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CHARACTERISTICS OF FORWARD- AND BACKWARD-TRAVELING-WAVE PARAMETRIC INTERACTIONS

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

By
Chung Yu, B.Eng., M.Sc.

*****

The Ohio State University
1973

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<td>$A_i(z)$</td>
<td>Real amplitude of the interacting wave at $\omega_i$</td>
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<td>$\omega_i$</td>
<td>Radian frequency of the respective interacting wave</td>
</tr>
<tr>
<td>$\sigma_i$</td>
<td>The coupling coefficient between the wave at $\omega_i$ and the remaining waves</td>
</tr>
<tr>
<td>$A_i^2(z)/\sigma_i$</td>
<td>A quantity proportional to the number of photons, phonons, or magnons etc., depending on the nature of the wave</td>
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<td>$z$</td>
<td>One dimensional spatial variable</td>
</tr>
<tr>
<td>$L$</td>
<td>Total interaction length</td>
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<td>FTW</td>
<td>Forward-traveling-wave in the positive $z$ direction</td>
</tr>
<tr>
<td>BTW</td>
<td>Backward-traveling-wave, in the direction opposite to FTW</td>
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<td>$n^2$</td>
<td>Elliptic function parameter in the range $n \leq 1$</td>
</tr>
<tr>
<td>$n^{-2}$</td>
<td>Elliptic function parameter in the range $n \geq 1$</td>
</tr>
<tr>
<td>$K$</td>
<td>Real quarter period of the elliptic function</td>
</tr>
<tr>
<td>$G$</td>
<td>Ratio of output signal amplitude to input signal amplitude. $1 \leq G &lt; \infty$ for amplification and oscillation; $0 \leq G \leq 1$ for absorption or conversion of signal in mixing</td>
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<td>$E$</td>
<td>The ratio of net converted signal photons to input pump photons, or the ratio of net converted idler photons to input pump photons, or the ratio of net converted pump photons to input pump photons. The terms of signal conversion efficiency, idler conversion efficiency and pump depletion are completely analogous</td>
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CHAPTER I
INTRODUCTION

The parametric process can generally be understood as a process in which some energy storage parameter of a resonant system is varied periodically so that different resonant modes are coupled together, leading to an exchange of energy among them. The principle is thus applicable to any resonant systems be they mechanical, acoustical, electromagnetic or any combinations. The theory to be developed here will be based on electromagnetic phenomena keeping in mind that the results obtained can have general application.

The parametric concept dates back to Lord Rayleigh and Faraday in connection with mechanical vibrations[1,2]. It has since been extensively used in electronics such as in the development of low noise microwave amplifiers[3]. The low noise property results from the fact that tube and transistor amplifiers are based on electron (hole) conduction and thus possess inherent thermal and shot noise, while diode parametric amplifiers make use of the variable reactance and are thus (in theory) noise free. The concept of a variable reactance naturally extends into the optical region where the energy storage parameter is the field dependent dielectric constant or equivalently a field dependent susceptibility[4]. This in turn gives rise to a nonlinear polarization. As in lumped circuits in which a
periodically varying capacitance mixes currents at different frequencies, here the nonlinear polarization mixes electromagnetic waves of different frequencies.

The most elementary nondegenerate parametric process in lumped circuits involves three frequencies pertaining to two tuned circuits called the signal and idler coupled through a variable reactance varying at the pumping frequency[5]. This is a nondegenerate system in that it relaxes the phase and frequency constraints on the pump as required in the degenerate case[3]. The frequency or time periodicity condition for cumulative effects is then \( \omega_3 = \omega_1 + \omega_2 \). This condition holds in the interaction of traveling waves with the addition of a spatial periodicity requirement, the so called phase matching condition \( \mathbf{k}_3 = \mathbf{k}_1 + \mathbf{k}_2 \), where \( k_i = 2\pi/\lambda_i \). This condition can be derived from the consideration of maximum cumulative effects[5] or other considerations [23]. The above constraints are then applied in the derivation of a set of coupled differential equations for the electric fields from the wave equation for the total field in the nonlinear, lossless medium [6,7] this is one example. Investigations in optical parametric interactions have so far been mostly concerned with forward traveling waves (FTW)[8,9]. Similar interactions involving backward traveling waves (BTW are still less well known despite the fact that BTW parametric amplification and oscillation have been observed in parametrid diodes [10] and in stimulated Brillouin scattering[11].

The present work is a general and more exact analysis of both FTW and BTW parametric interactions as compared to earlier reviews[12].
This is also an extension of recent papers on BTW parametric oscillators[7,13-16] to include those interaction characteristics in the amplification and mixing processes. The method of analysis is similar to that of Armstrong, et al[8] on steady-state FW interactions of plane waves, where limited physical interpretation of the results is given. This work differs from previous analyses in that emphasis is placed on the establishment and physical interpretation of the interaction characteristics with a unified viewpoint. Thus, the effectiveness of all the interactions will be gaged in terms of signal gain, conversion efficiency and required pump excitation. These characteristics will be presented graphically in such forms suitable for physical interpretation. The dynamics of the interaction processes can then be readily described.

Some of the important parameters will be defined here. The signal gain is defined as the ratio of the output signal amplitude over the input signal amplitude. The spatial variable is $z$ for collinear interactions so that for a forward traveling wave, the input will be at $z=0$ and the output at $z=L$, where $L$ is the total interaction length of the medium. On the other hand, a backward traveling wave will have input at $z=L$ and output at $z=0$. Signal gain can go to infinity indicating oscillation, it is greater than unity for amplification, unity for no interaction and less than unity for absorption in the case of mixing. Thus, signal gain is zero when there is complete depletion of the signal due to parametric up conversion.

Since the solutions of the wave amplitudes are in terms of elliptic functions[17], it is quite natural to try to relate the function
parameters to physical characteristics of the processes under study. We have found that the elliptic function parameter $n^2$ is related to the photon conversion efficiency $E$ and signal gain while pump excitation is associated with the independent variable of the elliptic function. $n^2$ has therefore been used as a measure of conversion efficiency in our earlier analysis[18,19]. However, $n^2$ does not in general represent photon conversion efficiency. The parameter photon conversion efficiency is therefore used for the direct measure of how efficiently are the pump photons converted into the signal and idler photons. $n^2$ is also retained both for its physical significance and as a computation parameter. The photon conversion efficiency is therefore defined as the ratio of the number of pump photons converted or depleted to the number of input pump photons. This ratio will be found to be identical to the ratio of the number of converted signal or idler photons to the number of input pump photons.

Some of the significant findings of our work are first of all a unified approach to the analysis of parametric interactions. It will be seen that the results of various interaction schemes bear remarkable similarities when expressed in terms of the parameters of signal gain, conversion efficiency and pump excitation. Some of the results even have parallel interpretation. Furthermore, new light has been shed on the question of complete depletion of the pump or maximum signal conversion. Thus, simultaneous complete depletion of the signal and pump is not possible at finite pump excitation in both FTW and BTW mixing. Complete pump depletion is also impossible at finite pump excitation in BTW oscillation and in the BTW amplification scheme
involving a FTW signal. A BTW signal is thus seen to be more effectively amplified than an FTW signal.

Another very important finding is the -2 slope for the asymptotic $\ln (1-E)$ vs pump excitation $u$ curve for BTW parametric oscillation. This indicates clearly the effectiveness of generating instabilities or oscillations.

Some of the salient features of parametric interactions can be obtained by considering, as an example, steady-state, electromagnetic traveling-wave interactions in a medium. The analysis is known[15,7] and the results are summarized below as background.

The driven vector wave equation for the total electric field in a dielectric medium is

\[ (1-1) \quad \nabla \times \nabla \times \vec{E}(\vec{r},t) + \mu_0 \frac{\partial^2}{\partial t^2} \vec{E}(\vec{r},t) = -\mu \frac{\partial^2}{\partial t^2} \vec{P}(\vec{r},t) \]

Nonlinearity of the medium is reflected by a second order nonlinear polarization.

Since in the steady-state, polarization $\vec{P}(t)$ is completely determined by the susceptibility tensor $\chi(\omega)$ at various frequencies [22], we can write Eq. (1-1) as

\[ (1-2) \quad \nabla \times \nabla \times \vec{E}(\vec{r},\omega) = \omega^2 \varepsilon \mu_0 (\omega) \vec{E}(\vec{r},\omega) \]

\[ = \omega^2 \varepsilon \mu_0 \int_{-\infty}^{\infty} \chi(\omega_1,\omega-\omega_1) \cdot \vec{E}(\vec{r},\omega)\vec{E}(\vec{r},\omega-\omega_1) d\omega_1, \]

where $\varepsilon(1) = \varepsilon_0 [\chi(1)(\omega)]$, $\varepsilon_0$ being the permittivity of free space.

If the total field in the medium is
\[ \mathbf{E}(\mathbf{r}, \omega) = \mathbf{E}_1(\mathbf{r}, \omega_1) + \mathbf{E}_2(\mathbf{r}, \omega_2) + \mathbf{E}_3(\mathbf{r}, \omega_3) + \text{c.c.} \]

then for parametric interactions of the type

\[ \omega_3 = \omega_1 + \omega_2, \quad \mathbf{k}_3 = \mathbf{k}_1 + \mathbf{k}_2 \]

we have

\[ (1-3a) \quad \nabla \times \nabla \times \mathbf{E}_3 - \omega_3^2 \mu_0 \varepsilon_0 \mathbf{\mathbf{E}}_3 = 2 \omega_3^2 \mu_0 \varepsilon_0 \chi^2(\omega_1, \omega_2) \mathbf{E}_1 \mathbf{E}_2^* \]

\[ (1-3b) \quad \nabla \times \nabla \times \mathbf{E}_2 - \omega_2^2 \mu_0 \varepsilon_0 \mathbf{\mathbf{E}}_2 = 2 \omega_2^2 \mu_0 \varepsilon_0 \chi^2(\omega_3, \omega_1) \mathbf{E}_3 \mathbf{E}_1^* \]

\[ (1-3c) \quad \nabla \times \nabla \times \mathbf{E}_1 - \omega_1^2 \mu_0 \varepsilon_0 \mathbf{\mathbf{E}}_1 = 2 \omega_1^2 \mu_0 \varepsilon_0 \chi^2(\omega_2, \omega_3) \mathbf{E}_3 \mathbf{E}_2^* \]

For weak coupling, we may assume the solution of any wave to be of the form

\[ (1-4) \quad \mathbf{E}_i(\mathbf{r}, t) = \mathbf{\mathbf{e}}_i A_i(\mathbf{r}) \cdot \exp j[\omega_i t - \mathbf{k}_i \cdot \mathbf{r} + \phi_i(\mathbf{r})] \]

where \( i = 1, 2, 3 \), \( A_i(\mathbf{r}) \), \( \phi_i(\mathbf{r}) \) are slowly varying wave amplitude and phase in space and \( \mathbf{\mathbf{e}}_i \) is the polarization vector. For collinear interaction, we can write \( \mathbf{k}_i \neq \mathbf{k}_i \mathbf{n} \), with \( \mathbf{n} \) as unit vector in the propagation direction. We are interested in variations of amplitude and phase in the coordinate \( z = \mathbf{n} \cdot \mathbf{r} \).

The only terms in Eqs. (1-3) which need evaluation are \( \nabla \times \nabla \times \mathbf{E}_i \). This is performed on wave of the form (1-4) omitting the time dependent factor \( \exp j(\omega_i t) \). Neglecting second order derivatives and noting \( \nabla (\mathbf{n} \cdot \mathbf{r}) = \mathbf{n} \) for a forward-traveling-wave, \( \nabla (-\mathbf{n} \cdot \mathbf{r}) = -\mathbf{n} \) for a backward-
traveling-wave and $\nabla f = \pm (af/az)\hat{n}$ for any spatial function $f$, we have for a forward traveling wave

$$
(1-5) \quad \nabla \times \nabla \times \mathbf{E}_i(r) = e^{-j(k_i r - \phi_i)} \hat{n} \times (\hat{n} \times \mathbf{E}_i) \left\{ -Ak_i^2 + 2A_ik_i \frac{\partial \phi_i}{\partial z} - 2jk_i \frac{\partial A_i}{\partial z} \right\}.
$$

Applying this formula to the set (1-3) and using the Fresnel equation [22], we arrive at the set of equations

$$
(1-6a) \quad -j \frac{\partial A_3}{\partial z} + A_3 \frac{\partial \phi_3}{\partial z} = \sigma_3 A_1 A_2 e^{j(\phi_1 + \phi_2 - \phi_3)},
$$

$$
(1-6b) \quad -j \frac{\partial A_2}{\partial z} + A_2 \frac{\partial \phi_2}{\partial z} = \sigma_2 A_1 A_3 e^{j(\phi_1 + \phi_2 - \phi_3)},
$$

$$
(1-6c) \quad -j \frac{\partial A_1}{\partial z} + A_1 \frac{\partial \phi_1}{\partial z} = \sigma_1 A_2 A_3 e^{j(\phi_1 + \phi_2 - \phi_3)},
$$

where

$$
(1-7) \quad \sigma_i = \frac{\omega^2 \mu \epsilon}{k_i^2} e_i \chi(2)(\omega_j, \omega_k) : e_j e_k.
$$

We note that from permutation symmetry of the susceptibility tensors[15]

$$
\mathbf{e}_3 \cdot \chi(2)(\omega_1, \omega_2) : \mathbf{e}_1 \mathbf{e}_2 = \mathbf{e}_2 \cdot \chi(2)(\omega_3, -\omega_1) : \mathbf{e}_3 \mathbf{e}_1
$$

$$
= \mathbf{e}_1 \cdot \chi(2)(\omega_3, -\omega_2) : \mathbf{e}_3 \mathbf{e}_2 = 0
$$
so that

\[ \sigma_i = \frac{\omega_i v_i}{c^2} p \]

with

\[ v_i = \frac{\omega_i}{k_i} \]

the phase velocity and \( c^2 = (\mu e_0)^{-1} \).

Taking \( \phi = \phi_3 - (\phi_1 + \phi_2) \), the set of Eqs. (1-6) transforms into two sets

(1-8a) \[ \frac{\partial A_3}{\partial z} = \sigma_3 A_1 A_2 \sin \phi \]

(1-8b) \[ \frac{\partial A_2}{\partial z} = -\sigma_2 A_1 A_3 \sin \phi \]

(1-8c) \[ \frac{\partial A_1}{\partial z} = -\sigma_1 A_2 A_3 \sin \phi \]

and

(1-9a) \[ A_3 \frac{\partial \phi_3}{\partial z} = \sigma_3 A_1 A_2 \cos \phi \]

(1-9b) \[ A_2 \frac{\partial \phi_2}{\partial z} = \sigma_2 A_1 A_3 \cos \phi \]
Combining Eqs. (1-9), we obtain

\[ \frac{\partial \phi}{\partial z} + \left( -\frac{\sigma_3 A_1 A_2}{A_3} + \frac{\sigma_2 A_1 A_3}{A_2} + \frac{\sigma_1 A_2 A_3}{A_1} \right) \cos \phi = 0 \]

The \(\sigma_i\)'s can be eliminated by using Eqs. (1-8). The resulting form of Eq. (10) immediately leads to the condition

\[ A_1(z) A_2(z) A_3(z) \cos \phi(z) = \text{const.} \]

The constant is identically equal to zero when one of the waves is assumed to be an idler so that \(A_i(0) = 0\) (no input). For condition (1-11) to hold for any \(z\), \(\cos \phi(z) = 0\), i.e., \(\phi(z) = \pi/2\). \(\sin \phi(z)\) is then chosen to be \(-1\) from physical considerations. Condition (1-11) also holds in the case one of the waves is a backward-traveling-wave.

The same treatment applies to cases of mixing by appropriate identification of subscripts with pump, signal and idler.

It is thus clear that only the amplitude equations need be considered in the analysis of parametric interactions since the phase relationship between the interacting waves will remain invariant in space when one of the waves is taken as an idler. The signs of the coupling coefficient \(\sigma_i\) then play the important role of determining whether we have amplification or mixing.

We also note that in the case of Eqs. (1-8)
This can be conceived as Manley-Rowe power relations since, if we take $z=v_it$, $A_{it}(\text{traveling wave}) = v_i \times A_{ic}(\text{cavity mode})$, and since $\sigma_i = \omega_i v_i p c^{-2}$ then

$$\frac{\partial}{\partial z} \left( \frac{A_i^2}{\sigma_i} \right) = \frac{c^2}{p} \frac{\partial}{\partial z} \left( \frac{A_{it}^2}{\omega_i v_i} \right) = \frac{c^2}{p} \frac{\partial}{\partial t} \left( \frac{v_i^2 A_{ic}^2}{\omega_i v_i} \right)$$

Thus Eqs. (1-12) become

$$\frac{p_3}{\omega_3} = -\frac{p_2}{\omega_2} = -\frac{p_1}{\omega_1}$$

or the Manley-Rowe relations for amplification.

We will begin the analysis with interactions involving three forward traveling waves. Such interactions are called FTW parametric interactions and treated in Chapter II. The cases of amplification and mixing will be discussed in separate section headings. Chapter III will be devoted to interactions involving one or more backward traveling waves. They are grouped under the title of BTW parametric interactions. The different cases of amplification, oscillation and mixing will be discussed under different section headings. Each section will be preceded by a section on theoretical analysis to provide the necessary mathematical formalism. The significance of the present work and possible improvements will be discussed in Chapter IV.
CHAPTER II
FORWARD-TRAVELING-WAVE (FTW) PARAMETRIC INTERACTIONS

2.1 FTW PARAMETRIC AMPLIFICATION THEORY

The frequency and phase-matching conditions for collinear three-wave interactions are

\[ \omega_3 = \omega_1 + \omega_2 \]
\[ k_3 = k_1 + k_2 \]

Thus, the system is pumped at \( \omega_3 \) by a forward-traveling-wave (FTW) for amplification at either \( \omega_1 \) (FTW) or \( \omega_2 \) (FTW). Analyses of both cases are identical so that only one case will be studied. The results obtained for one case are directly applicable to the other case with proper redefinition of signal and idler.

The coupled differential equations for the wave amplitudes are

\[ \frac{dA_1(z)}{dz} = \sigma_1 A_2(z) A_3(z) \]
\[ \frac{dA_2(z)}{dz} = \sigma_2 A_1(z) A_3(z) \]
\[ \frac{dA_3(z)}{dz} = -\sigma_3 A_1(z) A_2(z) \]

where \( A_i(z) \) are wave amplitudes and \( A_3(z) \) has been taken as the pump amplitude, and \( \sigma_i \) are nonlinear coupling coefficients. It follows
from Eqs. (2-1-2) that

\[ \frac{d}{dz} \left[ \frac{A_1^2(z)}{\sigma_1} + \frac{A_3^2(z)}{\sigma_3} \right] = 0 \]

(2-1-3a)

\[ \frac{d}{dz} \left[ \frac{A_2^2(z)}{\sigma_2} + \frac{A_3^2(z)}{\sigma_3} \right] = 0 \]

(2-1-3b)

In the case of \( A_1(z) \) as the signal and \( A_2(z) \) as the idler so that \( A_2(0) = 0 \) (idler has zero input), conservation relations are obtained by integrating Eqs. (2-1-3) with the proper boundary conditions. We thus have

\[ \frac{A_1^2(z)}{\sigma_1} + \frac{A_3^2(z)}{\sigma_3} = \frac{A_1^2(0)}{\sigma_1} + \frac{A_3^2(0)}{\sigma_3} \]

(2-1-4a)

\[ \frac{A_2^2(z)}{\sigma_2} + \frac{A_3^2(z)}{\sigma_3} = \frac{A_3^2(0)}{\sigma_3} \]

(2-1-4b)

Equation (2-1-2a) for the signal amplitude may then be transformed using Eqs. (2-1-4), resulting in

\[ \frac{dA_1(z)}{dz} = (\sigma_2 \sigma_3) [A_1^2(z)-A_1^2(0)] \times \]

\[ \times [A_1^2(0) + \frac{\sigma_1}{\sigma_3} A_3^2(0)-A_1^2(z)]^{1/2} \]

or
(2-1-5) \[ \left[ A_1^2(z) - A_1^2(0) \right] \left[ A_1^2(0) + \frac{\sigma_1}{\sigma_3} A_3^2(0) - A_1^2(z) \right]^{-1/2} dA_1(z) \]

\[ = (\sigma_2 \sigma_3)^{1/2} \, dz \]

By transforming the variable \( A_1(z) \), we can put Eq. (2-1-5) \[17\] in the standard elliptic integral form with proper boundary conditions. Thus,

(2-1-6) \[- \int_y^1 \frac{dy}{\sqrt{[(1-y^2)(1-n^2y^2)]^{1/2}}} = rz \]

where

(2-1-7) \[ y = \left\{ 1 + \frac{[A_1^2(0) - A_1^2(z)]/\sigma_1}{A_3^2(0)/\sigma_3} \right\}^{1/2} \]

(2-1-8) \[ n^2 = \left[ 1 + \frac{A_1^2(0)/\sigma_1}{A_3^2(0)/\sigma_3} \right]^{-1} \]

and

(2-1-9a) \[ r = \left[ \sigma_2 \sigma_3 A_1^2(0) + \sigma_1 \sigma_2 A_3^2(0) \right]^{1/2} \]

(2-1-9b) \[ = n^{-1} (\sigma_1 \sigma_2)^{1/2} A_3(0) \]

Upon further simplification, Eq. (2-1-6) reduces to

(2-1-10) \[ \int_0^y \frac{dy}{\sqrt{[(1-y^2)(1-n^2y^2)]^{1/2}}} = K - rz \]
where $K$ is the real quarter period of the elliptic function.

We readily derive from Eq. (2-1-10), the expression for the signal amplitude in $z$. Hence

$$A_1(z) = (1-n^2)^{-1/2} A_1(0) \ Dn(K-rz)$$  \hspace{1cm} (2-1-11)

We can then derive the spatial dependence of the remaining wave amplitudes simply using the conservation relations (2-1-4) The pump amplitude is

$$A_3(z) = A_3(0) \ Sn \ (K-rz) \hspace{1cm},$$

and the idler amplitude is

$$A_2(z) = \left( \frac{\sigma_2}{\sigma_3} \right)^{1/2} A_3(0) \ Cn \ (K-rz) \hspace{1cm}.$$
2.2 PHOTON CONVERSION EFFICIENCY IN FTW PARAMETRIC AMPLIFICATION

One of the main goals of our analysis is an attempt to relate mathematical variables to the physical parameters of amplification in the case of an amplifier, those of oscillation if an oscillator is possible and those of mixing in the case of frequency up conversion. The relevant quantities of the elliptic function are the independent variable defined as the product of the propagating factor $r$ and $z$, and the elliptic function parameter $n^2$. The pumping of the system, the interaction or conversion efficiency and signal gain can be expressed in terms of these two quantities and elliptic functions.

The signal gain is defined as the ratio of output signal amplitude to input signal amplitude. In this case, the signal gain at $\omega_1$ in an interaction length $L$ and pumping at $\omega_3$ is given by

\[(2-2-1a)\quad G = A_1(L)/A_1(0)\]

and in terms of the elliptic function from Eq. (2-1-11),

\[(2-2-1b)\quad G = [Dn (rL)]^{-1} \]

According to Eq. (2-1-8),

\[(2-2-2)\quad n^2 = \left[1 + \frac{A_1^2(0)/\sigma_1}{A_3^2(0)/\sigma_3}\right]^{-1}.\]

Since from Eq. (2-1-9b),

\[rL = n^{-1} (\sigma_1 \sigma_2)^{1/2} A_3(0) L\]
then the overall pumping factor is by definition

\[(2-2-3) \quad \eta = (\sigma_1 \sigma_2)^{1/2} A_3(0)L = n r L \]  

We obtain the Manley Rowe relation[20] from the conservation relations (2-1-4), i.e.,

\[(2-2-4) \quad \frac{A_3^2(0) - A_3^2(L)}{\sigma_3} = \frac{A_1^2(L) - A_1^2(0)}{\sigma_1} = \frac{A_2^2(L)}{\sigma_2} \]

so that the definition of conversion efficiency \( E \) is

\[(2-2-5a) \quad E = \frac{[A_1^2(L) - A_1^2(0)]/\sigma_1}{A_3^2(0)/\sigma_3} \]

or

\[(2-2-5b) \quad = \frac{A_3^2(0) - A_3^2(L)}{A_3^2(0)} \]

or

\[(2-2-5c) \quad = \frac{A_2^2(L)/\sigma_2}{A_3^2(0)/\sigma_3} \]

The quantity \( E \) can therefore be interpreted as signal conversion efficiency from Eq. (2-2-5a) or pump depletion from Eq. (2-2-5b), or idler conversion efficiency from Eq. (2-2-5c). The first two terms will be used interchangeably whichever appears to be more convenient. This identity exists because physically the converted pump photons must split into signal and idler photons simultaneously.
From the point of view of the signal, we have the signal conversion efficiency

\[ E = \frac{[A_1^2(L)-A_1^2(0)]/\sigma_1}{A_3^2(0)/\sigma_3} \]

Using Eqs. (2-2-1a) and (2-2-2), we can also express \( E \) as

\[ E = (n^{-2}-1)(G^2-1) \]

At first glance, it seems that \( E \) may be unbounded for \( G >> 1 \) in contrast to its definition. However, we note from Eq. (2-2-1b) that at a fixed \( n^2 \)

\[ G_{\text{max}} = (1-n^2)^{-1/2} \]

from the theory of elliptic functions so that at this value, \( E \) is maximum and is equal to unity as seen by substituting Eq. (2-2-7) in Eq. (2-2-6).

It is of interest to study the variation of conversion efficiency with pump excitation \( u \) at various input signal levels or \( n^2 \) values and at various gain. \( E \) vs \( u \) curves are therefore plotted in Fig. 1 at fixed \( n^2 \) values. A fixed value of \( n^2 \) indicates a certain signal input at a certain pump excitation as seen in Eq. (2-2-2). It immediately follows from the figure that the amount of conversion is also set. To be specific, let us consider the operation at a pump excitation value of 1 in Fig. 1. We first note that amplification is impossible with no input signal \( (n^2=1) \). Thus, for a very small input signal the operating point is at, say, \( a_1 \). Conversion or pump depletion is
negligible as expected. With a slight increase of signal we arrive at, say, point $a_2$ indicating a corresponding increase in conversion. This continues with further increase in the input signal until point $a_3$ is reached at which signal conversion is maximum $E=1$. The input pump is also completely depleted by the input signal and this parametric interaction scheme is complete.

To view the situation in terms of signal gain, we will resort to Fig. 2, in which conversion efficiency $E$ is plotted against the pump excitation. At unity along the abscissa, increasing signal will increase signal-conversion with simultaneous rapid decrease in signal gain by having the operating point moving upwards vertically.
Fig. 1. Photon conversion efficiency $E$ vs pump excitation $u$ at fixed $\eta^2$ in FTW parametric amplification.
Fig. 2. Photon conversion efficiency $E$ vs pump excitation $u$ at fixed signal gain in FTW parametric amplification.
2.3 PHYSICAL INTERPRETATION OF THE CHARACTERISTICS OF AMPLIFICATION

The amplification process for an FTW signal with an FTW pump at $\omega_3$ is best visualized in Fig. 3, in which signal gain is plotted as a function of the pump excitation with conversion efficiency as a running parameter. As to be expected, all curves begin at the value 1 along the ordinate indicating the condition that signal-in is equal to signal-out for an unpumped, lossless amplifying medium, i.e., the medium is linear and passive at this point. When pumping is introduced, any signal or noise is amplified first gradually with increasing pump, then more sharply at larger pump excitation. However, since there is no feedback in the system (such as that provided by a backward-traveling-wave), there is no oscillation, i.e., the system is inherently stable.

Let us consider the actual amplification process at an input pump excitation value of 3 along the abscissa. For negligibly small signal input as compared to the input pump excitation, the possible gain is 10 assuming that there is practically no pump depletion. Further increase in the signal level will then lower the possible gain since pump depletion begins to come into play. Thus, the finite pump depletion is equivalent to a finite signal conversion so that although the gain is lower the output signal is larger. We may then be operating on the $E=0.4$ curve with a corresponding gain of 9. Ultimately, a signal input level is reached at which the pump is completely depleted ($E=1.0$ curve) and the system settles to the lowest possible gain of
approximately 5.4. The theory of parametric interactions holds up to this point.

The inadequacy of using the parameter $n^2$ to characterize the amplification process is best seen in Fig. 4. Here, the signal gain is plotted against the pump excitation with $n^2$ as a running parameter. However, in this case, $n^2$ by definition specifies only the input signal condition and reflects nothing of the output condition (see Eq. (2-2-2)). Thus, $n^2 \to 0$ denotes the fact that the number of input signal photons greatly exceeds the input pump photon, while $n^2 + 1$ implies negligible input signal. A typical mode of operation is as follows. With very small signal input, $n^2$ is very nearly equal to unity so that the system gain settles at the point $a_1$. The gain quickly drops to the point $a_2$ with a slight increase of signal input and continues to decrease with increasing input signal until the point $a_3$. However, there is no apparent physical reason why the amplification process cannot proceed beyond this point. This figure thus does not define the upper limit. It does nevertheless provide a limit for the input signal level. Hence, point $a_3$ specifies the condition that at a pump excitation of $a$ the maximum allowable input signal corresponds to an $n^2$ value of 0.95. We also note that the dotted curve in Fig. 4 coincides with the $E=1$ curve in Fig. 3, since at fixed $n^2$, $G_{\text{max}}$ from Eq. (2-2-7) also corresponds to $E=1$ as seen in Eq. (2-2-6).
Fig. 3. Signal gain $G$ vs pump excitation $u$ at fixed $E$ in FTW parametric amplification.
Fig. 4. Signal gain G vs pump excitation u at fixed \( n^2 \) in FTW parametric amplification.
2.4 FTW PARAMETRIC MIXING THEORY

The frequency and phase matching conditions are:

\[ \omega_3 = \omega_1 + \omega_2 , \]
\[ k_3 = k_1 + k_2 , \]

where the system is pumped at \( \omega_1 \) with signal at \( \omega_2 \) (pump and signal are interchangeable in the analysis but are distinguishable in physical situations) and frequency up conversion at \( \omega_3 \). All interacting waves are forward traveling waves (FTW).

The coupled differential equations for the wave amplitudes are:

\[ \frac{dA_1(z)}{dz} = - \sigma_1 A_2(z) A_3(z) , \]  
\[ \frac{dA_2(z)}{dz} = - \sigma_2 A_1(z) A_3(z) , \]  
\[ \frac{dA_3(z)}{dz} = \sigma_3 A_1(z) A_2(z) . \]

This system of equations immediately leads to the set

\[ \frac{d}{dz} \left[ \frac{A_1^2(z)}{\sigma_1} + \frac{A_3^2(z)}{\sigma_3} \right] = 0 , \]
\[ \frac{d}{dz} \left[ \frac{A_2^2(z)}{\sigma_2} + \frac{A_3^2(z)}{\sigma_3} \right] = 0 , \]

which yields for the boundary condition \( A_3(0) = 0 \) the conservation rules
Using Eqs. (2-4-4) to express $A_1(z)$ and $A_3(z)$ in terms of $A_2(z)$, the differential equation for the signal amplitude Eq. (2-4-2b) can then be written as

\[
\frac{dA_2(z)}{dz} = \left\{ (\sigma_1^2) \left[ \frac{A_1^2(0) - A_2^2(0) + A_2^2(z)}{\sigma_1^2} \right] \times \left[ A_2^2(0) - A_2^2(z) \right] \right\}^{1/2}.
\]

or

\[
\{ \left[ \frac{A_1^2(0) - A_2^2(0) + A_2^2(z)}{\sigma_1^2} \left[ A_2^2(0) - A_2^2(z) \right] \right] \}^{-1/2} \quad \text{d}A_2(z)
\]

\[= - (\sigma_1 \sigma_2)^{1/2} \quad \text{d}z.
\]

There are two operating modes depending on whether $A_1^2(0)/\sigma_1$ is greater or smaller than $A_2^2(0)/\sigma_2$.

a) $\frac{A_1^2(0)}{\sigma_1} \geq \frac{A_2^2(0)}{\sigma_2}$

Equation (2-4-5) is put in the form of a standard elliptic integral with proper limits

\[
\int_0^y \frac{dy}{\left[ (1-y^2)(1-\eta^2y^2) \right]^{1/2}} = \operatorname{rz}.
\]
where

\[ y \equiv \left[ 1 - \frac{A_2^2(z)/A_2^2(0)}{A_1^2(0)/\sigma_1} \right]^{1/2} \]

and

\[ n^2 = \frac{A_2^2(0)/\sigma_2}{A_1^2(0)/\sigma_1} \leq 1 \]

Thus from Eq. (2-4-6) we obtain for the signal amplitude

\[ A_2(z) = A_2(0) \text{ Cn rz} \]

Using the conservation relations (2-4-4) and Eq. (2-4-8), we obtain expressions for the amplitudes of the remaining pump and up-converted waves. Thus,

\[ A_3(z) = \left( \frac{\sigma_3}{\sigma_2} \right)^{1/2} A_2(0) \text{ Cn rz} \]

and

\[ A_1(z) = A_1(0) \text{ Dn rz} \]

b) \[ \frac{A_1^2(0)/\sigma_1}{A_2^2(0)/\sigma_2} \leq 1 \]

The standard elliptic integral derived from Eq. (2-4-5) is

\[ \int_0^y \frac{dy}{\sqrt{(1-y^2)(1-n^2y^2)^{1/2}}} = (\sigma_1\sigma_3)^{1/2} \frac{A_2(0)z}{A_2(z)} \]
where

\[ y \equiv \left\{ \frac{[A_2^2(0) - A_2^2(z)]/\sigma_2}{A_1^2(0)/\sigma_1} \right\}^{1/2} \]

and

\[ n^2 \equiv \frac{A_1^2(0)/\sigma_1}{A_2^2(0)/\sigma_2} \leq 1 \]  \hspace{1cm} (2-4-11)

Equation (2-4-10) yields

\[ A_2(z) = A_2(0) \, Dn \left( n^{-1} r \xi \right) \]  \hspace{1cm} (2-4-12)

where \( r \) is identical to that defined in Eq. (2-4-8). From the above formula and the conservation rules (2-4-4), we derive the remaining wave amplitudes

\[ A_3(z) = \left( \frac{\sigma_3}{\sigma_1} \right)^{1/2} A_1(0) \, Sn \left( n^{-1} r \xi \right) \]

and

\[ A_1(z) = A_1(0) \, Cn \left( n^{-1} r \xi \right) \].
2.5 PHOTON CONVERSION EFFICIENCY IN FTW PARAMETRIC MIXING

The question of conversion in mixing is somewhat different from that in amplification. Here, not only pump depletion but also the simultaneous conversion of the signal must be taken into account. The case of simultaneous complete conversion of both the signal and the pump is of interest.

For a system pumped at $\omega_1$ with a signal at $\omega_2$ and total interaction length $L$, the signal gain is defined as

\[
G \equiv \frac{A_2(L)}{A_2(0)}
\]

Here, $0 < G < 1$. We note that signal gain in this case represents the conversion or absorption of the signal to the up-converted wave at $\omega_3$. Thus, $G=0$ corresponds to complete signal depletion, while $G=1$ corresponds to no depletion.

The pump excitation is defined in Eq. (2-4-9), i.e.,

\[
u \equiv \left(\sigma_2 \sigma_3\right)^{1/2} A_1(0)L = \pi L
\]

The photon conversion efficiency can be defined as pump depletion

\[
E \equiv \frac{[A_1^2(0) - A_1^2(L)]/A_1^2(0)}{A_1^2(0)/\sigma_1}
\]

or as signal conversion from the conservation relations in Eqs. (2-4-4).
(2-5-3b) \[ \frac{A_2^2(0)/\sigma_2}{A_1^2(0)/\sigma_1} = [1 - G^2] \]

E is thus interpreted physically as pump depletion but is expressed as a function of signal gain through Eq. (2-5-3b). We define \( n^2 \) for any arbitrary value

(2-5-4) \[ n^2 = \frac{A_2^2(0)/\sigma_2}{A_1^2(0)/\sigma_1} \]

so that Eq. (2-5-3b) becomes

(2-5-5) \[ E = n^2[1 - G^2] \]

E can then attain unity for any G. We note that \( n^2 \) values become significant only in evaluating G in terms of elliptic functions. Thus, from Eq. (2-5-3a), complete pump depletion implies that

\[ \frac{A_1^2(0)}{\sigma_1} + \frac{A_2^2(L)}{\sigma_2} = \frac{A_2^2(0)}{\sigma_2} \]

i.e., the number of input signal photons (\( A_i^2/\sigma_i \) is proportional to the number of photons at \( \omega_i \)) must be equal to the sum of the number of pump photons and the number of output signal photons. This condition requires the parameter \( n^2 \) to assume values exceeding unity as from formula (2-5-4)

\[ n^2 \equiv \frac{A_2^2(0)/\sigma_2}{A_1^2(0)/\sigma_1} = 1 + \frac{A_2^2(L)/\sigma_2}{A_1^2(0)/\sigma_1} \geq 1 \]
\( \eta^2 \) is unity only when the signal output is zero, i.e., this is the condition for complete signal depletion as well as pump depletion. This is the optimum operating mode in terms of efficiency. However, it will be shown that this condition is not attainable at finite pump excitation.

In the consideration of signal gain, \( \eta^2 \) values must then be split into two ranges. Thus, \( \eta^2 \) is used as the elliptic function parameter when it is less than unity, while \( \eta^{-2} \) is used in the \( \eta^2 \geq 1 \) range. The corresponding signal gain formulas are from Eqs. (2-4-9) and (2-4-12)

\[
\begin{align*}
(2-5-6a) \quad G &= Cn \quad \eta L = Cn(u, \eta) \quad \text{for} \quad \eta \leq 1, \\
(2-5-6b) \quad G &= Dn(\eta u, \eta^{-1}) \quad \text{for} \quad \eta > 1.
\end{align*}
\]

The photon conversion efficiency formulas corresponding to these gain formulas are from Eq. (2-5-3b)

\[
\begin{align*}
(2-5-7a) \quad E &= \eta^2 \quad Sn^2(u, \eta) \quad \text{for} \quad \eta \leq 1, \\
(2-5-7b) \quad E &= Sn^2(\eta u, \eta^{-1}) \quad \text{for} \quad \eta > 1.
\end{align*}
\]

The question of complete conversion is of interest. This requires

\[
\begin{align*}
(2-5-8a) \quad \eta^2 \quad Sn^2(u, \eta) &= 1 \quad \text{for} \quad \eta \leq 1, \\
(2-5-8b) \quad Sn^2(\eta u, \eta^{-1}) &= 1 \quad \text{for} \quad \eta > 1.
\end{align*}
\]
Hence, in the range \( n \leq 1 \), Eq. (2-5-8a) calls for \( n^2 = 1 \) so that

\[
    u = K ,
\]

where \( K \) is the real quarter period of the elliptic function and is dependent on \( n^2 \). It approaches infinity for \( n^2 = 1 \). This indicates that complete conversion is impossible at finite pump excitation in the range \( n \leq 1 \). However, in the \( n > 1 \) range where Eq. (2-5-8b) is applicable, we have for complete conversion

\[
    S_n(nu, n^{-1}) = 1 ,
\]

or

\[
    u = K/n .
\]

Here, \( K \) is finite since \( n > 1 \) and \( n^{-2} \) is the elliptic function parameter. Thus, finite pump excitation \( u \) exists for complete depletion.

Physically, \( n > 1 \) corresponds to the case when the input signal photons exceed the input pump photons in number. It is thus quite clear that all the pump photons can be depleted. However, as shown in Fig. 5, \( G \) is not zero in this \( n \) range so that not all the signal can be depleted. On the other hand, the input signal photons are less in number than the pump photons in the \( n < 1 \) range, so that obviously complete conversion or pump depletion is impossible although all the signal photons can be depleted giving a zero value of \( G \). The optimum operating condition of complete depletion of both pump and signal photons occurs at \( n^2 = 1 \) (pt. A, Fig. 5) and is thus impossible at finite pump excitation.
Fig. 5. Photon conversion efficiency $E$ vs $\eta^2$ at fixed $G$ in FTW parametric mixing.
2.6 PHYSICAL INTERPRETATION OF THE CHARACTERISTICS OF MIXING

The mixing process is best illustrated in Fig. 6, in which the signal gain $G$ is plotted against the pump excitation with conversion efficiency $E$ as a running parameter. It is at once clear that the $E=1$ curve approaches $G=0$ axis only asymptotically at infinite pump excitation $u$. In contrast, along the $E < 1.0$ curves signal can be completely depleted although it does not up convert all the pump energy. We note that there is a threshold for complete signal depletion $G=0$. Thus, for pump excitation value below $\pi/2$, no matter how small the input signal is, it cannot be totally depleted. It is interesting to mention here that this threshold value is also found in the case of BZW parametric mixing involving a FZW pump and both BZW signal and up-converted waves. However, the latter case has drastically different contours near complete pump depletion $E \rightarrow 1$.

It is necessary to further comment on the existence of a threshold in FZW mixing and the inability of such interactions to simultaneously completely deplete the pump and signal. Specifically, the dotted curve corresponds to $n^2=1$, signifying equal number of signal and pump photons. Thus at a fixed pump excitation value of $\pi/2$, reducing input signal causes the operating point to depart from point "a" and move vertically downwards reaching complete depletion on the us axis for a negligible or zero input signal. Raising the input signal of course moves the operating point vertically upwards from point "a". The process is complete as we reach the $E=1$ curve when the pump is completely depleted. Thus $\pi/2$ is the threshold in mixing for the complete
depletion of a negligible input signal. The E=0 curve corresponds to condition \( n^2 = 0 \), thus representing the lower boundary of the operating region when input signal is negligible. At higher pump excitation, such as E=0.5 curve, input signal does not have to be negligible for complete depletion. Thus the E=0.5 curve intersects the \( u \) axis at the point corresponding to \( n^2 = 0.5 \), i.e., the number of input signal photons is half the number of pump photons. This increase of possible signal input with complete depletion continues with increasing pump excitation until \( n^2 = 1 \) when input signal photons equal to pump photons in number. Complete signal depletion can now occur only at infinite pump excitation (see the dotted curve). The region bounded by this curve and the E=1 curve defines signal input level exceeding pump excitation. We see that everywhere complete pump depletion is possible by increasing the input signal level but complete signal depletion is impossible for the obvious reason that signal level is higher than the pump level. It is thus clear that simultaneous depletion of the pump and the signal can be approached with infinite pump only when \( n^2 = 1 \), i.e., when the number of input signal photons equals that of the pump.

We have thus found that mixing is not optimum either when signal photons greatly exceed pump photons in number or vice versa. Optimum mixing occurs when the numbers are equal (\( n^2 = 1 \)). In this case, simultaneous depletion of the signal and pump occurs at infinite pump and hence signal (since \( n^2 = 1 \) means number of signal photons equal that of the pump photons).
Fig. 6. Signal gain $G$ vs pump excitation $u$ at fixed $E$ in FTW parametric mixing
SUMMARY OF CHAPTER II

Spatial variations of the amplitudes of the interacting waves are given in terms of elliptic functions in Section 2.1 for the case of parametric amplification involving all forward traveling waves. Collinear and phase-matched interactions in a lossless medium are treated. Phase mismatch can be readily incorporated [6, 8]. Photon conversion efficiency in FTW parametric amplification is then considered in Section 2.2 using the definitions and results established in Section 2.1. Appropriate graphs are presented in Section 2.3 for physical interpretation.

Corresponding expressions for the amplitudes of the interacting waves are given in Section 2.4 for the case of FTW parametric mixing. The analysis is applicable to all levels of the input signal with appropriate gain formulas depending upon whether signal photons exceed pump photons in number of vice versa. Photon conversion efficiency of the pump in parametric FTW mixing is then considered in Section 2.5. Here, signal depletion or conversion must be considered simultaneously. Signal gain defined as output signal amplitude to input signal amplitude varies between 0 and 1 in this case. The signal gain in mixing should be interpreted as the absorption or up conversion of the signal to a high frequency output. Pump depletion is also taking place simultaneously. Physical interpretation is given in Section 2.6 for the mixing process, the presence of a threshold for complete signal depletion and optimum mixing with simultaneous complete depletion of the signal and pump.
3.1 BTW Parametric Amplification and Oscillation Theory

The frequency and phase matching conditions for collinear, three-wave interactions are

\[ \omega_3 = \omega_1 + \omega_2 , \]
\[ k_3 = k_1 - k_2 . \]

(3-1-1)

The system is pumped at \( \omega_3 \) for amplification of a FTW signal at \( \omega_1 \) (with a BTW idler at \( \omega_2 \)) or a BTW signal at \( \omega_2 \) (with a FTW idler at \( \omega_1 \)) or for simultaneous oscillation. Unlike the interactions involving all forward traveling waves where the amplification characteristics of both FTW signals are identical, here we have the two distinct cases of amplification depending upon whether the input signal is at \( \omega_1 \) or \( \omega_2 \).

In both amplification schemes, the coupled differential equations of the wave amplitudes are

\[ \frac{dA_1(z)}{dz} = \sigma_1 A_2(z) A_3(z) , \]

(3-1-2a)
\[ (3-1-2b) \quad \frac{dA_2(z)}{dz} = -\sigma_2 A_1(z) A_3(z), \]

\[ (3-1-2c) \quad \frac{dA_3(z)}{dz} = -\sigma_3 A_1(z) A_2(z), \]

which readily leads to

\[ (3-1-3a) \quad \frac{d}{dz} \left[ \frac{A_1^2(z)}{\sigma_1} + \frac{A_2^2(z)}{\sigma_2} \right] = 0, \]

\[ (3-1-3b) \quad \frac{d}{dz} \left[ \frac{A_1^2(z)}{\sigma_1} + \frac{A_3^2(z)}{\sigma_3} \right] = 0. \]

After integration for a total interaction length \( L \), these equations lead to the required conservation conditions

\[ (3-1-4a) \quad \frac{A_1^2(z)}{\sigma_1} + \frac{A_2^2(z)}{\sigma_2} = \frac{A_1^2(0)}{\sigma_1} + \frac{A_2^2(0)}{\sigma_2} \]

\[ = \frac{A_1^2(L)}{\sigma_1} + \frac{A_2^2(L)}{\sigma_2}, \]

\[ (3-1-4b) \quad \frac{A_1^2(z)}{\sigma_1} + \frac{A_3^2(z)}{\sigma_3} = \frac{A_1^2(0)}{\sigma_1} + \frac{A_3^2(0)}{\sigma_3} \]

\[ = \frac{A_1^2(L)}{\sigma_1} + \frac{A_3^2(L)}{\sigma_3}. \]

Case 1. Amplification of an FTW signal at \( \omega_1 \). Applying the boundary condition \( A_2(L) = 0 \) (the BTW idler has zero input) to the
conservation relations in Eqs. (3-1-4) and substituting the resulting expressions of \( A_2(z) \) and \( A_3(z) \) to Eq. (3-1-2a), we have for the signal amplitude

\[
\frac{dA_1(z)}{dz} = (\sigma_2\sigma_3)[A_1^2(L)-A_1^2(z)]
\times
\left[ A_1^2(0) + \frac{\sigma_1}{\sigma_3} A_3^2(0)-A_1^2(z) \right]^{1/2}.
\]

This can be written in the standard elliptic integral form

\[
(3-1-5) \quad \int_{y(0)}^{y(z)} \frac{dy}{[(1-y^2)(1-n^2y^2)]^{1/2}} = \frac{(\sigma_2\sigma_3 A_1^2(0) + \sigma_1\sigma_2 A_3^2(0))^{1/2}z}{},
\]

where

\[
y(z) = \frac{A_1(z)}{A_1(L)}, \quad y(0) = \frac{A_1(0)}{A_1(L)},
\]

\[
(3-1-6) \quad n^2 = \frac{A_1^2(L)/\sigma_1}{A_1^2(0)/\sigma_1 + A_3^2(0)/\sigma_3}.
\]

We note that this parameter so defined cannot exceed unity under any physical conditions since it is the ratio of the number of output signal photons to the total number of input photons.

It follows readily from Eq. (3-1-5) that
(3-1-7a) \[ A_1(z) = A_1(L) \, \text{Sn} \left\{ \int_0^y \frac{dy}{\sqrt{(1-y^2)(1-n^2y^2)^{1/2}}} + \right. \]
\[ \left. + \left[ \frac{\sigma_2 \sigma_3}{\sigma_3} A_1^2(0) + \sigma_1 \sigma_2 A_3^2(0) \right]^{1/2} z \right\} \]

(3-1-7b) \[ = n \left[ A_1^2(0) + \frac{\sigma_1}{\sigma_3} A_3^2(0) \right]^{1/2} \, \text{Sn} \,(r z + u_0) \]

where

(3-1-8) \[ u_0 = \int_0^y \frac{dy}{\sqrt{(1-y^2)(1-n^2y^2)^{1/2}}} \]

and

(3-1-9) \[ r = \left[ \frac{\sigma_2 \sigma_3}{\sigma_3} A_1^2(0) + \sigma_1 \sigma_2 A_3^2(0) \right]^{1/2} \]
\[ = \left( \sigma_1 \sigma_2 \right)^{1/2} A_3(0) \left[ 1 + \frac{A_1^2(0)/\sigma_1}{A_3^2(0)/\sigma_3} \right]^{1/2} \]

We can evaluate \( u_0 \) by setting \( z=L \) in Eq. (3-1-7a). Then

(3-1-10) \[ u_0 = K - rL \]

where \( K \) is the real quarter period of the elliptic function.

Equation (3-1-7b) can then be rewritten as

(3-1-11) \[ A_1(z) = n \left[ A_1^2(0) + \frac{\sigma_1}{\sigma_3} A_3^2(0) \right]^{1/2} \, \text{Sn}(K-rL + rz) \]
The amplitudes of the remaining pump and idler waves are derived from the conservation relations and Eq. (3-1-11). Thus,

\[ A_3(z) = \left( \frac{\sigma_3}{\sigma_1} \right) \left[ A_1^2(0) + \frac{\sigma_1}{\sigma_3} A_3^2(0) \right]^{1/2} \text{Dn}(K-rL + rz) , \]

and

\[ A_2(z) = \left( \frac{\sigma_2}{\sigma_1} \right) \left[ A_1^2(0) + \frac{\sigma_1}{\sigma_3} A_3^2(0) \right]^{1/2} \text{Cn}(K-rL + rz) . \]

We have from Eq. (3-1-11) for the FTW signal gain at output \( z=L \)

\[ G_f = \frac{A_1(L)}{A_1(0)} \]

(3-1-12) \[ = (\text{Sn}(K-rL))^{-1} \]

\[ = \frac{\text{Dn}(rL)}{\text{Cn}(rL)} . \]

The gain can also be incorporated in \( \eta^2 \) and \( rL \). Hence, we have from Eq. (3-1-6)

(3-1-13) \[ \eta^2 = G_f^2 \left[ 1 + \frac{A_3^2(0)/\sigma_3}{A_1^2(0)/\sigma_1} \right]^{-1} \]

and using this and Eq. (3-1-9) we also have

(3-1-14a) \[ rL = \left( 1 + \frac{A_1^2(0)/\sigma_1}{A_3^2(0)/\sigma_3} \right)^{1/2} (\sigma_1\sigma_2)^{1/2} A_3(0)L \]

or
It is apparent from the conservation relations in Eqs. (3-1-4) that signal conversion efficiency is again completely equivalent to pump depletion or idler conversion efficiency. It is defined as usual in terms of the signal

$$E_f = \frac{[A_1^2(L) - A_1^2(0)]/\sigma_1}{A_3^2(0)/\sigma_3}$$

and readily reduces to a formula in terms of $n^2$ and gain when Eq. (3-1-13) and the definition of gain are included. Therefore,

$$E_f = n^2 \left[ \frac{G_f^2 - 1}{G_f^2 - n^2} \right]$$

Case ii. Amplification of a BTW signal at $\omega_2$. The appropriate boundary condition to be used in this case is $A_1(0) = 0$ (the FTW idler has zero input). This is imposed on Eqs. (3-1-4), which are then used together with Eq. (3-1-2b) to give

$$\frac{dA_2(z)}{dz} = - \{(\sigma_1\sigma_3)[A_2^2(0) - A_2^2(z)] \times$$

$$\times \left[ \frac{\sigma_2}{\sigma_3} A_3^2(0) - A_2^2(0) + A_2^2(z) \right]\}^{1/2}.$$
1) \( \frac{\sigma_2}{\sigma_3} A_3^2(0) \geq A_2^2(0) \) or 2) \( \frac{\sigma_2}{\sigma_3} A_3^2(0) \leq A_2^2(0) \).

1) When \( (\sigma_2/\sigma_3) A_3^2(0) \geq A_2^2(0) \), Eq. (3-1-16) yields the standard elliptic integral

\[
(3-1-17) \quad \int_0^y \frac{dy}{\sqrt{(1-y^2)(1-n^2 y^2)^{1/2}}} = (\sigma_1 \sigma_2)^{1/2} A_3(0)z ,
\]

where

\[
y = \left[ 1 - \frac{A_2^2(z)}{A_2^2(0)} \right]^{1/2},
\]

and

\[
(3-1-18) \quad n = \frac{A_2^2(0)/\sigma_2}{A_3^2(0)/\sigma_3}.
\]

As prescribed by the operating condition, \( n^2 \) cannot exceed unity and this is consistent with the theory of elliptic functions. The signal amplitude is then derived from Eq. (3-1-17)

\[
(3-1-19) \quad A_2(z) = A_2(0) \text{Cn rz} ,
\]

where the pump excitation

\[
(3-1-20) \quad u = (\sigma_1 \sigma_2)^{1/2} A_3(0)L = rL .
\]

Using the conservation rules in Eqs. (3-1-4) with the boundary condition \( A_1(0)=0 \) and Eq. (3-1-19), we obtain the amplitudes for the remaining pump and idler waves. Thus,
\[ A_3(z) = A_3(0) \, Dn \, rz \ , \]

and

\[ A_1(z) = \sqrt{\sigma_1/\sigma_2} \, A_2(0) \, Sn \, rz \ . \]

The desired BTW signal gain at the output in terms of elliptic functions is

\[
(3-1-21a) \quad G_{b1} \equiv A_2(0)/A_2(L) \ ,
\]

\[
(3-1-21b) \quad = [Cn \, rL]^{-1} .
\]

The photon conversion efficiency defined as

\[
(3-1-22) \quad E_{b1} = \frac{[A_2^2(0) - A_2^2(L)]/\sigma_2}{A_3^2(0)/\sigma_3}
\]

can be expressed in terms of \( n^2 \) and gain directly from the definitions in Eqs (3-1-18) and (3-1-21a). This is therefore a general formula for any \( n^2 \) and gain values, i.e., it is not restricted by the constraints of elliptic functions. This fact will become clear, but for the time being we will use this formula in Case 1 only. Hence

\[
(3-1-23a) \quad E_{b1} = n^2 \left[ 1 - G_{b1}^{-2} \right]
\]

\[
(3-1-23b) \quad = n^2 \, Sn^2 rL \ .
\]

2) When \( (\sigma_2/\sigma_3) A_3^2(0) \leq A_2^2(0) \), we write Eq. (3-1-16) in the form
\[
\frac{dA_2(z)}{dz} = -((\sigma_1\sigma_3)[A_2^2(0)-A_2^2(z)] \times \\
\times [A_2^2(z)-A_2^2(0) + \frac{\sigma_2}{\sigma_3} A_3^2(0)])^{1/2},
\]

which leads to the corresponding elliptic integral

\[\int_0^y dy \frac{dy}{[(1-y^2)(1-\eta^2 y^2)]^{1/2}} = (\sigma_1\sigma_3)^{1/2} z\]

with

\[(3-1-25) \quad y = \left\{ \frac{[A_2^2(0)-A_2^2(z)]/\sigma_2}{A_3^2(0)/\sigma_3} \right\}^{1/2},\]

and

\[(3-1-26) \quad \eta^2 = \frac{A_3^2(0)/\sigma_3}{A_2^2(0)/\sigma_2}.\]

This parameter is seen to be the reciprocal of the parameter defined in Eq. (3-1-18). It is so defined here that it again cannot exceed unity as required by the theory of elliptic functions. Equation (3-1-24) gives the signal amplitude in this case

\[(3-1-27) \quad A_2(z) = A_2(0) \operatorname{Dn} rz,\]

where the pump excitation is

\[(3-1-28) \quad u = (\sigma_1\sigma_2)^{1/2} A_3(0) L \equiv \eta RL.\]
The remaining amplitude formulas follow from the conservation relations and Eq. (3-1-27). Hence, the idler amplitude is

\[ A_1(z) = \sqrt{\alpha_1/\alpha_2} \eta A_2(0) S_n r_L z \]

and the pump amplitude is

\[ A_3(z) = A_3(0) C_n Z L \]

The B.TW signal gain in this case is

(3-1-29a) \[ G_{b2} = \frac{A_2(0)}{A_2(L)} \]

(3-1-29b) \[ = [D_n r_L L]^{-1} \]

The photon conversion efficiency is

\[ E_{b2} = \frac{[A_2^2(0) - A_2^2(L)]/\sigma_2}{A_3^2(0)/\sigma_3} \]

and from Eq. (3-1-26) and the definition of gain, \( E_{b2} \) can be written as

(3-1-31a) \[ E_{b2} = \eta^{-2} [1 - G_{b2}^{-2}] \]

(3-1-31b) \[ = S n^2 r_L \]

We note that \( \eta^{-2} \geq 1 \). Considering Eqs. (3-1-23a) and (3-1-31a), we see that they are identical and together they cover any \( \eta^2 \) values. The reason of splitting the range of \( \eta^2 \) values will be discussed in detail in the next chapter in connection with the question of complete photon conversion.
3.2 PHOTON CONVERSION EFFICIENCY IN BTW PARAMETRIC AMPLIFICATION AND OSCILLATION

The effectiveness of backward-traveling-wave (BTW) parametric amplification and oscillation has previously been characterized by \( \eta^2 \), the elliptic function parameter[18] which approximates the conversion efficiency. The approximation was good at reasonably high gain. Exact relationship between \( \eta^2 \) and the conversion efficiency has also been given[19]. The approximation deteriorates with decreasing gain and complete photon conversion appears impossible in the case of amplification of a BTW signal. The present analysis will consider both cases of BTW amplification, one involving a forward-traveling-wave (FTW) signal (Case i in Section 3.1) and the other a BTW signal (Case ii in Section 3.1). It will be seen that complete conversion of the pump is impossible at finite pump excitation in FTW signal amplification. On the other hand, complete conversion is possible in BTW signal amplification at finite pump excitation.

To recapitulate Section 3.1, the system is taken to be pumped at \( \omega_3 \) for amplification of a FTW signal at \( \omega_1 \) or a BTW signal at \( \omega_2 \). \( A_i \) is the amplitude of the wave at frequency \( \omega_i \), \( \sigma_i \) is the respective nonlinear coupling coefficient and \( L \) represents the total interaction length. In FTW signal amplification, the signal input is at \( z=0 \) and output at \( z=L \), while in BTW signal amplification, signal input is at \( z=L \) and output at \( z=0 \).

Case i. FTW signal amplification. The FTW signal gain at \( \omega_1 \) at the output is defined as
(3-2-1) \[ G_f = \frac{A_1(L)}{A_1(0)} \]

with

(3-2-2) \[ n^2 = \frac{A_1^2(L)/\sigma_1}{A_1^2(0)/\sigma_1 + A_3^2(0)/\sigma_3} \]

The photon conversion efficiency is defined as

(3-2-3) \[ E_f = \frac{[A_1^2(L) - A_1^2(0)]/\sigma_1}{A_3^2(0)/\sigma_3} \]

and this can be expressed in terms of \( n^2 \) and gain directly using Eqs. (3-2-1) and (3-2-2). Thus,

(3-2-4) \[ E_f = n^2 \left[ \frac{G_f^2 - 1}{G_f^2 - n^2} \right] \]

It is immediately apparent that \( E_f = 1 \) when \( n^2 = 1 \) as seen in Eq. (3-2-4) and Fig. 7. \( n^2 \) as defined in Eq. (3-2-2) is in practice always less than unity since the number of output signal photons \( (A_i^2/\sigma_i) \) is proportional to the number of photons at \( \omega_i \) cannot exceed the total number of input photons. \( n^2 \) defined in Eq. (3-2-2) can thus be used as the elliptic function parameter. The condition for complete conversion of the pump is so far not obvious in its physical interpretation. This point will be clarified below.

The FTW signal gain at \( \omega_1 \) is given in Eq. (3-1-12)

\[ G_f = \left\{ \text{Sn}[K - cu, n] \right\}^{-1} \]

(3-2-5) \[ = Dn \left( cu, n \right)/Cn \left( cu, n \right) \]
where \( K \) is the real quarter period of the elliptic function and* from Eqs. (3-1-14)

\[
(3-2-6a) \quad c = \left[1 + \frac{A_1(0)/\sigma_1}{A_3(0)/\sigma_3}\right]^{-1/2},
\]

\[
(3-2-6b) \quad u = \left[1 - \frac{n^2}{G_f^2}\right]^{-1/2},
\]

while the overall pumping factor or pump excitation

\[
(3-2-6c) \quad u = (\sigma_1 \sigma_2)^{1/2} A_3(0) L.
\]

It follows from Eqs. (3-2-4) and (3-2-5) that \( E_f \) can also be expressed as

\[
(3-2-7) \quad E_f = n^2 Sn^2(cu,n).
\]

To achieve complete conversion, i.e., \( E_f = 1 \) (and \( n^2 = 1 \)), we must therefore require

\[ Sn(cu,n) = 1 \]

or

\[ u = K/c. \]

Since \( c \) as defined above is finite while \( K \) approaches infinity for \( n^2 = 1 \), it can therefore be concluded that complete conversion or depletion of the pump is impossible with a finite pump excitation.

* This factor is missing in Eq. (3) of Ref. 18.
The relationship between $E$ and $n^2$ is shown in Fig. 7. We thus see that for $n^2 + 1$ or reasonably high gain, $n^2$ is a good approximation of $E$.

Case ii. BTW signal amplification. The gain of a BTW signal at $\omega_2$ is defined as

$$G_b \equiv \frac{A_2(0)}{A_2(L)},$$

and

$$n^2 \equiv \frac{A_2^2(0)/\sigma_2}{A_3^2(0)/\sigma_3}.$$  

The photon conversion efficiency in this case is defined as

$$E_b = \frac{[A_2^2(0) - A_2^2(L)]/\sigma_2}{A_3^2(0)/\sigma_3},$$

which can be rewritten directly as

$$E_b = n^2 [1 - G_b^{-2}]$$

using Eqs. (3-2-8) and (3-2-9).

We note that so far we have not attached any physical significance to the definition of $n^2$ and thus it can presumably take on any value. However, a brief comment is necessary here in order to give the background for the splitting of $n^2$ values into two ranges.

The necessity of this split can be seen by considering the case of complete conversion, i.e., $E_b = 1$. It then follows from the definition of $E_b$ (Eq. (3-2-10)) that
Fig. 7. Photon conversion efficiency $E$ vs $\eta^2$ at fixed $G$ in BTW amplification involving a FTW signal.
\[(3-2-12) \quad \frac{A_2^2(0)/\sigma_2}{A_3^2(0)/\sigma_3} = \frac{A_2^2(L)/\sigma_2 + A_3^2(0)/\sigma_3}{A_3^2(0)/\sigma_3} \quad . \]

In other words, the number of output signal photons at \( \omega_2 \) must be equal to the sum of the number of input signal photons at \( \omega_2 \) and the number of input pump photons at \( \omega_3 \). Relationship (3-2-12) then requires the parameter \( n^2 \) to assume values exceeding unity, i.e., from Eq. (3-2-9)

\[
n^2 = \frac{\frac{A_2^2(0)/\sigma_2}{A_3^2(0)/\sigma_3}}{1 + \frac{A_2^2(L)/\sigma_2}{A_3^2(0)/\sigma_3}} \geq 1 \quad .
\]

The quantity \( n^2 \) is directly associated with the elliptic function parameter through the signal gain formula. Thus, in computing gain, \( n^2 \) is used as the elliptic function parameter when \( n \leq 1 \), while \( n^{-2} \) is used in the case when \( n \geq 1 \). The corresponding BTW signal gain formulas from Eqs. (3-1-21b) and (3-1-29b) are:

\[(3-2-13a) \quad G_b = [Cn(u,n)]^{-1} \quad \text{for} \quad n \leq 1 \quad ,
\]

\[(3-2-13b) \quad G_b = [Dn(\nu,n^{-1})]^{-1} \quad \text{for} \quad n \geq 1 \quad .\]

We note that at \( n=1 \)

\[Cn u = Dn u = \text{Sech} u\]

so that the above two equations join smoothly.

The photon conversion efficiency formulas corresponding to these gain formulas are from formula (3-2-11)
\[ (3-2-14a) \quad E_b = n^2 \text{Sn}^2(u, n) \quad \text{for} \quad n \leq 1 \]
\[ (3-2-14b) \quad E_b = \text{Sn}^2(\nu u, n^{-1}) \quad \text{for} \quad n \geq 1 \]

Let us consider the case of complete conversion. We therefore require

\[ (3-2-15a) \quad n^2 \text{Sn}^2(u, n) = 1 \quad \text{for} \quad n \leq 1 \]
\[ (3-2-15b) \quad \text{Sn}^2(\nu u, n^{-1}) = 1 \quad \text{for} \quad n \geq 1 \]

Thus, in the range \( n \leq 1 \), Eq. (3-2-15a) requires \( n^2 = 1 \) and
\[ \text{Sn}(u, n) = 1 \]
or
\[ u = K \]
which again approaches infinity for \( n^2 = 1 \) thus indicating that complete conversion is not achievable in the \( n \leq 1 \) range. However, in the \( n \geq 1 \) range where Eq. (3-2-15b) applies, we have for complete pump conversion
\[ \text{Sn}(\nu u, n^{-1}) = 1 \]
or
\[ (3-2-16) \quad u = K/n \]

For any \( n \) value except unity, \( K \) is finite (note, \( n^{-2} \) is the elliptic function parameter). Complete conversion is thus seen to be possible at finite pump excitation.

Our physical interpretation of the two ranges of \( n^2 \) values is as follows. The \( n \geq 1 \) range corresponds to a sufficiently large
number of input signal photons or phonons such that the number of amplified output BTW signal photons exceeds the input pump photons. This is hence a region where complete pump depletion is possible. In contrast, the \( n \leq 1 \) range refers to the number of input BTW signal photons such that the amplified output signal photons are less than the input pump photons in number. We have thus covered the entire operation of the BTW amplifier and oscillator.

We can thus conclude that a large BTW input signal can deplete the pump completely while a FTW signal can never do so at any signal level. These results are demonstrated in Fig. 8. Thus the points along the \( E_b=1 \) axis corresponds to \( n \) values that satisfy Eq. (3-2-16).

\[
(3-2-17) \quad n^2 = \frac{k^2}{u^2}
\]

It is clearly seen that in the \( n^2 \leq 1 \) range, \( E_b=1 \) can be achieved only at one point corresponding to \( n^2=1 \) or infinite pump excitation. On the other hand, in the \( n^2 > 1 \) range, \( E_b=1 \) is achievable at a finite pump excitation \( u \) since when \( n^2 > 1, K \) is finite and \( K > u \) for any value of the elliptic function parameter \( n^{-2} \).

The above analysis has led us to conclude that in this case \( n^2 \) should be regarded as a unique parameter for computational purposes rather than as a physical parameter. The physically significant parameter is the photon conversion efficiency which describes exactly the effectiveness of the parametric interaction.
Fig. 8. Photon conversion efficiency $E$ vs $\eta^2$ at fixed $G$ in BTW amplification involving a BTW signal.
3.3 ASYMPTOTIC BEHAVIOR OF PHOTON CONVERSION EFFICIENCY IN BTW PARAMETRIC AMPLIFICATION AND OSCILLATION

The question of complete photon conversion of the pump in backward-traveling-wave (BTW) parametric amplification and oscillation has been considered qualitatively. It was concluded that photon conversion efficiency can never be complete in the amplification involving a forward-traveling-wave (FTW) signal, while it is always possible to completely deplete the pump in the amplification with a BTW signal. It is therefore of interest to examine analytically the asymptotic behavior of photon conversion efficiency as a function of the pump excitation near the limit of complete conversion. The results will demonstrate further the clear advantage of the amplification scheme with a BTW signal.

The effectiveness of interaction is best reflected through the quantity \(1-E\) representing the fraction of residual transmitted pump. Thus, the behavior of this quantity with respect to the pump excitation \(u\) depicts the interaction efficiency at various pump excitation.

Variation of \(1-E\) with \(u\) is found to be describable to a good approximation by simple expressions in the range of large depletion \(E \gg 1\). The simple asymptotic relationship is demonstrated by the linear nature of the computed \(\log(1-E)\) vs \(u\) curves at reasonably high gain (see Figs. 9 and 10).

To further investigate these characteristics, general analytic expressions are sought for the slope of the \(\ln(1-E)\) vs \(u\) curves at fixed gain, i.e.,
For a system pumped at $\omega_3$ for amplification of a FTW signal at $\omega_1$ or a BTW signal at $\omega_2$, $A_i$ is the amplitude of the wave at $\omega_i$, $\sigma_i$ is the respective nonlinear coupling coefficient and $L$ is the total interaction length. The FTW signal has input at $z=0$ and output at $z=L$, while the BTW signal has input at $z=L$ and output at $z=0$.

Case 1. Amplification of a FTW signal.

The photon conversion efficiency is defined as

\[
(3-3-2) \quad E_f = \frac{\frac{A_1^2(L) - A_1^2(0)}{\sigma_1}}{\frac{A_3^2(0)}{\sigma_3}}
\]

and with signal gain

\[
(3-3-3) \quad G_f = \frac{A_1(L)}{A_1(0)}
\]

and

\[
(3-3-4) \quad n^2 = \frac{\frac{A_1^2(L)}{\sigma_1}}{\frac{A_1^2(0)}{\sigma_1} + \frac{A_3^2(0)}{\sigma_3}}
\]

we have for photon conversion efficiency

\[
(3-3-5) \quad E_f = n^2 \left[ \frac{G_f^2 - 1}{G_f^2 - n^2} \right]
\]

The FTW signal gain at $\omega_1$ can also be expressed in terms of elliptic function (see Eq. (3-2-5)). Thus,
\[(3-3-6) \quad G_f = \frac{Dn(u',n)}{Cn(u',n)} \]

where

\[(3-3-7) \quad u' = Cu = \left[1 - \frac{n^2}{G_f^2}\right]^{-1/2} (\sigma_1 \sigma_2)^{1/2} A_3(0)L \]

\[u\] being the overall pumping factor or pump excitation.

From Eqs. (3-3-1) and (3-3-7) the slope of the \(\ln(1-E_f)\) vs \(u\) curves at fixed gain is

\[(3-3-8) \quad \frac{a[\ln(1-E_f)]}{au} = -\left(c + u \frac{ac}{\partial u} \right) \left[1 - \frac{n}{G_f}\right]^{-1} \frac{aE_f}{au'} \frac{1}{G_f} \]

To evaluate this expression, we note that

\[(3-3-9) \quad \frac{ac}{\partial u} = \left(1 - u \frac{ac}{\partial u'} \right) \frac{1}{G_f} \]

and since from Eq. (3-3-7)

\[(3-3-10) \quad \frac{ac}{\partial u'} = \frac{G_f^n}{(G_f^2 - n^2)^{3/2}} \frac{\partial n}{\partial u'} \frac{1}{G_f} \]

we obtain from Eq. (3-3-9)

\[(3-3-11) \quad c + u \frac{ac}{\partial u} = \left(1 - \frac{G_f^n u}{(G_f^2 - n^2)^{3/2}} \frac{\partial n}{\partial u'} \frac{1}{G_f} \right)^{-1} \]

Furthermore, from Eq. (3-3-5), we have

\[(3-3-12) \quad \frac{aE_f}{au'} = 2n \frac{G_f^2}{G_f^2 - n^2} \frac{\partial n}{\partial u'} \frac{1}{G_f} \]
To evaluate the slope of $\eta$ vs $u'$ curves at fixed gain, we will have to begin with basic definitions. First noting the standard elliptic integral is defined as

\begin{equation}
(3-3-13) \quad u'(x,\eta) = \int_x^0 \frac{dt}{\sqrt{(1-t^2)(1-\eta^2t^2)^{1/2}}},
\end{equation}

and correspondingly,

\begin{equation}
(3-3-14) \quad Sn \ u' = x, \ Cn \ u' = [1-x^2]^{1/2}, \ Dn \ u' = [1-\eta^2x^2]^{1/2},
\end{equation}

then from Eq. (3-3-6), signal gain can also be expressed as

\[ G_f = \left[\frac{1-\eta^2x^2}{1-x^2}\right]^{1/2} \]

so that

\begin{equation}
(3-3-15) \quad x(G_f,\eta) = \left[\frac{G_f^2-1}{G_f^2-\eta^2}\right]^{1/2}.
\end{equation}

The upper limit of the integral in formula (3-3-13) is now given in terms of $G_f$ and $\eta$ as in Eq. (3-3-15), and accordingly the formula assumes the form

\begin{equation}
(3-3-16) \quad u'(G_f,\eta) = \int_0^{x(G_f,\eta)} \frac{dt}{\sqrt{(1-t^2)(1-\eta^2t^2)^{1/2}}}.
\end{equation}

By differentiating the integral above with respect to $\eta$ and keeping in mind Eq. (3-3-15) for $x$, we arrive at
\[ (3-3-17a) \quad \left. \frac{\partial u'}{\partial n} \right|_{G_f} = \int_0^x \left( \frac{\partial}{\partial n} \left[ \frac{1}{(1-n^2 t^2)^{1/2}} \right] \right) \frac{dt}{(1-t^2)^{1/2}} + \frac{1}{\left[ (1-x^2)(1-n^2 x^2)^2 \right]^{1/2}} \left. \frac{\partial x(G_f,n)}{\partial n} \right|_{G_f} \]

\[ (3-3-17b) \quad = n \int_0^x \frac{t^2 dt}{(1-t^2)^{1/2}(1-n^2 t^2)^{3/2}} + \frac{n}{(1-n^2)G_f} \frac{G_f^2-1}{G_f^2-n^2} \]

The integral in Eq. (3-3-17b) is known[17]

\[ (3-3-18) \quad n \int_0^x \frac{t^2 dt}{(1-t^2)(1-n^2 t^2)^{3/2}} = \frac{E(u')-(1-n^2)u'}{n(1-n^2)} \]

\[- \frac{n}{1-n^2} \frac{Sn \, u' \cdot Cn \, u'}{Dn \, u'} \]

where \( E(u') \) is an elliptic integral[17]. According to Eqs. (3-3-14) and (3-3-15),

\[ (3-3-19) \quad \frac{Sn \, u' \cdot Cn \, u'}{Dn \, u'} = \frac{1}{G_f} \left[ \frac{G_f^2-1}{G_f^2-n^2} \right]^{1/2} \]

Substituting Eq. (3-3-19) in the last term of Eq. (3-3-18), we find it to be identical to the last term in Eq. (3-3-17b) except for a minus sign. These two terms cancel so that after inverting Eq. (3-3-17b), we finally obtain

\[ (3-3-20) \quad \left. \frac{\partial n}{\partial u'} \right|_{G_f} = \frac{n(1-n^2)}{E(u')-(1-n^2)u'} \]
Substituting this expression in Eqs. (3-3-11) and (3-3-12), we obtain

\[
(3-3-21) \quad c + u \frac{\partial c}{\partial u} \bigg|_{G_f} = \left\{ 1 - \frac{G_f n^2 (1-n^2)u}{(G_f^2 - n^2)^{3/2}[E(u') - (1-n^2)u']} \right\}^{-1}
\]

and

\[
(3-3-22) \quad \frac{\partial E_f}{\partial u'} \bigg|_{G_f} = \frac{2n^2 (1-n^2)G_f^2 (G_f^2 - 1)}{(G_f^2 - n^2)^4[E(u') - (1-n^2)u']}
\]

so that the slope of \(\ln(1-E_f)\) vs \(u\) curves at fixed gain given in Eq. (3-3-8) reduces to

\[
(3-3-23) \quad \frac{\partial [\ln(1-E_f)\]}{\partial u} \bigg|_{G_f} = \left. -\frac{2n^2}{E(u') - (1-n^2)u'} \left[ \frac{G_f^2 - 1}{G_f^2 - n^2} \right] \right\} \times
\]

\[
\times \left\{ 1 - \frac{G_f^2 (1-n^2)u}{(G_f^2 - n^2)^{3/2}[E(u') - (1-n^2)u']} \right\}^{-1}
\]

To estimate the slope shown above for large depletion or \(E_f \to 1\), we first take note that \(n^2\) must also approach unity according to Eq. (3-3-5). Since \(E(u') \to 1\) for reasonably high gain, we have for \((1-n^2) \to 0\)

\[
(3-3-24) \quad \frac{\partial [\ln(1-E_f)\]}{\partial u} \bigg|_{G_f} \sim -2
\]

The asymptotic behavior of conversion efficiency near complete conversion as a function of the pump excitation for the amplification
of a FTW signal can thus be approximated by the expression

\[ \ln(1-E_f) \sim -2(u-u_g) \]  

valid for \( E_f \) sufficiently close to unity and at reasonably high gain. The gain dependence is reflected in this range only in the parameter \( u_g \).

This characteristic -2 slope of the operating curves in FTW signal amplification has considerable physical significance. Since the slope is practically gain independent in the asymptotic limit, it is a unique property and thus can serve as an indicator for this type of amplification. The approximation is best in the case of oscillation, for then \( u'=k, (1-\eta^2) \rightarrow 0, G_f \rightarrow \infty \) and \( E(K) \rightarrow 1 \) (see Eq. (3-3-23)). Thus, the effectiveness of generating instabilities or oscillations is apparent from the \( \omega \) curve in Fig. 9. For example, at a pump excitation three times the zero-depletion oscillation threshold of \( \pi/2 \), the fraction of residual transmitted pump drops approximately to a meager 0.1%. In other words, practically all the input pump has been transferred to the FTW and BTW signal waves at a relatively low pump excitation. Hence, in practice, if \( \omega_2 \) of the BTW signal is much higher than \( \omega_1 \) of the FTW signal (Type I interaction), then most of the energy of the converted pump will be contained in the BTW signal resulting in effectively total reflection of the pump. Further increase in the pump excitation will only reduce the penetration of the pump into the interaction medium. On the other hand, if the FTW signal has a much higher frequency (Type II interaction), then most of the pump will be converted to the FTW
signal and is thus transmitted. In this case, BTW parametric interaction has little effect on pump penetration into a medium. However, if a multiple 3-wave interaction scheme is possible, then this amplified high frequency FTW signal may in turn act as a new pump for a Type I interaction eventually causing all of its energy to be reflected. We note that in practice Type I interactions can occur in any media, while Type II interactions are possible only in anisotropic media as have been demonstrated in stimulated Brillouin scattering[11].

The asymptotic -2 slope of the \( \ln(1-E_f) \) vs \( u \) curves at any fixed gain value thus verifies the qualitative prediction of the impossibility of achieving complete conversion at finite pump excitation in BTW parametric amplification involving an FTW signal. Since at any level of depletion, oscillation requires the largest pump excitation (see Figs. 9 and 10), we may thus conclude that amplification is always more efficient.

**Case 2. Amplification of a BTW signal.**

The photon conversion efficiency is defined in this case as

\[
E_b = \frac{[A_2^2(0)-A_2^2(L)]/\sigma_2}{A_3^2(0)/\sigma_3},
\]

and with signal gain defined as

\[
G_b = A_2(0)/A_2(L),
\]

and
we have

\[ (3-3-29) \quad E_b = n^2 [1 - G_b^{-2}] \]

The BTW signal gain at \( \omega_2 \) can also be expressed in terms of elliptic functions (see Eqs. (3-1-21b) and (3-1-29b)). Thus

\[ (3-3-30a) \quad G_b = [Cn(u, \eta)]^{-1} \quad \text{for} \quad n \leq 1 \]

where \( n^2 \) is the elliptic function parameter, and

\[ (3-3-30b) \quad G_b = [Dn(u', n^{-1})]^{-1} \quad \text{for} \quad n \geq 1 \]

where \( u' \equiv n u \) and \( n^{-2} \) is the elliptic function parameter.

The corresponding \( \eta n(1-E_b) \) vs \( u \) curves at fixed gain for BTW signal amplification are presented in Fig. 10. An analytic expression for their slopes can be derived in an analogous manner. Following Case 1, we have for the slope

\[ (3-3-31) \quad \left. \frac{\partial \eta n(1-E_b)}{\partial u} \right|_{G_b} = \frac{-2n(1-G_b^{-2})}{1-n^2(1-G_b^{-2})} \left. \frac{\partial \eta}{\partial u} \right|_{G_b} \]

In the range \( n \leq 1 \) (see Eq. (3-3-30a)), a fixed \( G_b \) implies fixed \( x = Sn u \), which is the upper limit of the standard elliptic integral as in Eq. (3-3-13). The second term in the right hand side of Eq. (3-3-17a) is thus missing so that Eq. (3-3-18) gives the desired slope of \( \eta \) vs \( u \) curves at fixed gain for this case.
We derive from Eq. (3-3-30a),

\[
\frac{S_n u \cdot C_n u}{D_n u} = \frac{1}{G_b} (1-G_b^{-2})^{1/2} \left[1-n^2(1-G_b^{-2})\right]^{-1/2}
\]

Substituting this expression in Eq. (3-3-32) and inverting it, we obtain

\[
\frac{\partial n}{\partial u} = \frac{n(1-n^2)[1-n^2(1-G_b^{-2})]^{1/2}}{[E(u)-(1-n^2)u][1-n^2(1-G_b^{-2})]^{1/2}-n^2G_b^{-1}(1-G_b^{-2})^{1/2}}
\]

Using this equation in Eq. (3-3-31) leads to the desired slope

\[
\frac{\partial n(1-E_b)}{\partial u} = \frac{-2n^2(1-n^2)}{[1-n^2(1-G_b^{-2})][E(u)-(1-n^2)u]-n^2G_b^{-1}(1-G_b^{-2})^{1/2}}
\]

Acccording to Eq. (3-3-29), \(E \to 1\) when \(n \to 1\) and at reasonably high gain. The asymptotic form of Eq. (3-3-34) for \(G_b \gg 1\) and \(E(u) \to 1\) approaches the value -2. We will consider next the range \(n \geq 1\), where complete photon conversion is also attainable at finite pump excitation.

In the range \(n \geq 1\), we have from Eq. (3-3-30b)

\[
G_b = [Dn(u',n^{-1})]^{-1}
\]

\[
(3-3-35) \quad = \left[1-n^{-2}x^2\right]^{-1/2}
\]
where \( x = S_n u' \) and \( u' = \eta u \). It then follows that

\[
(3-3-36) \quad x = \eta (1-G_b^{-2})^{1/2},
\]

which is not fixed for fixed gain. Furthermore, \( x \) is the upper limit of the standard elliptic integral

\[
u' = \eta u = \int_0^x \frac{dt}{\sqrt{(1-t^2)(1-\eta^2 t^2)}}
\]
or

\[
(3-3-37) \quad u = \int_0^x \frac{dt}{\sqrt{(1-t^2)(\eta^2 t^2)}}
\]

Thus, differentiating \( u \) with respect to \( \eta \) at fixed \( G_b \), we obtain

\[
(3-3-38a) \quad \left. \frac{\partial u}{\partial \eta} \right|_{G_b} = \int_0^x \frac{\partial}{\partial \eta} \left[ \frac{1}{(n^2 t^2)^{1/2}} \right] \frac{dt}{(1-t^2)^{1/2}} + \\
+ \frac{1}{[(1-x^2)(n^2-x^2)]^{1/2}} \left. \frac{\partial u}{\partial \eta} \right|_{G_b} \\
\]
or

\[
(3-3-38b) \quad \left. \frac{\partial u}{\partial \eta} \right|_{G_b} = -\eta \int_0^x \frac{dt}{(n^2-t^2)\sqrt{[(n^2-t^2)(1-t^2)]^{1/2}}} + \\
+ \frac{(1-G_b^{-2})^{1/2}}{[(1-x^2)(n^2-x^2)]^{1/2}}
\]

The integral in Eq. (3-3-38b) is known[17]
\[ -n \int_0^x \frac{dt}{(n^2-t^2)[(n^2-t^2)(1-t^2)]^{1/2}} = -\frac{1}{(n^2-1)} \left[ E(u') - \frac{S_n u' C_n u'}{n^2 D_n u'} \right] \]

so that expressed in terms of \( n \) and \( G_b \), Eq. (3-3-38b) becomes

\[
(3-3-38c) \quad \left. \frac{\partial u}{\partial n} \right|_{G_b} = \frac{G_b(1-G_b^2)^{1/2}}{n(n^2-1)[1-n^2(1-G_b^2)]^{1/2}} \times \left[ \left(\frac{n^2-1}{1-n^2(1-G_b^2)}\right)^{1/2} - \frac{E(u')n[1-n^2(1-G_b^2)]^{1/2}}{G_b(1-G_b^2)^{1/2}} \right].
\]

For the asymptotic behavior, Eq. (3-3-38c) is given in the final form in terms of \( E_b \) and \( G_b \) using Eq. (3-3-29). Thus,

\[
(3-3-38d) \quad \left. \frac{\partial u}{\partial n} \right|_{G_b} = \frac{G_b(1-G_b^2)^{2(1-E_b)})^{1/2}}{E_b^{1/2}(E_b-1+G_b^2)^{1/2}} \left\{ \frac{E_b-1+G_b^2}{1-G_b^2} (2-E_b) - \frac{E(u')E_b^{1/2}(1-E_b)^{1/2}}{G_b(1-G_b^2)} \right\}
\]

Inverting this expression and substituting it in Eq. (3-3-31), we have

\[
(3-3-39) \quad \left. \frac{\partial [n(1-E_b)]}{\partial u} \right|_{G_b} = \frac{-2 E_b(E_b-1+G_b^2)^{1/2}}{(1-E_b)^{1/2}G_b(1-G_b^2)^{3/2}} \times \left\{ \frac{(E_b-1+G_b^2)(2-E_b)}{1-G_b^2} - \frac{E(u')E_b^{1/2}(1-E_b)^{1/2}}{G_b(1-G_b^2)} \right\}^{-1}
\]
Thus the slope at finite gain approaches $-\infty$ when $E_b \to 1$. This is clearly seen in Fig. 10, where the $\log(1-E_b)$ vs $u$ curves have practically vertical drops at sufficiently small $(1-E_b)$ values while pump excitation is finite.

It is thus apparent that the amplification scheme involving a BTW signal is more favorable than that involving a FTW signal. Hence, in Figs. 9 and 10, for instance, to deplete the pump to the $1-E = 0.001$ level at $G=1.5$, the amplification process involving a FTW signal requires more than twice the pump excitation as that used in the process involving a BTW signal. The departure increases at even higher pump depletion. We note briefly that the dashed curve in Fig. 9 represents the condition of equal numbers of input signal photons or phonons and pump photons. Its apparent parallelism to the $-2$ slope implies that infinite signal and pump are required for complete pump depletion. Further physical significance will be discussed in Section 3.4 on physical interpretation. There are two boundary curves of interest in Fig. 10. The dotted curve represents the $n^2=1$ condition, which says that the number of amplified output signal photons or phonons equals that of the pump photons. On the other hand, the dot-dash curve represents input conditions, i.e., equal numbers of input signal photons and pump photons. It clearly indicates that complete pump depletion is possible at finite pump excitation with finite input signal. Additional discussion of these two curves will be given in Section 3.4.
Fig. 9. Residual fractional transmitted pump (1-E) vs pump excitation $u$ at fixed $G$ in BTW amplification involving a FTW signal.
Fig. 10. Residual fractional transmitted pump $(E-1)$ vs pump excitation $u$ at fixed $G$ in BTW amplification involving a BTW signal.
3.4 PHYSICAL INTERPRETATION OF THE CHARACTERISTICS OF BTW AMPLIFICATION AND OSCILLATION

Backward-traveling wave (BTW) parametric amplification and oscillation have been discussed previously[18] using signal gain vs pump excitation curves with the elliptic function parameter \( n^2 \) as the running parameter. This parameter was interpreted approximately as the conversion efficiency or pump depletion at reasonably high gain. However, a more complete picture can be obtained by expressing the results directly in terms of the photon conversion efficiency.

For a system pumped at \( \omega_3 \) for amplification of a forward-traveling-wave (FTW) signal at \( \omega_1 \) or a BTW signal at \( \omega_2 \) or for simultaneous oscillations, there are two cases of amplification depending upon whether the signal input is at \( \omega_1 \) or \( \omega_2 \). The amplitude of the interacting wave at \( \omega_1 \) is \( A_i \) and \( \sigma_i \) is the respective nonlinear coupling coefficient, while \( L \) is the total interaction length.

Case 1. Amplification of a FTW signal at \( \omega_1 \).

The FTW signal gain for input at \( z=0 \) and output at \( z=L \) is, from Eq. (3-1-12)

\[
G_f = \frac{A_1(L)/A_1(0)}{= Dn(cu,E_f)/Cn(cu,E_f)}
\]

The photon conversion efficiency is defined as

\[
E_f = \frac{[A_1^2(L)-A_1^2(0)]/\sigma_1}{A_2^2(0)/\sigma_3}
\]

and from Eq. (3-1-14a)
Fig. 11. Signal gain \( G \) vs pump excitation \( u \) at fixed \( E \) in BTW amplification involving a FTW signal.
Figure 11 shows the amplification and oscillation characteristics of the FTW signal at $\omega_1$ based on Eq. (3-4-1). The process of amplification can be described as follows. In the absence of the pump, all the curves begin at the value $1$ along the ordinate indicating that signal-in equals signal-out. This holds for any input signal as the system is passive and no gain exists. However, with increasing pump excitation, system gain first rises gradually and then steeply as the value of $\pi/2$ along the abscissa is approached. This is the threshold for oscillation when signal gain approaches infinity and there is negligible output as $E \to 0$ or no pump depletion. This also means that input signal is negligible. Increasing the input signal level at this point will actually lower the gain of the system and thus quench the oscillation since pump depletion is no longer zero. We note that although system gain has dropped, the signal output is larger than that in the no depletion case since there is conversion of energy from the pump. Let us consider specifically the case when there is some input signal and the operating point is $a_1$. This point will move vertically downwards with increasing input signal resulting in increasing pump depletion. The dot-dash curve is the dividing line below which the input signal photons exceed pump photons in number. Ultimately, with much larger number input signal photons relative to the pump photons system gain approaches unity and there is nearly complete

\begin{align*}
\text{(3-4-3) } & \quad c = \left[1 + \frac{E_f}{G_f^2 - 1}\right]^{1/2} \\
\text{(3-4-4) } & \quad u = (\sigma_1 \sigma_2)^{1/2} A_3(0)L
\end{align*}
pump depletion. The situation remains essentially the same for larger pump excitation. We may thus conclude that the BTW scheme is capable of performing amplification of input signals at any state of pumping irrespective of the threshold of oscillation.

The significance of the threshold of oscillation is that in the region above \( \pi/2 \) on the abscissa and between the contours of \( E=0 \) and \( E \to 1 \), the system is inherently unstable.

The dot-dash curve in Fig. 11 establishes amplification in terms of input signal and pump levels. Thus, the region above the curve corresponds to the condition when the input signal photons are less than pump photons in number, while the region below corresponds to the condition when the number of input signal photons exceeds that of the pump photons. We notice that the latter region is much smaller.

This curve can be readily established using Eq. (3-2-2). Thus,

\[
\eta^2 = G_f^2 \left[ 1 + \frac{A_3^2(0)/\sigma_3}{A_1^2(0)/\sigma_1} \right]^{-1}
\]

reduces to

\[
(3-4-5) \quad 2\eta^2 = G_f^2
\]

when we impose the condition of equal numbers of input signal photons and pump photons. In this case, photon conversion efficiency from Eq. (3-2-4)

\[
E_f = \eta^2 \left[ \frac{G_f^2 - 1}{G_f^2 - \eta^2} \right]
\]

becomes
Fig. 12. Photon conversion efficiency $E$ vs pump excitation $u$ at fixed $G$ in BTW amplification involving a FTW signal.
(3-4-6) \[ E_f = G_f^2 - 1 = 2n^2 - 1 \]

According to Eq. (3-4-1)

\[ G_f = D_n(cu,n)/C_n(cu,n) \]

where \( c \) from Eq. (3-4-3) is in this case

\[ c = \sqrt{2} \]

The dotted-curve in Fig. 11 is obtained by plotting the signal gain vs. \( u \) for the condition of equal numbers of signal and pump photons.

Figure 12 gives a different viewpoint of these interactions. For oscillation, the curve of infinite gain, designated by "w", depicts the variation of pump depletion with pump excitation. For a finite input signal, oscillation would be quenched and the operating point would move upwards at a fixed pump excitation value indicating an increase in pump depletion and hence an increase in the amplified output signal even though there is a reduction in gain. It is clearly seen in this figure that complete conversion or depletion of the pump is not possible at finite pump excitation. The dot-dash curve here corresponds to that in Fig. 11. Thus, above this curve the number of input signal photons exceeds that of pump photons. Complete pump depletion can be approached with increasing input signal photons over pump photons. On the other hand, in the region below the curve, in which the input signal photons are less than pump photons in number, complete pump conversion can be achieved at infinite pump excitation.
Fig. 13. Signal gain $G$ vs pump excitation $u$ at fixed $E$ in BTW amplification involving a BTW signal.
Case 2. Amplification of a BTW signal at $\omega_2$.

The BTW signal gain for input at $z=L$ and output at $z=0$ is from Eqs. (3-1-21b) and (3-1-29b)

\[(3-4-7a) \quad G_b = \left[Cn(u,n)\right]^{-1} \quad \text{for} \quad n \leq 1 , \]
\[(3-4-7b) \quad G_b = \left[Dn(nu,n^{-1})\right]^{-1} \quad \text{for} \quad n \geq 1 . \]

The two expressions represent two segments of a continuous curve for a fixed $E_b$ or $G_b$ when the parameter $n$ is eliminated. Here,

\[(3-4-8) \quad n^2 = \frac{A_2^2(0)/\sigma_2}{A_3^2(0)/\sigma_3} , \]

and

\[(3-4-9) \quad E_b = \frac{[A_2^2(0) - A_2^2(L)]/\sigma_2}{A_3^2(0)/\sigma_3} \]
\[= n^2 [1 - G_b^{-2}] . \]

The pump excitation $u$ is the same as that defined in Eq. (3-4-4).

The description of the amplification and oscillation processes in this case is very similar to that given in Case 1. Only a brief account will be given here with emphasis on the differences between the two cases. Thus, in Fig.13 an $E=1.0$ contour exists indicating that total pump depletion or complete signal conversion is possible at finite pump excitation and with substantial gain. For example, at
a pump excitation value of 3, we can have total pump depletion with the minimum signal gain of 5, while in Case 1, the gain would be essentially unity, i.e., no gain. Another point of view is also taken in Fig. 14 and the conclusions drawn are similar to those given for Fig. 12. We note that in Figs. 13 and 14, the dotted lines indicate the boundary between the $n < 1$ and the $n > 1$ regions. The continuous nature of the curves is obvious.

The dotted curve for $n^2 = 1$ corresponds to the condition of equal numbers of output signal photons and pump photons. This condition is used basically for computation purposes. Following Case 1, we can establish the curve corresponding to the condition of equal numbers of input signal photons and pump photons. This condition is imposed on Eq. (3-4-8)

$$n^2 = G_b^2 \left[ \frac{A_2^2(L)/\sigma_2}{A_3^2(0)/\sigma_3} \right]$$

leading to the resulting expression

$$n^2 = G_b^2$$

(3-4-10)

The $n > 1$ region is of interest since $G_b > 1$. The signal gain $G_b$ is then given by Eq. (3-4-7b), and the conversion efficiency defined in Eq. (3-4-9) reduces to

$$E_b = n^2 - 1 = G_b^2 - 1$$

(3-4-11)
Fig. 14. Photon conversion efficiency $E$ vs pump excitation $u$ at fixed $G$ in BTW amplification involving a BTW signal.
The signal gain vs pump excitation curve in this case is given as the dot-dash curve in Fig. 13. We notice that in both regions above and below the curve corresponding respectively to conditions of input signal photons exceeding pump photons in number and vice versa, complete pump depletion is possible. The corresponding curve is also shown in Fig. 14 as the dot-dash curve. It is interesting to note in Figs. 11 and 13 the much smaller regions below the curves that correspond to the number of input signal photons exceeding the number of pump photons.

A comparison of the effectiveness of the BTW amplification process involving a FTW signal and that involving a BTW signal is best illustrated in Fig. 15. Here the required pump excitation at fixed gain and fixed conversion efficiency is plotted for the two cases. Thus, the two processes are equally effective when there is no pump depletion. This is shown as the 45° solid line. The two processes are also identical for oscillation since there is no external signal to distinguish the two processes. This is represented as the dotted 45° line in the figure. The break point between these two straight line segments occurs at point a that corresponds to a pump excitation values of \( \pi/2 \) along the ordinate and abscissa. This is, of course, the oscillation threshold for zero depletion as noted before. For finite gain or nonzero depletion, the amplification scheme with a BTW signal is found to be much more effective than the scheme with a FTW signal. This is especially true for large depletion and low gain.
Fig. 15. Pump excitation $u$ for FTW signal vs pump excitation $u$ for BTW signal at fixed $G$ and $E$ in BTW amplification.
The frequency and phase matching conditions are

\[ \omega_3 = \omega_1 + \omega_2, \]
\[ k_3 = k_1 - k_2. \]

There are two possible cases of operation. In both cases, the pump is the reference and taken as a forward-traveling-wave. Case 1 involves a FTW pump at \( \omega_1 \), a BTW signal at \( \omega_2 \) and an up-converted FTW output at \( \omega_3 \); Case 2 involves a FTW pump at \( \omega_2 \), a BTW signal at \( \omega_1 \) and an up-converted BTW output at \( \omega_3 \). As usual, the wave amplitudes are denoted by \( A_i \) at frequency \( \omega_i \) and the nonlinear coupling coefficient by \( \sigma_i \). The total interaction length is \( L \).

Case 1. The coupled differential equations for the wave amplitudes are:

\[ \frac{dA_1(z)}{dz} = -\sigma_1 A_2(z) A_3(z) \]
\[ \frac{dA_2(z)}{dz} = \sigma_2 A_1(z) A_3(z) \]
\[ \frac{dA_3(z)}{dz} = \sigma_3 A_1(z) A_2(z) \]

which lead immediately to the set

\[ \frac{d}{dz} \left[ \frac{A_1^2(z)}{\sigma_1} + \frac{A_2^2(z)}{\sigma_2} \right] = 0. \]
With boundary condition for the idler set as $A_3(0) = 0$, the equations (3-5-3) can be integrated to yield the conservation rules

$$\begin{align*}
(3-5-4a) & \quad \frac{A_1^2(z)}{\sigma_1} + \frac{A_2^2(z)}{\sigma_2} = \frac{A_1^2(0)}{\sigma_1} + \frac{A_2^2(0)}{\sigma_2}, \\
(3-5-4b) & \quad \frac{A_1^2(z)}{\sigma_1} + \frac{A_3^2(z)}{\sigma_3} = \frac{A_1^2(0)}{\sigma_1}.
\end{align*}$$

These relations enable the idler and signal amplitudes to be expressed in terms of the pump amplitude. The differential equation for the pump amplitude is the simplest and is solved first. Solutions of the other differential equations follow readily from the conservation relations. Thus, from Eq. (3-5-2a)

$$\begin{align*}
(3-5-5) & \quad \frac{dA_1(z)}{dz} = \frac{\sigma_1}{\sigma_2} \left( A_1^2(0) + \frac{\sigma_1}{\sigma_2} A_2^2(0) - A_1^2(z) \right) \times \\
& \quad \times \left[ A_1^2(0) - A_1^2(z) \right]^{1/2},
\end{align*}$$

which can be put in the form of the standard elliptic integral with the variable transformation $y = A_1(z)/A_1(0)$. Thus
\[ \frac{A_1(z)}{A_1(0)} \]

(3-5-6) \[ \int_0^1 \frac{dy}{[(1-y^2)(1-n^2y^2)]^{1/2}} \]

\[ = \int_0^1 \frac{dy}{[(1-y^2)(1-n^2y^2)]^{1/2}} - 
[\sigma_2\sigma_3 A_1^2(0) + \sigma_1\sigma_3 A_2^2(0)]^{1/2} z \]

\[ = K - [\sigma_2\sigma_3 A_1^2(0) + \sigma_1\sigma_3 A_2^2(0)]^{1/2} z \]

noting the definition for the real quarter period of the elliptic function K and

(3-5-7) \[ n^2 = \frac{A_1^2(0)/\sigma_1}{A_1^2(0)/\sigma_1 + A_2^2(0)/\sigma_2} \]

Finally,

\[ A_1(z) = A_1(0) \, \text{Sn}(K - rz) \]

where

(3-5-8) \[ \Gamma = [\sigma_2\sigma_3 A_1^2(0) + \sigma_1\sigma_3 A_2^2(0)]^{1/2} \]

\[ = n^{-1} \left[ \left(\sigma_2\sigma_3\right)^{1/2} A_1(0) \right] \]

It readily follows that

\[ A_3(z) = \left(\frac{\sigma_3}{\sigma_1}\right)^{1/2} A_1(0) \, \text{Cn}(k - rz) \]
and the signal amplitude

\[(3-5-9) \quad A_2(z) = \eta^{-1} \left( \frac{\sigma_2}{\sigma_1} \right)^{1/2} A_1(0) \ Dn(K - rz) \]

Case 2. The relevant differential equations of the wave amplitudes are:

\[(3-5-10a) \quad \frac{dA_1(z)}{dz} = \sigma_1 A_2(z) A_3(z) \]

\[(3-5-10b) \quad \frac{dA_2(z)}{dz} = - \sigma_2 A_1(z) A_3(z) \]

\[(3-5-10c) \quad \frac{dA_3(z)}{dz} = - \sigma_3 A_1(z) A_2(z) \]

It follows directly from this set that

\[(3-5-11a) \quad \frac{d}{dz} \left[ \frac{A_1^2(z)}{\sigma_1} + \frac{A_2^2(z)}{\sigma_2} \right] = 0 \]

\[(3-5-11b) \quad \frac{d}{dz} \left[ \frac{A_2^2(z)}{\sigma_1} - \frac{A_3^2(z)}{\sigma_3} \right] = 0 \]

which yield the conservation rules for the boundary condition \( A_3(L) = 0 \)

\[(3-5-12a) \quad \frac{A_1^2(z)}{\sigma_1} + \frac{A_2^2(z)}{\sigma_2} = \frac{A_1^2(0)}{\sigma_1} + \frac{A_2^2(0)}{\sigma_2} \]
(3-5-12b) \[ \frac{A_2^2(z)}{\sigma_2} - \frac{A_3^2(z)}{\sigma_3} = \frac{A_2^2(L)}{\sigma_2} = \frac{A_2^2(0)}{\sigma_2} - \frac{A_3^2(0)}{\sigma_3} \]

Using Eqs. (3-5-12), the pump amplitude equations follows from Eq. (3-5-10b)

(3-5-13) \[ \frac{dA_2(z)}{dz} = - \left\{ (\sigma_1\sigma_3)[A_2^2(0)-A_2^2(z)] + \frac{\sigma_2}{\sigma_1} A_1^2(0) \right\} \times \]
\[ \times \left[ A_2^2(z)-A_2^2(L) \right]^{1/2} \]

or in standard elliptic integral form[17]

(3-5-14) \[ \int^y_0 \frac{dy}{\left[ (1-y^2)(1-n^2 y^2) \right]^{1/2}} = \int^{y_0}_0 \frac{dy}{\left[ (1-y^2)(1-n^2 y^2) \right]^{1/2}} + rz \]

where

\[ y \equiv \left[ \frac{A_2^2(0) + (\sigma_2/\sigma_1) A_1^2(0)-A_2^2(z)}{A_2^2(0) + (\sigma_2/\sigma_1) A_1^2(0)-A_2^2(L)} \right]^{1/2} \]

\[ y_0 \equiv A_1(0) \left[ A_1^2(0) + (\sigma_1/\sigma_2) [A_2^2(0)-A_2^2(L)] \right]^{-1/2} \]

(3-5-15) \[ r \equiv \left[ \sigma_1\sigma_3 A_2^2(0) + \sigma_2\sigma_3 A_1^2(0) \right]^{1/2} \]

and

(3-5-16) \[ n^2 \equiv 1 - \frac{A_2^2(L)/\sigma_2}{A_2^2(0)/\sigma_2+A_1^2(0)/\sigma_1} \]

Thus,
The up-converted and signal wave amplitudes can be derived from Eq. (3-5-17) and the conservation relations (3-5-12). The signal gain can be first derived from basic definitions. This will also facilitate the evaluation of $u_0$ in Eq. (3-5-17). We thus begin with Eq. (3-5-16)

\begin{equation}
(3-5-19a) \quad n^2 = \frac{A_1^2(0) / \sigma_1 + [A_2^2(0) - A_2^2(L)] / \sigma_2}{A_1^2(0) / \sigma_1 + A_2^2(0) / \sigma_2},
\end{equation}

and from the conservation relations,

\begin{equation}
(3-5-19b) \quad n^2 = \frac{A_1^2(0) / \sigma_1 + [A_1^2(L) - A_1^2(0)] / \sigma_1}{A_1^2(0) / \sigma_1 + A_2^2(0) / \sigma_2},
\end{equation}

\begin{equation}
(3-5-19c) \quad = \left[\frac{A_1^2(0)}{A_1^2(L)} + \frac{\sigma_1}{\sigma_2} \frac{A_2^2(0)}{A_1^2(L)}\right]^{-1}.
\end{equation}

We define as usual signal gain as

\begin{equation}
(3-5-20) \quad G = A_1(0) / A_1(L),
\end{equation}

so that Eq. (3-5-19c) can be written as
\begin{equation}
(3-5-21) \quad n^2 = \left[ G^2 + \frac{\sigma_1}{\sigma_2} \frac{A_2^2(0)}{A_1^2(L)} \right]^{-1},
\end{equation}

or

\begin{equation}
(3-5-22) \quad \frac{\sigma_1}{\sigma_2} \frac{A_2^2(0)}{A_1^2(L)} = n^{-2}[1 - n^2 G^2].
\end{equation}

It then follows from the definition for photon conversion efficiency \( E \)

\begin{equation}
(3-5-23) \quad E = \frac{[A_1^2(L)-A_1^2(0)]/\sigma_1}{A_2^2(0)/\sigma_2}
\end{equation}

\begin{equation}
= \frac{\sigma_2}{\sigma_1} \frac{A_1^2(L)}{A_2^2(0)} [1 - G^2]^{-1},
\end{equation}

and Eq. (3-5-22) that

\begin{equation}
(3-5-24) \quad E = n^2(1-G^2)(1-n^2 G^2)^{-1}.
\end{equation}

To evaluate \( u_0 \) given in Eq. (3-5-18), we first use Eq. (3-5-22)

Thus, by rearranging and noting Eq. (3-5-20), we have

\begin{equation}
(3-5-25) \quad \frac{\sigma_2}{\sigma_1} \frac{A_1^2(0)}{A_2^2(0)} = n^2 G^2(1-n^2 G^2)^{-1}.
\end{equation}

The amplitude formula for the pump in Eq. (3-5-17) can then be written as

\begin{equation}
(3-5-26) \quad A_2(z) = (1-n^2 G^2)^{1/2} A_2(0) \text{Dn}(\tau z + u_0).
\end{equation}
By setting \( z = 0 \), we obtain immediately

\[
1 - n^2G^2 = [Dn u_0]^2,
\]

or

\[
(3-5-27) \quad G = Sn u_0.
\]

Using the conservation relations and Eq. (3-5-17), we can find the signal amplitude

\[
(3-5-28) \quad A_2^2(z) = A_1^2(L) - \frac{\sigma_1 A_2^2(0)}{\sigma_2 (1-n^2G^2)} [Dn^2 u_0 - Dn^2(rL+u_0)].
\]

Then

\[
G^2 \equiv \frac{A_1^2(0)}{A_1^2(L)} = 1 - \frac{\sigma_1 A_2^2(0)}{\sigma_2 A_1^2(L)} \frac{1}{1-n^2G^2} [Dn^2 u_0 - Dn^2(rL+u_0)],
\]

and from Eq. (3-5-22),

\[
(3-5-29) \quad G^2 = 1-n^{-2} [Dn^2 u_0 - Dn^2(rL+u_0)].
\]

Using Eq. (3-5-27), we have a relation from Eq. (3-5-29)

\[
Dn^2 u_0 - Dn^2(rL+u_0) = n^2 Cn^2 u_0,
\]

so that

\[
(3-5-30) \quad Dn(rL+u_0) = (1-n^2)^{1/2}.
\]
The right side of Eq. (3-5-30) is the minimum value of a Dn function, and this occurs at values of K of the independent variable (K is the real quarter period of the elliptic function). Thus,

\[ r_L + u_0 = K , \]

or

\[(3-5-31) \quad u_0 = K - r_L \]

We can now express G in terms of elliptic functions by using Eqs. (3-5-31) and (3-5-27).

\[(3-5-32a) \quad G = Sn(K - r_L) \]
\[(3-5-32b) = Cn \frac{r_L}{Dn} \frac{r_L}{r_L} \]

The photon conversion efficiency can now be expressed in terms of the elliptic function. Thus, from Eq. (3-5-24),

\[(3-5-33) \quad E = n^2 Sn^2(r_L) \]

From Eq. (3-5-15), it follows that

\[ r_L \equiv \left[ \sigma_1 \sigma_3 A_2^2(0) + \sigma_2 \sigma_3 A_1^2(0) \right]^{1/2} L \]

\[ = (\sigma_1 \sigma_3)^{1/2} A_2(0)L \left[ 1 + \frac{\sigma_2}{\sigma_1} \frac{A_1^2(0)}{A_2^2(0)} \right]^{1/2} \]

and from Eq. (3-5-25), we have

\[ r_L = (\sigma_1 \sigma_3)^{1/2} A_2(0)L \left[ 1 - n^2 G^2 \right]^{-1/2} \]
The overall pumping factor or pump excitation is now obtained.

\[(3-5-34a) \quad (\sigma_1 \sigma_3)^{1/2} A_2(0)L \equiv u = (1-n^2 G^2)^{1/2} r_L\]

It follows from Eq. (3-5-32a) that

\[(3-5-34b) \quad (\sigma_1 \sigma_3)^{1/2} A_2(0)L = (1-n^2)^{1/2} (Dn r_L)^{-1} (r_L)\]
3.6 PHOTON CONVERSION EFFICIENCY IN BTW PARAMETRIC MIXING

Following the treatment given in Section 2.5 for FTW mixing, we will also consider both signal and pump depletion simultaneously. Similarities and differences in the mixing characteristics for the two BTW mixing schemes are discussed. These two schemes will be analyzed separately.

Case 1. As in Section 2.5, we use the conservation relations (3-5-3) to define photon conversion efficiency

\[
E \equiv \frac{A_1^2(0) - A_1^2(L)}{A_1^2(0)}
\]

in terms of pump depletion, or equivalently,

\[
E \equiv \frac{[A_2^2(L) - A_2^2(0)]/\sigma_2}{A_1^2(0)/\sigma_1}
\]

in terms of signal conversion. E will then be interpreted as pump depletion while Eq. (3-6-1b) is used to relate pump depletion to signal gain. Here, the system has a FTW pump at \(\omega_1\), a BTW signal at \(\omega_2\) and an up-converted wave at \(\omega_3\). The total interaction length is \(L\).

Signal gain in mixing is defined as usual

\[
G \equiv A_2(0)/A_2(L)
\]

and in terms of the elliptic function from Eq. (3-5-9)

\[
G = Dn \sqrt{L}
\]
where

\[(3-6-3)\quad r = n^{-1} \left[ (\sigma_2 \sigma_3)^{1/2} A_1(0) \right], \]

while

\[(3-6-4)\quad n^2 = \left[ 1 + \frac{\sigma_1}{\sigma_2} \frac{A_2^2(0)}{A_1^2(0)} \right]^{-1}, \]

and \(\sigma_i\) is the nonlinear coupling coefficient. Thus, pump excitation is

\[(3-6-5)\quad u = (\sigma_2 \sigma_3)^{1/2} A_1(0)L = nRL.\]

Photon conversion efficiency can then be expressed as

\[(3-6-6a)\quad E = n^{-2}(1-n^2)(G^{-2}-1)\]

\[(3-6-6b)\quad = n^{-2}(1-n^2)[(DnRL)^{-2}-1],\]

using Eqs. (3-6-1), (3-6-2a), (3-6-4) and (3-6-2b). \(G\) again varies between 0 and 1 signifying respectively complete and zero conversion or depletion of the input signal. Equation (3-6-6a) is plotted in Fig. 16. Thus, the conversion efficiency \(E\) is not directly proportional to \(n^2\). The latter will still be used since it characterizes some features of the input signal and pump. Its role is better understood in Fig. 17, in which \(E\) is plotted against \(u\) at fixed \(G\) and \(n^2\). We first note from Eq. (3-6-4) that

\[(3-6-7a)\quad n^2 = \left[ 1 + G^2 \frac{A_2^2(L)/\sigma_2}{A_1^2(0)/\sigma_1} \right]^{-1}, \]
Fig. 16. Photon conversion efficiency $E$ vs $(1-\eta^2)$ at fixed $G$ in BTW mixing involving a FTW pump, a BTW signal and a FTW up-converted wave.
Fig. 17. Photon conversion efficiency $E$ vs pump excitation at fixed $G$ and $\eta^2$ in BTW mixing involving a FTW pump, a BTW signal and a FTW up-converted wave.
so that for equal number in input signal photons and pump photons

\[(3-6-7b) \quad \eta^2 = \left[1 + G^2\right]^{-1}.\]

Then, from Eq. (3-6-6a)

\[(3-6-8a) \quad E = 1 - G^2,\]

and from Eq. (3-6-2b)

\[(3-6-8b) \quad E = 1 - Dn^2\Gamma L.\]

Equation (3-6-8b) is also plotted against \(u\) as the dot-dash curve in Fig. 17. Along this curve there are equal number of input signal photons or phonons and pump photons. Above this curve input signal photons exceed pump photons in number and vice versa below this curve. Thus, complete pump depletion is possible at any finite pump excitation above the dot-dash curve. However, complete signal depletion is not possible at finite pump excitation.

**Case 2.** Following the same reasoning used in Case 1, the photon conversion efficiency or pump depletion is

\[(3-6-9) \quad E \equiv \frac{[A_1^2(L) - A_1^2(0)]/\sigma_1}{A_2^2(0)/\sigma_2},\]

where the system has a FTW pump at \(\omega_2\), a BTW signal at \(\omega_1\) and a BTW up-converted wave at \(\omega_3\). The total interaction length is \(L\). Signal gain is defined as usual

\[(3-6-10) \quad G \equiv A_1(0)/A_1(L),\]
with a range between 0 and 1. From Eq. (3-5-32b), it is also expressed as

\[ G = \frac{Cn}{rL/Dn} \frac{rL}{rL} \]

where \( rL \) is related to the pump excitation \( u \) as

\[ u = (\sigma_1 \sigma_3)^{1/2} A_2(0)L = (1-n^2G^2)^{1/2} rL \]

and

\[ n^2 = \left[ G^2 + \frac{\sigma_1 \sigma_2(0)}{\sigma_2 A_1^2(L)} \right]^{-1} \]

The photon conversion efficiency then follows from Eqs. (3-6-9), (3-6-10) and (3-6-13)

\[ E = n^2 \left(1-G^2\right) \left(1-n^2G^2\right)^{-1} \]

or from Eq. (3-6-11)

\[ E = n^2 Sn^2 rL \]

The range of \( G \) is again between 0 and 1.

The relationship between \( E \) and \( n^2 \) at fixed \( G \) is shown in Fig. 18. Here, \( n^2 \) approximates \( E \) well for near complete signal depletion \( G \to 0 \). Pump depletion plotted as a function of pump excitation at fixed \( G \) and \( n^2 \) is given in Fig.19. It is evident that there is a threshold for complete signal depletion at \( u = \pi/2 \). In contrast to Case 1, complete pump depletion is not possible in this case at finite pump excitation. To investigate further, for
Fig. 18. Photon conversion efficiency $E$ vs $\eta^2$ at fixed $G$ in BTW mixing involving a FTW pump, a BTW signal and a BTW up-converted wave.
Fig. 19. Photon conversion efficiency $E$ vs pump excitation $u$ at fixed $\eta^2$ and $G$ in BTW mixing involving a FTW pump, a BTW signal and a BTW up-converted wave.
equal numbers of input signal photons and pump photons, we have from Eq. (3-6-13)

\[(3-6-15) \quad n^2 = \left[1 + G^2 \right]^{-1}\]

so that Eq. (3-6-14a) becomes

\[(3-6-16a) \quad E = 1 - G^2\]

and from Eq. (3-6-11),

\[(3-6-16b) \quad E = S n^2 (T L) \left(1 - n^2 \right) \left[1 - n^2 \ S n^2 (T L) \right]^{-1}\]

Equation (3-6-16b) is also plotted in Fig. 19 as the dot-dash curve. Thus, above the curve input signal photons exceed pump photons in number and nearly complete pump depletion is possible at finite pump excitation. Below the curve the number of input signal photons is greater than the number of pump photons but complete signal depletion is only possible at values above \(\pi/2\). We note that the dot-dash curve approaches \(E=1\) only asymptotically thus indicating that complete simultaneous signal and pump depletion is possible only at infinite pump and signal.
3.7 PHYSICAL INTERPRETATION OF THE CHARACTERISTICS OF BTW PARAMETRIC MIXING

The BTW parametric mixing process is, as in FTW mixing, best visualized in the graphs in which the signal gain \( G \) is plotted as a function of the pump excitation \( u \) with the conversion efficiency or pump depletion \( E \) as a running parameter. The curves in Fig. 20 (Case 1) depict the case involving a FTW pump, a BTW signal and FTW up-conversion and those in Fig. 21 (Case 2) describe mixing involving a FTW pump and both BTW signal and up-conversion.

Case 1. The signal gain is defined as

\[
G = \frac{A_2(0)}{A_2(L)}
\]

and from Eq. (3-5-9),

\[
G = D n r L
\]

where pump excitation is from Eq. (3-5-8)

\[
u = n r L
\]

and

\[
\eta^2 = \left[ 1 + \frac{\sigma_1}{\sigma_2} \frac{A_2^2(0)}{A_1^2(0)} \right]^{-1}
\]

\( \sigma_1 \) being nonlinear coupling coefficients.

The photon conversion efficiency or pump depletion is then given by

\[
E = \frac{[A_1^2(0) - A_1^2(L)]}{A_1^2(0)}
\]
Fig. 20. Signal gain G vs pump excitation u at fixed E in BTW mixing involving a FTW pump, a BTW signal and a FTW up-converted wave.
Fig. 21. Signal gain $G$ vs pump excitation $u$ at fixed $E$ in BTW mixing involving a FTW pump, a BTW signal and a BTW up-converted wave.
The variation of $G$ with $u$ for fixed $E$ is shown in Fig. 20. The $E=0$ curve corresponds to operating conditions where there is negligible input signal. We notice that at finite pump excitation, complete signal depletion, i.e., $G=0$ is not possible even for negligible signal input. The dotted curve represents the condition when input signal photons equal pump photons in number. This corresponds to the dot-dash curve in Fig. 17. To be specific, let us consider the operating point $a$ corresponding to a pump excitation value of 2. When there is negligible signal, the operating point is at $a_1$. The fraction of signal not depleted is approximately $0.27$. With increasing signal to point "$a_2$", pump depletion increases although there is a larger fraction of undepleted signal ($0.35$). The increase in $G$ continues until point $a_3$ at which the pump is completely depleted and the fraction of undepleted signal is $0.45$. This is the limit of the mixing scheme. We observe that at very large pump excitation, signal depletion is nearly complete no matter how large the input signal is. This is indicated by the convergence of the curves at large $u$.

**Case 2.** The signal gain is defined as

\[
(3-7-5a) \quad G = A_1(0)/A_1(L) ,
\]

and from Eq. (3-5-32b),

\[
(3-7-5b) \quad G = C_{nR}R/L / D_{nR}R
\]
where the pump excitation is from Eq. (3-5-34b)

\[(3-7-6) \quad u \equiv (1 - \eta^2 G^2)^{1/2} rL \]

and

\[(3-7-7) \quad n^2 \equiv \left[ G^2 + \frac{\sigma_1 A_2^2(0)}{\sigma_2 A_1^2(0)} \right]^{-1} \]

\(\sigma_i\) being nonlinear coupling coefficients.

The photon conversion efficiency or pump depletion is

\[(3-7-8a) \quad E \equiv \frac{A_2^2(0) - A_2^2(L)}{A_2^2(0)} \]

\[(3-7-8b) \quad = n^2 S n^2 rL \]

The variation of \(G\) with \(u\) for fixed \(E\) is shown in Fig. 21. The dotted curve here corresponds to the dot-dash curve in Fig. 19. It thus represents the condition when the number of input signal photons equals the number of pump photons. Above this curve, signal photons exceed pump photons in number, but unlike Case 1, complete pump depletion is possible only with infinite signal input as \(E=1\) only when \(G=1\). On the other hand, complete signal depletion is possible at finite pump excitation above the threshold of \(\pi/2\). This is the region bounded by the \(E=0\) curve and the dotted curve.
SUMMARY OF CHAPTER III

Expressions for the amplitudes of the interacting waves are derived in Section 3.1 for the case of parametric amplification and oscillation involving two forward-traveling-waves and a backward-traveling-wave. There are two amplification schemes, one in which the pump and signal are forward traveling waves while the other signal wave is a backward-traveling-wave. In the second case for computational convenience, the region is split into two depending upon whether the number of signal photons exceeds that of the pump photons. The definitions are then generalized to cover all input signal levels. This generalized picture is presented in Section 3.2 on photon conversion efficiency. Emphasis of the discussion lies in the question of complete conversion. It is concluded that in BTW amplification involving a FTW signal, complete conversion is impossible at finite pump excitation. This also holds in BTW parametric oscillation. In contrast, complete conversion is always possible at finite pump excitation in BTW amplification involving a BTW signal. The latter is thus a more efficient scheme. Due to the physical significance of complete conversion, to further investigate the qualitative conclusions of Section 3.2, analytic asymptotic expressions are sought in Section 3.3 for the slope of the \( \ln(1-E) \) vs \( u \) curves at fixed gain. The asymptotic -2 slope of the curve for parametric oscillation is a most significant finding regarding the effectiveness of such interactions for instability generation. Description of the amplification and oscillation process is given with the help of appropriate curves in Section 3.4.
The theory of BTW parametric mixing is established in Section 3.5. There are two interaction schemes, one involving a FTW pump, a BTW signal and FTW up-converted wave and the other involving a FTW pump, a BTW signal and a BTW up-converted wave. The photon conversion efficiency of both cases are discussed in Section 3.6 and comparisons made. We then conclude the Chapter by giving a physical description of the mixing processes in Section 3.7. The existence of pump threshold for complete signal depletion as in FTW mixing and the inability of both mixing schemes to simultaneously deplete the signal and pump completely are some of the significant features.
CHAPTER IV
DISCUSSION

We have thus seen that the treatment of the pump, signal and idler on a par is most essential in the consideration of the efficiency of parametric interactions. At finite pump excitation the assumption of no pump depletion inherently restricts our interest to interactions involving relatively low signal input, while the inclusion of pump depletion is necessary when we consider the entire range of operation. A more fundamental aspect of pump depletion is that it takes into account the reaction of the signal and idler on the pump thus setting a physically necessary limit on the growth of these two waves.

The introduction of the concept of pump depletion in the description of the amplification, oscillation and mixing processes has brought about numerous features which are absent in the case of no pump depletion. Thus, for instance, in the case of amplification raising the input signal level inherently lowers the system gain if the pump excitation is fixed. The reaction of the amplified waves on the pump is therefore automatically incorporated. It is then not surprising to find that in the case of oscillation when system gain is infinite, introduction of a signal will actually quench oscillation, i.e., lower the system gain to a finite value. This seems to bear out qualitatively in our phonon experiments, involving
laser induced 75 MHz acoustic waves in a quartz crystal[11]. Thus, we have often observed that larger input signal does not mean higher gain as would be expected in the case of no-pump depletion.

Some processes that are identical in terms of efficiency when pump is assumed to be undepleted become distinct when pump depletion is nonzero. This is best demonstrated in BTW interactions. Thus, BTW amplification involving a FTW signal is equally efficient as BTW amplification involving a BTW signal when pump is assumed to be infinite. However, the latter scheme becomes increasingly more efficient as pump depletion increases. This may furnish a qualitative explanation for the experimentally observed traveling-wave phonon amplification. Thus, the 75 MHz acoustic waves are set up in the crystal from an external RF source with a pulse length sufficient to establish standing waves. Upon illumination with a ruby laser, only sharply defined amplified phonon echoes are observed. This may be an indication of selective amplification of one of the traveling wave components of the standing wave.

Since pump depletion is first conceived as a necessary mechanism to limit the indefinite growth of the amplified or generated waves in a laser cavity or a medium of infinite length, it must therefore lead to the establishment of some sort of inherent limiting factor. We have thus found that the fraction of residual transmitted pump decreases exponentially with pump excitation. This inherent limiting factor in pump penetration into a medium thus casts much doubt on the idea of enhancing the coupling of laser energy into an interaction medium by simply raising the incident laser power.
It is essential in the case of mixing to consider the depletion of both the signal and the pump since theoretically they are indistinguishable. The treatment of three interacting waves on equal footing thus lends itself naturally to the analysis of this situation. The important finding in mixing is that in both FTW and BTW mixing, simultaneous complete depletion of the pump and the signal can be achieved only at infinite input pump and signal.

This unified picture of elementary three-wave parametric interactions is intended to lay the basis for more complex multiple three-wave interactions, which may be found to be essential in the circumvention of the inherent limiting factors or the enhancement of desired interactions. The idealistic steady-state, plane-wave analysis may prove to be inadequate in solving the actual problem involving picosecond laser pulses and beams of gaussian shape. The possibility of very long interaction length in ionospheric interactions can support the phenomena described above if loss can be overcome. In this respect, it is necessary to include loss in future analysis. These are just a few of the topics that can be of immediate interest as an improvement of the present theory.
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