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Chen, Ching-Shung, Ph.D.
The Ohio State University, 1987
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UMI
AXISYMMETRIC INCOMPRESSIBLE NAVIER-STOKES
CALCULATIONS OF THE VORTEX WAKE
OF HOVERING ROTORS

DISSERTATION

Presented in Partial Fulfillment of the Requirement for
the Degree Doctor of Philosophy in the Graduate
School of the Ohio State University

By

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1987

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\( A \) = the radius of the far wake in Landgrebe's prescribed wake equations

\( A \) = tridiagonal matrix

\( C_T \) = thrust coefficient

\( d \) = distance from a point in the flow field to the origin

\( g_1 \) = axial velocity constant in the outer edge of the inboard vortex sheet before the first blade passage in Landgrebe's prescribed wake equations

\( g_2 \) = axial velocity constant in the outer edge of the inboard vortex sheet after the first blade passage in Landgrebe's prescribed wake equations

\( g_3 \) = axial velocity constant in the inner edge of the inboard vortex sheet in Landgrebe's prescribed wake equations

\( I \) = identity matrix

\( I_{mn} \) = moments of vorticity

\( k_1 \) = axial tip velocity constant before the first blade passage in Landgrebe's prescribed wake equations

\( k_2 \) = axial tip velocity constant after the first blade passage in Landgrebe's prescribed wake equations

\( N \) = number of grid points

\( p \) = pressure
\( Q \) = number of blade

\( R \) = blade radius

\( r \) = nondimensionalized radial coordinate

\( r_A \) = radial coordinate of the inboard vortex sheet in Landgrebe's prescribed wake equations

\( r_B \) = radial coordinate of the inboard vortex sheet in Landgrebe's prescribed wake equations

\( r_C \) = radial coordinate of the inboard vortex sheet in Landgrebe's prescribed wake equations

\( r_e \) = the core radius of a Lamb vortex

\( r_D \) = radial coordinate of the inboard vortex sheet in Landgrebe's prescribed wake equations

\( r_d \) = the radius from the center of a Lamb vortex

\( r_T \) = tip vortex radial coordinate in Landgrebe's prescribed wake equations

\( Re \) = the Reynolds number

\( t \) = nondimensionalized time

\( u \) = velocity in the radial direction

\( u_f \) = induced velocity in the radial direction due to a semi-infinite vortex cylinder
\( v \) = velocity in the azimuthal direction
\( w \) = velocity in the axial direction
\( w_I \) = induced velocity in the axial direction due to a semi-infinite vortex cylinder
\( w_s \) = the self induced velocity of a vortex ring
\( X_j \) = solution matrix along the \( j \)th grid row
\( Y_j \) = inhomogeneous terms for the \( j \)th grid row
\( z \) = nondimensionalized axial coordinate
\( z_T \) = tip vortex axial coordinate in Landgrebe's prescribed wake equations
\( z_{r=0} \) = inner edge axial coordinate of the inboard vortex sheet in Landgrebe's prescribed wake equations
\( z_{r=1} \) = outer edge axial coordinate of the inboard vortex sheet in Landgrebe's prescribed wake equations
\( \lambda \) = radial tip velocity constant in Landgrebe's prescribed wake equations
\( \psi_w \) = tip vortex age in Landgrebe's prescribed wake equations
\( \sigma \) = rotor solidity
\( \theta_L \) = linear twist angle in Landgrebe's prescribed wake equations
\( \varsigma \) = vorticity
\( \Gamma \) = circulation  
\( \nu \) = kinematic viscosity  
\( \theta \) = azimuthal coordinate in an inertial coordinate system  
\( \phi \) = azimuthal coordinate in a coordinate system rotating with the blade  
\( \Omega \) = rotational speed of the rotor  
\( \omega \) = relaxation factor of the relaxation scheme  
\( \psi \) = stream function  
\( \Delta t \) = time step of the difference equations  
\( \Delta r \) = grid size in the radial direction of the difference equations  
\( \Delta z \) = grid size in the axial direction of the difference equations  
\( \Delta \phi \) = grid size in the azimuthal direction of the difference equations  
\( \vec{i}, \vec{j}, \vec{k} \) = unit vectors of the Cartesian coordinate system  
\( \vec{e}_r, \vec{e}_\phi, \vec{e}_z \) = unit vectors of the cylindrical coordinate system  
\( \mathcal{E}(k) \) = the complete elliptic integral of the first kind  
\( K(k) \) = the complete elliptic integral of the second kind  

**SUPERSCRIPTS**

- * = nondimensionalized values  
- \( n \) = time level in the ADI method
\[ n + 1/2 = \text{time level in the ADI method} \]
\[ n + 1 = \text{time level in the ADI method} \]

**SUBSCRIPTS**

\[ i, j = \text{indices of radial and axial locations of a grid point in the difference equations} \]
Chapter 1

Introduction

1.1 Review of the Past Work

Hover capability is a critical design goal for helicopters and other vertical take off and landing (VTOL) aircraft, since these aircraft are designed to take off and land vertically, and to hover for a relatively long periods of time. Hover performance prediction is also essential owing of the low payload to gross-weight ratio of such aircraft. For example, if the payload is 25 percent of the gross weight, a hover performance deficiency of 1 percent in lift capability would result in a 4 percent reduction in payload. In predicting the lift of a helicopter rotor, the structure of the vortex wake shed by the blades is a crucial element because it stays close to the rotor plane and has a strong interaction with the blades. Fig. 1.1 shows the vortex wake shed by a one-bladed rotor. The vortex sheet shed by by a blade quickly breaks into an inboard sheet and a high strength rolled-up tip vortex. These two portions are convected downstream in different rates as well as being contracted. The rolled-up tip vortex moves downstream much slower than the inboard sheet. This results in that the tip vortex shed by the preceeding
Figure 1.1: Rotor wake structure.
blade stays very close to the following blade. This tip vortex induces upward velocity on the blade outboard of it and downward velocity inboard of it as shown in Fig. 1.2. These velocities alter the angles of attack of the blade and therefore affect the performance of the rotor. In fact, vortex wake is an important element in all helicopter problems including performance, structural loads, vibration, stability, and noise [1].

The approaches that have been used in investigating this problem can be classified under three broad groups:

1. classical-wake analysis

2. generalized-wake analysis

3. free-wake analysis

The classical-wake analysis was derived from the vortex theory for propellers. It was developed with the assumption that the total flow through the propeller is large compared with the induced velocity due to the vortex wake system. This assumption permitted the neglect of wake contraction. While this is a reasonable assumption for propellers, it is questionable in the case of the rotor in hover, since both vortex theory and momentum theory show that the velocity in the wake far downstream is twice that through the rotor plane and therefore the wake has to contract. Furthermore classical vortex theory assumes the rotor is an ideal rotor (actuator disk) such that the circulation on the blade and the induced velocity over the rotor plane are constant. The wake therefore consists of a vortex cylinder generated by the blade tips and a root vortex generated by the blade roots. The root vortex descends vertically at the hub because the blade tangential velocity
Figure 1.2: Velocities induced on the blade by the tip vortex shed by the preceding blade.
is zero there and, therefore, has no contribution to the induced velocity normal to the rotor plane. It was then assumed that the strength of the vortex cylinder generated by the tip vortex could be determined by the vortex spacing, and the vortex spacing was based on the inflow velocity, uniform over the rotor plane, generated by the vortex cylinder. However, the inflow velocity is far from uniform over the rotor plane. For a propeller, the free stream dominates the total inflow, and the above mentioned discrepancies are acceptable. They are clearly inadequate in the case of a hovering rotor where the flow through the rotor consists only of induced velocities from the vortex wake.

A generalized-wake analysis is to describe the wake geometry by a set of equations in terms of a few parameters, such as thrust coefficient, twist rate, rotor solidity, number of blades etc. These equations are generated by interpolation of experimental data base [2] [3]. If a rotor's configuration and operating conditions are within the interpolation range of the data base, its geometry can be fairly well predicted. This method was first developed by Landgrebe [2]. The generalized equations developed by Landgrebe will be described in Chapter 3. The wake geometry, rotor configurations, and operating conditions then are input to a lifting-line or lifting-surface code to obtain the rotor performance such as lift, drag, and torque. This method is the most widely used method by the helicopter industry nowadays. If the configuration and operating conditions of a rotor are within the interpolation range of the data base, the wake geometry predicted by this method and therefore the performance of the rotor generally agree well with the experimental data. This method is fairly accurate but requires expensive experiments to generate the data base. For modern rotors, which very often
involve combinations of twist, airfoil shapes, and blade tip shapes, it would be a formidable task to generate such a data base.

Attempts to model the actual wake structure and its interaction with the rotor blade surface has been approached through both analytical and numerical methods. Because of the complexity of the problem most recent emphasis has been placed on numerical solutions. The early methods [2] consisted of using a line vortex for the tip vortex, typically as a connected series of straight line segments, and several other series of line segments for the inboard vortex sheet. These segments were moved by their mutual interactions, through the Biot-Savart law,

\[
\vec{v} = -\frac{\Gamma}{4\pi} \int \frac{\vec{s} \times d\vec{l}}{s^3}
\]

(1.1) until the wake geometry converged. In this expression, \(\vec{v}\) is the induced velocity, \(\Gamma\) is the strength of the vortex, \(d\vec{l}\) is a differential element of the vortex segment, and \(\vec{s}\) is the position vector of the induced point and \(d\vec{l}\).

But the early attempts were not satisfactory due to the instabilities encountered. The difficulty was the singularity introduced by the Biot-Savart law, which was used to calculate the mutual induced velocity of the vortex segments. According to the Biot-Savart law, the induced velocity at a point due to a vortex segment is proportional to \(1/d\), where \(d\) is the perpendicular distance from the point to the vortex segment. When a vortex segment moves too close to a control point, it induces an unrealistic large velocity at the control point, and the instability starts. Finite core line vortex, such as the Rankine vortex, or others, were then used to remove the singularity. Yet the wake geometry would not converge unless numerical damping was used. The most often used numerical damping was to take the advantage of the axisymmetry of this problem [4]. For instance, for a
two bladed rotor, if there is a control point at one side of the rotor, there should be a corresponding point at the other side of the rotor. If the vortex wake is steady, these two points should have the same radial and axial coordinates, although there is a 180 degrees difference in their azimuthal coordinates. During the computational process, if the coordinates of these two points become different radially or axially, they would be redistributed to the same radial and axial coordinates through a ratio of their difference. This redistribution scheme is applied to every control point in the flow field until the wake geometry converges. Bliss et al. [5] questioned the use of numerical damping: "Researchers have tried to circumvent this stability problem by the use of elaborate models of the wake flow field, and sometimes by artificial suppression of the instability. Methods of suppression include periodically smoothing and imposing symmetry on the wake, and the introduction of substantial amounts of numerical damping. Unfortunately, such approaches may effectively impose forces on the wake, leaving open the question of whether equilibrium free motion conditions are really being satisfied. Depending on the type of suppression used, long computer run times may still be required to obtain convergence or to compute time-averaged results if some residual instability remains. The fundamental difficulty lies in the fact that the wake of a hovering rotor actually is unstable, as is also confirmed by experiment."

Panel methods were developed to answer unresolved problems with lifting-line theory, and are being used extensively in predicting the performance of rotors in hovering and forward flight. A number of the panel method solutions for hovering rotor include free wake calculations [6] [7] [8] [9] [10] [11]. In these methods, the flow was usually assumed inviscid and
irrotational outside the vortex wake. The vortex wake is modeled by panels of doublets (double layers). Across the vortex wake, pressure is assumed continuous but there are potential jumps due to the discontinuities of the tangential velocities across the wake. The magnitude of the potential jump determines the strength of the doublet. The potential jump across a panel is constant in time and is determined by the trailing edge condition. The Green's theorem is used to convert the partial differential equation (the unsteady Bernoulli equation or the Laplace equation) of the velocity potential into an integral equation. The velocity field is obtained by solving the integral equation, and the vortex wake is marched to the next time step. Panel methods have been successful in predicting the performance of hovering rotors with prescribed wake geometry, but their role in free-wake prediction was not satisfactory.

Bliss et al. [5] used a different approach to study this problem. Their method was based on the recognition that a hovering rotor possesses a self-preserving solution that appears steady in a coordinate frame rotating with the blades. The desired solution was found by a procedure which does not involve time stepping and does not require the flow field to be stable. In the analysis, the tip vortex was represented by the curved vortex elements [12] [13], and the solution procedure required that the problem be solved in a rotating coordinate frame. The velocity components were analyzed in crossflow planes normal to the curved vortex filaments (Fig. 1.3). These planes were located at the wake collocation points through which the curved vortex elements passed. The desired steady solution was reached when, at every crossflow plane, the crossflow velocity was zero, i.e., when the local resultant velocity was tangent to the filament. The solution method involved
Figure 1.3: Free wake represented by curved vortex elements showing the crossflow planes at the collocation points.
the determination of the influence coefficient matrix. Given an initial wake configuration, the effect of small displacements of the collocation points on the rest of the wake was determined. The influence coefficient matrix was constructed from these displacement effects. The matrix was used to predict how the collocation points should simultaneously be displaced to null the crossflow velocities. Several relaxation steps were required to obtain a converged solution. The authors considered the tip vortex only, i.e., the vortex wake was represented by the tip vortex and the inboard vortex sheet was not included.

Miller [14] [15] [16] [17] developed a fast free wake method for hovering rotors. Both two dimensional and axisymmetric (ring) models were used. The wake was assumed to have rolled up into three concentrated vortices (tip, inboard, and root), and a far wake was used after four revolutions. The average velocity between vortices was used to iterate the wake geometry.

Murman and Stremel [18] [19] [20] used a cloud-in-cell method [21] to calculate the wake of a hovering rotor. The wake was assumed two dimensional, and three infinite line vortices (tip, inboard, and root) for five revolutions, plus intermediate and far wake models, were used to represent the wake. For each time step, the concentrated vorticity was distributed to nearby grid points by a distribution scheme. The Laplace equation (with potential jumps on branch cuts to represent vorticity) was then solved in the Eulerian mesh and the velocities calculated. Using a Fast Poisson Solver to solve the Laplace equation requires only $O(N \ln N)$ operations, while direct integration of the Biot-Savart law requires $O(N^4)$ operations [22], where $N$ is a typical number of computational grid points in the radial or axial direction. Therefore, the cloud-in-cell method can be used effectively to
handle a large amount of discrete vortices. The velocity was interpolated
to the vortex positions, which were then integrated to the next time step.
The cloud-in-cell method is also called Eulerian-Lagrangian method, since
velocity potential was solved in an Eulerian mesh, while discrete vortex
elements were tracked in a Lagrangian frame. The distribution of vorticity
in an Eulerian mesh introduced an effective core size, on the order of the
grid size, which eliminates the velocity singularities. Also there was no
diffusion of vorticity, since vortex elements were tracked individually in a
Lagrangian frame.

Murman and Roberts [23] extended Murman and Streml's work to
calculate the wake geometry of a hovering rotor, using the cloud-in-cell
method and an axisymmetric (ring) model. The self-induced velocity of a
ring vortex is a function of its core size and strength, therefore it is grid
dependent, since the grid size determines the effective core size. This effect
was eliminated by using the analytical solution of the self-induced velocity
of a vortex ring and the experimental core size (from experimental data)
to calculate the self-induced velocity, and the cloud-in-cell accounted for
the influence of the other rings. When less than ten vortex rings were
used to model a vortex sheet, the solution converged and agrees fair with
experimental data; the solution diverged when more than ten rings were
used.

Cantaloube and Huberson [24] developed a vortex point method for
both hovering and forward flight. The blades were modeled by discretized
quadrangular elements with piecewise constant doublet distribution. The
vortex wake was modeled by vortex carrying particles emitted at the blade
trailing edge. The strength and location of these emitted vortex particles
were determined through an emission model. The induced velocity at the vortex carrying particles in the wake was calculated by the Green identity, and the wake was integrated to the next time step. The blade loads were calculated using incompressible lifting surface theory for a thin wing.

Liu [22] et al. solved the incompressible, viscous, Navier-Stokes equations for a hovering rotor wake. They used an unsteady, axisymmetric (ring) model of the wake. The vorticity distribution at the blade trailing edge was derived from the prescribed blade loading distribution, which was from a separate lifting-surface calculation. A new vorticity distribution was introduced, at intervals in time corresponding to the circumferential distance from one blade passage to another, and was superimposed on the existing vorticity distribution. The vertical separation of the superimposed vorticity distribution was obtained from experimental data. Computations were performed for up to 8 superpositions of initial vorticity distribution corresponding to 4 rotor revolutions for a two bladed rotor, with no far-wake model. Results showed little wake contraction and significant vorticity diffusion.

1.2 Overview of the Approach Used to Solve the Hover Wake

The purpose of this work is to develop a numerical method based on axisymmetric incompressible Navier-Stokes equations to predict the vortex wake shed by a hovering helicopter rotor. This numerical method, which will be called free-wake analysis, is coupled with a lifting-surface method. The lifting-surface method utilized (described in Chapter 5) was developed by Summa [6]. It provides the circulation distribution on a blade which de-
termines the strength of the vortex sheet shed by the blade as the starting condition for the free-wake analysis.

Initially, we assume a shape for the vortex wake (the details of how the vortex wake is modeled will be described in Chapter 2). The geometry of the vortex wake, the rotor configuration, and its operating conditions are input to the lifting-surface code. The lifting-surface code provides the circulation distribution on the blade under this wake geometry. This circulation distribution is then used as an input to the free-wake code to generate the vortex sheet shed by the blade. An iteration approach is used in the free-wake analysis to converge the shape of the vortex wake under the specified circulation distribution (details of the free-wake analysis will be described in Chapter 3). The converged vortex wake is again input to the lifting-surface code to obtain a new circulation distribution. This circulation distribution is input to the free-wake code again to converge the vortex wake under this circulation distribution. This process is repeated until both the circulation distribution on the blade and the shape of the vortex wake converge. The flow-chart of this coupling is shown in Fig. 1.4.

The lifting-surface method (described in detail in Chapter 5) assumes the flow field is inviscid and incompressible; therefore, the equation of motion is expressed in velocity potential form. The governing equation is the Laplace equation of the velocity potential. The Green's theorem is used to obtain the integral solution of the Laplace equation. This integral solution of the velocity potential is solved by a numerical procedure. Once the velocity potential of the flow field is known, the velocity in the vicinity of the rotor blades can be obtained, and therefore the lift force or the circulation distribution on the blade.
Figure 1.4: Flow-chart of the coupling of the free-wake method and the lifting-surface method.
The vortex sheet shed from a blade quickly breaks up into an inboard sheet and a rolled-up tip vortex and the two portions are convected downstream at different rates as well as being contracted. The rolled-up tip vortex convects downstream far more slowly than the inboard vortex sheet. The structure of the tip vortex is unstable and breaks down after two to three blade revolutions. The tip vortices are very unstable due to the hydrodynamic instability of the problem. A small disturbance can stimulate the instability as is confirmed by experiment. The tip vortex instability also occurred in numerical simulations. In order to solve this problem, the inboard vortex sheet and the tip vortex are treated separately. A relaxation scheme is used to relax the tip vortex positions at the time of blade passages to solve this problem (described in detail in Chapter 3).

Most the methods which have been used in investigating this problem assume the flow field is inviscid and the vortex wake is traced in a Lagrangian frame. In the present work, the flow field is assumed viscous and the vortex wake is traced in an Eulerian frame. The advantages and disadvantages of this approach compared with the other approaches is discussed in Chapter 6.
Chapter 2

Physical Model and Mathematical Formulation

2.1 The Physical Model of the Vortex Sheet Shed by a Blade

In setting up the initial condition of vortex strength along the blade, the classical lifting-line theory was utilized in modeling the vortex sheet shed by a blade. According to this theory, if there is a difference in the circulation (or lift force) in the neighboring two points on a wing or blade, then there will be a line vortex shed by the wing or blade and its strength is equal to the difference of circulations on these two points (Fig. 2.1). Since the circulation on the blade is not uniform, the vortex sheet is thus replaced by a row of discrete line vortices. After each line vortex is shed by the blade, it is locked into the rotor flow and moves in an axisymmetric fashion. The dominant transporting mechanism of vorticity in the vortex following its deposition in the flow is diffusion. Each line vortex as initially separated was chosen to be of the form of a Lamb vortex, described mathematically by the solution of the incompressible axisymmetric Navier-Stokes equations

16
Figure 2.1: The trailing vortex system from a lifting line.
18

as [25],

\[
\zeta(r_d) = \frac{\Gamma}{\pi r_c^2} \exp\left(\frac{-r_d^2}{r_c^2}\right)
\]

(2.2)

where \(\zeta(r_d)\) is the vorticity distribution, \(r_d\) is the radius from the center of each vortex, \(r_c\) is the assumed core radius which gives the location of the maximum velocity of the vortex. The Lamb vortex form was chosen so as to avoid the singularity such as that could arise with the use of a free vortex. The circulation strength of the vortex is \(\Gamma\), which is determined by the circulation difference of the neighboring points on the blade.

Fig. 2.2 shows the vorticity contour plot of two such line vortices behind the blade trailing edge. Each line vortex has its own vorticity distribution which is governed by Eq.(2.2). In our calculations, if there were \(N\) grid points along the blade, then there would be \(N - 1\) line vortices shed by the blade, each one had its own strength and vorticity distribution. The vortex sheet shed by a blade was modeled by the superposition of these line vortices. Fig. 2.3 shows the vorticity contour plot of a vortex sheet behind a blade after the superposition. It can be seen that the individual vortices inboard of the peak circulation have formed a sheet and the outboard vortices have rolled up into a discrete tip vortex.

### 2.2 Governing Equations

In the free-wake analysis, the flow field was assumed axisymmetric. The assumption of axisymmetry is based on the fact that the pitch angle of the vortex wake is quite small. For example, for a typical two-bladed rotor the vertical distance from the blade to the first tip vortex underneath it is about 5% of the blade radius, and this distance between the neighboring
Figure 2.2: The vorticity contour plot of line vortices behind a blade trailing edge.
Figure 2.3: The vorticity contour plot of a vortex sheet behind a blade trailing edge.
tip vortices beneath the first one is about 15%. If we consider the 15% situation, and assume the radius of the tip vortices is 85% of the blade radius. The pitch angle of the tip vortex helix will be about 3.2 degrees (see Fig. 2.4). This angle will be even smaller for a four or six-bladed rotor since the spacing between neighboring vortex sheets is smaller.

In the calculation of the vortex wake shed by a fixed-wing aircraft, it is often assumed that the streamwise gradients are small. In the case of a rotor in hover, an analogous assumption is made; that because the circumferential gradients are small the streamwise development of the vortex wake can be reduced to a time-dependent axisymmetric calculation.

The flow was assumed to be viscous and incompressible. The time coordinate was replaced by the azimuthal coordinate for convenience. The coordinate system was converted by the following conditions:

\[ r = r \]

\[ \phi = \theta - \Omega t \]

\[ z = z \]

\[ t = t \]

The rotor blade rotates with angular speed \( \Omega \) about the \( z \) axis (see Fig 2.5 for the coordinate system). The three-dimensional flow field was further reduced to an unsteady axisymmetric flow field by assuming (1) the circumferential velocity component and pressure gradient are negligible and (2) the rate of change of the velocity gradients in the circumferential direction
Vertical displacement of tip vortices:
0.15 (15% of the blade radius)

(assuming the radius of the tip vortex is 85% of the blade radius)

\[ \theta = \arctan \frac{0.15}{0.85 \times \pi} \]

\[ \approx 3.2^\circ \]

Figure 2.4: The calculation of the pitch angle of the tip vortex helix.
Figure 2.5: Coordinate system, contour plot of the vortex wake, and the boundary conditions of the computation.
is small

\[ v = \frac{1}{r} \frac{\partial p}{\partial \phi} = \frac{1}{r^2} \frac{\partial^2 u}{\partial \phi^2} = \frac{1}{r^2} \frac{\partial^2 w}{\partial \phi^2} = 0 \]  

(2.3)

This assumption is a straightforward extension of a practice commonly used in fixed wing analysis to reduce the steady three-dimensional vortex wake roll-up calculation to an unsteady two-dimensional calculation. The development over time of the two-dimensional (in the case of fixed-wing) or axisymmetric (in the case of a rotor in hover) flow field represents the streamwise development of the three dimensional flow. The resulting axisymmetric continuity and momentum equations are given below

\[ \frac{1}{r} \frac{\partial (ru)}{\partial r} + \frac{\partial w}{\partial z} = 0 \]  

(2.4)

\[ \Omega \frac{\partial u}{\partial \phi} - u \frac{\partial u}{\partial r} - w \frac{\partial u}{\partial z} = \frac{1}{\rho} \frac{\partial p}{\partial r} - \nu (\nabla^2 u - \frac{u}{r^2}) \]  

(2.5)

\[ \frac{1}{\rho r} \frac{\partial p}{\partial \phi} = 0 \]  

(2.6)

\[ \Omega \frac{\partial w}{\partial \phi} - u \frac{\partial w}{\partial r} - w \frac{\partial w}{\partial z} = \frac{1}{\rho} \frac{\partial p}{\partial z} - \nu \nabla^2 w \]  

(2.7)

where \( u, v, \) and \( w \) are velocities in the \( r, \theta, \) and \( z \) directions, and the axisymmetric Laplacian operator is

\[ \nabla^2 = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{\partial^2}{\partial z^2} \]

Note that the above equations are time-like in the variable \( \phi \) representing circumferential development of the flow. Equations (2.4)-(2.7) can be efficiently solved in vorticity-stream-function form

\[ \Omega \frac{\partial \zeta}{\partial \phi} - \frac{\partial (u \zeta)}{\partial r} - \frac{\partial (w \zeta)}{\partial z} = -\nu (\nabla^2 \zeta - \frac{\zeta}{r^2}) \]  

(2.8)
and

$$\nabla^2 \psi - \frac{2}{r} \frac{\partial \psi}{\partial r} = -\zeta r \quad (2.9)$$

where the vorticity was defined as

$$\zeta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial r} \quad (2.10)$$

and the velocities were defined as

$$u = -\frac{1}{r} \frac{\partial \psi}{\partial z} \quad (2.11)$$

$$w = \frac{1}{r} \frac{\partial \psi}{\partial r} \quad (2.12)$$

The analysis was also assumed being limited to locations outside the boundary layer of the blades. Equations (2.8) and (2.9) together with the initial and boundary conditions constituted an initial-boundary-value problem. The initial condition corresponds physically to the vorticity distribution at a small distance circumferentially downstream of the rotor-blade trailing edge. The initial vorticity distribution must be obtained by other means, such as from experimental data or numerical lifting-surface (or lifting-line) calculation. A lifting surface calculation [6] of the blade load distribution was used to determine the initial vorticity distribution in the present work.

In order to generalize the problem, the governing equations need to be nondimensionalized. The coordinates $r$ and $z$ were nondimensionalized by the radius of the blade, $R$.

$$r^* = \frac{r}{R}$$

$$z^* = \frac{z}{R}$$
and the variables in the governing equations were nondimensionalized by the following

\[ u^* = \frac{u}{\Omega R} \]

\[ v^* = \frac{v}{\Omega R} \]

\[ w^* = \frac{w}{\Omega R} \]

\[ \zeta^* = \frac{\zeta}{\Omega} \]

\[ \psi^* = \frac{\psi}{\Omega R^3} \]

\[ \phi^* = \frac{\phi}{2\pi} \]

where values without asterisk are dimensional, and values with asterisk are nondimensional. \( \Omega \) is the rotational speed of the rotor. After nondimensionalization, the governing equations became

\[ \frac{\partial \zeta}{2\pi \partial \phi} - \frac{\partial (u \zeta)}{\partial r} - \frac{\partial (w \zeta)}{\partial z} = -\frac{1}{Re} (\nabla^2 \zeta - \frac{\zeta}{r^2}) \]  \hspace{1cm} (2.13)

\[ \nabla^2 \psi - \frac{2}{r} \frac{\partial \psi}{\partial r} = -\zeta r \]  \hspace{1cm} (2.14)

where asterisks were dropped for convenience, and \( Re \) is the Reynolds number defined by

\[ Re = \frac{\Omega R^3}{\nu} \]
2.3 Boundary Conditions

A lot of problems encountered in fluid dynamics deal with concentrated vortices. In these problems, vorticity only concentrates in certain regions, outside these regions the flow can be assumed inviscid and irrotational. Far away from the vorticity concentrated regions, vorticity can be assumed null, since vorticity decays rapidly with distance. The present problem is one of this category, outside the blade and the vortex wake the flow can be approximated as inviscid and irrotational flow.

Two sets of boundary values were required in the calculations. One set was the boundary values of vorticity for the vorticity transport equation. The other was the boundary values of the stream function for the Poisson equation. The vortices shed by the blade were modeled by the Lamb vortex (page 17). For each Lamb vortex, the vorticity decays exponentially from its center (Eq. (2.2)),

\[ \zeta = O(\exp(d^{-2})) \quad \text{as} \quad d = (r^2 + z^2)^{1/2} \to \infty \]

Therefore, zero vorticity was used on the boundary and is considered as a reasonable assumption as long as the boundary is far enough (Fig. 2.5).

For circular vortices in an unbounded domain with cylindrical coordinates \((x, r, \phi)\) in which fluid particles are at rest far away from the vortices the stream function can be obtained by [25] (see Fig. 2.6)

\[
\psi(x, r, t) = \frac{r}{4\pi} \int_{-\infty}^{\infty} \int_{0}^{\infty} \int_{0}^{2\pi} \frac{r' \zeta(x', r', t)}{\xi} \cos \theta dz' dr' d\theta
\]

\[ = \frac{r}{4\pi} \int_{-\infty}^{\infty} dz' \int_{0}^{\infty} r' dr' \zeta(x', r', t) \int_{0}^{2\pi} \frac{\cos \theta}{\xi} d\theta \quad (2.15) \]
Figure 2.6: Coordinate system and geometry for obtaining the boundary values of stream function
where

\[ \xi = \left[ (x - x')^2 + r^2 + r'^2 - 2rr' \cos \theta \right]^{1/2} \]

\[ \theta = \phi' - \phi \]

and \( \zeta(x', r', t) \) is the value of the vorticity at position \((x', r')\) and time \(t\).

To obtain the boundary values of the stream function, each vortex sheet was divided into several subdivisions. In this work, the length of each subdivision was 5% blade radius. The total strength and centroid of the vorticities inside each subdivision were calculated. When calculating the boundary values of the stream function by direct integration of Eq.(2.15), the contributions of the vorticities in each subdivision were represented by a concentrated vortex with its strength and location equal to the total strength and centroid of the vorticities in the subdivision.
Chapter 3

Computational Procedures of the Free-Wake Analysis

3.1 Computational Procedures of the Free-Wake Analysis

The computation of the free-wake code starts by guessing a shape for the vortex wake. In the present work, the vortex wake consists of five (five for a two-bladed rotor and seven for a four-bladed rotor) free-to-move vortex sheets and a far-wake. The strength distribution of each vortex sheet is determined by the circulation distribution on the blade provided by the lifting-surface code. The initial geometry of the vortex wake is described by Landgrebe's prescribed wake equations. Although arbitrary geometry can be used as the initial geometry, since an iteration approach is used, Landgrebe's prescribed wake equations provides a reasonable starting point. These equations were derived from interpolations of the data base of a set of rotor configurations subjected to the following restrictions:
1. uniform blade: the same chord length and airfoil shape throughout the entire blade

2. no twist or linearly twisted

3. no sweep in the tip region

In the equations, the wake geometry is prescribed by rotor thrust coefficient, solidity, and twist rate. Thrust coefficient, $C_T$, and solidity, $\sigma$, are defined correspondingly by

$$C_T = \frac{T}{\rho \pi R^2 (\Omega R)^2}$$
$$\sigma = \frac{Q C}{\pi R}$$

where $T$ is the rotor thrust, $\rho$ is the density of the air, $\Omega$ is the rotational speed of the rotor, $Q$ is the number of the blade, $C$ is the blade chord length, and $R$ is the radius of the blade. Among these three parameters, solidity and twist rate are known from the rotor configuration, and thrust coefficient is provided by the lifting-surface code along with the circulation distribution.

The equations are described below (see Fig. 3.1 for the coordinates):

**Tip Vortex Axial Coordinates:**

$$z_T = k_1 \psi_w \quad \text{for} \quad 0 \leq \psi_w \leq \frac{2\pi}{Q} \quad (3.16)$$

$$z_T = k_1 \frac{2\pi}{Q} + k_2 (\psi_w - \frac{2\pi}{Q}) \quad \text{for} \quad \psi_w \geq \frac{2\pi}{Q} \quad (3.17)$$
where

\[ k_1 = -0.25\left(\frac{C_T}{\sigma} + 0.001\theta_t\right) \]

\[ k_2 = -(1.41 + 0.0141\theta_t)\sqrt{C_T/2} \]

Tip Vortex Radial Coordinates:

\[ r_T = A + (1 - A) \exp(-\lambda\psi_w) \quad (3.18) \]

where

\[ A = 0.78 \]

\[ \lambda = 0.145 + 27C_T \]

Inboard vortex sheet axial coordinates:

\[ z_{r=1} = g_1\psi_w \quad \text{for} \quad 0 \leq \psi_w \leq \frac{2\pi}{Q} \quad (3.19) \]

\[ z_{r=1} = g_1\frac{2\pi}{Q} + g_2(\psi_w - \frac{2\pi}{Q}) \quad \text{for} \quad \psi_w \geq \frac{2\pi}{Q} \quad (3.20) \]

where

\[ g_1 = -2.2\sqrt{C_T/2} \]

\[ g_2 = -2.7\sqrt{C_T/2} \]
and

\[ \varepsilon_{r=0} = 0 \quad \text{for} \quad 0 \leq \psi_w \leq \frac{\pi}{2} \quad (3.21) \]

\[ \varepsilon_{r=0} = g_3(\psi_w - \frac{\pi}{2}) \quad \text{for} \quad \psi_w \geq \frac{\pi}{2} \quad (3.22) \]

where

\[ g_3 = \left[ \frac{\theta_0}{128} (0.45\theta_0 + 18) \right] \sqrt{C_T/2} \]

Inboard vortex sheet radial coordinates:

\[ \mathbb{F}_A = \frac{\mathbb{F}_C}{\mathbb{F}_B} \quad (3.23) \]

In the above equations, \( \psi_w \) is the age of the vortex after it was generated by the blade, and \( \theta_0 \) is the rate of change of local blade pitch angle due to built-in linear twist with respect to blade spanwise direction, positive when tip section is twisted leading-edge up relative to root section, degrees. It was assumed that the vortex sheet cross sections were linear, and the vortex sheet boundary was equivalent to the boundary formed by the locus centers of the tip vortex cross sections (equivalently, tip vortex streamline). \( \mathbb{F}_D \) is the point of maximum circulation on the blade, \( \mathbb{F}_C \) is its corresponding point some where downstream, \( \mathbb{F}_B \) is an arbitrary point on the blade, and \( \mathbb{F}_A \) is its corresponding point downstream (Fig. 3.1). A typical wake geometry prescribed by the above equations is shown in Fig. 3.2.

A far-wake was placed underneath the tip vortex of the last free-to-move vortex sheet, since the vortex wake would not stop abruptly. One
Figure 3.1: Schematic of wake cross section, showing wake coordinate system.
Figure 3.2: The cross section of the vortex wake plus the far-wake.
also has to acknowledge that there are uncertainties about the far wake. The evidences gathered from experiments are that the tip vortex diffuses rapidly after it was formed and bursts after two to three rotor revolutions. Although the burst of the tip vortex makes the far-wake very difficult to detect, one certainly is sure that this far-wake exists and must be included in the model. A semi-infinite vortex cylinder and a stack of vortex rings were both used as the far wake model. The radius of the far wake was equal to the radius of the tip vortex of the last free-to-move vortex sheet, and the axial spacing of the far wake was assumed equal to that of the last two tip vortices belonging to the corresponding last two free-to-move vortex sheets. The difference in the results due to the use of the two different far wake models were negligible as long as the far wake extends beyond five blade radii.

Fig. 3.3 shows the flow-chart of the free-wake analysis. At the time corresponding to a blade passage, a new vortex sheet, identical to the one initially placed on the rotor plane was added to the rotor plane to represent the vortex sheet shed by the passing blade. Meanwhile, the last free-to-move vortex sheet (an inboard vortex sheet plus a tip vortex) was removed from the computational domain, keeping the number of free-to-move vortex sheets constant. Between blade passages, the flow field was force-free, i.e., the vortex wake moved by their mutual interactions.

The vortex wake is integrated step by step between blade passages. At each time step, the computational procedure is:

1. Calculate the boundary values of the stream function by Eq. (2.15).

2. Solve the Poisson equation, Eq. (2.14), to obtain the stream function.
Solve the vorticity transport equation

Is another blade passing by?

Is the vortex wake converged?

Circulation distribution on the blade from the lifting-surface code

Guessing a shape for the vortex wake

Calculate the boundary values of vorticity and stream function

Solve the Poisson equation to obtain the velocity field

Solve the vorticity transport equation

Add a new vortex sheet on the rotor plane and remove the last vortex sheet out from the computational domain

Stop

Figure 3.3: The flow-chart of the free-wake analysis.
3. Obtain the velocity field from the stream-function field.

4. Integrate the vortex wake one step forward by the vorticity transport equation, Eq. (2.13). The time step is 3°; therefore 120 time steps correspond to a rotor revolution.

This process is repeated until another blade passes by.

3.2 Relaxation

It was found that the tip vortices are quite unstable. The tip vortex instability grows rapidly in computational process. This is owing to the hydrodynamic instability of the problem, as is also confirmed by experiment. In order to solve this problem, a relaxation scheme was developed and applied to the numerical scheme to stabilize the tip vortices. More about this instability problem will be discussed in Chapter 6. The relaxation scheme is described in the following paragraph.

This relaxation scheme was applied at the time of blade passages. When a blade just passed by, the positions of the tip vortices were relaxed by the following formulas:

\[
r_T^{\text{new}} = r_T^{\text{old}} + \omega(r_T^{\text{present}} - r_T^{\text{old}}) \quad (3.24)
\]

\[
z_T^{\text{new}} = z_T^{\text{old}} + \omega(z_T^{\text{present}} - z_T^{\text{old}}) \quad (3.25)
\]

where \(r_T^{\text{new}}\) and \(z_T^{\text{new}}\) represent the new coordinates for the next loop of iteration, \(r_T^{\text{present}}\) and \(z_T^{\text{present}}\) represent the coordinates at the end of the present loop, \(r_T^{\text{old}}\) and \(z_T^{\text{old}}\) represent the coordinates at the beginning of
the present loop, and $\omega$ is the relaxation factor. A relaxation factor of 0.5 and 0.25 were used for the calculations of two- and four-bladed rotor correspondingly.

The idea of the relaxation scheme used in the present method is similar to those ideas used in the other relaxation methods, such as the over relaxation method for solving the elliptic differential equations. First, a unique solution is assumed to exist, then the searching is started by running a proposed initial solution, and the final solution is approached by reducing the errors between successive iterations. In order to apply the relaxation scheme to the tip vortices, each free-to-move vortex sheet was separated into two portions, a tip vortex and an inboard vortex sheet. The separation point was the maximum circulation on the blade, since it was assumed that the tip vortex rolls up from the blade tip to the position of maximum circulation. The tip vortex and inboard vortex sheet were stored in different arrays. Whenever the relaxation scheme was applied, the centroid and strength of each tip vortex was calculated by the Simpson rule [26]. The centroids were used as the locations of tip vortices in the relaxation scheme.

3.3 Reconcentration

Another problem encountered before meaningful results were obtained was the diffusion (or artificial viscosity) problem. This problem originates from the large grid size, 1% of the blade radius, used in the mesh system. To solve this problem, a reconcentration scheme was used. This scheme was applied at the time of blade passages. Each tip vortex was reconcentrated as a Lamb vortex (page 17), its strength was unchanged, the core radius was assumed 2% of the blade radius, and the location was the centroid of the
original tip vortex. Each inboard vortex sheet was divided into subdivisions along the r-direction, typically 5% of the blade radius was used as the width of a subdivision. The centroid and strength of the vorticities inside each subdivision were calculated by the Simpson rule [26]. The vorticities inside each subdivision were assumed to be represented by a Lamb vortex, its strength equals the sum of the vorticities in the subdivision, its core radius was assumed to be 2% of the blade radius, and its location was the centroid of the vorticities in the subdivision. The inboard vortex sheet was then represented by the superposition of these corresponding Lamb vortices. The discussion of the details of this reconcentration will be presented in Chapter 6.

3.4 Correction of the Self-Induced Velocity

A correction to the self-induced velocity of a vortex ring was added to the velocity field of each tip vortex to account for the deficiency caused by the assumption of a large core radius. For a vortex ring, the analytical solution of the self-induced velocity is [27]

\[ w_s = \frac{\Gamma}{4\pi r} \left[ \ln \frac{8r}{a} - \frac{1}{4} \right] \]  

(3.26)

where \( \Gamma \) is the strength of the vortex ring, \( r \) and \( a \) are its radius and core radius correspondingly.

The core radius assumed in the free-wake analysis was 2% of the blade radius. But the tip vortex core radius of a two-bladed rotor at 60° after it was shed by a blade is about 0.5% of the blade radius [28]. There is a significant difference in the magnitude of the self-induced velocity especially of the tip vortices. Yet, self-induced velocity is an important mechanism in
the motion of the tip vortices. A corrective term

\[
\omega_c = \frac{\Gamma}{4\pi r} \left[ \ln \frac{8r}{a_{\text{real}}} - \ln \frac{8r}{a_{\text{assumed}}} \right]
\]  

(3.27)
is added to the velocity field of each tip vortex at each time-step to resolve this deficiency. A value of 0.5\% is used for the \( a_{\text{real}} \). We were able to do so since we treated tip vortex and inboard vortex sheet separately. They shared the same velocity field (except this corrective self-induced velocity), but each one was independent from the others when marched by the vorticity transport equation. For instance, assuming there were five tip vortices and five inboard sheets, then we had to calculate the vorticity transport equation ten times.
Chapter 4
Computational Methods

4.1 Computational Method of the Poisson Equation

In computational fluid dynamics, the Poisson equation is solved frequently. Very often, the computational cost of a calculation depends strongly on the efficiency in solving the Poisson equation in the finite difference form. If the geometry or the boundary condition of a problem is complicated, the difference equations would have to be solved by an iterative approach. However, when problems with rectangular geometry and simple boundary conditions are considered, the direct solution methods, much faster than the iterative or the Gaussian elimination methods, are available [29]. Among several fast direct solution methods, probably the cyclic reduction method, which was suggested by Buzbee [30], devised by Hockney [31] [32], and later improved by Buneman [33], is most widely used.

After carrying out the Laplacian operator of Eq. (2.9), the Poisson equation has the following form

\[
\frac{\partial^2 \psi}{\partial r^2} + \frac{\partial^2 \psi}{\partial z^2} - \frac{1}{r} \frac{\partial \psi}{\partial r} = -r \xi
\]  

(4.28)
and the finite difference form of the Poisson equation after some arrangement is

\[
\left( -\frac{A^2}{2r_{i,j}^2} - \frac{A^2}{r^2} \right) \psi_{i-1,j} + 2\left( \frac{A^2}{r^2} + 2 \right) \psi_{i,j} \\
+ \left( -\frac{A^2}{r^2} + \frac{A^2}{2r_{i,j}^2} \right) \psi_{i+1,j} - \psi_{i,j+1} - \psi_{i,j-1}
\]

\[= r_{i,j} A^2 \xi_{i,j} \quad (4.29)\]

In order to apply the cyclic reduction method, the grid size of the mesh system in the \( z \) direction must be constant (but this is not necessary in \( r \) direction). The boundary conditions of the Dirichlet type, the Neumann type, and the cyclic boundary condition are all acceptable to the present method. With \( I \) columns times \( J \) rows of grid points excluding those on the boundaries, the finite difference equations for the Poisson equation with the Dirichlet boundary conditions become a block-tridiagonal matrix. The block-tridiagonal matrix becomes particularly simple due to the uniform mesh spacing in \( z \) direction. The nonnull off-diagonal blocks are all identity matrices, and the diagonal blocks are all identical tridiagonal matrices. One restriction is that the number of the blocks, namely, \( J \), must be equal to \( 2^k - 1 \), where \( k \) is an integer. For example, if \( k = 3 \), then \( J = 2^3 - 1 = 7 \),
then the difference equation may be written as

\[
\begin{pmatrix}
A & -I \\
-I & A & -I \\
-I & A & -I \\
-I & A & -I \\
-I & A & -I \\
-I & A & -I \\
\end{pmatrix}
\begin{pmatrix}
X_1 \\
X_2 \\
X_3 \\
X_4 \\
X_5 \\
X_6 \\
\end{pmatrix}
=
\begin{pmatrix}
Y_1 \\
Y_2 \\
Y_3 \\
Y_4 \\
Y_5 \\
Y_6 \\
\end{pmatrix}
\]

(4.30)

where \( A \) is a tridiagonal matrix of order \( J \), \( I \) is an identity matrix of the same order, \( X_j \) represents the solution along the \( j \)th grid row, and \( Y_j \) represents the inhomogeneous terms for the \( j \)th grid row. If the even rows are multiplied by \( A \) and the two adjacent rows are added to each even row, the difference equation becomes

\[
\begin{pmatrix}
A & -I \\
0 & A^{(2)} & 0 & -I \\
-I & A & -I \\
-I & 0 & A^{(2)} & 0 & -I \\
-I & A & -I \\
-I & 0 & A^{(2)} & 0 \\
\end{pmatrix}
\begin{pmatrix}
X_1 \\
X_2 \\
X_3 \\
X_4 \\
X_5 \\
X_6 \\
\end{pmatrix}
=
\begin{pmatrix}
Y_1 \\
Y_2^{(2)} \\
Y_3 \\
Y_4^{(2)} \\
Y_5 \\
Y_6^{(2)} \\
\end{pmatrix}
\]

(4.31)

where

\[
A^{(2)} = A^2 - 2I
\]

(4.32)
The even rows are independent of the odd rows, by applying the same technique the even rows can be further reduced to a single row. This single equation is solved and the solutions to the remaining rows are found by solving the reduced equations appropriate to each level of reduction. This algorithm avoids any need for Fourier analysis and leads to a short computer program. There is a problem one has to be aware of: Factorization is used during the computation, and the central coefficient of the factorization grows rapidly on the order of $4^m$ ($m = 2^k, m - 1$ gives the grid points in $z$ direction) [32]. For $k = 7$, which would occur on a mesh with 128 vertical grid points, the central coefficient is of the order $10^{78}$ and overflow will occur on some computers. There are some ways to resolve this problem (for detail see [30]).

4.2 Computational Methods of the Vorticity Transport Equation

The alternating direction implicit method (ADI method) [34] was used to solve the vorticity transport equation,

$$\frac{\partial \zeta}{2\pi \partial \phi} = u \frac{\partial \zeta}{\partial r} + w \frac{\partial \zeta}{\partial z} - \frac{1}{Re} \left( \frac{\partial^2 \zeta}{\partial r^2} + \frac{1}{r} \frac{\partial \zeta}{\partial r} + \frac{\partial^2 \zeta}{\partial z^2} - \frac{\zeta}{r^2} \right)$$  \hspace{1cm} (4.34)

The alternating-direction-implicit method, or the ADI method, was first introduced in companion papers by Peacemen, Rachford, and Douglas (1955). Also known as the method of variable direction, this method makes use of a splitting of the time step to obtain a multi-dimensional implicit method which requires only the inversion of a tridiagonal matrix. The
advancement of the vorticity transport equation over $\Delta t$ is accomplished in two steps, as

$$\frac{\phi_{i,j}^{n+1/2} - \phi_{i,j}^n}{\pi \Delta \phi} = u \left( \frac{\phi_{i+1,j}^{n+1/2} - \phi_{i-1,j}^{n+1/2}}{2\Delta r} \right) + w \left( \frac{\phi_{i,j+1}^{n+1/2} - \phi_{i,j-1}^n}{2\Delta z} \right)$$

$$- \frac{1}{Re} \left( \frac{\phi_{i+1,j}^{n+1/2} - 2\phi_{i,j}^{n+1/2} + \phi_{i-1,j}^{n+1/2}}{\Delta r^2} \right) \frac{1}{r} + \frac{1}{2\Delta r} \left( \phi_{i+1,j}^{n+1/2} - \phi_{i,j}^{n+1/2} \right)$$

$$+ \frac{\phi_{i,j+1}^{n+1} - 2\phi_{i,j}^n + \phi_{i,j-1}^n}{\Delta z^2} \frac{\phi_{i,j}^{n+1/2} - \phi_{i,j}^n}{r_{i,j}^2} \right)$$

(4.35)

$$\frac{\phi_{i,j}^{n+1} - \phi_{i,j}^{n+1/2}}{\pi \Delta \phi} = u \left( \frac{\phi_{i+1,j}^{n+1/2} - \phi_{i-1,j}^{n+1/2}}{2\Delta r} \right) + w \left( \frac{\phi_{i,j+1}^{n+1} - \phi_{i,j-1}^n}{2\Delta z} \right)$$

$$- \frac{1}{Re} \left( \frac{\phi_{i+1,j}^{n+1/2} - 2\phi_{i,j}^{n+1/2} + \phi_{i-1,j}^{n+1/2}}{\Delta r^2} \right) \frac{1}{r} + \frac{1}{2\Delta r} \left( \phi_{i+1,j}^{n+1/2} - \phi_{i,j}^{n+1/2} \right)$$

$$+ \frac{\phi_{i,j+1}^{n+1} - 2\phi_{i,j}^{n+1} + \phi_{i,j-1}^n}{\Delta z^2} \frac{\phi_{i,j}^{n+1/2} - \phi_{i,j}^{n+1/2}}{r_{i,j}^2} \right)$$

(4.36)

or the $r - z$ permutation [34].

The advantage of this approach over the fully implicit methods is that each equation, although implicit, is only tridiagonal. The stability of this two-dimensional method is unconditional, as in the fully implicit method. The accuracy of this method is second order in both space and time, $O(\Delta t^2, \Delta r^2, \Delta z^2)$. 
Chapter 5

A Lifting-Surface Method, AMI Hover, for Rotors in Hover or Climb

5.1 Mathematical Formulation

This chapter is paragraphed from [6] by Summa for the completeness of this document and the convenience of the readers. For detail descriptions of the method, please refer to [6].

The flow field is assumed inviscid and incompressible in this method. Therefore, the governing partial differential equation for the velocity potential is Laplace's equation. The velocity potential for this problem is subject to the following boundary conditions: the infinity condition, the flow tangency condition, the Kutta-Joukowski condition, the dynamic free-surface condition, and the geometric free-surface condition. It is assumed that the motion has continued for some time so that the fluid motion is independent of time, i.e., steady flow. The dimensionless velocity at some arbitrary point, P, in the external fluid domain with respect to the blade-
fixed coordinate frame is (see Fig. 5.1)

\[ \vec{v} = \vec{v}_o(P) + \nabla \phi(P) \]  

(5.37)

where

\[ \vec{v} = \text{Dimensionless velocity} = \frac{\vec{v}}{\bar{U}} \]

\[ \vec{v}_o = \text{Dimensionless onset velocity} = \frac{-\vec{v}_0 + \vec{v}_\Omega}{\bar{U}} \]

\[ \nabla = \text{Dimensionless gradient operator} \]

\[ \phi = \text{Dimensionless disturbance velocity potential} = \frac{\phi}{\bar{U}} \]

\[ \vec{V}_b = \text{Rotor climbing velocity} \]

\[ \Omega = \text{Rotor rotational speed} \]

\[ \vec{r} = \text{Position vector of P} \]

\[ R = \text{Rotor radius} \]

If the blade thickness is neglected, then the integral solution for \( \phi \) can be written as [6] (see Fig. 5.1)

\[ \phi(P) = \frac{1}{4\pi} \sum_{i=1}^{b} \int \int_{S_i + W_i} \sigma \frac{\partial}{\partial n} \left( \frac{1}{\xi} \right) dS \]  

(5.38)

Here,

\[ b = \text{Number of blades} \]

\[ S_i = \text{Blade surface} \]

\[ W_i = \text{Vortex wake surface} \]
Figure 5.1: Primary blade upper camber surface and projected planform surface.
\[ \sigma = \text{Doublet strength per unit area} \]

\[ \xi = \text{The directed distance from } P \text{ to the integration point} \]

\[ \vec{n} = \text{Running unit normal} \]

Eq. (5.38) states that the velocity potential (integral solution of Laplace's equation) is the calculated influence of the unknown doublet distributions on the blade and wake surfaces. Because the flow field is a body of revolution (each blade will see the same local flow field, wake patterns, etc.), the doublet distribution on each of the blades and wakes are identical and the flow tangency condition is enforced on the primary blade as follows

\[ \frac{\partial \phi}{\partial n}(x, y, z) = -\vec{v}_o \cdot \vec{n} \quad \text{on } S_i \]  

(5.39)

If the twist angles, coning angle, angles of attack, etc., and the blade displacements from the rotor plane are all small, then a linearized lifting-surface representation of Eqns. (5.38) and (5.39) can be written as

\[ \phi(P) \approx \frac{1}{4\pi} \sum_{i=1}^{k} \left( \int_{S_i} \sigma \frac{\partial}{\partial \xi} \frac{1}{\xi} dS + \int_{W_i} \sigma_w \frac{\partial}{\partial n} \frac{1}{\xi} dS \right) \]  

(5.40)

which is subject to

\[ \frac{\partial \phi}{\partial z}(x, y, 0) \approx -\vec{v}_o \cdot \vec{n} \quad \text{on } S_i' \]  

(5.41)

and the remaining boundary conditions. As illustrated in Fig. 5.1), \( S_i' \) is the blade planform area projected into the rotor disk plane while the wake surface, \( W_i' \), is obtained by preserving the relative distances between \( W_i \) and \( S_i \). Of Course, this approximation greatly simplifies the computation of the blade surface contribution to \( \phi \), since the effects of camber, twist, and coning are now carried only by the right-hand side of Eq. (5.41).
5.2 Numerical Representation

The solutions for the doublet distributions are obtained by representing the planform and wake surfaces by discrete quadrilateral panels and assuming a constant doublet distribution over individual panels. Therefore, the discrete doublet strengths of these panels become the basic unknowns of the problem. If each blade is divided into $N_R$ rows and $N_C$ columns of panels $(N = N_R \times N_C)$ then,

$$
\phi(P) = \frac{1}{4\pi} \sum_{k=1}^{N} \sum_{t=1}^{N} \int \int_{\Delta s_{k,t}} \frac{\partial}{\partial z}(\frac{1}{\xi_{k,t}})ds
$$

$$
+ \sum_{m=1}^{N} \sigma_{w(m)} \int \int_{\Delta w_{m,t}} \frac{\partial}{\partial n_{m,t}}(\frac{1}{\xi_{m,t}})ds
$$

where $\Delta s_{k,t}$ is the $k^{th}$ panel surface of blade $t$ and $\Delta w_{m,t}$ is the $m^{th}$ wake surface column of blade $t$ (see Fig. 5.2). The axisymmetric nature of the fluid motion has been included in Eq. (5.43) in that

$$
\sigma_{k,i} = \sigma_{k,j} = \sigma_k
$$

and

$$
\sigma_{w(m,i)} = \sigma_{w(m,j)} = \sigma_{w(m)}
$$

Also, the wake dynamical boundary condition (or, equivalently, the Kelvin-Helmholtz Theorem) has been used in that the doublet strength per unit area of an entire wake column is a constant value. Of course, since the lattice and wake surfaces are of zero thickness, the doublet strength is equal to the potential jump across the vortex sheet, i.e., $\sigma = \Delta \phi = \phi_u - \phi_l$. 
Figure 5.2: Lifting surface by discrete panels.
The surface tangency boundary condition equation is written in numerical form as

\[ A_{ij} B \sigma_j + A_{im} W \sigma_{wm} = C_i \]  \hspace{1cm} (5.45)

where

- \( A_{ij} B \) = Blade influence coefficient matrix, each element giving the dimensionless normal velocity on panel \( i \) (blade \( \ell \)) induced by panel \( j \) of all blades

- \( A_{im} W \) = Blade influence coefficient matrix, each element giving the dimensionless normal velocity on panel \( i \) (blade \( \ell \)) induced by the \( m^{th} \) wake column of all wakes

- \( C_i = -(\bar{u}_i) \cdot \bar{n}_i \)

Of course this is further simplified upon enforcing the Kutta condition. This boundary condition is satisfied by requiring that the wake doublet strength of a column equal the doublet strength of the adjacent trailing-edge panel, i.e.,

\[ (\sigma_w)_m = \sigma_{TE} \hspace{0.5cm} \text{at} \hspace{0.5cm} m^{th} \text{ column} \] \hspace{1cm} (5.46)

As a consequence, wake contributions to the aerodynamic matrix can be added to the blade matrix appropriately to give symbolically

\[ A_{ij} \sigma_j = C_i \] \hspace{1cm} (5.47)

Of course, the right-hand side is known so that, once the wake geometry is specified, the aerodynamic matrix can be generated and, finally, Eq. (5.47) is inverted to obtain the unknown panel doublet (vortex ring) strengths. Once the doublet values are known, the calculation of the rotor performance properties proceeds.
Chapter 6

Results and Discussions of the Free-Wake Calculations

6.1 Tip Vortex Instability

Early calculations of the free-wake analysis were done without applying the relaxation scheme to the tip vortices. The results were not satisfactory due to the tip vortex instability encountered. Fig. 6.1 through Fig. 6.15 show a typical development of the instability. The model rotor was a Huey 1/7-scale two-bladed rotor [28]. The blade parameters and operating conditions are listed on Table 6.1. Five free-to-move vortex sheets were used to model the portion of the vortex wake near the rotor plane, and its configuration was described initially by Landgrebe's prescribed wake equations (Chapter 3) as shown in Fig. 6.1. A far-wake which was consisted of thirty vortex rings was placed underneath the tip vortex ring of the last free-to-move vortex sheet (or tip vortex 1 in Fig. 6.1). The radius of the vortex rings in the far-wake was identical with that of the last free-to-move tip vortex ring (tip vortex 1); the vertical spacing between rings was equal to that of the last two free-to-move tip vortex rings (tip vortex 1 and 2).
At the time corresponding to a blade passage, a new vortex sheet, identical to the one initially placed on the rotor plane, was added to the rotor plane to represent the vortex sheet just shed by the passing blade. Meanwhile, the last free-to-move vortex sheet, a tip vortex plus an inboard vortex sheet, was removed out from the domain keeping the number of the free-to-move sheets constant. The reconcentration scheme was applied to the vortex sheets at every 45°, a quarter blade passage or one eighth of a rotor revolution, to prevent the excessive numerical diffusion.

The evolution of the wake is shown at every 45°, corresponding to one eighth of a rotor revolution. The configuration of the vortex wake remains well defined from 0° to 180°. The beginning of the instability can be seen at 180° or right before the first blade passage (Fig. 6.5). The tip vortex initially on the rotor plane, tip vortex 5, moved too much inward after the first blade passage. Its radius was less than that of the tip vortex, tip vortex 4, beneath it. From 180° to 360°, tip vortex 4 and tip vortex 5 locked together and started rotating around each other. The cause of the pairing of tip vortex 4 and 5 is that these two vortices are locked together through their mutual interaction. The induced velocities due to this mutual interaction are stronger than that from the rest of the flow field; therefore one is confined by another like being locked together. This rotating continued through 540°, corresponding to three blade passages. At 540°, their positions were reversed.

Fig. 6.16 through Fig. 6.19 show the development of instability of a linearly twisted four-bladed rotor [35], rotor 4 in Table 6.1, from the first to the second blade passage, corresponding to 90° to 180° in azimuthal direction. The pairing and rotating of tip vortices can be seen clearly in
Table 6.1: Model rotor blade parameters and operating conditions.

<table>
<thead>
<tr>
<th>Blade Parameter</th>
<th>Rotor 1</th>
<th>Rotor 2</th>
<th>Rotor 3</th>
<th>Rotor 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>No. of Blades</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>Tip Speed (ft/sec)</td>
<td>449.</td>
<td>700.</td>
<td>700.</td>
<td>351.</td>
</tr>
<tr>
<td>Collective Pitch (deg)</td>
<td>8</td>
<td>8</td>
<td>8</td>
<td>10</td>
</tr>
<tr>
<td>Linear Twist (deg)</td>
<td>none</td>
<td>none</td>
<td>-8.</td>
<td>-8.3</td>
</tr>
<tr>
<td>Aspect Ratio, AR</td>
<td>13.7</td>
<td>18.2</td>
<td>18.2</td>
<td>15.</td>
</tr>
<tr>
<td>Radius, R (ft)</td>
<td>3.428</td>
<td>2.229</td>
<td>2.229</td>
<td>2.461</td>
</tr>
<tr>
<td>Chord, c (ft)</td>
<td>0.250</td>
<td>0.122</td>
<td>0.122</td>
<td>0.164</td>
</tr>
<tr>
<td>Airfoil Section</td>
<td>NACA</td>
<td>NACA</td>
<td>NACA</td>
<td>FRENCH</td>
</tr>
<tr>
<td>Root Cutout (%R)</td>
<td>14.4</td>
<td>14.8</td>
<td>14.8</td>
<td>20.</td>
</tr>
<tr>
<td>Taper</td>
<td>none</td>
<td>none</td>
<td>none</td>
<td>none</td>
</tr>
</tbody>
</table>

Rotor 1: Huey 1/7-scale two-bladed rotor
Rotor 2: UTRC's untwisted two-bladed rotor
Rotor 3: UTRC's twisted two-bladed rotor
Rotor 4: ONERA's linearly twisted four-bladed rotor
UTRC: United Technologies Research Center
ONERA: Office National d'Etudes et de Recherches Aerospatiales
Figure 6.1: Vorticity contour plot for the Huey two-bladed rotor at $T=0^\circ$. 
Figure 6.2: Vorticity contour plot for the Huey two-bladed rotor at $T=45^\circ$. 
Figure 6.3: Vorticity contour plot for the Huey two-bladed rotor at $T=90^\circ$. 
Figure 6.4: Vorticity contour plot for the Huey two-bladed rotor at $T=135^\circ$. 
Figure 6.5: Vorticity contour plot for the Huey two-bladed rotor at $T=180^\circ$. 
Figure 6.6: Vorticity contour plot for the Huey two-bladed rotor at $T=225^\circ$. 
Figure 6.7: Vorticity contour plot for the Huey two-bladed rotor at $T=270^\circ$. 
Figure 6.8: Vorticity contour plot for the Huey two-bladed rotor at T=315°.
Figure 6.9: Vorticity contour plot for the Huey two-bladed rotor at $T=360^\circ$. 
Figure 6.10: Vorticity contour plot for the Huey two-bladed rotor at \(T=405^\circ\).
Figure 6.11: Vorticity contour plot for the Huey two-bladed rotor at $T=450^\circ$. 
Figure 6.12: Vorticity contour plot for the Huey two-bladed rotor at $T=495^\circ$. 
Figure 6.13: Vorticity contour plot for the Huey two-bladed rotor at $T=540^\circ$. 
Figure 6.14: Vorticity contour plot for the Huey two-bladed rotor at $T=585^\circ$. 
Figure 6.15: Vorticity contour plot for the Huey two-bladed rotor at $T=630^\circ$. 
Figure 6.16: Vorticity contour plot for the French linearly twisted four-bladed rotor at $T=90^\circ$. 
Figure 6.17: Vorticity contour plot for the French linearly twisted four-bladed rotor at $T=120^\circ$. 
Figure 6.18: Vorticity contour plot for the French linearly twisted four-bladed rotor at $T=150^\circ$. 
Figure 6.19: Vorticity contour plot for the French linearly twisted four-bladed rotor at $T=180^\circ$. 
the figures.

Once the instability started, continuation of the calculation made the vortex wake even more unstable. Several different model rotors were tested; all the results developed tip vortex instability rapidly, within two to three blade passages.

A parametric study was undertaken trying to find the possible causes of the instability, such as changing the number of the free-to-move vortex sheets in modeling the wake, changing the radius and spacing of the vortex rings in the far-wake, changing the grid size etc.. But in each case the instability developed rapidly. No improvements were observed. It was concluded that the instability might be characteristics of the present problem.

The vortex wake of a hovering rotor is extremely unstable, the instability can be stimulated by disturbances easily, as is also confirmed by experiment. The stability of small perturbations to the vortex wake of a hovering rotor was investigated by Bliss et al [5]. They converted the stability problem into an eigenvalue problem and solved it numerically. They found that most of the eigenvalues were complex, and many had positive real parts, which indicates numerous unstable modes.

Some of the assumptions used in the present method introduced disturbances into the computational domain during the computational process. These could be as the exclusion of the inboard vortex sheet in the far wake, the neglect of the effect of the circulation on the blade to the vortex wake, and the assumption of axisymmetry. It can be expected that these disturbances would be amplified in the calculations, because the vortex wake is inherently unstable. But if a vortex wake is undisturbed the vortex wake preserves a well defined configuration. Based on this assumption, a relax-
ation scheme was developed to solve the instability problem (as described in Chapter 3).

6.2 Results by Applying Relaxation and Reconcentration Schemes

6.2.1 Huey 1/7-scale untwisted two-bladed rotor

Satisfactory results were obtained after applying the relaxation and reconcentration schemes. The calculations of four different rotors had been completed. The first rotor studied was the Huey 1/7-scale untwisted two-bladed rotor, rotor 1 in Table 6.1. The grid size was 1% of the blade radius. The number of grid points used was 151 along r direction and 601 along z direction. The boundary values of the stream function were obtained by direct integration of Eq. (2.15). The far wake was a stack of 30 vortex rings, each of them had the same strength and radius of the tip vortex of the last free-to-move vortex sheet, and the vertical spacing between neighbouring rings was identical with that of the last two tip vortices belonging to the corresponding last two free-to-move vortex sheets. 30 vortex rings extended to about 5 blade radii below the rotor plane, and beyond this extent the far wake has little effects on the results.

An uncontracted vortex wake was assumed as the initial configuration of the vortex wake. Fig 6.20 shows the configuration in which local centroids of the vortex sheets calculated by the reconcentration scheme were plotted out and not the vorticity contour. The use of an uncontracted vortex wake as the initial configuration is of academic interest but certainly will result in a slower convergence of the results. If one wants to converge the solution
Figure 6.20: The model of the vortex wake for the initial condition.
faster, a closer initial configuration can be assumed.

The uncontracted wake, the rotor configuration, and the operating conditions were used as the input to the lifting surface code, AMI Hover [6] (see Chapter 5), to obtain the circulation distribution on the blade. The output, circulation distribution on the blade, was used as the input to the free-wake calculation. The free-wake calculation terminated when the configuration of the vortex wake converged under this specified circulation distribution. The new configuration of the vortex wake was again used as the input to the lifting surface code. This process was repeated until both the circulation distribution on the blade and the configuration of the vortex wake converged. It took seven repetitions to converge the results in this case, and the CPU time required was about 2 hours in Cray X-MP/48.

Fig. 6.21 and Fig. 6.22 show the converging process of the configuration of the vortex wake. In these two figures, only the locations of the tip vortices are shown, and the inboard vortex sheets were not shown. Fig. 6.23 and Fig. 6.24 show the converging process of the circulation distribution on the blade. From the above figures we observe that when an uncontracted vortex wake was first assumed; the peak circulation was relatively flat, since the tip vortex nearest to the rotor plane located further away from it and had a weaker interaction with the blade. This flatter circulation distribution resulted in a vortex wake which stayed closer to the rotor plane.

The peak value of the circulation on the blade is the dominant element in determining the configuration of the vortex wake. This is owing to the fact that the tip vortex rolls up from the blade tip to the location of the maximum circulation, and its strength equals to the maximum circulation on the blade. The tip vortex nearest to the rotor plane has a decisive
LEGEND

• Iteration 1

○ Iteration 2

△ Iteration 3

▼ Iteration 4

Figure 6.21: Converging process of the configuration of the vortex wake for the Huey 1/7-scale two-bladed rotor.
Figure 6.22: Converging process of the configuration of the vortex wake for the Huey 1/7-scale two-bladed rotor.
Figure 6.23: Converging process of the circulation on the blade for the Huey 1/7-scale two-bladed rotor.
Figure 6.24: Converging process of the circulation on the blade for the Huey 1/7-scale two-bladed rotor.
influence on the circulation distribution on the blade, since it stays very close to the rotor plane. In some cases, the separating distance between the blade and the tip vortex underneath it is less than a half of the blade chord. For some highly twisted blades, the blade could cut through the tip vortex generated by the preceding blade.

In general, if the tip vortex nearest to the blade stays close to the rotor plane, it reduces the rotor thrust coefficient, through inducing downward velocities to the inboard portion of the blade. These downward velocities reduce the angles of attack in the inboard portion of the blade and thus reduce the thrust. The tip vortex nearest to the rotor plane does induce upward velocities to the portion of the blade outboard of it. But this outboard portion is small, around 10% of the blade radius, and can not compensate the loss in the inboard portion of the blade.

The tip vortices of the free-to-move vortex sheets below the tip vortex nearest to the rotor plane all push it upward, since they are located further inboard than the nearest one (see Fig. 6.22, the tip vortices rotate counterclockwise). The far wake pushes the tip vortex nearest to the rotor plane down, but its effect is limited, about the order of the upward induction from the second or the third tip vortex below the nearest one. The dominant mechanism that pushes the nearest tip vortex down is its self-induced velocity, which is a function of its strength and core size, and in turn is determined by the peak circulation on the blade.

The previous paragraph explains why a flatter circulation distribution results in a wake which stays closer to the rotor plane. Curve 2 of Fig. 6.23 shows the effects of this closer wake on the circulation distribution on the blade. There is a high peak in the circulation distribution which was re-
sulted from the closer wake. The tip vortex immediately below the rotor plane stayed very close to the rotor plane and induced strong upward velocities to the outboard portion of the blade. But the thrust coefficient actually was lower than that of the previous iteration due to the higher loss of lift force in the inboard portion of the blade.

The configuration of the vortex wake and the circulation distribution on the blade both converge quite well. This can be seen from the last two curves of Fig. 6.22 and Fig. 6.24, which are almost identical. Fig. 6.25 shows the comparison of the numerical vortex wake with the experimental data [28] (only the tip vortex locations are available in the experimental data). The agreement is fairly good considering the uncertainties and difficulties involved in both hovering test and numerical simulation. Fig. 6.26 shows the configuration of the converged vortex wake. The numerical thrust coefficient is 0.0035 compared with the experimental value of 0.0037.

Fig. 6.27 shows the streamline plot of the converged vortex wake. The difference of the stream function values in every two consecutive streamlines is constant. The circles at radial coordinate of about 0.78 are the tip vortices. The density of the streamlines is an indication of the volume flow rate, or mass flow rate since the flow is incompressible. The denser the streamlines are, the higher the volume flow rate is. One can see that the volume flow rate is low in the center region, and becomes higher in the regions close to the tip vortices. This is because the tip vortices have high strength and therefore induce higher velocities to fluid particles near by. One can also see that some of the streamlines extend to the right hand side boundary. This is owing to fluid particles are sucked in from the boundary by the vortex rings in the far-wake.
Figure 6.25: The comparison of numerical results with experimental data for the Huey 1/7-scale two-bladed rotor.
Figure 6.26: The configuration of the converged vortex wake for the Huey 1/7-scale two-bladed rotor.
Figure 6.27: The streamline plot of the converged vortex wake for the Huey 1/7-scale two-bladed rotor.
6.2.2 UTRC's untwisted two-bladed rotor.

The second rotor studied was a UTRC's (United Technologies Research Center) untwisted two-bladed rotor used by Landgrebe and his colleague [2] in their hover test. The blade parameters and operating conditions are listed in Table 6.1. Fig. 6.28 through Fig. 6.33 show the results of the calculations. The comparison of the numerical results and the experimental data is shown in Fig. 6.32. The calculated thrust coefficient was 0.0030, and the experimental value was 0.0028. The difference is about 8%.

6.2.3 UTRC's linearly twisted two-bladed rotor.

The third rotor studied was a UTRC's linearly twisted two-bladed rotor (see Table 6.1). Fig. 6.34 through Fig. 6.39 show the numerical results. The numerical results are compared with test data [2] in Fig. 6.38. The numerical thrust coefficient was 0.0031, and the test value was 0.0028. The difference is about 9%.

6.2.4 ONERA's linearly twisted four-bladed rotor.

The last rotor studied was a ONERA's (Office National d'Études et de Recherches Aerospatiales) linearly twisted four-bladed rotor [35] (see Table 6.1). Fig. 6.40 through Fig. 6.45 show the numerical results. The numerical results are compared with test data [35] in Fig. 6.44.

The relaxation factor used in the relaxation scheme in this case was 0.25, since the solution diverged with a relaxation factor of 0.5. Also, it took more iterations, nine compared with seven for two-bladed rotor. The divergence was probably owing to the spacing between vortex sheets is smaller and therefore is less stable and more sensitive to disturbances.
The numerical thrust coefficient was 0.0075 and the experimental value was 0.0078. In Fig. 6.42 and Fig. 6.43 the tip vortex trajectories are fitted by straight lines, because if a spline fit is used the trajectories move above the rotor plane. According to the numerical results, the tip vortex nearest to the rotor plane stayed very close to the rotor plane for a short time (after it was generated by the passing blade), and then started moving down; it did not move above the rotor plane.

Fig. 6.46 shows the comparison of the circulation distribution on the blade obtained by the present method with the results obtained by Steinhoff [36] and the test data. Fig. 6.47 shows the comparison of the configurations of the vortex wakes (only the tip vortex locations are shown).

6.3 The Effects of the Extent of the Computational Boundaries

The extent of the boundaries sometimes are very important in some fluid dynamics problems, especially those impose boundary conditions at infinity, such as boundary layer problems, jet problems, flow due to moving bodies etc.. In solving these problems by computational fluid dynamics methods, a finite domain is used, and the infinity conditions are imposed on the finite boundaries, since an infinite long boundary very often is impossible in numerical simulation (coordinate transformation in some cases stretches the boundary to near infinity, but sometimes it is not obtainable).

In the present problem, boundary values of vorticity and stream function need to be obtained to solve the problem. For vorticity, which was concentrated only in certain regions, it decays to zero when the boundary approaches infinity. It decays exponentially with the square of the distance;
Figure 6.28: Converging process of the configuration of the vortex wake for UTRC's untwisted two-bladed rotor.
Figure 6.29: Converging process of the configuration of the vortex wake for UTRC’s untwisted two-bladed rotor.
Figure 6.30: Converging process of the circulation on the blade for UTRC's untwisted two-bladed rotor.
Figure 6.31: Converging process of the circulation on the blade for UTRC's untwisted two-bladed rotor.
Figure 6.32: The comparison of numerical results with experimental data for UTRC's untwisted two-bladed rotor.
Figure 6.33: The configuration of the converged vortex wake for UTRC's untwisted two-bladed rotor.
Figure 6.34: Converging process of the configuration of the vortex wake for UTRC's linearly twisted two-bladed rotor.
Figure 6.35: Converging process of the configuration of the vortex wake for UTRC's linearly twisted two-bladed rotor.
Figure 6.36: Converging process of the circulation on the blade for UTRC's linearly twisted two-bladed rotor.
Figure 6.37: Converging process of the circulation on the blade for UTRC's linearly twisted two-bladed rotor.
Figure 6.38: The comparison of numerical results with experimental data for UTRC's linearly twisted two-bladed rotor.
Figure 6.39: The configuration of the converged vortex wake for UTRC's linearly twisted two-bladed rotor.
Figure 6.40: Converging process of the configuration of the vortex wake for ONERA's linearly twisted four-bladed rotor.
Figure 6.41: Converging process of the configuration of the vortex wake for ONERA's linearly twisted four-bladed rotor.
Figure 6.42: Converging process of the circulation on the blade for ON-ERA's linearly twisted four-bladed rotor.
Figure 6.43: Converging process of the circulation on the blade for ONERA's linearly twisted four-bladed rotor.
Figure 6.44: The comparison of numerical results with experimental data for ONERA's linearly twisted four-bladed rotor.
Figure 6.45: The configuration of the converged vortex wake for ONERA's linearly twisted four-bladed rotor.
Figure 6.46: The comparison of the circulation distribution on the blade for ONERA's linearly twisted four-bladed rotor.
Figure 6.47: The comparison of the tip vortex locations for ONERA's linearly twisted four-bladed rotor.
therefore a relatively short boundary can be used with the assumption that vorticities are zero on the boundary.

For stream function, an analytical expression [25], Eq. (2.15), was used to obtain the values of stream function on the boundary. Local centroids of the vorticity were regarded as point vortices in applying the expression. Maskew [37] indicates that the resulting error is small when the boundary is a distance \(1.5H\) away from the nearest local centroid where \(H\) is the maximum distance between adjacent local centroids. The maximum distance between local centroids in the present problem is that of adjacent tip vortices which in all four cases solved was less than 20% of the blade radius. The computational boundary was always kept at least 40% away from the nearest local centroid.

From the above discussions, one concludes that the error introduced by the usage of finite boundary should be small. Yet, a numerical experiment was executed to prove this conclusion. In the experiment, two cases with identical initial condition but different boundary lengths were calculated. The model rotor was the Huey 1/7-scale two-bladed rotor as used in the previous section. The first case had a computational domain of 1.5 by 3, 1.5 in radial direction and 3 in axial direction, and both were nondimensionalized by the blade radius. The second case had a domain of 2 by 6. The calculations were run for ten blade passages or five rotor revolutions. The calculations were run by the free-wake code only, there were no interactions with the lifting-surface code, since the purpose of this experiment was to explore the sensitivity of the results to the boundary length in our free-wake analysis.

Fig. 6.48 shows the comparison of the configurations of the vortex wakes
after five rotor revolutions. The points representing the inboard vortex sheets are the local centroids. The tip vortices and the first, second, and third inboard sheets (counted from the rotor plane down) in the two cases are almost overlapped with each other. There are small differences in the axial locations of the fourth and fifth inboard sheets. These differences are due to the accumulation of the errors. A vortex sheet stayed in the computational domain for five blade passages after it was generated, since five free-to-move vortex sheets were used to model the vortex wake near the rotor plane. And the differences shown on the figure are due to the accumulation of errors introduced by the usage of different boundaries, through the highly nonlinear interaction of the vortex wake.

The radial and axial velocities at the blade tip at different time steps were also printed out to examine their differences due to different boundary lengths. Their differences are well within the accuracy of the numerical schemes. The accuracy of the numerical schemes, discussed in Chapter 4, is on the order of $O(\Delta t^2, \Delta r^2, \Delta z^2)$. The grid size used was 0.01 and the time step was 0.035; therefore the accuracy of the numerical results were on the order of $O(10^{-3})$. Table 6.2 and 6.3 show the comparison of velocities at the blade tip from the beginning to the first blade passage. The differences are on the order of $O(10^{-4})$, well within the accuracy of the numerical schemes used.

The conclusion of the study in this section is that the usage of different boundary lengths introduces small differences to the configuration of the vortex wake resulted. These differences are due to the accumulation of errors through the highly nonlinear interaction of the vortex wake. But they are very minor. Their effects on the results of the lifting-surface calculation,
Figure 6.48: The comparison of the configurations of the vortex wake due to different sizes of computational domains.
**Table 6.2:** The comparison of radial velocities at the blade tip at different time steps from the results of two different computational domains.

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<th>2 by 6 domain</th>
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One $\pi \approx 3.1416$, corresponds to the time of a blade passage.
Table 6.3: The comparison of axial velocities at the blade tip at different time steps from the results of two different computational domains.

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<th>TIME</th>
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<th>2 by 6 computational domain</th>
</tr>
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</table>

One $\pi$, 3.1416, corresponds to the time of a blade passage.
when the configuration of the vortex wake is used as an input to the lifting-surface code, are negligible.

### 6.4 The Effects of the Grid Size

The vortex sheet shed by the blade was modeled by the superposition of a series of line vortices, and each line vortex was modeled by the Lamb vortex, Eq. (2.2). The grid size used affects directly the core radius that is assumed. For example, if a grid size of 0.02 is used, one can not use a core radius of 0.01, because the total vorticity will not be conserved. In the present work, the core radius was assumed twice of the grid size. That would give enough grid points to cover the vortex core, and approximately conserved the total vorticity. Therefore, the usage of different grid sizes change the core radius and thus change the vorticity distribution of the vortex sheet. A larger grid size results in a flatter vorticity distribution and a more spreading vortex sheet.

The thickness of the vortex sheet shed by the blade is very small, the smaller grid size, the better. But there are restrictions imposed by the computer memory and CPU time. For instance, according to the experimental data [28], the radius of the tip vortex core of a two-bladed rotor with a tip Mach number of 0.44 at age of 60° (60° after it was shed by the blade) was about 0.3-0.5% of the blade radius. according to Rai [38], at least 16 grid points across a Lamb-type vortex core is necessary to resolve the vortex and to prevent excessive numerical diffusion. For hover problems, assuming a rotor with an aspect ratio of 10, and the thickness of the vortex sheet to be captured by 16 grid points is initially 0.5% of the blade radius, the number of grid points needed in a rectangle with dimensions of 2 radius wide by 5
radius long is on the order of:

\[ \frac{2}{0.0016} \times \frac{5}{0.0016} \approx 100 \text{ million points.} \quad (6.48) \]

Of course, using suitably clever grid clustering, particularly near the vortex sheets, the number of the grid points needed can be greatly reduced. Still, this type of calculation is not feasible in the supercomputers of the present time.

An experiment was carried out to investigate the effects of the grid size to the results of the free-wake calculations. Three cases with identical initial condition and computational domain — they all had a 1.5 by 3 domain, but with different grid sizes, were calculated. The model rotor was the Huey 1/7-scale two-bladed rotor as used in the previous section. The three grid sizes were — 0.01, 0.0125, and 0.015. The calculations were terminated at the tenth blade passage, or five rotor revolutions.

Fig. 6.49 shows the comparison of the configurations of the vortex wakes of two different grid sizes, 0.01 and 0.0125; Fig. 6.50 shows the comparison of the vortex wakes of 0.01 and 0.015. Notice that the data points shown on the figures are local centroids of the vortex sheets. For different cases, the numbers of the local centroids representing a vortex sheet were different. The larger grid size case had less centroids, since there were less grid points in the computational domain.

From Fig. 6.49 and Fig. 6.50, one can see that the locations of the tip vortices are almost identical except the last tip vortex in Fig. 6.49. There are minor differences in the configurations of the inboard vortex sheets. There are very little differences in the first two inboard sheets. This is because these two sheets stayed in the computational domain for one and two blade passages only. There are more differences in the third, fourth,
Figure 6.49: The comparison of the configurations of the vortex wake due to different grid sizes.
Figure 6.50: The comparison of the configurations of the vortex wake due to different grid sizes.
and fifth inboard sheets, with the fifth sheet has the largest differences. It is believed due to the accumulation of the differences through the highly nonlinear interaction of the vortex wake. But the global configurations of the vortex wakes remain consistent.

The conclusion drawn from this grid size experiment is that the usage of different grid sizes changed the vorticity distribution or structure of the vortex sheet. These changes resulted in minor differences in the configurations of the vortex wakes. But the global configurations are consistent. When the configurations were used as input to the lifting-surface code, the effects of these minor changes on the circulation distribution on the blade are negligible. This is expectable, first, the differences are minor, secondly, most differences occur in the fourth and fifth inboard sheets which are further away from the rotor plane. Therefore, their influences on the rotor plane are small.
Chapter 7

Conclusions and Recommendations

7.1 Conclusions

A numerical method based on the axisymmetric incompressible Navier-Stokes equations was developed and coupled with a lifting-surface method to predict the vortex wake of hovering rotors. The calculations of four different hovering rotors were completed. The results were compared with test data and the agreement is fairly good considering the difficulties and uncertainties involved both in test and numerical simulation.

Past researchers have modeled the flow field as being inviscid. The vortex sheet was modeled as a series of line vortices or vorticity markers which were then traced in a Lagrangian frame. This method is simple and requires a reasonable amount of computer time. Numerical damping schemes are used on the vorticity markers to prevent the motion from chaotic. The introduction of the viscosity in our method was chosen to stabilize the vortex sheets. The motion of the vortex sheets is quite smooth, with no ad hoc damping scheme required. It was found that there is still instability
with the tip vorticies, but it is believed that the tip vortex instability is owing to the basic hydrodynamic instability of the hover flow problem. A relaxation scheme was applied to the tip vortices at the time of blade passages, and the inherent instability was smoothed out so that a pragmatic solution technique could be obtained.

The disadvantage of our method is that it requires more computer memory and CPU time. Since we trace the vortex wake in the Eulerian frame and also treat each inboard vortex sheet and tip vortex separately (in order to apply the relaxation scheme to the tip vortices), more memory is required to store them separately. The vorticity transport equation needs to be solved on the whole computational domain to march the vortex wake, but in fact the vortex sheets only concentrate on certain regions and outside these regions the flow field is vorticity free. Thus some computer time is wasted in solving the vorticity free regions.

A fast vortex ring method which models a vortex sheet by three concentrated vortex rings and requires a small amount of computer time was developed at the beginning of this work as a preliminary study tool. It was later used in the first three iterations of in the free-wake analysis when coupling with the lifting-surface code. The derivations and the limitations of this method are described in Appendix A.

7.2 Recommendations

A uniform grid mesh and a relatively large grid size were used in all the calculations of the present work. The large grid size introduces a severe numerical diffusion. Yet, vorticity only exists in the vortex sheet which is very thin in terms of the thickness. In order to prevent the excessive
numerical diffusion, a relatively small grid size is required to model the vortex sheet, but outside the vortex sheet a relatively large grid size can be used. Also, the location of the vortex sheet is changing with time. A natural candidate for investigating this problem would be an adaptive algorithm with the ability of providing refined grids at the vortex sheet and capturing the vortex sheet as it moves with time. More work in this area is needed.

The assumption of axisymmetry is thought reasonable in the present work, because the pitch angle of the vortex sheet is small. But if a higher degree of accuracy is required and with the advancement of supercomputers, complete three dimensional simulations of the flow field would be preferable.

A semi-infinite vortex cylinder or a stack of vortex rings extending beyond 5 blade radii below the rotor plane were used as the far wake. In fact, during model rotor flow visualization test the tip vortex diffuses very rapidly (because of the involvement of turbulent diffusion), and then bursts after one to two blade passages. An organized far-wake, as the semi-infinite vortex cylinder proposed in the present work, has never been observed. Actually, only three to four tip vortices are usually seen clearly during test. In some cases, the tip vortex expands outward after it reaches a maximum contraction [3] leaving open the entire question of the model of the far wake. Further investigation in this area certainly is desirable.
Bibliography


Appendices
Appendix A

The Fast Vortex Ring Method

A.1 Mathematical Formulations

Complete calculation of the flow field is complicated and expensive, a simpler method, the fast vortex ring method, which requires little computer time, was developed as a complementary tool of the free-wake calculations. It helps understand the problem and guide the complex and expensive free-wake calculations. For example, the far wake model and the core size of the tip vortex both have strong influences on the solution, the fast vortex ring calculations give comprehensive information about their effects.

All the flow fields considered in the present work are axisymmetric, i.e., the velocity in the azimuthal direction vanishes. At first this fast vortex ring method considered only the tip vortex, the inboard vortex sheet was not included, since the tip vortex is the dominant element in the vortex wake of a hovering rotor. The vortex wake was divided into two parts, a force free near wake and a semi-rigid far wake. The near wake consisted of 5-8 vortex rings, these vortex rings moved under their mutual interactions plus the induction from the far wake. The far wake consisted of a semi-
infinite vortex cylinder or a stack of vortex rings (the number is flexible). The radius of the far wake was the same as the radius of the last vortex ring in the near wake, and the vertical spacing was the spacing of the last two rings in the near wake.

The induced velocity at a vortex ring had two components, one was the self-induced velocity and the other was due to the influences of all other rings and the far wake. Once the induced velocities at every ring were calculated, the wake was then integrated to the next time step. Typically, 3 degrees (180 degrees corresponds to a blade passage for a two bladed rotor) was used as the time step. Although a larger time step could be used, a smaller time step gave better simulation of the vortex wake, since the mutual interaction of the vortex wake is highly nonlinear. The tip vortex was assumed to roll up from the tip to the position of the maximum circulation on the blade. The strength of the tip vortex therefore is,

$$\Gamma_0 = -\int_\alpha^1 \frac{d\Gamma}{d\eta} \, d\eta$$  \hspace{1cm} \text{(A.1)}$$

where $$\alpha$$ is the position (nondimensionalized by the blade radius) of the maximum circulation on the blade. The tip vortex was assumed to roll up immediately after the vortex sheet had been shed by the blade. The center of the tip vortex right behind a blade was determined by the conservation of the first moment of vorticity,

$$\eta^* \Gamma_0 = \int_\alpha^1 \eta \frac{d\Gamma}{d\eta} \, d\eta$$  \hspace{1cm} \text{(A.2)}$$

The solution procedure started by guessing an initial configuration for the vortex wake. At the time corresponding to a blade passage, a new vortex ring was added to the rotor plane and the last vortex ring in the
near wake was removed. The number of vortex rings in the near wake was kept constant during the iteration.

In order to execute the computation, the self-induced velocity of each vortex ring, the induced velocity from a vortex ring to another vortex ring, and the induced velocity from a semi-infinite vortex cylinder to each vortex ring needed to be known. The self-induced velocity of a vortex ring has an analytical solution [27],

$$w_s = \frac{\Gamma}{4\pi r} \left[ \ln \frac{8r}{a} - \frac{1}{4} \right]$$  \hspace{1cm} (A.3)

where $r$ is the radius of the vortex ring, $a$ is the radius of its core, and $\Gamma$ is its strength.

The induced velocity from a vortex ring to another vortex ring is derived below. The induced velocity at a point $A$, its coordinates described by $\vec{x}$, due to a differential element of a vortex ring of strength $\Gamma$ at $B$, its coordinates described by $\vec{y}$, can be derived from the Biot-Savart law (Fig. A.1),

$$\vec{v} = -\frac{\Gamma}{4\pi} \int \frac{\vec{s} \times d\vec{\ell}}{s^5}$$  \hspace{1cm} (A.4)

where

$$\vec{s} = \vec{x} - \vec{y}$$

$$= -(rR - \eta R \cos \phi) \vec{i} + \eta R \sin \phi \vec{j} + zR \vec{k}$$

and

$$d\vec{\ell} = rR \, d\phi \vec{j}$$

$$\vec{s} \times d\vec{\ell} = -rzR^2 d\phi \vec{i} + rR(\eta R \cos \phi - rR) d\phi \vec{k}$$
Figure A.1: The geometry for the derivation of the induced velocity from a vortex ring to another vortex ring.
The \((\mathbf{i}, \mathbf{j}, \mathbf{k})\) coordinate system is moving with the differential vortex element \(d\mathbf{\ell}\). The unit vectors \((\mathbf{i}, \mathbf{j}, \mathbf{k})\) are in the same directions as unit vectors \((\mathbf{e}_r, \mathbf{e}_\phi, \mathbf{e}_z)\). In the above equations, \(z, r, \) and \(\eta\) are nondimensional values (nondimensionalized by the blade radius), and \(R\) is the radius of the blade.

Change the \((\mathbf{i}, \mathbf{j}, \mathbf{k})\) coordinate system to a cylindrical coordinate system \((\mathbf{e}_r, \mathbf{e}_\phi, \mathbf{e}_z)\) by

\[
\mathbf{e}_r = \cos \phi \mathbf{e}_r - \sin \phi \mathbf{e}_\phi
\]
\[
\mathbf{e}_\phi = \mathbf{e}_z
\]

The velocity in the \(\mathbf{e}_\phi\) (or \(\mathbf{j}\)) direction vanishes. Then,

\[
\mathbf{e} \times d\mathbf{\ell} = r(\eta \cos \phi - r)d\phi R^2 \mathbf{e}_z - rz \cos \phi R^2 d\phi \mathbf{e}_r
\]
\[+rz \sin \phi R^2 d\phi \mathbf{e}_\phi\]

and

\[
s^3 = (z^2 + \eta^2 + r^2 - 2r\eta \cos \phi)^{3/2} R^3
\]

Substitute the above two expressions into the Biot-Savart law and carry out the integration,

\[
\bar{v} = -\frac{\Gamma}{4\pi R} \int \frac{r(\eta \cos \phi - r)d\phi \mathbf{e}_z - rz \cos \phi d\phi \mathbf{e}_r}{(z^2 + \eta^2 + r^2 - 2r\eta \cos \phi)^{5/2}}
\]  \(\text{(A.5)}\)

It can be seen from the above equation that the induced velocity in \(\mathbf{e}_\phi\) direction vanishes, since the flow field is axisymmetry. Therefore, the vertical induced velocity at a point \((\eta R, z_1 R)\) due to a vortex ring at \((rR, z_2 R)\) is

\[
\omega_i = -\frac{\Gamma}{4\pi R} \int_0^{2\pi} \frac{r(\eta \cos \phi - r)d\phi}{(z^2 + \eta^2 + r^2 - 2r\eta \cos \phi)^{3/2}}
\]
where

\[ z = z_1 - z_2 \]

\[ k^2 = \frac{4r\eta}{(r + \eta)^2 + z^2} \]

\[ A = \frac{4(r^2 + r\eta)}{[z^2 + (r + \eta)^2]^{3/2}} \]

\[ B = \frac{8r\eta}{[z^2 + (r + \eta)^2]^{3/2}} \]

\( \mathcal{E}(k) \) and \( \mathcal{K}(k) \) are the complete elliptic integral of the first and the second kind as defined below:

\[ \mathcal{E}(k) = \int_0^{\pi/2} (1 - k^2 \sin^2 \theta)^{1/2} d\theta \]

\[ \mathcal{K}(k) = \int_0^{\pi/2} \frac{d\theta}{(1 - k^2 \sin^2 \theta)^{1/2}} \]

The radial induced velocity at a point \((rR, z_1R)\) due to a vortex ring at \((rR, z_2R)\) is

\[ u_1 = \frac{\Gamma}{4\pi R} \int_0^{2\pi} \frac{rz \cos \phi \, d\phi}{(z^2 + \eta^2 + r^2 - 2r\eta \cos \phi)^{3/2}} \]

\[ = \frac{\Gamma}{4\pi R} \left[ \frac{C(2 - k^2)}{2k^2(1 - k^2)} \mathcal{E}(k) - \frac{1}{k^2} \mathcal{K}(k) \right] \tag{A.7} \]

where

\[ C = \frac{8rz}{[z^2 + (r + \eta)^2]^{3/2}} \]
The induced velocity at a vortex ring due to a semi-infinite vortex cylinder underneath also needed to be known, since a semi-infinite vortex cylinder was used as one of the far wake models. The radial and vertical induced velocities at a point \((\eta R, z_1 R)\) due to a semi-infinite vortex cylinder, with a radius of \(rR\) and starting at \(z_3 R\), are derived below. The radial induced velocity is

\[
\begin{align*}
uf &= -\frac{1}{4\pi R} \frac{d\Gamma}{dz} \int_{0}^{2\pi} \int_{-\infty}^{-s} \frac{rz \cos \phi d\phi dz}{(z^2 + \eta^2 + r^2 - 2r\eta \cos \phi)^{3/2}} \\
 &= \frac{1}{4\pi R} \frac{d\Gamma}{dz} \int_{0}^{2\pi} \frac{r \cos \phi d\phi}{(z^2 + \eta^2 + r^2 - 2r\eta \cos \phi)^{1/2}} \\
 &= \frac{1}{4\pi R} \frac{d\Gamma}{dz} D \left[ \left(\frac{1}{k^2} - \frac{1}{2}\right) K(k) - \frac{E(k)}{k^2} \right] \quad (A.8)
\end{align*}
\]

where

\[
D = \frac{8r}{\left[z^2 + (r + \eta)^2\right]^{1/2}}
\]

and the vertical induced velocity is

\[
\begin{align*}
w_f &= -\frac{1}{4\pi R} \frac{d\Gamma}{dz} \int_{0}^{2\pi} \int_{-\infty}^{-s} \frac{r(r - \eta \cos \phi)d\phi dz}{(z^2 + \eta^2 + r^2 - 2r\eta \cos \phi)^{3/2}} \\
 &= \frac{1}{4\pi R} \frac{d\Gamma}{dz} \int_{0}^{2\pi} \frac{r^2 + r\eta - 2r\eta \sin \phi^3}{(\eta + r)^2 - 4r\eta \sin \phi^3} \\
 &= \frac{1}{(1 - k^2 \sin \phi^2)^{1/2}} \int_{0}^{2\pi} \frac{dz}{\left[z^2 + (r + \eta)^2\right]^{1/2}} d\phi \quad (A.9)
\end{align*}
\]
Although the above equation could be converted into a combination of the elliptic integral of the first, second, and third kind, but calling these elliptic integrals from the computer mathematics library several times to calculate its value would not be faster than direct integration. Therefore the Simpson rule [26] was used to perform the direct integration of Eq. (1.10).

The value of \( d\Gamma/dz \) was determined by the following way: Supposing a rotor has \( Q \) blades and the strength of the tip vortex shed by the blade is \( \Gamma \). When it rotates one revolution there will be \( Q\Gamma \) tip vortices shed by the rotor, and the distance travels by a tip vortex will be \( v \cdot \frac{2\pi}{\Gamma} \), \( v \) is the descending velocity of the tip vortex and \( \Omega \) is the rotational speed of the rotor. Therefore,

\[
\frac{d\Gamma}{dz} = \frac{Q\Gamma\Omega}{2\pi v}
\]  

(A.11)

A concentrated vortex, with strength equal to a fraction of the tip vortex, was also used to represent the inboard vortex sheet to study the effect of the inboard vortex sheet. It turns out that an adequate model of the inboard vortex sheet is necessary if a successful prediction of the vortex wake is to be obtained, as explained in the next section.

### A.2 Results of the Fast Vortex Ring Method

The fast vortex ring method was first used to determine the effects of some of the influential elements in the problem. The first one was the effects of the core size of the tip vortex. The rotor studied was the Huey 1/7-scale two-bladed rotor which was used in Chapter 6 [28]. There was one tip vortex on the rotor plane and were 7 tip vortex rings below the rotor plane, the inboard vortex sheet at first was not included (Fig. A.2). These 8 vortex rings were free to move. A stack of 30 vortex rings was used as
Figure A.2: The vortex wake model of the Huey 1/7-scale two-bladed rotor for the fast vortex ring method.
the far wake. Their radii were equal to the radius of the last free-to-move ring, and the vertical spacing of neighbouring rings was equal to that of the last two free-to-move rings. The strength of all the vortex rings was the maximum circulation on the blade, for which an experimental value was used, since the target here was to determine the significance of the core size of the tip vortex.

The Biot-Savart law was used to determine the velocity field as derived in the previous section. The time step used was 3 degrees; therefore 60 time steps correspond to a blade passage. Although a larger time step could be used, a smaller time step is able to better simulate the motion of the vortex wake, since the motion is highly nonlinear. The calculation was terminated when the configuration of the vortex wake converged.

Three different core radii, 0.1, 0.05, and 0.005 (nondimensionalized by the blade radius), were tested. The results are shown in Fig. A.3. One can see that the size of the tip vortex core does have significant effects on the results. The vortex wake with a smaller tip vortex core contracts less. The spacing between the blade and the tip vortex from the preceding blade, which is critical for the performance of the rotor as discussed in Chapter 6, is larger.

The second effect studied was the influences of the inboard vortex sheet. Eight vortex rings were used to represent the eight inboard vortex sheets associated with the eight tip vortices. Each inboard vortex sheet was modeled by a vortex ring. The strength of these inboard vortices was a half of the maximum circulation on the blade; their radius was assumed 0.56R, and the axial spacing of the neighboring rings was 0.15R initially. After the calculation started, the locations of the inboard vortices were determined
Figure A.3: The effects of the core size of the Huey 1/7-scale two-bladed rotor calculated by the fast vortex ring method.
by the mutual interactions of the vortex wake. The vortex core radius used in both tip and inboard vortices was 0.005.

Fig. A.4 shows the converged configuration of the vortex wake with and without the inboard vortex sheets. The inboard vortex sheets had decisive effects on the configuration of the vortex wake. An adequate model of the inboard vortex sheets is necessary for successful prediction of the vortex wake.

Finally, the effect of the far wake was investigated. Four different far wake models,

1. A stack of 5 vortex rings.
2. A stack of 30 vortex rings.
3. A stack of 60 vortex rings.

were used to test the effect of the far wake. The inboard vortex sheet was not included, and the core radius of the tip vortices was 0.005. Fig. A.5 shows the results. The far wake had only a small impact on the wake when it extended beyond certain extent. The numerical results show that beyond five blade radii below the rotor plane, the far-wake effect is negligible.

**A.3 The Effects of the Circulations on the Blade on the Vortex Wake**

The bound circulations on the blade have some effects on the vortex wake. This was not included in all the calculations. These effects are thought to be of second order. Since although the vortex sheet shed by a blade
Figure A.4: The effects of the inboard vortex sheet of the Huey 1/7-scale two-bladed rotor calculated by the fast vortex ring method.
Figure A.5: The effects of the far wake of the Huey 1/7-scale two-bladed rotor calculated by the fast vortex ring method.
receives downward inductions from the circulations on the originating blade, it receives a upward inductions from the following blade when it moves close to the following blade. These two inductions tend to cancel each other out. Even without the cancellation, the downward induced velocities received by the vortex sheet from the circulations on the originating blade are small.

A simplified calculation, which calculated the downward induced velocity on a tip vortex from the circulations on its originating blade, was executed to prove this argument. The model rotor was the Huey 1/7-scale two bladed rotor. The blade was considered as a lifting line. The tip vortex was assumed to stay on the rotor plane after it had been generated, since before the next blade passage the axial distance traveled by it is small. The calculation was based on the derivations in the previous section. The downward induced velocities at different azimuthal angle, from 0° to 90° or a half blade passage, was calculated by integrating the influence of the circulations from the blade root to tip. The induced velocities from 90° to 180° were not considered, since the tip vortex was further away from its originating blade and their values become negligible as will be seen from the results.

Table A.1 shows the results. The induced velocity drops very rapidly. The distance traveled by the tip vortex due to these induced velocities, from 0° to 90°, is about 0.0042 (nondimensionalized by the blade radius). This is about one thirteenth of the distance that would have been travelled by the tip vortex in the hover test. If the upward velocities induced by the circulations on the following blade were taken into account, the effect would be negligible.
Table A.1: The downward induced velocity on a tip vortex due to the circulation on the originating blade for the Huey 1/7-scale two-bladed rotor.

<table>
<thead>
<tr>
<th>Tip vortex age (radian)</th>
<th>Induced velocity (nondimensionalized by the blade tip speed)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0685</td>
<td>0.0183</td>
</tr>
<tr>
<td>0.1185</td>
<td>0.0115</td>
</tr>
<tr>
<td>0.1685</td>
<td>0.0082</td>
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<td>0.0062</td>
</tr>
<tr>
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<td>0.0050</td>
</tr>
<tr>
<td>0.3185</td>
<td>0.0041</td>
</tr>
<tr>
<td>0.3685</td>
<td>0.0034</td>
</tr>
<tr>
<td>0.4185</td>
<td>0.0029</td>
</tr>
<tr>
<td>0.4685</td>
<td>0.0026</td>
</tr>
<tr>
<td>0.5185</td>
<td>0.0023</td>
</tr>
<tr>
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<td>0.0020</td>
</tr>
<tr>
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</tr>
<tr>
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<tr>
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</tr>
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