market are extensive, majority of them look at stock options and suffer severe data deficiency (see Galai [1983], Cox and Rubinstein [1985]). Only a few empirical studies of foreign-currency options appear in the literature and all of them focus on the Black-Scholes model. Shastri and Tandon (1985) and Bodurtha and Courtadon (1986a) focus on the put-call parity and other boundary conditions. In a simulation study, Shastri and Tandon (1986a) show that the Black-Scholes model performs well for a European call option for foreign currency but is less accurate when used to price put options. Shastri and Tandon (1986b) formally tests a European call option that is based on the Black-Scholes formula for foreign-currency

options. However, they do not test any alternative model. Even worse, the data they use in their studies suffer from non-simultaneity in the observed currency price and option price. This calls for an empirical tests for an alternative option-pricing model.

Because fundamental determinants of foreign-currency prices, such as monetary policy, fiscal policy are highly variable and that government interventions are unsteady (Levich [1979]), Jarrow and Rudd (1983, p. 160) point out that a mixed diffusion-Poisson process appear to be fairly realistic description of foreign-currency price movements. This makes very promising Merton’s (1976a) option-pricing model that assumes stock prices follows a mixed Diffusion-Poisson process. However, because multiple interest rate are involved in foreign-currency options, the stock-option pricing model developed by Merton needs to be modified. No adaptation of Merton’s model to foreign-currency options has yet appeared in the literature.

Summing up, this study has six novel features:

First, existing empirical tests of option markets (except for Beckers [1983]) assume that the underlying distribution belongs to a particular class (typically a lognormal process), without testing this assumption. This study is the first to test whether the Black-Scholes distributional assumptions are violated in the foreign currency markets before a formal test on the Black-Scholes model for foreign-currency options is performed. We find that a lognormal process that
the Black-Scholes model assumes does not fit the observed foreign-currency price changes as well as a mixed diffusion-Poisson process that the Merton's model assumes.

Second, this study is the first to derive an arbitrage-free pricing model for foreign-currency options by using Merton(1976a)'s techniques and assumptions. Because foreign interest rates are involved, foreign-currency options have features that differ from stock options. The extension of Merton's model for stock options to foreign-currency options is not trivial.

Third, although there have been some attempts to test Merton's model in stock options market, most of them stop in the initial stage of estimating the relevant parameters. This study marks the first extensive test for an option-pricing model that is based on a mixed diffusion-Poisson process in any options markets, not only for foreign-currency options market. We propose several detailed techniques for estimating the parameters for a mixed diffusion-Poisson process and for estimating foreign-currency option prices implied in the modified Merton's model. We find that the modified Merton's model can eliminate some of the bias that the Black-Scholes model for foreign-currency options exhibits. The relatively better performance from the modified Merton model as compared to the modified Black-Scholes formula appears to relate to the underlying distribution for foreign-currency price changes. The more seriously the underlying currency price changes depart from a lognormal process, the better the modified Merton model performs.
Fourth, because a mixed diffusion-Poisson process can be approximated by a mixed normal process and a truncated Poisson mixture of normal process, this study propose some new methods to estimate the parameters for a mixed diffusion-Poisson process. We further the study of parameter estimation for Poisson mixtures of normal process initiated by Press (1967) and expanded by Ball and Torous (1983;1985). Empirical results obtained in this paper in some ways differ from theirs. By relaxing parameter restrictions, we also present some more general results.

Fifth, this study re-examines ad-hoc approaches to pricing an option. Gastineau (1979) notes that practitioners often use an ad-hoc approach to obtain an price estimate for an option. This study marks the first extensive review of the past literature that uses an ad-hoc approach to pricing an option. We establish several criteria for judging an ad-hoc model. Ten mathematical properties are derived by which to ascertain whether these ad-hoc models are conceptually deficient. We summarize past ad-hoc models into three models for foreign-currency options.

Sixth, this study is the first to test ad-hoc models in listed option markets and compare their performances with arbitrage-free models. Among the three ad-hoc models, we find that the modified Kassouf model, the only ad-hoc model that is free of any conceptual deficiency, performs best. Using the performance of the modified Kassouf model as a benchmark, we do not see evidence that the two arbitrage-free option-pricing models systematically predict market
option prices better than an ad-hoc model. In fact, in-sample analysis of the edited data shows that the modified Kassouf model systematically fits observed market prices better than the arbitrage-free models. However, out-of-sample analysis is not conclusive.

1.2 Logical Flow of Study

This study is organized as follows.

Chapters 2 and 3 examine underlying probability models that describe the observed price changes for foreign-currency.

Chapter 2 checks whether the Black-Scholes' distributional assumptions, which involve two separate hypotheses, hold for foreign-currency price changes. Section 2.2 tests whether price ratios for foreign currency are lognormal. Section 2.3.1 tests whether price changes for foreign currency are correlated. Section 2.3.2 tests whether the variance of log price changes for foreign currency is proportional to the time period observed. In Section 2.4, we conclude that in modeling the price changes for foreign currency, the Black-Scholes' stationary lognormal model is inadequate.

One stochastic process that proves consistent with the empirical price changes we observe in Chapter 2 for foreign currency is a mixed diffusion-Poisson process. Chapter 3 tests whether price changes for foreign currency follow a mixed diffusion-Poisson process. Two procedures, the cumulant-matching method and the maximum-likelihood estimation, are proposed for estimating the parameters for a mixed
diffusion-Poisson process. Because a mixed diffusion-Poisson process would reduce to a stationary lognormal process if its jump component disappears, we conduct a likelihood-ratio test for these two processes. The results of the likelihood-ratio tests show that the jump components of the foreign-currency prices differ significantly from zero. This finding suggests that Merton's (1976a) option-pricing model that assumes a mixed diffusion-Poisson process is a promising alternative model for pricing foreign-currency options.

Chapter 4 presents an introduction to option-pricing theories. We review the past literature of option-pricing models that incorporate an assumed probability distribution for the underlying asset's price changes. Section 4.3 reviews these probability option-pricing models developed before Black-Scholes. Section 4.4 discusses the Black-Scholes call option-pricing model. Section 4.5 surveys some alternative option-pricing models developed in the wake of Black-Scholes.

Chapter 5 develops two arbitrage-free models for foreign-currency options using techniques similar to Black and Scholes (1973) and Merton (1976a). Foreign-currency options have features that distinguish them from options on common stock. Because multiple interest rates are involved, the Black-Scholes model and Merton's (1976a) model for stock options both need some modification. Section 5.2 adapts the Black-Scholes model to price foreign-currency options by assuming that foreign-currency price changes follow a stationary lognormal process. Section 5.3 assumes that price changes for foreign
currency follows a mixed diffusion-Poisson process. In deriving the second model, we use Merton's assumption that the jump component of the currency prices represents "nonsystematic risk" that can be diversified away. The formula we derive in this chapter for a foreign-currency option subject to a mixed diffusion-Poisson process is new to the literature.

Chapter 6 uses an ad-hoc approach to study foreign-currency options. We assume that the relation between an option's price and the price of the underlying asset has been and remains stationary. This assumption justifies interpreting econometrically fitted functional forms to past data as descriptions of the relation between these prices and their potential determinants. Section 6.2 review past literature that uses an ad-hoc approach to estimate option prices. Besides examining the explanatory variables these models might contain, we study the mathematical properties of their proposed functional forms and explore implications these properties have in the context of option pricing. Ten new mathematical theorems are proved to evaluate these functional forms. Only the functional form proposed by Kassouf (1965) for stock warrants emerges as a form that is free of conceptual deficiencies. Several technical errors in Kassouf's model are corrected. A new estimation technique is introduced to develop price estimates for options based on this form. This form and two other forms that are taken from prior literature are used to estimate option prices for foreign currency in section 6.3.
Chapter 7 tests the ability of the option-pricing models derived in Chapter 5 and 6 to explain the pricing of foreign-currency options. To do this, we calculate the call values implied by each of the proposed option-pricing models. The implied call values are compared with observed market prices for call options with models' relative success judged by the size of the gap between the figures. The smaller is this gap, the better the model. Section 7.2 explains the data and proxies that are used to conduct empirical tests for foreign-currency options. Section 7.3 discusses computational problems for each model to be tested, particularly for the option-pricing model that is based on a mixed diffusion-Poisson process. Section 7.4 details the specific tests conducted for each model. Test results are presented in section 7.5. Section 7.6 concludes the study.
CHAPTER II
THE BEHAVIOR OF FOREIGN-CURRENCY PRICES

2.1 Introduction

This chapter examines the behavior of foreign-currency prices. In particular, we test the distributional assumptions that are used in the Black-Scholes option-pricing model. Because we use the transactions surveillance report compiled daily by the Philadelphia Stock Exchange (PHLX) to conduct empirical tests for foreign-currency options in Chapter 7, to make the study consistent, daily closing prices for foreign currency from PHLX is used. The daily closing prices for foreign currency from PHLX are taken along with the foreign-currency option prices from the Wall Street Journal. Five foreign-currency prices from PHLX are reported in the Wall Street Journal: British pound (BP), Japanese yen (JY), Deutsche marks (DM), Swiss franc (SF) and Canadian Dollars (CD). Because trading in the CD is infrequent\(^6\), this study investigates the first four currencies only. The period studied in this chapter runs from February 28, 1983 to June 27, 1985, the same period covered by the transactions surveillance report.

\(^6\) The number of observations for the CD call option is only 1415, while the number for the BP, JY, SF and DM are 4109, 3861, 4944, 5901.
We denote $S_t$ as the foreign-currency spot price at time $t$ in units of U.S. dollars, $x_t$ as the first differences of the natural logarithms of $S_t$, i.e.,

$$x_t = \ln S_t - \ln S_{t-1}.$$  

(2.1)

We note that, (2.1) implies

$$x_t = \ln \left( \frac{S_t}{S_{t-1}} \right),$$

$$\Rightarrow \left( \frac{S_t}{S_{t-1}} \right) = \exp(x_t),$$

$$\Rightarrow S_t = S_{t-1} \exp(x_t).$$  

(2.2)

Equation (2.2) shows that $x_t$ is also the yield, with continuous compounding, from holding the security for one period.

In deriving their path-breaking option-pricing model, Black and Scholes (1973) make strong assumptions about the stochastic process the underlying asset prices must follow. Their distributional assumptions actually involve two separate hypotheses:

(1) asset price ratios conform to a lognormal distribution, i.e., $x_t$ follows a normal process.

(2) price changes are independent. To be specific, the successive logarithms price changes are independent, i.e., $x_t$ and $x_{t-1}$ are independent for any $t$. Two implications follow from this hypothesis. If $x_t$ and $x_{t-1}$ are independent, then

(a) $x_t$ and $x_{t-1}$ are uncorrelated.
(b) the variance of \( x_t \) is proportional to the time period observed, i.e., \( \text{Var}(x_{t+\Delta t}) = \sigma^2 \Delta t \), where \( \sigma^2 \) is an instantaneous variance for \( x_t \) and \( \Delta t \) is the time period observed.

We examine each of these hypotheses in detail in Section 2.2, 2.3.1, and 2.3.2. Section 2.4 concludes the results in this chapter.

2.2 Are Price Ratios for Foreign Currency Lognormal?

This section tests the simple null hypothesis that the unknown distribution function (d.f.) \( F(x) \) of \( x_t \) is a normal distribution against the general alternative of nonnormality. We denote \( F^*(x) \) as a normal distribution whose population mean is \( \mu^* \) and variance is \( \sigma^2 \). We denote \( \bar{x} \) and \( s^2 \) as a sample mean and sample variance for \( x_t \) series.

The hypotheses at issue are:

\[
H_0: F(x) = F^*(x) \quad \text{for } -\infty < x < \infty,
\]

\[
H_1: \text{The hypothesis } H_0 \text{ is not true.} \tag{2.3}
\]

If the null hypothesis in (2.3) is true, then \( E(\bar{x}) = \mu_* \), \( E(s^2) = \sigma^2 \), and \( \bar{x} \) would be a normal variate even in small samples. This problem is a nonparametric problem, because the unknown distribution from which the random sample is taken might be any continuous distribution.
One simple way of analyzing the null hypothesis in (2.3) is to construct frequency distributions for daily currency prices. For each foreign-currency prices observed, the proportions of price changes within given standard deviations of the mean change can be computed and compared with what would be expected if the distributions were exactly normal.

If $x_t$ is normally distributed, then $z = \frac{x_t - \bar{x}}{s}$ is asymptotically a standardized normal variate because both $\bar{x}$ and $s$ are consistent estimators for $\mu_*$ and $\sigma_*$. Given $\bar{x}$ and $s$, we can construct empirical frequency distribution for every range. For convenience, we compute the cumulative frequency distribution within a given number of sample standard deviations around the sample mean. For example, for any positive constant $c$, we compute the cumulative frequency that a given set observation is within $c$ sample standard deviation around the sample mean, i.e.,

$$
Pr(-c < z < c) = Pr(-c < \frac{x_t - \bar{x}}{s} < c) = Pr(\bar{x} - cs < x_t < \bar{x} + cs). \tag{2.4}
$$

Table 1 shows the cumulative probability given by equation (2.4) for $c = .5, 1.0, 1.5, 2.0, 2.5, 3.0, 4.0, 5.0$ and $c > 5.0$. For example, Table 1 shows that the total frequency within 0.5 standard deviations of from the mean for the British Pound is .4838. A casual look at Table 1 indicates that, the frequency distribution of log price changes for foreign currency differs significantly from a normal
distribution. For each currency the empirical distributions are more peaked in the center and have longer tails than a normal distribution. The proportions of a central observations, e.g., those within 0.5, 1.0, 1.5 standard deviations, all exceed what should be expected from the unit normal. Except for the DM, proportions of observations within 2.5, 3.0, 4.0 and 5.0 standard deviation are all less than what would be expected from the unit normal. Even for the DM, the total frequency within 3.0, 4.0 and 5.0 standard deviation are less than the unit normal.

Figure 1 plots an empirical frequency bar chart for \( x_t \). The bar charts for every currency prices are bell-shaped and more peaked in the center. At first glance, the presence of leptokurtosis in Figure 1 is not obvious when compared to the peak. However, this is a wrong impression. Because the normal hypothesis implies that the total predicted relative frequency beyond three, four and five standard deviations are only \( .27\% \), \( .0062\% \), \( .00006\% \) respectively. The slight leptokurtosis shown in the Figure 1 has a big impact on the relative size of the tails. For example, the actual excess frequency for the BP after 5 standard deviations in Table 1 is 2,839 times larger than the total expected frequency for a normal process after 5 standard deviations. If one considers the frequency after 5 standard deviations unreliable for our sample size, he may look at the British Pound in Table 1, which the actual excess frequency after 3 and 4 standard deviation is 6.9 and 109 times larger than the total expected
frequency for a normal process. The departure from a normal process is still very clear.

A casual look at Table 1 and Figure 1 give some hints of the departure from a normal process for currency price changes, but we need a formal test to check whether the difference between the empirical distribution and a normal distribution is significant. Three test statistics are used in this section to test the null hypothesis in (2.3): the Kolmogorov-Smirnov D test, the Studentized range (SR) test, and the stable-process $\alpha$ test$^7$.

We define $D^*_n$ as the maximum difference between the sample d.f. $F_n(x)$ (with sample-size $n$) and the hypothesized d.f. $F^*(x)$, i.e.,

$$D^*_n = \sup_{x} |F_n(x) - F^*(x)|.$$ (2.5)

---

7. Another test statistic that is often reported for testing distribution is the Shapiro-Wilk (SW) test suggested in Shapiro and Wilk (1968). If a sample size is small, the powers of SW test, D test, SR test and $\alpha$ test differ. With a relatively large sample size, differences for these tests are small (see Fama and Roll [1971], Shapiro, Wilk and Chen [1968]). The SW statistic is based on a comparison of two estimates of the population variance: the usual variances estimator $s^2$ and the estimator $\sigma$. The estimator $\sigma$ is obtained by least squares estimates of the slope, when the ordered observation $x_t$ series are plotted against expected values of order statistics from a standard normal distribution. From a practical viewpoint, this procedure has some disadvantages. For each sample size $n$, a different set of coefficients is required for estimating $\sigma$. These required coefficients are hard to obtain for $n>50$ (see Shapiro and Wilk [1968], Stephens [1974]). Furthermore, in a simulation study, Fama and Roll (1971) report that for a sample size $n\geq99$, both SR and SW have almost perfect power against the alternative Paretoan stable process. For smaller samples, they find that the advantage of SW over SR is marginal. For these reasons, we do not try the SW test.
In (2.5), \( \sup f(x) \) denote the lowest upper bound for all the value of \( f(x) \). If the null hypothesis \( H_0 \) in (2.3) is true, then for any positive constant \( t \), Kolmogorov (1933) proves that:

\[
\lim_{n \to \infty} \Pr((n^{1/2}D_n^* \leq t)) = 1 - 2\sum_{i=1}^{\infty} (-1)^{i-1} e^{-2i^2t^2}.
\] (2.6)

Thus, if the null hypothesis \( H_0 \) is true, then as \( n \to \infty \) the d.f. of \( n^{(1/2)}D_n^* \) will converge to the d.f. given by (2.6). The value of \( n^{(1/2)}D_n^* \) will tend to be small if the null hypothesis is true, and will tend to be larger if the actual d.f. \( F(x) \) is different from \( F^*(x) \). A test procedure which rejects \( H_0 \) when \( n^{(1/2)}D_n^* > c \) is called a Kolmogorov-Smirnov test.\(^8\)

We denote \( x_j \) as the \( j \)th order statistic from a sample of size \( n \). The Studentized range is defined as

\[
SR = \left( \frac{x_n - x_1}{\sqrt{\frac{1}{n-1} \sum_{j=1}^{n} (x_j - \bar{x})^2}} \right)^{1/2} R/s. \tag{2.7}
\]

Thus, the Studentized range \( SR \) is the ratio of the range \( R \) to the sample standard deviation \( s \). Both \( R \) and \( s \) are measures of variation being calculated on the same sample of \( n \) observations. While the first measure can play a useful part in providing rapid tests in the

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8. The reader is referred to DeGroot (1975, p.465) for more details about the Kolmogorov and Smirnov test.
analysis of variance, the second may be useful in detecting heterogeneity of data or departure from normality. David, Hartley and Pearson (1954) show that the exact moments of the distribution of SR can be derived simply from the known moments of R and s and the ratio might be used to provide a quick assessment of the homogeneity or normality of data.

Another test for normality is a test, which is based on a special property of a stable process. A stable distribution is a four-parameter family (see Appendix A for an introduction). The location parameter is $\delta$, the characteristic exponent $\alpha$ determines the height of total probability contained in the tail of the distribution, the scale parameter is $\gamma=c^{\alpha}$, and the skewness parameter is $\beta$. Models based on normality assumptions are commonly justified by appeals to the Central Limit Theorem, because the variables under consideration are often treated as sums of random variables. But as explained in Appendix A, the normal process is just one member of the class of limiting distributions, of which the characteristic exponent $\alpha = 2$. We denote $F(S; \alpha, \beta, c, \delta)$ as the distribution function for a stable process with parameter $\alpha, \beta, c$ and $\delta$. We assume $F$ to be symmetric, so $\beta=0$. If $x_t$ series come from a symmetric stable distribution, then the estimate for $\alpha$ must be close to 2. The hypothesis at issue in (2.3) can be rewritten for this particular case:

$H_0: \alpha = 2$,
Several methods are designed to estimate the parameters $\alpha$ and the scale factor $c$ for a symmetric stable distribution. We use the simplest consistent one that is proposed by McCulloch (1984), which is an improved version of the method initiated by Fama and Roll (1968, 1971). We use the method of .5 truncated means as suggested by Fama-Roll (1968; p.827) to compute the location parameter $\delta$.

Table 2 reports the results of the Kolmogorov-Smirnov D test, the Studentized ranged SR test and the stable process $\alpha$ estimate test. The
evidence overwhelmingly indicates that the $x_t$ series are not generated by a normal process. Both the D statistics and the SR test are significant at 1 percent level for all the currency price changes. All the characteristic exponent $\alpha$ estimates are around 1.5 and 1.6 indicating a serious deviation from the normal case 2.0.

2.3 Are Price Changes for Foreign Currency Independent?

In statistical terms, if two random variables $x_{t-1}$ and $x_t$ are intertemporal independent, then the covariance of these two variables

$$\text{Cov}(x_{t-1}, x_t) = 0.$$  \hspace{1cm} (2.9)

The variables that have zero covariance are called linearly uncorrelated. It is important to note that while independence necessarily implies zero covariance, the converse is not true. Thus, if we find that $x_{t-1}$ and $x_t$ are linearly correlated, then we can reject the hypothesis that $x_{t-1}$ and $x_t$ are independent. However, if we find that $x_{t-1}$ and $x_t$ are not linearly correlated, we cannot claim that $x_{t-1}$ and $x_t$ are independent. Because $x_t$ and $x_{t-1}$ may be nonlinearly correlated, to further establish the property of $x_t$ series, we complement by checking another implication of independence.

If $x_{t-1}$ and $x_t$ are intertemporal independent, the variance of $x_t$ series is proportional to the time period observed. Thus, by examining
whether the variance of $x_t$ series is proportional to the time period observed, we can reject the null hypothesis that $x_{t-1}$ and $x_t$ are independent.

### 2.3.1 Are Price Changes for Foreign Currency Correlated?

This section tests the null hypothesis that $x_t$ series are independent by testing whether $x_t$ series are serially correlated. If $x_t$ series are serially correlated, we can conclude that $x_t$ series is not independent. If the $x_t$ series proves uncorrelated, then we cannot reject the null hypothesis that $x_t$ series are independent. Two statistics are used to test the null hypothesis of no serial correlation for $x_t$ in this section: the von Neumann Ratio test, and the Pearson correlation coefficient test.

The Von Neumann Ratio (VNR) is the ratio of mean-square successive differences to the variance, and is related algebraically to the serial correlation coefficient. We define

$$
\delta^2 = \frac{1}{n-1} \sum_{t=1}^{n-1} (x_{t+1} - x_t)^2.
$$

(2.10)

$$
s^2 = \frac{1}{n} \sum_{t=1}^{n} (x_t - \bar{x})^2.
$$

(2.11)
In (2.11), \( x = \frac{\sum_{t=1}^{n} x_t}{n} \). If \( x_t \) series are independent, von Neumann (1941) proves that for large \( n \), the ratio \( \delta^2/s^2 \) is normally distributed with mean and variance given by

\[
E(\delta^2/s^2) = \frac{2n}{n-1}, \quad (2.12)
\]

\[
V(\delta^2/s^2) = \frac{4n^2(n-2)}{(n+1)(n-1)^3}, \quad (2.13)
\]

In (2.12) and (2.13), \( E(x) \) denotes the expected value of \( x \), \( V(x) \) denotes the variance of \( x \). For small samples, e.g., \( n \leq 60 \), the ratio is not normal. However, Hart (1962) prepares a table that can be used in this case. Thus, if \( x_t \) series are independent, the von Neumann ratio \( \delta^2/s^2 \) is a normal variable for large \( n \). If the von Neumann ratio deviates from a normal variable significantly, then \( x_t \) series must not be independent.

Table 3 reports the von Neumann ratios for all the currency prices. Only the DM exhibits a significant test value. The von Neumann ratio tests for BP, JY and SF are not significant. These findings suggest that the DM are serially correlated. Because the von Neumann ratio only tests against the first-order autocorrelation, however, these findings do not enable us to conclude that BP, JY and SF are not

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10. This can be seen from equation (2.10) which calculates only the successive differences for \( x_t \), i.e., the first-order difference for \( x_t \). We also note that, the first-order autoregression coefficient estimates, which are reported in Table 3, are consistent with those of the von Neumann ratio test. Only the DM shows a significant first-order autocorrelation.
serially correlated. To conclude that whether $x_t$ series are not correlated, we need to check whether $x_t$ series exhibit a higher-order autocorrelation or a multi-period autocorrelation.

To investigate the multi-period autocorrelation for $x_t$ series, Pearson Correlation Coefficients tests are run for $x_t$ series against the lagged-$k$ series $x_{t-k}$, for $k=1$ to $9$. The pearson correlation coefficient is computed as:

$$r = \frac{SS_{x_{t-k} \cdot x_t}}{(SS_{x_{t-k}} \cdot SS_x)^{1/2}}$$

for $k=1,...,9$. (2.14)

In (2.14), $SS_{x_{t-k} \cdot x_t} = \frac{1}{n-k} \sum_{t=k+1}^{n} (x_{t-k} - \bar{x})(x_t - \bar{x})$,

$$SS_{x_{t-k}} = \frac{1}{n-k} \sum_{t=k+1}^{n} (x_{t-k} - \bar{x})^2,$$

$SS_x = \frac{1}{n} \sum_{t=1}^{n} (x_t - \bar{x})^2$.

The range of the Pearson correlation coefficient is $-1 < r < 1$. The variance of the Pearson correlation coefficient is known and can be used to test whether the correlation coefficient of two series differs significantly from zero. Table 4 presents the Pearson correlation coefficients between $x_t$ series and $x_{t-k}$ series for $k$ from 1 to 9. Except for the noted first-period autocorrelation case from the DM, no currency price changes exhibits a multi-period autocorrelation at even 5 percent significant level. Thus, except for the DM, we cannot reject

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11. The reader is referred to Snedecor and Cochran (1980) for a proof of the property associated with the Pearson correlation coefficient.
the null hypothesis that the price changes for foreign currency are uncorrelated.

2.3.2 Tests of the Variance-Time Function

This section examines whether the variance of $x_t$ series is proportional to the time period observed. Assuming the distributional function $F(x)$ is stationary and intertemporally independent, then the variance of $x_t$ series is proportional to the time period observed. Thus, by testing whether the variance of $x_t$ series is proportional to the time period observed, we also test whether $x_t$ series are intertemporally independent [see Working (1949), Mandelbrot and Wallis (1969), Poole (1967), Young (1971), Schwartz and Whitcomb (1981)].

We define $x_{it}$ as the $i$th differences of the natural logarithms of $S_t$, i.e.,

$$x_{it} = \ln S_t - \ln S_{t-i}. \quad (2.15)$$

Under the definition of (2.15), $x_t$, the original data defined by (2.1), becomes a special case for $x_{it}$, i.e., $x_{1t}$. If $x_{it}$ series are independent and has a stationary variance, then the variance of $x_{it}$ series must equal $i$ times the variance of the $x_{1t}$ series. This can be proved as follows:

$$x_{it} = \ln S_t - \ln S_{t-i}.$$
\[ (\ln S_t - \ln S_{t-1}) + (\ln S_{t-1} - \ln S_{t-2}) + \cdots + (\ln S_{t-i+1} - \ln S_{t-i}), \]

\[ = \sum_{j=0}^{i-1} x_1(t-j). \]  

(2.16)

Applying the variance operator to (2.16), we obtain:

\[ \nu(x_{1t}) = \sum_{t=1}^{i} \sum_{u=1}^{i} \sigma_t \sigma_u \rho_{t,u}. \]  

(2.17)

In (2.17), \( \sigma_t \) and \( \sigma_u \) are the standard deviation of \( x_{1t} \) series at time \( t \) and \( u \) respectively; \( \rho_{t,u} \) is the correlation between \( x_{1t} \) and \( x_{1u} \).

Assuming that the variance parameter of \( F(x) \) that generates \( x_{1t} \) series is stationary, we have: (1) \( \sigma_t = \sigma_u \) and hence \( \sigma_t \sigma_u = \text{var}(x_{1t}) = \sigma^2 \) for all \( t,u \); (2) \( \rho_{t,u} \) is the same for any given distance between \( x_t \) and \( x_u \). We denote \( s \) as the period distance between \( t \) and \( u \), i.e., \( s = |t-u| \).

Assuming \( F(x) \) is stationary, we have \( \rho_{t,u} = \rho_{1,1+s} \). Thus, separating the variance and covariance terms, equation (2.17) becomes:

\[ \nu(x_{1t}) = i \cdot \sigma^2 + 2 \sum_{s=1}^{i} \sigma^2 \rho_{1,1+s}. \]  

(2.18)

Assuming that \( x_{1t} \) series are independent, then the correlation between \( x_t \) and \( x_u \) are zero, i.e., \( \rho_{1,1+s} = 0 \) for all \( s \). Equation (2.18) reduces to:

\[ \nu(x_{1t}) = i \cdot \sigma^2 = i \cdot \nu(x_{1t}), \]

\[ \Rightarrow \quad R_s = \frac{\nu(x_{1t})}{\nu(x_{1t})} = 1. \]  

(2.19)
If \( x_{1t} \) series is not independent, e.g., \( x_{1t} \) series exhibits autocorrelation for the series \( x_{st} \), then \( \rho_{1,1+s} \neq 0 \) and \( \rho_{1,1+k} = 0 \) for all \( k \neq s \). The general time-variance relationship, equation (2.18) becomes:

\[
v(x_{1t}) = i \cdot \sigma^2 + 2 \sigma^2 \rho_{1,1+s}.
\] (2.20)

And the variance ratio \( R_i \) in equation (2.19) becomes,

\[
R_i = \frac{\hat{v}(x_{1t})}{\hat{v}(x_{1t})} = i(1+2\rho_{1,1+s}).
\] (2.21)

We denote \( \hat{v} \) as a consistent estimator for population variance \( v \), and \( \hat{R}_i = \frac{\hat{v}(x_{1t})}{\hat{v}(x_{1t})} \) as the sample variance ratio, then

\[
\text{Plim}(R) \equiv \text{Plim} \left[ \frac{\hat{v}(x_{1t})}{\hat{v}(x_{1t})} \right] = i(1+2\rho_{1,1+s}).
\] (2.22)

In (2.22), Plim is the probability-limit operator.

Thus, if the original \( x_{1t} \) series is independent, \( \hat{R}_i \), the ratio of estimated variance of \( i \)-periods difference to the estimated variance of \( 1 \)-period difference, must be roughly equal to \( i \). Systematic differences in these estimates as a function of the differencing interval may indicate the nature of any serial dependence. If \( \hat{R}_i \) is systematically less than \( i \), the series may exhibit some sort of negative autocorrelation and vice versa.

We note that \( \hat{R}_i \) equal \( i \) only in probability limit. Even if the original \( x_{1t} \) series is independent, the sample variance ratio \( \hat{R}_i \) would differ from the ideal ratio \( i \). How far can the sample variance ratio
differ from the ideal ratio for us to conclude that \( x_{1t} \) series are not generated from an iid process? To understand the sampling behavior of the nonoverlapping sample variance ratio \( \hat{R}_i \), we conduct a simulation study. We draw 500 samples each with 587 iid normal variates. We calculate simulated sample variance ratio \( \hat{R}_i \) for this artificial series with non-overlapping differencing intervals of 1, 2, 3, 4, ..., to 20. Table 5 reports the simulation result. Although simulation results are only suggestive, the exercise does give us a benchmark against which to compare the sample variance ratio from foreign-currency price changes.

Because all the simulated samples are independently drawn from a normal distribution, each sample variance ratio is independent. Applying the Central limit theorem, the distribution of the average of the simulated sample variance ratio \( \hat{R}_i \), converges to a normal process with a mean of \( i \). To examine the simulated sampling behavior of \( \hat{R}_i \), we calculate the standard deviation for \( \hat{R}_i \) based on our simulated 500 samples and report the range of 1.96 standard deviation from the mean of the simulated sample variance ratio. These ranges are used as a confidence interval for testing the variance-time relation. If the variance-time ratios \( \hat{R}_i \) we compute from daily currency prices are beyond the simulated reference ratio-range, then \( x_{1t} \) series is not
likely to be independent. Otherwise, we cannot reject the null hypothesis that $x_{1t}$ series is independent.

To use table 5 to conduct a test for variance-time ratio, we must be aware of two problems. First, as the distance between periods increases, the sample size for each sample sharply decreases. For example, the sample size of the series $x_{20t}'$, which is the currency price series for 20-period difference, shrinks from 587 to 28. This small sample makes the 1.96 standard deviation interval reported in Table 5 very wide and not very useful. Second, there are $i$ different ways to count the new series $x_{it}$ from $x_{1t}$ series. For example, depending on the base period chosen, there are three ways to construct $x_{3t}$ from $x_{1t}$. If we choose $x_1$ as the base period, then $x_1, x_4, x_7, \ldots, x_{k+3}, \ldots$ is one of the series $x_{3t}$. If we use $x_2$ as the base period, then $x_2, x_5, x_8, \ldots$ is another series for $x_{3t}$. Choosing $x_3$ as the base period, $x_3, x_6, \ldots$ is another series for $x_{3t}$. If the original series $x_{1t}$ is independent, all the different counting series for $x_{1t}$ with an $i$-period difference must be independent. They can be treated as different drawings from the same distribution. To use all the information we have from $x_{1t}$ series, we construct variance-time ratio for the observed currency price changes $x_{1t}$ series based on all $i$ methods. The average of the variance ratio $\hat{R}_i$ is reported in table
6. Because the first method always produces a largest sample size among the \( i \) methods to construct \( x_{1t} \), we also report the variance \( \hat{R}_i \) that is based on the first method. Results are presented in Table 6.

In comparison to the simulation results in Table 5, except for the DM the variance-time ratios \( \hat{R}_i \) all stay within their 1.96 standard deviation interval. This indicates that except for the DM, the variance-time relation holds for currency price changes. We cannot reject the null hypothesis that \( x_{1t} \) series is intertemporally independent except for the DM. This result is consistent with that of section 2.3.1 but differs from those reported in Poole (1967). Poole (1967) finds that the variance-time ratio for foreign-currency price changes rises more than linearly with longer differencing intervals and suspects the existence of positive serial dependence in ten series. The data Poole (1967) uses consist of daily currency prices for ten industrial country from 1950-62. Recognizing that the same currency in a different period could exhibit different nature, our result in no way invalidate those of Poole's. However, we must point out two shortcomings that are associated with Poole's result. First, Poole (1967) does not recognize that there are \( i \) different ways to calculate the variance ratio \( \hat{R}_i \). Thus, his result must be based only on the first method we discuss earlier. By doing so, he discards some information from his data and his estimator \( \hat{R}_i \) is not as efficient as
ours. Second, no reference interval for $R_i$ is developed in Poole's study. Thus, Poole's conclusion that the variance ratios differ significantly from the ideal ratio is too casual.

2.4 Conclusion for the Behavior of Foreign-Currency Prices

From the evidence presented in this chapter, we find that the Black and Scholes (1973) assumption that the underlying asset price changes follow a stationary lognormal process does not hold for foreign-currency price changes. Specifically, we find that:

(1) currency price ratios do not conform to a lognormal distribution.
(2) except for the DM, we cannot reject the null hypothesis that price changes for foreign currency are independent. This conclusion follows from two test results from Section 2.3.1 and Section 2.3.2:

(a) except for the DM, foreign-currency price changes are not linearly correlated.
(b) except for the DM, the variance of $x_t$ is roughly proportional to the time period observed.
(3) the empirical distribution for currency price changes are more peaked in the center and have longer tails than a normal distribution.

One stochastic process that is consistent with the facts of (1), (2) and (3) is a mixed Diffusion-Poisson process. In Chapter III, we propose a mixed diffusion-Poisson process to model price changes for foreign currency.
CHAPTER III
THE DIFFUSION-POISSON MODEL OF FOREIGN-CURRENCY PRICE CHANGES

3.1 Introduction

In Chapter II, we find that empirical foreign-currency price changes tend to be more peaked around their mean and to have fatter tails than the distribution of a comparable lognormal variate. One elegant model capable of generating these observations is a mixed diffusion-Poisson model originally proposed by Press (1967). In his paper on the distribution of stock price returns, Press argues that the behavior of log returns can be portrayed as the sum of two independent components: (1) a continuous diffusion component which is responsible for the usual day-to-day price movement and is rendered by traditional normal process; and (2) a discontinuous jump component which captures the arrival of important (i.e., disruptive) new information to the market.

This model is useful in describing foreign-currency price movements. The diffusion component results from temporary imbalances between supply and demand and only marginally affects the value of the foreign-currency price. Important information causes significant price changes and is modelled by a jump process. Compared to stock prices, whose movement is very vulnerable to insider trading and whose jump component may be subject to non-random influences, foreign-currency prices are much more difficult for individual parties to manipulate.
It is generally agreed that the fundamental determinants of foreign-currency prices, such as monetary policy and fiscal policy are highly variable and that government interventions are unsteady (Levich [1979]). Thus, the arrival of this new information on government policies can be treated as random.

Formally, we assume that the total percentage price changes for foreign currency are composed of two types of changes: a normal vibration in price and an abnormal vibration in price. The normal vibration in price is described by a stationary normal process. The abnormal vibration in price is modelled by a Poisson process. We also assume that the Poisson-driven events are independent and identically distributed. This lets us partition the probability of an event occurring during a time interval of length $h$ as:

(a) $\text{prob\{the event does not occur once in the time interval}\,(t,t+h)\} = 1 - \lambda h + O(h)$,

(b) $\text{prob\{the event occurs once in the time interval}\,(t,t+h)\} = \lambda dt + O(h)$,

(c) $\text{prob\{the event occurs more than once in the time interval}\,(t,t+h)\} = O(h)$,

where $O(h)$ is the asymptotic order symbol defined by $\#(h) = O(h)$ if $\lim_{h \to 0} [\#(h)/h] = 0$, i.e., $O(h)$ is of a much smaller order of magnitude than $h$, and $\lambda$ is the mean number of jump arrivals per unit time. Thus during time interval $h$ the expected number of jump arrivals is $\lambda h$. 

Whenever the Poisson event occurs, a jump is observed in the foreign-currency price. We let $Y$ be the random variable describing the jump size. Neglecting the continuous part of the process, the currency price at time $t+h$, $S(t+h)$, becomes a random variable $S(t+h)=S(t)Y$, given the condition that exactly one such arrival occurs between $t$ and $(t+h)$. Given that the Poisson event occurs, $(Y-1)$ becomes the random variable describing the percentage change in the foreign-currency price. The posited currency price returns can be written formally as a stochastic differential equation:

$$dS/S = [\alpha+(1/2)\sigma^2]dt + \sigma dZ + (Y-1)d\Pi. \quad (3.1)$$

In (3.1), $\alpha$ is the instantaneous expected return on the currency return; $\sigma^2$ is the instantaneous variance of the return, conditional on the nonoccurrence of the Poisson event; $dZ$ is a standard Gauss-Wiener process; $d\Pi(t)$ is a standardized Poisson random variable; and $dZ$ and $d\Pi$ are assumed to be independent.

We assume $Y_i$, the random variable describing the jump size of the $i$th Poisson event, follows a stationary lognormal process. This means that $y_i=\ln(Y_i)$ is a normal variate possessing the following characteristics:

$$E(y_i) = \mu, \quad V(y_i) = \delta^2, \quad E(Y_i) = E(\exp(y_i)) = \exp(\mu+\delta^2/2). \quad (3.2)$$
An interesting special case occurs, which we assume hereafter, when \( \alpha, \sigma, \lambda \) and \( \kappa \) are constants. This allows (3.1) to be written in an alternative but equivalent form:
\[
S_T = S_t \exp\left( s + \sum_{i=1}^{\pi_1(\tau)} y_i \right).
\] (3.3)

In (3.3), \( s \) is a normally distributed random variable with \( \mu(s) = \alpha \sigma \) and \( \sigma^2 \tau \), and \( \pi_1(\tau) \) is a Poisson random variable describing the total number of jump events occurs during the time interval \( \tau \). Taking a natural logarithm on both sides of (3.3), we obtain:
\[
\ln \left( \frac{S_T}{S_t} \right) = s + \sum_{i=1}^{\pi_1(\tau)} y_i.
\] (3.4)

Equation (3.4) shows that the log price ratio \( \ln \frac{S_T}{S_t} \) is a Poisson mixture of normal variables. For \( \lambda = 0 \), the total number of jump events \( \pi_1(\tau) \) is zero and this model reduces to the Black-Scholes stationary

---

12. We ignore the jump part of (3.1) and let \( X_t = \ln(S_t) \), i.e., \( S_t = \exp(X_t) \). Applying Ito's Lemma to \( X = \ln(S) \), we obtain:
\[
dX = \alpha dt + \sigma dz.
\]
This implies \( X_T = X_t + \alpha \tau + \sigma Z(\tau) \) by Ito's integration (the opposite operation of Ito's lemma), where \( \tau = T - t \). Using the fact that \( S = \exp(X) \), this implies \( S_T = S_t \exp[\alpha \tau + \sigma Z(\tau)] \). Putting the jump part into the former equation and noting that the jump part \( Y(n) = \sum_{j=1}^{\pi_1(\tau)} y_j = \exp(\Sigma_j y_j) \), we obtain
\[
S_T = S_t \exp(\alpha \tau + \sigma Z) \times \prod_{j=1}^{\pi_1(\tau)} y_j
\]
Because \( Z(\tau) \) is a standard Gauss-Wiener process, \( \alpha \tau + \sigma Z \) is a normal random variable with mean = \( \alpha \tau \) and variance = \( \sigma^2 \tau \).
lognormal assumption. The probability density for the log price ratio in (3.4) can be described as:

$$\ln \left[ \frac{S_T}{S_t} \right] \sim \sum_{n=0}^{\infty} \frac{e^{-\lambda T}(\lambda T)^n}{n!} N(x; \alpha T + nu, \tau^2 + n \delta^2).$$  \hspace{1cm} (3.5)

In (3.5), $\tau = T - t$, $x = \ln \left[ \frac{S_T}{S_t} \right]$ and $N(x; \mu, \nu^2)$ is a probability density for a normal variable $x$ with mean $\mu$ and variance $\nu^2$, i.e.,

$$N(x; \mu, \nu^2) = (2\pi \nu^2)^{-1/2} \exp\left( -\frac{(x-\mu)^2}{2\nu^2} \right).$$

Because we use daily currency price data in this chapter, we assume that $\tau = T - t = 1$ and $x = \ln(S_T) - \ln(S_{t-1})$, which is same as equation (2.1). In this case, the density for $x_t$ is

$$f(x) = \sum_{n=0}^{\infty} \frac{e^{-\lambda T}(\lambda T)^n}{n!} N(x; \alpha + nu, \sigma^2 + n \delta^2).$$ \hspace{1cm} (3.6)

We denote $\phi_x(t) = \mathbb{E}[\exp(itx)]$ as the characteristic function for $f(x)$ and $k_i$ as the $i^{th}$ population cumulant. The population cumulant $k_i$ for a density function is defined as the coefficient of $(it)^i/i!$ in $\log \phi_x(t)$. Press (1967) finds the population cumulants that are implied in equation (3.6). In particular, he shows that

$$k_1 = \mathbb{E}(x) = \lambda \mu + \alpha,$$

$$k_2 = (\sigma^2 + \lambda \delta^2 + \lambda \mu^2),$$

13. The reader is referred to Kendall and Stuart (1977a, p.65) for more details about cumulants.
Using the known cumulants in (3.7), the coefficients of kurtosis $K$ and skewness $SK$ may be defined by\textsuperscript{14}:

\begin{align*}
K = & \frac{K_4}{(K_2)^2}. \\
SK = & \frac{K_3}{(K_2)^{3/2}}.
\end{align*}

(3.8) \hspace{1cm} (3.9)

Because the denominator of $K$ is always positive, the sign of $K$ depends on $K_4$. For a normal process, $\lambda=0$, $K_4$ of (3.7) becomes zero. Thus, the coefficient of kurtosis $K$ for a normal process is zero. Any positive number for $K$ indicates a fatter tail. For the mixed diffusion-Poisson process that is defined in (3.6), $K_4 = \lambda(3\delta^4 + 6\mu^2 \delta^2 + \mu^4)$ is always positive. The sign of $SK$ depends on $K_3$. For a normal process, $\lambda=0$, $K_3$ of (3.7) becomes zero. Thus, the coefficient of skewness $SK$ for a normal process is zero. Because $K_3 = \lambda\mu(3\delta^2 + \mu^2)$ for a mixed diffusion-Poisson process, if $\mu=0$, then $K_3=0$. Thus, if $\mu=0$, a mixed diffusion-Poisson process is symmetric.

Thus, we show that a mixed diffusion-Poisson process is leptokurtic, i.e., has fatter tails in the end. Moreover, it could be

\textsuperscript{14} The coefficients for kurtosis and skewness we use here are consistent with those of Press (1967), Kendall and Stuart (1977a), Beckers (1980).
symmetric. Therefore, a mixed diffusion-Poisson process might better describe foreign-currency price changes than the pure lognormal model (which can be considered to be the special case where \( \lambda = 0 \)). Difficulties arise with the empirical implementation and verification of this stochastic process. In Section 3.2, we examine past research on estimating the parameters for this model, particularly the cumulant-matching method. In Section 3.3, we discuss how to obtain maximum-likelihood parameters for this model. Parameter estimates and test results are presented in Section 3.4.

3.2 Estimating the Parameters of a Diffusion-Poisson Model

Unless we make special assumptions, five unknown parameters need to be estimated for a mixed diffusion-Poisson process (3.6): \( \alpha, \delta, \sigma, \mu, \lambda \). Two different estimators are often used to estimate unknown parameters for a known probability process: a maximum-likelihood estimator (MLE) and a cumulant-matching estimator (CME). Maximum-likelihood estimators are designed to find the parameters that maximize the observed samples' likelihood function. A maximum-likelihood estimator is consistent and asymptotically efficient\(^{15}\). The cumulant-matching method relies upon the theoretical relation between the population cumulants and the parameters of the distribution. This method equates the population cumulants to their observed sample

\(^{15}\) The reader is referred to Wilks (1962) for the proof.
cumulants and solve the resulting equations for parameter estimates. A CME is consistent because a sample cumulant is a consistent estimator for a population cumulant. However, compared to a MLE, a CME is not efficient. Thus, a MLE is a preferred estimator.

A preliminary analysis of (3.6), however, reveals maximum-likelihood estimation to be computationally impractical. First, first-order conditions for likelihood maximization are highly nonlinear and require the inclusion of an indefinite large number of terms. Second, the likelihood function is unbounded in parameter space\(^\text{16}\). Attempts to find a global maximum lead to inconsistent estimates. Using the cumulant-matching method is an alternative. Because there are five unknown parameters involved, we can equate the first five population cumulants to their observed sample cumulants and solve the resulting equations for the parameter estimates. However, solving the unrestricted system of equations which relates the sample cumulants to the parameter values does not give satisfactory results. In most cases the equations are nonlinear, leading to parameter estimates in the form of complex numbers and useless. Therefore, Press (1967), a priori constrains \(\alpha\), the instantaneous expected rate of return for the diffusion component to be zero. This simplifies the density of (3.6) to:

\[
f(x) = \sum_{n=0}^{\infty} \frac{e^{-\lambda n}}{n!} N(x;n\mu, \sigma^2 + n\delta^2).
\]

\[\text{\text{(3.10)}}\]

\(^{16}\) We prove this point in Section 3.4.
Press (1967) derives the characteristic function for $f(x)$ in (3.10) as:

$$\ln \phi(t) = -t^2 \sigma^2 / 2 + \lambda \{ \exp [itu - (t^2 \delta^2 / 2)] - 1 \}. \quad (3.11)$$

Using this simplified form, the first four population cumulants in (3.7) reduce to:

$$K_1 = \lambda \mu,$$
$$K_2 = \sigma^2 + \lambda (\mu^2 + \delta^2),$$
$$K_3 = \lambda \mu (\mu^2 + 3 \delta^2),$$
$$K_4 = \lambda (\mu^4 + 6 \mu^2 \delta^2 + 3 \delta^4). \quad (3.12)$$

Because the population cumulants are unknown, the corresponding sample cumulants (denoted by $\hat{K}$) are substituted into (3.12). The system of equations (3.12) is then solved for the vector parameter estimates $\hat{\mu}, \hat{\sigma}, \hat{\delta}, \hat{\lambda}$:

---

17. To understand how (3.11) is derived, we note that the probability process can be expressed as equation (3.4)

$$\ln (S_t / S_0) = s + \sum_{i=1}^{\hat{n}(\tau)} y_i,$$

where $s$ is a normal variable with $E(s) = \alpha \tau$ and $V(s) = \sigma^2 \tau$, $y_i$ is another normal variable with $E(y_i) = \mu$, $V(y_i) = \delta^2$ and $\hat{n}(\tau)$ is a Poisson variable describing the total number of jump events occurs during the time interval $\tau$. The characteristic function for normal variables $s$ and $y$ are $\exp [i \alpha t - (1/2) \sigma^2 t^2]$ and $\exp [i \mu t - (1/2) \delta^2 t^2]$. The characteristic function for the Poisson variable $\hat{n}$ is $\exp \{ \lambda [\exp (it) - 1] \}$. Using these three characteristic functions, equation (3.4), and the fact that $\alpha = 0$, we can obtain (3.11).
\[
\hat{\mu}^4 - 2 \left( \frac{K_3}{K_1} \right) \hat{\mu}^2 + \left( \frac{3K_4}{2K_1} \right) \hat{\mu} - \frac{K_3^2}{2K_1} = 0,
\]

\[
\lambda = \frac{K_4}{\mu},
\]

\[
\hat{\delta}^2 = \frac{(K_3 - \hat{\mu}^2 K_1)}{3K_1}.
\]

\[
\hat{\sigma}^2 = K_2 - \left( \frac{K_1}{\mu} \right) \left( \hat{\mu}^2 + \left( \frac{K_3 - \hat{\mu}^2 K_1}{3K_1} \right) \right).
\]

Because an exact relation holds between the cumulants and moments of a probability distribution, the sample cumulants in (3.13) can be calculated based on sample moments. We denote \( m_r \) as population moment of order \( r \) about 0. Kendall-Stuart (1977a) establish

\[
K_1 = m_1,
\]

\[
K_2 = m_2 - m_1^2,
\]

\[
K_3 = m_3 - 3m_1 m_2 + 2m_1^3,
\]

\[
K_4 = m_4 - 3m_2 m_3 + 12m_1^2 m_2 - 6m_1^4,
\]

\[
K_5 = m_5 - 6m_3 m_2 - 15m_4 m_1 + 30m_1^2 m_3 + 10m_2^2 - 120m_1 m_2 m_3 + 120m_1^2 m_2 - 270m_2 m_1^2 + 360m_2 m_1^4 - 120m_1^6.
\]

Using a sample of monthly returns from ten NYSE-listed common stocks over the period 1926 through 1960, Press calculates the parameter estimates based on (3.13) and (3.14). Press reports that the estimates for the variance parameters \( \sigma^2 \) and \( \delta^2 \) are frequently to be negative. Press attributes the problem to his relatively small
samples. Beckers (1981), however, argues that the estimation method does not yield parameters which are descriptive of the true underlying distribution\(^\text{18}\).

In modifying the Press procedure, Beckers (1981) sets the mean logarithmic jump size equal to zero, \(\mu = 0\), rather than \(\alpha = 0\). This allows the security price returns to experience positive or negative jumps at random intervals, but requires jumps to average zero. In this special case, the probability density in (3.6) reduces to:

\[
f(x) = \sum_{n=0}^{\infty} \frac{(e^{-\lambda} \lambda^n / n!) N(x; \alpha, \sigma^2 + n\delta^2)}{n!} = \mathcal{N}(x; \alpha, \sigma^2 + n\delta^2) \tag{3.15}
\]

We note that with \(\mu = 0\), the coefficient of skewness in (3.9) is zero. Thus, this distribution is symmetric. Based on the equation for population cumulants in (3.7), this implies that odd cumulants vanish after the first one. This also means that, to use the cumulant-matching method to solve for the four parameters, the sixth cumulant has to be included. Beckers shows

\[
K_1 = \mathbb{E}(x) = \alpha\tau,
\]

\[
K_2 = \text{Var}(x) = \sigma^2 \tau + \lambda\tau\delta^2,
\]

\[
K_3 = K_5 = 0,
\]

18. Using the formulas for kurtosis and skewness given in (3.8) and (3.9), Beckers (1981) uses the parameter estimates from (3.13) to compute the coefficients of kurtosis and skewness. Beckers compares these "implicit" values for kurtosis and skewness to the ones reported for the original sample and finds that the parameter values reported by Press do not characterize the underlying distribution.
\[ K_4 = 3\delta^4 \lambda \tau, \]
\[ K_6 = 15\delta^6 \lambda \tau. \]  

(3.16)

Solving this system of equations Beckers obtain:
\[ \hat{\lambda} = 25 \frac{K_4^3}{3K_6^2}, \]
\[ \hat{\alpha} = \frac{K_4}{K_6}, \]
\[ \hat{\delta}^2 = \frac{K_6}{5K_4}, \]
\[ \hat{\sigma}^2 = \frac{K_2 - 5K_4^2}{3K_6}. \]  

(3.17)

Even this model provides no guarantee that the variance estimates have a positive sign: the sign of \( \hat{\delta}^2 \) is completely determined by the sign of \( K_6 \) given that \( K_4 \) is positive; and \( \hat{\sigma}^2 \) is positive only if \( K_2 > \frac{5K_4^2}{3K_6} \). It is disappointing that Beckers obtain a negative variance in 30 cases out of 47 NYSE listed common stocks each with 500 daily return observations.

Ball and Torous (1983) follow Beckers and, a priori, set the mean logarithmic jump size equal to zero, guaranteeing a symmetric return distribution. A Bernoulli mixture of normal distributions is used to approximate the Poisson mixture of normal distributions. In this case, the daily security return density becomes
\[ b(x) = (1-\lambda)N(x; \alpha, \sigma^2) + \lambda N(x; \alpha, \sigma^2 + \delta^2). \]  

(3.18)
Both models of (3.15) and (3.18) are symmetric and allow for discontinuous jumps. Moreover, if returns were computed for smaller time intervals, the Bernoulli model would converge to the Poisson model. In fact, we can express the exact relation between (3.15) and (3.18) as
\[ b(x) = f(x) + O(\lambda^2). \] (3.19)
Equation (3.19) shows that using a Bernoulli model \( b(x) \) to approximate a Poisson model \( f(x) \), the approximation error is at most a function of \( \lambda^2 \). For small values of \( \lambda \), \( f(x) \) and \( b(x) \) are practically indistinguishable. Again using the theoretical relation between cumulants and population moments, we can obtain:

\[
\begin{align*}
K_1 &= \alpha, \\
K_2 &= \sigma^2 + \lambda \delta^2, \\
K_3 &= K_5 = 0, \\
K_4 &= 3\delta^4 \lambda (1-\lambda), \\
K_6 &= 15\delta^6 \lambda (1-\lambda)(1-2\lambda).
\end{align*}
\] (3.20)
Substituting \( K_2 \) of (3.20) into \( K_4 \) and \( K_6 \) and solving for \( \lambda \), we obtain a linear equation with second-order in terms \( \lambda \):
\[
(100K_4^2 + 3K_6^2)\lambda^2 - (100K_4^2 + 3K_6^2)\lambda + 25K_4^2 = 0.
\]

\[ aX^2 + bX + c = 0. \]  (3.21)

In (3.21), \( a = (100K_4^2 + 3K_6^2), \) \( b = -(100K_4^2 + 3K_6^2), \) \( c = 25K_4^2. \) The solution for \( \lambda \) in (3.21) is:

\[
\lambda = \frac{b \pm (b^2 - 4ac)^{1/2}}{2a},
\]

\[ = \frac{1}{2} \left[ \frac{3K^*}{(3K^* + 100)} \right]^{1/2}, \text{ where } K^* = (K_6/K_4)^2. \]  (3.22)

We note that, Ball and Torous (1983, p.58) reports a solution for \( \lambda \) as

\[
\lambda = \frac{1}{2} \left[ \frac{3K^*}{(3K^* + 100)} \right]^{1/2}, \text{ where } K^* = (K_6/K_4)^2. \]  (3.23)

We think that their solution for \( \lambda \) as given in (3.23) is wrong. It is not clear whether this error is simply a typographical error from their paper or a real computational mistake. If Ball and Torous (1983) do use (3.23) as a solution for \( \lambda \) in their empirical test, the comparative results that are presented as Table 1 in Ball and Torous (1983) are meaningless. Given the solution for \( \lambda \) in (3.22), we can solve for all other parameters. Equating sample cumulants with population cumulants, estimators \( \hat{\lambda}, \hat{\sigma}^2, \hat{\delta}^2, \) and \( \hat{\alpha} \) are derived:

\[
\hat{\lambda} = \left( 1 + \frac{3K^*}{3K^* + 100} \right)^{1/2}, \text{ where } \hat{K}^* = (\hat{K}_6/\hat{K}_4)^2,
\]

\[
\hat{\sigma}^2 = \hat{K}_2 - \hat{\lambda} \hat{\delta}^2,
\]

\[
\hat{\delta}^2 = \hat{K}_6 / (\hat{K}_4(5(1-2\hat{\lambda}))),
\]

\[
\hat{\alpha} = \hat{K}_1. \]  (3.24)
To ensure that $\delta^2$ is positive a '+' is taken in $\lambda$ if $K_6<0$ and a '−' if $K_6>0$. We note that in assuming a Bernoulli jump process, $\lambda$ is implicitly confined to be greater or equal to zero and less than or equal to 1. This restriction is not part of a general Poisson jump process. The cumulating Bernoulli process would converge to a Poisson jump process, however, if $\tau$ (the fixed period of time over which the Bernoulli process is assumed to govern) is small, and $n$ (the number of the independent Bernoulli process) is large. With daily data in hand, $\tau=1$, and with 590 daily sample $n=590$, so that these two conditions are approximately satisfied.

One attractive property of a Bernoulli jump process is it allows us to use maximum-likelihood estimation for its parameters. When Ball-Torous use a maximum-likelihood estimation procedure on Beckers' sample, they obtain uniformly positive variances. Ball and Torous (1985) go one step further by approximating the Poisson density with a truncated Poisson density. This removes the trouble of dealing with an infinite series. For small $\lambda$, the Poisson distribution could converge very quickly. Ball-Torous set the truncation at $N=10$ and find no evidence of truncation error even for larger $\lambda$. Formally they consider a truncated version of (3.15), i.e.,

$$f(x) = \sum_{n=0}^{M} \frac{e^{-\lambda/n!}}{n!} N(x; \alpha, \sigma^2 + n\delta^2).$$

(3.25)

In (3.25), $M$ is some positive constant, e.g., $M=10$. 

This chapter continues to study estimation procedures for a mixed diffusion-Poisson process (3.6). Five sets of estimation procedures are employed: Beckers' method of cumulants; a parallel method of cumulants for the Bernoulli jump process; maximum-likelihood estimation for a Bernoulli mixture; maximum-likelihood estimation for a truncated version of Poisson mixtures of normal processes; and maximum-likelihood estimation of a Poisson mixture of normal processes without restricting the jump mean. To apply the cumulants-matching method to obtain parameter estimates for a probability process is straightforward. We first use equation (3.14) to compute the values of sample cumulants based on sample moments. Then we apply the values of sample cumulants to solutions (3.17) and (3.24). However, finding maximum-likelihood estimates (MLE) for a model which does not admit explicit solutions is not a trivial pursuit. Thus, we need to digress to clarify some background issues first.

3.3 MLE and Parameter Estimation for a Mixture of Normal Processes

We denote vector \( x=(x_i, i=1,2,...,n) \) as \( n \) observed samples, e.g., \( n \) daily logarithm foreign-currency price relatives and \( f(x_i) \) as the probability density function that generates \( x_i \), vector \( \Theta=(\theta_1,\theta_2,\theta_3,\theta_4) \) as the unknown parameters in \( f(x_i) \), e.g., \( \Theta=(\lambda,\sigma^2,\delta^2,\alpha) \). The joint
probability of the observations, regarded as a function of \( \theta \), is called the Likelihood Function (LF) of the sample and is written as:

\[
L(x|\theta) = f(x_1|\theta)f(x_2|\theta)\cdots f(x_n|\theta).
\] (3.26)

In (3.26), \( f(x_i|\theta) \) is the probability density value for \( x_i \) given the parameter vector \( \theta \). A MLE directs us to take as estimators of \( \theta \) the value \( \hat{\theta} \) within the admissible range of \( \theta \) which makes the LF as large as possible. That is, we choose \( \theta \) so that for any admissible value \( \theta' \),

\[
L(x|\theta') > L(x|\theta).
\] (3.27)

If the LF is a twice-differentiable function of \( \theta \) throughout its range and there is no terminal maximum of the LF at the extreme permissible values of \( \theta \), MLE will be given by some roots of

\[
L'(x|\theta) = \frac{3L(x|\theta)}{\theta_r} = 0, \text{ where } r = 1, 2, 3, 4.
\] (3.28)

A sufficient (though not a necessary) condition for those roots to be local maxima is that the matrix \( L''(x|\theta) \) be negative definite. In practice, it is often simpler to work with the logarithm of the LF than with the function itself. Of course, as monotonic transformations of each other, \( x \) and \( \log x \) reach their maxima together. Maximum-likelihood estimators are asymptotically normally distributed and are asymptotically efficient estimators\(^\text{20}\).

\(\text{20. The reader is referred to Wilks (1962) or Kendall and Stuart (1977b, ch. 18) for a proof.}\)
To compute maximum-likelihood estimates for a Bernoulli mixtures or Poisson mixtures of normal processes, we need to solve the equations implied by (3.28). Problems arise in finding a solution since the likelihood equations (3.26) and the associated first-order conditions (3.28) for a Bernoulli mixtures and Poisson mixtures of normal processes are nonlinear and have no closed-form solutions. To find a MLE for (3.15) and (3.18), Ball and Torous (1983, 1985) apply a multidimensional Newton-Raphson procedure, which provides a numerical solution algorithm. Unfortunately, they fail to provide details for their estimation techniques so that we cannot replicate their work. Instead, we use a different numerical technique to find MLE for (3.15) and (3.18).

An explicit solution of the first-order conditions (3.28) is not necessary in finding a maximum-likelihood estimate by a numerical technique. However, an explicit solution is helpful and reduces computer-time required for calculation. Therefore, we make some effort to simplify the first-order conditions implied by (3.28) for both (3.15) and (3.18). We find a simplified first-order condition for a Bernoulli mixture of normal processes. Unfortunately, for a Poisson mixture of normal processes, we are unable to state explicitly its derivatives. Solutions for a Bernoulli mixture of normal processes are found by the following procedure.

The logarithm of the likelihood-function for a Bernoulli mixture of normal processes (3.18) is
\[ \ln L(x; \theta) = \sum_{i=1}^{n} \ln b(x_i; \theta) \]
\[ = \sum_{i=1}^{n} \ln \{(1-\lambda)N(x_i; \mu, \sigma^2) + \lambda N(x_i; \mu, \sigma^2 + \delta^2)\}. \quad (3.29) \]

Necessary conditions for a maximum-likelihood estimator at \( \hat{\theta} \) are provided by

\[ \frac{\partial \ln L(x|\theta)}{\partial \theta} = 0. \quad (3.30) \]

First, we define \( c = \delta^2 / \sigma^2 \), so that \( \delta^2 = \sigma^2 (1+c) \). The objective function (3.29) is then simplified as:

\[ L(x; \tau) = \prod_{i=1}^{n} \frac{1}{(2\pi\sigma^2)^{1/2}} \exp\left\{ -\frac{1}{2} \frac{(x_i - u)^2}{\sigma^2} \right\} \]
\[ + \lambda \frac{1}{(1+c)^{1/2}} \exp\left\{ -\frac{1}{2} \frac{(x_i - u)^2}{\sigma^2 (1+c)} \right\}, \]
\[ = \frac{2\pi\sigma^2}{\prod_{i=1}^{n} (1-\lambda) \exp\{-\frac{(x_i - u)^2}{2\sigma^2}\}} \]
\[ + \lambda \frac{(1+c)^{1/2}}{\prod_{i=1}^{n} [(1-\lambda) \exp\{-\frac{(x_i - u)^2}{2\sigma^2}\}]}, \]
\[ = \ln L(x; \tau) = -n/2 [\ln(2\pi \cdot \sigma^2)] + \sum_{i=1}^{n} \ln \{(1-\lambda) e_{1i} + \lambda e_{2i}\}, \]
\[ = -n/2 \ln 2\pi - (n/2) \ln \sigma^2 + \sum_{i=1}^{n} \left\{ -\frac{(x_i - u)^2}{2\sigma^2} + \ln[(1-\lambda) \exp\{-\frac{(x_i - u)^2}{2\sigma^2}\} \right\} \]
\[ + \lambda \frac{1}{(1+c)^{1/2}} \cdot e^{1/(1+c)}. \quad (3.31) \]

In (3.31), \( e_{1i} = \exp\{-\frac{(x_i - u)^2}{2\sigma^2}\}; \)
\[ e_{2i} = \frac{1}{(1+c)^{1/2}} \exp\{-\frac{(x_i - u)^2}{[2\sigma^2 (1+c)]}\}. \]
Equation (3.31) is the simplified likelihood function the MLE maximizes. If a maximum-likelihood estimate exists, it should satisfy the first-order conditions (3.30) and its corresponding second-order condition. The first-order condition (3.30) can also be simplified:

\[
\frac{\partial \ln L}{\partial \lambda} = \sum_{i=1}^{n} \frac{e_{2i} - e_{1i}}{(1-\lambda)e_{1i} + \lambda e_{2i}} = 0,
\]

\[
\frac{\partial \ln L}{\partial \mu} = \sum_{i=1}^{n} \frac{(1-\lambda)e_{1i} \cdot (1/\sigma^2) \cdot (x_i - \mu) + \lambda e_{2i} (x_i - \mu) \cdot (1/2\sigma^2(1+c))}{(1-\lambda)e_{1i} + \lambda e_{2i}} = 0,
\]

\[
= \sum_{i=1}^{n} \frac{(1-\lambda)e_{1i} \cdot (x_i - \mu) + \lambda e_{2i} (x_i - \mu) \cdot (1/(1+c))}{(1-\lambda)e_{1i} + \lambda e_{2i}} = 0,
\]

\[
\frac{\partial \ln L}{\partial \sigma^2} = -\frac{n}{2} \sigma^{-2} + \sum_{i=1}^{n} \frac{(1-\lambda)e_{1i} \cdot ((x_i - \mu)^2 / 2) \cdot (1/\sigma^4) + \lambda e_{2i} \cdot ((x_i - \mu)^2 / 2) \cdot (1/(1+c)\sigma^4)}{(1-\lambda)e_{1i} + \lambda e_{2i}} = 0
\]

\[
= -n \sigma^{-2} + \sum_{i=1}^{n} \frac{(1-\lambda)e_{1i} \cdot (x_i - \mu)^2 + \lambda e_{2i} \cdot (x_i - \mu)^2 \cdot (1/(1+c))}{(1-\lambda)e_{1i} + \lambda e_{2i}} = 0,
\]

\[
\frac{\partial \ln L}{\partial c} = \sum_{i=1}^{n} \frac{\lambda e_{2i} \cdot [\frac{1}{2} \cdot (1+c)^{-1} \cdot (1+c)^{-2} \cdot (x_i - u)^2 / 2\sigma^2]}{(1-\lambda)e_{1i} + \lambda e_{2i}} = 0,
\]

\[
= \sum_{i=1}^{n} \frac{\lambda e_{2i} \cdot [(1-\lambda)^{-1} \cdot (x_i - u)^2 / 2\sigma^2]}{(1-\lambda)e_{1i} + \lambda e_{2i}} = 0. \quad (3.32)
\]

Given the likelihood function (3.31) and first derivatives in (3.32), MLEs for a Bernoulli mixture of normal processes can be found by
numerical techniques. For a Poisson mixture of normal processes, the logarithm of the corresponding LF (3.26) can written explicitly as:

\[
\ln L(x | \theta) = \sum_{i=1}^{N} \ln f(x_i | \theta),
\]

\[
= \sum_{i=1}^{N} \ln (\frac{\lambda^n}{n!} N(x; \alpha, \sigma^2 + n\delta^2)).
\]  \hspace{1cm} (3.33)

Unfortunately, we cannot derive an explicit form for its derivatives. A numerical technique is needed. Another problem is, the LF (3.33) contains an infinite sum in a Poisson probability density. To find MLEs from (3.33), an adequate approximation for a Poisson probability density is needed. We can approximate an infinite sum by a finite sum wherever its remainder terms are negligible. We assume that the infinite sum in (3.33) may be safely truncated at Mth term with the resultant approximation error denoted by B(M). Ball and Torous (1985) show that

\[
B(M) < (2\pi\sigma^2)^{-1/2} \cdot \frac{\lambda^{M+1}}{(M+1)!} .
\]  \hspace{1cm} (3.34)

Inequality (3.34) makes the truncation error a function of \(\sigma^2\), \(\lambda\) and \(M\). Given the value of \(\sigma^2\), \(B(M)\) is a function of \((\lambda^{M+1}/(M+1)!))\). For given \(M\), the smaller is \(\lambda\), the smaller the truncation error. Because we use daily currency price data, the value of \(\lambda\) is believed to be less than 1. The empirical results in Chapter 2 suggest that the value of \(\sigma^2\) is less than 1. In this particular case, we find

\[
B(M) = B(10) \leq (2\pi)^{-1/2}/(11!) = 0, \text{ for } \lambda=1 \text{ and } \sigma^2=1.
\]  \hspace{1cm} (3.35)
Inequality (3.35) shows that if we approximate (3.25) by summing the series from 0 to 10, the resultant approximation error is negligible for our samples.

Because likelihood functions (3.31) and (3.33) are complicated, MLEs cannot be obtained in explicit form. This means that suitable iterative methods must be employed. To find an MLE by an iterative procedure, we ordinarily start from a trial value $k$. If we choose $k$ so that it is likely to be in the neighborhood of the true MLE, the Newton-Raphson iterative process can be used. This method estimates $\theta$ by:

$$
\theta = k - \frac{\partial \log L}{\partial \theta} \bigg|_k \left/ \frac{\partial^2 \log L}{\partial \theta^2} \right|_k.
$$

(3.36)

The most common method for choosing the initial value $k$ is to take it as the value of some (preferably simply-calculated) consistent estimates of $\theta$. To calculate MLE for a Bernoulli mixture of normal processes, we start with the values generated by the cumulants-matching method. Because a Bernoulli mixture is a good approximation to a Poisson mixture, its MLE provides good starting values for a Poisson mixture process. We also take the value of cumulants-matching method (3.17) as the starting value and compare likelihood values. The MLEs obtained are then used as starting values with $u$ initialized at 0 for the MLE of an unconstrained Poisson mixture of normal processes.

We use G0OPT3, a self-contained computer package for numerical optimization of functions, to obtain MLEs for a Bernoulli mixture and
Poisson mixture. GQOPT\textsuperscript{21} contains a set of \textsc{fortran} algorithms designed to optimize general nonlinear functions supplied by the user. The algorithm we use is DFP. The DFP variable-metric algorithm is a numerical procedure developed by Davidon (1959), Fletcher and Powell (1963) and expanded by Powell (1971, p. 27). It applies to a "quasi-Newton" iterative algorithm in (3.36) that involves linear searches and employs first derivatives and an approximation to the second partial derivatives. Instead of solving the likelihood functions, this algorithm maximizes it outright.

3.4 Comparison of the Parameter Estimates

To test whether the presence of a jump in foreign-currency price changes is significant, we use the likelihood-ratio statistic:

\[ A = -2[\ln L(x; \theta^0) - \ln L(x; \hat{\theta})] \]  

(3.37)

In (3.37), \( \hat{\theta} \) is the unrestricted maximum-likelihood estimate vector, and \( \theta^0 \) is the restricted MLE vector that generate a local maximum for its LF when \( \lambda = 0 \) (which implies that no jump structure is present). We note that Ball and Torous (1983; 1985) incorrectly tabulate the likelihood-ratio statistic as

\[ A = -2[\ln L(x; \hat{\theta}) - \ln L(x; \theta^0)] \]

\[ 21. \text{ The reader is referred to Goldfeld and Quandt (1976, Introduction) for more details about GQOPT.} \]
Unless this is a typographical error, their results should be interpreted with suspicion.

Under the null hypothesis that currency price changes follow a pure diffusion process without a jump structure, $\Lambda$ is asymptotically distributed as Chi-squared with two degrees of freedom\textsuperscript{22}. Results are reported in Table 7.

Row (1) and (2) of Table 7 report the parameter estimates by using the cumulant-matching method (3.17) and (3.24). Row (3) and Row (4) report the MLEs for (3.15) and (3.18). Row (5) reports MLEs for the unrestrictive mixed diffusion-Poisson process (3.6). Parenthetical values under (3) and (4) in $\lambda$-estimate panel give likelihood-ratio statistics.

As expected, the cumulant-matching method for Poisson mixtures sometimes produces negative variance estimates. We find that JY and SF have negative $\delta^2$ estimates and BP has a negative $\sigma^2$ estimate. In contrast, using a Bernoulli mixture of normal processes to approximate a Poisson mixture and applying the cumulant-matching method to estimate the parameters, all the parameter estimates in Row (2) achieve the right sign. This is consistent with Ball and Torous

\textsuperscript{22} The number of degrees of freedom can be counted in two equivalent ways: the number of restrictions or the difference between the number of parameters implied in the maintained hypothesis and the number of parameters implied in the alternative hypothesis. The maintained hypothesis constraints the value of $\lambda$ to zero and only two parameters need to be estimated: $\alpha$ and $\sigma^2$. Four parameters are required for the alternative hypothesis: $\alpha$, $\sigma^2$, $\lambda$, and $\delta^2$. Therefore, the number of degrees of freedom for the likelihood-ratio test is 2.
Also the MLEs for Bernoulli mixtures using the cumulant-method parameter estimates as starting values are close to their cumulant-matching-method counterparts.

For Poisson mixtures of normal processes, two different starting points were used to obtain MLEs with truncating number set at M=10, 15 and 20. We start with the parameter values obtained from the cumulant-matching method (3.17) and also start with the MLEs of Bernoulli mixtures. Comparing the MLEs obtained and their likelihood values, in most cases the two starting values give the same MLE. In some cases, the MLEs obtained differ. We find that the MLEs that have a higher likelihood value are the ones produced by using MLE as starting values. Table 7 reports only the MLEs that generates the higher likelihood-function value. As expected, truncating the Poisson series at M=10, 15 or 20 converge to the same MLEs.

The likelihood-ratio statistics are all significant for \( \lambda \) at 1 percent. This suggests that the \( \lambda \) estimates for both the Bernoulli mixture and Poisson mixture differ significantly from zero. Thus, the maintained hypothesis that foreign-currency price changes follow a stationary lognormal process must be rejected in favor a mixed diffusion-Poisson process.

Row (5) of Table 7 also reports MLEs for a mixed diffusion-Poisson process with no restriction placed on the mean of its jump size \( \mu \). We use the MLEs obtained by row (4) as the starting value and initialize \( \mu \) as zero. The MLEs obtained are very close to the values for the constrained Poisson version. For the JY and the DM, the MLEs
in row (4) and row (5) are identical. This indicates that the constraint \( \mu = 0 \) is not unrealistic.

The results in Table 7 show that the maximum-likelihood estimation procedure provides consistently positive estimates of variance for a Bernoulli mixture as well as for a Poisson mixture of normal processes. Our results in general agree with those of Ball and Torous (1983; 1985). In particular, we find that a mixed diffusion-Poisson process fits the observed currency-price changes better than a stationary lognormal process. However, several of our findings differ from theirs. In addition to the technical errors cited earlier, we find some claims are not convincing.

First, besides being asymptotically more efficient, Ball and Torous (1983; 1985) claim that MLE is better than the cumulant-matching method because it achieves the right sign for variance estimates. However, we must interpret this claim with reserve because an MLE does not always produce the right sign for variance estimates. For example, if we do not place a priori positive restriction on variance parameters, then a negative, but near-zero variance could make the likelihood function (3.31) or (3.33) infinite. Thus, the positiveness of the variance is a result by design rather than a result that comes naturally.

Second, Ball and Torous may not be aware of the fact that the likelihood functions (3.31) and (3.33) are both unbounded in parameter space. To see that the likelihood function (3.31) is unbounded, we can
choose \( \alpha \) such that \( x_i^{-\alpha} = 0 \) for some value of \( i \). Then as \( \sigma \) approaches zero, the \( i \)th term in (3.31) becomes indefinitely large. Thus the likelihood function (3.31) becomes unbounded in parameter space. The same arguments can be applied to LF (3.33). Because the likelihood function is unbounded, attempts to find the global MLE may result in inconsistent estimates. This difficulty would vanish if the variance ratio \( c = \sigma_2/\sigma^2 \) is known a priori. This is one of the reasons why we estimate \( c \) first in likelihood function (3.31). Fortunately, in the absence of such a priori knowledge, LF (3.31) and (3.33) frequently possess interior local maxima with reasonable properties in small samples. For relevant discussion, refer to Goldfeld and Quandt (1976; Ch. 1).

Third, Ball and Torous (1983;1985) report that the values of their MLE estimates are all close to the cumulant-matching-method counterparts, and claim that the estimates from CME are good. As we just pointed out in the last paragraph, due to the unboundness of the likelihood functions, any attempt to find a global maximum will lead to an inconsistent estimate. However, if we believe that the true MLE is close to some other consistent estimates, then an interior local maximum would possess reasonable properties. That the MLEs Ball and Torous obtain are close to their starting values does not mean the starting values are good. It only confirms that if the starting value is a consistent estimate, then the MLEs they obtain possess good properties.
CHAPTER IV
AN INTRODUCTION TO OPTION PRICING THEORIES

4.1. Introduction

An option is a security that gives its owner the right to buy or sell an asset. A call option conveys the right to buy a specified number units of a specified asset at a fixed price on or before a specific date. The asset featured in the option contract is called the underlying asset. The price at which the asset may be exchanged is called the exercise price or the striking price. The last date on which the option may be exercised is called its expiration date or the maturity date. A European option is an option that can be exercised only at the time of maturity. An option which can be exercised at any time up to and including maturity is called an American option.

The "intrinsic value" of a call option is defined as zero or the difference between the underlying asset price and striking price, whichever is greater. On the expiration date, a call option must be worth its intrinsic value. Before the expiration date, the price of a call option may exceed its intrinsic value. The difference between the underlying asset price and a call option's intrinsic value is called the "time value" of a call option.

Before the expiration date, the value of a call option depends crucially on the probability that on the expiration date the underlying asset price will lie above the striking price. Knowledge of
the probability distribution for the underlying asset price allows one to estimate the expected terminal value of a call option. The discounted expected terminal value of a risky asset is often used as a basis for calculating a risky asset's price. However, unless investors are neutral toward risk, the discounted expected terminal value of a risky asset will not equal a risky asset's price. The return on a risky asset, such as a call option, must include compensation for risk-bearing. Therefore, any price at which the call option might trade during its life will reflect market estimates of the option's probable terminal value and risk-reward. Option pricing theories attempt to explain an option's market price and to derive a unique "fair value" for an option. The fair value of an option is the option's price that permits a fair game for an option. In other words, the fair value of an option is the option price that eliminates arbitrage opportunities. If an option is priced according to its fair value, then neither an option's buyers nor an option's sellers can consistently obtain abnormal profit.

Depending on whether they incorporate the probability distribution of the underlying asset's price changes, option pricing models can be classified into two categories: (1) ad hoc models and (2) probability models. An ad hoc model assumes that the past

23. The fair value for an option does allow its investor to earn abnormal profit even though the probability for such an occurrence is very small. Therefore, only a value which can "consistently" eliminate abnormal profit counts as the fair value for an option.
relation between an option's price and the price of the underlying asset is stationary. This assumption justifies interpreting econometrically fitted functional forms to past data as descriptions of the relation between these prices and their potential determinants. Since these models use econometric techniques to fit past option data, they are a subclass of "econometric models." Such models' principal weakness is that the value they generate is only an expected option price, based on past relation between selected variables, of what the price of a call option might be, not the fair value, based on the behavior pattern of the underlying asset, of what the value should be.

Probability models make a priori assumptions about the probability distributions for the underlying asset price changes. They also make strong assumptions about the expected return on an option and its underlying asset. Based on these distributional assumptions, they calculate the expected value of an option. Then they typically assume that investors are risk-neutral or risk-averse and arrive at a numerical estimate of the fair value of an option.

Ad hoc models are reviewed in chapter 6. This chapter reviews the development of probability models\(^\text{24}\). Section 4.2 introduces the terminology employed in options trading and develops the basic relation between the value of an option and its underlying asset, using only the mild assumption that investors prefer more to less. The

\(^{24}\) The review of this chapter has benefited from three excellent review articles: Samuelson (1972), Smith (1976), and Cox and Rubinstein [chapter 1 of Brenner (1983)].
general relations developed in this section do not provide numerical solutions to the option-pricing problem, but they do define acceptable regions within which admissible solutions must fall to avoid arbitrage opportunities. Explicit solutions to the call option problem are developed in Section 4.3, 4.4 and 4.5. Section 4.3 reviews the probability option-pricing models developed before Black-Scholes. Section 4.4 discusses the Black-Scholes call option-pricing model. Section 4.5 surveys some alternative option-pricing models developed in the wake of Black-Scholes. This chapter lays a foundation for the next chapter. In the next chapter, the Black-Scholes' model and one of the post-Black-Scholes' option-pricing techniques reviewed in this chapter are used to price options on foreign currency.

4.2 Terminology and some fundamental constraints on option prices

The symbols used are:

- \( t \) - current date,
- \( T \) - expiration date of the option,
- \( \tau \) - time to expiration \((T-t)\),
- \( S(t) \) - asset price or stock price at \( t \),
- \( C(S,\tau;X) \) - price of an American call option at \( t \), when its underlying asset's price is \( S(t) \), time to expiration is \( \tau \), exercise price is \( X \).
- \( c(S,\tau;X) \) - price of a European call option at \( t \), when its underlying asset's price is \( S(t) \), time to expiration date is \( \tau \), exercise
price is X.

X - exercise price of the option,

r - risk-free interest rate,

B(t) - price of a default-free pure discount bond with a face value of one dollar, \( B(t) = \exp(-rt) \) if \( r \) is constant,

\( \alpha \) - expected average rate of growth in the underlying asset price \( \mathbb{E}[S(T)/S(t)] = \exp(\alpha t) \),

\( \beta \) - expected average rate of growth in the call price, i.e. \( \mathbb{E}[C(T)/C(t)] = \exp(\beta t) \),

\( V_A \) - value of portfolio A at t.

Merton (1973b) derives an exhaustive set of equilibrium restrictions that can be placed on call option prices without making distributional assumptions. Merton makes no assumptions about the process generating the asset price over time. The restrictions he derives depend only on dominance arguments. Portfolio A dominates portfolio B over some given time interval if during that interval the return to A never falls below the return to B in any anticipated state of the world, and the return to A is strictly greater than the return to B in at least one relevant state of the world. Whenever a dominant portfolio of securities exists, everyone would prefer to hold that portfolio. The price of that portfolio would be bid up until the dominance disappeared. In this chapter, we use dominance arguments to obtain some basic boundary conditions for call options.
Since a call option is a right and not an obligation to buy designated securities or commodities, the price of an option must always be nonnegative. Hence, the prices of both American and European calls cannot be negative:

\[ C(S(t),\tau;X) > 0 \] [American call],
\[ c(S(t),\tau;X) > 0 \] [European call].  \( (4.1) \)

It is trivially true that when the price of an underlying asset is zero, the option must be worthless. This allows us to write:

\[ c(0,\tau;X) = C(0,\tau;X) = 0. \]  \( (4.2) \)

At the expiration date, \( T \), the call has either of two values: the difference between the asset price and the exercise price, \( S(T)-X \), or zero. This allows us to write:

\[ c(S,0;X) = C(S,0;X) = \text{Max}(0,S(T)-X). \]  \( (4.3) \)

At any date before the maturity date, an American call option is potentially more valuable than a European one. Because an American call can be exercised at any time before the expiration date, its price must be at least the difference between the asset's current price and the exercise price:

\[ C(S,\tau;X) \geq \text{Max}(0,S-X). \]  \( (4.4) \)

This condition does not apply for a European call except at \( T \).

The price of a call option cannot exceed the value of the underlying asset. This is because to obtain a unit of underlying asset by means of a call option, a non-negative exercise price must be expended, therefore

\[ c(S,\tau;X) \leq S. \]
\[ C(S, \tau; X) \leq S. \] \hspace{1cm} (4.5)

If two American calls differ only in their expiration date, then the one with the longer term to maturity, \( \tau_2 \), cannot sell for less than that of the shorter term to maturity, \( \tau_1 \). At the expiration date of the shorter option, its price must equal the maximum of zero and the difference between the asset price and the exercise price as stated in (4.3). By (4.4), this sets a minimum price for the longer option. To prevent dominance,

\[ C(S, \tau_2; X) \geq C(S, \tau_1; X). \] \hspace{1cm} (4.6)

Restrictions (4.1)-(4.6) set basic constraints on the price of any call option. Figure 2 applies these restrictions to the case where the underlying asset is common stock. The figure illustrates the five constraints in a coordinate system spanned by call option prices and stock prices. The 45-degree line from the origin which represents the upper bound on a call option price set by restriction (4.5). The 45-degree line from the striking price \( X \) and the bottom line \( 0-X \) are the the lower bounds set by restriction (4.1), (4.2), (4.3) and (4.4). Lines \( V_1, V_2, V_3 \) represent the value of an option for successively shorter maturities. The shorter the period to maturity, the more closely the price of a call option approaches the lower bound. This tendency is expressed in restriction (4.6).

---

25. Merton (1973b) or Smith (1976) give additional restrictions. Further restrictions are not crucial to address the major issues of this chapter.
While these restrictions do not pin down the exact price at which a call option must trade, they place important bounds on feasible call prices. Any option-pricing model which violates these restrictions must be described as conceptually deficient.

4.3 Probability Models of Call Option Pricing Developed Prior to Black-Scholes

Option-pricing theory has a long history. The first option-pricing model appeared in 1900, authored by a French mathematician, Louis Bachelier. In his thesis on the Theory of Speculation, Bachelier (1900) makes two distributional assumptions: that the stock price is a random variable and that absolute price changes are independent and identically distributed (iid.). From these assumptions, Bachelier deduces that the density function of the asset price must be normal:

\[ F[S(T)-S(t); \tau] = \Phi \left( \frac{S(T)-[S(t)+\alpha \tau]}{\sigma \sqrt{\tau}} \right), \]

where \( F \) is the cumulative distribution function (cdf.) of the stock's price changes from \( t \) to \( T \), \( \alpha \) is the mean expected price change per time period, \( \sigma^2 \) the variance per time period, and \( \Phi \) the cumulative distribution function of the standard normal distribution.

26. One version of the Central Limit Theorem assures that the limiting distribution for the sums of iid. random variables is normal if the variance of the iid. variable is finite. Under Bachelier’s assumptions, price changes can be represented as the sum of (possibly infinitely many) iid. variables. Hence, with additional assumption that the variance of price changes is finite, what Bachelier deduces reduces to this version of the Central Limit Theorem.
standard normal distribution. The above equation says that the probability that the stock price \( T \) periods in the future \( (S(T)) \) is less than the current stock price \( (S(t)) \) can be expressed as a function of the standard normal distribution. Bachelier next simplifies the model by assuming that \( \alpha \) equals zero. He also assumes that the call is priced to make the current call price the expected terminal call value. Therefore Bachelier's model holds that

\[
c(t) = E[C(T)] = \int_{\mathbb{R}} (S(T) - X) N'(S) dS, \tag{4.7}
\]

where \( N'(S) \) is the normal density function for \( S(T) \).

Bachelier's model has had two profound effects on subsequent developments in option-pricing theory. First, subsequent researchers have usually retained the assumption that the price change can be expressed as the sum of iid. random variables. However, these authors have recognized that assuming that a normal process describes expected price movements implies a positive probability of observing negative prices for the underlying security. Second, the assumption that the call is priced according to its expected terminal value is also used in some modern option-pricing models. But in Bachelier's model this assumption requires both zero interest rates and risk neutrality.

Instead of assuming absolute price changes are iid., Sprenkle (1964) assumes that relative price changes (i.e., price ratios) are independent and identically distributed. This assumption implies that asset prices are log-normally distributed. This succeeds in explicitly ruling out the possibility of negative prices for the underlying
securities. Further, he allows for positive drift in the random walk, thus requiring positive interest rates. Under Sprenkle's model, the stochastic process of the asset price has the following cdf:

\[ F[S(T)/S(t); t] = L[S(T); S(t), \alpha, \sigma^2 \tau], \]
\[ = N[\ln(S(T)/S(t)) - (\alpha - \sigma^2/2)\tau]. \tag{4.8} \]

In (4.8), \( L \) is the cumulative log-normal distribution function for \( S(T) \) given these parameters: a current stock price of \( S(t) \), a mean expected price change per time period of \( \alpha \), and a variance per time period of \( \sigma^2 \).

In Sprenkle's model the expected value of the call option at the expiration date is:

\[ E[C(T)] = E[\max(0, S(T) - X)] = \int_{-\infty}^{\infty} (S(T) - X) L'(S) dS. \tag{4.9} \]

In (4.9), \( L'(S) \) is a log-normal density function. Sprenkle develops a theorem useful for solving integrals involving the log-normal distribution:

**Theorem.** If \( S(T) \) is log-normally distributed, then

\[ E[\max(0, aS(T) - bX)] = e^{\alpha \tau} aS(t) \cdot N\left[ \frac{\ln(S(t)/X) + \alpha \tau + (\sigma^2/2)\tau}{\sigma \sqrt{\tau}} \right] - bX \cdot N\left[ \frac{\ln(S(t)/X) + \alpha \tau - (\sigma^2/2)\tau}{\sigma \sqrt{\tau}} \right]. \tag{4.10} \]

---

27. This theorem, taken from Smith (1976), is more general than the one provided in the appendix of Sprenkle (1964). But the proof in Sprenkle (1964) can be used to prove this theorem.
In (4.10), \(a\) and \(b\) are arbitrary parameters, \(\alpha\) is the expected average rate of growth in the security \(S\) [i.e., \(\exp(\alpha t) = E(S(T)/S(t))\)], and \(N\) is the cumulative standard normal distribution function.

Using Sprenkle’s theorem [equation (4.10)], and setting \(a=b=1\), (4.9) has the following solution:

\[
E[(c(T))] = \exp(\alpha T)S - N(d_1) - X \cdot N(d_2), \quad (4.11)
\]

where

\[
d_1 = \frac{\ln (S/X) + \left[\alpha + (\sigma^2/2)\right]T}{\sigma \sqrt{T}},
\]

\[
d_2 = \frac{\ln (S/X) + \left[\alpha - (\sigma^2/2)\right]T}{\sigma \sqrt{T}}.
\]

Sprenkle correctly observes that an investor would not willingly pay a price for the option exactly equal to its expected value. At the same time, Sprenkle incorrectly asserts that if the investor were neutral to risk, he would accept the expected value of the option as option’s price. This assertion is correct for a risk-neutral investor only if interest rates are zero. The final form of Sprenkle’s model contains a modification for risk-averse pricing using a parameter \(K\):

\[
c = \exp(\alpha T)S \cdot N(d_1) - (1-K)X \cdot N(d_2), \quad (4.12)
\]

and \(K\) parameterizes the degree of market risk aversion.

Boness (1964) allows for positive interest rates, thus avoiding Sprenkle’s error. Boness assume that investors are indifferent to risk and the expected yields on all securities are equal, i.e., \(\alpha = \beta\). This

---

28. In the context of contingent claims, \(a\) and \(b\) are contractual parameters specifying contingent claims’ pay-off structures. For a simple contingent claim, e.g., an option, \(a=1\) and \(b=1\).
assumption suggests that the price of a call option is the discounted present value of the expected value of the call at the expiration date:

\[ c = \exp(-\alpha t)E[c(T)], \]
\[ = S \cdot N(d_1) - \exp(-\alpha t)X \cdot N(d_2). \]  \hspace{1cm} (4.13)

The main objection to Boness' model is that his assumptions are counterfactual: investors are not indifferent to risk and expected yields on all securities are not equal. Samuelson (1965) allows the expected yield on assets and options to differ. Samuelson assumes that an option price grows at the rate $\beta$,

\[ E[c(T)/c(t)] = \exp(\beta t). \]  \hspace{1cm} (4.14)

With this assumption, the value of the option becomes:

\[ c = \exp(-\beta t)E[c(T)], \]
\[ = \exp[(\alpha - \beta) t]S \cdot N(d_1) - \exp(-\beta t)N(d_2). \]  \hspace{1cm} (4.15)

Samuelson correctly observes that $\beta$ should exceed $\alpha$. Because an option's price may be more volatile than its underlying stock, maximizers of concave utility functions might require that $\beta > \alpha$. However, Samuelson's assumption that $\beta$ or $\beta - \alpha$ is a constant turns out to be inappropriate.$^{29}$

$^{29}$ Because the more closely an option approaches to its expiration date, the less its time value will be. This precludes $\beta$ from being constant. Of course, for a perpetual option, which already has an indefinitely large life, its time value would not change much as time proceeds. It is no coincidence that Samuelson can solve an explicit option-pricing formula only for a perpetual option.
Samuelson and Merton (1969) relax the assumption that $\beta - \alpha$ is constant and offer an alternative theory of option pricing based upon utility maximization. The unique feature of this model is that it is based on what the authors call a 'util-prob' or combined utility and probability distribution. Unfortunately this unique feature makes their model mathematically intractable\textsuperscript{30}.

Samuelson and Merton do not rule out the fat-tailed, infinite-variance Paretian stable distribution. Samuelson (1972; p. 15) does say that Paretian stable distribution is mathematically intractable, and that he is "inclined to believe in Merton's conjecture that a strict Levy-Pareto distribution on $\log(S_{t+1}/S_t)$ would lead, with $1 < \alpha < 2$, to a 5-minute warrant or call being worth 100 percent of the common!".

If a stock price follows a log-stable process, then the expected payoff on the call option is ordinarily infinite. This is proved in Appendix A. But Samuelson's support of Merton's conjecture proves unfortunate because the market price of even a risky asset that has an infinite expected value can be finite. For example, in 1738 Daniel Bernoulli resolved the most elementary version of this "St. Petersburg Paradox" merely by assuming logarithmic utility.

\textsuperscript{30} The solution to this model requires: (1) utility function specified in wealth, and (2) iterated integrals with their util-prob distribution. Samuelson and Merton (1969) give a complete solution only in the easy case of the binomial process with Bernoulli logarithmic utility. For other more general cases, they are not able to give an explicit solution.
Black and Scholes (1973) dismiss the option pricing models reviewed above as incomplete. Their objection is that, to make their models tractable, the authors each impose one or another arbitrary parameters on investors' preferences.

4.4 The Black-Scholes Option-Pricing Model

Black and Scholes (1973) focus on the possibility of creating a riskless hedge by forming a portfolio containing stock and European call options. Since the return on such a "hedge portfolio" is riskless, to eliminate dominance opportunities, the portfolio must earn the riskless interest rate. The value of the fair price of a call option is then implicit in the value of the hedge portfolio.

The value of this hedge portfolio, $V_h$, can be expressed as the stock price times the number of shares of stock plus the call price times the number of calls in the hedge,

$$V_h = S \cdot w_1 + c \cdot w_2,$$

(4.16)

where $w_1$ is the quantity of stock and $w_2$ the quantity of calls.

The change in the value of the hedge, $dV_h$, is the total derivative of (4.16):

$$dV_h = w_1 dS + w_2 dc.$$  (4.17)

Black and Scholes assume that the stock price, $S$, follows a lognormal process as described in (4.8). Black and Scholes also assume
that the call price is a function of the stock price and the current
date t, i.e.,

\[ c(t) = c(S(t), t). \] (4.18)

Ito's Lemma provides a technique by which a specific class of
functions of Wiener processes may be differentiated\(^\text{31}\). Using Ito's
lemma, the infinitesimal change of stock price whose time path is
described in (4.8) can be expressed as:

\[ \frac{dS}{S} = \alpha dt + \sigma dW(t), \] (4.19)

\(^{31}\) Any stationary normal or lognormal process can be described as a
function of a Wiener process. A Wiener process \( W(t) \) is a stochastic
process with the following properties: (1) every increment of a Wiener
process \( dW(t) \) is an independent random variable, and (2) \( dW(t) \) is normally distributed with mean 0 and variance
proportional to the time interval observed, i.e. \( \sigma^2 dt \), where \( \sigma^2 \) is the
variance for the Wiener process for one unit of time interval. Because
of its close relation with a normal process, which is also called a
Gauss process, a Wiener process is commonly referred to as a "Gauss-
Wiener" process. If we assume \( \sigma = 1 \), then \( W(t) \) is a standard Gauss-
Wiener process (i.e., \( dW = \xi dt \), \( \xi \) is a normal, time-independent random
variable with mean 0 and variance 1). A Wiener process is a special
class of diffusion process. Assuming \( X \) follows a diffusion process
and its dynamics can be described as

\[ dX = \alpha dt + \sigma dz, \]

where \( z(t) \) is a standard Wiener process. Ito's lemma states that, if
\( Y = F(X, t) \), where \( F(\cdot) \) is a twice-differentiable function, then \( Y \) will
be a diffusion process with

\[ dY = (\frac{\partial F}{\partial X}) dX + (\frac{\partial F}{\partial t}) dt + (1/2) (\frac{\partial^2 F}{\partial X^2}) (dX)^2. \]

The calculation of the term \( (dX)^2 \) is governed by the following
multiplication rules: \( (dt)^2 = 0 \), \( (dt)(dz) = 0 \), \( (dz)^2 = dt \). Ito's lemma and
its associated multiplication rules can be intuitively understood by
expanding in Taylor's series for \( F(X + dX, t + dt) \) and by using the fact
\( dz = \xi dt \), and any terms involving a finite quantity multiplied by a
power of \( dt \) greater than 1 are of a much smaller order of magnitude
than \( dt \) so that they are effectively zero. More simply this is a
quadratic approximation to Taylor series in \( dX \), given \( dt = 0 \). For a
detailed treatment, the reader may consult Malliaris (1982).
where $W(t)$ is a standard Gauss-Wiener process at time $t$.

Assuming $c$ is a twice-differentiable function, Ito's lemma can be employed to express $dc$ as

$$dc = c_S dS + c_t dt + \frac{1}{2} c_{SS} \sigma^2 S^2 dt.$$  \hfill (4.20)

In (4.20), the subscripts denote first partial derivatives, e.g., $c_S = \partial c / \partial S$ and $c_{SS} = \partial^2 c / \partial S^2$.

Substituting the expression given for $dc$ in (4.20) for $dc$ in (4.17) yields:

$$dV_h = w_1 dS + w_2 [c_S dS + c_t dt + \frac{1}{2} c_{SS} \sigma^2 S^2 dt].$$

$$=(w_1 + w_2 c_S) dS + w_2 [c_t + \frac{1}{2} c_{SS} \sigma^2 S^2] dt. \hfill (4.21)$$

Note that the only stochastic term in the expression for $dV_h$ is $dS$. All other terms are strictly deterministic. To make $dV_h$ riskless, we need to have:

$$w_1 + w_2 c_S = 0. \hfill (4.22)$$

Indefinitely many solutions exist to equation (4.22). One simple solution is to set $w_1 = 1$ and $w_2 = -1 / c_S$. This yields:

$$dV_h = -1/c_S [c_t + \frac{1}{2} c_{SS} \sigma^2 S^2] dt. \hfill (4.23)$$

Because the hedge is riskless, its return must equal the risk-free rate $r$. This implies:

$$dV_h / V_h = r dt. \hfill (4.24)$$
Substituting (4.16) and (4.23) into (4.24) defines a differential equation for the value of the option:

\[(1/2)c_{ss}\sigma^2 S^2 + c_s S r - r c + c_t = 0.\] (4.25)

Of course, this differential equation must conform to the boundary condition (4.3). The differential equation (4.25) can be solved for the equilibrium call price. Black and Scholes transform the equation into the heat-exchange equation from physics to find the solution.

A more-intuitive solution technique is suggested by Cox and Ross (1975) and expanded by Harrison and Krep (1979). Note that in generating the hedge, the sole assumption made about investor preferences is that two assets which are perfect substitutes must earn the same equilibrium rate of return. This assumption rules out the existence of a dominant security. No assumption about risk preferences is employed. We note that the differential equation is a function only of five variables \(r, S, \tau, \sigma, X\). It does not involve investor preferences, nor does it require knowledge of the expected return of stock. This suggests that if a solution to the problem can be found which assumes a particular preference structure, it must also provide the solution to the differential equation for any preference structure that is consistent with the existence of equilibrium. This permits us to focus on the structure that promises to be the most tractable mathematically. The most tractable case is that of risk-neutrality, in which the market is composed only of risk-neutral investors.
Assuming risk-neutrality, the equilibrium rate of return to all assets equal the risk-free rate $r$. Thus, the stock price dynamics of (4.19) reduce to:
\[ \frac{dS}{S} = rdt + \sigma dW(t). \] (4.26)
The expected average rate of growth in the value of the option must also be $r$. To eliminate arbitrage opportunities, the current call price must be the discounted value of the expected terminal price:
\[ c = \exp(-rt)E[c(T)], \]
\[ = \exp(-rt) \int_{X}^{S(T)-X} L'(S) dS. \] (4.27)
In (4.27), $dS$ follows (4.26). Applying the lognormal theorem (4.10), equation (4.27) yields the Black-Scholes solution to the call pricing problem:
\[ c = S N(p_1) - \exp(-rt) X N(p_2), \] (4.28)
where
\[ p_1 = \frac{\ln(S/X) + [r + (\sigma^2/2)]T}{\sigma \sqrt{T}}, \]
\[ p_2 = \frac{\ln(S/X) + [r - (\sigma^2/2)]T}{\sigma \sqrt{T}}. \]
Equation (4.28) allows us to calculate $c_s$, $c_t$, and $c_{ss}$. Substituting these values into (4.25), it can be proved that equation (4.28) satisfies the Black-Scholes differential equation (4.25).

The Black-Scholes call-option pricing equation is a function of only five variables: the stock price, the exercise price, the time to maturity of the option, the risk-free rate, and the instantaneous variance rate on the stock price. The first four of these variables are directly observable so that only the variance rate must be
estimated. One remarkable thing about Black-Scholes pricing equation is that it does not require many other unobservable variables, such as the expected rate of return on the stock, the expected rate of return on the option, or a measure of market risk aversion. Furthermore, the Black-Scholes equation satisfies restriction (4.1)-(4.6). Its general property can be summarized in Figure 3. The Black-Scholes call option price lies below the maximum possible value, $C=S$ and above the minimum value, $C=\text{Max}[0, S - X \exp(-rt)]^{32}$. We note that as the stock price increases, the curve relating the Black-Scholes call price to the stock price asymptotically approaches the $C=\text{Max}[0, S - X \exp(-rt)]$ line.

Although widely used by option traders, the Black-Scholes formula is often reported to produce model values that differ in systematic ways from market prices (Galai [1983], Rubinstein [1985]). These reports have stimulated interest in alternative option-pricing formulas. While several assumptions underlying the Black-Scholes analysis have been questioned, subsequent research has focused on the distributional assumption (4.19).

4.5 Post-Black-Scholes Option Pricing Models

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32. Black-Scholes (1973) solve for the call option price assuming that stock does not pay dividends. For a security paying no dividends, applying dominance arguments it can be proved [see Merton (1973)] that $c(S, t; X) \geq \text{Max}[0, S - X \exp(rt)]$. This restriction implies that an American call on a non-dividend-paying stock will not be exercised before the expiration date.
The stock-price equation (4.19) that led to the Black-Scholes formula possesses two important properties. First, because parameters $\alpha$ and $\sigma$ are constants, the possible percentage changes in the stock price over any period does not depend on the level of the stock price at the beginning of the period. Second, because percentage changes in the stock price over time are a function of a normal process, over a very small interval of time, the size of the change in stock prices is also small.

If we relaxed the first assumption to allow stock-price variance to depend on the beginning-of-the-period stock price, as well as on the date, the Black-Scholes model would require some modification. Two option-pricing models are based on relaxing this assumption: Cox (1975)'s Constant-Elasticity-of-Variance Formula (CEV) and Geske (1979)'s Compound Options Model.

The second assumption implies that past stock prices, if observed in continuous time, can be graphed without lifting the pencil from the paper. Such a process is called a "continuous-time diffusion" and is said to have a continuous-sample path. This continuity allows the construction of an instantaneous, perfectly hedged portfolio employing only the option and the stock on which the derivation in Section 4.4 is based. But diffusion processes are merely one of two general types of continuous-time stochastic process. The second type of continuous-time stochastic process is called a "jump process." Two option-pricing models are based on jump process (or mixed jump processes): Cox and
Ross (1976)'s Pure-Poisson-Process model and Merton (1976a)'s Mixed Diffusion-Poisson-Process model.

Cox (1975) derives his model by explicitly assuming that the instantaneous volatility $\sigma^2(S,t)$ has the form:

$$\sigma^2(S,t) = k^2 S^{2p-2}, \quad (4.29)$$

where $0 < p < 1$ and $k$ is a positive constant.

Differentiating (4.29) with respect to $S$, we see that

$$\frac{\partial \sigma^2}{\partial S} = (2p-2) k^2 S^{2p-3} < 0.$$  

This means that the variance of stock prices, $\sigma^2(S,t)$, varies inversely with stock prices, a feature that several studies have found to characterize actual stock-price movements (e.g., Black [1976]). We may observe that the elasticity of the instantaneous variance (which indicates the percentage change in variance for a one-percentage-point change in the stock price) equals a negative constant:

$$\left(\frac{\partial \sigma^2}{\partial S}(S/\sigma^2)\right) = (2p-2) k^2 S^{2p-3} \left(\frac{S}{k^2 S^{2p-2}}\right) = 2p - 2 < 0.$$  

This family of processes can be labeled as constant-elasticity-of-variance (CEV) diffusions. The Black-Scholes case corresponds to $p=1$.

Substituting (4.29) into the fundamental partial differential equation (4.25) produces the following differential equation for the CEV specification:

$$c_t = rc - rSc_S - \frac{1}{2} \left(\frac{\partial^2 c}{\partial S^2}\right) k^2 S^{2p}, \quad (4.30)$$
again subject to the boundary condition (4.3). The solution to (4.30) has been found and has a complicated form\textsuperscript{33}.

The primary objection to the CEV model is that an economic reason needs to be given to justify the assumption that the variance of stock prices would vary inversely with stock prices. One rationalization proceeds through the capital structure of the underlying firm. Let us assume that a firm has m shares of stock outstanding and borrows B dollars in bonds. Let us assume further that all bonds are pure discount debt that matures at date $T^<T$ with face value B. On the maturity date $T_1$ for bonds, the value of the firm's stock depends on the value of the firm $V(T_1)$\textsuperscript{34}. If $V(T_1)$ is less than B, the firm's stock is worthless. If $V(T_1)$ is greater than B, then the firm's stock price $S=[V(T_1)-B]/m$. This analysis shows that the stock itself may be likened to a call option on the value of the firm with striking price B and time to expiration $T_1$. From this perspective, a call option on the stock becomes an option on an option, which may be termed a "compound option". Assuming that $\sigma_v$, the volatility of the value of

\textsuperscript{33} The reader is directed to Cox (1975) or Jarrow and Rudd (1983; p.152) for the solution and its proof.

\textsuperscript{34} The value of the firm is the value of the total assets owned by the firm. The assets the firm might own may be federal bonds, other firms' stocks or some intangible assets. What form of assets the firm might own does not matter in this example.
the firm's assets, is constant, it can be shown (see Geske [1979; p. 63-81]) that the standard deviation for the stock price is:

\[ \sigma_S = N(h)\sigma_v / m > 0, \]

where \( h = [\ln(V(t)/X) + (r-(1/2)\sigma_v^2)T] / \sigma_v \). This in turn implies \( \sigma_S / \sigma_S < 0, \) so that the variance of the stock price varies inversely with the stock price. Unlike the CEV model, which remains consistent with this restriction on \( \sigma_S / \sigma_S \), the compound-option model offers an explicit mechanism to support it.

If the value of the compound option is \( c^m \), then \( c^m \) will satisfy the fundamental differential equation (4.25), so that:

\[ c_t^m = r c^m - r S c_S^m (1/2)(\sigma^2 c / \sigma S^2) \sigma_S^2 S^2, \]  \hspace{1cm} (4.31)

where \( 0 \leq t \leq T_1 \), and subject to \( c^m_{T_1} = \max[V(T_1)-B,0]/m \), and

\[ \sigma_S / \sigma_S = r S - r V(\sigma_S / \sigma V) - (1/2)(\sigma^2 S / \sigma V^2) \sigma_S^2 \sigma V^2. \]  \hspace{1cm} (4.32)

Using successive substitution, a closed-form solution to (4.31) has been found and has a complicated form.\(^{35}\)

The compound-option formula generalizes the Black-Scholes formula by considering the effects of a firm's capital structure on the volatility of its stock. Instead of regarding a stock's volatility as

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\(^{35}\) The reader is directed to Geske (1979; p. 63-81) for the solution and its proof.
fixed, the model moves one step deeper to regard the firm's asset-value volatility as fixed. The Black-Scholes formula emerges as the special case if either $B=0$ or $T_1=\infty$. In either case, the firm effectively has no debt and $V=mS$.

Cox and Ross (1976) demonstrate that if the stochastic part of the stock-price movement is defined as a simple Poisson process, and the Poisson-jump is in only one direction and of a given amplitude, then a risk-free hedge, like the Black-Scholes hedge, can be created. A simple jump process can be written analogous to (4.19) as:

$$dS/S = \lambda dt + (Y-1)d\pi.$$ (4.33)

In (4.33), $\pi$ is a continuous-time standard Poisson process, $\lambda$ is the intensity of the process and $Y-1$ is the jump amplitude. Equation (4.33) says that the percentage change in the value of the stock over the interval from $t$ to $t+dt$ is composed of a drift term, $\lambda dt$, and a term, $(Y-1)d\pi$, which will jump the percentage stock's price change to $Y-1$ with probability $\lambda dt$. By letting $\lambda \to \infty$ and $Y=1$, i.e., the jump events occur continuously and its jump sizes are very small, the movement of the stock price approximates that of equation (4.19). In this case, the diffusion equation in (4.19), in which information arrives continuously and has only a differential impact, becomes the limit of a pure Poisson process (4.33).

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36. A standard Poisson process is a Poisson process which has a value of 1 if a jump event occurs and a value of 0 if a jump does not occur.
A corollary to Ito's lemma can be used to differentiate functions of Poisson processes [see Malliaris (1982; p. 121)]. Cox and Ross (1976) use the technique employed by Black and Scholes to derive the option price. Under different parameter specifications, Cox and Ross derive five alternative pure Poisson-process option-pricing formulas.

Merton (1976a) examines a more general specification of the stock price movements, letting both a stationary lognormal process and a Poisson-jump process be present. He demonstrates that hedging both against the lognormal changes and the Poisson-jump changes is not possible, so that a Black-Scholes type of risk-free hedge cannot be created. However, as long as the jumps are uncorrelated across securities, the risk associated with the jump in the Black-Scholes type of hedge portfolio is an unsystematic risk. Such a risk, is diversifiable by holding a portfolio of hedges. If the equilibrium return to a security or portfolio of securities is determined by its nonderecomposible risk, as suggested by the Sharp-Lintner-Mossin Capital Asset Pricing Model (CAPM), then a Black-Scholes type of hedge portfolio can be made riskless. This renders the fair price of a call option implicit in the price of the riskless hedge portfolio.

Merton assumes that the percentage change in the stock price may be decomposed into two types of changes. He defines "normal" vibrations in price as those that are due, for example, to a temporary imbalance between supply and demand. This component may be modeled by a diffusion-normal process. He defines "abnormal" vibrations in price
as those that are due to the arrival of important new information about the stock. This component is modeled by a Poisson process. Whether these mixed processes can more accurately describe the path of asset prices than a pure diffusion process or pure Poisson process is an open empirical question. However, as pointed out by Jarrow and Rudd (1983; p. 160) and empirically examined in chapter 3, these processes appear to be fairly realistic descriptions of foreign-currency price movements.

Merton assumes that the arrivals of the Poisson-driven events are independent and identically distributed. Given that the Poisson event occurs (i.e., that some important information on the stock arrives), he envisions a conditional "drawing" from another distribution to determine the impact of this information on the stock price. We let $Y$ be the random variable describing the outcome of the subsequent drawing. Neglecting the continuous part of the process, the stock price at time $t+dt$, $S(t+dt)$, becomes a random variable $S(t+dt)=S(t)Y$, given the condition that exactly one such arrival occurs between $t$ and $(t+dt)$. Letting $(Y-1)$ be the random variable describing the percentage change in the stock price given that the Poisson event occurs, Merton defines $\kappa=E(Y-1)$, i.e., $\kappa$ is the expected value for stock's percentage price change if exactly one Poisson jump occurs. The posited stock price return can be written formally as a stochastic differential equation:

$$\frac{dS}{S}=(\alpha-\lambda\kappa)dt+\sigma dZ+(Y-1)d\pi.$$  (4.34)
In (4.34), $\alpha$ is the instantaneous expected return on the stock; $\sigma^2$ is the instantaneous variance of the return, conditional on the nonoccurrence of Poisson event; $\pi(t)$ is an independent standard Poisson process with parameter $\lambda$, which gives the mean number of arrivals per unit time; $dZ$ is a standard Gauss-Wiener process; and $d\pi$ and $dz$ are assumed to be independent.

Note (4.34) is a mixed form of pure Poisson process (4.33) and pure Diffusion process (4.19). The model is frequently called a mixed Diffusion-Poisson process, a mixed Diffusion-Jump process or simply a mixed jump process.

Applying the CAPM, and using a method similar to Black-Scholes, Merton derives a differential equation which an arbitrage-free option pricing function $c^*$ under mixed diffusion-jump processes must satisfy:

$$\frac{1}{2}\sigma^2 S^2 c_s^* + (r - \lambda \kappa) Sc^*_s - c^*_t - rc^* + \lambda E[c^*(S_T, \tau) - c^*(S, \tau)] = 0, \quad (4.35)$$

subject to the boundary condition (4.3). (4.35) reduces to the Black-Scholes equation (4.25) if $\lambda = 0$. While (4.35) is in some respects formally the same type of equation as (4.25), it is a mixed difference-differential equation. We may note that even though the jumps represent diversifiable risk, the jump component does affect the equilibrium option price.  

37. Note that $\lambda E[c^*(S_T, \tau) - c^*(S, \tau)]$, which represents the component of expected option-price changes due to a Poisson jump event, is part of the differential equation (4.35).
Merton assumes that the random variable $Y$ has a stationary lognormal distribution. This means that $y = \ln(Y)$ is a normal variate possessing the following characteristics:

$$E(y) = u, \quad V(y) = \delta^2, \quad E(Y) = E(\exp(y)) = \exp(u + \delta^2/2), \quad \text{and}$$

$$K = E(Y-1) = \exp(u + \delta^2/2) - 1.$$ 

Merton shows that when exactly $n$ Poisson jumps occur during the life of the option, the average variance per unit time for stock price is $V_n^2 = \sigma^2 + n\delta^2/\tau$. Merton defines $\gamma = \ln(1+K)$, $r_n = r - \lambda K + n\gamma/r$ and uses a series of complicated substitutions to develop the following solution to (4.35):

$$c^*(S, r, \sigma, \tau, X) = \sum_{n=0}^{\infty} \frac{\exp(-\lambda'\tau)(\lambda'\tau)^n}{n!} c(S, r_n, V_n, \tau, X),$$  \hspace{1cm} (4.36)$$

where $\lambda' = \lambda(1+K)$ and $c$ is the Black-Scholes pricing equation (4.28).

Remembering $V_n^2$ is the average variance per unit time, $c(S, r_n, V_n, \tau, X)$ may be seen to be the value of the option, conditional on knowing that exactly $n$ Poisson jumps will occur during the life of the option. The actual value of the option, $c^*$, is just the weighted sum of each of these prices where each weight equals the probability that a Poisson random variable with characteristic parameter $\lambda'\tau$, takes on the value $n$.

The main objection to Merton's model is that formula (4.36) requires two potentially falsifiable assumptions. First, the model assumes that securities are priced so as to satisfy the Capital Asset
Pricing model. Second, the model requires the jump component of a security’s return to be uncorrelated with the market. While the CAPM has been extensively tested, its validity as a description of equilibrium returns remains open. Even if the CAPM were validated, the assumption of zero correlation between the jump component of stocks’ return and the market return, needs additional empirical justification.

The post-Black-Scholes option-pricing models reviewed in this chapter use Black-Scholes’ zero-arbitrage restrictions. Since the specified stochastic processes can converge to a stationary lognormal process, the option-pricing formulas they derive incorporate the Black-Scholes’ formula as a special case. It remains to be seen whether these more-general option-pricing formulas perform better than the Black-Scholes’ formula. The performance of the richer option-pricing models may depend on how more closely the new stochastic processes can describe the underlying asset’s price movements. Thus, it is important to examine the pricing behavior of the underlying asset to see if the distributional assumptions of one particular model hold. But we should not dismiss a model merely because its simplifying assumptions are invalid. Whether a pricing model is useful or not must be judged on its ability to explain and forecast the real market-price movements.

Empirical tests to be reported in chapter 3 indicate that Merton’s mixed-jump stochastic processes may provide a fairly realistic description of foreign-currency price movements. Because
multiple interest rates are involved in foreign currency options, both the Black-Scholes' model and Merton's model need to be modified. An adaptation of Merton's model to foreign-currency option has not previously appeared in the literature. We make the first attempt in the next chapter to apply Merton's jump option model to foreign-currency options. For purposes of comparison, we simultaneously adapt the Black-Scholes model to foreign-currency options as well. Both Black-Scholes' formula and Merton's jump-option formula for foreign-currency options are examined empirically in chapter 7.
CHAPTER V
TWO ARBITRAGE-FREE MODELS OF FOREIGN-CURRENCY OPTIONS

5.1 Introduction

Most asset-pricing models embody conditions designed to eliminate opportunities for arbitrage. This chapter develops no-arbitrage equations that price call options on foreign currency. Two alternatives are stated, one using Black-Scholes' assumptions and the other using Merton's assumptions. Because multiple interest rates are involved in foreign-currency options, the stock-option pricing models discussed in chapter 4 need to be modified. Some studies examine foreign-currency option pricing in a Black-Scholes' setting. In assuming that foreign-currency returns follow a stationary lognormal process, Garman and Kohlhagen (1983), Biger and Hull (1983) and Grabbe (1983) derive similar foreign-currency options formulas. These studies differ in the assumptions they make about domestic and foreign interest rates. Garman and Kohlhagen (1983) and Biger and Hull (1983) derive the same foreign-currency option formula by assuming interest rates are constant. Grabbe (1983) derives a more general foreign-currency option formula based on stochastic interest rates. The two currency-option models we derive in this chapter use the constant-interest-rate assumption.

The first model (derived in section 5.2) assumes that foreign-currency returns follow a stationary lognormal process. The second
model (derived in section 5.3) assumes that the foreign-currency
return follows a mixed diffusion-jump process. In deriving the second
model, we use Merton's assumption (1976a) that the jump component of
the currency return represents "nonsystematic risk" that can be
diversified away. Merton's model (1976a) has been suggested by Jarrow
and Rudd (1983; p. 160) as an alternative model which might explain the
biases expressed in the Black-Scholes formula. Nevertheless, no
adaptation of Merton's model to foreign-currency options has yet
appeared in the literature. The formula we derive in this chapter for
a foreign-currency option subject to a mixed jump process is new to
the literature.

The symbols used are adapted from those of chapter 4. Some
symbols need to be given new meanings in foreign-currency options
markets. The adaptations are:

\[ S(t) \] - the spot domestic-currency price of a unit of foreign exchange
at time \( t \).

\[ c(t) \] - the domestic-currency price at time \( t \) of a European call
option written on one unit of domestic currency.

\[ r_d \] - domestic risk-free interest rate.

\[ r_f \] - foreign risk-free interest rate.

\[ B(t, \tau) \] - the domestic-currency price of a pure discount bond which pays
one unit of domestic currency at time \( t + \tau = T \). If the risk-free
domestic interest rate is constant, then \( B(t) = \exp(-r_d \tau) \).

\[ B^*(t, \tau) \] - the foreign-currency price of a pure discount bond which pays
one unit of foreign exchange at time $t+\tau=T$. If the risk-free foreign interest rate is constant, then $B^*(t)=\exp(-r_f\tau)$.

$X$ - the domestic-currency exercise price of an option on foreign currency.

5.2 A Foreign-Currency Option Model with a Pure-diffusion process.

This section derives a foreign-currency option model in a Black-Scholes' setting. We make three important assumptions:

(a) the price of one unit of foreign currency follows a stationary lognormal process, i.e.,

$$dS/S = \alpha dt + \sigma dZ(t). \quad (5.1)$$

In (5.1), $Z(t)$ is a standard Gauss-Wiener process at time $t$; $\alpha$ is the instantaneous expected return; $\sigma^2$ is the instantaneous variance of the foreign-currency returns.

(b) foreign exchange and options markets operate continuously without transaction costs or taxes.

(c) the risk-free interest rates in both the foreign country and the home country remain constant during the life of the option.

Foreign-currency options and stock options differ as to whether or not a role is given to foreign interest rates. An investor in stock options considers holding stocks as an alternative to stock options. However, in a foreign-currency options market an investor who wishes
to hold a foreign currency must prefer interest-earning foreigncurrency bonds to holding foreign currency in a non-interest-bearing account. Black and Scholes (1973) demonstrate that it is possible to create a riskless hedge by forming a portfolio containing European stock call options and stocks. A similar Black-Scholes type hedge in foreign-currency options market, therefore, must contain European foreign-currency call options and foreign discount bonds \( G \) instead of foreign currencies.

The foreign discount bond is defined formally as:

\[ G = S \cdot \exp(-r_f T). \]  \( (5.2) \)

\( G \) is the dollar-value of a foreign discount bond. On maturity date \( T \), \( G \) will be worth one unit of foreign currency. Applying Ito's lemma to \( (5.2) \) and using the fact that \( dt = -dt \), we obtain:

\[ \frac{dG}{G} = (\alpha + r_f) dt + \sigma dZ. \]  \( (5.3) \)

We assume that the foreign-currency call price \( c \) is a function of foreign discount bond \( G \) and a time index \( t \), so that:

\[ c(t) = c(G(t), t). \]  \( (5.4) \)

We also assume that \( c \) is a twice-differentiable function. Ito's lemma can be employed to express \( dc \) as

\[ dc = \left\{ \frac{1}{2} \sigma^2 G^2 c_{GG} + (\alpha + r_f) G c_c + c_t \right\} dt + c_G \sigma G \, dz, \]

and

\[ \frac{dc}{c} = \left\{ \frac{1}{2} \sigma^2 G^2 c_{GG} + (\alpha + r_f) G c_c + c_t \right\} / c \, dt + c_G \sigma G / c \, dz. \]
\[ \dot{u}_c = u_c \dot{t} + \sigma_c \dot{z}. \quad (5.5) \]

In (5.5), subscripts on \( c \) denote partial derivatives and
\[ u_c = \left[ (1/2) \sigma^2 c_{GG} + \left( \alpha + r_c \right) c_G + c_t \right] / c, \quad \sigma_c = \sigma / c. \]

In the spirit of the Black-Scholes formulation, we focus on the possibility of creating a riskless hedge by forming a portfolio \( (h) \) composed of foreign discount bonds \( G \), foreign currency options \( c \), and domestic bonds \( B \). The aggregate initial investment \( V_h \) in a hedged portfolio required to equal zero. Since returns on a hedge portfolio are riskless, to eliminate arbitrage opportunities, the portfolio must yield a zero return. This condition makes the value of the fair price of a call option implicit in the value of the hedge portfolio.

We define \( W_1, W_2, W_3 \) as the (instantaneous) number of dollars of the portfolio invests in \( G \), \( c \) and \( B \). The value of this hedge portfolio \( V_h \) can be expressed as:
\[ V_h = W_1 \frac{G}{G} + \frac{W_2}{c} c + \frac{W_3}{B} B, \quad (5.6) \]

In (5.6), \( \nu_1 \equiv \frac{W_1}{G}, \nu_2 \equiv \frac{W_2}{c}, \nu_3 \equiv \frac{W_3}{B} \), i.e., \( \nu_1, \nu_2, \nu_3 \) are the number of units of security that portfolio \( h \) invests in \( G \), \( c \) and \( B \). The change in the value of the hedge, \( dV_h \), is the total derivative of
Using Ito's Lemma and the fact that $V_3 = -W_1 - W_2$, $dV_h$ can be written as:

$$dV_h = v_1 dG + v_2 dc + v_3 dB,$$

$$= W_1 \left( \frac{dG}{G} \right) + W_2 \left( \frac{dc}{c} \right) + (-W_1 - W_2) r_d dt,$$

$$= [W_1 (\alpha + r_f - r_d) + W_2 (u_c - r_d)] dt + [W_1 \sigma + W_2 \sigma_c] dZ. \quad (5.7)$$

Note that the only stochastic term in the expression for $dV_h$ is $dZ$. All other terms are deterministic. To make $dV_h$ riskless, we need to make the coefficients for $dZ$ equal zero, i.e.,

$$W_1 \sigma + W_2 \sigma_c = 0. \quad (5.8)$$

Let us suppose a strategy $\hat{W}_j = \hat{W}_j^*$ can be chosen such that condition (5.8) is satisfied and the hedge portfolio is riskless. Since the portfolio uses no wealth, to eliminate arbitrage opportunities it must yield a zero return. This implies that the coefficients of $dt$ in (5.7) should sum to zero:

$$W_1 (\alpha + r_f - r_d) + W_2 (u_c - r_d) = 0. \quad (5.9)$$

Equation (5.8) and (5.9) constitute a system of homogeneous equations with two unknown variables. To have a nontrivial solution, the determinant of the two equations' coefficients matrix must be zero. This implies:

$$\sigma (u_c - r_d) = \sigma_c (\alpha + r_f - r_d). \quad (5.10)$$
Substituting \( u_c \) and \( \sigma_c \) of (5.5) into (5.10) defines a differential equation for the value of a foreign currency option:

\[
\frac{1}{2} \frac{\partial^2 c}{\partial G^2} G^2 \sigma^2 + \frac{\partial c}{\partial G} G r_d \sigma_c + c_t = 0.
\]  

(5.11)

Of course, this differential equation must conform to the boundary condition 
\( c(T) = \text{Max}[0, S(T) - X] \).

The differential equation (5.11) for foreign currency options has a form identical to the Black-Scholes' differential equation (4.25) for stock options. The only difference between (5.11) and (4.25) is that \( S \) and \( r \) of (4.25) are replaced by \( G \) and \( r_d \) of (5.11). To solve the riskless hedge equation (5.11), we note that equation (5.11) does not involve investor preferences, therefore, we can focus on the "risk-neutrality" case to find a solution. Assuming risk-neutrality, the equilibrium rate of return to all assets in domestic dollars equals the domestic risk-free rate \( r_d \). For a foreign discount bond \( G \) which is denominated in domestic dollars, price dynamics of (5.3) must reduce to:

\[
dG/G = r_d dt + \sigma dz.
\]  

(5.12)

From (5.2), we can express \( S \) in terms of \( G \) as:

\[
S = G \cdot \exp(r_f \tau).
\]  

(5.13)

Applying Ito's lemma to (5.13) with the restriction that \( dG \) follows (5.12), the price dynamics of foreign currency (5.1) is changed to:

\[
dS/S = (r_d - r_f) dt + \sigma dz.
\]  

(5.14)
The expected average rate of growth in the value of a foreign-currency option must also be $r_d$. To eliminate arbitrage opportunities, the current foreign-currency call price must the discounted value of the expected terminal price:

$$c(t) = \exp(-r_d \tau) \mathbb{E}[\text{Max}(S(T) - X, 0)].$$ (5.15)

In (5.15), $S(T)$ follows (5.14). Applying the lognormal theorem (4.10) to equation (5.15) with $a = \exp(-r_d \tau)$ and $\alpha = (r_d - r_f)$ yields the Black-Scholes solution to the foreign-currency call pricing problem:

$$c(t) = e^{(r_d - r_f) \tau} \cdot e^{-r_d \tau} \cdot S \cdot N \left( \frac{\ln(S/X) + (r_d - r_f) \tau + (\sigma^2/2) \tau}{\sigma \sqrt{\tau}} \right) - e^{-r_d \tau} \cdot X \cdot N \left( \frac{\ln(S/X) + (r_d - r_f) \tau - (\sigma^2/2) \tau}{\sigma \sqrt{\tau}} \right) = S \cdot e^{-r_f \tau} \cdot N(d_1) - X \cdot e^{-r_d \tau} \cdot N(d_2),$$ (5.16)

where $d_1 = \frac{\ln(S/X) + (r_d - r_f) \tau + (1/2) \sigma^2 \tau}{\sigma \sqrt{\tau}}$, $d_2 = \frac{\ln(S/X) + (r_d - r_f) \tau - (1/2) \sigma^2 \tau}{\sigma \sqrt{\tau}}$. We note that both the foreign interest rate $r_f$, and the interest-rate differential, $r_d - r_f$, play distinct roles in the solution (5.16).

5.3 A Foreign Currency Option Model with a Mixed Diffusion–Poisson Process

The critical assumption in deriving the pricing model of Chapter 5.2 is that the underlying foreign-currency return dynamics can be
described by the diffusion process (5.1). If the return dynamics of the basic securities follow a mixed diffusion-jump process, can we form a riskless hedge portfolio? Merton (1976a) explores this question for stock options and finds the answer to be negative. Since the derivation of the pricing equation for foreign-currency options in the section 5.2 parallels that of stock options under Black-Scholes' assumptions, Merton's answer can be applied to foreign-currency options. This means that, if foreign-currency returns follow mixed diffusion-jump processes, the hedge portfolio described in (5.6) cannot be made to be riskless. Thus the differential equation (5.11) would no longer define the fair price of a call option for a foreign currency.

Merton (1976a) assumes the jump component of underlying security returns is 'diversifiable' so that a Black-Scholes type of risk-free hedge can be constructed. If the equilibrium return to a security is determined by its nondiversifiable risk, as suggested by the CAPM, the riskless hedge portfolio must command the return of a risk-free rate. This renders the fair price of a call option implicit in the price of the riskless hedge portfolio. This approach has most of the attractive features of the original Black-Scholes model in that it does not depend on investor preferences or knowledge of the expected return on the underlying security. In this section we use this approach to price foreign-currency options that are subject to a mixed jump process.

We assume that the total percentage change in the foreign-currency price is the composition of two types of changes: a normal
vibration in price and an abnormal vibration in price. The normal vibration in price is described by a stationary normal process. The abnormal vibration in price is modelled by a Poisson process. We also assume that the Poisson-driven events are independent and identically distributed. This lets us partition the probability of an event occurring during a time interval of length $h$ as:

(a) $\text{prob}\{\text{the event does not occur once in the time interval}(t,t+h)\} = 1 - \lambda h + o(h)$,

(b) $\text{prob}\{\text{the event occurs once in the time interval}(t,t+h)\} = \lambda dt + o(h)$,

(c) $\text{prob}\{\text{the event occurs more than once in the time interval}(t,t+h)\} = o(h)$,

where $o(h)$ is the asymptotic order symbol defined by $\Phi(h) = o(h)$ if

$$\lim_{h \to 0} \frac{\Phi(h)}{h} = 0,$$

i.e., $o(h)$ is of a much smaller order of magnitude than $h$, and $\lambda$ is the mean number of jump arrivals per unit time. Thus during time interval $h$ the expected number of jump arrivals is $\lambda h$.

Whenever the Poisson event occurs, a jump is observed in the foreign-currency price. We let $Y$ be the random variable describing the jump size. Neglecting the continuous part of the process, the currency price at time $t+h$, $S(t+h)$, becomes a random variable $S(t+h) = S(t)Y$, given the condition that exactly one such arrival occurs between $t$ and $(t+h)$. Given that the Poisson event occurs, we denote $(Y-1)$ as the random variable describing the percentage change in the foreign-currency price and $\kappa = E(Y-1)$ as the expected value of $(Y-1)$. The
posited currency price returns can be written formally as a stochastic differential equation as in equation (4.34):

\[ \frac{dS}{S} = (\alpha - \lambda \kappa) dt + \sigma dz + (\gamma - 1) d\pi, \]  

(5.17)

In (5.17), \( \alpha \) is the instantaneous expected return on the currency return; \( \sigma^2 \) is the instantaneous variance of the return, conditional on the nonoccurrence of the Poisson event; \( dz \) is a standard Gauss-Wiener process; \( d\pi(t) \) is a standardized Poisson random variable; and \( dz \) and \( d\pi \) are assumed to be independent.

Following Merton (1976), we assume \( Y_i \), the random variable describing the jump size of the \( i \)th Poisson event, follows a stationary lognormal process. This means that \( y_i = \ln(Y_i) \) is a normal variate possessing the following characteristics:

\[ E(y_i) = \mu, \quad V(y_i) = \sigma^2, \quad E(Y_i) = E(\exp(y_i)) = \exp(\mu + \sigma^2/2), \quad \text{and} \]

\[ \kappa = E(Y - 1) = \exp(\mu + \sigma^2/2) - 1. \]

An interesting special case occurs, which we assume hereafter, when \( \alpha, \sigma, \lambda \) and \( \kappa \) are constants. This allows (5.17) to be written in an alternative but equivalent form:

\[ X(T) = X(t) + (\alpha - \lambda \kappa - (\sigma^2/2)) T + \sigma z(T) \]

(5.18)

This implies \( X(T) = X(t) + (\alpha - \lambda \kappa - (\sigma^2/2)) T + \sigma z(T) \) by Ito’s integration (the opposite operation of Ito’s lemma). Using the fact that \( S = \exp(X) \), this implies \( S(T) = S(t) \exp[(\alpha - \lambda \kappa - (\sigma^2/2)) T + \sigma z(T)] \). Putting the jump part into (Footnote continues on next page)
\[ S(T) = S(t) \exp(x + \sum_{i=1}^{\Pi(t)} Y_i) \]  
\text{(5.18)}

In (5.18), \( x \) is a normally distributed random variable with \( \mu(x) = (\alpha - \lambda \kappa - \sigma^2/2) \tau \) and \( \sigma(x) = \sigma \tau \), and \( \Pi \) is a Poisson random variable describing the total number of jump events occurs during the time interval \( \tau \). Equation (5.18) shows that the variable \( \ln[S(T)] \) follows a mixture of normal process. This form proves useful later in solving the call option's value with mixed jump processes.

We note that because \( \mu(d\Pi(t)) = \lambda dt \), the expected return on the foreign currency over the interval \((t, t+dt)\) implied by (5.17) is \( \alpha \):

\[ \mu(dS/S) = (\alpha - \lambda \kappa) dt + \kappa(\lambda dt) = \alpha. \]

But the expected forgone return from holding a foreign-currency option is not \( \alpha \). As we explain in section 5.2, because investors prefer holding an interest-bearing foreign bond to a foreign currency, the opportunity cost of holding a currency option must be the expected return on a foreign discount bond \( G \).

Applying Ito's lemma to equation (5.2) for foreign discount bonds and noting that \( dt = -d\tau \), we obtain:

(Footnote continued from previous page) the former equation and noting that the jump part \( Y(n) = \sum_{j=1}^{\Pi} Y_j = \exp(\sum_{j} Y_j) \), we obtain

\[ S(T) = S(t) \exp\left[ (\alpha - \lambda \kappa - (\sigma^2/2)) \tau + \sigma Z \right] Y(n), \]
\[ = S(t) \exp\left[ ((\alpha - \lambda \kappa - (\sigma^2/2)) \tau + \sigma Z) + \sum_{j} Y_j \right]. \]

Because \( Z(t) \) is a standard Gauss-Wiener process, \( (\alpha - \lambda \kappa - (\sigma^2/2)) \tau + \sigma Z \) is a normal random variable with mean \( (\alpha - \lambda \kappa - (\sigma^2/2)) \tau \) and variance \( \sigma^2 \tau \).
dG/G=(\alpha+r_f-\lambda \kappa)dt+\sigma dz+ (Y-1)d\pi(t). \hspace{1cm} (5.19)

To be consistent with section 5.2's derivation, we assume the foreign-currency option-pricing function \( c \) is a function of \( G \) and \( t \) and is twice-differentiable. Using Ito's lemma for diffusion processes and an analogous lemma for Poisson processes, option's pricing-dynamics can be written as

\[
d c = \left[ \frac{1}{2} \sigma^2 \frac{\partial^2 c}{\partial G^2} + (\alpha + r_f - \lambda \kappa) \frac{\partial c}{\partial G} + c \frac{\partial r}{\partial t} + \frac{\partial c}{\partial t} \right] dt + \sigma \frac{\partial c}{\partial G} dz + \left[ c(G*Y) - c(G) \right] d\pi(t),
\]

where \( c(G*Y) \) is the new option price if the Poisson event occurs. The equation can be rewritten as,

\[
d c/c = \left[ \frac{1}{2} \sigma^2 \frac{\partial^2 c}{\partial G^2} + (\alpha + r_f - \lambda \kappa) \frac{\partial c}{\partial G} + c \frac{\partial r}{\partial t} + \frac{\partial c}{\partial t} - \frac{\lambda \kappa c}{c} \right] dt + \sigma \frac{\partial c}{\partial G} dz + \left[ \frac{c(G*Y) - c(G)}{c} \right] d\pi(t)
\]

\[=(u_c - \lambda \kappa)dt + \sigma_c dz + (Y_c - 1)d\pi(t). \hspace{1cm} (5.20)\]

In (5.20), subscripts on \( c \) denote partial derivatives. The four newly-defined parameters on the option-pricing dynamics are:

\( Y_c \equiv c(G*Y)/c(G) \), the new option price relative if the Poisson event occurs;

\( \kappa_c \equiv \frac{\frac{\partial}{\partial t} \lambda \kappa}{\frac{\partial c}{\partial t}} = \frac{\big\{ c(G*Y) - c(G) \big\}}{c} \), the expected percentage change in the option price if the Poisson event occurs;

\( u_c \equiv \frac{1}{2} \sigma^2 \frac{\partial^2 c}{\partial G^2} + (\alpha + r_f - \lambda \kappa) \frac{\partial c}{\partial G} + c \frac{\partial r}{\partial t} + \frac{\partial c}{\partial t} + \lambda \kappa c \), the instantaneous expected return on the option, and \( \sigma_c \equiv \frac{\sigma c}{c} \), the
instantaneous variance of the return on the option, conditional on no occurrence of a Poisson event.

We again focus on the possibility of creating a hedge portfolio now termed \( j \), that is composed of \( G \), \( c \) and \( B \). The hedge portfolio \( j \) is similar to portfolio \( h \) described in (5.6). As in section 5.2, we denote \( W_1, W_2, W_3 \) as the (instantaneous) number of dollars of the portfolio invests in \( G \), \( c \) and \( B \). We also assume the initial investments on \( j \) is zero. The value of this hedge portfolio, \( V_j \), can be expressed as

\[
V_j = W_1^G + W_2^c + W_3^B = 0,
\]

\[
=(W_1/G)G + (W_2/c)c + (W_3/B)B,
\]

\[
=v_1G + v_2c + v_3B. \tag{5.21}
\]

In (5.21), \( v_1 = W_1/G \), \( v_2 = W_2/c \), \( v_3 = W_3/B \), i.e., \( v_1 \), \( v_2 \), \( v_3 \) are the number of respective's units of each security that portfolio \( j \) contains.

Using Ito's Lemma and the fact that \( W_3 = -W_1 - W_2 \), the change in the value of the hedge \( dV_j \) is:

\[
dV_j = W_1(dG/G) + W_2(dc/c) + (-W_1 - W_2)rdt,
\]

\[
=[W_1(\alpha r - rd) + W_2(u - rd) - W_1^\lambda K - W_2^\lambda K]dt + [W_1^{\sigma + W_2^{\sigma}}]dz + \{W_1^{(Y-1)} + W_2^{(Y - 1)}\}d\pi(t),
\]

\[
=(\alpha r - \lambda K)dt + \sigma d\pi + [W_1^{(Y-1)} + W_2^{(Y - 1)}]d\pi(t). \tag{5.22}
\]
In (5.22), \( \alpha_v = \mu_v (\alpha - r_f - r_d) + \mu_w (u_c - r_d) \), is the instantaneous expected return on the portfolio; \( \sigma_v = \sigma_1 + \sigma_2 \), is the instantaneous standard deviation of the return, conditional on the Poisson event not occurring and \( \kappa_v = \mu_1 \kappa_1 + \mu_2 \kappa_2 = \mu_1 E(Y-1) + \mu_2 E(Y_c-1) \) is the random variable describing the percentage change in the portfolio's value if the Poisson event occurs.

Two stochastic terms appear in the infinitesimal portfolio-value change (5.22): in \( dZ \) and \( dn(t) \). To make this portfolio nonstochastic, we must pick \( \mu \) such that the coefficients of each term sum to zero. Let us denote this \( \mu \)-vector by \( \mu^* \). By construction,

\[ \mu_1^* \sigma + \mu_2^* \gamma = 0, \]  

(5.23)

and

\[ \mu_1^* (Y-1) + \mu_2^* (Y_c-1) = 0. \]  

(5.24)

In the analysis of section 5.2, where \( \lambda = 0 \), the stochastic term from Poisson processes \( dn(t) \) of equation (5.22) disappears. In this case, the portfolio return could be made riskless by picking \( \mu_1 - \mu_1^* \) and \( \mu_2 - \mu_2^* \) such that (5.23) is satisfied. But in the presence of the jump process, \( dn \), the return on such a Black-Scholes type hedge portfolio with dollar investments \( \mu_1^* \) and \( \mu_2^* \) is not riskless. The change in the value of such a hedge is:

\[ dV_j = (\alpha_v - \lambda \kappa_v) dt + [\mu_1 (Y-1) + \mu_2 (Y_c-1)] dn(t). \]  

(5.25)
Equation (5.25) shows that this hedge would still be risky because of the exposure to the Poisson process $d\pi$. As noted by Merton (1976), there does not exist a set of portfolios $(W_1, W_2)$ that eliminates all stochastic parts of (5.22). Therefore, we follow Merton's (1976) solution in assuming that the jump component of the foreign currency return represents a "nonsystematic risk" that can be diversified away. Thus the jump component of the portfolio return is uncorrelated with the market. That means the "beta" of this portfolio is zero. If the Capital Asset Pricing model holds, then the expected return on all zero-beta securities must equal the riskless rate. For a portfolio having zero initial investment, to eliminate arbitrage opportunities, the drift term in (5.22) must sum to zero. This implies:

$$W_1^*(\alpha - r_f - r_d) + W_2^*(u_c - r_d) = 0. \quad (5.26)$$

Together with the generalized Black-Schole's hedged condition (5.23),

$$W_1^* \sigma + W_2^* \sigma_c = 0$$

constitute a system of homogeneous equations with two unknown variables. To have a nontrivial solution, the determinant of the two equations' coefficient matrix must be zero. This implies:

39. We note that the term $(c_{GG})$ of the second-degree approximation to $dc$ given by Taylor series expansion in (5.20) is not zero. Thus, the option price is a nonlinear function of the underlying asset price, while portfolio mixing is a linear operation. Therefore, for any $W_1$ and $W_2$, $W_1(Y-1) + W_2(Y_c - 1)$ of $dV_j$ assumes nonzero values for some possible $Y$ and $Y_c$. 


Substituting for $u_c$ and $\sigma_c$ from (5.20), (5.27) can be written as

$$(1/2)\sigma^2G^2c_{GG}+(r_d-\lambda \kappa)Gc_G-c_{\tau}r_d c +\lambda E\{c(GY,\tau)-c(G,\tau)\}=0. \quad (5.28)$$

The arbitrage-free option-pricing function $c$ must satisfy the differential equation (5.28) and be subject to the boundary condition $c(T)=\text{Max}[0,S(T)-X]$. Instead of using a series of complicated substitutions to develop a solution to (5.28) as Merton (1976a) does for stock options, we develop a more-intuitive solution. We note that (5.28) depends on neither the mean security return $\alpha$ nor investor preferences. This lets us employ the risk-neutrality argument as in section 5.2. In a world of risk-neutrality, every security in domestic dollars must earn the domestic risk-free rate $r_d$. This implies,

$$dG/G=(r_d-\lambda \kappa)dt+\sigma dz+(Y-1)dn. \quad (5.29)$$

Applying Ito's lemma to $S=G\cdot \exp(r_f \tau)$ from (5.2), the foreign-currency pricing equation (5.17) reduces to

$$dS/S=(r_d-r_f-\lambda \kappa)dt+\sigma dz+(Y-1)dn. \quad (5.30)$$

(5.30) implies that (5.18) reduces to:

$$S(T)=S(t)\exp(x+\sum_{i=1}^{\pi(\tau)} y_i). \quad (5.31)$$

In (5.31), $x$ is a normally distributed random variable with

$$E(x)=(r_d-r_f-\lambda \kappa-\sigma^2/2)\tau$$

and $V(x)=\sigma^2 \tau$. 

$$(\alpha-r_f-r_d)/\sigma=(u_c-r_d)/\sigma_c. \quad (5.27)$$
We use $c^*$ to denote the call option-pricing function subject to mixed jump processes. Assuming risk-neutrality, the expected average rate of growth in the value of $c^*$ must also be $r_d$. To eliminate arbitrage opportunities, $c^*(t)$ must be the discounted value of the expected terminal call price:

$$c^*(t) = \exp(-r_d t) \mathbb{E}[\max(S(T)-X,0)]. \quad (5.32)$$

In (5.32), $S(T)$ follows equation (5.31). To solve (5.32), we first must solve $\mathbb{E}[\max(S(T)-X,0)]$. Using (5.31) and taking conditional expectations on $\max(S(T),0)$ gives:

$$\mathbb{E}[\max(S(T)-X,0)] = \mathbb{E}[\max(S(t)\exp(x+\sum_{i=1}^{\infty} y_i)-X,0)],$$

$$= \mathbb{E}[\mathbb{E}[\max(S(t)\exp(x+\sum_{i=1}^{\infty} y_i)-X,0) | n(\tau)=n]],$$

$$= \sum_{n=0}^{\infty} \frac{\exp(-\lambda \tau)(\lambda \tau)^n}{n!} \mathbb{E}[\max(S(t)\exp(y^*+\sum_{i=1}^{\infty} y_i)-X,0)],$$

$$= \sum_{n=0}^{\infty} \frac{\exp(-\lambda \tau)(\lambda \tau)^n}{n!} \mathbb{E}[\max(S(t)\exp(y^*)-X,0)]. \quad (5.33)$$

In (5.33), $y^*=x+\sum_{i=1}^{\infty} y_i$ is a random variable describing the growth of the currency price. Let us define $x_{n}=\sum_{i=1}^{n} y_i$ so that $y^*=x+x_n$. Because normal variables are closed under addition, and since $y_i$ is a normal variable, $x_n$ must also be a normal variable and $\mathbb{E}[\exp(x_n)]=\exp(ny)$
where \( \gamma = \log(1+\kappa) \). Because \( x \) and \( x_n \) are both normally distributed, \( y^* \) is also normally distributed with mean and variance as follows\(^4\):

\[
E(y^*) = (r_d - r_f - \lambda \kappa - (\sigma^2/2) + \nu n) \tau = r_n \tau,
\]

(5.34)

where \( r_n = r_d - r_f - \lambda \kappa + n \gamma / \tau \).

\[
V(y^*) = (\sigma^2 + \nu n \delta^2 / \tau) \tau = V_n \tau,
\]

(5.35)

where \( V_n^2 = \sigma^2 + n \delta^2 / \tau \). Thus, \( \exp(y^*) \) is a lognormal variable with:

\[
E[\exp(y^*)] = E[\exp(x) \exp(x_n)] = E[\exp(x)] E[\exp(x_n)],
\]

\[
= \exp[ (r_d - r_f - \lambda \kappa) \tau ] \exp(\nu n) = \exp[ (r_d - r_f - \lambda \kappa + n \gamma / \tau) \tau ],
\]

\[
= \exp(r_n \tau).
\]

(5.36)

In the world of risk-neutrality when exactly \( n \) Poisson jumps occur during the period, equation (5.35) and (5.36) shows that \( V_n^2 \) can be interpreted as the average variance per unit time and \( \gamma_n \) the average drift per unit time for foreign-currency prices.

---

\(^4\) To see how we obtain \( E[\exp(x_n)] = \exp(n \gamma) \), we note that

\[
V(x_n) = \sum_{i=1}^{n} V(y_i) = n \delta^2.
\]

\[
E(\exp(x_n)) = \prod_{i=1}^{n} E(\exp(y_i)) = (\exp(u + \delta^2/2))^n, \quad \text{since} \quad E(y_i) = \exp(u + \delta^2/2) = 1 + \kappa.
\]

\[
= e^n (u + \delta^2/2) = e^n \gamma, \quad \text{since} \quad \gamma = \log(1 + \kappa) = u + \delta^2/2.
\]

To see how we obtain (5.34), we note that ignoring the jump part, equation (5.31) implies \( x = \ln(S) \). Applying Itô's Lemma to \( x = \ln(S) \) and using the fact that \( dS \) follows (5.30), we obtain

\[
dx = [r_d - r_f - \lambda \kappa - (\sigma^2/2)] dt + \sigma dz.
\]
Once the number of jump events is fixed, we have just shown that 
$S(T)$ follows a lognormal process. Applying the lognormal theorem 
(4.10), with the drift term $\omega = r_d - r_f - \lambda \nu / \tau = r_n$, variance term 
$\nu_n^2 = \sigma^2 + \nu^2 / \tau$ and with parameters $a = 1$, $b = 1$, we can evaluate 

$E[\text{Max}(S(t)e^{y^* - X}, 0)]$:

$$
E[\text{Max}(S(t)e^{y^* - X}, 0)]
= e^{(\rho \tau)SN[ln(S/X) + \rho \tau + (\nu^2 / 2) \tau / \sqrt{\tau}]} - X \cdot N[ln(S/X) + \rho \tau + (\nu^2 / 2) \tau / \sqrt{\tau}].
$$

(5.37)

Substituting (5.32) for (5.37) we obtain:

$$
c^*(t) = \exp(-r_d \tau)E[\text{Max}(S(T)-X, 0)],
$$

$$
= \sum_{n=0}^{\infty} \frac{\exp(-\lambda \tau)(\lambda \tau)^n}{n!} e^{(\rho \tau) \cdot e^{-r_d \tau} S \cdot N[ln(S/X) + \rho \tau + (\nu^2 / 2) \tau / \sqrt{\tau}] - e^{-r_d \tau} S \cdot N[ln(S/X) + \rho \tau + (\nu^2 / 2) \tau / \sqrt{\tau}].
$$

(5.38)

In (5.38), $r_f^* = (r_f + \lambda \nu / \tau)$, $\rho = r_d - r_f^*$, and

$$
d_1^* = [ln(S/X) + (r_d - r_f^*) \tau + (\nu_n^2 / 2) \tau / \sqrt{\tau}],
$$

$$
d_2^* = [ln(S/X) + (r_d - r_f^*) \tau - (\nu_n^2 / 2) \tau / \sqrt{\tau}].
$$

We note that except for the front parts, the mixed-jump solution (5.38) has a form similar to that of (5.16), which gives
a Black-Scholes solution for a call value with a pure diffusion process:

\[ c(S, r_d, r_f, \sigma, \tau, X) = S \cdot e^{-r_f \tau} N(d_1) - X \cdot e^{-r_d \tau} N(d_2), \]

where

\[ d_1 = \frac{\log(S/X) + (r_d - r_f)\tau + (1/2)\sigma^2\tau}{\sigma\sqrt{\tau}}, \]

\[ d_2 = \frac{\log(S/X) + (r_d - r_f)\tau - (1/2)\sigma^2\tau}{\sigma\sqrt{\tau}}. \]

Expressing \( c^* \) in terms of \( c \), we obtain:

\[ c^*(S, r_d, r_f, \sigma, \tau, X) = \sum_{n=0}^{\infty} \frac{\exp(-\lambda\tau)(\lambda\tau)^n}{n!} c(S, r_d, r_f^*, V_n^2, \tau, X). \]  

Since \( V_n^2 \) is the average variance per unit time, \( c(S, r_d, r_f^*, V_n^2, \tau, X) \) may be viewed as the value of a foreign currency option, conditional on knowing that exactly \( n \) Poisson jumps are going to occur during the life of the option. Equation (5.39) shows that the actual value of the option, \( c^* \), is just the weighted sum of each of these conditional prices where each weight equals the probability that exactly \( n \) Poisson jumps occur and the foreign risk-free rate is replaced by the risk-adjusted rate \( r_f^* + \lambda \kappa - n\gamma / \tau \).

We note that the solution (5.39) for a foreign-currency option differs from Merton (1976)'s solution (4.36) for a stock option. However, we can transform the solution (5.38) to a form similar to (4.36). We note that instead of adjusting the foreign risk-free rate to the new level \( r_f^* \) from (5.38), we can leave the foreign rate...
unchanged at $r_f$ and redefine a risk-adjusted equivalent of domestic risk-free rate to generate a new form for $c^*$. We note that the interest-rate differential $r_d^*-r_f^* = r_d^*-r_f^* - \lambda \kappa + n \gamma / \tau = \rho$ is used inside the normal distribution of equation (5.38). Defining a new domestic risk-free rate $r_d^* = r_d - \lambda \kappa + n \gamma / \tau$, then $r_d^*-r_f^* = r_d - r_f^*$. Therefore, the old definition of interest-rate differential $r_d^*-r_f^*$ can be replaced by the new definition of interest-rate differential $r_d^* - r_f^*$ in equation (5.38). This implies $d^*(r_d^*, r_f^*) = d^*(r_d, r_f)$ and $d^*(r_d, r_f^*) = d^*(r_d^*, r_f^*)$.

Using the definition for $r_d^*$, (5.38) can be rewritten as:

$$c^* = \sum_{n=0}^{\infty} \frac{\exp(-\lambda \tau) \lambda^n}{n!} \left[ e^{-r_f^* \tau} e^{-(\lambda \kappa - n \gamma / \tau) \tau} S \cdot N(d_1^*) - e^{-r_d^* \tau} X \cdot N(d_2^*) \right] ,$$

$$= \sum_{n=0}^{\infty} \frac{\exp(-\lambda \tau) \lambda^n}{n!} \left[ e^{-r_f^* \tau} \cdot S \cdot N(d_1^*) - e^{-r_d^* \tau} \cdot X \cdot N(d_2^*) \right] ,$$

$$= \sum_{n=0}^{\infty} \frac{\exp(-\lambda \tau - \lambda \kappa \tau) \lambda^n}{n!} \left[ e^{-r_f^* \tau} \cdot S \cdot N(d_1^*) - e^{-r_d^* \tau} \cdot X \cdot N(d_2^*) \right] ,$$

$$= \sum_{n=0}^{\infty} \frac{\exp(-\lambda \tau - \lambda \kappa \tau) \lambda^n}{n!} \left[ e^{-r_f^* \tau} \cdot S \cdot N(d_1^*) - e^{-r_d^* \tau} \cdot X \cdot N(d_2^*) \right] ,$$

(note: $e^{\tau} = e^{(1+\kappa) \tau} \approx 1 + \kappa$)

$$= \sum_{n=0}^{\infty} \frac{\exp(-\lambda \tau - \lambda \kappa \tau) \lambda^n}{n!} c(S, r_d^*, r_f^*, V_n, \tau, X) .$$

(5.40)
In (5.40), $\lambda' = \lambda(1 + \kappa)$.

The solution (5.40) has a form similar to Merton's solution (4.36) for stock options. It also has a similar interpretation to that of (4.36): the actual value of the foreign-currency option given a mixed jump processes is the weighted sum of the values of the options that exactly $n$ Poisson jumps occur during the life of the option, where each weight equals the probability that a Poisson random variable with characteristic parameter $\lambda' \tau$, takes on the value $n$. 
6.1 Introduction

Chapter V develops two arbitrage-free models that price call options on foreign currency. If the assumptions underlying these models hold, the values generated by these models represent fair prices that eliminate arbitrage opportunities. If the underlying assumptions do not hold, however, model values may not represent fair prices for options. In this case, model prices could deviate systematically from fair prices without creating arbitrage opportunities.

This chapter uses an ad-hoc approach to studying foreign-currency options. We assume that the relation between an option's price and the price of the underlying asset has been and remains stationary. This assumption justifies interpreting econometrically fitted functional forms to past data as descriptions of the relation between these prices and their potential determinants. Such models' principal weakness is that the value they generate is only an expected option price. These values express the past relation between selected variables of what the price of a call option might be, not the fair value, based on the behavior pattern of the underlying asset, of what the value should be. If the past relation between an option's price and the price of the underlying asset is not stationary, applying the
stationary model at a later date would develop predicted prices that differ consistently from market prices.

Fitted curves based on past options data are called empirical pricing equations. Despite its weakness, an empirical pricing equation can give useful results, particularly in estimating the expected market price of an option. Furthermore, an empirical pricing equation can be used to study the impact of potential determinants on options prices. Even the assumption that the past relation between an option's price and the price of the underlying asset is not stationary does not hold, empirical pricing equations can give insight into the differential behavior of options prices between two different periods.

To estimate an empirical pricing equation for options, we must decide two things. First, we need to specify the determinants that potentially impact on option prices. Second, we need to specify an appropriate functional form that can capture the relation between these potential determinants and option prices. As with any empirical model, economic theory offers many suggestions on how to answer the first question but offers few hints for answering the second question. Restrictions developed in Chapter IV suggest that at least the following variables must be included in an empirical pricing equation for foreign-currency options: $S(t)$, $X$, $\tau$, $r_d$, $r_f$. However, these restrictions do not offer any suggestion about the functional form into which these variables must fit. Nevertheless, the analysis in
Chapter IV does establish six basic restrictions that any good functional form should satisfy.

Six basic constraints on the price of any call option established in section 4.2 are:

\[ C(S(t), \tau; X) > 0. \] (6.1)
\[ C(0, \tau; X) = 0. \] (6.2)
\[ C(S, 0; X) = \max(0, S(T) - X). \] (6.3)
\[ C(S, \tau; X) \geq \max(0, S(t) - X). \] (6.4)
\[ C(S, \tau; X) \leq S(t). \] (6.5)
\[ C(S, \tau_2; X) \geq C(S, \tau_1; X), \text{ for } \tau_2 \geq \tau_1. \] (6.6)

To eliminate arbitrage opportunities, the price of any American call option must satisfy restrictions (6.1)–(6.6). Any empirical pricing equation which violates these restrictions must be described as conceptually deficient. In the next section, we review past literature that uses an ad-hoc approach to estimate option prices. Besides examining the explanatory variables these models might contain, we evaluate flexible functional forms for empirical pricing equations for options consistent with these ad-hoc models. Specifically, we study the mathematical properties of the proposed functional forms and explore implications these properties have in the context of option pricing. Our criteria in evaluating these functional forms are: (1) to check whether or not these functional forms satisfy restrictions (6.1)–(6.6); and (2) to check whether or not these functional forms are flexible enough to span all feasible option prices that satisfy
these six restrictions. Only one functional form emerges as the form that survives these tests. This form and two other forms are used to price foreign-currency options in section 6.3.

6.2 Ad-hoc Models Revisited

All ad-hoc models reviewed in this section focus on evaluating warrants on common stocks. A warrant is similar to a call option in the following respect: the holder of a warrant has the right to exercise the warrant on or before the expiration date at a predetermined exercise price. However, unlike call options, warrants are issued by business corporations. They involve cashflows between corporations and investors, and not between two investors as with call or put options. Also unlike options which usually expire in less than 12 months, warrants can have maturities of several years. Thus, a warrant is an investment that has some characteristics that are similar to a call option. The difference between call options and warrants are subtle and are considered unimportant for evaluating warrants by ad-hoc models. Therefore, in this section we use "warrants" and "call options" interchangeably. Also, to be consistent with these models' original presentation, we assume the underlying asset is common stocks. The symbols used in this section are those used in reviewing probability models in section 4.2.

In the Analysts Journal of 1947, Paul Hallingby makes a close study of price movements of individual warrants during several market
swings. He finds that a perpetual or long-term warrant generally sells at the same ratio to its stock whenever the stock reaches a given price level. For example, he finds that when the stock is selling at one quarter of the distance between its high and low, the option normally also sell at one quarter of the distance between the option's high and low. Thus, he proposes the following condition as one that the "correct price" of a call option \( C \) must satisfy:

\[
\frac{\text{Max}S - S(t)}{\text{Max}S + \text{Min}S} = \frac{\text{Max}C - C(t)}{\text{Max}C + \text{Min}C} \tag{6.7}
\]

In (6.7), \( \text{Max}S \) and \( \text{Min}S \) are respectively the maximum and minimum in the possible stock prices; \( \text{Max}C \) and \( \text{Min}C \) are respectively the maximum and minimum of the possible option prices.

To use formula (6.7), a big task is to find nontrivial limiting prices for stocks and options. Hallingby suggests that we can use "previous highs and lows as limiting values." This certainly is not acceptable because a call option derives its time value (the difference between the price of a call option and a call option's intrinsic value) from its potential to rise beyond the exercise price. If we knew for sure that stock prices would never rise beyond their previous highs, the time value of a call option would be very limited or completely disappear. Thus, Hallingby's use of previous high and low as limiting values is not justified. Another problem for
Hallingby's model is that his formula does not take into consideration the time to the expiration date. Thus this model can only be applied for the perpetual options or an option that has a long life. Perhaps the most serious flaw for Hallingby's model is that no economic reason is given to justify that condition (6.7) should be satisfied. Condition (6.7) is proposed mainly for an ad-hoc reason that it describes the past relation between "some" warrant's prices and stock prices. Therefore, we conclude that Hallingby's formula is conceptually deficient. Nevertheless, the assumption that relative price changes in options must match relative price changes in common stocks is used in some subsequent models.

Hallingby's observation that relative option-price changes approximately equal relative stock-price changes is rephrased by Morrison (1957) as a principle of "equal gain or loss." Morrison shifts the focus from relative ranges of variation to that expected returns on an option at the maturity must equal that anticipated on a stock. His equal-gain principle implies that:

\[
\frac{S(T)}{S(t)} = \frac{C(T)}{C(t)} = \frac{[S(T) - X]}{C(t)}. \tag{6.8}
\]

Given that condition (6.8) holds, Morrison focuses his analysis on the question: "What price would the stock have to reach at the options' expiration to give equal gain to option holder and stockholder?". Morrison's version of the correct stock price \(S^*(T)\) at the option's expiration date can be calculated from (6.8):

\[
S^*(T) = \frac{X}{[1 - C(t)/S(t)]}. \tag{6.9}
\]
Therefore, if the stock price on the option's expiration date is expected to exceed \( S^*(T) \), then options are a better purchase than the stocks. Morrison avoids the Hallingby's trouble of estimating the limiting values for stocks and options, but implicitly assumes an equality in riskiness. He retains Hallingby's idea of "equal gain" for stock and option prices. He also retains the conceptual deficiencies of Hallingby's model.

Hallingby's and Morrison's models only state conditions that the correct price of an option might satisfy. They do not offer an explicit empirical pricing equation for call options. The first explicit empirical pricing equation is offered by Guynemer Guguere in the Analysts Journal of November 1958. Giguere's formula derives from the observation that the past relation between a perpetual warrant's price and its underlying stock price in a log-log paper is a fixed constant. Giguere selects two corporations that issue perpetual warrants. He draws charts depicting the relation between the stock prices and warrant prices for these two corporations. In both charts, he finds that the perpetual warrant's empirical pricing function can be described as a parabola:

42. The two corporations Giguere selects are Tri-Continental Corporation and Atlas Corporation. Giguere (1958, p.17) states the following reasons why he choose the warrants issued by these corporations: 'these securities are well seasoned market-wise and have a relatively long history of wide price fluctuations, permitting wide-range charts to be drawn. Moreover, these warrants are perpetual and their "exercising price" is constant.' Thus these warrants represent 'the highest quality securities, where the number of variables (that impact warrant prices) is at a minimum.'
\[
C(t) = \begin{cases} 
S^2(t)/4X, & \text{for } S < 2X. \\
S(t) - X, & \text{for } S \geq 2X.
\end{cases}
\] (6.10)

Equation (6.10) implies that the time value for a perpetual warrant disappears completely when the price of stock reaches twice the exercise price. When the price of stock is less than twice the exercise price, the value of a perpetual warrant equals the price of the stock squared, divided by four times the exercise price. The curve which this formula describes is illustrated in Figure 4. The curve is a parabola with a value of zero when the stock price is zero. It gives very small values for C when S is near X, increasing gradually to meet the intrinsic value curve C=S-X.

Because formula (6.10) is originally developed for evaluating a perpetual warrant, the time to the expiration date is considered unimportant in determining the warrant's price. The relevant restrictions to check for a perpetual warrant are restrictions (6.2), (6.4) and (6.5). Formula (6.10) trivially satisfies restriction (6.2) because \( C(S=0) = 0/4X = 0 \). We find that Giguere's formula also satisfies the other two restrictions.

Property 6.1. Formula (6.10) satisfies restriction (6.4).

Proof:
(a) For \( S(t) \geq 2X \), formula (6.10) implies \( C(t) = S(t) - X \geq \text{Max}(0, S(t) - X) \).

(b) For \( S(t) < 2X \), formula (6.10) implies \( C(t) = S(t)^2/4X \). To prove \( C(t) > \text{Max}(0, S - X) \), we need to show that \( C-(S-X) > 0 \):

\[
C-(S-X) = [S^2/4X] - S + X = [S^2 - 4SX + 4X^2]/4X = (S-2X)^2/4X > 0.
\]
The last step is justified because \((S-2X)^2\) is positive for \(S<2X\).

From (a) and (b) we conclude that formula (6.10) \(\geq\) Max(0, S-X). Q.E.D.

**Property 6.2.** Formula (6.10) satisfies restriction (6.5).

**Proof:**
(a) For \(S \geq 2X\), \(C(t) = S(t) - X \leq S(t)\).

(b) For \(S < 2X\), \(C(t) = \frac{S}{2}/4X\).

Thus, if \(C(t) \cdot \frac{2X}{S} > C(t)\), since \(2X/S > 1\) for \(S < 2X\).

Thus, if \(C(t) \cdot \frac{2X}{S} < S\), then \(C(t) < S(t)\). This can be established because

\[
C(t) \cdot \frac{2X}{S} = \frac{S}{4X} \cdot \frac{2X}{S} = S/2 < S.
\]

From (a) and (b) we conclude \(C(t) \leq S(t)\). Q.E.D.

Another way to analyze formula (6.10) is to plot the function \(C(t) = \frac{S^2}{4X} \) on a log-log paper. Taking the natural logs of both sides of equation (6.10), we obtain:

\[
\ln[C(t)] = -\ln(4X) + 2 \ln[S(t)].
\]

Equation (6.11) implies that in a log-log paper, the relation of option prices and stock prices is linear with a slope of 2. The slope of a log-log equation may be viewed as an elasticity. For options, this slope states the percentage change in option prices associated with a one-percentage-point change in the stock price. Thus, Giguere's formula implies that the elasticity of the option price is always 2. This is a very strong assertion, which can be justified or refuted by empirical tests.
Everything else being equal, a perpetual warrant must command a greater time value than a warrant that has a finite life. Because equation (6.10) is designed to apply to a perpetual warrant, it establishes an upper limit on the price of a warrant that has a finite life. For a warrant with a life of one to five years, Giguere suggests modifying formula (6.11) as the lower limit:

\[ C(t) = \left[ \frac{S^2(t)}{4X} \right] - \frac{X}{16}, \quad \text{for } S < 2X. \quad (6.12) \]

Unfortunately, Giguere fails to disclose the reason why he establishes the lower limit for a warrant price at (6.12). Giguere (1958,p.25) simply states that equation (6.12) 'can be tested and shown to be very conservative.'

Giguere plots the values of six corporations' warrants and stock prices at different times in their market history. He finds that the relation of warrant price and stock price fit his formula closely. Of course, Giguere's conclusion is questionable. Because the samples he uses are rather arbitrary, we must question how representative Giguere's samples are. It is not surprising that Samuelson (1965) reports that Giguere's formula postulates warrant prices that generally prove too low. Shelton (1967) also finds that Giguere's formula leads to values that are lower than most of the quotations actually observed during 1959-1962. Some deviation for market warrant prices from Giguere's model warrant prices is inevitable. Besides suffering a sampling problem, Giguere's model ignores a number of factors which potentially could affect warrants' values, including
among others, the time to maturity, dividend yields, volatility of the
common stock.

Shelton (1967) presents an ad-hoc model which, in its most
general form, can value call options with any period of life.
Giguere's formula is difficult to adjust for variations in an option's
life, dividend yields, and other parameters. In contrast, the Shelton
model can be modified to handle virtually every kind of option
contracts. Like Giguere, Shelton sets the value of a long-term warrant
at its intrinsic value when the price of the stock exceeds a "critical
level." Largely from examining the relation between warrant prices and
stock prices, Shelton sets the critical level of stock price at four
times the exercise price instead of two times the exercise as Giguere
suggests.

As Hallingby and Giguere do, Shelton argues that minimum and
maximum theoretical values exist for any option. Unlike Giguere, who
uses equation (6.12) to be a lower limit, Shelton argues that the
minimum value of a call option with any period of life remaining is
its intrinsic value. Shelton's lower limit is more reasonable because
it agrees with restriction (6.4). When the stock price is less than
the critical value, Shelton sets the maximum value of the warrant at
\((3/4)S(t)\), 75 percent of the stock price. Because \((3/4)S < S\), this upper
limit satisfies restriction (6.5). Shelton's upper limit for the
warrant price is established using the principle of "equal percentage
gain or loss" to the holder of either the warrant or the stock. This
idea apparently borrows from Hallingby (1947) and Morrison (1957). The
principle of equal percentage gain or loss to the warrant and stock implies:

\[ \frac{C(t_2)}{C(t_1)} = \frac{S(t_2)}{S(t_1)} \]  
for \( t_1 < t_2 \).

\[ C(t_1) = \frac{[C(t_2)S(t_1)]}{S(t_2)} \]  
(6.13)

How can equation (6.13) establish an upper bound \((3/4)S\) for the price of a call option? Shelton fails to provide a proof. To complete his analysis, we supply such a proof right now.

Property 6.3.

The principle of equal percentage gain or loss to the warrant and stock sets an upper bound for the warrant’s price at \((3/4)S\).

Proof:

Because equation (6.13) is valid for any time \( t_2 \), we can set \( t_2 = T \), the expiration date for the warrant. Thus, \( C(t_2) = S(t_2) - X \).

\[ C(t_1) = \frac{[C(t_2)S(t_1)]}{S(t_2)} \], \text{ by (6.13)}

\[ = \frac{[S(t_2) - X]S(t_1)}{S(t_2)} \], \text{ by using the fact } C(t_2) = S(t_2) - X.

\[ = \frac{[S(t_2) - X]}{S(t_2)} S(t_1). \]

But, \( \frac{[S(t_2) - X]}{S(t_2)} = 1 - \frac{X}{S(t_2)} < 1 - \frac{X}{[S(t_2)/4X]} = 1 - 1/4 = 3/4. \)

(note: \( S(t_2)/4X < 1 \) for \( S(t_2) < 4X \))

Thus, \( C(t_1) = \frac{[S(t_2) - X]}{S(t_2)} S(t_1) < \frac{3}{4} S(t_1) \). Q.E.D.

We find that Shelton’s upper bound \((3/4)S\) always exceeds his lower bound \((S - X)\). This property, established below as property 6.4, is important if Shelton’s formula is to be useful.
Property 6.4.

\( \frac{3}{4}S > (S-X) \) for \( S < 4X \).

Proof:

\( \frac{3}{4}S - (S-X) = X - \left( \frac{S}{4} \right) \).

Because \( S < 4X \Rightarrow \frac{S}{4} < X \), thus \( \frac{3}{4}S - (S-X) > 0 \Rightarrow \frac{3}{4}S > (S-X) \). Q.E.D.

Once the upper bound and lower bound for a call option are specified, many factors could push a call price up near the top or down toward the lower limit of the plausible-price zone. Shelton uses "stepwise multiple regression" to identify the set of potential factors. In the first step he converts observed warrant prices during the 1959–62 period into a percentage according to the following formula:

\[
\text{Percent}(t) = \left( \frac{C(t) - (S(t) - X)}{\left( \frac{3}{4}S(t) - (S(t) - X) \right)} \right) \times 100\%.
\]  

(6.14)

In (6.14), "Percent(t)" is the percentage location of warrant price \( C(t) \) in the plausible-price zone. If the warrant price is at the lower edge of the zone \( (S-X) \), Percent(t) is 0%. If the warrant is priced at the upper limit of the range of values \( \frac{3}{4}S \), Percent(t) is 100%.

The next step is to see what factors influence the time value for a specific warrant. Shelton uses multiple regression to check if Percent(t) could be "explained" by the following six independent variables: \( \tau \), the time to expiration (measured in months); \( \text{Yield} \), the ratio of the foregone dividend yield of the associated stock to the current price of stock; \( \text{Listed} \), a dummy variable indicating whether the warrant is listed on the American Stock Exchange or traded over-the-counter; \( B \), another dummy variable denoting whether the warrant
sold for more or less than $5.00 (because at that time, if the price
of warrant is below $5.00, most member firms will not extend margin
credit); v, the past stock's volatility (measured by averaging the
ratios of the annual high divided by its annual low for each of the
three preceding years); and Trend, the recent trend of the stock price
(measured by the percentage change of the stock price over the past
year). His multiple regression can be written as:

\[
\text{Percent}(t) = \alpha + \beta_1 \tau + \beta_2 \text{Yield} + \beta_3 \text{Listed} + \beta_4 B + \beta_5 v + \beta_6 \text{Trend} + \epsilon_t. \quad (6.15)
\]

In (6.15), \( \epsilon_t \) is a stochastic variable assumed to have a zero mean.
Shelton uses "F" tests to check which (if any) of the six independent
variables in equation (6.15) proves significant at 1 percent
significant level. He finds that only the following three variables
are statistically significant (in order of importance): Yield; Listed
(whether the warrant was listed) and \( \tau \). His final equation to explain
(or estimate) the relative-price percentage of a long-term warrant is
given by the following expression:

\[
\text{Percent}(t) = (\tau/72)^{1/4} \left( .47 - 4.25 \text{ Yield} + .17 \text{ Listed} \right). \quad (6.16)
\]

In (6.16), \( \text{Percent}(t) \) is the estimated relative-price percentage of a
warrant. Once \( \text{Percent}(t) \) is calculated, the estimated warrant price

---

43. The sample size that produces this equation is 99. Shelton
(1967, p. 89) fails to explain how the samples are drawn but does
states that 'prices were obtained for all warrants (and their related
stocks) quoted in Standard and Poor's Stock Guide at five points in
time; the last trading day of the year in 1959, 1961, and 1962;
November 30, 1960; and August 31, 1962.'
can be computed using formula (6.14). Shelton's model is illustrated in Figure 5.

Unlike Giguere's formula, Shelton's formula adjusts explicitly for the effect of the dividend rate on the common stock and other potential factors. A major objection to Shelton's formula is that the formula makes no adjustment for the volatility of the stock. However, volatility is absent from Shelton's formulation not because he ignores it, but because he concludes by multiple regression analysis that volatility is not significant in explaining the price of long-term warrants. The volatility measure he uses is based on the average of annual high-low ratios. Of course, this measure is too simple to accommodate all forms of variation in the underlying distribution. Thus, his conclusion about the impact of volatility on the warrant may be premature. A better volatility measure to test would be sample variance. Three subsequent models, Kassouf (1969), Van Horne (1969) and Parkinson (1972), employ sample variance as volatility measure and find volatility to be significant in explaining the price of warrants.

Shelton uses equation (6.16) to do an out-of-sample test for 20 widely traded warrants as of November 18, 1963, July 1, 1966, and May 4, 1967--dates subsequent to the period during which equation (6.16) is based. He finds that the predicted prices calculated by the equation (6.14) and (6.16) are close enough to the actual prices to support the belief that the relations of option prices and their determinants in the 1959-62 period appear to be stable over time. However, because the way Shelton draws samples to test his formula is
rather arbitrary, we must interpret his conclusion with reserve. In an unpublished test, Gastineau (1979; p. 232) claims that Shelton's model gives values for options that appear to be too high and "casual observation of listed option trading suggests that any premium above intrinsic value disappears long before the stock price reaches four times the exercise price."

Besides omitting a volatility factor, Shelton's equation (6.16) could violate (6.4). For example, we may consider a nonlisted warrant (thus Listed=0) whose stock has Yield=.12 (i.e., annual dividend equals 12% of the stock price, not an unreasonable assumption in recent years). The estimated percent(t) by equation (6.16) becomes negative. A negative percent(t) in (6.14) implies that an estimated warrant price is below its minimum intrinsic value. This result, besides being self-contradictory in Shelton's setting, also violates restriction (6.4). This kind of self-contradictory result seems inevitable for a regression equation (6.15). The dependent variable in regression (6.15) is assumed to be within 0% and 100%. However, because no a priori constraints are imposed on the equation's coefficients, the estimated relative-price percentage may bust its assumed bounds, i.e., be less than 0% or greater than 100%. Of course, we can use ex post adjustments to restrict the range of Percent(t) in (6.15) within the upper bound and lower bound. For example, in section 6.3 we modify Shelton's regression equation (6.15) with a loglinear regression to price foreign-currency options. This adaptation makes
the estimated percent location remain positive, and avoids Shelton's problem. But there is a more elegant way to avoid violating the general restrictions.

We define \( \text{c} = C/X \) as the price of a standardized call option; \( s = S/X \), as the standardized stock price. Given this new notation, restrictions (6.1)-(6.6) reduce to:

\[
\begin{align*}
\text{c}(s(t), \tau; 1) &> 0. \quad (6.1)' \\
\text{c}(0, \tau; 1) &= 0. \quad (6.2)' \\
\text{c}(s, \tau; 1) &= \text{Max}(0, s(t) - 1). \quad (6.3)' \\
\text{c}(s, \tau; 1) &\geq \text{Max}(0, s(t) - 1). \quad (6.4)' \\
\text{c}(s, \tau; 1) &\leq s(t). \quad (6.5)' \\
\text{c}(s, \tau_2; 1) &> \text{c}(s, \tau_1; 1), \text{ for } \tau_2 > \tau_1. \quad (6.6)' 
\end{align*}
\]

Kassouf (1965, 1968, 1969) proposes a family of functions:

\[
\text{c}(z, s) = (s^z + 1)^{1/z} - 1, \text{ for } 1 \leq z \leq \infty. \quad (6.17)
\]

In (6.17), \( z \) is a parameter designed to allow the empirical call pricing function \( \text{c}(z) \) to differ in curvature according to the potential determinants of option prices. Kassouf points out that given suitable \( z \) value, \( \text{c}(z) \) in equation (6.17) can reduce to \( \text{c}(z) = 0 \) or \( \text{c}(z) = s - 1 \), the lower bound of option prices indicated by (6.4)' . Given another suitable \( z \) value, \( \text{c}(z) \) can reduce to \( \text{c}(z) = s \), the upper bound of option prices suggested in (6.5)' . Thus the functional form in (6.17) can be used to construct an empirical pricing function for a call option. However, a functional form \( \text{c}(z) \) which can reduce to the theoretical lower bound and upper bound of an option's price might
also violate restrictions (6.1)-(6.6) for some suitable z. To claim
that the functional form specified in (6.17) is free of this
conceptual deficiency, we must prove that it satisfies restrictions
(6.1)'-(6.6)'. Unfortunately, Kassouf neglects to provide any
mathematical proof of this property. We provide the missing proofs
below. Because equation (6.17) is used later in this chapter to price
foreign-currency options, we also explore other mathematical
properties of this function.

Property 6.5.
The standardized call price c in equation (6.17) is a strictly
decreasing function with respect to z.

Proof:
To prove c(z) is monotonically decreasing w.r.t. z, we need to show
c'(z)<0 for 1≤z≤∞. To prove this property, we define a new variable.
We begin by recalling that:

\[ c(z) = (s^z + 1)^{1/z} - 1, \]

which implies that c+1=(s^z+1)^{1/z}.

We define a new variable \( Y = \ln(c+1) \).

Clearly, \( Y = \ln(c+1) = \ln (s^z+1)^{1/z} = (1/z) \ln (s^z+1) \).

Differentiating this last expression by the product rule, we obtain:

\[ Y'(z) = -\frac{1}{nz^2} \ln (s^z+1) + (1/z)(s^z \ln s/(s^z+1)). \]

Next, we note that \( c = e^Y - 1 \) since \( y = \ln (c+1) \),

\[ c'(z) = e^Y \cdot Y'(z), \]
\[ Y'(z) = (c+1) \cdot Y(z), \text{ (since } e^Y = c+1) \]
\[ = (s^2+1)^{1/z} \cdot \frac{1}{z} \cdot \ln \left( s^2+1 \right) + \left( \frac{1}{z} \right) \ln \left( s^2+1 \right) \left( \frac{s}{s^2+1} \right), \text{ by (6.17)} \]
\[ = (s^2+1)^{1/z} \cdot \left( \frac{1}{z} \left[ \ln \left( s^2+1 \right) + \ln s^2+1 \right] \right) \]
\[ = (s^2+1)^{1/z} \cdot \left( \ln s^2+1 \right) \left( \frac{1}{z} \right) \]
\[ = \left( s^2+1 \right) \left( \frac{1}{z} \right) \cdot \ln \left( s^2+1 \right) \left( \frac{s}{s^2+1} \right) \]

Note, since \( s > 0 \) and \( z > 1 \), \( (s^2+1)^{1/z} > 0 \) and \( (1/z) > 0 \), thus the sign of \( c'(z) \) depends only on the bracketed term:
\[ h = \frac{\ln s^2+1}{1+(1/s^2)} - \frac{\ln(s^2+1)}{z} \]

We evaluate \( h \) separately over four branches:

(a) for \( s = 0 \), \( \lim_{s \to 0} \ln s = -\infty \), \( \ln(s^2+1) = 0 \). This means that the first term in \( h \) is negative and the second term is zero, so that \( h \) must be negative, i.e., \( c'(z) < 0 \) for \( s = 0 \).

(b) for \( s < 1 \) and \( s > 0 \), \( \ln s < 0 \), \( \ln(s^2+1) > 0 \). This means that the first term in \( h \) is negative and the second term is positive, so that \( h \) must be negative, i.e., \( c'(z) < 0 \) for \( s < 1 \) and \( s > 0 \).

(c) for \( s = 1 \), \( \ln s = 0 \), \( \ln(s^2+1) > 0 \). This means that the first term in \( h \) is zero and the second term is positive, so that \( h \) must be negative, i.e., \( c'(z) < 0 \) for \( s = 1 \).

(d) for \( s > 1 \),
\[ h = \left( \frac{\ln s}{1+(1/s^2)} - \frac{\ln(s^2+1)}{z} \right) \ln s - \frac{\ln(s^2+1)}{z}, \text{ [since } 1+(1/s^2) > 1 \]
Thus, $h < 0$ for $s > 1$, and $c'(z) < 0$ for $s > 1$.

From (a), (b), (c) and (d) we conclude that $c'(z) < 0$.

**Property 6.6.**

The standardized call price $c$ in equation (6.17) satisfies constraints (6.1)', (6.2)', (6.4) and (6.5)' for $1 \leq z \leq \infty$.

**Proof:**

(a) When $s = 0$, $c(z) = 1 - 1 = 0$ so that restriction (6.2)' is satisfied.

(b) By property 6.5, $c$ is monotonically decreasing w.r.t. $z$. To prove that $c$ satisfies (6.5)'), we need to prove the lowest higher bound of $c(z)$

$$\sup_{\mathbb{Z}}(c(z)) = s;$$

$$\sup_{\mathbb{Z}}(c(z,s)) = c(z=1,s), \text{ (by property 6.5)}$$

$$= (s+1)-1 = s.$$ Thus restriction (6.5)' is satisfied.

(c) To prove that $c$ satisfies (6.4)', we need to prove that the greatest lower bound of $c(z)$, $\inf[c(z)]$ equals $\max(0,s-1)$:

$$\inf_{\mathbb{Z}}(c(z,s)) = c(z=\infty,s) = \lim_{s \to \infty} (s^z+1)^{1/z} - 1, \text{ (by property 6.5).}$$

We proceed by evaluating this function over two branches:

1. For $s \leq 1$,

$$c(z=\infty) = \lim_{s \to \infty} (s^z+1)^{1/z} - 1 = 0 = \max(0,s-1), \text{ since } s-1 \leq 0 \text{ for } s \leq 1.$$

2. For $s > 1$,

$$c(z=\infty) = \lim_{s \to \infty} (s^z+1)^{1/z} - 1 = \lim_{z \to \infty} y(z)-1, \text{ by (6.18).}$$
But, $\lim_{z \to 1} \ln y(z) = \lim_{z \to 1} \ln (s^z + 1)^{1/z}
= \lim_{z \to 1} \ln(s^z + 1), \quad \text{by (L'Hopital's rule)}
= \lim_{z \to 1} \frac{s^z \ln s}{s^{z+1}}.
= \lim_{z \to 1} \frac{\ln s}{1 + (1/s^z)} = \ln s.

Thus, $\lim_{z \to 1} y(z) = \lim_{z \to 1} \exp[\ln (y(z))] = \exp(\lim_{z \to 1} \ln y(z)) = \exp(\ln s) = s,$
therefore,

c(z=1) = \lim_{z \to 1} y(z) - 1 = s - 1 = \text{Max}(0, s - 1) \quad \text{for } s > 1 \quad \text{since } s - 1 > 0.

Taken together, (1) and (2) prove that $\inf[c(z)] = \text{Max}(0, s - 1).$ Thus, restriction (6.4)' is satisfied.

(d) From (c) we establish that $\inf[c(z)] = 0$ for $s \leq 1$ and $\inf[c(z)] > 0$ for $z > 1.$ Thus, restriction (6.1)' (which holds that $c \geq 0$) is satisfied.
Q.E.D.

Thus, property 6.5 and 6.6 imply that $c(z=1) = s,$ and $\lim_{z \to 1} c(z) = 0$
when $s \leq 1,$ $\lim_{z \to 1} c(z) = s - 1$ otherwise. That is, $c(z=1)$ is the upper
boundary of the rhomboidal region to which the model is restricted and
$c(z=\infty)$ is the lower boundary (see Figure 6). Every point in the region
lies on a unique $z$ curve. In other words, $c(z,s)$ can span the whole
feasible area for options prices.

Property 6.7.
The standardized call price $c$ in equation (6.17) is a strictly convex
function with respect to $z,$ i.e., $c''(z) > 0.$
Proof:

Property 6.5 establishes \( c'(z) = (s^z+1)^{1/z} \cdot \left[ \frac{1}{z} \cdot \frac{\ln s}{1+(1/s^z)} \cdot \frac{\ln(s^z+1)}{z} \right] \)

Differentiating this last expression by the product rule, we obtain:

\[
c''(z) = (s^z+1)^{1/z} \cdot \left[ \frac{1}{z} \cdot \frac{\ln s}{1+(1/s^z)} \cdot \frac{\ln(s^z+1)}{z} \right] + \left( \frac{1}{z} \cdot \frac{\ln s}{1+(1/s^z)} \cdot \frac{\ln(s^z+1)}{z} \right) \]

\[
\frac{\partial}{\partial z} \left[ (s^z+1)^{1/z} \right] = A \cdot B + C \cdot D,
\]

where \( A = (s^z+1)^{1/z} \),

\[
B = \frac{1}{z} \cdot \frac{\ln s}{1+(1/s^z)} \cdot \frac{\ln(s^z+1)}{z},
\]

\[
C = \frac{1}{z} \cdot \frac{\ln s}{1+(1/s^z)} \cdot \frac{\ln(s^z+1)}{z},
\]

\[
D = \frac{\partial}{\partial z} \left[ (s^z+1)^{1/z} \right].
\]

We first evaluate the last term \( D \)

\[
\frac{\partial}{\partial z} \left[ (s^z+1)^{1/z} \right] = (s^z+1)^{1/z} \cdot Y'(z) < 0.
\]

To understand how we arrive this conclusion, we recall that in the proof of Property 6.5, we have shown that \( c'(z) = (c+1) \cdot Y'(z) < 0 \). This implies \( Y'(z) < 0 \). Because \( Y'(z) < 0 \), together with the fact that \( (s^z+1)^{1/z} > 0 \), lead us to establish that \( D < 0 \). Also in the proof of Property 6.5, we have shown that \( C = \left[ \frac{1}{z} \cdot \frac{\ln s}{1+(1/s^z)} \cdot \frac{\ln(s^z+1)}{z} \right] < 0 \). Thus, three terms out of four terms in \( c''(z) \) have the known signs, i.e.,
A>0, C<0, and D<0. Thus, to prove \( c''(z) > 0 \), it is sufficient to show that \( B > 0 \):

\[
B = \frac{\ln s}{1+(1/s^z)} \ln(s^z+1) \left( \ln(sz+1) \right)
\]

\[
= (-1/z^2) \left( \ln s \right) \ln(sz+1) + (1/z^3) \left( \ln s \right) \ln(s^z+1).
\]

Because in the proof of Property 6.5, we have shown that

\[
\left[ \ln s \left( \frac{\ln(s^z+1)}{z} \right) \right] < 0,
\]

Together with the fact that \(-1/z^2 < 0\), we have shown that the first term in \( B \) is positive. Because \((1/z) > 0\), the sign of \( B \) reduces to the last term

\[
\frac{\partial}{\partial z} \left( \ln s \left( \frac{\ln(s^z+1)}{z} \right) \right) = \frac{\partial}{\partial z} \left( \ln s \right) \frac{\partial}{\partial z} \ln(s^z+1) = E - F,
\]

where, \( E = \frac{\partial}{\partial z} \left( \ln s \left( \frac{\ln(s^z+1)}{z} \right) \right) \), and \( F = \frac{\partial}{\partial z} \left( \ln(s^z+1) \right) \).

If \((E-F)\) is positive, then \( B \) is positive. We find that the term \( E \) is positive because

\[
E = \frac{\partial}{\partial z} \left( \ln s \left( \frac{\ln(s^z+1)}{z} \right) \right) = (\ln s)^2 \cdot s^{-z} > 0.
\]

Thus, if the term \( F \) is negative, we can prove the last term in \( B \) is positive and \( B > 0 \).
\[ F = \frac{\ln(s^2+1)}{z} \left[ \frac{z^2}{(s^2+1)} \right] \cdot \left[ z \cdot \ln(s) \right] - \ln(s^2+1), \]

Because \(0 < \frac{z^2}{(s^2+1)} < 1\), \(z \cdot \ln(s) < \ln(s^2+1)\) implies

\[ \frac{z^2}{(s^2+1)} \cdot [z \cdot \ln(s)] < \ln(s^2+1) \text{ and } F < 0. \]

Thus, to prove \(F < 0\), it is sufficient to prove \(z \cdot \ln(s) < \ln(s^2+1)\). But to prove this last inequality, it is sufficient to prove the inequality of the monotonical transforms on both sides hold, e.g.,

\[ \exp[z \cdot \ln(s)] < \exp[\ln(s^2+1)]. \]

This can be established because

\[ \exp[z \cdot \ln(s)] - \exp[\ln(s^2+1)] = s^2 - (s^2+1) = -1 < 0. \]

Thus, \(\exp[z \cdot \ln(s)] < \exp[\ln(s^2+1)]\). This implies \(z \cdot \ln(s) < \ln(s^2+1)\) and \(F < 0\). Because \(F < 0\), we have proved the last term in \(B\) is positive and \(B > 0\).

Finally, because \(c''(z) = A \cdot B + C \cdot D\) and we have shown \(A > 0\), \(B > 0\), \(C < 0\), \(D < 0\), we conclude that \(c''(z) > 0\).

**Property 6.8.**

(a) \(c(s, z)\) in equation (6.17) is continuous with respect to \(s\).

(b) \(c(s, z)\) is a strictly increasing function with respect to \(s\).

(c) For \(s \neq 0\) and \(z \neq 1\), \(c(s, z)\) is strictly convex with respect to \(s\). For all the feasible value of \(s\) and \(z\), \(c(s, z)\) is weakly convex in \(z\).

**Proof:**

(a) Given \(z\), \(c(s)\) is a polynomial function. Because any polynomial function is a continuous function, \(c(s, z)\) must be a continuous function.
(b) \[ c'(s) = \frac{1}{2} [ (s^z + 1) \left( \frac{1}{z} \right) (s + 1/z) s^{-1} (z - 1) - 1 ] (y/z)s(z_1) = [c(s)+1] \left( \frac{1}{s+z+1} \right) s^{1-z} \]

= \[ [c(s)+1] \left( \frac{1}{s+z+1} \right) s^{1-z} > 0, \quad \text{for } s > 0. \]

Because \( s > 0 \), \( c'(s) = 0 \), and \( c'(s) > 0 \) for \( s > 0 \), we conclude that \( c'(s) > 0 \).

(c) Because \( c'(s) > 0 \), to prove \( c \) is strictly convex in \( z \), we need to prove \( c''(s) > 0 \).

\[ c''(s) = c'(s) \left( \frac{1}{s+z+1} \right) + [c(s)+1] \left( -\frac{1}{s+z+1} \right) \]

= \[ [c(s)+1] \left( \frac{1}{s+z+1} \right) + [c(s)+1] \left( -\frac{1}{s+z+1} \right) \]

= \[ [c(s)+1] \left( \frac{1}{s+z+1} \right) \left( z-1 \right) \]

Because \( [c(s)+1] > 0 \), the sign of \( c''(s) \) depends on \( \left( \frac{1}{s+z+1} \right) \left( z-1 \right) \).

This term is positive if \( s \neq 0 \) and \( z \neq 1 \). This term is nonnegative if \( s = 0 \) or \( z = 1 \). Thus \( c''(s) > 0 \) for \( s \neq 0 \) and \( z \neq 1 \) and \( c''(s) \geq 0 \) for all the feasible value of \( s \) and \( z \). Q.E.D.

Property 6.9.

\( E[c(s)] \geq c(u_s) \), where \( E[c(s)] \) is the expected value of a call option, and \( u_s = E(s) \) is the standardized mean stock price.

Proof:
Taylor's Theorem asserts that any function $f(x)$ whose $(n+1)$th derivative $f^{(n+1)}(x)$ exists can be expressed as

$$f(x) = f(0) + f'(0)x + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \frac{f^{(n+1)}(t)}{(n+1)!}x^{n+1}.$$

By Rolle's Theorem, $t$ is a number strictly between 0 and $x$.

Expanding the Taylor series for $c(s)$ around $u_s$, and setting $n=1$, we obtain:

$$c(s) = c(u_s) + c'(u_s)(s-u_s) + c''(k)(s-u_s)^2/2, \quad u_s < k < s.$$

We note that the above expansion is exact because the second Taylor remainder of $c(s)$ is $c''(k)(s-u_s)^2/2$, where $k$ is a number strictly between $u_s$ and $s$. Because $(s-u_s)^2 \geq 0$ and by property 6.8, we know that $c''(s) \geq 0$, we establish the following inequality:

$$c(s) \geq c(u_s) + c'(u_s)(s-u_s).$$

Taking expectation on this last inequality, we obtain:

$$E[c(s)] \geq E[c(u_s)] + c'(u_s)(u_s-u_s) = c(u_s). \text{ Q.E.D.}$$

We note that unless $s=0$ or $z=1$ (i.e., $c=s$), the inequality in the property 6.9 is exact. This means, in normal cases where $s \neq 0$ and $c \neq s$, (6.17) implies that the expected returns on holding an option today exceeds those of holding a common stock today and convert it to an option tomorrow. We note that Black-Scholes call option-pricing
formula (4.28) also shares this property. In other words, a call pricing function that possesses property 6.9 asserts that the expected return from holding a call option always exceeds that of holding an underlying asset. If the classical capital asset pricing theory holds, this means that an option always possess more undiversifiable risk than its underlying asset. This property is consistent with the observation in options markets that options can allow more leverage than their underlying assets.

Property (6.5) points out that c(z) in equation (6.17) is a monotonically decreasing function in z. Thus, given observation s(t), c(z)=(s^2+1)^{1/z}-1 is a strictly decreasing function in z. Because a strictly decreasing function c(z) is a one-to-one mapping of c(z) and z, given observations on c(t), we can obtain a unique z. Because the value of z cannot be expressed explicitly in terms of c(t) and s(t), to find the value of z given c and s, we must use some indirect approximation method. We note that c(z) is a strictly decreasing function in z, a natural way to estimate z (given c and s) is to compute the value of c(s,z) by iterating z from z=1, to ∞. The best estimate for z (given c and s) is the value of z that produce a value of c(z,s) that is most close to the observed c. Besides being inefficient, this type of iteration would encounter problems in

44. To prove that Black-Scholes formula (4.28) also possess property 6.8, we need to show that the second derivative of (4.28) w.r.t. S is positive:

\[ c'(S)=N(p_1)>0; \quad c''(S)=1/(S\sigma\sqrt{T})N'(p_1)>0. \]
practice because as \( z \) exceeds 143, \( (s^z+1)^{1/z} \) would generate an underflow (the value becomes less than a computer can represent) in most computers (unless one has an access to a supercomputer). To approximate \( z \), we need to use some other indirect method. Kassouf (1965; p.72) uses the Newton-Raphson method to obtain the value of \( z \). Unfortunately, he does not use it correctly. We note that (6.17) implies:

\[
(c+1) = (s^z+1)^{1/z}.
\]

Passing both sides of the equation to the \( z \) power, we obtain:

\[
(c+1)^z = s^z + 1 + s^z + 1 - (c+1)^z = 0.
\]

Defining a new function \( f(z) \) to express the equation, we have:

\[
f(z) = s^z + 1 - (c+1)^z = 0. \tag{6.19}
\]

We note that equation \( f(z) \) in (6.19) is derived from equation (6.17). Thus, to approximate \( z \) in (6.17) is equivalent to approximate a zero of the function \( f(z) \) in (6.19). Unfortunately, instead of approximating a zero of (6.19), Kassouf (1965) searches for a zero in \( f(z) = s^z + 1 - c^z \). This flaw makes his subsequent empirical work deficient. To approximate a zero of (6.19), the Newton-Raphson method employs the first derivative:

\[
f'(z) = s^z \ln s - (c+1)^z \ln(c+1). \tag{6.20}
\]

45. The reader is referred to Ellis and Gulick (1982, p.161) for details.
Using the first derivative (6.20), the Newton-Raphson method focuses on the recursive relation:

\[ z_{n+1} = z_n - \frac{f(z)}{f'(z)}. \]  

(6.21)

In (6.21), \( z_n \) is the initial value of \( z \) and \( z_{n+1} \) is the new value of \( z \). Given the initial value for \( z_n \), observation of \( c \) and \( s \), we can use (6.21) to obtain a new \( z_{n+1} \). Because Kassouf is incorrect in the first step, his iteration equation is also wrong.

As can be seen from (6.21), the effectiveness of the Newton-Raphson method in finding a zero of a function \( f \) depends on \( f \) and \( f' \). In particular, (6.21) works only if \( f'(z) \neq 0 \). Normally (6.21) is most effective if \( f'(z) \) is neither too close to 0 nor too large in absolute value (Ellis and Gulick[1982, p.162]). We find a better algorithm (in terms of computational efficiency) to approximate the implicit \( z \) in (6.17) than the Newton-Raphson method. Instead of applying the Newton-Raphson method that requires a first derivative, we apply the Newton's binomial formula to (6.17). We note that (6.17) implies:

\[ c(z)+1 = (s^z+1)^1/z, \]
\[ = (c(z)+1)^z = s^z+1, \]
\[ = s^z = (c(z)+1)^z-1, \]
\[ = 1 + \left( \frac{z}{1} \right)c + \left( \frac{z}{2} \right)c^2 + \ldots + \left( \frac{z}{i} \right)c^i + \ldots - 1, \]  \( \text{(Newton's binomial formula)} \)

46. The reader is referred to Feller (1967, p. 51) for some applications that use the Newton's binomial formula.
\[ = \binom{2}{1}c + \binom{2}{2}c^2 + \ldots + \binom{2}{i}c^i + \ldots, \tag{6.22} \]

In (6.22), the last two steps can be justified by applying the Newton's binomial formula because \( c < 1 \) in practice, i.e., the price of an option is not higher than its exercise price. From (6.17) and (6.22), each set of observations of \( c \) and \( s \) give rise to a synthetically unique observed value of \( z \). Because of its computational efficiency and ease, formula (6.22) is used instead of (6.20) in Chapter 7 for empirical studies. \(^{47}\)

Given observation on \( s \) and \( c \), we can obtain a unique \( z \) from (6.21) or (6.22). Like Shelton (1967), Kassouf (1965, 1969) uses multiple regression to check how well \( z \) can be "explained" by these potential determinants in a given sample of data. Besides \( s_t \) and \( X \), Kassouf uses the following five regressors: \( 1/\tau \), the inverse of the time to expiration (measured in months); Yield (same as Shelton); \( D \), the number of outstanding warrants per number of outstanding shares (i.e.,

\( ^{47} \) We experiment with both (6.21) and (6.22) to compute observed value of \( z \) for foreign-currency options. The Newton binomial formula (6.22) is used to iterate \( z \) from 2 to 4000 and the Newton-Raphson formula (6.21) is used to iterate \( z \) to zero function (6.19) \( f(z) = 0.005 \). On average, the Newton-Raphson method takes three times as much CPU time as the Newton binomial method does. Furthermore, the estimated error \( |\hat{z} - z| \) by the Newton-Raphson method is sometimes as high as 50 while the maximum estimated error \( |\hat{z} - z| \) by the Newton binomial method is at most 0.5. This result is not surprising because the denominator \( f'(z) \) of the Newton-Raphson method (6.21) is close to zero. This fact makes the zero function \( f(z) \) very slow to converge to \( f(z) = 0.005 \) and the value of \( z \) estimate not very accurate. Thus, we conclude that the new computation method (6.22) is better than the Newton-Raphson method (6.21) in obtaining the observed value of \( z \).
potential dilution ratio); $E_1$, the slope of the least squares line fitted to logarithms of the monthly mean price for common for the previous eleven months; and $\sigma_t$, the standard deviation of logarithms of the monthly mean price for the common for the previous eleven months. The model Kassouf proposes for $z$ is:

$$z = \alpha + \beta_1/t + \beta_2 \text{Yield} + \beta_3 D + \beta_4 E_1 + \beta_5 \sigma_t + \beta_6 s_t + \beta_7 X + \varepsilon_t.$$  (6.23)

To use Kassouf's model to obtain a forecasted value for a call option, we need to estimate regression equation (6.23) first. The estimated equation (6.23) can be used to estimate $z$ for the forecasted option. The estimate for $z$ can then be inserted in equation (6.17) to obtain the forecasted option's price. We note that since $z$ enters (6.17) in a nonlinear form, even if the estimate $z$ obtained from (6.23) is unbiased, the resulting estimate of $c$ is biased upward. However, if the estimate $z$ is consistent, the resulting estimate of $c$ is consistent. We prove this as Property 6.10.

Property 6.10.

(a) The estimator $\hat{c} = c(\hat{z})$ is a biased estimator for $c$ even if $\hat{z}$ is an unbiased estimator.

(b) The estimator $\hat{c} = c(\hat{z})$ is a consistent estimator if $\hat{z}$ is a consistent estimator.

Proof:

(a) This can be proved by Jensen's inequality. Because property 6.7 establishes $c''(z) > 0$, Jensen's inequality implies
Thus, even $\hat{z}$ is unbiased (i.e., $E(\hat{z}) = z$), the estimated warrant price is biased, i.e., $E(c) = E(c(\hat{z})) > c(E(\hat{z})) = c(z)$.

(b) We note that $c$ is a continuous function in $z$ and $s$ (property 6.5 and 6.8). If $\hat{z}$ is a consistent estimator, Slutsky theorem\textsuperscript{48} establishes that any continuous function of a consistent estimator is itself a consistent estimator. Thus, although $\hat{c} = c(\hat{z})$ is not an unbiased estimator, it could be a consistent estimator, if $\hat{z}$ is consistent. Q.E.D.

If the error term $\epsilon$ in regression equation (6.23) satisfies the basic assumptions of the classical normal linear regression\textsuperscript{49}, the least-squares regression estimator $\hat{z}$ is a consistent estimator. This means that using the least-squares regression to obtain the estimator of $z$, the resultant call price estimator $\hat{c} = c(\hat{z})$, although not unbiased, could be a consistent estimator.

Kassouf's model has at least three important advantages over Shelton's. Firstly, it explicitly considers more variables. Any factor that has a potential impact on option's price can enter the equation (6.23). Secondly, the Kassouf model explicitly considers the effect of stock price volatility using a measure that is unbiased for the stock

\textsuperscript{48} For the proof the reader is referred to Wilks (1962; p.102-103).
\textsuperscript{49} The reader is referred to Kmenta (1974, ch. 8) for details.
price variance. Finally, Kassouf's functional form is more flexible in that the empirical pricing function $c$ does not tie to any particular curve. Although the basic price relation (6.17) is fixed, the exponent $z$ in equation (6.17) is allowed to drift from one curve to another within the admissible class based on the options' hypothesized potential determinants. This feature ties the model's price to a wider class of pricing function and expands potentially the forecasting ability of the model.

There are three weaknesses in Kassouf's pattern of inference. First, he fails to present the mathematical proofs that his proposed functional form is free of various conceptual deficiencies. Second, his failure to use the Newton-Raphson method correctly makes his empirical results questionable. Third, his proposed linear regression equation (6.23) for $z$ does not allow $z$ to be a nonlinear function of its potential determinants, nor does it guarantee that the estimates of $z$ exceed their logical lower bound of one. In section 6.3, we modify Kassouf's model for foreign-currency options. Among other things, we correct equation (6.23) in a loglinear regression that assures that the estimates of $z$ remain within their logical bounds.

After Kassouf (1969), several authors use regression techniques to test the impact of potential determinants on warrant prices. Van Horne (1969) uses a linear regression equation to study the impact of several variables on warrant prices, using cross-sectional and time-series data. The novel feature in his study is to test the relation between warrant prices and the value of funds to investors. He uses
the Treasury-bill rate as the proxy for the value of funds to investors and finds that it varies directly with the market prices for warrants. Miller (1971) experiments with different regression models to test the effects of longevity on the value of stock warrants. He finds that the loglinear regression produces the best goodness of fit as compared to a second-degree and a third-degree polynomial regressions. In a similar study, Rush and Melicher (1974) report that a third-degree polynomial model "fits" both pooled and cross-section sample data better than the linear or quadratic models do. Their results are consistent with those of Miller (1971) and Kassouf (1969). Parkinson (1972) minimize the squared difference between the market prices and predicted prices to find the best values for the parameters in Kassouf's equation (6.23), which he treats as a chi-square variable. He reports that the modified formula predicts warrant prices within about 15 percent of observed values. Kassouf (1976) uses a loglinear form to test whether warrant prices tend to lag common stock prices. Based on daily closing prices, he finds a positive correlation between warrant prices and past stock prices.

Summing up, the following variables are considered important in determining stock warrant prices in the past literature: the time to the expiration date, the current stock prices, the exercise price, the last stock prices, the volatility of stock prices, the dividend yields of common stocks, the interest rate, the trend of stock prices and the dilution ratio (the ratio of warrants outstanding to common stock outstanding). Except for the last variable, which is irrelevant for
foreign-currency options, these variables can be adapted in section 6.3 to study the behavior of foreign-currency options.

Four types of functional forms were used in past ad-hoc studies for the warrant prices: (1) a simple linear regression form, used by Kassouf (1965), Van Horne (1969) and Rush and Melicher (1974); (2) a log-linear regressional form initiated by Giguere (1958) and used by Miller (1971), Kassouf (1976); (3) a stepwise regression form initiated by Shelton (1967) and (4) an indirect nonlinear form originated by Kassouf (1969). Because option prices are always nonnegative (restriction 6.1), and the relation between option prices and stock prices are nonlinear, the simple linear regression is not an appropriate method to construct an empirical pricing function for options. By transforming the original price data in a logarithm form, a loglinear regression can satisfy the restriction (6.1) and address the nonlinear feature of an empirical pricing function. However, a loglinear form fails to incorporate contractual constraints inherent in a call option. By converting the option prices into the time value of an option, a stepwise regression can be used to estimate option prices within their plausible pricing zone. Nevertheless, without prior constraints on coefficients, the out-of-sample estimated option

50. All the functional forms that are reviewed in this section are the reduced form of the option-pricing model. These reduced form equations show explicitly how the endogenous variables are jointly dependent on the predetermined variables and the disturbances of the system. Because these forms are not derived from economics theory, they are not the structural form of the option-pricing model.
price may violate restrictions (6.1) and (6.4). The most elegant functional form to estimate an empirical pricing function is the nonlinear form originated by Kassouf (1965). Six mathematical properties are proved to show that this form satisfies restrictions (6.1)-(6.5) and is consistent with some of the properties associated with the Black-Scholes formula. In the following section, we modify the functional forms (2)-(4) to develop estimatable pricing functions for foreign-currency options.

6.3 Three Empirical Pricing Equations for Foreign-Currency Options

This section develops an empirical pricing equation for foreign-currency options. The symbols used follow those of Chapter 5. The empirical pricing equation $C$ for foreign-currency options is hypothesized to be a function of options' potential determinants. Based on the summary in section 6.2, we assume $C$ can be written as:

$$C(t) = C(\tau, S(t), X, S(t-1), V, \tau_d, r_f, Trend) \text{ for } t < T. \quad (6.24)$$

In (6.23), $V$ is a general volatility measure for the underlying foreign currency and Trend is a variable measuring any general trend for foreign-currency prices. We use the standard deviation $\sigma_t$ (estimated from the past data) to represent $V$. Trend is proxied by $S_t/S$, the relative ratio of the present foreign-currency price to the average currency price of the last two months.
Corresponding to the dividend yields for $C$ we employ the foreign interest rate $r_f$ in (6.24). An investor who chooses to hold a foreign-currency option instead of a foreign currency, foregoes the potential gain from foreign-currency interest rate. Thus, a foreign-currency option can be analogous to an option on a stock paying a continuous dividend. Therefore, dividend effects on a stock's warrant price is similar to the foreign interest effect on a foreign-currency option price.

We note that the Black-Scholes formula (5.16) for foreign-currency options $C$ is homogeneous of degree 1 with respect to $X$ and $S$. This can be proved easily: let $S$ and $X$ double, $C(2S,2X)=2C(S,X)$ by formula (5.16). In this section, we "assume" that the empirical pricing function $C$ for foreign-currency option possesses this homogeneous property. This is equivalent to assume that option investors are free of "share price illusion". As in the last section, we define $c=C/X$ as a standardized option price and $s=S/X$ as the underlying foreign-currency price. The homogeneity assumption for $C$ in $S$ and $X$ in (6.24) implies:

$$c=C/X=C(\tau,S(t)/X,X/X,S(t-1)/X,V,r_d,r_f,Trend)$$

51. Samuelson (1965, p.18) and Kassouf (1969, p.686) both explicitly make this assumption. Samuelson's justification for this assumption is that this is a 'property of competitive arbitrage ... a property that says no more than that two shares always cost just twice one share.' In foreign-currency option market, it is hard to imagine two $5$ bills to be worth more than one $10$ bill. Thus, the assumption that investors in the foreign-currency option market are free of share price illusion can be justified.
\( C(\tau, s(t), s(t-1), V, r_d, r_f, \text{Trend}), \)

\( =c(\tau, s(t), s(t-1), V, r_d, r_f, \text{Trend}). \quad (6.25) \)

In (6.24), only the variables that use \( S \) and \( X \) are divided by \( X \). We note that the variable Trend is unchanged because

\[
\text{Trend} = \frac{S(t)}{\bar{S}} = \frac{S(t)}{S/X}/[\bar{S}/X].
\]

Once the standardized empirical pricing function \( c \) is obtained, the original pricing function \( C \) can be restored by the following formula:

\[
C(t) = X \cdot c(\tau, s(t), s(t-1), V, r_d, r_f, \text{Trend}). \quad (6.26)
\]

Thus, the parameters of \( C \) can be derived easily from estimates of the standarized empirical pricing equation (6.25).

To illustrate the issue of estimating a standardized empirical pricing equation, we place \( s \) on the horizontal axis and \( c \) on the vertical axis in Figure 6. To eliminate arbitrage opportunities, \( c \) must lie below the 45 degree line from the origin and lie above the 45 degree line from 1 by restrictions (6.1)' and (6.2)'. By restriction (6.3)', \( c \) must also pass through the origin. We note that these three conditions together preclude the graph \( c \) in the \( c-s \) plane from being linear unless \( c=s \). The true form of the nonlinearity between \( c \) and \( s \) is not known. A simple hypothesis is that \( c \) is a multiplicative model and can be estimated by taking logarithms. Because \( c \) and \( s \) are always nonnegative, taking logarithms is always possible and can guarantee the estimate of \( c \) to be nonnegative.
If the pricing function is not a multiplicative model, however, the loglinear regression would not produce a consistent estimates of \( c \). Because as \( \tau \) approaches 0, the option price must approach to its intrinsic value \( \text{Max}(0, S - X) \). This suggests that the pricing function is at least not in a multiplicative form with respect to \( \ln s \) and \( \tau \).

Assuming that the function \( c \) is sufficiently smooth in \( \ln s \) and \( \tau \) so that it can be represented by a truncated Taylor series polynomial in \( \ln s \) and \( \tau \). This suggests that we put the cross-products term \( \tau \cdot \ln s \) as interaction variables in the regression. With the term \( \tau \cdot \ln s \), the slope of a pricing equation with respect to \( \ln s \) can vary with \( \tau \).

Given this consideration, the first empirical pricing equation proposed for foreign-currency option is a simple loglinear model:

\[
\ln c_t = \alpha + \beta_1 \ln s_t + \beta_2 \tau + \beta_3 r_t + \beta_4 r_f + \beta_5 \sigma_t + \beta_6 \tau \cdot \ln s_t + \beta_7 \ln (S_t / S_t^-) + \beta_8 \ln s_{t-1} + \varepsilon_t.
\]

In (6.26), \( \varepsilon_t \) is a stochastic variable assumed to have a zero mean. Because time to expiration and the current asset price are believed to be the most important determinants, the cross-product term \( \tau \cdot \ln(s) \) is included to address the problem of nonlinearity in \( c \). We note that seven predetermined variables are hypothesized to be important in pricing function \( c \). To include all the cross-product terms, the polynomial regression might have as many as 42 cross-product terms and as many coefficients to estimate. This is certainly not a feasible solution for nonlinearity. In leaving out all other cross-product
terms, we implicitly assume that these terms do not have a significant impact on \( c \).

The second empirical pricing equation we propose for foreign-currency options adapts Shelton's (1967) stepwise regression model. The first step is to convert an observed option price to an option's time value, the premium over intrinsic value. The intrinsic value for a stock option is defined in equation (6.4)' as the lower bound for \( c: s(t)-1 \). However, because foreign-currency options are analogous to stock options with continuous dividends, this feature enables us to obtain a different lower bound from restriction (6.4) for foreign-currency options:

\[
C(t) > \max(0, S_t e^{-r_f \tau} - X e^{-r_d \tau}), \text{ for } t \leq T. \tag{6.4a}
\]

Using the standardized call-pricing function \( c \) and foreign currency price \( s \), restriction (6.4a) reduces to:

\[
c(t) > \max(0, s_t e^{-r_f \tau} - e^{-r_d \tau}), \text{ for } t \leq T. \tag{6.4a}'
\]

The lower bound in (6.4a) or (6.4a)' is the intrinsic value for foreign-currency options. By subtracting the intrinsic value from the option price, we obtain the time value of an option. The time

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52. The reader is referred to Merton (1973b) or Grabbe (1983) for the proof of (6.4a).

Strictly speaking, the intrinsic value means not the amount of money one can earn merely from exercising a foreign-currency option immediately. Rather, we focus on the earnings one can accrue by setting up an appropriate portfolio composed of a foreign-currency call option, a domestic bond and a foreign bond.
value of an option is the "real premium" on an option when we conceive an option as an insurance policy. We denote "Prem" as the real premium of a call option. By definition, we have:

\[
\text{Prem}(t) = c(t) - (s_t e^{-r_d t} - e^{-r_f t}).
\]  

(6.28)

The real premium is the true price for maintaining an option. Thus any determinants that could impact on the option price must affect its real premium first. Once past real-premium data are obtained, we may use the following loglinear regression to express Prem as a function of its potential determinants:

\[
\ln \text{Prem} = \alpha + \beta_1 \ln s_t + \beta_2 r_d + \beta_3 r_f + \beta_4 \sigma_t^2 + \beta_5 \tau t + \beta_6 \ln (S_t / S) + \beta_7 \ln (S_t / S) + \beta_8 \ln s_{t-1} + \epsilon_t.
\]

(6.29)

The third pricing equation we propose for foreign-currency options uses the nonlinear functional form (6.17) originated by Kassouf (1965). In the first step, we convert the past option prices and currency prices into a series of implicit observations on \( z \) according to (6.17). Once the values of \( z \) are obtained by the Newton binomial formula (6.22), we use a loglinear regression to explain the value of \( z \) as a function of its potential determinants:

\[
\ln z = \alpha + \beta_1 \ln (s_t / S) + \beta_2 r_d + \beta_3 r_f + \beta_4 \sigma_t^2 + \beta_5 \tau t + \beta_6 \ln (s_t / S) + \beta_7 \ln (S_t / S) + \beta_8 \ln s_{t-1} + \epsilon_t.
\]

(6.30)

The three proposed pricing equations (6.27), (6.29) and (6.30) for foreign-currency options are estimated empirically in Chapter 7.
These estimated empirical pricing equations let us test whether or not each of an option's potential determinants have a statistically significant impact on option prices. We also use the estimated pricing equations to predict the out-of-sample option prices. The out-of-sample performance of the empirical pricing equation is used to establish a benchmark criterion for evaluating an arbitrage-free option pricing model developed in Chapter 5.
CHAPTER VII

SOME EMPIRICAL EVIDENCE OF FOREIGN-CURRENCY OPTION VALUATION MODELS

7.1 Introduction to Empirical Tests of Option Valuation Models

In this chapter, we test the ability of the option-pricing models that are derived in Chapter 5 and 6 to explain the pricing of foreign-currency options. To do this, we calculate the call values implied by each of the option-pricing models. The implied call values are compared with observed market prices for call options with models' relative success judged by the size of the gap between the two figures. The smaller is this gap, the better the model. This criterion parallels the use of goodness-of-fit tests in judging a regression model.

Two groups of option-pricing models are tested: empirical pricing equations and arbitrage-free option valuation models. We perform the following tests on each empirical pricing equation:

(A) We run the multiple regression and check whether the coefficients for each model's hypothesized explanatory variables differ significantly from zero.

(B) We examine a goodness-of-fit measure, e.g., R-squared, to check how well the model's implied prices fit the observed market prices.

(C) We forecast option prices using out-of-sample data and compare predicted prices with observed market prices.
The performance of empirical pricing equations is used to establish a benchmark criterion for evaluating arbitrage-free models. Specifically, we test whether an arbitrage-free model can systematically predict market option prices better than an empirical pricing equation. We also test the mixed-jump call-option valuation model for foreign currency against the modified Black-Scholes call-option valuation model.

We note that goodness-of-fit tests cannot unambiguously identify the structure of an arbitrage-free formula or empirical pricing equation. Our tests evaluate the applicability of a "joint hypothesis", which inseparably combines three subhypotheses:

1. an assumed mathematical structure for the pricing formula;
2. specific measurements of each formula inputs and outputs;
3. either the efficiency of the foreign-currency options market (for an arbitrage-free model) or (for an empirical pricing equation) the stationarity of the relation between option prices and their potential determinants.

In this chapter we assume (2) and (3) hold. This enables us to interpret the goodness-of-fit test for (1) unambiguously.

The chapter is organized as follows. In Section 7.2, we explain the data and proxies that are used to conduct empirical tests for foreign-currency options. In Section 7.3, we discuss computational problems for each models to be tested, particularly for the mixed-jump option-pricing model. In Section 7.4, we detail the specific tests
conducted for each formula. Test results are presented in Section 7.5. In Section 7.6, we conclude the results in this study.

7.2 The Data and Proxies For Empirical Tests in Foreign-Currency Option Markets

To estimate any of the foreign-currency option-pricing formulas that are derived in Chapter 5 and 6, we need at least the following information: simultaneous observations on foreign-currency option prices, foreign-currency spot prices, domestic and foreign interest rates for maturities matching those of the option contracts, variance of the underlying currency. The source for foreign-currency spot prices and option prices is the transactions surveillance report compiled daily by the Philadelphia Stock Exchange (PHLX). This report summarizes exchange on the PHLX from February 28, 1983 to June 27, 1985. Five currencies are listed on the exchange: British Pound (BP), Japanese Yen (JY), Deutsche Mark (DM), Swiss Franc (SF) and Canadian Dollars (CD). Because trading in the CD is infrequent\(^{53}\), this study investigates the first four currencies only. The reader is referred to Appendix B for the detailed description of the data. The procedures used to edit the data are also described in Appendix B.

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\(^{53}\) The number of observations for the CD call option is only 1415, while the number for the BP, JY, SF and DM are 4109, 3861, 4944, 5901.
For domestic interest rates, observations that are precisely simultaneous with foreign-currency option prices are not available. Daily observations on Treasury bill rates for the U.S. Treasury bill (T-bill) maturing closest to the option maturity date are used as a proxy. For maturities under six months, this is the Treasury bill that matures the day after the foreign-currency option expires. For longer maturities, the Treasury bill with the closest maturity may mature as early as two weeks before or as late as two weeks after the option expiration date. Among other data, The Wall Street Journal reports the following information about the T-bill: the bid discount (B), the asked discount (A) and the maturity date. Given the maturity date for the T-Bill, it is easy to calculate the number of calendar days to maturity (n) for any T-Bill. The current closing discount price of T-bill B(t) (with $10,000 face value) is computed via the following formula:

\[ B(t) = 10,000[1 - .01\text{discount}(n/360)] \]  

(7.1)

In (7.1), discount could be bid discount or asked discount. Assuming option investors can buy or sell the T-bill at the average of the bid and ask discount, formula (7.1) gives the current closing price of a T-bill:

\[ B(t) = 10,000[1 - .01((B+A)/2)(n/360)] \]  

(7.2)

Formula (7.2) states that if an investor can buy T-bill at the average of bid-ask discount, then an investment of B(t) today results in a certain return of $10,000 in n days. Given B(t) and n, we can
calculate the continuously-compounded per-annum rate $r_d$ implied on the T-bill. By construction, $r_d$ is the interest rate that satisfies the following equation:

$$B(t)\exp(r_d \tau) = 10,000.$$  \hspace{1cm} (7.3)

In (7.3), $\tau = n/365$ is the maturity of T-bill in terms of years. Substituting (7.2) into (7.3), we obtain the proxy for domestic interest rate:

$$r_d = \frac{-\ln[1 - .01((B+A)/2)(n/360)]}{\tau}. \hspace{1cm} (7.4)$$

Observations on foreign interest rates precisely simultaneous with foreign-currency options are not available either. In this case, the international interest-rate-parity theorem suggests a way to construct a proxy series. The theorem holds that in equilibrium, the percentage difference between forward and spot exchange rate must equal the difference between interest rates in the two countries (see Giddy [1983]). Specifically, the interest-rate-parity theorem suggests that:

$$\ln \frac{F(t)}{S(t)} = (r_d - r_f)\tau.$$  

Taking the antilog of both sides gives:

$$F(t) = S(t) \cdot e^{(r_d - r_f)\tau}. \hspace{1cm} (7.5)$$

In (7.5), $F(t)$ is the forward domestic currency price of a unit of foreign exchange at time $t$ with execution date $\tau$ years away and
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\[ T = n/365. \] Given \( F(t), r_d, \) and \( S(t), \) an implicit foreign interest rate \( r_f \) can be derived from (7.5):

\[ r_f(t) = r_d(t) - (\ln F(t, \tau) - \ln S(t)) \tag{7.6} \]

Thus, given the domestic interest rate and foreign-currency spot price, a proxy for foreign interest rates \( r_f \) is generated by (7.6).

Unfortunately, forward prices matching the maturity of currency options are not available. Instead, we use the prices of currency futures. Foreign-currency futures are traded on the International Money Market (IMM) and are closely related in maturity to the currency options traded on the PHLX. Although futures prices are very close to forward prices\(^{54}\), we are aware that the implied foreign interest rates derived by applying futures prices to (7.6) could still be inaccurate if interest-rate parity does not hold. Thus, as explained in section 7.1, the empirical evidence presented in this chapter must be interpreted as a result of joint test of the mathematical structure of the formula and the hypothesis of interest-rate parity. To study the extent of this ambiguity, one could use daily Eurocurrency data for daily Eurocurrency interest rates. The Eurocurrency data is published in the London Financial Times. This is a topics for further research.

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\(^{54}\) Besides being different in performance guarantee, if domestic and foreign interest rate are stochastic, future prices may differ from forward prices because of the marking to market of the futures contracts. However, several studies, e.g., Cornell and Reinganum (1981), have shown that the difference is small due to transaction cost and tax differential effect.
The final parameter we need to estimate is the variance of the price changes for the underlying foreign currency. The variance and other parameter estimation problems for a mixed diffusion-Poisson process are complicated and are treated separately in Chapter 3. For variants of the Black-Scholes formula (5.16), estimation of variance is easier. Because formula (5.16) assumes that currency prices are lognormally distributed, the natural logarithm of the price relative over any period has a normal distribution, with mean and variance proportional to the length of the period. A simple estimate comes from treating past foreign-currency prices as independent samples from a lognormal distribution. This allows us to apply standard statistical techniques to estimate the parameters of a normal distribution with unknown variance. An unbiased estimator for the currency-price variance \( \sigma^2 \) would be its sample variance \( s^2 \):

\[
s^2 = \frac{\sum_{t=1}^{m} (x_{t-1} - \bar{x})^2}{m-1}.
\]

In (7.7), \( x_t = \log(S_t/S_{t-1}) \), \( \bar{x} \) is the sample mean for \( x_t \) and \( m \) is the number of observations used to construct the variance estimate. What is the appropriate number for \( m \)? Using daily data for estimating variance, the question becomes: how many past days of data should we use to calculate a variance estimate for today's options? The more past daily prices we use to estimate the variance, the more efficient the estimator is if all past daily prices come from the same distribution with the same variance parameter. However, good reasons
exist to expect currency variance to change over time. Therefore, the further past data we use, the more likely that the data are not relevant in forming today's variance estimate. Obviously, some trade-off must be made for choosing an appropriate days for m. Following Shastri and Tandon (1986), we use the latest 40 currency prices relative (approximately the data generated from the last two months' observations) to obtain the daily variance estimate:

\[ s_1^2 = \frac{\sum_{i=1}^{40} (x_{t_i} - \bar{x})^2}{39}, \quad (7.8) \]

Trades do not occur every day and evidence exists that asset returns are generated by a process operating closer to trading time rather than to calendar time (French [1984]). Hence, a yearly variance estimate \( s_y^2 \) is generated as the product of the daily variance \( s_1^2 \) and the number of trading days. We assume that the number of trading days in a year is 252 so that:

\[ s_y^2 = s_1^2 \cdot (252). \quad (7.9) \]

55. To see the impact of different variance estimators on model prices, we experiment with two different variance estimators: one using the last 40 observations, another one using the last 23. We found that the variance estimator formed by the lastest 40 observations produces the lower sum of squared of forecasting errors for modified Black-Scholes formula.

56. As a sensitivity experiment, we developed two alternative yearly variance estimate by using 255 and 365 days. The resultant variance estimates are used to estimate the modified B-S formula. The variance estimates obtained by multiplying daily variance by 252 produces the lowest sum of squared of forecasting errors.
An alternative estimate for currency variance is the variance implied in observed option prices. This implied variance is obtained based on the assumption that investors price options according to a particular model. One such model is the modified B-S formula (5.16) (Latane and Rendleman [1976], Whaley [1982]). According to Whaley's method, the daily implied variance estimate for a currency should be chosen to minimize the sum of squared deviations between the market prices and model prices of the option trades for the same currency the previous day. For two reasons, we think that this kind of implied variance estimate is inappropriate. First, given all the yesterday's market option prices with different exercise prices and maturity dates, the implied variance estimates may differ. We need a criterion for choosing among these estimates. One solution is to pick the implied variance derived from the options contract which has features closer to the one we want to estimate. But doing this introduces bias into the estimation procedure in favor of the option with that particular feature. Second, if no trade occurs or trades are very thin the previous day, we would have to use the more distant option prices to obtain the implied volatility. However, because the more distant option prices may not be relevant in forming today's variance estimate, doing this would introduce another bias into the estimation procedure.

7.3 Models to Be Tested and Some Computational Considerations
For convenience, we begin by listing the formulas we intend to test. The three empirical pricing equations we estimate are the loglinear regression model (6.27), the real-premium stepwise regression model (6.29), and the modified Kassouf model (6.30):

\[
\ln c_t = \alpha + \beta_1 \ln s_t + \beta_2 \tau + \beta_3 r_t + \beta_4 \tau f + \beta_5 \sigma t + \beta_6 \tau \ln s_t
\]

\[
+ \beta_7 \ln \left( \frac{S_t}{S} \right) + \beta_8 \ln s_{t-1} + \varepsilon_t, \tag{7.10}
\]

where \( \varepsilon_t \) is a stochastic variable assumed to have a zero mean.

\[
\ln \text{Prem} = \alpha + \beta_1 \ln s_t + \beta_2 \tau + \beta_3 r_t + \beta_4 \tau f + \beta_5 \sigma t + \beta_6 \tau \ln s_t + \beta_7 \ln \left( \frac{S_t}{S} \right)
\]

\[
+ \beta_8 \ln s_{t-1} + \varepsilon_t. \tag{7.11}
\]

\[
\ln z = \alpha + \beta_1 \ln s_t + \beta_2 \tau + \beta_3 r_t + \beta_4 \tau f + \beta_5 \sigma t + \beta_6 \tau \ln s_t
\]

\[
+ \beta_7 \ln \left( \frac{S_t}{S} \right) + \beta_8 \ln s_{t-1} + \varepsilon_t. \tag{7.12}
\]

Two arbitrage-free pricing formulas are also tested in this chapter: the modified Black-Scholes formula (5.16) and the modified Merton's formula (5.40). The modified Black-Scholes formula (5.16) for foreign-currency options is

\[
c = e^{-r_f \tau} S \cdot N(d_1) - X e^{-r_d \tau} \cdot N(d_2). \tag{7.13}
\]

In (7.13), \( d_1 = \frac{(\ln(S/X) + (r_d - r_f + \frac{1}{2} \sigma^2) \tau)}{\sigma \sqrt{\tau}} , \ d_2 = D_1 - \sigma \sqrt{\tau} . \)

Using the interest-rate-parity theorem (7.5), formula (7.13) can be rewritten as

\[
c = e^{-r_d \tau} [F \cdot N(D_1) - X \cdot N(D_2)]. \tag{7.14}
\]
In (7.14), $D_1 = \left\{ \ln(F/X) + \left( \frac{1}{2} \sigma^2 \right) \tau \right\}/\sigma/\tau = d_1$, $D_2 = D_1 - \sigma/\tau = d_2$. With this substitution, the call value depends only on forward rate $F$ and domestic interest rate $r_d$. The call value does not depend directly on the foreign interest rate or the spot exchange rate because this information is already impounded into the forward rate.

The mixed-jump option-pricing formula that we derive in chapter 5 can be written in two forms: (5.39) and (5.40). Formula (5.39) is:

$$c^* = \sum_{n=0}^{\infty} \frac{\exp(-\lambda \tau)(\lambda \tau)^n}{n!} c(S, r_d^*, r_f^*, V_n, \tau, X).$$  \hspace{1cm} (7.15)

In (7.15), $c$ is the modified Black-Scholes call option-pricing formula (7.13) and $r_f^* = (r_f + \lambda \kappa - \eta \gamma/\tau)$, $\kappa = \exp(\mu + (\delta^2/2)) - 1$, $\gamma = \sigma^2 + n \delta^2/\tau$. Formula (5.40) is:

$$c^* = \sum_{n=0}^{\infty} \frac{\exp(-\lambda' \tau)(\lambda' \tau)^n}{n!} c(S, r_d^*, r_f^*, V_n, \tau, X).$$  \hspace{1cm} (7.16)

In (7.16), $\lambda' = \lambda(1+\kappa)$, and $r_d^* = r_d - \lambda \kappa + \eta \gamma/\tau$.

We note that formulas (7.15) and (7.16) are mathematically equivalent. Thus, given the same input data, they should produce identical price estimates. Because formula (7.15) is more mathematically straightforward than (7.16), we use formula (7.15) to test the mixed-jump option-pricing model.

Estimation of the empirical pricing equations (7.10)-(7.12) and Black-Scholes type formulas (7.13)-(7.14) is straightforward once input data are collected. However, because more parameters are needed
and infinite summation is involved, the computational task for the jump-option pricing formula (7.15) is more difficult.

First, besides the data required for the modified Black-Scholes formula, the jump option-pricing formula (7.15) requires five additional parameters. Parameter estimation for a mixed diffusion-Poisson process is discussed in Chapter 3. Table 7 of Chapter 3 reports parameter estimates for a mixed diffusion-Poisson process using PHLX daily foreign-currency closing prices. Parameter estimates in the first four rows are based on assuming zero expected logarithmic jump size, i.e., \( \mu = 0 \). Given this particular parameter restriction, the value of the parameters that are used in formula (7.15) can be derived as:

\[
\begin{align*}
\kappa &= \exp(\delta^2/2) - 1, \\
\gamma &= \ln(1 + \kappa) = \ln[\exp(\delta^2/2)] = \delta^2/2, \\
r^*_d &= r_d + (n(\delta^2/2)/\tau) - \lambda(\exp(\delta^2/2) - 1), \\
r^*_f &= r_f + \lambda(\exp(\delta^2/2) - 1) - n(\delta^2/2)/\tau. 
\end{align*}
\]

Second, to compute option prices by formula (7.15) we need to figure out a way to add the infinite series \( c = \sum_{n=0}^{\infty} \frac{\exp(-\lambda \tau) (\lambda \tau)^n}{n!} c \), where \( \lambda \) is the mean of jump intensity. Obviously, unless this infinite series converges uniformly, formula (7.15) is useless. Fortunately, the sum of this infinite series does converge uniformly. We prove this assertion as theorem (7.18):
Theorem. \( c^*(t)=\sum_{n=0}^{\infty} \frac{\exp(-\lambda t)(\lambda t)^n}{n!} c(t) \) converges uniformly. \( (7.18) \)

Proof:

We use the Weierstrass M-test to prove the infinite series \( c^* \) converges uniformly. Foulks[1978,p.508] states the Weierstrass M-test as follows: given any sequence \( u_n \) and a sequence of positive constants \( M_n \) with \( |u_n| \leq M_n \) for all \( n \). If the series \( \sum M_n \) converges, then the series \( \sum u_n \) converges uniformly.

To use Weierstrass M-test, we let \( u_n=\frac{\exp(-\lambda t)(\lambda t)^n}{n!} c(t) \) and \( M_n=\frac{\exp(-\lambda t)(\lambda t)^n}{n!} S(t) \). Because the modified Black-Scholes option price \( c(t) \leq S(t) \), we have \( |u_n| \leq M_n \) for all \( n \). Also because \( \sum_{n=0}^{\infty} \frac{\exp(-\lambda t)(\lambda t)^n}{n!} =1 \), we have \( \sum M_n = \sum_{n=0}^{\infty} \frac{\exp(-\lambda t)(\lambda t)^n}{n!} S(t) = S(t) \sum_{n=0}^{\infty} \frac{\exp(-\lambda t)(\lambda t)^n}{n!} = S(t) \), and the series \( \sum M_n \) converges. Thus, we can apply the Weierstrass M-test and conclude that the series \( c^* = \sum_{n=0}^{\infty} u_n \) converges uniformly. Q.E.D.

Because Theorem 7.18 shows that the infinite series \( c^*=\sum_{n=0}^{\infty} u_n \) converges uniformly, we are justified to use its partial sum \( c^*_K = \sum_{n=0}^{K} u_n \) to approximate \( c^* \) as long as the sum of the truncated terms are negligible. We denote the sum of the terms truncated at \( k \) as \( E(K) \).
Because \( S(t) \) is a constant at time \( t \), equation (7.19) shows that the truncation error \( E(K) \) is bounded and is a function of mean jump \( \lambda t \). We would like to find a cut-off point \( K \) which can assure that \( E(K) \) is negligible for practical purposes. We define \( P(\lambda t; n) = \frac{\exp(-\lambda t)(\lambda t)^n}{n!} \) as the probability that \( n \) Poisson jumps will occur, given jump intensity \( \lambda t \). We note that \( P(\lambda t; n) \) is a decreasing function in \( n \) for \( n > \lambda t \). This can be proved using a ratio test:

\[
P(\lambda t; n+1)/P(\lambda t; n) = (\lambda t)/n < 1 \text{ if } n > \lambda t.
\]

Equations (7.20) and (7.16) suggest that we can approximate \( c^* \) by adding the series \( u_n \) from 0 up to \( n = K \) for \( K > \lambda t \) and \( P(\lambda t; K) = 0 \). The parameter estimates for \( \lambda \) obtained in Chapter 3 are based on the daily closing spot rates reported by the PHLX. The value of \( \lambda t \) represents the mean number of jump occurrences during the life of options contract. Therefore, the problem is to find a cut-off number \( K \) for every option that satisfies two conditions for \( K \): (1) \( K > \lambda t \) and (2) \( P(\lambda t; K) = 0 \).

We define \( \Delta \) as the option life in terms of number of calendar days and \( \tau \) as the option life in terms of number of trading days. Obviously \( \tau < \Delta \) and \( \tau = \Delta \cdot (252/365) \). Our solution for the cut-off days \( K \) is this: For short-lived options whose expiration days \( \Delta \leq 100 \) calendar days
days, we set the cut-off days $K = 120$. For longer-life options whose expiration days $\Delta > 100$ calendar days, we set the cut-off days $K = \Delta \cdot 1.2$. Because the largest value of $\lambda$ we obtain in table 7 is 0.9019 and the fact that $\tau < \Delta$, we have $K = 120 > \lambda \tau$ for short-lived options and $K = 1.2 \cdot \Delta > \lambda \tau$ for longer-life options. Thus, the cut-off number $K$ we propose for both types of options satisfies the first condition. It also satisfies the second condition as we explain below:

For short-lived options whose $\Delta \leq 100$, the expiration in business days $\tau \leq 70$. For such options, if $\lambda = 0.9019$ (the largest $\lambda$ value we estimated from Chapter 3), then $p(0.9019 \cdot 70; 120)$, the probability that the total number of jump events exceeds 120, is about $0.277 \cdot 10^{-9}$, which is close to zero. Thus the second condition is also satisfied for these short-lived options.

For an option whose life $\Delta$ exceeds 100 days, the search for a suitable $K$ is not simple. The longest option life in our data is 346 days. Of course, if we set a cut-off days $K$ large enough, e.g., 450,

57. Given mean occurrences $\lambda \tau$, and a number $K$ that denotes the number of Poisson occurrences, $p(\lambda \tau; K)$ can be obtained using the subroutine MDTPS that is contained in IMSL package or using mathematical function POISSON in SAS package.

58. To see how different $K$ can have different option values for such options. We experiment the computational process for such short-life options by summing the infinite series from 0 to $K = 70$, 100 and 120. We found that all the mixed-jump option prices converge to within $0.5 \cdot 10^{-75}$ error tolerance even to $K = 70$. Two reasons can explain this quick convergence. First, the majority of the modified Black-Scholes option value $c$ is very small (less than $\$10$). Second, the majority of $\lambda$ value is less than 0.5. For a $\lambda$ that is less than 0.5, $p(\lambda \tau; 70)$ can be as close to zero as $p(\lambda \tau; 120)$ for $\lambda = 0.9019$. 
for such options, this $K$ would also be suitable for other shorter-lived options. However, such a solution would be very inefficient computationally because a shorter-lived options would not need $K=450$ to converge. This is why our approach is to let $K$ vary proportionally to $\Delta$. For example, for an option whose life $\Delta$ is 270 calendar days, $\tau=190$ trading days. For such an option, if $\lambda=0.9019$, then $p(0.9019 \cdot 190;270 \cdot 1.2)=0.535E-21\approx0$. For the option whose option life is longest, our algorithm takes $p(0.9019 \cdot 238;346 \cdot 1.2)=0.122E-32\approx0$.

For other options, our algorithm would always produce a cut-off probability $p(\lambda \tau;K)<p(0.9019 \cdot 70;120)$. Thus, our approach to $K$ also satisfies the second condition $^{59}$.

7.4 Structure of the Tests

A simple way to compare different models is to see how well each model explains a given set of observed data. This amounts to a test for goodness of fit. This is particularly appropriate for a simple empirical pricing equation that uses direct linear regression. But for an empirical pricing equation that indirectly use linear regression or

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$^{59}$ To check the truncation error from using this method, we compute the option value by summing the series from 0 to $K=1.2 \cdot \Delta$ as well as to $K=1.5 \cdot \Delta$. We found that all option values we obtain by summing the series from 0 to $K=1.2 \cdot \Delta$ are identical to those that we compute by summing to $K=1.5 \cdot \Delta$. This is not surprising because the error tolerance by the first method shows that $p(\lambda \tau;K)<0.5E-75$ for $K \geq 1.2 \cdot \Delta$. In other words, the probability that the number of jumps exceeds $1.2 \cdot \Delta$ during the option life can be considered zero.
an arbitrage-free model that does not use regression, goodness-of-fit tests warrant some explanations.

Table 8, 9 and 10 report regression results for the three empirical pricing equations (7.10), (7.11) and (7.12)\(^{60}\). The relevant Durbin-Watson D critical values are presented in table 11 for comparison. The R-squareds of the loglinear regression (7.10) and the stepwise regression (7.11) lie in the range .84 to .86. R-squareds from the regression for the indirect modified Kassouf model (7.12) lie in the range .74 to .78. As measured by this version of goodness of fit, the modified Kassouf model is inferior to the other two empirical pricing equations. But this is misleading, because unadjusted R-squared is not a meaningful statistic for an indirect regression model. Because the dependent variables for the three equations differ, comparisons among these three equations need some adjustment. We note that our goal is to explain market option prices rather than the dependent variables ln c, ln prem or ln z, a more meaningful statistic is the sum of squared forecasting errors (or pricing errors) \(\sum (\hat{C} - C)^2\), where \(\hat{C}\) denote the estimated (or predicted) call price within samples (or out of samples). Dividing the sum of squared forecasting errors by

\(^{60}\) The modified Kassouf model (7.12) is tested with the regressor ln(s\(_{t-1}\)) included and excluded. As expected from the autocorrelation results developed in Chapter 2, only DM finds a significant t value for ln(s\(_{t-1}\)). To save space, except for the DM, we report regression results from equation (7.12) without ln(s\(_{t-1}\)) as a regressor.
the variance of market prices, we obtain the ratio \( \Sigma(C - \hat{C})^2 / \Sigma(C - \bar{C})^2 \) that represents the portion of market price movements which cannot be explained by the model. Subtracting this ratio from one, we obtain a measure of the portion of market price movements which is explained by the model. Because this measure has a meaning parallel to the \( R^2 \) from a regression (Kmenta [1974, p.365]), we call it as the model's "implied \( R^2 \)”, i.e.,

\[ \text{The model's implied } R^2 = 1 - \frac{\Sigma(C - \hat{C})^2}{\Sigma(C - \bar{C})^2}. \]

Other measures that enable us to examine how closely a model explains the observed data include: dollar-pricing errors \( Y \), relative-pricing errors \( V \) and Davidson and MacKinnon's (1981) model-specification test. A dollar-pricing error \( Y \) measures the dollar difference between the market price and model price and is defined as:

\[ Y(t) = C(t) - \hat{C}(t). \quad (7.21) \]

A relative-pricing error \( V \) measures the percent difference between the model's implied price and the market price. \( V \) is defined as:

\[ V(t) = \frac{[C(t) - \hat{C}(t)]}{C(t)}. \quad (7.22) \]

Tables 12-15 report dollar-pricing errors, relative-pricing errors, the sum of squared forecasting errors and each model's implied \( R^2 \). We note that the results reported in the rows marked (1) \( \lambda=0 \) in these tables can be treated either as a special case for jump-option price for \( \lambda=0 \) or as the modified Black-Scholes price using spot-rate formula (7.13). Concerning the dollar-pricing error, each table reports the
average absolute dollar-pricing error and the average dollar-pricing error on the first two columns. They are followed in parentheses by their relative-pricing errors.

Results in tables 12-15 show how closely each model's prices fit market prices. Based on these results, one can rank all the competing models in the order of their implied R-squared or pricing errors. However, the implied R-squared or the pricing errors per se do not enable us to establish the true model. Presumably, the model that has the significantly largest implied R-squared or has the significantly smallest pricing errors should be the model supported by the evidence. Unfortunately, to establish that the implied R-squared of a model significantly exceeds that of another model is a difficult task. Davidson and MacKinnon (1981) propose several procedures to test the specification of a model against other models. They define $y_t$ as an observation of the dependent variable (e.g., market price). The issue is to test a null hypothesis that $f$, a possibly nonlinear regression model, is the underlying model that generates $y_t$. An alternative hypothesis is that $y_t$ is generated by an alternative, non-nested model $g$. To test the null hypothesis $H_0$ as against the alternative hypothesis $H_1$, Davidson and MacKinnon suggest running the following regression:

$$y_t = (1-\alpha)\hat{f}_t + \alpha \hat{g}_t + \varepsilon_t,$$
\[ y_t - \hat{f}_t = \alpha (\hat{g}_t - \hat{f}_t) + \epsilon_t. \] (7.23)

In (7.23), \( \hat{f}_t \) and \( \hat{g}_t \) are model estimates. If \( H_0 \) is true, then the true value of \( \alpha \) is zero. If \( H_1 \) is true, the estimates of \( \alpha \) should converge asymptotically to one. Davidson and MacKinnon prove that the t-statistic for the coefficient estimate \( \hat{\alpha} \) is asymptotically normally distributed, and can be used as a significance test to reject \( H_0 \). If one wants to test \( H_1 \), the simplest procedure is simply to reverse the roles of \( H_0 \) and \( H_1 \) and carry out the test again. This test is remarkably simple both conceptually and computationally. However, two shortcomings for this test should be mentioned. First, it is conceivable that both hypotheses may be rejected, or that neither may be rejected. This would leave the choice of true model undecided. Second, Davidson and MacKinnon (1981, p. 783) warn that their procedures are really designed for testing model specification, not for choosing among competing models. For this reason, we apply their test procedure (7.23) only to the three regression models: the loglinear model (7.10), denoted by \( H_1 \), the stepwise regression model (7.11), denoted by \( H_2 \) and the modified Kassouf model (7.12), denoted by \( H_3 \). Tables 16-19 present the results of pairwise tests of each model, \( H_1 \) through \( H_3 \), against each of the other models.
To examine the source of pricing errors, we break down the pricing errors and each model's implied $R^2$ into two alternative partitions: by maturity classes and by in-the-money, at-the-money, and out-of-money classes similar to Rubinstein (1985). We consider a call option in the money if $S/X$, the ratio of the spot price to the exercise price, is greater than 1.02; a call option at the money if $S/X$ lies between .98 and 1.02; and a call option out of the money if this ratio is less than .98. Tables 20-23 report the results.

7.5 Tests Results

7.5.1 The Behavior of Foreign-Currency Option Prices.

Table 8 reports coefficient estimates and other summary statistics for the loglinear regression model (7.10). Without exception, four variables are statistically significant at 1 percent for all currency options: the time to expiration $\tau$, the foreign-currency spot price $\ln(s)$ (that is simultaneous with foreign-currency option), the cross-product term $\ln(s) \cdot \tau$ and the recent sample standard deviation of underlying currency price $\sigma$. The fundamental restrictions established in Chapter 4 and the past empirical works on warrants surveyed in Chapter 6 suggest that variables $\tau$, $\ln s$, $\sigma$ should have positive impact on option prices. Table 8 shows that the coefficients
for these three variables are significantly positive for all foreign currencies. This, of course, is expected.

Table 8 shows that the cross-product term $\tau \cdot \ln(s)$ has a significant negative impact on foreign-currency option prices. We recall that the cross-product term is put in the regression (7.10) to restrict the implied price to stay within the feasible-pricing zone of Figure 6. For example, as $\tau$ approaches 0, the option price must approach its intrinsic value $\text{Max}(0, S-X)$. This suggests that the slope of a pricing equation with respect to $\ln s$ should vary with $\tau$. The significant relation of the interative term $\tau \cdot [\ln(s)]$ with $\ln(c)$ confirms our conjecture.

We note that the signs of the coefficients for the domestic interest rate $r_D$ are always opposite to those of the coefficients for the foreign interest rate. Restriction (6.4a) places a lower bound on any foreign-currency option price at $S e^{-r_F \tau - X - r_D \tau}$. This suggests that the higher the domestic interest rate is, the higher the option price. On the other hand, the higher the foreign interest rate is, the lower the option price. Our estimates show that the coefficients on domestic interest rate $r_D$ are significantly positive and the coefficients on foreign interest rate $r_F$ are significantly negative. Again, our results are consistent with the theoretical restriction.

Table 9 reports coefficient estimates and other summary statistics for the stepwise regression model (7.11). The variables
that are significant for regression (7.10) in Table 8 are almost the same as those for regression (7.11) in Table 9. It seems that both models produce similar results. However, one difference exists between equation (7.10) and (7.11).

We recall that the regressand in (7.10) is foreign-currency prices ln(c), while the regressand in (7.11) is an option's time value ln Prem. Because the price of an option is the sum of an option's intrinsic value and an option's time value, other things being equal, a factor that increases an option's time value also increases an option's price. However, a factor that causes an option's price to increase does not necessarily imply that an option's time value also increases. A given increase in an option's price may be simply caused by its intrinsic value. This difference is suggested by the coefficient of ln(s) in both models. Significant positive relations are found for ln(s) and option prices in (7.10). But the relations between ln(s) and the options' time value in (7.11) turn out to be significantly negative. This result suggests that the positive partial correlation between option prices and currency prices reflects the effect on the option's intrinsic value. Furthermore, as the underlying currency price increases, the time value of an option decreases. This finding is consistent with Gugerere (1958), Samuelson (1965) and Shelton (1967), who report empirically that the time value of a perpetual warrant completely disappears when the underlying stock price is high enough.
Two variables, $\ln(s_{t-1})$ and $\ln(S_t/S)$, are designed to test whether the recent price history of the foreign currency has a significant impact on the foreign-currency option prices. The two arbitrage-free models we derive in chapter 5 assume that the distributions that govern the underlying asset's price movement are Markov processes. Karlin and Taylor (1975, p. 29) gives the following definition for a Markov process: "A Markov process is a stochastic process with the property that, given the value of $X_t$, the values of $X_s$, $s > t$, do not depend on the values of $X_u$, $u < t$; that is, the probability of any particular future behavior of the process, when its present state is known exactly, is not altered by additional knowledge concerning its past behavior." If the foreign-currency price follows a Markov process, the price of foreign-currency option should not depend on the price history of the exchange rates. Significant relations between the price of a foreign-currency option and the price history of a foreign currency, i.e., $\ln(S_t/S)$ and $\ln(s_{t-1})$, would suggest that option investors do not quite believe that foreign-currency price changes follow a Markov process.

Table 8 shows that the coefficients for $\ln(S_t/S)$ are significant at 1 percent for all the currency options. Although the coefficients on $\ln s_{t-1}$ for the BP, JY and SF are not significant. However, the DM
does show a significant coefficient on $\ln s_{t-1}$. This suggests that option investors do not quite believe that foreign-currency price changes follow a Markov process.

There are at least two possible effects that the recent price history of the foreign currency can impact foreign-currency option prices. The first effect can be called as "the correction-effect". If option investors believe on average that currency prices that have moved up are due for a correction (or that currency prices that have fallen are due to rise), then they will adjust the amount they pay for a foreign-currency option accordingly. If this hypothesis is true, the proxy variable for recent trends $\ln(S_t/S)$ would have a negative impact on foreign-currency option prices. The second effect is called "the lagged-market effect". Under this hypothesis, foreign-currency option prices may take time to adjust to changes in the underlying currency prices. If the currency prices move up and it takes time for foreign-currency option traders to act on this information, the price changes of the foreign-currency option would lag the underlying currency price changes. If this hypothesis is true, the recent trends would have a positive effect on foreign-currency option prices. Both the correction

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61. We recall that in Chapter II, only the DM exhibits a significant autocorrelation for its spot price changes. The results of the coefficients on $\ln(s_{t-1})$ suggest that unless the spot price changes exhibit a significant autocorrelation, the previous day's currency closing price does not have a significant impact on foreign-currency option prices.
effect and the lagged-market effect conflict with the efficient market hypothesis that arbitrage-free models assume.

The results in Table 8 suggest that both explanations could have contributed the significant effect of \( \ln(S_t/S) \) on foreign-currency option prices. The coefficients for \( \ln(S_t/S) \) are negative for the BP, SF, DM and positive for the JY. This suggests that during the sample period, the correction effect dominates the lagged-market effect for the BP, SF and DM while the lagged-market effect dominates the correction effect for the JY.

7.5.2 Comparisons of the Three Empirical Pricing Equations.

Tables 12-15 suggest why unadjusted R-squared is not a good criterion by which to rank an indirect regression model. Despite its lowest in unadjusted regression R-squared, the modified Kassouf model (7.12) has the highest implied R-squared among the three empirical pricing equations. If one use the mean of dollar-pricing error, the mean of relative-pricing error and the sum of squared of forecasting errors as criterion, the modified Kassouf model again fits the data better for every currency options.

Tables 16-19 present the results of the Davidson and MacKinnon pairwise tests (7.23) for model specification on the three regression models. If the maintained hypothesis \( H_1 \) is true, then the true value
of $a$ in equation (7.23) is zero. By testing whether the coefficient estimate $a$ differs significantly from zero, we can reject the maintained hypothesis $H_1$. For every currency option, we reject the maintained hypothesis that the true model is the loglinear regression or the stepwise regression in favor of the modified Kassouf model. When we pair other two models against the null hypothesis that the true model is the modified Kassouf, we cannot reject the null hypothesis for the BP and JY with 1% significance level. The coefficient estimates $a$ for the SF and DM lie in the range $-0.04$ to $0.03$. These coefficient estimates for $a$, although relatively small as compared to other pairwise tests, differ significantly from zero. This means, for the BP and JY, we reject the loglinear model and the stepwise regression model in favor of the modified Kassouf model. However, for the SF and DM, the Davidson and Mackinnon pairwise tests reject all the three regression models. This leaves the choice of the true model undecided for the SF and DM.

62. That the Davidson and MacKinnon pairwise test cannot distinguish between the three regression models for the SF and DM by no means suggests that the three empirical pricing equations perform equally well for the SF and DM. For the SF and DM, Tables 12-15 show that the modified Kassouf model consistently generates the lowest mean of dollar-pricing error, the lowest mean of relative-pricing error and the lowest sum of squared of forecasting errors. Moreover, the magnitudes of the pricing errors generated by the other two models are at least twice as high as those generated by the modified Kassouf model. Thus, clearly if one is to choose a best model among the three, the modified Kassouf model must be the best even for the SF and DM.
7.5.3 Comparisons of the Two Arbitrage-free Models with the Modified Kassouf Model

Tables 12-15 show that the modified Black-Scholes formula when it is estimated from futures prices (i.e., equation [7.14]), consistently generates a higher implied R-squared than the same formula when it is estimated from spot rates as in (7.13). If international interest-rate parity holds in the foreign-exchange market, formula (7.13) and (7.14) should produce similar results. The discrepancy between the results for these two formulas suggests that international interest-rate parity might not hold. This interpretation, however, is premature for two reasons. First, international interest-rate parity is strictly valid only for forward prices, not the futures price data which we use for estimation here. Although the empirical findings by Cornell and Reinganum (1981) suggest that the difference between forward price and futures prices might be small, it is not clear whether the difference is small in our sample periods. Second, perhaps the more serious problem is that, the futures price data we use in estimating formula (7.14) are daily closing prices reported by IMM. These futures prices are not transaction data. Thus, these futures prices do not possess the simultaneity that is required for estimating option prices while the spot currency prices we use in estimating (7.13) possess the

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63. The differences in implied R-squared are .023, .177, .178 and .121 respectively for the BP, JY, SF and DM in favor of equation (7.14).
require simultaneity. For example, the simultaneous futures prices for the options that are traded early in the day may not be the closing futures prices we use here. On the other hand, the spot currency price we use in estimating (7.13) is the prevailing average of the spot bid and ask at the time of the option trade. Thus, we think that the results of the modified Black-Scholes formula when it is estimated from spot currency prices are more typical than those of the same formula when it is estimated from futures prices.

The largest implied R-squared for British Pound and Swiss Franc is produced by the modified B-S formula (7.13) using futures prices. The top slot as measured by implied R-squared for Japanese Yen and Deutsche Mark belongs to the jump-option pricing formula using MLEs under the assumption that exchange rates follow Bernoulli mixture of normal processes. The implied R-squared from the modified Kassouf model are consistently placed second for all currency options. If the result from the modified B-S formula using futures prices are discarded due to non-simultaneity, the modified Kassouf model would produce largest R-squared for British Pound and Swiss Franc.

The implied R-squareds from the modified B-S formula (7.13) for each currency options ranges from .61 to .93. In contrast, the implied R-squareds generated by the modified Kassouf model range only from .87 to .95. The performance of the mixed-jump option model differs not

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64. The reader is referred to Bodurtha (1984) for the detailed description on how the spot rates contained in the PHLX surveillance report are recorded.
only among currency options but also among the ways its parameters are estimated. We note that, except for the BP, which has the largest R-squared in row (4) and second largest R-squared in row (3), row (3) produces the top R-squared for every currency option. The parameter estimates in row (3) are the MLEs for the Bernoulli mixtures of normal processes. If the jump-option model is correct, this suggests that the MLEs estimated by assuming Bernoulli mixtures are the one closer to the true parameters.

Because the modified B-S formula is a special case of the mixed jump formula, it appears that the mixed jump formula should always outperform the modified B-S formula. Tables 12-15 shows that this is not true here. The mixed jump formula has a significantly higher R-squared than the modified B-S formula only for the JY and DM, despite the fact that the likelihood tests performed in Chapter 3 show that the underlying currency prices are generated by a process closer to the mixed jump than a lognormal process. Although the implied R-squareds generated by the mixed jump for the BP and SF (.862, .676) are lower than those generated by the modified B-S (.938, .760 respectively), the differential implied R-squareds between these two formulas are small for the BP and SF. In contrast, the implied R-squareds (row 3) generated by the mixed jump formula for the JY and DM (.95 and .94) are a lot higher than those generated by the modified B-S formula (.61 and .69).

Tables 12-15 also report the average dollar pricing errors and relative pricing errors for the three models. We note that the average
absolute pricing errors generated by the mixed-jump option-pricing model for the JY and DM are .135 and .164 while those generated by the modified Black-Scholes model for the JY and DM are .424 and .341. This result together with the result of higher implied R-squareds, although not tested for significance, do demonstrate that the mixed jump options model fit the observed option prices better than the modified Black-Scholes model for the JY and DM.

Why does the mixed-jump option-pricing model perform better for the JY and DM? We recall the behavior of the underlying currency price changes in Chapter 3. Table 7 shows that the likelihood-ratio statistics, which test whether the jump intensity \( \lambda \) differs significantly from zero, for the JY and DM prove higher (965.87 and 845.58) than the numbers achieved for the BP and SF (62.80 and 216.24). This means that the departure from a lognormal process for the JY and DM are more serious than that for the BP and SF. The relatively better performance from the mixed jump option-pricing model for the JY and DM suggests that the more seriously the underlying currency price changes depart from a lognormal process, the better the mixed jump option-pricing model performs.

The average dollar pricing errors for the modified B-S formula are all negative. This shows that on average the modified B-S formula underprices call options. To see more clearly how the modified B-S formula misprices call options, tables 20-23 present pricing errors separately for in-the-money, at-the-money and out-of-the-money classes and for maturity classes. The modified B-S formula consistently
underprices out-of-the-money options relative to in-the-money options. For the BP, the relative-pricing errors generated by the modified B-S formula are about -25 percent for out-of-the-money options but go down to about -2 percent for in-the-money options. For the JY, SF, and DM, the underpricing patterns for the modified B-S formula are similar. For these currency options, the relative-pricing errors generated by the modified B-S formula for out-of-the-money options are about -70 percent while those for in the money options is only -5 percent. The magnitude of underpricing by the modified B-S formula appears to be independent of maturity class.

When we present the results according to the maturity class and S/X level in Tables 20-23, we find that the modified Kassouf model emerges as the best model on every category. Whether one uses the mean of absolute dollar-pricing error, the mean of absolute relative-pricing error or the sum of squared forecasting errors, the modified Kassouf model fits the data better for 47 out of 48 groups. If one a priori assumes that the three models fit the data equally well and the probability that one model performs better than the other two models follows a Bernoulli process with $p=1/3$. Then we can consider each category as an independent observation, the probability that the modified Kassouf model’s sum of squared forecasting errors is smallest among the three models for one category is 1/3. Using the binomial

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65. The only one group that the modified Kassouf model is not better is the group of options from the DM that are at the money options and whose expiration days are less than 30 days.
distribution, we can calculate the probability that the modified Kassouf model's sum of squared forecasting errors is smaller in 47 out of 48 categories:

\[ \frac{48!}{(47! \cdot 1!)} \left( \frac{1}{3} \right)^{47} \left( \frac{2}{3} \right)^1 = 1.20231E-21 = 0. \]  \hspace{1cm} (7.24)

Equation (7.24) suggests that if the three models reported in Tables 20-23 fit the data equally well, then the probability that the modified Kassouf model is smallest in sum of squared forecasting errors is almost zero. The superiority of the modified Kassouf model is more pronounced for an option whose life exceeds 90 days. The relative-pricing errors for these options are less than 3 percent in most cases.

When we test the modified Merton model against the modified Black-Scholes model in Tables 20-23, we find that the modified Merton price estimates based on the MLEs of Bernoulli mixture perform better than the modified B-S price estimates for the JY and DM. The superiority of modified Merton’s price estimates is more pronounced for at the money and in-the-money options with more than 90 days maturity.

We note that the average of the absolute values of the relative pricing errors for the arbitrage-free models is a decreasing function of the ratio of \( S/X \) as well as the time to maturity\(^66\). In contrast,

\(^66\) We ran the following regression to see whether the biases observed from the modified B-S formula are statistically related to some explanatory variables:

(Footnote continues on next page)
the average of the absolute values of the relative pricing errors for
the modified Kassouf model is a decreasing function only of the ratio
of S/X.

The results in Tables 12-23 let us compare the performance of
each competing model. The modified Kassouf model proves best as
measured by the Davidson and MacKinnon test among the three empirical
pricing equations. Using the edited data, the modified Kassouf model
also performs better than the two arbitrage-free models as measured by
the sum of squared forecasting errors. However, the Davidson-MacKinnon
test and the sum of squared forecasting errors presented in tables 12-
23 employ in-sample data. To evaluate an empirical pricing equation,
it is also useful to employ out-of-sample data to test its forecasting
performance. For a regression model, this out-of-sample test is
equivalent to testing the stationarity of its parameters. Since we are
interested in forecasting option prices, we would like a long period
to estimate the regressions and a large-enough number of samples to
examine their out-of-sample forecasts. To do this, all the empirical
pricing equations are re-estimated with data from Feb. 28, 1983 to May
31, 1985. These results are used to forecast foreign-currency option
prices in the month between June 1, 1985 to June 27, 1985.

(Footnote continued from previous page)

\[ C_t - C_t = \alpha + \beta_1 s_t + \beta_2 \tau + \beta_3 \sigma_t + \beta_4 \ln \left( \frac{S_t}{S} \right) + \varepsilon_t. \]

We find that for all currency options, coefficients for \( s_t \) and \( \tau \) are
significantly negative while coefficients for \( \sigma_t \) and \( \ln(S_t/S) \) are
significantly positive.
Tables 24-27 present the summary of out-of-sample forecasting errors for the data that are not separated by maturity classes and S/X ratio. We find that out-of-sample forecasts from the modified Kassouf model produce an implied $R^2$ close to the in-sample implied $R^2$ (Tables 12-15) for all foreign currency options. As measured by the out-of-sample implied $R^2$, the mixed-jump option-pricing model is best for the JY. The mixed-jump option-pricing model can also claim to forecast best for the DM, if the modified B-S formula that is estimated by using futures prices is discarded due to non-simultaneity. The out-of-sample implied $R^2$ from the modified B-S formula is the worst for the JY. But the modified B-S model performs best for the BP, SF. The out-of-sample results in these table are consistent with those in-sample results reported in tables 12-15.

Because the modified Kassouf model performs better with the edited data (according to maturity class and S/X ratio) as demonstrated by comparison of Tables 12-15 with Tables 16-19, the out-of-sample results reported in Tables 24-27 represent lower bounds for the modified Kassouf model. The preliminary out-of-sample result shows that the modified Kassouf model, although it does not produce the highest implied $R^2$s on the unedited out-of-sample data, is the most consistent model, predicting option prices within about 11

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67. Tables 12-15 show that the in-sample implied $R^2$s produced by the modified Kassouf model are .952, .924, .876 and .914 respectively for the BP, JY, SF and DM while Tables 24-27 show that out-of-sample implied $R^2$s for each currency are .926, .919, .895 and .890 respectively.
percent of observed values. Although the mixed-jump option-pricing model and the modified Black-Scholes model sometimes forecast option prices better, their performances are not consistent acrosss the different currencies.

7.6 Conclusion

We have shown in this study that the modified Black-Scholes model exhibits a systematic bias for a foreign-currency option. The modified Merton model that is based on a mixed diffusion-Poisson process can eliminate some of the bias shown by the modified Black-Scholes model for the JY and DM. However, for the BP and SF, both the modified Black-Scholes and the modified Merton model exhibit the same bias. The relatively better performance of the modified Merton model as compared to the modified Black-Scholes formula appears to relate to the underlying distribution for foreign-currency price changes. Because departures from a lognormal process for the JY and DM are more serious than for the BP and SF, it suggests that the more seriously the underlying currency price changes depart from a lognormal process, the better the modified Merton model performs.

Among the three ad-hoc models, the modified Kassouf model performs best. Chapter 6 shows that only the functional form used by the modified Kassouf model is free of any conceptual deficiency. Results in this chapter confirm our conjecture that if one want to use
an ad-hoc model to estimate option prices, the modified Kassouf model produces the best result.

Using the performance of the modified Kassouf model as a benchmark criterion, we do not see evidence that the arbitrage-free option-pricing models systematically predict market option prices better than an empirical pricing equation. In fact, in-sample results with edited data show that the modified Kassouf model systematically fits observed market prices better than the arbitrage-free models. However, out-of-sample results are not conclusive. To reach more definite conclusions about comparative forecasting ability of ad-hoc empirical pricing equations and arbitrage-free models, we need to develop additional out-of-sample data to perform additional tests.
APPENDICES

Appendix A. Option Value with a Stable Distribution

A.1 Introduction to a Stable Distribution

We define $U_n$ as an independent, identically distributed random variables with a constant mean $\mu = E(U_n)$ and a finite variance $\sigma^2 = E(U_n - \mu)^2$ for $n=1, \ldots, N$. The classical central limit theorem asserts that

$$\lim_{N \to \infty} \frac{\sum_{n=1}^{N} (U_n - \mu)}{\sqrt{N} \cdot \sigma} = \text{a standard normal variable.} \quad (A.1)$$

Because $\sigma$ and $\mu$ are constant, the infinite sum in (A.1) can be expressed as:

$$\lim_{N \to \infty} \left( \sum_{n=1}^{N} U_n - \frac{\sqrt{N}}{\sigma} \cdot \mu \right) = A(N) \sum_{n=1}^{N} U_n - B(N). \quad (A.2)$$

In (A.2), $A(N)$ and $B(N)$ are two real functions of sample size $N$. Thus, the essence of the Central Limit Theorem is the following assertion:

There exists two real functions $A(N)$ and $B(N)$ such that as $N \to \infty$, the weighted sum $A(N) \sum_{n=1}^{N} U_n - B(N)$ has a finite limit and is a well-defined random variable. \quad (A.3)
Doeblin and Gnedenko, this limiting distribution of sums leads to a broader class of distribution called a stable distribution of which a normal distribution is only a special case.

We denote \( x \) as a stable random variable. A stable distribution can be described by its characteristic function:

\[
\psi(t) = \log E[\exp(itx)] = \log f(t),
\]

\[
= i\delta t - c^{\alpha} t^\alpha (1-i\beta \text{sgn}(t) \tan(\pi\alpha/2)), \quad \text{for } \alpha \neq 1. \quad (A.4)
\]

In (A.4), \( t \) is any real number; \( \alpha, \beta, \delta, c \) are parameters, \( \text{sgn}(t) = 1 \) if \( t > 0 \) and \( \text{sgn}(t) = -1 \) if \( t < 0 \). In contrast to a normal distribution, a stable distribution is a four-parameters family. The location parameter is \( \delta \), and if \( \alpha \) is greater than one, \( \delta \) is the expectation or the mean of the distribution. The scale parameter is \( \gamma = c^\alpha \), while the parameter \( \beta \) is an index of skewness. When \( \beta = 0 \), the distribution is symmetric. When \( \beta > 0 \), the distribution is skewed to the right (has a long right hand tail).

The characteristic exponent \( \alpha \) determines the height of, or total probability contained in the tail of the distribution. The value of \( \alpha \) must be greater than 0, but less or equal to 2. When \( \alpha = 2 \), the distribution becomes normal with variance \( 2c^2 \). When \( \alpha < 2 \), the total probability in the extreme tails increases as \( \alpha \) moves toward 0.

The three most important properties of stable distributions are:

\[68\] The reader is referred to Fama (1965, p. 103) for these conditions.
(1) the asymptotically Paretian nature of the far tail area, i.e., for any constant u,
\[
\Pr(x>u) \rightarrow (x/U_1)^{-\alpha}, \quad \text{as } u \rightarrow \infty ,
\]
(A.5)
\[
\Pr(x<u) \rightarrow (|x|/U_2)^{-\alpha}, \quad \text{as } u \rightarrow \infty .
\]
(A.6)
In (A.5) and (A.6), \(U_1\) and \(U_2\) are constants and are defined by
\[
\beta=(U_1^\alpha-U_2^\alpha)/(U_1^\alpha+U_2^\alpha).
\]
(A.7)
From expressions (A.5) and (A.6), it is possible to define an approximate density for the extreme tail areas of a Paretian stable process.

(2) the stability or invariance of these distributions under addition. This property can be demonstrated by deriving the characteristic function for \(N\) iid stable variables.
\[
\log(E[\exp(iNxt)])=N \log(E[\exp(ixt)])
\]
\[
=i(N\delta)t-(N^{1/\alpha}c)^\alpha(1-i\beta \sgn(t) \tan (\pi \alpha/2)), \quad \alpha \neq 1 .
\]
(A.8)
Equation (A.8) shows that the distribution of the sum of \(N\) iid stable variables is, except for origin and scale, exactly the same as the distribution of the individual summands. This property also holds for the independent stable variables that have different location and scale parameters. We denote \(S(X; \alpha, \beta, c, \delta)\) as the stable distribution with parameter \(\alpha, \beta, c, \delta\). Assuming \(x_1\) is drawn from \(S(X_1; \alpha_1, \beta_1, c_1, \delta_1)\) and \(x_2\) is an independent drawing from \(S(X_2; \alpha_2, \beta_2, c_2, \delta_2)\), then their sum
\[ X_3 = X_1 + X_2 \] is again a stable process with exponent \( \alpha \) and parameters \( \beta_3, c_3 \) and \( \delta_3 \), where

\[ c_3^\alpha = c_1^\alpha + c_2^\alpha \]  
\[ \beta_3 c_3^\alpha = \beta_1 c_1^\alpha + \beta_2 c_2^\alpha \]  
\[ \delta_3 = \delta_1 + \delta_2, \alpha \neq 1. \]

We note that when \( \beta_1 = \beta_2 \), (A.9) and (A.10) imply that \( \beta_3 = \beta_1 = \beta_2 \).

(3) a stable distribution is the only possible limiting distribution for the sum of iid random variables\(^{69}\).

For most stable distribution \( x \), the long upper tail makes its expected value \( E(e^x) \) infinite. However, when \( \beta = -1 \), that is, when the distribution is negatively skewed, \( E(e^x) \) is finite. Zolotarev (1983) shows that when \( \beta = -1 \)

\[ \log E(e^x) = \delta - \alpha c \sec(\pi \alpha/2), \alpha \neq 1. \]  

A.2 The Expected Value of An Option with a log-stable Process

We denote \( S \) as asset price that follows a log-stable process. This implies \( s = \log(S) \) is a stable random variable. The expected payoff on the asset is ordinarily infinite because \( E(S) = E[\exp(s)] = \infty \) unless the stable process has a parameter \( \beta = -1 \) or \( \alpha = 2 \). This led

\(^{69}\) The reader is referred to Gnedenko and Kolmogorov (1954, p. 162-163) for a proof.
Samuelson (1973, p. 15) to 'believe in Merton's conjecture that a strict
Levy-Pareto distribution on log(S_{t+1}/S_t) would lead, with 1<\alpha<2, to a
5-minute warrant or call being worth 100 percent of the common!'.
Merton's conjecture can be demonstrated as follows:

Because \( (S_t/S_t) \) is assumed to be a log-stable random variable,
\( \alpha=\ln(S_T/S_t) \) is a non-normal stable-Paretian variable and \( S_T/S_t=\exp(\alpha) \).
This implies that the expected value of \( S_T/S_t \) is

\[
E(S_T/S_t) = \int_0^\infty e^{\alpha} f(\alpha) d\alpha. \tag{A.13}
\]

In (A.13), \( f(\alpha) \) is the non-normal stable-Paretian density function.
We note that, in (A.13) the limit of integration is given as 0 rather
than \( -\infty \), because of the important phenomenon of limited liability for
the asset holder. Equation (A.13) can be written as the sum of two
terms separated by the integration point \( \alpha \):

\[
E(S_T/S_t) = \int_0^\alpha e^{\alpha} f(\alpha) d\alpha + \int_\alpha^\infty e^{\alpha} f(\alpha) d\alpha. \tag{A.14}
\]
Choose \( \alpha \) so that \( e^{\alpha} \) is greater than \( \alpha^2 \). Then the second integral of
(A.13) is larger than the same integral with \( e^{\alpha} \) replaced by \( \alpha^2 \), i.e.,

\[
\int_\alpha^\infty \alpha^2 f(\alpha) d\alpha < \int_\alpha^\infty e^{\alpha} f(\alpha) d\alpha. \tag{A.15}
\]

70. Smith (1976, p.15) states a similar demonstration to prove Merton's
assertion. The readers are cautioned that there are some notational
mistakes in Smith (1976)'s demonstration, which is credited to
Merton.
Because the stable-Paretian is not squared summable if $\beta \neq 1$ or $\alpha \neq 2$, $\int_{a}^{\infty} \alpha^2 f(\alpha) d\alpha$ is infinite. From (A.14), the last integration of (A.13) is infinite. Thus, the expected value of $(S_T/S_t)$ is infinite unless $\beta = -1$ or $\alpha = 2$.

We define $\rho$ as the positive drift over time for the expected asset return, i.e.,

$$E(S_T/S_t) = e^{\rho T}. \quad (A.16)$$

Because $E(S_T/S_t)$ is infinite, equation (A.16) implies that $\rho T$ is infinite. If $\rho T$ is infinite, then in equilibrium Merton conjectured that $rT$ would have to be infinite. This implies that $\exp(-rT) = 0$ and $\max[0, S-X\cdot\exp(-rT)] = \max[0, S-0] = S$. However, to eliminate arbitrage opportunities, the price of a call option must satisfies the following condition:

$$S > C(S, T; X) \geq \max[0, S-X\cdot\exp(-rT)],$$

$$\Rightarrow S \geq C \Rightarrow C = S.$$

This is why Samuelson states that 'a 5-minute warrant or call being worth 100 percent of the common'.

The fallacies of the above proof are: (1) If asset prices follow a stable process with $\alpha < 2$, the expected payoff for the asset must be infinite. (2) For an asset that has an infinite expected payoff, the

71. The reader is referred to section 4.2 or Merton (1973b) for a proof.
equilibrium interest rate must be infinite. To see Merton's conjecture is unfounded, we note that

(1) When $\beta=-1$, $E(S_T)$ is finite and is defined in (A.11) even if $\alpha<2$.

(2) A risky asset that has an infinite expected payoff will not lead to an infinite interest rate because the market price of the asset can still be finite. For example, in 1738 Daniel Bernoulli solved the most elementary type of "St. Petersburg Paradox" by assuming logarithmic utility.

Even Merton's conjecture is unfounded, to find a fair price for a call option under a log-stable process is a difficult, if not an impossible, task. For one thing, the expected value of a call option under a log-stable uncertainty is infinite. This can be proved easily as follows:

$$E[C(T)] = \int_X^\infty (S_T-X)f(\alpha)d\alpha,$$

$$= \int_X^\infty S_Tf(\alpha)d\alpha - \int_X^\infty xf(\alpha)d\alpha.$$

Because the first integral in (A.18) is infinite if $\beta\neq-1$ and the second integral is finite, the expected value of a call option ordinarily would be infinite. With an infinite expected value for a call option, there is no way we can form a riskless Black-Scholes type hedge portfolio. Thus, the Black-Scholes arbitrage-free option-pricing approach is not applicable here.
Appendix B. The Data Base for Foreign-Currency Option Prices

The source for foreign-currency spot prices and option prices is the transactions surveillance report compiled daily by the Philadelphia Stock Exchange (PHLX). This report summarizes exchange trades on the PHLX from February 28, 1983 to June 27, 1985. It reports the following information for each set of option trade: date and time of trade; currency; maturity; exercise price; number of contracts traded; option prices; and the actual spot bid, ask and trade prices. Each set of data covers five foreign-currencies listed on the exchange: British Pound (BP), Japanese Yen (JY), Deutsche Mark (DM), Swiss Franc (SF) and Canadian Dollars (CD). The number of observations for each currency options in this report are shown in the following table:

<table>
<thead>
<tr>
<th></th>
<th>BP</th>
<th>JY</th>
<th>SF</th>
<th>DM</th>
<th>CD</th>
</tr>
</thead>
<tbody>
<tr>
<td>Call Options</td>
<td>4109</td>
<td>3861</td>
<td>4944</td>
<td>5901</td>
<td>1415</td>
</tr>
<tr>
<td>Put Options</td>
<td>2441</td>
<td>1721</td>
<td>2541</td>
<td>3051</td>
<td>1426</td>
</tr>
</tbody>
</table>

The above table shows that, compared to other currencies, trading in the CD is relatively infrequent. Thus, this study investigates only the first four currencies.

For each currency option, we use the following procedures to construct our data base used for empirical studies in Chapter 7:

1. According to Barron’s (1987, p. 716), the expiration date for a foreign-currency option is the Saturday before the third Wednesday of
each expiration month. Using this information, together with the trading date and expiration month information from the PHLX reports, we can obtain the number of expiration days $\tau$ for each traded option. If $\tau$ is not positive, the trade information must contain some errors and the data is not used.

(2) In section 4.2, we state that because an American call can be exercised at any time before the expiration date, its price must be at least the difference between the asset's current price and the exercise price, i.e., its intrinsic value:

$$C(S, \tau; X) \geq \text{Max}(0, S(t) - X).$$  \hspace{1cm} (B.1)

We calculate the intrinsic value for each traded foreign-currency option. If the intrinsic value is less than the traded price for a foreign-currency option, the data is not used in Chapter 7. We need to drop this kind of data for two reasons. First, the modified Kassouf model uses a functional form that inherently requires restriction (B.1). To estimate an empirical pricing option using this functional form, a data that violates (B.1) cannot be used. Second, any option price that violates restriction (A.1) admits an arbitrage opportunity. Although we do not know the real reason why such an arbitrage opportunity can exist, we have a good reason to suspect that such data must either (a) contains some errors or (b) the reported option price is a rare case, i.e., an outlier. In either case the dropping of such data can be justified.
Figure 1 Frequency Distribution for PXL Foreign-currency Price Changes
Option Price

V1, V2, V3 = Values of options for successively shorter maturities

Figure 2 The General Stock-Option Relationship

Figure 3 Diagram of Black-Scholes Call Option Prices for Different Stock Prices
Option Price

$C = S$

$C = S^2 / 4X$

$C = \text{Max}(0, S-X)$

$X = 2X$

$S = X$

Figure 4 Guynemer Giguere's Stock-Option Relationship

Option Price

$C = \text{Max}(0, S-X)$

Shelton's equation

$C = (3/4)S$

$X = 4X$

Figure 5 Diagram of John Shelton's Call Option Prices for Different Stock Prices
Option Price

\[ c(z=1) = s \]

\[ c(z=\infty) = \text{Max}(0, s-1) \]

Kassouf's Stock-Option Relationship

\[ c = f(s, \tau, \cdots) \]

Figure 6 Sheen Kassouf's Stock-Option Relationship
### Table 1
Cumulative Frequency Distributions Around the Sample-Mean

<table>
<thead>
<tr>
<th>Foreign Currency</th>
<th>Interval</th>
<th>Unit Normal</th>
<th>British Pound</th>
<th>Japanese Yen</th>
<th>Swiss Franc</th>
<th>Deutsche Mark</th>
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</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>(1)</td>
<td>(2)</td>
<td>(3)</td>
<td>(4)</td>
<td>(5)</td>
</tr>
<tr>
<td></td>
<td>0.5S</td>
<td>1.0S</td>
<td>1.5S</td>
<td>2.0S</td>
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<td></td>
<td>0.3830</td>
<td>0.6826</td>
<td>0.8664</td>
<td>0.9545</td>
<td>0.9876</td>
<td>0.9973</td>
</tr>
<tr>
<td></td>
<td>0.4838</td>
<td>0.7529</td>
<td>0.8994</td>
<td>0.9608</td>
<td>0.9761</td>
<td>0.9812</td>
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<tr>
<td></td>
<td>0.4701</td>
<td>0.7478</td>
<td>0.8824</td>
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<tr>
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<td>0.6081</td>
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<td>0.9574</td>
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### Table 2
Test of Normality Using PHLX Daily Closing Prices

<table>
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<tr>
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<th>Japanese Yen</th>
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</tr>
</thead>
<tbody>
<tr>
<td>N</td>
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<td>587</td>
<td>587</td>
<td>587</td>
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<td>D:normal</td>
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<td>.061575</td>
<td>.12693</td>
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<td>Prob&gt;D</td>
<td>&lt;.01</td>
<td>&lt;.01</td>
<td>&lt;.01</td>
<td>&lt;.01</td>
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<tr>
<td>Range</td>
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<td>.25376</td>
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<tr>
<td>Std Dev</td>
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<td>.004816</td>
<td>.007250</td>
<td>.01027</td>
</tr>
<tr>
<td>Prob&gt;SR</td>
<td>&lt;.005</td>
<td>&lt;.005</td>
<td>&lt;.005</td>
<td>&lt;.005</td>
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<tr>
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<td>1.65970</td>
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<td>γ</td>
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<td>.002700</td>
<td>.004383</td>
<td>.004106</td>
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<td>-.000589</td>
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### Table 3
Test of First Serial Correlations

<table>
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</tr>
</thead>
<tbody>
<tr>
<td>N</td>
<td>587</td>
<td>587</td>
<td>587</td>
<td>587</td>
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<td>VNR Mean</td>
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<td>2.00341</td>
<td>2.00341</td>
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<td>VN Ratio</td>
<td>1.96981</td>
<td>2.09482</td>
<td>1.95051</td>
<td>2.43042</td>
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<td>VNR S.D.</td>
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<td>VNR Test</td>
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<tr>
<td>Autoregression Parameter</td>
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<td>-.04581</td>
<td>.02592</td>
<td>-.21319</td>
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<tr>
<td>Prob &gt; P.</td>
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<td>.2678</td>
<td>.5312</td>
<td>.0001</td>
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### Table 4
Pearson Correlation Coefficients / (Prob > |R| under H0: p=0)

<table>
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<th>Japanese Yen</th>
<th>Swiss Franc</th>
<th>Deutsche Mark</th>
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<td>X_{lt}</td>
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<td>X_{lt}</td>
<td>X_{lt}</td>
<td>X_{lt}</td>
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<tr>
<td>Lag 1</td>
<td>.01712</td>
<td>-.04581</td>
<td>.02592</td>
<td>-.21319</td>
</tr>
<tr>
<td></td>
<td>(.6792)</td>
<td>(.2678)</td>
<td>(.5312)</td>
<td>(.0001)*</td>
</tr>
<tr>
<td>Lag 2</td>
<td>.02918</td>
<td>-.04585</td>
<td>-.03665</td>
<td>-.05085</td>
</tr>
<tr>
<td></td>
<td>(.4811)</td>
<td>(.2678)</td>
<td>(.3763)</td>
<td>(.2194)</td>
</tr>
<tr>
<td>Lag 3</td>
<td>.03647</td>
<td>-.00733</td>
<td>.02573</td>
<td>.03977</td>
</tr>
<tr>
<td></td>
<td>(.3790)</td>
<td>(.8398)</td>
<td>(.5348)</td>
<td>(.3374)</td>
</tr>
<tr>
<td>Lag 4</td>
<td>-.07260</td>
<td>-.03665</td>
<td>-.03911</td>
<td>-.04174</td>
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<tr>
<td></td>
<td>(.0798)</td>
<td>(.3771)</td>
<td>(.3458)</td>
<td>(.3143)</td>
</tr>
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<td>Lag 5</td>
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<td>.03154</td>
<td>-.00920</td>
<td>.00749</td>
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<td></td>
<td>(.2542)</td>
<td>(.4475)</td>
<td>(.8247)</td>
<td>(.8569)</td>
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<td>Lag 6</td>
<td>.07504</td>
<td>.06771</td>
<td>.05094</td>
<td>.00827</td>
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<td></td>
<td>(.0707)</td>
<td>(.1030)</td>
<td>(.2202)</td>
<td>(.8424)</td>
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<td>Lag 7</td>
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<td>-.01878</td>
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<td>(.5017)</td>
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<td>(.6518)</td>
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<td>Lag 8</td>
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<td>(.5602)</td>
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<td>Lag 9</td>
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<td>.06433</td>
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<tr>
<td></td>
<td>(.3120)</td>
<td>(.9346)</td>
<td>(.1224)</td>
<td>(.5760)</td>
</tr>
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</table>

Note:
* indicates that t-value is significant at 1 percent level.
Table 5
Simulation Results of Variance-Time Function for Selected Differencing Interval

<table>
<thead>
<tr>
<th>Sample Size</th>
<th>587 normal variables with mean=0 and variance=2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 period Avg Ratio</td>
<td>1</td>
</tr>
<tr>
<td>(R-1.96s,R+1.96s)</td>
<td>(1,1)</td>
</tr>
<tr>
<td>2 periods Avg Ratio</td>
<td>1.99</td>
</tr>
<tr>
<td>(R-1.96s,R+1.96s)</td>
<td>(1.75,2.22)</td>
</tr>
<tr>
<td>3 periods Avg Ratio</td>
<td>2.99</td>
</tr>
<tr>
<td>(R-1.96s,R+1.96s)</td>
<td>(2.50,3.47)</td>
</tr>
<tr>
<td>4 periods Avg Ratio</td>
<td>3.98</td>
</tr>
<tr>
<td>(R-1.96s,R+1.96s)</td>
<td>(3.19,4.76)</td>
</tr>
<tr>
<td>5 periods Avg Ratio</td>
<td>4.99</td>
</tr>
<tr>
<td>(R-1.96s,R+1.96s)</td>
<td>(3.92,6.06)</td>
</tr>
<tr>
<td>6 periods Avg Ratio</td>
<td>6.01</td>
</tr>
<tr>
<td>(R-1.96s,R+1.96s)</td>
<td>(4.54,7.48)</td>
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<tr>
<td>7 periods Avg Ratio</td>
<td>7.00</td>
</tr>
<tr>
<td>(R-1.96s,R+1.96s)</td>
<td>(5.04,8.96)</td>
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<tr>
<td>8 periods Avg Ratio</td>
<td>7.98</td>
</tr>
<tr>
<td>(R-1.96s,R+1.96s)</td>
<td>(5.74,10.22)</td>
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<tr>
<td>9 periods Avg Ratio</td>
<td>8.99</td>
</tr>
<tr>
<td>(R-1.96s,R+1.96s)</td>
<td>(6.03,11.95)</td>
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<tr>
<td>10 periods Avg Ratio</td>
<td>10.02</td>
</tr>
<tr>
<td>(R-1.96s,R+1.96s)</td>
<td>(6.53,13.51)</td>
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<tr>
<td>12 periods Avg Ratio</td>
<td>12.07</td>
</tr>
<tr>
<td>(R-1.96s,R+1.96s)</td>
<td>(7.49,16.65)</td>
</tr>
<tr>
<td>14 periods Avg Ratio</td>
<td>14.09</td>
</tr>
<tr>
<td>(R-1.96s+R+1.96s)</td>
<td>(7.89,20.29)</td>
</tr>
<tr>
<td>16 periods Avg Ratio</td>
<td>16.08</td>
</tr>
<tr>
<td>(R-1.96s,R+1.96s)</td>
<td>(8.54,23.63)</td>
</tr>
<tr>
<td>18 periods Avg Ratio</td>
<td>18.15</td>
</tr>
<tr>
<td>(R-1.96s,R+1.96s)</td>
<td>(8.85,27.46)</td>
</tr>
<tr>
<td>20 periods Avg Ratio</td>
<td>20.04</td>
</tr>
<tr>
<td>(R-1.96s,R+1.96s)</td>
<td>(9.50,30.58)</td>
</tr>
<tr>
<td>Days</td>
<td>British Pound</td>
</tr>
<tr>
<td>------</td>
<td>--------------</td>
</tr>
<tr>
<td>1</td>
<td>.000064</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>195</td>
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<tr>
<td>4</td>
<td>146</td>
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<td>6</td>
<td>97</td>
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<tr>
<td>7</td>
<td>83</td>
</tr>
<tr>
<td>8</td>
<td>(73,72)</td>
</tr>
</tbody>
</table>

Avg Var(R) | .000130(2.041) | .000044(1.914) | .000108(2.057) | .000167(1.578) |
1st Var(R) | .000118(1.852) | .000041(1.786) | .000095(1.814) | .000094(0.886) |

Avg Var(R) | .000201(3.146) | .000064(2.768) | .000160(3.046) | .000217(2.058) |
1st Var(R) | .000194(3.034) | .000065(2.784) | .000168(3.200) | .000337(3.191) |

Avg Var(R) | .000227(4.332) | .000084(3.607) | .000215(4.094) | .000277(2.621) |
1st Var(R) | .000244(3.812) | .000078(3.832) | .000188(3.577) | .000190(1.800) |

Avg Var(R) | .000344(5.380) | .000101(4.367) | .000266(5.054) | .000327(3.094) |
1st Var(R) | .000405(6.332) | .000115(4.976) | .000306(5.821) | .000608(5.757) |

Avg Var(R) | .000405(6.334) | .000121(5.197) | .000316(6.012) | .000379(3.588) |
1st Var(R) | .000356(5.569) | .000092(3.978) | .000259(4.926) | .000272(2.574) |

Avg Var(R) | .000476(7.452) | .000144(6.190) | .000373(7.088) | .000434(4.108) |
1st Var(R) | .000513(8.028) | .000154(6.638) | .000415(7.893) | .000412(3.906) |

Avg Var(R) | .000547(8.564) | .000165(7.127) | .000428(8.133) | .000493(4.671) |
1st Var(R) | .000545(8.535) | .000160(6.896) | .000418(7.954) | .000393(3.720) |

- cont -
Table 6 (continued)

<table>
<thead>
<tr>
<th></th>
<th>British Pound</th>
<th>Japanese Yen</th>
<th>Swiss Franc</th>
<th>Deutsche Mark</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>N</strong></td>
<td>587</td>
<td>587</td>
<td>587</td>
<td>587</td>
</tr>
<tr>
<td><strong>Days</strong></td>
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<td>9</td>
<td>9</td>
<td>9</td>
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<tr>
<td><strong>Sample Size</strong></td>
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<td>(65,64)</td>
<td>(65,64)</td>
<td>(65,64)</td>
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<tr>
<td>Avg Var(R)</td>
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<td>.000188(8.122)</td>
<td>.000490(9.318)</td>
<td>.000557(5.278)</td>
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<td>1st Var(R)</td>
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<td>.000173(7.438)</td>
<td>.000488(9.278)</td>
<td>.000434(4.108)</td>
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<tr>
<td><strong>Days</strong></td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td><strong>Sample Size</strong></td>
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<td>(58,57)</td>
<td>(58,57)</td>
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<tr>
<td>Avg Var(R)</td>
<td>.000717(11.21)</td>
<td>.000212(9.121)</td>
<td>.000560(10.65)</td>
<td>.000626(5.933)</td>
</tr>
<tr>
<td>1st Var(R)</td>
<td>.000926(14.48)</td>
<td>.000240(10.36)</td>
<td>.000680(12.92)</td>
<td>.000653(6.188)</td>
</tr>
<tr>
<td><strong>Days</strong></td>
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<td>12</td>
<td>12</td>
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<td><strong>Sample Size</strong></td>
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<td>48</td>
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<tr>
<td>Avg Var(R)</td>
<td>.000866(13.54)</td>
<td>.000256(11.03)</td>
<td>.000673(12.79)</td>
<td>.000752(7.129)</td>
</tr>
<tr>
<td>1st Var(R)</td>
<td>.000680(10.64)</td>
<td>.000203(8.763)</td>
<td>.000572(10.87)</td>
<td>.000596(5.648)</td>
</tr>
<tr>
<td><strong>Days</strong></td>
<td>14</td>
<td>14</td>
<td>14</td>
<td>14</td>
</tr>
<tr>
<td><strong>Sample Size</strong></td>
<td>41</td>
<td>41</td>
<td>41</td>
<td>41</td>
</tr>
<tr>
<td>Avg Var(R)</td>
<td>.001002(15.68)</td>
<td>.000298(12.84)</td>
<td>.000774(14.72)</td>
<td>.000873(8.273)</td>
</tr>
<tr>
<td>1st Var(R)</td>
<td>.000712(11.15)</td>
<td>.000260(11.20)</td>
<td>.000558(10.60)</td>
<td>.000699(6.623)</td>
</tr>
<tr>
<td><strong>Days</strong></td>
<td>16</td>
<td>16</td>
<td>16</td>
<td>16</td>
</tr>
<tr>
<td><strong>Sample Size</strong></td>
<td>(36,35)</td>
<td>(36,35)</td>
<td>(36,35)</td>
<td>(36,35)</td>
</tr>
<tr>
<td>Avg Var(R)</td>
<td>.001162(18.18)</td>
<td>.000346(14.91)</td>
<td>.000900(17.11)</td>
<td>.000972(9.209)</td>
</tr>
<tr>
<td>1st Var(R)</td>
<td>.001048(16.41)</td>
<td>.000360(15.50)</td>
<td>.000759(14.44)</td>
<td>.000785(7.437)</td>
</tr>
<tr>
<td><strong>Days</strong></td>
<td>18</td>
<td>18</td>
<td>18</td>
<td>18</td>
</tr>
<tr>
<td><strong>Sample Size</strong></td>
<td>(32,31)</td>
<td>31</td>
<td>31</td>
<td>31</td>
</tr>
<tr>
<td>Avg Var(R)</td>
<td>.001315(20.58)</td>
<td>.000392(16.92)</td>
<td>.001015(19.30)</td>
<td>.001097(10.39)</td>
</tr>
<tr>
<td>1st Var(R)</td>
<td>.001374(21.50)</td>
<td>.000391(16.85)</td>
<td>.000960(18.26)</td>
<td>.000944(8.94)</td>
</tr>
<tr>
<td><strong>Days</strong></td>
<td>20</td>
<td>20</td>
<td>20</td>
<td>20</td>
</tr>
<tr>
<td><strong>Sample Size</strong></td>
<td>28</td>
<td>28</td>
<td>28</td>
<td>28</td>
</tr>
<tr>
<td>Avg Var(R)</td>
<td>.001459(22.83)</td>
<td>.000437(18.82)</td>
<td>.001117(21.25)</td>
<td>.001203(11.39)</td>
</tr>
<tr>
<td>1st Var(R)</td>
<td>.001406(22.00)</td>
<td>.000436(17.48)</td>
<td>.001071(20.36)</td>
<td>.001090(10.33)</td>
</tr>
</tbody>
</table>
Table 7
Parameter Estimates For Mixed Jump Processes

<table>
<thead>
<tr>
<th></th>
<th>British Pound</th>
<th>Japanese Yen</th>
<th>Swiss Franc</th>
<th>Deutsche Mark</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda(1)$</td>
<td>5.921957</td>
<td>2.588947</td>
<td>0.619594</td>
<td>0.032151</td>
</tr>
<tr>
<td></td>
<td>(2)</td>
<td>0.399370</td>
<td>0.648375</td>
<td>0.768091</td>
</tr>
<tr>
<td></td>
<td>(3)</td>
<td>0.278809</td>
<td>0.742999</td>
<td>0.330770</td>
</tr>
<tr>
<td>L.R. for (3)</td>
<td>(62.80)*</td>
<td>(965.87)*</td>
<td>(216.24)*</td>
<td>(845.58)*</td>
</tr>
<tr>
<td></td>
<td>(4)</td>
<td>0.094166</td>
<td>0.901909</td>
<td>0.218018</td>
</tr>
<tr>
<td>L.R. for (4)</td>
<td>(87.96)*</td>
<td>(59.60)*</td>
<td>(51.98)*</td>
<td>(403.58)*</td>
</tr>
<tr>
<td>(5)</td>
<td>0.084165</td>
<td>0.901909</td>
<td>0.231817</td>
<td>0.024817</td>
</tr>
</tbody>
</table>

For each estimate, the first row (1) provides parameter estimates for Poisson mixtures of normal processes as in (3.15) using the cumulant-matching method (3.17). The second row (2) provides the results for the Bernoulli mixtures as in (3.18) using the cumulant-matching method (3.24), while rows (3) and (4) gives maximum-likelihood estimates for the Bernoulli mixtures and the Poisson mixtures of normal processes, respectively. Row (5) gives the maximum-likelihood estimates for Poisson mixtures of normal processes with no restriction on the jump mean $\mu$. Parenthetical values under (3) and (4) in $\lambda$ give likelihood-ratio statistics.

* indicates that L.R. test is significant at 1% level.
Table 8
Regression Results for Econometric Model 1
Dependent Variable: ln c \textsubscript{t} in equation (7.10)

<table>
<thead>
<tr>
<th>VARIABLE</th>
<th>British Pound</th>
<th>Japanese Yen</th>
<th>Swiss Franc</th>
<th>Deutsche Mark</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intercept</td>
<td>-5.434(--51.26)</td>
<td>-5.551(--70.05)</td>
<td>-5.264(--61.95)</td>
<td>-5.295(--72.38)</td>
</tr>
<tr>
<td>ln s\textsubscript{t}</td>
<td>25.943( 26.99)</td>
<td>37.046( 23.44)</td>
<td>30.151( 31.50)</td>
<td>13.607( 25.58)</td>
</tr>
<tr>
<td>\tau</td>
<td>2.054( 47.19)</td>
<td>2.705( 62.08)</td>
<td>2.833( 62.26)</td>
<td>2.551( 64.46)</td>
</tr>
<tr>
<td>r\textsubscript{d}</td>
<td>11.378( 1.11)</td>
<td>119.696( 7.58)</td>
<td>20.452( 2.59)</td>
<td>-3.185( -0.43)</td>
</tr>
<tr>
<td>r\textsubscript{f}</td>
<td>-9.658( -0.94)</td>
<td>-116.37(-7.29)</td>
<td>-18.443(-2.32)</td>
<td>9.005( 1.21)</td>
</tr>
<tr>
<td>\sigma\textsubscript{t}</td>
<td>82.939( 23.03)</td>
<td>109.43(16.05)</td>
<td>69.641( 20.48)</td>
<td>42.270( 21.19)</td>
</tr>
<tr>
<td>\tau \cdot \ln s\textsubscript{t}</td>
<td>-0.063(-32.55)</td>
<td>-0.103(-33.74)</td>
<td>-0.088(-39.73)</td>
<td>-0.073(-39.09)</td>
</tr>
<tr>
<td>ln(S\textsubscript{t}/\bar{S})</td>
<td>-1.408( -4.41)</td>
<td>1.706( 3.39)</td>
<td>-1.089( -3.53)</td>
<td>-0.880( -3.35)</td>
</tr>
<tr>
<td>ln s\textsubscript{t-1}</td>
<td>0.342( 0.37)</td>
<td>-1.987( -1.30)</td>
<td>-0.387(-0.42)</td>
<td>13.087(27.84)</td>
</tr>
</tbody>
</table>

Observations 3907 3775 4880 5779
R-SQUARED 0.840 0.860 0.853 0.843
Dubin-Watson D 1.817 1.851 1.718 1.563

Table 9
Regression Results for Econometric Model 2
Dependent Variable: ln prem Equation (7.11)

<table>
<thead>
<tr>
<th>VARIABLE</th>
<th>British Pound</th>
<th>Japanese Yen</th>
<th>Swiss Franc</th>
<th>Deutsche Mark</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intercept</td>
<td>-5.217(--74.83)</td>
<td>-5.007(--85.72)</td>
<td>-4.440(97.27)</td>
<td>-4.588(-96.06)</td>
</tr>
<tr>
<td>ln s\textsubscript{t}</td>
<td>-20.073(-31.76)</td>
<td>-26.050(-22.36)</td>
<td>-17.639(-34.30)</td>
<td>-14.117(-40.65)</td>
</tr>
<tr>
<td>\tau</td>
<td>1.832( 63.99)</td>
<td>2.499( 77.78)</td>
<td>2.306( 94.35)</td>
<td>2.421( 93.69)</td>
</tr>
<tr>
<td>r\textsubscript{d}</td>
<td>-6.858( -1.02)</td>
<td>93.724( 8.05)</td>
<td>4.603( 1.08)</td>
<td>-20.361( -4.26)</td>
</tr>
<tr>
<td>r\textsubscript{f}</td>
<td>11.652( 1.72)</td>
<td>-91.223(-7.75)</td>
<td>-3.792(-0.88)</td>
<td>22.978( 4.76)</td>
</tr>
<tr>
<td>\sigma\textsubscript{t}</td>
<td>55.562( 23.46)</td>
<td>55.472( 11.04)</td>
<td>24.439( 13.38)</td>
<td>16.586( 12.73)</td>
</tr>
<tr>
<td>\tau \cdot \ln s\textsubscript{t}</td>
<td>0.054( 41.98)</td>
<td>0.098( 43.64)</td>
<td>0.062( 52.25)</td>
<td>0.068( 56.37)</td>
</tr>
<tr>
<td>ln(S\textsubscript{t}/\bar{S})</td>
<td>-0.527( -2.51)</td>
<td>0.799( 2.15)</td>
<td>-0.413(-2.49)</td>
<td>-0.140(-0.82)</td>
</tr>
<tr>
<td>ln s\textsubscript{t-1}</td>
<td>-1.080( -1.80)</td>
<td>-0.175(-0.15)</td>
<td>-0.814(-1.66)</td>
<td>-5.170(-16.84)</td>
</tr>
</tbody>
</table>

Observations 3907 3775 4880 5779
R-SQUARED 0.856 0.836 0.854 0.841
Dubin-Watson D 1.959 1.960 1.877 1.860
### Table 10
Regression Results for Econometric Model 3

<table>
<thead>
<tr>
<th>VARIABLE</th>
<th>PARAMETER (t)</th>
<th>PARAMETER (t)</th>
<th>PARAMETER (t)</th>
<th>PARAMETER (t)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>ESTIMATE</td>
<td>ESTIMATE</td>
<td>ESTIMATE</td>
<td>ESTIMATE</td>
</tr>
<tr>
<td>Intercept</td>
<td>4.777( 97.62)</td>
<td>4.710(111.83)</td>
<td>4.324(111.92)</td>
<td>4.254(122.10)</td>
</tr>
<tr>
<td>ln s_t</td>
<td>1.028( 6.88)</td>
<td>-0.019(-0.09)</td>
<td>0.190( 1.40)</td>
<td>3.132( 12.36)</td>
</tr>
<tr>
<td>(\tau)</td>
<td>-1.652(-82.22)</td>
<td>-2.236(-96.54)</td>
<td>-2.261(-109.2)</td>
<td>-2.099(-111.3)</td>
</tr>
<tr>
<td>(r_d)</td>
<td>-0.716(-0.15)</td>
<td>-80.457(-9.65)</td>
<td>-13.462(-3.77)</td>
<td>17.551( 5.03)</td>
</tr>
<tr>
<td>(r_f)</td>
<td>-2.810(-0.59)</td>
<td>77.130( 9.16)</td>
<td>12.234( 3.40)</td>
<td>-20.941(-5.95)</td>
</tr>
<tr>
<td>(\sigma^2)</td>
<td>-67.223(-40.81)</td>
<td>-71.889(-19.97)</td>
<td>-46.458(-30.21)</td>
<td>-23.897(-25.16)</td>
</tr>
<tr>
<td>(\tau\cdot\ln s_t)</td>
<td>-0.001(-1.28)</td>
<td>-0.008(-5.03)</td>
<td>-0.005(-5.59)</td>
<td>-0.007(-8.59)</td>
</tr>
<tr>
<td>ln s_t-1</td>
<td>-2.634(-11.76)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>ln(S_t/S)</td>
<td>0.738( 5.21)</td>
<td>-1.370(-5.29)</td>
<td>0.571( 4.19)</td>
<td>-0.062(-0.50)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Observations</th>
<th>3907</th>
<th>3775</th>
<th>4880</th>
<th>5779</th>
</tr>
</thead>
<tbody>
<tr>
<td>R-SQUARE</td>
<td>0.767</td>
<td>0.778</td>
<td>0.781</td>
<td>0.741</td>
</tr>
<tr>
<td>Dubin-Watson D</td>
<td>1.731</td>
<td>1.754</td>
<td>1.676</td>
<td>1.456</td>
</tr>
</tbody>
</table>

### Table 11
Significance Points of \(d_1\) and \(d_u\) for Durbin-Watson D

<table>
<thead>
<tr>
<th></th>
<th>(d_1)</th>
<th>(d_u)</th>
<th>4-(d_u)</th>
<th>4-(d_1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5%</td>
<td>1.57</td>
<td>1.78</td>
<td>2.22</td>
<td>2.43</td>
</tr>
<tr>
<td>2.5%</td>
<td>1.51</td>
<td>1.72</td>
<td>2.28</td>
<td>2.49</td>
</tr>
<tr>
<td>1%</td>
<td>1.44</td>
<td>1.65</td>
<td>2.35</td>
<td>2.56</td>
</tr>
</tbody>
</table>
Table 12
Summary of Pricing Errors and Implied $R^2$ for British Pound:

|                      | Mean of $|\hat{c} - c|$ | Mean of $(\hat{c} - c)$ | Sum of $(\hat{c} - c)^2$ | $R^2$ |
|----------------------|--------------------------|-------------------------|---------------------------|-------|
| Loglinear Model      | 1.633(0.526)             | 0.571(0.219)            | 155336.07                 | *****|
| Premium Model        | 1.016(3.855)             | 0.249(3.318)            | 27730.08                  | 0.452 |
| Modified Kassouf     | 0.501(0.366)             | -0.013(0.166)           | 2385.45                   | 0.952 |
| BS Future            | 0.523(0.303)             | -0.190(-.110)           | 1968.74                   | 0.961 |
| (1) $\lambda=0$, BS Spot | 0.614(0.294)             | -0.055(-.098)           | 3048.76                   | 0.938 |
| (2) $\lambda=0.39$, $\mu=0$ | 1.805(0.712)             | -1.403(0.059)           | 31994.61                  | 0.368 |
| (3) $\lambda=0.27$, $\mu=0$ | 0.904(0.452)             | -0.503(0.108)           | 6956.55                   | 0.862 |
| (4) $\lambda=0.09$, $\mu=0$ | 0.826(0.377)             | -0.022(-.022)           | 5728.29                   | 0.886 |
| (5) $\lambda=0.08$, $\mu=0$ | 3.166(0.826)             | -3.156(-.770)           | 82381.38                  | *****|

Note:
1. The number of observations is 3907. Prices are quoted in cents per BP.
2. The entries in parentheses denote relative pricing errors; ***** in $R^2$ denotes that sum of $(\hat{c} - c)^2$ is greater than sum of $(c - \hat{c})^2$.
3. Parameters estimates for jump-option pricing formula in row (2)-(5) are taken from (2)-(5) of Table 7.

Table 13
Summary of Forecasting Errors and Implied $R^2$ for Japanese Yen:

|                      | Mean of $|\hat{c} - c|$ | Mean of $(\hat{c} - c)$ | Sum of $(\hat{c} - c)^2$ | $R^2$ |
|----------------------|--------------------------|-------------------------|---------------------------|-------|
| Loglinear Model      | 0.289(0.426)             | 0.076(0.172)            | 3098.07                   | *****|
| Premium Model        | 0.228(2.497)             | 0.028(2.060)            | 794.14                    | 0.698 |
| Modified Kassouf     | 0.146(0.349)             | 0.008(0.149)            | 199.62                    | 0.924 |
| BS Future            | 0.220(0.347)             | -0.209(-.530)           | 293.72                    | 0.888 |
| (1) $\lambda=0$, BS Spot | 0.424(0.368)             | -0.423(-.347)           | 1022.28                   | 0.611 |
| (2) $\lambda=0.64$, $\mu=0$ | 0.627(0.607)             | -0.618(-.505)           | 2931.57                   | 0.577 |
| (3) $\lambda=0.74$, $\mu=0$ | 0.135(0.369)             | 0.046(0.302)            | 108.10                    | 0.958 |
| (4) $\lambda=0.90$, $\mu=0$ | 0.748(0.773)             | -0.748(-.764)           | 3652.41                   | *****|
| (5) $\lambda=0.90$, $\mu=0$ | 0.748(0.773)             | -0.748(-.764)           | 3652.39                   | *****|

Note:
1. The number of observations is 3775. Prices are quoted in 1/100 cent per JY.
2. The entries in parentheses denote relative pricing errors; ***** in $R^2$ denotes that sum of $(\hat{c} - c)^2$ is greater than sum of $(c - \hat{c})^2$.
3. Parameters estimates for jump-option pricing formula in row (2)-(5) are taken from (2)-(5) of Table 7.
Table 14
Summary of Forecasting Errors and Implied $R^2$ for Swiss Franc:

|                  | Mean of $|\hat{c} - c|$ | Mean of $(\hat{c} - c)$ | Sum of $(\hat{c} - c)^2$ | $R^2$ |
|------------------|-------------------------|--------------------------|--------------------------|-------|
| Loglinear        | 0.383(0.527)            | 0.095(0.217)             | 6357.95                  | *****|
| Premium Model    | 0.308(5.622)            | 0.072(5.105)             | 3626.50                  | 0.226 |
| Modified Kassouf | 0.190(0.415)            | 0.005(0.192)             | 579.96                   | 0.876 |
| BS Future        | 0.175(0.300)            | -0.064(-.172)            | 289.28                   | 0.938 |
| (1)$\lambda=0, BS Spot$ | 0.343(0.527)            | -0.319(-.422)            | 1122.99                  | 0.760 |
| (2)$\lambda=0.76, \mu=0$ | 0.486(0.496)            | -0.481(-.449)            | 2442.88                  | 0.478 |
| (3)$\lambda=0.33, \mu=0$ | 0.392(0.429)            | -0.386(-.376)            | 1516.77                  | 0.676 |
| (4)$\lambda=0.21, \mu=0$ | 0.505(0.531)            | -0.503(-.508)            | 2473.92                  | 0.472 |
| (5)$\lambda=0.23, \mu=0.0035$ | 1.010(0.948)           | -1.010(-.942)            | 9549.53                  | ****  |

Note:
1. The number of observations is 4880. Prices are quoted in cents per SF.
2. The entries in parentheses denote relative pricing errors; **** in $R^2$ denotes that sum of $(\hat{c} - c)^2$ is greater than sum of $(c - \hat{c})^2$.
3. Parameters estimates for jump-option pricing formula in row (2)-(5) are taken from (2)-(5) of Table 7.

Table 15
Summary of Forecasting Errors and Implied $R^2$ for Deutsche Mark:
Econometric Models vs. Arbitrage Models

|                  | Mean of $|\hat{c} - c|$ | Mean of $(\hat{c} - c)$ | Sum of $(\hat{c} - c)^2$ | $R^2$ |
|------------------|-------------------------|--------------------------|--------------------------|-------|
| Loglinear        | 0.292(0.507)            | 0.029(0.213)             | 2173.68                  | 0.535 |
| Premium Model    | 0.319(5.246)            | 0.072(4.702)             | 4901.98                  | *****|
| Modified Kassouf | 0.173(0.424)            | 0.006(0.199)             | 401.37                   | 0.914 |
| BS Future        | 0.200(0.388)            | -0.069(-.182)            | 866.67                   | 0.814 |
| (1)$\lambda=0, BS Spot$ | 0.341(0.521)            | -0.260(-.362)            | 1432.01                  | 0.693 |
| (2)$\lambda=0.02, \mu=0$ | 0.265(0.878)            | 0.220(0.857)             | 636.50                   | 0.863 |
| (3)$\lambda=0.76, \mu=0$ | 0.164(0.338)            | -0.048(0.102)            | 279.35                   | 0.940 |
| (4)$\lambda=0.02, \mu=0$ | 0.194(0.334)            | -0.123(0.051)            | 437.63                   | 0.906 |
| (5)$\lambda=0.02, \mu=0$ | 0.194(0.334)            | -0.123(0.051)            | 437.63                   | 0.906 |

Note:
1. The number of observations is 5779. Prices are quoted in cents per DM.
2. The entries in parentheses denote relative pricing errors; **** in $R^2$ denotes that sum of $(c - \hat{c})^2$ is greater than sum of $(c - \hat{c})^2$.
3. Parameters estimates for jump-option pricing formula in row (2)-(5) are taken from (2)-(5) of Table 7.
### Table 16
**Davidson and Mackinnon Pairwise Tests for \( H_1 \) through \( H_3: \) BP**

<table>
<thead>
<tr>
<th>Alternative Hypothesis</th>
<th>( H_1 )</th>
<th>( H_2 )</th>
<th>( H_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maintained Hypothesis: ( H_1 )</td>
<td>0.8542</td>
<td>1.0053</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(146.55)</td>
<td>(500.90)</td>
<td></td>
</tr>
<tr>
<td>( H_2 )</td>
<td>0.1457</td>
<td></td>
<td>0.9894</td>
</tr>
<tr>
<td></td>
<td>(25.01)</td>
<td></td>
<td>(203.85)</td>
</tr>
<tr>
<td>( H_3 )</td>
<td>-0.0053</td>
<td>0.0106</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(-2.66)</td>
<td></td>
<td>(2.18)</td>
</tr>
</tbody>
</table>

**Note:**
1. \( H_1 \): the loglinear model (7.10), \( H_2 \): the stepwise regression model (7.11), \( H_3 \): the modified Kassouf model (7.12).
2. The first element in each off-diagonal entry is the value of coefficient in regression (7.23), the ordinary t statistic is followed in the parenthesis.

### Table 17
**Davidson and Mackinnon Pairwise Tests for \( H_1 \) through \( H_3: \) JY**

<table>
<thead>
<tr>
<th>Alternative Hypothesis</th>
<th>( H_1 )</th>
<th>( H_2 )</th>
<th>( H_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maintained Hypothesis: ( H_1 )</td>
<td>0.8234</td>
<td>1.0077</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(115.50)</td>
<td>(234.19)</td>
<td></td>
</tr>
<tr>
<td>( H_2 )</td>
<td>0.1765</td>
<td></td>
<td>0.9758</td>
</tr>
<tr>
<td></td>
<td>(24.76)</td>
<td></td>
<td>(106.14)</td>
</tr>
<tr>
<td>( H_3 )</td>
<td>-0.0077</td>
<td>0.0241</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(-1.80)</td>
<td></td>
<td>(2.63)</td>
</tr>
</tbody>
</table>

**Note:**
1. \( H_1 \): the loglinear model (7.10), \( H_2 \): the stepwise regression model (7.11), \( H_3 \): the modified Kassouf model (7.12).
2. The first element in each off-diagonal entry is the value of coefficient in regression (7.23), the ordinary t statistic is followed in the parenthesis.
### Table 18
Davidson and Mackinnon Pairwise Tests for $H_1$ through $H_3$: SF

<table>
<thead>
<tr>
<th>Alternative Hypothesis</th>
<th>$H_1$</th>
<th>$H_2$</th>
<th>$H_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maintain Hypothesis: $H_1$</td>
<td>0.6514 (85.88)</td>
<td>1.0324 (221.67)</td>
<td></td>
</tr>
<tr>
<td>$H_2$</td>
<td>0.3485 (45.95)</td>
<td>0.9633 (160.82)</td>
<td></td>
</tr>
<tr>
<td>$H_3$</td>
<td>-0.0324 (-6.96)</td>
<td>0.0366 (6.12)</td>
<td></td>
</tr>
</tbody>
</table>

Note:
1. $H_1$: the loglinear model (7.10), $H_2$: the stepwise regression model (7.11), $H_3$: the modified Kassouf model (7.12).
2. The first element in each off-diagonal entry is the value of coefficient in regression (7.23), the ordinary $t$ statistic is followed in the parenthesis.

### Table 19
Davidson and Mackinnon Pairwise Tests for $H_1$ through $H_3$: DM

<table>
<thead>
<tr>
<th>Alternative Hypothesis</th>
<th>$H_1$</th>
<th>$H_2$</th>
<th>$H_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maintain Hypothesis: $H_1$</td>
<td>0.2895 (43.89)</td>
<td>1.0477 (160.63)</td>
<td></td>
</tr>
<tr>
<td>$H_2$</td>
<td>0.7104 (107.69)</td>
<td>0.9860 (254.84)</td>
<td></td>
</tr>
<tr>
<td>$H_3$</td>
<td>-0.0477 (-7.31)</td>
<td>0.0139 (3.59)</td>
<td></td>
</tr>
</tbody>
</table>

Note:
1. $H_1$: the loglinear model (7.10), $H_2$: the stepwise regression model (7.11), $H_3$: the modified Kassouf model (7.12).
2. The first element in each off-diagonal entry is the value of coefficient in regression (7.23), the ordinary $t$ statistic is followed in the parenthesis.
### Table 20

Summary of Pricing Errors for the British Pound According to In, At and Out of the Money and to Maturity Classes

<table>
<thead>
<tr>
<th>Maturity:</th>
<th>Out of the Money</th>
<th>At the Money</th>
<th>In the Money</th>
</tr>
</thead>
<tbody>
<tr>
<td>Less than 30 Days</td>
<td>#=114</td>
<td>160</td>
<td>51</td>
</tr>
<tr>
<td>Mean of (c-c)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1. BS Spot</td>
<td>-.068(-.365)</td>
<td>-.042(-.025)</td>
<td>.009 (.002)</td>
</tr>
<tr>
<td>2. BS Future</td>
<td>-.071(-.364)</td>
<td>-.069 (.049)</td>
<td>-.090 (-.019)</td>
</tr>
<tr>
<td>3. Mixed Jump(3)</td>
<td>-.042 (.304)</td>
<td>.183 (.417)</td>
<td>-.015 (.008)</td>
</tr>
<tr>
<td>4. Mixed Jump(4)</td>
<td>-.114(-.053)</td>
<td>.023 (.216)</td>
<td>-.069(-.004)</td>
</tr>
<tr>
<td>5. Modified Kassouf</td>
<td>-.047 (.209)</td>
<td>-.026 (.144)</td>
<td>-.020(-.002)</td>
</tr>
<tr>
<td>Sum of (c-c)</td>
<td>26.371</td>
<td>14.658</td>
<td>3.110</td>
</tr>
</tbody>
</table>

| Between 31690 Days | #=540 | 331 | 209 |
| Mean of (c-c) | | | |
| 1. BS Spot | -.023(-.206) | .007 (.013) | .157 (.017) |
| 2. BS Future | -.042(-.216) | -.011 (.019) | -.128(-.014) |
| 3. Mixed Jump(3) | .067 (.559) | .377 (.306) | -.353(-.034) |
| 4. Mixed Jump(4) | -.096 (.166) | .108 (.172) | -.449(-.054) |
| 5. Modified Kassouf | -.029 (.111) | -.022 (.026) | -.044(-.007) |
| Sum of (c-c) | 89.119 | 87.775 | 76.069 |

Note: 1. The entries in parentheses denote relative pricing errors.
2. Parameters estimates for mixed jump-option price (3), (4) are taken from row (3), (4) of Table 14.
Table 20 (continued)

Summary of Pricing Errors for the British Pound
According to In, At and Out of the Money and to Maturity Classes

<table>
<thead>
<tr>
<th>Maturity:</th>
<th>Out of the Money</th>
<th>At the Money</th>
<th>In the Money</th>
</tr>
</thead>
<tbody>
<tr>
<td>Between 91 &amp; 180 Days</td>
<td># = 760</td>
<td>409</td>
<td>261</td>
</tr>
<tr>
<td>Mean of $c$</td>
<td>S.D. of $c$</td>
<td>Mean of $</td>
<td>c-c</td>
</tr>
<tr>
<td>1. BS Spot</td>
<td>0.446 (.371)</td>
<td>0.673 (.173)</td>
<td>.939 (.111)</td>
</tr>
<tr>
<td>2. BS Future</td>
<td>0.392 (.348)</td>
<td>0.602 (.161)</td>
<td>.652 (.080)</td>
</tr>
<tr>
<td>3. Mixed Jump(3)</td>
<td>0.570 (.513)</td>
<td>0.819 (.224)</td>
<td>1.193 (.341)</td>
</tr>
<tr>
<td>4. Mixed Jump(4)</td>
<td>0.551 (.426)</td>
<td>0.815 (.210)</td>
<td>1.131 (.126)</td>
</tr>
<tr>
<td>5. Modified Kassouf</td>
<td>0.258 (.245)</td>
<td>0.393 (.111)</td>
<td>0.250 (.032)</td>
</tr>
</tbody>
</table>

| More than 180 Days | # = 509 | 322 | 241 |
| Mean of $c$ | S.D. of $c$ | Mean of $|c-c|$ | Mean of $(c-c)$ | Sum of $(c-c)^2$ |
| 1. BS Spot | 0.806 (.338) | 1.172 (.467) | 1.307 (.131) | 2.596 (.146) | 5.478 (.143) | 10.507 (.371) |
| 2. BS Future | 0.730 (.329) | 0.977 (.451) | 0.916 (.098) | 1.313 (.329) | 1.977 (.451) | 3.290 (.098) |
| 3. Mixed Jump(3) | 0.971 (.341) | 1.740 (.482) | 2.856 (.272) | 2.568 (.341) | 4.578 (.482) | 7.146 (.272) |
| 4. Mixed Jump(4) | 0.965 (.337) | 1.569 (.460) | 2.294 (.222) | 2.568 (.337) | 4.578 (.460) | 7.146 (.222) |
| 5. Modified Kassouf | 0.370 (.169) | 0.490 (.372) | 0.335 (.037) | 0.335 (.169) | 0.490 (.372) | 0.335 (.037) |

Note: 1. The entries in parentheses denote relative pricing errors.
2. Parameters estimates for mixed jump-option prices (3) and (4) are taken from row (3) and (4) of Table 14.
Table 21

Summary of Pricing Errors for the Japanese Yen

<table>
<thead>
<tr>
<th>Maturity:</th>
<th>Out of the Money</th>
<th>At the Money</th>
<th>In the Money</th>
</tr>
</thead>
<tbody>
<tr>
<td>Less than 30 Days</td>
<td>4=129</td>
<td>178</td>
<td>75</td>
</tr>
</tbody>
</table>


Mean of $|c-c|$ |
1. BS Spot | .042 (.847) | .090 (.455) | .088 (.059) |
2. BS Future | .037 (.794) | .073 (.341) | .134 (.075) |
3. Mixed Jump(3) | .062 (1.754) | .127 (.824) | .068 (.048) |
4. Mixed Jump(4) | .041 (.809) | .092 (.374) | .106 (.071) |
5. Modified Kassouf | .017 (.446) | .044 (.237) | .032 (.020) |

Mean of $(c-c)$ |
1. BS Spot | -.042 (-.847) | -.085 (-.157) | -.088 (-.059) |
2. BS Future | -.037 (-.793) | -.050 (-.140) | -.087 (-.054) |
3. Mixed Jump(3) | .057 (1.567) | .119 (.784) | .003 (.019) |
4. Mixed Jump(4) | -.041 (-.789) | -.090 (-.217) | -.106 (-.071) |
5. Modified Kassouf | -.004 (.153) | -.001 (.078) | -.005 (.002) |

Sum of $(c-c)$ |
1. BS Spot | .436 | 2.401 | .889 |
2. BS Future | .327 | 1.981 | 4.745 |
3. Mixed Jump(3) | 1.015 | 3.932 | .550 |
4. Mixed Jump(4) | .460 | 2.744 | 1.251 |
5. Modified Kassouf | .083 | .755 | .148 |

Between 31 & 90 Days | 550 | 409 | 179 |

Mean of $c$ [S.D. of $c$] | .224 [.164] | .817 [.320] | 2.116 [.763] |

Mean of $|c-c|$ |
1. BS Spot | .145 (.740) | .285 (.385) | .311 (.159) |
2. BS Future | .115 (.635) | .160 (.235) | .123 (.066) |
3. Mixed Jump(3) | .108 (.757) | .155 (.267) | .107 (.050) |
4. Mixed Jump(4) | .188 (.854) | .446 (.547) | .603 (.301) |
5. Modified Kassouf | .044 (.252) | .077 (.112) | .057 (.030) |

Mean of $(c-c)$ |
1. BS Spot | -.145 (-.740) | -.285 (-.385) | -.311 (-.159) |
2. BS Future | -.111 (-.624) | -.146 (-.216) | -.104 (-.057) |
3. Mixed Jump(3) | .088 (.694) | .122 (.238) | -.045 (-.008) |
4. Mixed Jump(4) | -.188 (-.852) | -.446 (-.547) | -.603 (-.301) |
5. Modified Kassouf | -.003 (.060) | -.004 (.011) | -.004 (-.001) |

Sum of $(c-c)^2$ |
1. BS Spot | 16.927 | 39.646 | 19.759 |
2. BS Future | 10.483 | 14.231 | 4.036 |
4. Mixed Jump(3) | 30.082 | 99.950 | 73.941 |
5. Modified Kassouf | 2.109 | 3.764 | .980 |

Note: 1. The entries in parentheses denote relative pricing errors.
2. Parameters estimates for mixed jump-option prices (3) and (4) are taken from row (2) and (3) of Table 14.
Table 21 (continued)

Summary of Pricing Errors for the Japanese Yen

<table>
<thead>
<tr>
<th>Maturity:</th>
<th>Out of the Money</th>
<th>At the Money</th>
<th>In the Money</th>
</tr>
</thead>
<tbody>
<tr>
<td>Between 91&amp;180 Days</td>
<td>746</td>
<td>533</td>
<td>203</td>
</tr>
<tr>
<td>Mean of c [S.D. of c]</td>
<td>0.545 [0.302]</td>
<td>1.311 [0.351]</td>
<td>2.541 [0.770]</td>
</tr>
<tr>
<td>Mean of $</td>
<td>c-c</td>
<td>$</td>
<td></td>
</tr>
<tr>
<td>1. BS Spot</td>
<td>0.363 (.721)</td>
<td>0.568 (.454)</td>
<td>0.624 (.259)</td>
</tr>
<tr>
<td>2. BS Future</td>
<td>0.243 (.523)</td>
<td>0.270 (.232)</td>
<td>0.210 (.091)</td>
</tr>
<tr>
<td>3. Mixed Jump(3)</td>
<td>0.135 (.386)</td>
<td>0.130 (.120)</td>
<td>0.175 (.065)</td>
</tr>
<tr>
<td>4. Mixed Jump(4)</td>
<td>0.505 (.932)</td>
<td>1.053 (.799)</td>
<td>1.535 (.620)</td>
</tr>
<tr>
<td>5. Modified Kassouf</td>
<td>0.061 (.025)</td>
<td>0.082 (.074)</td>
<td>0.081 (.034)</td>
</tr>
<tr>
<td>Mean of $(c-c)$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1. BS Spot</td>
<td>-0.363 (-.721)</td>
<td>-0.564 (-.436)</td>
<td>-0.624 (-.259)</td>
</tr>
<tr>
<td>2. BS Future</td>
<td>-0.243 (-.520)</td>
<td>-0.257 (-.202)</td>
<td>-0.173 (-.076)</td>
</tr>
<tr>
<td>3. Mixed Jump</td>
<td>0.108 (.353)</td>
<td>0.056 (.075)</td>
<td>-0.130 (-.039)</td>
</tr>
<tr>
<td>4. Mixed Jump(4)</td>
<td>-0.505 (-.932)</td>
<td>-1.053 (-.797)</td>
<td>-1.535 (-.620)</td>
</tr>
<tr>
<td>5. Modified Kassouf</td>
<td>-0.005 (.025)</td>
<td>-0.005 (.008)</td>
<td>-0.002 (-.000)</td>
</tr>
<tr>
<td>Sum of $(c-c)$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1. BS Spot</td>
<td>123.473</td>
<td>188.153</td>
<td>85.463</td>
</tr>
<tr>
<td>2. BS Future</td>
<td>57.068</td>
<td>51.442</td>
<td>15.336</td>
</tr>
<tr>
<td>4. Mixed Jump(4)</td>
<td>250.574</td>
<td>648.699</td>
<td>528.712</td>
</tr>
<tr>
<td>5. Modified Kassouf</td>
<td>5.561</td>
<td>6.591</td>
<td>2.113</td>
</tr>
<tr>
<td>More than 180 Days</td>
<td>369</td>
<td>303</td>
<td>101</td>
</tr>
<tr>
<td>Mean of c [S.D. of c]</td>
<td>1.020 (.362)</td>
<td>1.815 (.350)</td>
<td>2.758 (.455)</td>
</tr>
<tr>
<td>Mean of $</td>
<td>c-c</td>
<td>$</td>
<td></td>
</tr>
<tr>
<td>1. BS Spot</td>
<td>0.680 (.694)</td>
<td>0.897 (.501)</td>
<td>0.948 (.349)</td>
</tr>
<tr>
<td>2. BS Future</td>
<td>0.404 (.429)</td>
<td>0.359 (.210)</td>
<td>0.248 (.093)</td>
</tr>
<tr>
<td>3. Mixed Jump(3)</td>
<td>0.147 (.179)</td>
<td>0.165 (.089)</td>
<td>0.226 (.078)</td>
</tr>
<tr>
<td>4. Mixed Jump(4)</td>
<td>1.006 (.985)</td>
<td>1.748 (.962)</td>
<td>2.470 (.898)</td>
</tr>
<tr>
<td>5. Modified Kassouf</td>
<td>0.077 (.079)</td>
<td>0.087 (.049)</td>
<td>0.090 (.033)</td>
</tr>
<tr>
<td>Mean of $(c-c)$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1. BS Spot</td>
<td>-0.680 (-.694)</td>
<td>-0.897 (-.501)</td>
<td>-0.948 (-.349)</td>
</tr>
<tr>
<td>2. BS Future</td>
<td>-0.399 (-.424)</td>
<td>-0.348 (-.204)</td>
<td>-0.230 (-.087)</td>
</tr>
<tr>
<td>3. Mixed Jump(3)</td>
<td>0.059 (.118)</td>
<td>-0.090 (-.036)</td>
<td>-0.211 (-.071)</td>
</tr>
<tr>
<td>4. Mixed Jump(4)</td>
<td>-1.006 (-.985)</td>
<td>-1.748 (-.962)</td>
<td>-2.470 (-.898)</td>
</tr>
<tr>
<td>5. Modified Kassouf</td>
<td>-0.002 (.008)</td>
<td>-0.000 (.003)</td>
<td>-0.000 (.001)</td>
</tr>
<tr>
<td>Sum of $(c-c)$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1. BS Spot</td>
<td>188.695</td>
<td>260.101</td>
<td>93.335</td>
</tr>
<tr>
<td>2. BS Future</td>
<td>72.047</td>
<td>52.408</td>
<td>9.618</td>
</tr>
<tr>
<td>4. Mixed Jump(4)</td>
<td>363.786</td>
<td>962.837</td>
<td>631.711</td>
</tr>
</tbody>
</table>

Note: 1. The entries in parentheses denote relative pricing errors.
2. Parameters estimates for mixed jump-option prices (3) and (4) are taken from row (2) and (3) of Table 14.
Table 22
Summary of Pricing Errors for the Swiss Franc
According to In, At and Out of the Money and to Maturity Classes

<table>
<thead>
<tr>
<th>Maturity:</th>
<th>Out of the Money</th>
<th>At the Money</th>
<th>In the Money</th>
</tr>
</thead>
<tbody>
<tr>
<td>Less than 30 Days</td>
<td># = 236</td>
<td>218</td>
<td>71</td>
</tr>
</tbody>
</table>

| | | | |
| Mean of | c-c | | |
| 1. BS Spot | .039 (.729) | .087 (.262) | .101 (.063) |
| 2. BS Future | .035 (.689) | .085 (.264) | .205 (.121) |
| 3. Mixed Jump(3) | .039 (.698) | .090 (.315) | .117 (.071) |
| 4. Mixed Jump(4) | .043 (.706) | .098 (.304) | .131 (.080) |
| 5. Modified Kassouf | .022 (.471) | .062 (.240) | .035 (.022) |

| | | | |
| Mean of | | c-c | | |
| 1. BS Spot | -.026 (-.569) | -.062 (-.150) | -.077 (-.044) |
| 2. BS Future | -.019 (-.491) | -.019 (-.048) | -.031 (-.015) |
| 3. Mixed Jump(3) | -.033 (.535) | -.036 (.006) | -.117 (-.071) |
| 4. Mixed Jump(4) | -.041 (-.628) | -.065 (-.088) | -.131 (-.080) |
| 5. Modified Kassouf | -.005 (.193) | .000 (.067) | -.003 (-.001) |

| Sum of (c-c)^2 | | | |
| 1. BS Spot | .624 | 2.434 | 1.173 |
| 2. BS Future | .517 | 2.954 | 5.425 |
| 3. Mixed Jump(3) | .830 | 2.894 | 1.463 |
| 4. Mixed Jump(4) | 1.041 | 3.531 | 1.797 |
| 5. Modified Kassouf | .249 | 1.559 | .159 |

| | | | |
| Between 31 & 90 Days | # = 956 | 386 | 182 |
| Mean of c [S.D. of c] | .253 [.225] | .995 [.358] | 2.545 [.926] |

| | | | |
| Mean of | c-c | | |
| 1. BS Spot | .117 (.606) | .287 (.317) | .220 (.100) |
| 2. BS Future | .079 (.469) | .137 (.154) | .151 (.067) |
| 3. Mixed Jump(3) | .111 (.510) | .237 (.221) | .362 (.156) |
| 4. Mixed Jump(4) | .133 (.568) | .313 (.298) | .432 (.188) |
| 5. Modified Kassouf | .045 (.304) | .060 (.067) | .050 (.021) |

| | | | |
| Mean of | (c-c) | | |
| 1. BS Spot | -.098 (-.522) | -.260 (-.289) | -.191 (-.085) |
| 2. BS Future | -.041 (-.318) | -.057 (-.080) | .061 (.025) |
| 3. Mixed Jump(3) | -.098 (-.375) | -.233 (-.212) | -.362 (-.156) |
| 4. Mixed Jump(4) | -.130 (-.526) | -.312 (-.297) | -.432 (-.188) |
| 5. Modified Kassouf | -.006 (.098) | -.002 (.003) | -.003 (-.002) |

| Sum of (c-c)^2 | | | |
| 1. BS Spot | 22.545 | 39.490 | 14.985 |
| 2. BS Future | 12.252 | 11.301 | 6.674 |
| 3. Mixed Jump(3) | 27.223 | 33.349 | 26.898 |
| 5. Modified Kassouf | 5.727 | 2.299 | .809 |

Note: 1. The entries in parentheses denote relative pricing errors.
2. Parameters estimates for mixed jump-option prices (3) and (4) are taken from row (2) and (3) of Table 14.
### Table 22 (continued)

**Summary of Pricing Errors for the Swiss Franc According to In, At and Out of the Money and to Maturity Classes**

<table>
<thead>
<tr>
<th>Maturity:</th>
<th>Out of the Money</th>
<th>At the Money</th>
<th>In the Money</th>
</tr>
</thead>
<tbody>
<tr>
<td>Between 91 to 180 Days</td>
<td># = 1275</td>
<td>455</td>
<td>168</td>
</tr>
<tr>
<td>Mean of $c$ [S.D. of $c$]</td>
<td>0.622 [0.404]</td>
<td>1.604 [0.396]</td>
<td>3.061 [0.849]</td>
</tr>
<tr>
<td>Mean of $</td>
<td>c-c</td>
<td>$</td>
<td>1. BS Spot: 0.306 (0.565)</td>
</tr>
<tr>
<td></td>
<td>2. BS Future: 0.164 (0.337)</td>
<td>0.231 (0.160)</td>
<td>0.218 (0.075)</td>
</tr>
<tr>
<td></td>
<td>3. Mixed Jump(3): 0.281 (0.443)</td>
<td>0.576 (0.354)</td>
<td>0.819 (0.275)</td>
</tr>
<tr>
<td></td>
<td>4. Mixed Jump(4): 0.370 (0.601)</td>
<td>0.750 (0.461)</td>
<td>1.039 (0.349)</td>
</tr>
<tr>
<td></td>
<td>5. Modified Kassouf: 0.085 (0.183)</td>
<td>0.087 (0.068)</td>
<td>0.081 (0.027)</td>
</tr>
<tr>
<td>Mean of $(c-c)$</td>
<td>1. BS Spot: -0.287 (-0.542)</td>
<td>-0.516 (-0.330)</td>
<td>-0.507 (-0.178)</td>
</tr>
<tr>
<td></td>
<td>2. BS Future: -0.090 (-0.215)</td>
<td>-0.087 (-0.039)</td>
<td>0.035 (0.007)</td>
</tr>
<tr>
<td></td>
<td>3. Mixed Jump(3): -0.280 (-0.437)</td>
<td>-0.569 (-0.335)</td>
<td>-0.819 (-0.275)</td>
</tr>
<tr>
<td></td>
<td>4. Mixed Jump(4): -0.370 (-0.601)</td>
<td>-0.745 (-0.448)</td>
<td>-1.039 (-0.349)</td>
</tr>
<tr>
<td></td>
<td>5. Modified Kassouf: -0.007 (0.035)</td>
<td>-0.005 (0.010)</td>
<td>-0.002 (0.001)</td>
</tr>
<tr>
<td>Sum of $(c-c)^2$</td>
<td>1. BS Spot: 176.367</td>
<td>171.299</td>
<td>72.872</td>
</tr>
<tr>
<td></td>
<td>2. BS Future: 54.651</td>
<td>34.898</td>
<td>12.253</td>
</tr>
<tr>
<td></td>
<td>5. Modified Kassouf: 17.402</td>
<td>7.559</td>
<td>1.898</td>
</tr>
<tr>
<td>More than 180 Days</td>
<td># = 556</td>
<td>262</td>
<td>115</td>
</tr>
<tr>
<td>Mean of $c$ [S.D. of $c$]</td>
<td>1.256 [0.474]</td>
<td>2.293 [0.418]</td>
<td>3.730 [0.982]</td>
</tr>
<tr>
<td>Mean of $</td>
<td>c-c</td>
<td>$</td>
<td>1. BS Spot: 0.592 (0.508)</td>
</tr>
<tr>
<td></td>
<td>2. BS Future: 0.313 (0.268)</td>
<td>0.364 (0.163)</td>
<td>0.310 (0.089)</td>
</tr>
<tr>
<td></td>
<td>3. Mixed Jump(3): 0.699 (0.553)</td>
<td>1.120 (0.483)</td>
<td>1.501 (0.411)</td>
</tr>
<tr>
<td></td>
<td>4. Mixed Jump(4): 0.900 (0.717)</td>
<td>1.460 (0.631)</td>
<td>1.977 (0.540)</td>
</tr>
<tr>
<td></td>
<td>5. Modified Kassouf: 0.112 (0.098)</td>
<td>0.106 (0.047)</td>
<td>0.105 (0.029)</td>
</tr>
<tr>
<td>Mean of $(c-c)$</td>
<td>1. BS Spot: -0.552 (-0.479)</td>
<td>-0.858 (-0.382)</td>
<td>-0.764 (-0.216)</td>
</tr>
<tr>
<td></td>
<td>2. BS Future: -0.120 (-0.118)</td>
<td>-0.165 (-0.080)</td>
<td>0.062 (0.013)</td>
</tr>
<tr>
<td></td>
<td>3. Mixed Jump(3): -0.699 (-0.553)</td>
<td>-1.120 (-0.483)</td>
<td>-1.501 (-0.411)</td>
</tr>
<tr>
<td></td>
<td>4. Mixed Jump(4): -0.900 (-0.717)</td>
<td>-1.460 (-0.631)</td>
<td>-1.977 (-0.540)</td>
</tr>
<tr>
<td></td>
<td>5. Modified Kassouf: -0.002 (0.010)</td>
<td>-0.001 (0.002)</td>
<td>-0.008 (0.001)</td>
</tr>
<tr>
<td>Sum of $(c-c)^2$</td>
<td>1. BS Spot: 254.641</td>
<td>257.323</td>
<td>109.232</td>
</tr>
<tr>
<td></td>
<td>2. BS Future: 79.889</td>
<td>51.852</td>
<td>16.610</td>
</tr>
<tr>
<td></td>
<td>5. Modified Kassouf: 11.467</td>
<td>4.609</td>
<td>2.172</td>
</tr>
</tbody>
</table>

**Note:**
1. The entries in parentheses denote relative pricing errors.
2. Parameters estimates for mixed jump-option prices (3) and (4) are taken from row (3) and (4) of Table 14.
Table 23
Summary of Pricing Errors for the Deutsche Mark
According to In, At and Out of the Money and to Maturity Classes

<table>
<thead>
<tr>
<th>Maturity:</th>
<th>Out of the Money</th>
<th>At the Money</th>
<th>In the Money</th>
</tr>
</thead>
<tbody>
<tr>
<td>Less than 30 Days</td>
<td>239</td>
<td>197</td>
<td>139</td>
</tr>
</tbody>
</table>

Mean of c [S.D. of c] | .059 [.063] | .403 [.247] | 1.901 [.889]

Mean of |c-c|
1. BS Spot | .041 (.874) | .097 (.291) | .076 (.049)
2. BS Future | .039 (.861) | .096 (.310) | .130 (.073)
3. Mixed Jump(3) | .088 (.809) | .083 (.339) | .061 (.038)
4. Mixed Jump(4) | .031 (.836) | .069 (.253) | .057 (.035)
5. Modified Kassouf | .026 (.589) | .073 (.326) | .046 (.031)

Mean of (c-c)
1. BS Spot | -.025(-.503) | -.056(-.206) | -.070(-.043)
2. BS Future | -.020(-.447) | -.024(-.100) | -.027(-.010)
3. Mixed Jump(3) | .007(.126) | .046(.205) | -.031(-.111)
4. Mixed Jump(4) | .011(.572) | -.001(.034) | -.036(-.181)
5. Modified Kassouf | -.005(.257) | -.007(.110) | -.012(-.006)

Sum of (c-c)
1. BS Spot | 2.592 | 5.718 | 1.371
2. BS Future | 2.655 | 5.911 | 4.090
3. Mixed Jump(3) | .696 | 2.156 | .842
4. Mixed Jump(4) | .549 | 1.546 | .809
5. Modified Kassouf | .541 | 2.209 | .563

Between 31 & 90 Days | 892 | 400 | 315

Mean of c [S.D. of c] | .228 [.189] | .854 [.301] | 2.128 [.800]

Mean of |c-c|
1. BS Spot | .172 (.862) | .273 (.321) | .295 (.156)
2. BS Future | .148 (.779) | .167 (.200) | .184 (.099)
3. Mixed Jump(3) | .096 (.694) | .159 (.243) | .145 (.071)
4. Mixed Jump(4) | .090 (.673) | .149 (.211) | .170 (.083)
5. Modified Kassouf | .075 (.458) | .116 (.148) | .087 (.048)

Mean of (c-c)
1. BS Spot | -.034(-.319) | -.181(-.233) | -.149(-.071)
2. BS Future | .003(-.178) | -.041(-.069) | .053(.030)
3. Mixed Jump(3) | .021(.326) | .060(.149) | -.108(-.044)
4. Mixed Jump(4) | .006(.360) | -.003(.063) | -.152(-.069)
5. Modified Kassouf | -.007(.168) | -.002(.019) | -.003(.000)

Sum of (c-c)^2
1. BS Spot | 120.220 | 60.044 | 57.694
2. BS Future | 125.832 | 46.246 | 47.070
5. Modified Kassouf | 9.574 | 10.279 | 3.769

Note: 1. The entries in parentheses denote relative pricing errors.
2. Parameters estimates for mixed jump-option prices (3) and (4) are taken from row (3) and (4) of Table 14.
---
Table 23 (continued)
Summary of Pricing Errors for the Deutsche Mark
According to In, At and Out of the Money and to Maturity Classes

<table>
<thead>
<tr>
<th>Maturity:</th>
<th>Out of the Money</th>
<th>At the Money</th>
<th>In the Money</th>
</tr>
</thead>
<tbody>
<tr>
<td>Between 91&amp;180 Days</td>
<td>#=1375</td>
<td>577</td>
<td>373</td>
</tr>
</tbody>
</table>

Mean of $|c-c|$:
1. BS Spot .301 (.680) .479 (.356) .468 (.190)
2. BS Future .211 (.514) .228 (.170) .213 (.086)
3. Mixed Jump(3) .142 (.419) .181 (.141) .240 (.091)
4. Mixed Jump(4) .139 (.377) .212 (.153) .326 (.124)
5. Modified Kassouf .114 (.296) .121 (.092) .112 (.047)

Mean of $(c-c)$:
1. BS Spot -.194 (-.526) -.429 (-.327) -.372 (-.152)
2. BS Future -.078 (-.323) -.126 (-.106) .005 (-.001)
3. Mixed Jump(3) .020 (.204) -.042 (-.003) -.215 (-.077)
4. Mixed Jump(4) -.029 (.105) -.144 (-.080) -.321 (-.121)
5. Modified Kassouf -.008 (.086) -.002 (.009) -.002 (.000)

Sum of $(c-c)$:
1. BS Spot 327.318 186.083 134.792
2. BS Future 294.933 78.790 74.577
3. Mixed Jump(3) 46.413 28.295 31.155
4. Mixed Jump(4) 47.079 39.813 53.682
5. Modified Kassouf 45.645 25.136 8.526

More than 180 Days | #=718 | 383 | 171 |
| Mean of $c$ [S.D. of $c$] | .993 [.419] | 1.847 [.336] | 2.970 [.596] |

Mean of $|c-c|$:
1. BS Spot .497 (.562) .654 (.356) .587 (.206)
2. BS Future .297 (.352) .267 (.152) .208 (.073)
3. Mixed Jump(3) .212 (.249) .270 (.139) .448 (.148)
4. Mixed Jump(4) .266 (.272) .410 (.210) .668 (.223)
5. Modified Kassouf .124 (.151) .099 (.055) .099 (.034)

Mean of $(c-c)$:
1. BS Spot -.453 (-.528) -.652 (-.364) -.586 (-.206)
2. BS Future -.193 (-.273) -.185 (-.110) -.027 (-.014)
3. Mixed Jump(3) -.082 (-.001) -.214 (-.101) .148 (-.146)
4. Mixed Jump(4) -.211 (-.143) -.404 (-.205) .223 (-.223)
5. Modified Kassouf -.000 (.030) -.006 (.000) -.001 (.002)

Sum of $(c-c)^2$:
1. BS Spot 254.879 201.791 79.505
2. BS Future 134.241 39.783 12.538
3. Mixed Jump(3) 48.874 41.931 42.176
4. Mixed Jump(4) 80.580 89.383 87.157
5. Modified Kassouf 46.028 6.254 3.511

Note: 1. The entries in parentheses denote relative pricing errors.
2. Parameters estimates for mixed jump-option prices (3) and (4) are taken from row (3) and (4) of Table 14.
### Table 24
Summary of Out-of-sample Forecasting Errors and Implied $R^2$: BP

|                | Mean of $|c-c|$ | Mean of $(c-\hat{c})$ | Sum of $(c-\hat{c})^2$ | $R^2$  |
|----------------|-------------|---------------------|-----------------------|-------|
| Loglinear      |             |                     |                       |       |
| Model 1        | 4.070       | 2.334               | 25991.11              | ****  |
| Premium        |             |                     |                       |       |
| Model 2        | 1.085       | -0.868              | 255.06                | 0.926 |
| Modified       |             |                     |                       |       |
| Kassouf        | 0.936       | -0.634              | 170.61                | 0.951 |
| BS Spot        | 0.390       | 0.350               | 38.29                 | 0.989 |
| BS Future      | 0.581       | -0.540              | 87.89                 | 0.974 |
| $\lambda=0.09$|             |                     |                       |       |
| Best Jump      | 1.608       | -1.607              | 521.30                | 0.850 |

Note:
1. The number of observations is 155.
2. Prices are quoted in cents per British Pound.
3. **** in $R^2$ denotes that sum of $(c-\hat{c})^2$ is greater than sum of $(c-c)^2$.

### Table 25
Summary of out-of-sample Forecasting Errors and Implied $R^2$: JY

|                | Mean of $|c-c|$ | Mean of $(c-\hat{c})$ | Sum of $(c-\hat{c})^2$ | $R^2$  |
|----------------|-------------|---------------------|-----------------------|-------|
| Loglinear      |             |                     |                       |       |
| Model 1        | 0.229       | -0.090              | 11.30                 | 0.804 |
| Premium        |             |                     |                       |       |
| Model 2        | 0.222       | -0.117              | 9.24                  | 0.839 |
| Modified       |             |                     |                       |       |
| Kassouf        | 0.161       | -0.056              | 4.65                  | 0.919 |
| BS Spot        | 0.390       | -0.390              | 23.99                 | 0.584 |
| BS Future      | 0.317       | -0.315              | 16.04                 | 0.722 |
| $\lambda=0.74$|             |                     |                       |       |
| Best Jump      | 0.154       | 0.152               | 3.45                  | 0.941 |

Note:
1. The number of observations is 122.
2. Prices are quoted in 1/100 cent per Japanese Yen.
3. **** in $R^2$ denotes that sum of $(c-\hat{c})^2$ is greater than sum of $(c-c)^2$. 
Table 26
Summary of out-of-sample Forecasting Errors and Implied $R^2$: SF

<table>
<thead>
<tr>
<th>Model</th>
<th>Mean of</th>
<th>Mean of</th>
<th>Sum of</th>
<th>$R^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Loglinear</td>
<td>0.600</td>
<td>0.310</td>
<td>297.91</td>
<td>*****</td>
</tr>
<tr>
<td>Premium</td>
<td>0.214</td>
<td>-0.046</td>
<td>16.20</td>
<td>0.909</td>
</tr>
<tr>
<td>Modified Kassouf</td>
<td>0.223</td>
<td>0.000</td>
<td>18.60</td>
<td>0.895</td>
</tr>
<tr>
<td>BS Spot</td>
<td>0.131</td>
<td>-0.108</td>
<td>5.66</td>
<td>0.968</td>
</tr>
<tr>
<td>BS Future</td>
<td>0.114</td>
<td>0.045</td>
<td>4.06</td>
<td>0.977</td>
</tr>
<tr>
<td>$\lambda=0.33$</td>
<td></td>
<td></td>
<td>18.60</td>
<td>0.895</td>
</tr>
<tr>
<td>Best Jump</td>
<td>0.517</td>
<td>-0.517</td>
<td>70.25</td>
<td>0.605</td>
</tr>
</tbody>
</table>

Note:
1. The number of observations is 185.
2. Prices are quoted in cents per Swiss Franc.
3. ***** in $R^2$ denotes that sum of $(\hat{c}-\hat{c})^2$ is greater than sum of $(c-c)^2$.

Table 27
Summary of Out-of-sample Forecasting Errors and Implied $R^2$: DM

<table>
<thead>
<tr>
<th>Model</th>
<th>Mean of</th>
<th>Mean of</th>
<th>Sum of</th>
<th>$R^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Loglinear</td>
<td>0.406</td>
<td>-0.051</td>
<td>144.02</td>
<td>0.160</td>
</tr>
<tr>
<td>Premium</td>
<td>0.302</td>
<td>-0.089</td>
<td>35.22</td>
<td>0.794</td>
</tr>
<tr>
<td>Modified</td>
<td>0.249</td>
<td>-0.070</td>
<td>18.79</td>
<td>0.890</td>
</tr>
<tr>
<td>BS Spot</td>
<td>0.171</td>
<td>-0.166</td>
<td>10.98</td>
<td>0.936</td>
</tr>
<tr>
<td>BS Future</td>
<td>0.088</td>
<td>-0.039</td>
<td>3.16</td>
<td>0.981</td>
</tr>
<tr>
<td>$\lambda=0.02$</td>
<td></td>
<td></td>
<td>18.79</td>
<td>0.890</td>
</tr>
<tr>
<td>Best Jump</td>
<td>0.135</td>
<td>0.105</td>
<td>5.55</td>
<td>0.970</td>
</tr>
</tbody>
</table>

Note:
1. The number of observation is 229.
2. Prices are quoted in cents per Deutsche Mark.
3. ***** in $R^2$ denotes that sum of $(\hat{c}-\hat{c})^2$ is greater than sum of $(c-c)^2$. 
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