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Optimal designs for treatment-control comparison

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The Ohio State University, 1987
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OPTIMAL DESIGNS FOR TREATMENT-CONTROL COMPARISON

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of the Ohio State University

By

Chao-Ping Ting, B.C.

The Ohio State University

1987

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To My Parents
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\[ \sum_{i=1}^{l} \left( \frac{1}{p-i} \right) \left( \frac{1}{d} \right) \left( \frac{1}{j} \right) \left( \frac{1}{m} \right) \left( \frac{1}{p} \right) + T \left( \frac{1}{m} \right) + T \left( \frac{1}{d} \right) \]

2. Values of $r_0$ which minimize $J$. In the block design setting.

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Chapter 1
Introduction

In many experimental situations, for example industrial, agricultural and biological experiments, it is desired to compare several test treatments, say \( p \) test treatments, simultaneously with a control (the so-called multiple comparisons with a control problem). In this dissertation, research has been done in two different settings; the first setting is the row-column design of two-way heterogeneity, and the second setting is a similar and simpler extension to the usual block design of one-way heterogeneity.

1.1 Row-column design setting

Let the \( p+1 \) treatments be indexed by \( 0,1,2,\ldots,p \), with \( 0 \) denoting the control treatment and \( 1,2,\ldots,p \) denoting the test treatments. Suppose the \( p+1 \) treatments are to be applied to \( RC \) plots arranged in \( R \) rows and \( C \) columns where \( p,R,C \geq 2 \). Assume that only one treatment can be applied in each plot. Let \( Y_{ijk} \) denote the observation to be taken where treatment \( i (0 \leq i \leq p) \) is assigned to the plot located in row \( j (1 \leq j \leq R) \) and column \( k (1 \leq k \leq C) \). We assume the usual additive linear model without interactions.
where \( \mu \) is the overall mean, \( \alpha_i \) is the effect of treatment \( i \), \( \beta_j \) is the effect of row \( j \), \( \tau_k \) is the effect of column \( k \), and the \( \epsilon_{ijk} \) are assumed to be uncorrelated random variables with mean 0 and common variance \( \sigma^2 \).

We also impose the usual side conditions that 
\[ \Sigma_{i=1,p} \alpha_i = \Sigma_{j=1,R} \beta_j = \Sigma_{k=1,C} \tau_k = 0. \]

Most of the design literature for model (1.1) and the later (1.2) assumes one wishes to estimate all possible contrasts among \( \alpha_0, \alpha_1, \ldots, \alpha_p \).

In this dissertation, however, we focus attention on the situation where one is only interested in the \( p \) treatment-control contrasts \( \alpha_0 - \alpha_i \), \( 1 \leq i \leq p \).

These contrasts are to be estimated by their best linear unbiased estimates (B.L.U.E.'s). Our goal is to find an experimental design (an allocation of control and test treatments to plots of experimental units) which will give the best, in some sense, set of estimates among all possible designs.

For given values of \( R \), \( C \), and \( p \), let \( D(p,R,C) \) denote the set of all possible row-column designs for model (1.1) where \( p, R, C \geq 2 \). For a design \( d \in D(p,R,C) \) let \( r_{ij}(d) \) be the number of plots design \( d \) assigns to treatment \( i \) (\( 0 \leq i \leq p \)) in row \( j \) (\( 1 \leq j \leq R \)), let \( s_{ik}(d) \) be the number of plots design \( d \) assigns to treatment \( i \) in column \( k \) (\( 1 \leq k \leq C \)), let \( r_i(d) = \Sigma_{j=1,R} r_{ij}(d) = \Sigma_{k=1,C} s_{ik}(d) \) be the number of replications of treatment \( i \) in the design \( d \), and let

\[ \lambda_{ij}(d) = \Sigma_{h=1,R} r_{ih}(d) r_{jh}(d) \]
\[ \mu_{ij}(d) = \sum_{h=1}^{s} s_{ih}(d) s_{jh}(d) \]

where \( \lambda_{ij}(d) \) represents the total number of times treatment \( i \) and \( j \) appear together in the \( R \) rows and \( \mu_{ij}(d) \) represents the total number of times treatment \( i \) and \( j \) appear together in the \( C \) columns. Whenever it is clear which design \( d \) is being referred to we shall drop the \( (d) \) to simplify notation.

For \( d \in D(p,R,C) \), it is well known (see for example Kiefer (1958)) that the information matrix \( M(d) = ((\mu_{ij})) \) of \( (\alpha_0, \alpha_1, \ldots, \alpha_p)' \), where primes on vectors indicate the transpose, is given by

\[
\mu_{ij} = \begin{cases} 
 r_i (1/C) \lambda_{ii} - (1/R) \mu_{jj} + (1/RC) r_i^2 & \text{if } i=j \\
-(1/C) \lambda_{ij} - (1/R) \mu_{ij} + (1/RC) r_i r_j & \text{if } i\neq j 
\end{cases}
\]

where \( 0 \leq i, j \leq p \). The row and column sums of \( M(d) \) are known to be 0. By a similar argument to that in the appendix of Bechhofer and Tamhane (1981) one can show that the information matrix \( M(d) = ((\mu_{ij})) \) for estimating \( (\alpha_0 - \alpha_1, \alpha_0 - \alpha_2, \ldots, \alpha_0 - \alpha_p)' \) is the \( pxp \) matrix whose \( i,j^{th} \) entry \( \mu_{ij} = \mu_{ij} \), namely

\[
\mu_{ij} = \begin{cases} 
 r_i (1/C) \lambda_{ii} - (1/R) \mu_{jj} + (1/RC) r_i^2 & \text{if } i=j \\
-(1/C) \lambda_{ij} - (1/R) \mu_{ij} + (1/RC) r_i r_j & \text{if } i\neq j 
\end{cases}
\]

where \( 1 \leq i, j \leq p \). \( M(d) \) is a nonegative definite matrix and is nonsingular if and only if all \( \alpha_0 - \alpha_i, \ 1 \leq i \leq p, \) are estimable. If this is the case then the variance-covariance matrix of \( (\hat{\alpha}_0 - \hat{\alpha}_1, \hat{\alpha}_0 - \hat{\alpha}_2, \ldots, \hat{\alpha}_0 - \hat{\alpha}_p)' \) is proportional to the inverse of \( M(d) \).
1.2 Block design setting

Suppose there are \( p+1 \) treatments, one of which is a control treatment indexed by 0 and the remaining treatments are \( p \) test treatments indexed by 1, 2, ..., \( p \), and \( b \) blocks of common size \( k \) where \( p, b, k \geq 2 \).

Let \( Y_{ijh} \) denote the observation when treatment 1 (\( O \leq i \leq p \)) is assigned to the \( h \)th plot (\( 1 \leq h \leq k \)) of the \( j \)th block (\( 1 \leq j \leq b \)). We also assume the usual additive linear model without interactions

\[
Y_{ijh} = \mu + \alpha_i + \beta_j + \epsilon_{ijh}
\]

where \( \mu \) is the overall mean, \( \alpha_i \) is the effect of treatment \( i \), \( \beta_j \) is the effect of block \( j \), and the \( \epsilon_{ijh} \) are assumed to be uncorrelated random variables with mean 0 and common variance \( \sigma^2 \). We impose a similar side conditions as in sec.1.1, namely \( \Sigma_{i=1}^p \alpha_i = \Sigma_{j=1}^b \beta_j = 0 \).

For given values of \( b, k \) and \( p \) let \( C(p,b,k) \) denote the set of all possible block designs with \( b \) blocks of size \( k \) (\( p, b, k \geq 2 \)). For a design \( d \in C(p,b,k) \) let \( r_{ij}(d) \) be the number of plots design \( d \) assigns to treatment \( i \) in block \( j \). Also let \( r_i(d) = \Sigma_{j=1}^b r_{ij}(d) \) be the number of replications of treatment \( i \) in the design \( d \). Whenever it is clear which design \( d \) is being referred to we shall drop the \((d)\) to simplify notation.
For \( d \in \mathbb{C}(p,b,k) \), one can also see from eq (3.1) of Kiefer (1958) that
the information matrix \( M^*(d) = ((m^*_{ij})) \) of \( (\alpha_0, \alpha_1, \ldots, \alpha_p)' \) is given by

\[
m^*_{ij} = \begin{cases} 
(\gamma_i - (1/k) \sum_{j=1}^b \gamma_{ij})^2 & \text{if } i=j \\
(-1/k) \sum_{h=1}^b \gamma_{ih} \gamma_{jh} & \text{if } i=j
\end{cases}
\]

where \( 0 \leqslant i, j \leqslant p \). The row and column sums of \( M^*(d) \) are also known to be 0. By the argument in the appendix of Bechhofer and Tamhane (1981) one can see that the information matrix \( M(d) \) for estimating

\( (\alpha_0 - \alpha_1, \alpha_0 - \alpha_2, \ldots, \alpha_0 - \alpha_p)' \)

is the \( p \times p \) matrix whose \( i,j \)th entry \( m_{ij} = m^*_{ij} \), namely

\[
m_{ij} = \begin{cases} 
(\gamma_i - (1/k) \sum_{j=1}^b \gamma_{ij})^2 & \text{if } i=j \\
(-1/k) \sum_{h=1}^b \gamma_{ih} \gamma_{jh} & \text{if } i=j
\end{cases}
\]

where \( 1 \leqslant i, j \leqslant p \). \( M(d) \) is also a nonnegative definite matrix and nonsingular if and only if all the \( \alpha_0 - \alpha_1 \) are estimable. Then the variance-covariance matrix of \( \hat{\alpha}_0 - \hat{\alpha}_1, 1 \leqslant i, j \leqslant p \) is proportional to the inverse of \( M(d) \).
1.3 Optimality Criteria

Let us formulate more explicitly our goal of finding a design \( d \in D(p,R,C) \) or \( C(p,b,k) \) which yields the 'best' in some sense, set of estimates. Following the work of Kiefer (Kiefer 1958, 1959, 1971, 1975, and 1975) a design \( d \in D(p,R,C) \) (or \( C(p,b,k) \)) is said to be \( \Phi \)-optimal if it minimizes \( \Phi(\mathbf{M}(d)) \) for some real-valued function \( \Phi \) over \( D(p,R,C) \) (or \( C(p,b,k) \)). Restricting ourselves to non-singular designs, i.e. \( \mathbf{M}^{-1}(d) \) exists, some common examples of \( \Phi \) are

\[ \Phi_{0}(\mathbf{M}(d)) = \det[\mathbf{M}^{-1}(d)] \] (so called D-optimality), \[ \Phi_{1}(\mathbf{M}(d)) = \text{tr}[\mathbf{M}^{-1}(d)] \] (so called A-optimality), and \[ \Phi_{\infty}(\mathbf{M}(d)) = \text{maximum eigenvalue of } \mathbf{M}^{-1}(d) \] (so called E-optimality). In the structure of treatment-control comparison, A-optimality has an appealing statistical interpretation, i.e. of minimizing \( \sum_{i=1}^{p} \text{Var} (\hat{\alpha}_0 - \hat{\alpha}_i) \) over all designs, where \( \hat{\alpha}_0 - \hat{\alpha}_i \) is the B.L.U.E. of \( \alpha_0 - \alpha_i \). D- and E-optimality, however, can not be interpreted as functions of the variances of the elementary treatment-control contrasts \( \alpha_0 - \alpha_i \) only.

In most of the aforementioned papers by Kiefer, as well as those of other researchers, one was interested in an orthonormal basis of treatment contrasts. Research focused on finding good designs for estimating \( \mathbf{P}\alpha \) where \( \alpha = (\alpha_0, \alpha_1, \ldots, \alpha_p)' \) is the vector of all \( p+1 \) treatment effects and \( \mathbf{P} \) is a \( px(p+1) \) matrix with orthonormal rows and zero row sums. Until recently not much has been done for the case when the contrasts are not mutually orthogonal.
In row-column design setting, Notz (1985) gave conditions under which a design is A-optimal. Some examples are as follow. For any integer \( m \geq 1 \), a design has \( p = m^2 \), \( R = C = m^2 + m \), and each test treatment is applied once in each row and once in each column, and the control is applied \( m \) times in each row and \( m \) times in each column is A-optimal. This kind of designs can be constructed from a \( m^2 + m \) by \( m^2 + m \) Latin Square by changing treatment labels \( m^2 + 1, m^2 + 2, \ldots, m^2 + m \) to 0 (control).

Majumdar (1986) gave a generalized version of this. For any integer \( m, \alpha, \phi \geq 1 \), a design has \( p = m^2 \), \( R = \alpha(m^2 + m) \) and \( C = \phi(m^2 + m) \), and each test treatment is applied \( \phi \) times in each row and \( \alpha \) times in each column, and the control is applied \( \phi m \) times in each row and \( \alpha m \) times in each column is A-optimal.

Jacroux (1984) applied the techniques of Hall (1935) and Agarwal (1966) and proved the following result. A design is A-optimal if (1) it is an A-optimal block design with columns as blocks, and (2) the \( r_{ij} \) are all equal for \( 1 \leq i \leq p, 1 \leq j \leq R \), and the \( r_{0j} \) are all equal for \( 1 \leq j \leq R \).

In the incomplete block design setting, i.e. \( k \leq p \), Cox (1958) first noted that a BIB (balanced incomplete block) design might not be the best design for treatment-control comparison problem because of the special role played by the control. He suggested that one augments a BIB design by applying the control an equal number of times in each block. He did not give any justification for his suggestion. Pesek (1974) gave analytical details for one of the cases suggested by Cox (1958), i.e. the
control is applied only once in each block; he showed that this design is more efficient than a BIB design in all the p+1 treatments, test treatments and control.

Hedayat and Majumdar (1985) have actually shown that a BIB design with p test treatments in b blocks of common size $k-1$ each augmented by one replication of the control is A-optimal whenever $k$ and $p$ satisfy $(k-1)^2 + 1 \leq p \leq (k-1)^2$. Such designs are denoted as $ABIB(p, b, k-1, 1)$. An example of an A-optimal design for $p = 3$, $b = 6$, and $k = 3$ is given below, where the columns represent the blocks.

```
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
1 1 1 1 2 2
2 2 3 3 3 3
```

Stufken (1986) has extended the above idea to a more general case, i.e. an $ABIB(p, b, k-t; t)$, where $t$ is any positive integer, is A-optimal whenever $k$, $t$, and $p$ satisfy $(k-t-1)^2 + 1 \leq t^2p \leq (k-t)^2$. An example of an A-optimal design for $p = 8$, $b = 28$, $k = 8$, and $t = 2$ is given below.

```
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
3 2 2 2 2 2 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
4 4 3 3 3 3 4 3 3 3 3 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2 2
5 5 5 4 4 4 4 5 5 4 4 4 5 4 4 4 4 3 3 3 3 3 3 3 3 3 3 3
6 6 6 6 5 5 6 6 6 5 5 6 5 5 6 5 5 6 5 5 5 4 4 4 4 4 4
```
Instead of using an ABIB design only, one can combine two BIB designs to construct an A-optimal design for this problem. The following result is from Cheng, Majumdar, Stufken, and Ture (1986). For \( p = \alpha^2 - 1 \), \( b = \phi(\alpha + 2)(\alpha^2 - 1) \), and \( k = \alpha \), the union of an ABIB \((p, \phi(\alpha + 1)(\alpha^2 - 2), \alpha - 1; 1)\) and a BIB design in all the \( p+1 \) treatments in \( 6\alpha(\alpha + 1) \) blocks of common size \( k \) is A-optimal whenever \( \alpha \) is a prime power, and \( \phi \) is any positive integer.

There are many different combinations of the values \( b, k, \) and \( p \) which do not satisfy any of the aforementioned conditions but one can still find an optimal design by using the result of Majumdar and Notz (1983). They gave conditions for a design to be optimal among all designs in the incomplete block design setting for a large class of optimality criteria where A-optimal designs can be derived as a special case. Hedayat and Majumdar (1984) gave an algorithm and a complete list of A-optimal designs by this result when \( 2k \leq 8, k \geq 30, \) and \( p \leq 50 \).

Bechhofer and Tamhane (1981) proposed a new class of designs which are called Balanced Treatment Incomplete Block (abbreviated as BTIB) designs. That is, each test treatment must appear with the control together in the same block the same total number of times over the design, and two different test treatments are paired together the same total number of times over the design. The optimality criteria they
considered was not A-optimality, but of obtaining optimal simultaneously confidence interval and of using a new concept of admissibility of designs to eliminate the 'inferior' design from consideration.
Suppose $\Phi$ is of the form

$$\Phi(M(d)) = \sum_{i=1}^{p} f(\lambda_i(d))$$

where $f$ is a real-valued possibly infinite function on the set of all non-negative numbers which is continuous on the set of all positive numbers, has $f' \leq 0$ and $f'' \geq 0$ (here primes denote differentiation), and $\lambda_i(d)$, $1 \leq i \leq p$ are the eigenvalues of $M(d)$. The rationale for finding a way to simplify the verification of optimality is obvious. We do not want to compute $M^{-1}(d)$, or even $\lambda_i(d)$, for every competing design, and would hope to find an easier computational method which will suffice.

**Theorem 2.1.** Suppose that $M = (m_{ij})$ is a $p \times p$ positive definite matrix, $\lambda_1, \ldots, \lambda_p$ are its eigenvalues and $f$ is defined as above. Let

$$m_{i.} = \Sigma_{j=1}^{p} m_{ij} = \text{ith row sum of } M$$

$$m_{..} = \Sigma_{i=1}^{p} m_{i.} = \Sigma \Sigma_{i,j=1}^{p} m_{ij} = \text{sum of all entries of } M.$$  

Then

$$(p-1) f \left( \sum_{i=1}^{p} m_{ij}/(p-1) - m_{..}/p(p-1) \right) + f(m_{..}/p)$$
\[ ((p-1)/p) \sum_{j=1, p} f \left( (p/p-1) m_{jj} - \frac{2}{p-1} m_j + m_j/p(p-1) \right) + f(m_j/p) \leq \sum_{i=1, p} f(\lambda_i) \]  \hspace{1cm} (2.2)

**Proof.** Let \( P = ((p_{ij})) \) be a \( p \times p \) orthogonal matrix such that
\[ PMP' = \text{diag}(\lambda_1, \ldots, \lambda_p), \]
then
\begin{align*}
(1) & \quad \sum_{i=1, p} p_{ij}^2 = 1 \quad j=1,\ldots,p \\
(2) & \quad \sum_{j=1, p} p_{ij}^2 = 1 \quad i=1,\ldots,p \\
(3) & \quad \sum_{i=1, p} p_{ij} p_{ik} = 0 \quad \text{if } j \neq k
\end{align*}

This is due to the fact that \( P'P = PP' = I \). Since
\[ PMP' = \text{diag}(\lambda_1, \ldots, \lambda_p), \]
then
\[ M = P \text{ diag}(\lambda_1, \ldots, \lambda_p) P', \] whose
\[ i,j \] entry is
\[ m_{ij} = \sum_{e=1, p} p_{ei} p_{ej} \lambda_e \] \hspace{1cm} (2.3)

\[ (p/(p-1)) m_{jj} - (2/(p-1)) m_j + m_j/(p(p-1)) \]

\[ = (p/(p-1)) (\sum_{i=1, p} p_{ij}^2 \lambda_1) - (2/(p-1)) (\sum_{k=1, l=1, p} p_{ij} p_{lk} \lambda_l) + (1/(p(p-1))) (\sum_{e,k=1, l=1, p} p_{ie} p_{lk} \lambda_l) \]

\[ = \sum_{i=1, p} \left( (p/(p-1)) p_{ij}^2 + (1/p-1) (1/p) \sum_{k=1, l=1, p} p_{ie} p_{lk} - 2 \sum_{k=1, p} p_{ij} p_{lk} \right) \lambda_i \] \hspace{1cm} (2.4)

Notice
(1/p) \((\Sigma e_{k=1,p} P_{le} P_{lk})\) - 2 \((\Sigma k=1,p P_{lj} P_{lk})\)

= (1/p) \((\Sigma k=1,p P_{lk})^2\) - 2 \(p_{lj}\) \((\Sigma k=1,p P_{lk})\)

= \((1/\sqrt{p}) \Sigma k=1,p P_{lk} - (\sqrt{p}) P_{lk}\)^2 - \(p_{lj}^2\)

Substituting this into (2.4) yields

\((p/(p-1)) p_{lj}^2 + (1/p-1) \((1/p) \((\Sigma e_{k=1,p} P_{le} P_{lk})\)

- 2 \((\Sigma k=1,p P_{lj} P_{lk})\)\]

= (1/p-1) \((1/\sqrt{p}) \(\Sigma k=1,p P_{lk}\) - (\sqrt{p}) P_{lj}) \geq 0 \quad (2.5)

Also notice

\(\Sigma i=1,p \((p/(p-1)) p_{lj}^2 + (1/p-1) \((1/p) \Sigma e_{k=1,p} P_{le} P_{lk}\)

- 2 \((\Sigma k=1,p P_{lj} P_{lk})\)\]

= (p/(p-1)) \(\Sigma i=1,p p_{lj}^2\) + (1/p(p-1)) \(\Sigma e_{k=1,p} P_{le} P_{lk}\)

- (2/p-1) \(\Sigma e_{k=1,p} P_{lj} P_{lk}\)

= (p/(p-1)) + (1/p-1) - (2/p-1)

= 1 \quad (2.6),

since by properties (I), (II), and (III) of the \(p_{lj}\) given at the outset of this proof,

\(\Sigma e_{k=1,p} P_{lj} P_{lk} = \Sigma k=1,p \((\Sigma i=1,p p_{lj} P_{lk})\) = \Sigma k=1,p \((\delta_{k,j}) = 1\)
and \( \delta_{k,j} = 1 \) if \( k=j \) is the so-called Kronecker delta.

Thus we see (2.4) is a convex combination of the \( \lambda_i \), then by (2.5), (2.6), and the convexity of \( f \)

\[
f \left[ \frac{(p-1)}{p} m_{jj} - \frac{2}{p-1} m_{.j} + m_{..}/(p(p-1)) \right]
\]
\[
= f \left( \sum_{l=1,p} \left[ \frac{(p-1)}{p} p_{lj}^2 + \frac{1}{p-1} \left( \frac{1}{p} \right) \sum_{k=1,p} \delta_{l,k} \right] \right)
\]
\[
= \sum_{l=1,p} \left[ \frac{(p-1)}{p} p_{lj}^2 + \frac{1}{p-1} \left( \frac{1}{p} \right) \sum_{k=1,p} \delta_{l,k} \right] \right] f \left( \lambda_l \right)
\]

Summing over \( j \) and multiplying both sides by \((p-1)/p\) yields

\[
\frac{(p-1)}{p} \sum_{j=1,p} f \left[ \frac{(p-1)}{p} m_{jj} - \frac{2}{p-1} m_{.j} + m_{..}/(p(p-1)) \right]
\]
\[
\leq \frac{(p-1)}{p} \sum_{j,l=1,p} \left[ \frac{(p-1)}{p} p_{lj}^2 + \frac{1}{p-1} \left( \frac{1}{p} \right) \sum_{k=1,p} \delta_{l,k} \right] \right] f \left( \lambda_l \right)
\]

The right-hand side of the inequality satisfies

\[
\frac{(p-1)}{p} \sum_{j,l=1,p} \left[ \frac{(p-1)}{p} p_{lj}^2 + \frac{1}{p-1} \left( \frac{1}{p} \right) \sum_{k=1,p} \delta_{l,k} \right] \right] f \left( \lambda_l \right)
\]
\[
= \frac{(p-1)}{p} \sum_{j=1,p} \left[ \frac{(p-1)}{p} m_{jj} - \frac{2}{p-1} m_{.j} + m_{..}/(p(p-1)) \right] f \left( \lambda_l \right)
\]


\[ + (1/p(p-1)) (\Sigma \Sigma_{j,e,k=1,p} P_{ij} P_{ik}) \]

\[ - 2 \left( \Sigma \Sigma_{j,k=1,p} P_{ij} P_{ik} \right) \left( \lambda_i \right) \]

\[ = (p-1)p \Sigma_{i=1,p} \left[ (p/p-1) + (1/p-1) (\Sigma \Sigma_{e,k=1,p} P_{ie} P_{ik}) \right] \]

\[ - 2 \left( \Sigma \Sigma_{j,k=1,p} P_{ij} P_{ik} \right) \left( \lambda_i \right) \]

\[ = \Sigma_{i=1,p} \left[ 1 - (1/p) \Sigma \Sigma_{j,k=1,p} P_{ij} P_{ik} \right] \left( \lambda_i \right) \]

\[ = \Sigma_{i=1,p} \left( \lambda_i \right) - (1/p) \Sigma_{i=1,p} \left( \Sigma_{j=1,p} P_{ij} \right)^2 \left( \lambda_i \right) \]

Since \((l) (\Sigma_{j=1,p} P_{ij})^2 \geq 0\) for all \(l\),

\[ (l) \Sigma_{i=1,p} \left( \Sigma_{j=1,p} P_{ij} \right)^2 = \Sigma \Sigma_{j,i,k=1,p} P_{ij} P_{ik} = \Sigma_{j=1,p} 1 = p \]

we see \((1/p) \Sigma_{i=1,p} \left( \Sigma_{j=1,p} P_{ij} \right)^2 = 1\)

By the convexity of \(f\), we therefore have

\[ f \left( \left( (1/p) \Sigma_{i=1,p} \left( \Sigma_{j=1,p} P_{ij} \right)^2 \lambda_i \right) \right) \leq (1/p) \Sigma_{i=1,p} \left( \Sigma_{j=1,p} P_{ij} \right)^2 \left( \lambda_i \right) \]

Then upon substituting this into \((2.7)\)

\[ ((p-1)/p) \Sigma_{j=1,p} f \left\{ (p/p-1) m_{ij} - (2/p-1) m_j, + m_{..} / (p(p-1)) \right\} \]

\[ \leq ((p-1)/p) \Sigma \Sigma_{j,i=1,p} \left( (p/p-1) P_{ij}^2 \right) \]

\[ + (1/p-1) ((1/p) (\Sigma \Sigma_{e,k=1,p} P_{ie} P_{ik}) \]

\[ - 2 \left( \Sigma_{k=1,p} P_{ij} P_{ik} \right) \left( \lambda_i \right) \]

\[ = \Sigma_{i=1,p} \left( \lambda_i \right) - (1/p) \Sigma_{i=1,p} \left( \Sigma_{j=1,p} P_{ij} \right)^2 \left( \lambda_i \right) \]
\[ \sum_{i=1,p} f(\lambda_i) - f \left( \frac{1}{p} \sum_{i=1,p} \left( \sum_{j=1,p} \rho_{ij} \lambda_i \right) \right) \]

\[ = \sum_{i=1,p} f(\lambda_i) - f \left( \frac{1}{p} \sum_{i=1,p} \sum_{j=1,p} \rho_{ij} \rho_{ik} \lambda_i \right) \]

\[ = \sum_{i=1,p} f(\lambda_i) - f \left( \frac{m_{..}}{p} \right) \]

\[ \Rightarrow \left( \frac{p-1}{p} \right) \sum_{j=1,p} f \left( \frac{(p/p-1) m_{jj} - (2/p-1) m_{jj} + m_{..}/(p(p-1))}{p} \right) + f \left( \frac{m_{..}}{p} \right) \]

\[ \leq \sum_{i=1,p} f(\lambda_i) \]

And the first inequality of (2.2) is just due to the convexity of \( f \).

**Lemma 2.1.** The inequalities of (2.2) become equality when \( M \) is completely symmetric, i.e. \( M = a_{1p} + b_{Jp} \) where \( I_p \) is the \( p \times p \) identity matrix and \( J_p \) is the \( p \times p \) matrix all of whose entries are 1.

**Proof.** The proof follows immediately from the form of the eigenvalues for a completely symmetric matrix and the direct substitution in (2.2) in the completely symmetric case.

(2.2) is an extension of Kiefer's widely used inequality (1975, (2.9)), but without the restriction that the row and column sums are zero. If one adds the restriction in, (2.2) will reduce to

\[ \left( \frac{p-1}{p} \right) \sum_{j=1,p} f \left( \frac{(p/p-1) m_{jj}}{p} \right) \leq \sum_{i=1,p-1} f(\lambda_i) \]

which is exactly Kiefer's inequality. In fact the following analog to Proposition 3 of Kiefer (1975) is immediate.
Lemma 2.2. If, \( d^* \in D(p,R,C) \) or \( d^* \in C(p,b,k) \). Suppose \( \Phi \) is of the form (2.1), and if \( M(d^*) \) is completely symmetric and \( d^* \) minimizes either

\[
(p-1) f \left( \sum_{i=1}^{p} m_{ii}/(p-1) - m../p(p-1) \right) + f(m../p), \quad \text{or}
\]

\[
((p-1)/p) \sum_{j=1}^{p} f \left( (p/p-1) m_{jj} - (2/p-1) m_j + m../p(p-1) \right) + f(m../p)
\]

then \( d^* \) is \( \Phi \)-optimal over \( D(p,R,C) \) or \( C(p,b,k) \).
Chapter 3

A-optimal Designs in the Row-Column Design Setting

In this chapter we shall state a series of lemmas, with respect to the inequalities of (2.2), culminating in a general theorem from which A-optimal designs can be obtained as a special case. Our plan is to find conditions under which either

\[(p-1) f \{(\sum_{j=1,p} m_{ij}/(p-1) - m_{..}/p(p-1)) + f(m_{..}/p)\}

or \[(p-1)/p \Sigma_{j=1,p} f \{(p/p-1) m_{jj} - (2/p-1) m_{j..} + m_{..}/p(p-1)) + f(m_{..}/p)\]

will be a minimum.

3.1 Notation and definitions

Let (1) \[A_1 = (p/p-1) m_{..} - (2/p-1) m_{..} + m_{..}/p(p-1)\]

\[= (p/p-1) (r_i - (1/C) \Sigma_{j=1,R} r_{ij}^2) - (1/R) \Sigma_{k=1,C} s_{ik}^2 + r_{i1} r_{ij} + (1/R) (1/C) \Sigma_{j=1,R} r_{ij} r_{oj} + (1/R) \Sigma_{k=1,C} s_{ik} s_{ok} - r_{o1} r_{o}/RC) + (1/p(p-1)) (r_o - (1/C) \Sigma_{j=1,R} r_{oj}^2) - (1/R) \Sigma_{k=1,C} s_{ok}^2 + r_{o}/RC)\]
(II) \( g(r_1) = p(p+1) A_1 \)
\[ = p^2 m_{11} - 2p m_i + m_j \]
\[ = p^2 r_1 + r_0 + (p r_i + r_0)^2 / RC \]
\[ - (1/C) \Sigma_{j=1,R} (p r_{ij} + r_{oj})^2 \]
\[ - (1/R) \Sigma_{k=1,C} (p s_{ik} + s_{ok})^2 \]

(III) \( \Delta (r_1) = g(r_1+1) - g(r_1) \)

(iv) \( m_{11}(r_1) = m_{11} \)
\[ = r_1 - (1/C) \Sigma_{j=1,R} r_{ij}^2 - (1/R) \Sigma_{k=1,C} s_{ik}^2 + r_1^2 / RC \]

(v) \( \Delta^0(r_1) = m_{11}(r_1+1) - m_{11}(r_1) \)

(vi) \( N = \{ n \; ; \; 0 \leq n \leq RC - r_0 \} \), \( n = aR \) or \( aC \), a integer

**Definition 3.1.** (Kiefer (1975)) If \( A, B \in \mathbb{N} \), \( A < B \), and no integer between \( A \) and \( B \) is in \( \mathbb{N} \), we call \([A, B]\) an elementary interval.

**Definition 3.2.** (Kiefer (1975)) If \([A_0, B_0]\) is the elementary interval containing the value \((RC - r_0)/p\), where \((RC - r_0)/p \in \mathbb{N}\), we call \([A_0, B_0]\) the basic interval.

**Definition 3.3.** \([A^0, B^0]\) is the elementary interval containing the value \(RC/p\), where \(RC/p \in \mathbb{N}\).
3.2 Conditions for A-optimal designs

Lemma 3.1. If \( d \in D(p,R,C) \) and \( \Phi \) is of the form (2.1), then for fixed values of \( r_0 \), the \( r_{ij} \), the \( s_{ik} \), and the \( r_1 \),

\[
 f ((1/p-1) \sum_{i=1}^{p} m_{ii} - m../(p(p-1)))
\]

is minimized when the \( r_{ij} \) are as equal as possible and the \( s_{ik} \) are as equal as possible, i.e. \( r_{ij} = \lfloor r_{ij}/R \rfloor \) or \( \lfloor r_{ij}/R \rfloor + 1 \), \( s_{ik} = \lfloor r_{i}/C \rfloor \) or \( \lfloor r_{i}/C \rfloor + 1 \) for all \( 1 \leq i \leq p \), \( 1 \leq j \leq R \), and \( 1 \leq k \leq C \).

Proof. Since \( (1/p-1) \sum_{i=1}^{p} m_{ii} - m../(p(p-1)) \)

\[
= (1/p-1) (r_1 - (1/C) \sum_{j=1}^{R} r_{ij}^2 - (1/R) \sum_{k=1}^{C} s_{ik}^2
\]

\[+ r_1^2/RC) - (1/p(p-1)) (r_0 - (1/C) \sum_{j=1}^{R} r_{0j}^2
\]

\[- (1/R) \sum_{k=1}^{C} s_{0k}^2 + r_0^2/RC) \]

For fixed values of \( r_0 \), the \( r_{0j} \), and the \( s_{0k} \), \( r_0 - (1/C) \sum_{j=1}^{R} r_{0j}^2 \)

\[= (1/R) \sum_{k=1}^{C} s_{0k}^2 + r_0^2/RC \]

is thus fixed. Then for fixed \( r_1 \),

\[r_1 - (1/C) \sum_{j=1}^{R} r_{ij}^2 - (1/R) \sum_{k=1}^{C} s_{ik}^2 + r_1^2/RC \]

will be a maximum for all \( i \) if \( \sum_{j=1}^{R} r_{ij}^2 \) and \( \sum_{k=1}^{C} s_{ik}^2 \) are as small as possible. It is easy to show that for fixed \( r_1 \), \( \sum_{j=1}^{R} r_{ij}^2 \) is minimized by choosing \( r_1 - R \lfloor r_{i}/R \rfloor \) of the \( r_{ij} \) to be \( \lfloor r_{i}/R \rfloor + 1 \) and the remaining \( R(1+\lfloor r_{i}/R \rfloor) - r_1 \) of the \( r_{ij} \) to be \( \lfloor r_{i}/R \rfloor \). Also \( \sum_{k=1}^{C} s_{ik}^2 \) is minimized by choosing \( r_1 - C \lfloor r_{i}/C \rfloor \) of the \( s_{ik} \) to be \( \lfloor r_{i}/C \rfloor + 1 \) and the remaining \( C(1+\lfloor r_{i}/C \rfloor) - r_1 \) of the \( s_{ik} \) to be \( \lfloor r_{i}/C \rfloor \). Since \( f' < 0 \),
\[ f \left( \left( \frac{1}{p} - 1 \right) \sum_{i=1}^{p} m_{ii} - \frac{m_{..}}{p(p-1)} \right) \] is minimized when the \( r_{ij}, 1 \leq j \leq P \), are either \( \lceil r_{i}/R \rceil \) or \( \lfloor r_{i}/R \rfloor + 1 \) and the \( s_{ik}, 1 \leq k \leq C \), are either \( \lceil r_{i}/C \rceil \) or \( \lfloor r_{i}/C \rfloor + 1 \) for all \( i \).

In the following two lemmas we are looking at two different cases. In Lemma 3.2 we are dealing with the case when \( (R - r_{0})/p \in \mathbb{N} \), while in Lemma 3.3 we then look at the case when \( (R - r_{0})/p \notin \mathbb{N} \) to see what the 'best' \( r_{i}, 1 \leq i \leq p \), will be.

**Lemma 3.2.** For fixed values of \( r_{0} \), the \( r_{ij} \), and the \( s_{ik} \). Suppose \( \Phi \), the \( r_{ij} \) and \( s_{ik} \) are as in Lemma 3.1, and \( (RC - r_{0})/p \notin \mathbb{N} \). Then

\[ f \left( \left( \frac{1}{p} - 1 \right) \sum_{i=1}^{p} m_{ii} - \frac{m_{..}}{p(p-1)} \right) \]

is minimized when the \( r_{1} \) are in the same elementary interval.

**Proof.** Suppose \( r_{1} > r_{2} \) and \( r_{1}, r_{2} \) are \( t \geq 1 \) elementary intervals apart, i.e. if \( r_{2} = aR + \alpha = bC + \beta, 0 \leq \alpha, \beta \leq R - 1, 0 \leq \beta \leq C - 1 \), then

\[ r_{1} = (a+\delta)R + \tau = (d+y)C + \delta, 1 \leq \tau \leq R - 1, 1 \leq \delta \leq C - 1, \] with \( x + y = t \).

If either \( \tau \) or \( \delta \) equal 0 then \( t \) elementary intervals apart means that \( x + y = t + 1 \). Let \( r_{1}^{*} = r_{1} - b, r_{2}^{*} = r_{2} + b \) where \( b \) is the smallest integer such that \( r_{1}^{*} \) and \( r_{2}^{*} \) are \( (t-1) \) elementary intervals apart. Then

\[
\begin{align*}
    m_{11}^{*} + m_{22}^{*} &- (m_{11} + m_{22}) \\
    &= m_{11}(r_{1}^{*}) + m_{22}(r_{2}^{*}) - (m_{11}(r_{1}) + m_{22}(r_{2}))
\end{align*}
\]
\[ e_{22}(r_2^2 b) - m_{22}(r_2) - (m_{11}(r_1) - m_{11}(r_1 - b)) \]
\[ = m_{22}(r_2^2 b) - m_{22}(r_2^2 b - 1) + m_{22}(r_2^2 b - 1) - m_{22}(r_2^2 b - 2) + \ldots \]
\[ - m_{22}(r_2^2 + 1) + m_{22}(r_2^2 + 1) - m_{22}(r_2) - (m_{11}(r_1) - m_{11}(r_1 - b)) \]
\[ + m_{11}(r_1 - 1) - m_{11}(r_1 - 2) + \ldots + m_{11}(r_1 - b + 1) - m_{11}(r_1 - b) \]
\[ = \Delta^0(r_2^2 b - 1) + \Delta^0(r_2^2 b - 2) + \ldots + \Delta^0(r_2) - (\Delta^0(r_1 - 1) + \Delta^0(r_1 - 2) + \ldots \]
\[ + \Delta^0(r_1 - b) \]
\[ = \left( \Delta^0(r_2) - \Delta^0(r_1 - b) \right) + \left( \Delta^0(r_2^2 + 1) - \Delta^0(r_1 - b + 1) \right) + \ldots \]
\[ + \left( \Delta^0(r_2^2 b - 1) - \Delta^0(r_1) \right) \quad (3.1) \]

From sec. 3.1 (v)
\[ \Delta^0(r_1) = m_{11}(r_1 + 1) - m_{11}(r_1) \]
\[ = (1/RC) \left( RC - R - C + 1 - 2R \left\{ r_1/R \right\} - 2C \left\{ r_1/C \right\} + 2r_1 \right) \]

Thus \[ \Delta^0(r_2 + a) - \Delta^0(r_1 - b + a) \quad \text{for a = 0, 1, 2, ..., b-1} \]
\[ = (2/RC) \left\{ r_2 + a - r_1 + b - a + R \left\{ (r_1/R) - (r_2/R) \right\} \right. \]
\[ + C \left( (r_1/C) - (r_2/C) \right) \]
\[ = (2/RC) \left\{ r_2 + b - r_1 + R \left\{ (r_1/R) - (r_2/R) \right\} \right. \]
\[ + C \left( (r_1/C) - (r_2/C) \right) \]
\[ = (2/RC) \left\{ r_2 + b - r_1 + t \cdot \min(R, C) \right\} \]
Substituting this into (3.1) yields

\[ m_{11}^* + m_{22}^* \leq m_{11} + m_{22}. \]

We can conclude that \( \max \Sigma_{i=1,p} m_{ii} \) over \( r_i \) occurs when the \( r_i \) are in the same elementary interval. Now since \( r_0, r_0^i, \) and the \( s_{0k} \) are all fixed, \( m.. \) is hence fixed. And from (2.1) one sees that \( f < 0 \), Lemma 3.2 is thus proved.

**Corollary 3.2.** For fixed values of \( r_0, \) the \( r_0^i \) and \( s_{0k} \). Suppose \( \Phi \), the \( r_{ij} \) and \( s_{lk} \) are as in Lemma 3.1. Then

\[ f \left( \left( \frac{1}{p-1} \right) \Sigma_{i=1,p} m_{ii} - m../p(p-1) \right) \]

is minimized when the \( r_i \) are all in \( [A_0, B_0] \), the basic interval.

**Proof.** This corollary follows immediately from Lemma 3.2, and the fact that \( \Sigma_{i=1,p} r_i = RC - r_0 \).

**Lemma 3.3.** For fixed values of \( r_0, \) the \( r_0^i \), and the \( s_{0k} \). Suppose \( \Phi \), the \( r_{ij} \), and the \( s_{lk} \) are as in Lemma 3.1, and \( (RC - r_0)/p \in \mathbb{N} \). Then

\[ f \left( \left( \frac{1}{p-1} \right) \Sigma_{i=1,p} m_{ii} - m../p(p-1) \right) \]

is minimized when \( r_i = (RC - r_0)/p \) for all \( i \).

**Proof.** Suppose \( (RC - r_0)/p = bC \), then
\[ \Sigma_{i=1}^{p} m_{i1} = \Sigma_{i=1}^{p} (r_i - (1/C) \Sigma_{j=1}^{R} r_{ij}^2 - (1/R) \Sigma_{k=1}^{C} s_{ik}^2 + r_i^2/RC ) = RC - r_0 - (1/C) \Sigma_{i=1}^{p} \Sigma_{j=1}^{R} r_{ij}^2 + (1/R) \Sigma_{i=1}^{p} (r_i^2/C - \Sigma_{k=1}^{C} s_{ik}^2) \]

Notice \( r_i^2/C - \Sigma_{k=1}^{C} s_{ik}^2 \) is nonpositive for all \( i \), and is zero when \( r_i/C \) is an integer. Also \( \Sigma_{i=1}^{p} \Sigma_{j=1}^{R} r_{ij}^2 \) remains constant when \( aR \leq r_i \leq (a+1)R \) for all \( i \), and is minimized when \( a \) satisfies \( aR \leq (RC - r_0)/p \leq (a+1)R \). The proof is as follow. Let \( r_1 = a_1R + b_1 \), \( r_2 = a_2R + b_2 \), where \( a_1 \geq a_2 \), and \( 0 \leq b_1, b_2, \leq R - 1 \). Suppose \( b_1 \leq b_2 \), let \( r_1^* = r_1 - b_1 \), \( r_2^* = r_2 + b_1 \). Then

\[ \Sigma_{j=1}^{R} r_{ij}^2 - ( \Sigma_{j=1}^{R} r_{ij}^* )^2 \]

\[ = (a_1 + 1)^2 b_1 + a_1^2 (R - b_1) + (a_2 + 1)^2 b_2 + a_2^2 (R - b_2) - (a_1^2 R + (a_2 + 1)^2 (b_1 + b_2) + a_2^2 (R - b_1 - b_2)) \]

\[ = b_1 ((a_1 + 1)^2 - a_1^2 + a_2^2 - (a_2 + 1)^2) \]

\[ = b_1 (2a_1 - 2a_2) \geq 0 \text{ since } a_1 \geq a_2, \text{ with equal sign if } a_1 = a_2. \]

A similar proof holds if \( b_1 \leq b_2 \). Consequently, \( r_1 = (RC - r_0)/p = bC \), \( 1 \leq sp \), maximizes \( \Sigma_{i=1}^{p} m_{i1} \), and thus maximizes \( ((1/p - 1) \Sigma_{i=1}^{p} m_{i1} \)
Lemma 3.4. Suppose $\Phi$ is as in Lemma 3.1. Suppose $d \in \mathcal{D}(p,R,C)$ has the $r_i$, the $r_{ij}$, and the $s_{ik}$ are as in Lemma 3.1 and Corollary 3.2 (or Lemma 3.3) and has $r_0 > RC/2$. Then there exists $d^* \in \mathcal{D}(p,R,C)$ having the $r_i^*$, the $r_{ij}^*$, and the $s_{ik}^*$ are as in Lemma 3.1 and Corollary 3.2 (or Lemma 3.3), with $r_0^* \leq RC/2$, and satisfying

$$f\left(\frac{(1/p) \sum_{i=1}^{p} m_{i} - m_{..}/p(p-1)}{m_{..}/p(p-1)}\right)$$

\[\leq f\left(\frac{(1/p) \sum_{i=1}^{p} m_{i} - m_{..}/p(p-1)}{m_{..}/p(p-1)}\right)\]

for $p \geq 4$, $R, C \geq 3$ and either $R \geq 4$ or $C \geq 4$.

Proof. Let $d^*$ to be the 'design' with $r_0^* = RC - r_0$, $r_{0j}^* = C - r_{0j}$, $1 \leq j \leq R$, $s_{0k}^* = R - s_{0k}$, $1 \leq k \leq C$, and $r_i^* = r_i$ if $r_i > (RC - r_0^*)/p$, otherwise $r_i \leq r_i^* \leq \lfloor (RC - r_0^*)/p \rfloor + 1 \leq B_0^*$, and the $r_{ij}^*$ are as equal as possible, and the $s_{ik}^*$ are as equal as possible. In here the 'design' we mean is really a specification of $r_0$, the $r_i$, the $r_{ij}$, and the $s_{ik}$, such that $\sum_{j=1}^{R} r_{ij} = \sum_{k=1}^{C} s_{ik} = r_i$, and $r_0 + \sum_{i=1}^{p} r_i = RC$. Such a specification does not necessarily mean we can arrange treatments in rows and columns with these values of $r_0$, the $r_i$, the $r_{ij}$, and the $s_{ik}$. Notice
\[ m_{..} = r_0 - (1/C) \Sigma_{j=1,R} r_{0j} x^2 - (1/R) \Sigma_{k=1,C} s_{0k} x^2 + r_0 x^2 / RC \]

\[ = RC - r_0 - (1/C) \Sigma_{j=1,R} (C - r_{0j})^2 - (1/R) \Sigma_{k=1,C} (R - s_{0k})^2 \]

\[ + (RC - r_0)^2 / RC \]

\[ = r_0 - (1/C) \Sigma_{j=1,R} r_{0j} x^2 - (1/R) \Sigma_{k=1,C} s_{0k} x^2 + r_0^2 / RC \]

\[ = m_{..} , \]

and

\[ \Sigma_{l=1,p} m_{II}^{**} = \Sigma_{l=1,p} (r_{l}^* - (1/C) \Sigma_{j=1,R} r_{lj} x^2 - (1/R) \Sigma_{k=1,C} s_{lk} x^2 \]

\[ + r_{l} x^2 / RC ) \]

\[ = \Sigma_{l=1,p} m_{II}(r_{l}^*) \]

Let us examine some properties of \( m(r) \) from Kiefer (1975).

(i) \( m(r) \) is a convex quadratic on each elementary interval \([A, B]\), since

\[ \Delta^0(r) = m(r+1) - m(r) \]

\[ = (1/RC)(RC - R - C + 1 + 2r - 2R[r/R] - 2C[r/C]) \]

is linear in \( r \) and increasing for \( A \leq r < B \).

(ii) \( m(r) \) is symmetric about \( RC/2 \).

(iii) \( m(r) \) is increasing in each elementary interval \([A, B]\) with

\[ B \leq B^0, \text{ and } p \geq 4, \text{ R, C } \geq 3 \text{ and either } R \geq 4 \text{ or } C \geq 4. \]
Proof. Assume \( A = aR, B = bC \) and \( aR \leq r < bC \), where \( a \) and \( b \) are nonnegative integers (a similar proof works for \( A \) a multiple of \( C \) and/or \( B \) a multiple of \( R \)). Then

\[
\Delta^0(r) = \frac{1}{RC} \left( RC - R - C + 1 + 2r - 2R \left\lfloor \frac{r}{R} \right\rfloor - 2C \left\lfloor \frac{r}{C} \right\rfloor \right)
\]

\[
= \frac{1}{RC} \left( RC - R - C + 1 + 2r - 2aR - 2(b-1)C \right)
\]

\[
\geq \left( \frac{1}{RC} \right) \left( RC - R + C + 1 - 2bC \right)
\]

Recall \([A^0, B^0]\) is the elementary interval containing \( RC/p \). Since \( B^0 \) is either an integer multiple of \( R \) or \( C \), we can assert that

\[
B^0 \leq C \left( \frac{R}{p} + \frac{(p-1)}{p} \right) = C \left( \frac{(R-1)}{p} + 1 \right).
\]

Thus

\[
bC = B \leq B^0 \leq C \left( \frac{(R-1)}{p} + 1 \right),
\]

and hence

\[
b \leq \frac{(R-1)}{p} + 1.
\]

Therefore

\[
\Delta^0(r) \geq \frac{1}{RC} \left( RC - R + C + 1 - 2 \left\lfloor \frac{(R-1)}{p} + 1 \right\rfloor \right)
\]

\[
= \left( \frac{1}{RC} \right) \left( 1 + \left( \frac{(p-2)}{p} \right) RC - R - C + 2C/p \right)
\]

\[
\geq \left( \frac{1}{RC} \right) \left( 1 + RC/2 - R - C \right),
\]

provided \( p \geq 4 \).
If \( r = bC \) notice

\[
\Delta^0(r) = \frac{1}{RC}(RC - R - C + 1 - 2aR)
\]

Since \( aR = A \leq A^0 \leq RC/p \) we see once again

\[
\Delta^0(r) \geq \frac{1}{RC}\left(\frac{p-2}{p}\right)RC - R - C + 1
\]

\[
\geq \frac{1}{RC}\left(1 + \frac{RC}{2} - R - C\right)
\]

If \( p \geq 4 \).

Since \( 1 + \frac{RC}{2} - R - C \geq 0 \) if \( R, C \geq 3 \) and either \( R \geq 4 \) or \( C \geq 4 \)
we see that (III) follows.

Now \( r_i^{**} = r_1 \) or \( r_1 < r_i^{**} \leq B^{**} \). Thus for \( p \geq 4, R, C \geq 3 \) and either \( R \geq 4 \)
or \( C \geq 4 \) (III) above implies \( m_{ll}(r_i^{**}) \geq m_{ll}(r_i) \). Therefore

\[
\Sigma_{i=1,p} m_{ll} = \Sigma_{i=1,p} m_{ll}(r_i) \leq \Sigma_{i=1,p} m_{ll}(r_i^{**}) = \Sigma_{i=1,p} m_{ll}^{**}
\]

Since \( f' < 0 \) it follows that

\[
f\left(\frac{1}{p-1}\right)\Sigma_{i=1,p} m_{ll}^{**} - \frac{m_{ll}^{**}}{p(p-1)}
\]

\[
\leq f\left(\frac{1}{p-1}\right)\Sigma_{i=1,p} m_{ll} - \frac{m_{ll}}{p(p-1)}
\]

Finally if \( d^{**} \) is not of the form given in Corollary 3.2 (or Lemma 3.3),
then there exists \( d^{**} \) with \( r_0^{***} = r_0^{**} \leq RC/2 \) in the form of Corollary
3.2 (or Lemma 3.3) (this follows from Corollary 3.2) which yields a
value of $f(\cdot)$ less than or equal to that produced by $d^*$. The lemma is now proved.

Lemma 3.5. For fixed value of $r_0 < RC/2$, and $p, RC \geq 4$. Suppose $\Phi$, the $r_i$, the $r_{ij}$, and the $s_{ik}$ are as in Lemma 3.1 and Corollary 3.2, and $(RC - r_0)/p \in \mathbb{N}$. Then

$$(p-1)f\left(\frac{1}{p-1} \sum_{i=1}^{p} m_{ij} - \frac{m_i}{p(p-1)}\right) + f\left(\frac{m_i}{p}\right)$$

is minimized when the $r_{ij}$ are as equal as possible and the $s_{ik}$ are as equal as possible, i.e. $r_{ij} = [r_0/R] \pm 1$, $s_{ik} = [r_0/C] \pm 1$ for $1 \leq j \leq R$ and $1 \leq k \leq C$.

Proof. Since $f$ is a real valued possibly infinite function on the set of all nonnegative numbers which is continuous on the set of all positive numbers, has $f' < 0$ and $f'' > 0$, then $f$ has the property that if $\mu_1 \leq \mu_2 = \mu_3 = \cdots = \mu_p$, $\nu_1 \leq \nu_2 = \nu_3 = \cdots = \nu_p$, $\mu_1 \geq \nu_1$ and if $\Sigma_{i=1}^{p} \mu_i = \Sigma_{i=1}^{p} \nu_i$, then $\Sigma_{i=1}^{p} f(\mu_i) \leq \Sigma_{i=1}^{p} f(\nu_i)$. This is a result of the Schur-convexity of $\Sigma_{i=1}^{p} f(x_i)$.

Let us examine $(p-1)f\left(\frac{1}{p-1} \sum_{i=1}^{p} m_{ij} - \frac{m_i}{p(p-1)}\right) + f\left(\frac{m_i}{p}\right)$. Since for designs of the form in Lemma 3.1 and Corollary 3.2,
\[(p-1) \left( \frac{1}{p-1} \Sigma_{l=1,p} m_{ll} - m../p(p-1) \right) + m../p \]

\[= \Sigma_{l=1,p} m_{ll} - m../p + m../p \]

\[= \Sigma_{l=1,p} m_{ll} \]

\[= \Sigma_{l=1,p} \left( r_1 - (1/C) \Sigma_{j=1,R} \frac{r_j^2}{R} - (1/R) \Sigma_{k=1,C} S_{lk}^2 + \frac{r_i^2}{RC} \right) \]

\[= \Sigma_{l=1,p} \left( r_1 - (1/C) \left( r_1 + (2r_1-R) [r_1/R] - R \left[ r_1/R \right]^2 \right) \right. \]

\[\left. - (1/R) \left( r_1 + (2r_1-C) [r_1/C] - C \left[ r_1/C \right]^2 \right) + \frac{r_i^2}{RC} \right) \]

\[= (1 - 1/C - 1/R) (RC-r_0) \]

\[- (1/C) \Sigma_{l=1,p} \left( 2r_1-R \right) [r_1/R] - R \left[ r_1/R \right]^2 \]

\[- (1/R) \Sigma_{l=1,p} \left( 2r_1-C \right) [r_1/C] - C \left[ r_1/C \right]^2 \]

\[+ (1/RC) \Sigma_{l=1,p} r_i^2 \]

= constant for fixed \( r_0 \) and fixed values of the \( r_i \).

If one can show that \((1/p-1) \Sigma_{l=1,p} m_{ll} - m../p(p-1) \geq m../p\) then by the aforementioned property of \( f \), to find conditions under which

\[(p-1) f ((1/p-1) \Sigma_{l=1,p} m_{ll} - m../p(p-1)) + f (m../p) \]

is a minimum is the same as finding the conditions under which \( m../p \) is a maximum. Now

\[
(1/p-1) \Sigma_{l=1,p} m_{ll} - m../p(p-1) - m../p
\]

\[
= (1/p-1) \Sigma_{l=1,p} m_{ll} - p m../p(p-1)
\]
\[ (1/p-1) (\Sigma_{i=1,p} m_{ii} - m. ) \]

\[ = \frac{1}{p-1} \left[ \Sigma_{j=1,R} \frac{r_j^2}{RC} - \frac{1}{C} \Sigma_{j=1,R} \frac{r_j^2}{RC} - \frac{1}{R} \Sigma_{k=1,C} \frac{s_{ik}^2}{RC} \right] \]

\[ = \frac{1}{p-1} \left[ \Sigma_{i=1,p} \left( r_i - \frac{1}{C} \Sigma_{j=1,R} \frac{r_j^2}{RC} - \frac{1}{R} \Sigma_{k=1,C} \frac{s_{ik}^2}{RC} \right) \right] \]

\[ = \frac{1}{p-1} \left[ \Sigma_{i=1,p} \left( r_i - \frac{1}{C} \Sigma_{j=1,R} \frac{r_j^2}{RC} - \frac{1}{R} \Sigma_{k=1,C} \frac{s_{ik}^2}{RC} \right) \right] \]

\[ = \frac{1}{p-1} \left[ \Sigma_{i=1,p} \left( r_i - \frac{1}{C} \Sigma_{j=1,R} \frac{r_j^2}{RC} - \frac{1}{R} \Sigma_{k=1,C} \frac{s_{ik}^2}{RC} \right) \right] \]

\[ = \frac{1}{p-1} \left[ \Sigma_{i=1,p} \left( r_i - \frac{1}{C} \Sigma_{j=1,R} \frac{r_j^2}{RC} - \frac{1}{R} \Sigma_{k=1,C} \frac{s_{ik}^2}{RC} \right) \right] \]

\[ = \frac{1}{p-1} \left[ \Sigma_{i=1,p} \left( r_i - \frac{1}{C} \Sigma_{j=1,R} \frac{r_j^2}{RC} - \frac{1}{R} \Sigma_{k=1,C} \frac{s_{ik}^2}{RC} \right) \right] \]

\[ = \frac{1}{p-1} \left[ \Sigma_{i=1,p} \left( r_i - \frac{1}{C} \Sigma_{j=1,R} \frac{r_j^2}{RC} - \frac{1}{R} \Sigma_{k=1,C} \frac{s_{ik}^2}{RC} \right) \right] \]

\[ = \frac{1}{p-1} \left[ \Sigma_{i=1,p} \left( r_i - \frac{1}{C} \Sigma_{j=1,R} \frac{r_j^2}{RC} - \frac{1}{R} \Sigma_{k=1,C} \frac{s_{ik}^2}{RC} \right) \right] \]

\[ = \frac{1}{p-1} \left[ \Sigma_{i=1,p} \left( r_i - \frac{1}{C} \Sigma_{j=1,R} \frac{r_j^2}{RC} - \frac{1}{R} \Sigma_{k=1,C} \frac{s_{ik}^2}{RC} \right) \right] \]

\[ = \frac{1}{p-1} \left[ \Sigma_{i=1,p} \left( r_i - \frac{1}{C} \Sigma_{j=1,R} \frac{r_j^2}{RC} - \frac{1}{R} \Sigma_{k=1,C} \frac{s_{ik}^2}{RC} \right) \right] \]

\[ = \frac{1}{p-1} \left[ \Sigma_{i=1,p} \left( r_i - \frac{1}{C} \Sigma_{j=1,R} \frac{r_j^2}{RC} - \frac{1}{R} \Sigma_{k=1,C} \frac{s_{ik}^2}{RC} \right) \right] \]

\[ = \frac{1}{p-1} \left[ \Sigma_{i=1,p} \left( r_i - \frac{1}{C} \Sigma_{j=1,R} \frac{r_j^2}{RC} - \frac{1}{R} \Sigma_{k=1,C} \frac{s_{ik}^2}{RC} \right) \right] \]

\[ = \frac{1}{p-1} \left[ \Sigma_{i=1,p} \left( r_i - \frac{1}{C} \Sigma_{j=1,R} \frac{r_j^2}{RC} - \frac{1}{R} \Sigma_{k=1,C} \frac{s_{ik}^2}{RC} \right) \right] \]

By Corollary 3.2 that \( A_0 \leq r_1, r_2, \ldots, r_p \leq B_0 \), and it is easy to show that \( B_0 \leq B^0 \) where \( [A_0, B_0] \), by definition, is the basic interval and \( [A^0, B^0] \) is the elementary interval that covers \( RC/p \). Also from the
property (iii) of \( m(r) \) that \( m(r) \) is increasing in each elementary interval \([A, B]\) for \( B \leq B^0\), one can obtain the following

\[
\Sigma_{i=1,p} m_{ij} = \Sigma_{i=1,p} m_{ij}(r_i) \geq \Sigma_{i=1,p} m_{ij}(r_i^*) = \Sigma_{i=1,p} m_{ij}(r_i^*)
\]

where \( r_i^* = A_0 \) for all \( i \).

Suppose \( A_0 = a R, a \) integer, then \( r_{ij}^* = a \) for all \( 1 \leq i \leq p, 1 \leq j \leq R \) and substituting the \( r_{ij}^* \) and the \( s_{ik}^* \) into (3.2) yields

\[
(1/p-1) (\Sigma_{i=1,p} m_{ij} - m_.)
\]

\[
\geq (1/p-1) \left\{ \left(1/C\right) \Sigma_{j=1,R} \left( \Sigma_{i=1,p} r_{ij}^* \right)^2 - (1/C) \Sigma_{i=1,p} \Sigma_{j=1,R} r_{ij}^* \right\}
\]

\[
+ \frac{1}{R} \left( \Sigma_{k=1,C} \left( \Sigma_{i=1,p} s_{ik}^* \right)^2 - (1/R) \Sigma_{i=1,p} \Sigma_{k=1,C} s_{ik}^* \right)
\]

\[
+ \left(1/RC\right) \Sigma_{i=1,p} r_{ij}^* - (1/RC) \left( \Sigma_{i=1,p} r_{ij}^* \right)^2 \right\}
\]

\[
= (1/p-1) \left\{ \left(1/C\right) \left( R p^2 a^2 - R p a^2 \right) \right\}
\]

\[
+ \left(1/R\right) \left( \Sigma_{k=1,C} \left( \Sigma_{i=1,p} s_{ik}^* \right)^2 - (1/R) \Sigma_{i=1,p} \Sigma_{k=1,C} s_{ik}^* \right)
\]

\[
+ \left(1/RC\right) \left( p a^2 R^2 - p^2 a^2 R^2 \right) \}
\]

\[
= (1/p-1) \left\{ \left(1/R\right) \Sigma_{k=1,C} \left( \Sigma_{i=1,p} s_{ik}^* \right)^2 - (1/R) \Sigma_{i=1,p} \Sigma_{k=1,C} s_{ik}^* \right\}
\]

\[
\geq 0
\]

(3.3)

And do the same thing for \( \theta_0 = b C, b \) integer, then \( s_{ik}^* = b \) for all \( 1 \leq i \leq p, 1 \leq k \leq C \) and substituting the \( r_{ij}^* \) and the \( s_{ik}^* \) into (3.2) yields

\[
(1/p-1) (\Sigma_{i=1,p} m_{ij} - m_.)
\]
\[ 2 \left( \frac{1}{p-1} \right) \left[ \left( \frac{1}{C} \right) \Sigma_{j=1,R} \left( \Sigma_{i=1,p} r_{ij}^* \right)^2 - \left( \frac{1}{C} \right) \Sigma_{i=1,p} \Sigma_{j=1,R} s_{ik}^* \right]^2 \\
+ \left( \frac{1}{RC} \right) \left( \frac{1}{C} \right) \Sigma_{j=1,R} \left( \Sigma_{i=1,p} r_{ij}^* \right)^2 - \left( \frac{1}{RC} \right) \left( \Sigma_{i=1,p} \Sigma_{j=1,R} r_{ij}^* \right)^2 \\
= \left( \frac{1}{p-1} \right) \left[ \left( \frac{1}{C} \right) \Sigma_{j=1,R} \left( \Sigma_{i=1,p} r_{ij}^* \right)^2 - \left( \frac{1}{C} \right) \Sigma_{i=1,p} \Sigma_{j=1,R} r_{ij}^* \right]^2 \\
\geq 0 \]  
\( (3.4) \)

From (3.3) and (3.4) one can see that \( \frac{m_*}{p} \) is the smallest one among \( \left( \frac{1}{p-1} \right) \left( \Sigma_{i=1,p} m_{ii} - m_*/p(p-1) \right) \), . . . . .

\( \left( \frac{1}{p-1} \right) \left( \Sigma_{i=1,p} m_{ii} - m_*/p(p-1), m_*/p \right) \). So the next step will be finding conditions under which \( m_* = \frac{r_0}{p} - \left( \frac{1}{C} \right) \Sigma_{j=1,R} r_{0j}^2 - \left( \frac{1}{RC} \right) \Sigma_{k=1,C} s_{0k}^2 \)

+ \( \frac{r_0^2}{RC} \) is a maximum. For fixed \( r_0, m_* \) is maximized when

\( \Sigma_{j=1,R} r_{0j}^2 \) and \( \Sigma_{k=1,C} s_{0k}^2 \) are minimized, i.e. when the \( r_{0j}, 1 \leq j \leq R \), are either \( \left[ r_0/R \right] \) or \( \left[ r_0/R \right] + 1 \) and the \( s_{0k}, 1 \leq k \leq C \), are either \( \left[ r_0/C \right] \) or \( \left[ r_0/C \right] + 1 \). The lemma follows.

**Corollary 3.5.** For fixed value of \( R_0 \leq RC/2 \) and \( p, R, C \geq 4 \). Suppose \( \Phi \), the \( r_{ij} \) and the \( s_{ik} \) are as in Lemma 3.1, and \( r_i = (RC - r_0)/p \in N \) for all \( i \). Then

\[ (p-1) \left( \frac{1}{p-1} \right) \left( \Sigma_{i=1,p} m_{ii} - m_*/p(p-1) \right) + \left( \frac{m_*/p}{p} \right) \]

is minimized when the \( r_{0j}, 1 \leq j \leq R \), are either \( \left[ r_0/R \right] \) or \( \left[ r_0/R \right] + 1 \) and the \( s_{0k}, 1 \leq k \leq C \), are either \( \left[ r_0/C \right] \) or \( \left[ r_0/C \right] + 1 \).
Proof. Following the same proof as for Lemma 3.5, but with
\[ \Sigma_{i=1}^{\mu} m_{il} = \Sigma_{i=1}^{\mu} m_{il}^* \].

Lemma 3.6. For fixed value of \( r_0 \). Suppose the \( r_{ij} \), the \( s_{ik} \), the \( r_{0j} \),
and the \( s_{0k} \) are as in Lemma 3.1 and 3.5. Then \( g(r_i) \) is increasing in \( r_i \)
provided that \( r_i \leq B_0 \) and \( p \geq 6, R, C \geq 4, \) or \( p = 5, R, C \geq 5, \) or \( p = 4, R, C \)
\( \geq 6 \).

Proof. \( \Delta(r_i) = (p/RC) \left[ p \left( RC - R - C + 1 - 2R \left[ r_i/R \right] - 2C \left[ r_i/C \right] + 2r_i \right) \right. \)
\[ -2 \left[ R r_{0m} + C s_{on} - r_0 \right] \] \]
Let \( r_0 = aR + b = eC + d \) where \( 0 \leq b \leq R - 1, 0 \leq d \leq C - 1. \)
\[ R r_{0m} + C s_{on} - r_0 \leq aR + R + eC + C - r_0 \]
\[ = aR + b + eC + d + R - b + C - d - r_0 \]
\[ = r_0 + R + C - b - d \quad (3.5) \]
Substituting this into \( \Delta(r_i) \),
\[ \Delta(r_i) \geq (p/RC) \left[ p \left( RC - R - C + 1 - 2R \left[ r_i/R \right] - 2C \left[ r_i/C \right] + 2r_i \right) \right. \]
\[ -2 \left[ R r_{0m} + R + C - b - d \right] \]
\[ \geq (p/RC) \left[ p \left( RC - R - C + 1 - 2R \left[ r_i/R \right] - 2C \left[ r_i/C \right] + 2r_i \right) \right. \]
\[ -2 \left( R r_{0m} + R + C \right) \]
= \Delta^x(r_i)$ say, where $\Delta^x(r_i)$ is increasing in $r_i$ in each elementary interval.

> \geq \left(\frac{p}{RC}\right) \left[ p \left( RC - R - C + 1 - 2r_i - 2r_i + 2r_0 \right) - 2 \left( r_0 + R + C \right) \right]

= \left(\frac{p}{RC}\right) \left[ p \left( RC - R - C + 1 - 2r_i \right) - 2 \left( r_0 + R + C \right) \right]

> \geq 0

If $p \left( RC - R - C + 1 - 2r_i \right) - 2 \left( r_0 + R + C \right) \geq 0$

or if $r_i \leq \left(\frac{1}{2p}\right) \left[ p \left( RC - R - C + 1 \right) - 2 \left( r_0 + R + C \right) \right]

= \left(\frac{1}{2p}\right) \left( pRC - (p+2) (R+C) + p - 2r_0 \right)

= \left(\frac{1}{p}\right) \left( pRC/2 - (p+2) (R+C)/2 + p/2 - r_0 \right)

Now, if we can show that

\[ \frac{1}{p} \left( pRC/2 - (p+2) (R+C)/2 + p/2 - r_0 \right) \geq (RC - r_0)/p, \]

then $\Delta^x(r_i) \geq 0$ when $r_i \leq (RC - r_0)/p$, and since $\Delta^x(r_i)$ is increasing in $r_i$ in each elementary interval, one can show that $\Delta^x(r_i) \geq 0$ when $r_i \leq B_0 - 1$. Also, since $\Delta(r_i) \geq \Delta^x(r_i)$, the lemma thus follows. Now

\[ \frac{1}{p} \left( pRC/2 - (p+2) (R+C)/2 + p/2 - r_0 \right) - (RC - r_0)/p \]

= \left(\frac{1}{2p}\right) \left( (p-2) RC - (p+2) (R+C) + p \right).

Thus the lemma is true provided

\[ (p-2) RC - (p+2) (R+C) + p \geq 0. \]
Some cases in which this is true (proof is straightforward) are:

\[ p = 4, \ R, C \geq 6; \ p = 5, \ R, C \geq 5; \ p \geq 6, \ R, C \geq 4. \]

**Lemma 3.7.** For fixed value of \( r_0 \leq RC/2 \), suppose \( \Phi \), the \( r_i \), the \( r_{ij} \), the \( s_{ik} \), the \( r_{ij} \), and the \( s_{ok} \) are as in Lemma 3.1, 3.5, and Corollary 3.2. Then \( f(A_i) \), where \( f(x) = 1/x \), and \( A_i = (p/p-1) m_{ij} - (2/p-1) m_i + m_{ij}/p(p-1) \) (by definition), is convex in \( r_i \) in \([A_0, B_0]\) for

(i) \( p, R, C \geq 6; \)

(ii) \( p = 5, \ R, C \geq 10; \)

(iii) \( p = 4, \ R, C \geq 30. \)

**Proof.** Instead of using \( A_i \) we can use \( h(r_i) = RC g(r_i) = RC p(p-1) A_i \) to prove this lemma for the sake of convenience. In order to show the convexity of \( 1/h(r_i) \) in the basic interval, we have to show that

\[ (1/2)(1/h(r_i + 1) - 2/h(r_i) + 1/h(r_i - 1)) \geq 0 \quad (3.6) \]

for \( A_0 \leq r_i \leq B_0 \). Notice \( g(0) = m_{00}(0) \geq m_{00}(0) = 0 \) by the properties of \( m_{ij}(r_i) \) in the proof of Lemma 3.4. Also by Lemma 3.6, \( g(r_i) \geq g(0) \geq 0 \), hence \( h(r_i) \geq 0 \). So, showing (3.6) is true is equivalent to showing that

\[ \Gamma(r_i) = (1/2) h(r_i) h(r_i + 1) h(r_i - 1) (1/h(r_i + 1) - 2/h(r_i) + 1/h(r_i - 1)) \geq 0 \]

By definition:

\[ \Delta(r_i) = g(r_i + 1) - g(r_i) \]
where the increment from \( r_1 \) to \( r_1^1 \) takes place at the \( m^{th} \) row and the \( n^{th} \) column. Let us assume from the time being that the same \( r_{1m} \) and \( s_{in} \) are continuously increased, even though this makes the \( r_{ij} \) and the \( s_{ik} \) no longer as equal as possible. (Remark: After showing the convexity of \( f(A_1) \) and the \( r_1 \) are as equal as possible, we can apply Lemma 3.1 to get the desired result.)

Notice

\[
\Delta(r_1^1) - \Delta(r_1) = \frac{2p}{RC},
\]

and

\[
\Gamma(r_1^1) - \Gamma(r_1) = (1/2) \left[ h(r_1) h(r_1^1) - 2 h(r_1) h(r_1^2) + h(r_1^1) h(r_1^2) - h(r_1^2) h(r_1^1) \right]
\]

\[
= (1/2) \left[ h(r_1^2) (h(r_1) - 2 h(r_1) + h(r_1^1)) - h(r_1^1) (h(r_1^2) - 2 h(r_1^1) + h(r_1)) \right]
\]

\[
= (1/2) \left[ h(r_1^2) RC (\Delta(r_1) - \Delta(r_1^1)) - h(r_1^1) RC (\Delta(r_1) + \Delta(r_1^1)) \right]
\]

\[
= (1/2) RC \left( \frac{2p}{RC} \right) \left( h(r_1^2) - h(r_1^1) \right)
\]
since \( h(r_j) \) or \( RC g(r_1) \) is increasing in \([A_0, B_0]\). Hence \( \Gamma(r_j) \) is increasing in \( r_1 \) in \([A_0, B_0]\). So, if one can show that \( \Gamma(A_0+1) \geq 0 \) then \( \Gamma(r_j) \geq 0 \) for \( A_0 < r_1 \leq B_0 \) and therefore the lemma follows.

Now \( h(r_j) = RC g(r_j) \)

\[
= p^2 RC (r_1 - (1/C) \sum_{j=1,R} r_{ij}^2 - (1/R) \sum_{k=1,C} s_{ik}^2 + r_1^2/RC)
- 2p RC \left( (1/C) \sum_{j=1,R} r_{ij} r_{oj} + (1/R) \sum_{k=1,C} s_{ik} s_{ok} - r_j r_0/RC \right)
+ RC \left( r_0 - (1/C) \sum_{j=1,R} r_{oj}^2 - (1/R) \sum_{k=1,C} s_{ok}^2 + r_0^2/RC \right).
\]

By Lemma 3.1,

\[
\sum_{j=1,R} r_{ij}^2 = r_1 + (2r_1 - R) [r_1/R] - R [r_1/R]^2
\]

\[
\sum_{k=1,C} s_{ik}^2 = r_1 + (2r_1 - C) [r_1/C] - C [r_1/C]^2
\]

Substituting this into \( h(r_j) \) yields

\[
h(r_j) = p^2 RC \left( r_1 - (1/C) \left( r_1 + (2r_1 - R) [r_1/R] - R [r_1/R]^2 \right) \right)
- (1/R) \left( r_1 + (2r_1 - C) [r_1/C] - C [r_1/C]^2 \right) + r_1^2/RC
- 2p RC \left( (1/C) [r_1/R] r_0 + (1/C) \sum_{j=1,R} r_{oj} + (1/R) [r_1/C] r_0 \right)
- (1/R) \sum_{k=1,C} s_{ok} - r_j r_0/RC
+ RC \left( r_0 - (1/C) \sum_{j=1,R} r_{oj}^2 - (1/R) \sum_{k=1,C} s_{ok}^2 + r_0^2/RC \right)
\]
where $\Sigma^* r_{0j} =$ sum of those $r_{0j}$'s whose corresponding \
$\Sigma^* s_{0k} =$ sum of those $s_{0k}$'s whose corresponding \
$s_{1k} = [r_i/C] + 1.$

$$h(T|) = \frac{p^2 r_1^2}{2} + (\frac{p^2}{2} (RC - R - C - 2R[r_i/R] - 2C[r_i/C] + 2r_0) r_1$$

$$+ \frac{p^2}{2} RC ((1/C)(r_i[R] + r_i/R^2)$$

$$+ (1/R)(C[r_i/C] + C[r_i/C]^2))$$

$$- 2pRC ((1/C)((r_i/R)r_0 + \Sigma^* r_{0j})$$

$$+ (1/R)((r_i/C)r_0 + \Sigma^* s_{0k}))$$

$$+ RC (r_0 - (1/C) \Sigma_{j=1,R} r_{0j}^2 - (1/R) \Sigma_{k=1,C} s_{0k}^2 + r_0^2/RC).$$

Hence $h(r_1)$ can be written as

$$h(r_1) = \frac{p^2 r_1^2}{2} + \beta r_1 + \alpha,$$

where $\beta = \frac{p^2}{2} (RC - R - C - 2R[r_i/R] - 2C[r_i/C] + 2r_0$

$$\alpha = \frac{p^2}{2} RC ((1/C)(r_i[R] + r_i/R^2)$$

$$+ (1/R)(C[r_i/C] + C[r_i/C]^2))$$

$$- 2pRC ((1/C)((r_i/R)r_0 + \Sigma^* r_{0j})$$

$$+ (1/R)((r_i/C)r_0 + \Sigma^* s_{0k}))$$

$$+ RC (r_0 - (1/C) \Sigma_{j=1,R} r_{0j}^2 - (1/R) \Sigma_{k=1,C} s_{0k}^2 + r_0^2/RC).$$
By the definition of $\Delta(r_1)$, $\Gamma(r_1)$ can be rewritten as

$$\Gamma(r_1) = (1/2) h(r_1) RC (\Delta(r_1 - 1) - \Delta(r_1)) + (RC)^2 (\Delta(r_1 - 1) \Delta(r_1))$$

where $RC \Delta(r_1 - 1) = RC (g(r_1) - g(r_1 - 1))$

$$= h(r_1) - h(r_1 - 1)$$

$$= p^2 r_1^2 + \beta r_1 + \alpha - \left( p^2 (r_1 - 1)^2 + \beta (r_1 - 1) + \alpha + 2p \delta \right)$$

$$= 2p^2 r_1 - p^2 + \beta - 2p \delta$$

where $\delta = R r_m + C s_n$.

and $RC \Delta(r_1) = RC (g(r_1 + 1) - g(r_1))$

$$= h(r_1 + 1) - h(r_1)$$

$$= p^2 (r_1 + 1)^2 + \beta (r_1 + 1) + \alpha - 2p \delta - \left( p^2 r_1^2 + \beta r_1 + \alpha \right)$$

$$= 2p^2 r_1 + p^2 + \beta - 2p \delta.$$

Also

$$RC (\Delta(r_1 - 1) - \Delta(r_1))$$

$$= 2p^2 r_1 + \beta - p^2 - 2p^2 r_1 - \beta - p^2$$

$$= -2p^2$$

$$R^2C^2 (\Delta(r_1 - 1) \Delta(r_1))$$

$$= (2p^2 r_1 + \beta - 2p \delta - p^2) (2p^2 r_1 + \beta - 2p \delta + p^2)$$
\[ 4 p^4 r_1^2 + \beta^2 + 4 p^2 \delta^2 + 4 p^2 \beta r_1 - 8 p^3 \delta r_1 - 4 p \beta \delta - p^4 \]

Substituting this into \( \Gamma(r_1) \), one has

\[
\Gamma(r_1) = \left( \frac{1}{2} \right) \left( p^2 r_1^2 + \beta r_1 + \alpha \right) (-2 p^2) \\
+ 4 p^4 r_1^2 + \beta^2 + 4 p^2 \delta^2 + 4 p^2 \beta r_1 - 8 p^3 \delta r_1 - 4 p \beta \delta - p^4 \\
- p^4 r_1^2 - p^2 \beta r_1 - p^2 \alpha + 4 p^4 r_1^2 + \beta^2 + 4 p^2 \beta r_1 - p^4 + 4 p^2 \delta^2 \\
- 8 p^3 \delta r_1 - 4 p \beta \delta \\
- 9 p^4 r_1^2 + 3 p^2 \beta r_1 - p^2 \alpha + \beta^2 - p^4 + 4 p^2 \delta^2 - 8 p^3 \delta r_1 \\
- 4 p \beta \delta \]

(3.7)

Now, suppose \( A_0 = B_1 R \geq B_2 C = [A_0/C]C \). A similar argument works if \( A_0 = B_1 C \). Substituting \( r_1 = A_0 + 1 \) into \( h(r_1) \) and \( \alpha, \beta \) yield

\[
h(A_0+1) = p^2 \left( RC (A_0 + 1) - R (A_0 + 1 - RB_1^2 + (2A_0 + 2 - R) B_1) \\
- C (A_0 + 1 - CB_2^2 + (2A_0 + 2 - C) B_2) + (A_0 + 1)^2 \right) \\
- 2p (RB_1 R_0 + CB_2 R_0 + R r_{0q} + C \Sigma^k s_{ok} - (A_0 + 1) R_0) \\
+ A \\
\]

where \( A = RC (r_0 - (1/C) \Sigma_{j=1,R} r_{0j}^2 - (1/R) \Sigma_{k=1,C} s_{ok}^2 + r_{0j}^2/RC) \)

\( r_{0q} \) whose corresponding \( r_{1q} = [(A_0 + 1)/R] + 1 = B_1 + 1 \)

\( \Sigma^k s_{ok} = \) sum of those \( s_{ok}'s \) whose corresponding \( s_{lk} = [r_1/C] + 1 = [(A_0 + 1)/C] + 1 = B_2 + 1 \).
Thus \( \beta = p^2 (RC - R - C - 2RB_1 - 2CB_2) + 2pr_0 \)

\( \alpha = p^2 (R^2 B_1 (B_1 + 1) + C^2 B_2 (B_2 + 1)) - 2p (RB_1 r_0 + CB_2 r_0) \)

\[ + \hat{A}^* \]

where \( \hat{A}^* = A - 2pR r_{01} - 2pC \hat{S} s_{01} \).

Substituting \( \alpha \) and \( \beta \) into (3.7) yields

\[ \Gamma(A_0 + 1) = 3p^4 (A_0 + 1)^2 + 3p^2 (A_0 + 1) \left[ p^2 (\pi - \sigma - 2A_0 - 2CB_2) \right. \]

\[ + 2p r_0 \right] - p^2 \left[ p^2 (A_0^2 + RA_0 + C^2 B_2^2 + C^2 B_2) \right. \]

\[ - 2p (A_0 r_0 + CB_2 r_0) + \hat{A}^* \]

\[ + \left. \left\{ p^2 (\pi - \sigma - 2A_0 - 2CB_2) + 2p r_0 \right\}^2 - p^4 + 4p^2 \delta^2 \right. \]

\[ - 8p^3 \delta (A_0 + 1) - 4p \delta \left( p^2 (\pi - \sigma - 2A_0 - 2CB_2) \right. \]

\[ + 2p r_0 \right) \]

where \( \pi = RC, \sigma = R + C \).

\[ \Gamma(A_0 + 1) = p^4 \left[ 3 (A_0^2 + 2A_0 + 1) + (2/p) (A_0 r_0 + CB_2 r_0) \right] - \hat{A}^*/p^2 \]

\[ + 3 (A_0 + 1) (\pi - \sigma - 2A_0 - 2CB_2 + 2r_0/p) - (A_0^2 + RA_0 \]

\[ + C^2 B_2^2 + C^2 B_2) + (\pi - \sigma - 2A_0 - 2CB_2)^2 \]

\[ + (4r_0/p) (\pi - \sigma - 2A_0 - 2CB_2) + 4r_0^2/p^2 - 1 \]

\[ + 4 \delta^2/p^2 - 8 \delta (A_0 + 1)/p - (4 \delta/p) (\pi - \sigma - 2A_0 \]

\[ - 2 CB_2) - 8 \delta r_0/p^2 \]
\[
\begin{align*}
&= p^4 \left( 3A_0^2 + 6A_0 + 3 + 3A_0 (\pi - \sigma - 2A_0 - 2CB_2 + 2\gamma_0/p) \right) \\
&\quad + 3 (\pi - \sigma - 2A_0 - 2CB_2 + 2\gamma_0/p) - (A_0^2 + RA_0 + C^2 B_2^2) \\
&\quad + C^2 B_2) + (2/p) (A_0 \gamma_0 + CB_2 \gamma_0) - A^2/p^2 + \pi^2 + \sigma^2 \\
&\quad + 4 A_0^2 + 4 C^2 B_2^2 - 2\pi \sigma - 4\pi A_0 - 4 CB_2 \pi + 4 \sigma A_0 \\
&\quad + 4 CB_2 \sigma + 8 CB_2 A_0 + (4 \gamma_0/p) (\pi - \sigma - 2A_0 - 2CB_2) \\
&\quad + 4\gamma_0^2/p^2 - 1 + 4 \delta^2/p^2 - 8 \delta A_0/p - 8 \delta/p \\
&\quad - (4 \delta/p)(\pi - \sigma) + 8 \delta A_0/p + 8 \delta CB_2/p - 8 \delta\gamma_0/p^2 \\
&= p^4 \left( A_0^2 (3 - 6 - 1 + 4) + A_0 (6 + 3(\pi - \sigma) - 6CB_2 \\
&\quad + 6\gamma_0/p - 6 - R + 2\gamma_0/p - 4\pi + 4 \sigma + 8CB_2 - 6\gamma_0/p + 3 \\
&\quad + 3(\pi - \sigma) - 6CB_2 + 6 \gamma_0/p - C^2 B_2^2 - C^2 B_2 \\
&\quad + (2CB_2 \gamma_0/p) - A^2/p^2 + (\pi - \sigma)^2 + 4 C^2 B_2^2 - 4CB_2 \pi \\
&\quad + 4CB_2 \sigma + (4 \gamma_0/p)(\pi - \sigma - 2CB_2) + 4\gamma_0^2/p^2 - 1 \\
&\quad + 4 \delta^2/p^2 - (4 \delta/p)(2 + \pi - \sigma) + 8 \delta CB_2/p - 8 \delta\gamma_0/p^2 \\
&= p^4 (A_0 (2CB_2 - (\pi - \sigma) - R) - CB_2 (6 - 3CB_2 + C \\
&\quad + 6\gamma_0/p + 4(\pi - \sigma) - 8\delta/p + 3 + 3(\pi - \sigma) \\
&\quad + (\pi - \sigma)^2 + 6 \gamma_0/p + (4 \gamma_0/p)(\pi - \sigma) + 4\gamma_0^2/p^2 \\
&\quad - 1 - A^2/p^2 + 4 \delta^2/p^2 - (4 \delta/p)(2 + \pi - \sigma) \\
\end{align*}
\]
\[-8 \delta r_0/p^2\] 

\[2 \ p^4 \left[ A_0 \left( 2CB_2 - (\pi - \sigma) - R \right) - CB_2 \left( 6 - 3CB_2 + C \right.\right.\]
\[+ 6 r_0/p + 4(\pi - \sigma) - 8 \delta/p) + 3 + 3(\pi - \sigma)\]
\[+ (\pi - \sigma)^2 + 6 r_0/p + (4 r_0/p)(\pi - \sigma)\]
\[+ 4 r_0^2/p^2 - 1 - (1/p^2)(\pi r_0 - r_0^2) + 48^2/p^2\]
\[-(4 \delta/p)(2 + \pi - \sigma) - 8 \delta r_0/p^2\]

since \[A^* \leq A \leq RC \left( r_0 - r_0^2/RC \right) = \pi r_0 - r_0^2\]

\[= \ p^4 H(A_0, CB_2, r_0)\]

Next, let us examine the relationship between \[H(A_0, CB_2, r_0)\] and \[CB_2\].

\[H(A_0, CB_2+1, r_0) - H(A_0, CB_2, r_0)\]

\[= 2 \left( CB_2 + 1 \right) A_0 - \left( CB_2 + 1 \right)(6 - 3(CB_2 + 1) + C + 6 r_0/p\]
\[+ 4(\pi - \sigma) - 8 \delta/p) - \left\{ 2CB_2 A_0 - CB_2 (6 - 3 CB_2 + C + 6 r_0/p\]
\[+ 4(\pi - \sigma) - 8 \delta/p \right\}\]
\[= 2 A_0 + 3 CB_2 - 6 + 3 CB_2 + 3 - C - 6 r_0/p - 4(\pi - \sigma) + 8 \delta/p\]
\[= 2 A_0 + 6 CB_2 - 3 - C - 6 r_0/p - 4(\pi - \sigma) + 8 \delta/p\]
\[\leq 8 A_0 - 3 - C - 6 r_0/p - 4(\pi - \sigma) + 8 \delta/p\]
\[= (1/p)(8 \delta - 6 r_0) - 4(\pi - \sigma) - C - 3 + 8 A_0\]
\[ p \geq \frac{8 \delta - 6 r_0}{(4(\pi - \sigma) + C + 3 - 8A_0)} \]

\[ = \frac{8(R_{on} + C_{on}) - 6r_0}{(4(\pi - \sigma) + C + 3 - 8A_0)} \quad (3.8) \]

One can verify that (3.8) is satisfied when \( p, R, C \geq 4 \). Thus \( H(A_0, CB_2, r_0) \) is nonincreasing in \( CB_2 \) if \( p \) satisfies (3.8), i.e. if \( p \) satisfies (3.8) then

\[ H(A_0, CB_2, r_0) \geq H(A_0, A_0, r_0) \]

Now

\[ H(A_0, A_0, r_0) \]

\[ = A_0 (2A_0 - (\pi - \sigma) - R) - A_0 (6 - 3A_0 + C + 6r_0/p + 4(\pi - \sigma) - 8\delta/p) \]
\[ + 3 + 3(\pi - \sigma) + (\pi - \sigma)^2 + 6r_0/p + (4r_0/p)(\pi - \sigma) + 4r_0^2/p^2 - 1 \]
\[ + 4\delta^2/p^2 - (4\delta/p)(2 + \pi - \sigma) - 8\delta r_0/p^2 - (1/p^2)(\pi r_0 - r_0^2) \]
\[ = 5A_0^2 + A_0 (- (\pi - \sigma) - R - 6 - C - 6r_0/p - 4(\pi - \sigma) + 8\delta/p) \]
\[ + 3 + 3(\pi - \sigma) + (\pi - \sigma)^2 + 6r_0/p + (4r_0/p)(\pi - \sigma) + 4r_0^2/p^2 - 1 \]
\[ + 4\delta^2/p^2 - (4\delta/p)(2 + \pi - \sigma) - 8\delta r_0/p^2 - (1/p^2)(\pi r_0 - r_0^2) \]
\[ = 5A_0^2 + (4\delta - 5\pi - 6 - 6r_0/p + 8\delta/p)A_0 + 3 + 3(\pi - \sigma) + (\pi - \sigma)^2 \]
\[ + 6r_0/p + (4r_0/p)(\pi - \sigma) + 4r_0^2/p^2 - 1 + 4\delta^2/p^2 - (4\delta/p)(2 + \pi - \sigma) \]
\[ - 8\delta r_0/p^2 - (1/p^2)(\pi r_0 - r_0^2) \]
Let us look at the relationship between \( H(A_0, A_0, r_0) \) and \( \delta \). Notice

\[
\frac{\partial}{\partial \delta} \left( 88 A_0^2 / p + 4 \delta^2 / p^2 - (4 \delta / p) (2 + \pi - \sigma) - 8 \delta r_0 / p^2 \right)
\]

\[
= \left(1 / p^2 \right) \left[ 8p A_0 + 8 \delta - 4p (2 + \pi - \sigma) - 8 r_0 \right]
\]

\[
= \left(4 / p^2 \right) \left[ 2 \delta + 2p A_0 - \delta (2 + \pi - \sigma) - 2 r_0 \right]
\]

\[
= \left(4 / p^2 \right) \left[ 2 \delta - 2 r_0 - p (2 + \pi - \sigma - 2 A_0) \right]
\]

\( \leq 0 \)

If \( \delta \leq r_0 + (p/2)(2 + \pi - \sigma - 2 A_0) \) \hspace{1cm} \( (3.9) \)

Let \( r_0 = a_1 R + b_1 = a_2 C + b_2 \), where \( 0 \leq b_1 < R, 0 \leq b_2 < C \).

\[
\delta = R r_{om} + C s_{on} \leq R (a_1 + 1) + C (a_2 + 1)
\]

\[
= a_1 R + R + a_2 C + C
\]

\[
= a_1 R + b_1 + R - b_1 + a_2 C + b_2 + C - b_2
\]

\[
= 2 r_0 + R - b_1 + C - b_2.
\]

To show \( (3.9) \) is satisfied, notice

\[
r_0 + (p/2)(2 + \pi - \sigma - 2 A_0) - \delta
\]

\[
\geq r_0 + (p/2)(2 + \pi - \sigma - 2 A_0) - 2 r_0 - (R + C - b_1 - b_2)
\]

\[
= p + (p/2)(\pi - \sigma) - p A_0 - r_0 - (R + C - b_1 - b_2)
\]

\[
\geq p + (p/2)(\pi - \sigma) - p ((\pi - r_0)/p) - r_0 - (R + C - b_1 - b_2)
\]
\[ p + \frac{p}{2} (\pi - \sigma) - \pi - \sigma + (b_1 + b_2) \]
\[ = p + \frac{p}{2} - 1 \pi - (\frac{p}{2} + 1) \sigma + b_1 + b_2 \]
\[ \geq 0 \quad \text{provided} \quad p \geq \frac{(\pi + \sigma)}{(1 + \pi/2 - \sigma/2)} \]

\[ = \frac{RC + R + C}{(1 - \frac{RC}{2} - \frac{R}{2} - \frac{C}{2})} \quad (3.10) \]

\[ \Rightarrow H(A_0, A_0, r_0) \text{ is nonincreasing in } \delta \text{ provided (3.10) holds.} \]

Hence if (3.10) holds

\[ H(A_0, A_0, r_0) \geq H(A_0, A_0, r_0) \mid \delta = 2r_0 + R + C - b_1 - b_2 \]

\[ = 5A_0^2 + (4\sigma - 5\pi - 6 - 6r_0/p + (8/p)(2r_0 + R + C) \\
- b_1 - b_2))A_0 + 3 + 3(\pi - \sigma) + (\pi - \sigma)^2 + 6r_0/p \\
+ (4r_0/p)(\pi - \sigma) + 4r_0^2/p^2 - 1 \\
+ (4/p^2)(2r_0 + R + C - b_1 - b_2)^2 \\
- (4/p)(2 + \pi - \sigma)(2r_0 + R + C - b_1 - b_2) \\
- (8r_0/p^2)(2r_0 + R + C - b_1 - b_2) \\
- (1/p^2)(\pi r_0 - r_0^2) \]

\[ = 5A_0^2 + (4\sigma - 5\pi - 6 - 6r_0/p)A_0 + (8/p)(2r_0 + Q)A_0 \\
+ 3 + 3(\pi - \sigma) + (\pi - \sigma)^2 + 6r_0/p + (4r_0/p)(\pi - \sigma) \\
+ 4r_0^2/p^2 - 1 + (4/p^2)(2r_0 + Q)^2 \]
\[-(4/p)(2 + \pi - \sigma)(2r_0 + Q) - (8r_0/p^2)(2r_0 + Q)\]
\[-(1/p^2)(\pi r_0 - r_0^2)\]

where \(R + C - b_1 - b_2 = Q, \) and \(Q = 0\) if \(r_0 = 0\)

\[= G(A_0, r_0), \text{ say} \]

Then let us examine the relationship between \(G(A_0, r_0)\) and \(r_0\) when \(r_0\) is on a range where changes in \(r_0\) do not change \(A_0\) and \(B_0\). This means \(RC - pB_0 \leq r_0 \leq RC - pA_0\). Then

\[\frac{d}{dr_0} G(A_0, r_0)\]
\[= -6A_0/p + 16A_0/p + 6/p + 4(\pi - \sigma)/p + 8r_0/p^2\]
\[+ 16(2r_0 + Q)/p^2 - (8/p)(2 + \pi - \sigma) - 32r_0/p^2 - 8Q/p^2\]
\[-\pi/p^2 + 2r_0/p^2\]
\[= 10A_0/p - 10/p - 4(\pi - \sigma)/p + 10r_0/p^2 + 8Q/p^2 - \pi/p^2\]
\[= (1/p^2)(10pA_0 - 10p - 4p(\pi - \sigma) + 10r_0 + 8Q - \pi)\]
\[\leq (1/p^2)(10p(\pi - r_0)/p - 10p - 4p(\pi - \sigma) + 10r_0 + 8Q - \pi)\]
\[= (1/p^2)(9\pi - 10p - 4p(\pi - \sigma) + 8Q)\]
\[\leq (1/p^2)(9\pi - p(10 + 4(\pi - \sigma)) + 8Q)\]
\[\leq 0\]

If \(p \geq (9\pi + 8\sigma)/(10 + 4(\pi - \sigma))\) \hspace{1cm} (3.11)
\[ \begin{align*}
\Rightarrow & \quad \min G(A_0, r_0) \text{ over } r_0 \\
& = G(\ A_0, \ RC - pA_0) \text{ provided that (3.11) holds.}
\end{align*} \]

And

\[ G(\ A_0, \ RC - pA_0) \]

\[ = 5A_0^2 + (4\sigma - 5\pi - 6)A_0 - (6\ A_0/p)(\pi - pA_0) \]
\[ + (8A_0/p)(2(\pi - pA_0) + Q) + 3 + 3(\pi - \sigma) + (\pi - \sigma)^2 \]
\[ + (6/p)(\pi - pA_0) + (4/p)(\pi - pA_0)(\pi - \sigma) - 1 \]
\[ + (4/p^2)(\pi - pA_0)^2 + (4/p^2)(2(\pi - pA_0) + Q)^2 \]
\[ - (4/p)(2 + \pi - \sigma)(2(\pi - pA_0) + Q) \]
\[ - (8/p^2)(\pi - pA_0)(2(\pi - pA_0) + Q) \]
\[ - (1/p^2)(\pi(\pi - pA_0) - (\pi - pA_0)^2) \]

\[ = 5A_0^2 + (4\sigma - 5\pi - 6)A_0 - 6\pi A_0/p + 6 A_0^2 + 16\pi A_0/p - 16 A_0^2 \]
\[ + 8Q A_0/p + 6\pi/p - 6 A_0 + (4/p)(\pi^2 - \pi \sigma) - 4\pi A_0 + 4 \sigma A_0 \]
\[ + 4\pi^2/p^2 + 4 A_0^2 - 8\pi A_0/p + 16\pi^2/p^2 + 16 A_0^2 - 32\pi A_0/p \]
\[ + 4Q^2/p^2 + 16\pi^2/p^2 - 16Q A_0/p - (4Q/p)(2 + \pi - \sigma) \]
\[ - (8/p)(2\pi + \pi^2 - \pi \sigma) + 16 A_0 + 8\pi A_0 - 8 \sigma A_0 \]
\[ - (8/p^2)(\pi Q + 2 \pi^2) + 8Q A_0/p - 16 A_0^2 + 32\pi A_0/p + A_0^2 \]
\[ - 2\pi A_0/p + \pi A_0/p + 3 + 3(\pi - \sigma) + (\pi - \sigma)^2 - 1 \]
\[ A_0 \left( \frac{\pi}{p} + 4 - \pi \right) + \frac{4}{p} \left( \pi^2 - \pi \sigma \right) + 6 \frac{\pi}{p} + 20 \pi^2/p^2 + 4 Q^2/p^2 \]
\[ + 16Q \pi^2/p^2 - \left( \frac{4Q}{p} \right) \left( 2 + \pi - \sigma \right) - \left( \frac{8}{p} \right) \left( 2 \pi + \pi^2 - \pi \sigma \right) \]
\[- \left( \frac{8}{p^2} \right) \left( \pi Q + 2 \pi^2 \right) + 3 + 3 \left( \pi - \sigma \right) + (\pi - \sigma)^2 - 1 \]

Let us look at the relationship between \( G(A_0, R, C - pA_0) \) and \( A_0 \). Notice
\[ \frac{\partial}{\partial A_0} G(A_0, R, C - pA_0) = \left( \frac{1}{p} - 1 \right) \pi + 4 \]
\[ \leq 0 \quad \text{if} \quad p \geq 2, \quad R, C \geq 3. \]

\[ \Rightarrow \quad G(A_0, R, C - pA_0) \text{ is nonincreasing in } A_0. \]

In the following we will discuss cases when \( p \geq 6, \ p = 5, \) and \( p = 4 \) separately, where \( p, R, C \) satisfy (3.6) to (3.11).

Case 1. For \( p \geq 6, \ R, C \geq 6, \) then \( A_0 \leq \left( \frac{\pi}{6} \right), \) and since
\[ G(A_0, R, C - pA_0) \text{ is nonincreasing in } A_0. \]

\[ G(A_0, r_0) \geq G(A_0, R, C - pA_0) \]
\[ \geq G(\pi/6, 0) \]
\[ = 5 \left( \frac{\pi}{6} \right)^2 + (4\sigma - 5\pi - 6)(\pi/6) + 3 + 3(\pi - \sigma) \]
\[ + (\pi - \sigma)^2 - 1 \]
\[ = \left( 11/36 \right) \pi^2 + (2 - (4/3)\sigma) \pi + (\sigma - 1)(\sigma - 2) \]
\[ \geq 0 \]
Case 2. For $p = 5$, $R, C \geq 10$, then $A_0 \leq (\pi/5)$, and since

$G(A_0, RC - pA_0)$ is nonincreasing in $A_0$

$G(A_0, r_0) \geq G(A_0, RC - pA_0)$

$\geq G(\pi/5, 0)$

$= 5(\pi/5)^2 + (4\sigma - 5\pi - 6)(\pi/5) + 3 + 3(\pi - \sigma)$

$+ (\pi - \sigma)^2 - 1$

$= (1/5)x^2 + (9/5 - (6/5)\sigma)\pi + (\sigma - 1)(\sigma - 2)$

$\geq 0$

Case 3. For $p = 4$, $R, C \geq 30$, then $A_0 \leq (\pi/4)$, and since

$G(A_0, RC - pA_0)$ is nonincreasing in $A_0$

$G(A_0, r_0) \geq G(A_0, RC - pA_0)$

$\geq G(\pi/4, 0)$

$= 5(\pi/4)^2 + (4\sigma - 5\pi - 6)(\pi/4) + 3 + 3(\pi - \sigma)$

$+ (\pi - \sigma)^2 - 1$

$= x^2/16 + x(3/2 - \sigma) + (\sigma - 1)(\sigma - 2)$

$\geq 0$
Lemma 3.8. For a fixed value of \( r_0 < RC/2 \), suppose \( \Phi \), the \( r_i \), the \( r_{ij} \), the \( s_{ik} \), the \( s_{kj} \), and the \( s_{ok} \) are as in Lemma 3.1, 3.5, and Corollary 3.2, and \( p, R, C \) are as in Lemma 3.7. Then \( \Sigma_{i=1}^{p} f(A_i) \), where \( f(x) = 1/x \), is minimized when the \( r_i \) are as equal as possible, i.e. \( r_i = \left( \frac{RC-r_0}{p} \right) \) or \( \left( \frac{RC-r_0}{p} \right) + 1 \) for \( 1 \leq i \leq p \).

Proof. From Lemma 3.2 one knows that the \( r_i \) are in \( [A_0, B_0] \) for all \( i \) and in Lemma 3.7 it is shown that \( f(A_i) = f(g(r_i)/(p-1)) \) is convex in \( r_i \) in \( [A_0, B_0] \) for all \( i \). One thus can show that \( \Sigma_{i=1}^{p} f(A_i) \) is Schur convex and hence \( \min \Sigma_{i=1}^{p} f(A_i) \) over \( r_i \) occurs when the \( r_i \) are as equal as possible by adjusting the \( r_{im} \), \( 1 \leq i \leq p \) (i.e. make all adjustments in row \( m \)), and by adjusting the \( s_{in} \), \( 1 \leq i \leq p \) (i.e. make all adjustments in column \( n \)). In this way the same \( r_{om} \) and \( s_{on} \) come into play and so the resulting 'design' may not be of the form given in Lemma 3.1 and/or Lemma 3.5. But one can then apply Lemma 3.1 and/or 3.5 to show that there exists a design \( d^\star \) with the \( r_i^\star \) are as equal as possible and is in the form of Lemma 3.1 and Lemma 3.5 which yields a value of

\[
(p-1) f \left( (1/p-1) \Sigma_{i=1}^{p} m_{ii} - m_{ij}/(p-1) \right) + f(m_{ij}/p)
\]

\[
= (p-1) f \left( (1/p-1) \Sigma_{i=1}^{p} \left( r_i - (1/C) \Sigma_{j=1}^{R} r_{ij}^2 - (1/R) \Sigma_{k=1}^{C} s_{ik}^2 + r_i^2/RC \right) - (1/p(p-1)) \left( r_0 - (1/C) \Sigma_{j=1}^{R} r_{0j}^2 - (1/R) \Sigma_{k=1}^{C} s_{0k}^2 \right) \right)
\]
\[ + r_0^2/RC \right) + f \left\{ (1/p) \left( r_0 - (1/C) \Sigma_{j=1,R} r_{0j}^2 - (1/R) \Sigma_{k=1,C} s_{ok}^2 + r_0^2/RC \right) \right\} \]
\[ = (p-1) f \left\{ (1/p-1) \left( (RC - r_0 - (1/C) \Sigma_{i=1,p} r_{ij}^2 \right) \right\} \]
\[ - (1/R) \Sigma_{i=1,p} r_{ij}^2 \right\} - (1/p(p-1)) (r_0 \]
\[ - (1/C) \Sigma_{j=1,R} r_{0j}^2 - (1/R) \Sigma_{k=1,C} s_{ok}^2 + r_0^2/RC \right) \right\} \]
\[ + f \left\{ (1/p) \left( r_0 - (1/C) \Sigma_{j=1,R} r_{0j}^2 - (1/R) \Sigma_{k=1,C} s_{ok}^2 + r_0^2/RC \right) \right\} \]

(3.12)

where

\[ \Sigma_{j=1,R} r_{0j}^2 = r_0 + (2 r_0 - R)[r_0/R] - R[r_0/R]^2 \]
\[ \Sigma_{k=1,C} s_{ok}^2 = r_0 + (2 r_0 - C)[r_0/C] - C[r_0/C]^2 \]
\[ \Sigma_{i=1,p} r_{ij}^2 = \frac{RC - r_0 + 2 (RC - r_0 - p)[(RC - r_0)/p]}{-p[(RC - r_0)/p]^2} \]
\[ \Sigma_{j=1,R} r_{ij}^2 = r_1 + (2 r_1 - R)[r_1/R] - R[r_1/R]^2 \]
\[ \Sigma_{i=1,p} \Sigma_{j=1,R} r_{ij}^2 \]
\[ = \Sigma_{i=1,p} \left( r_1 + (2 r_1 - R)[r_1/R] - R[r_1/R]^2 \right) \]
\[ = RC - r_0 + 2 \Sigma_{i=1,p} (r_1/R) (2 r_1 - R - R[r_1/R]) \]
\[ = RC - r_0 + (p - RC + r_0 + p[(RC - r_0)/p])[(RC - r_0)/p]/R \]
\[
\begin{align*}
(2 \left(\frac{RC - r_0}{p}\right) - R - R \left[\left(\frac{RC - r_0}{p}\right)/R\right]) \\
+ \left(\frac{RC - r_0 - p \left(\frac{RC - r_0}{p}\right)}{1/R}\left(\frac{RC - r_0}{p} + 1\right)\right) \\
(2 \left[\left(\frac{RC - r_0}{p}\right)/p\right] + 1) - R - R \left[\left(\frac{RC - r_0}{p}\right)/p\right] + 1)\right)
\end{align*}
\]

\[
\Sigma_{k=1,c} s_{lk}^2 = r_1 + 2 r_1 - C \left[r_1/C\right] - C \left[r_1/C\right]^2
\]

\[
\Sigma_{l=1,p} \Sigma_{k=1,c} s_{lk}^2
\]

\[
= \Sigma_{l=1,p} \left(r_1 + 2 r_1 - C \left[r_1/C\right] - C \left[r_1/C\right]^2\right)
\]

\[
= RC - r_0 + \Sigma_{l=1,p} \left[r_1/C\right] \left(2 r_1 - C \left[r_1/C\right]\right)
\]

\[
= RC - r_0 + p - RC + r_0 + p \left[\left(\frac{RC - r_0}{p}\right)/p\right] \left[\left(\frac{RC - r_0}{p}\right)/p\right] + 1\right)
\]

\[
(2 \left[\left(\frac{RC - r_0}{p}\right)/p\right] - C - C \left[\left(\frac{RC - r_0}{p}\right)/p\right]\right)
\]

\[
+ \left(\frac{RC - r_0 - p \left(\frac{RC - r_0}{p}\right)}{1/R}\left(\frac{RC - r_0}{p} + 1\right)\right)
\]

\[
(2 \left[\left(\frac{RC - r_0}{p}\right)/p\right] + 1) - C - C \left[\left(\frac{RC - r_0}{p}\right)/p\right] + 1)\right)
\]

is a function of \( r_0 \) only, say \( F(r_0) \).

**Theorem 3.1.** If \( d \in D(p,R,C) \) is a completely symmetric design, i.e. \( M(d) \) is completely symmetric, satisfying

(i) The \( r_1 \) are as equal as possible, i.e.

\[
r_1 = \left[\left(\frac{RC - r_0}{p}\right)/p\right] \text{ or } \left[\left(\frac{RC - r_0}{p}\right)/p\right] + 1 \quad \text{for } 1 \leq i \leq p,
\]

(ii) The \( r_{ij} \) are as equal as possible, i.e.
\[ r_{ij} = \lceil r_i/R \rceil \text{ or } \lceil r_i/R \rceil + 1 \quad \text{for } 1 \leq i \leq p, 1 \leq j \leq R, \]

(iii) The \( s_{ik} \) are as equal as possible, i.e.

\[ s_{ik} = \lceil r_i/C \rceil \text{ or } \lceil r_i/C \rceil + 1 \quad \text{for } 1 \leq i \leq p, 1 \leq k \leq C, \]

(iv) The \( r_{oj} \) are as equal as possible, i.e.

\[ r_{oj} = \lceil r_o/R \rceil \text{ or } \lceil r_o/R \rceil + 1 \quad \text{for } 1 \leq j \leq R, \]

(v) The \( s_{ok} \) are as equal as possible, i.e.

\[ s_{ok} = \lceil r_o/C \rceil \text{ or } \lceil r_o/C \rceil + 1 \quad \text{for } 1 \leq k \leq C, \]

(vi) \( r_o \) is the nonnegative integer, \( 0 \leq r_o \leq RC/2 \), which minimized

\[ F(r) \text{ where } F(r) \text{ is given in (3.12).} \]

Then \( d \) is \( A \)-optimal over \( D(p,R,C) \).
Chapter 4
A-Optimal Designs in the Block Design Setting

In this chapter attention is focused on complete block designs, i.e. designs whose experimental units are to be arranged in $b$ blocks each of size $k$ where $p, k \geq 2$. Notice, this is a generalization of the result in Majumdar and Notz (1983) in which they require $k \leq p$. We begin with a series of lemmas, with respect to the inequalities (2.2), resulting in a general theorem from which A-optimal designs can be obtained as a special case. As in chapter 3, our plan is to find conditions under which either

\[(p-1) f \left\{ \sum_{i=1}^{p} m_{ij} / (p-1) - m../p(p-1) \right\} + f (m../p) \]

or \[((p-1)/p) \sum_{j=1}^{p} f \left\{ (p/p-1) m_{ij} - (2/p-1) m_{.j} + m../p(p-1) \right\} + f (m../p) \]

will be a minimum.

4.1 Notation and definitions

Let (i) $r_{ij}(d) =$ number of plots design $d$ assigns to treatment $i$ in block $j$.

(ii) $r_{i}(d) =$ number of replication of treatment $i$ in the design $d$.  

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(III) \( p = \) number of treatments,
\( b = \) number of plots,
\( k = \) block size.

(iv) \( M(d) = \{(m_{ij})_{p\times p}\} \) the information matrix of
\[ \begin{pmatrix} \alpha_0 - \alpha_1, & \ldots, & \alpha_0 - \alpha_p \end{pmatrix}, \]
\[ m_{i.} = \sum_{j=1}^{p} m_{ij}, \]
\[ m_{..} = \sum_{i=1}^{p} \sum_{j=1}^{p} m_{ij}. \]

(v) \( A_1 = (p/p-1) m_{11} - (2/p-1) m_{1.} + m_{..}/p(p-1) \)
\[ = (p/p-1) (r_1 - (1/k) \sum_{j=1}^{b} r_{ij}^2) \]
\[ - (2/k(p-1)) \sum_{j=1}^{b} r_{ij} r_{0j} \]
\[ + (1/p(p-1)) (r_0 - (1/k) \sum_{j=1}^{b} r_{0j}^2) \]

(vi) \( g(r_1) = p(p-1) A_1 \)
\[ = p^2 m_{11} - 2p m_{1.} + m_{..} \]
\[ = p^2 r_1 + r_0 - (1/k) \sum_{j=1}^{b} (p r_{ij} + r_{0j})^2 \]

(vii) \( \Delta(r_1) = g(r_1+1) - g(r_1) \)

(viii) \( m_{11}(r_1) = m_{11} \)
\[ = r_1 - (1/k) \sum_{j=1}^{b} r_{ij}^2 \]

(ix) \( \Delta^0(r_1) = m_{11}(r_1+1) - m_{11}(r_1) \)

(x) \( N = \{ n; 0 \leq n \leq bk - r_0, n = ab, a \text{ integer} \} \)
Definition 4.1. If $A, B \in \mathbb{N}$, $A < B$, and no integer between $A$ and $B$ is in $\mathbb{N}$, we call $[A, B]$ an elementary interval.

Definition 4.2. If $[A_0, B_0]$ is the elementary interval containing the value $(kb - r_0)/p$, where $(kb - r_0)/p \in \mathbb{N}$, we call $[A_0, B_0]$ the basic interval.

Definition 4.3. $[A^0, B^0]$ is the elementary interval containing the value $kb/p$, where $kb/p \in \mathbb{N}$.

4.2 Conditions for A-optimal designs.

Lemma 4.1. If $d \in \mathcal{C}(p,b,k)$, where $\mathcal{C}(p,b,k)$ denotes the set of all possible block designs with $b$ blocks of size $k$, and $\Phi$ is of form (2.1), then for fixed values of $r_0$, the $r_{ij}$, and the $r_i$,

$$f \left( (1/p-1) \sum_{i=1}^{p} m_{ij} - m. / p(p-1) \right)$$

is minimized when the $r_{ij}$ are as equal as possible, i.e. $r_{ij} = \lfloor r_i/b \rfloor$ or $\lfloor r_i/b \rfloor + 1$ for $1 \leq i \leq p$, and $1 \leq j \leq b$.

Proof. For fixed values of $r_0$ and the $r_{ij}$, $m.$ is thus fixed. One can see that $m_{ij} = r_i - (1/k) \sum_{k=1}^{b} r_{ij}^2$ will be a maximum for fixed $r_i$. If $\sum_{j=1}^{b} r_{ij}^2$ is as small as possible for all $i$. Using the same argument as in Lemma 3.1, one can conclude that for fixed value of $r_i$, $\sum_{j=1}^{b} r_{ij}^2$ is
minimized when the \( r_{ij} \) are as equal as possible, i.e. choosing
\[
\frac{r_1 - b}{r_i/b} \text{ of the } r_{ij} \text{ to have value } \lfloor \frac{r_1}{b} \rfloor + 1, \text{ and the remaining}
\]
\[
b \left( 1 + \left\lfloor \frac{r_1}{b} \right\rfloor \right) - r_1 \text{ of the } r_{ij} \text{ to have value } \left\lfloor \frac{r_1}{b} \right\rfloor, 1 \leq b \leq b. \text{ Since}
\]
f' < 0, the lemma follows.

Lemma 4.2. For fixed values of \( r_0 \) and the \( r_{0j} \), suppose \( \Phi \) and the
\( r_{ij} \) are as in Lemma 4.1 and \((kb-r_0)/p \in \mathbb{N}\). Then
\[
f((1/p-1) \Sigma_{i=1}^p m_{1i} - m_{..}/p(p-1))
\]
is minimized when the \( r_i \) are in the same elementary interval.

Proof. Suppose \( r_1 > r_2 \) and \( r_1, r_2 \) are \( t \geq 1 \) elementary intervals
apart, i.e. if \( r_1 = a_1 b^2 + d_1, r_2 = a_2 b^2 + d_2, 1 \leq d_1 \leq b-1, 0 \leq d_2 \leq b-1, \)
and \( a_1 - a_2 = t \); if \( d_1 = 0 \) then \( t \) elementary intervals apart means
\( a_1 - a_2 = t+1 \). Let \( r_1^* = r_1 - d, r_2^* = r_2 + d \) where \( d \) is the smallest
integer such that one of the \( r_1^* \) or \( r_2^* \) is in \( \mathbb{N} \) and \( r_1^*, r_2^* \) are
\( (t-1) \) elementary intervals apart. Then
\[
m_{11}^* + m_{22}^* - (m_{11} + m_{22})
\]
\[
= m_{11}(r_1^*) + m_{22}(r_2^*) - (m_{11}(r_1) + m_{22}(r_2))
\]
\[
= m_{22}(r_2+d) - m_{22}(r_2) - (m_{11}(r_1) - m_{11}(r_1-d))
\]
\[
= \Delta \Phi(r_2) + \Delta \Phi(r_2+1) + \ldots + \Delta \Phi(r_2+d-1)
\]
\[
-(\Delta \Phi(r_1-1) + \Delta \Phi(r_1-2) + \ldots + \Delta \Phi(r_1-d))
\]
\[ = \{ a^0(r_2) - a^0(r_1-d)\} - \{ a^0(r_2+1) - a^0(r_1-d+1)\} + \ldots \]
\[ + \{ a^0(r_2,d-1) - a^0(r_1)\} \]  
(4.1)

From sec 4.1 (v), one has

\[ a^0(r_1) = m_{11}(r_1+1) - m_{11}(r_1) \]
\[ = (1/k)(k-1 - 2\lfloor r_1/b \rfloor) \]

Thus \[ a^0(r_2+a) - a^0(r_1-d+a) \]
\[ \quad = (2/k)(\lceil r_1/b \rceil - \lceil r_2/b \rceil) \]
\[ \quad = (2/k)(a_1 - a_2) \]
\[ \quad \geq 0 \]

Substituting this into (4.1) yields

\[ m_{11}^* + m_{22}^* \geq m_{11} + m_{22} \]

Hence, one can conclude that \[ \max \Sigma_{i=1,p} m_{11} \] over the \( r_1 \) occurs when the \( r_1 \) are in the same elementary interval. Since \( f' < 0 \), the lemma follows.

**Corollary 4.2.** For fixed values of \( r_0 \) and the \( r_{ij} \). Suppose \( \Phi \) and the \( r_{ij} \) are as in Lemma 4.1, and \( (kb-r_0)/p \in \mathbb{N} \). Then

\[ f((1/p-1)\Sigma_{i=1,p} m_{11} - m_{..}/p(p-1)) \]
is minimized when the \( r_i \) are in the basic interval.

**Proof.** This corollary is an immediate result of Lemma 4.2, and the fact that \( \Sigma_{i=1,p} r_i = kb - r_0 \).

Let us consider the case when \( (kb - r_0)/p = ab \), a integer, i.e. \( (kb - r_0)/p \in \mathbb{N} \). From Lemma 4.2 and Corollary 4.2 we might guess that if this is case then \( f \left( \frac{1}{p-1} \Sigma_{i=1,p} m_{ii} - m_+/(p-1) \right) \) is minimized when \( r_i = (kb - r_0)/p = ab \) for all \( i \), which is correct from the next lemma.

**Lemma 4.3.** For fixed values of \( r_0 \) and the \( r_{ij} \), suppose \( \Phi \), the \( r_{ij} \) are as in Lemma 4.1 and \( (kb - r_0)/p \in \mathbb{N} \). Then

\[
f \left( \frac{1}{p-1} \Sigma_{i=1,p} m_{ii} - m_+/(p-1) \right)
\]

is minimized when \( r_i = (kb - r_0)/p \), \( 1 \leq i \leq p \).

**Proof.** Using a proof analogous to that used in Lemma 4.2, one can conclude that \( f \left( \frac{1}{p-1} \Sigma_{i=1,p} m_{ii} - m_+/(p-1) \right) \) is minimized when all \( r_i \) satisfy

\[
(kb - r_0)/p - (b-1) \leq r_i \leq (kb - r_0)/p + (b-1)
\]

Now suppose \( r_1 > (kb - r_0)/p \) and \( r_2 < (kb - r_0)/p \). Let \( r_1^* = r_1 - 1 \), \( r_2^* = r_2 + 1 \)

\[
m_{11}^* + m_{22}^* = (m_{11} + m_{22})
\]
\[ m_{11}(r^*_1) + m_{22}(r^*_2) - (m_{11}(r_1) + m_{22}(r_2)) \]
\[ = m_{22}(r^*_2 - 1) - m_{22}(r_2) - (m_{11}(r_1) - m_{11}(r_1 - 1)) \]
\[ = \Delta^0(r_2) - \Delta^0(r_1 - 1) \]
\[ = (2/k) \left( [(r_1 - 1)/b] - [r_2/b] \right) \]
\[ \geq 0 \]

Hence one can conclude that \( \max \sum_{i=1}^{p} m_{ii} \) over the \( r_i \) occurs when \( r_i = (kb - r_0)/p \) for all \( i \). Since \( f' < 0 \), the lemma is proved.

**Lemma 4.4.** Suppose \( \Phi \) is of form (2.1). Suppose \( d \in C(p,k,b) \) has the \( r_i \) and the \( r_{ij} \) as in Lemma 4.1 and Corollary 4.2 (or Lemma 4.3), and has \( r_0 > kb/2 \). Then there exists \( d^* \in C(p,k,b) \) having the \( r^*_i \) and the \( r^*_{ij} \) as in Lemma 4.1 and Corollary 4.2 (or Lemma 4.3), with \( r^*_0 \leq kb/2 \), and satisfying

\[ f \left( (1/p - 1) \sum_{i=1}^{p} m_{ii}^* - m_*^*/p(p-1) \right) \]

\[ \leq f \left( (1/p - 1) \sum_{i=1}^{p} m_{ii} - m_*/p(p-1) \right) \]

**Proof.** Let \( d^* \) to be the 'design' with \( r^*_0 = kb - r_0 \), \( r^*_{0j} = k - r_{0j} \), \( 1 \leq j \leq b \), and \( r^*_i = r_i \) if \( r_i > (kb - r^*_0)/p \), otherwise \( r^*_i \leq r^*_i \)
\[ \leq \left( (kb - r^*_0)/p \right) + 1 \leq B^0, \] and the \( r^*_{ij} \) are as equal as possible. Here by 'design' we mean a specification of \( r_0 \), the \( r_i \), and the \( r_{ij} \), such that \( \sum_{j=1}^{b} r^*_{ij} = r^*_1 \) and \( r_0 + \sum_{i=1}^{p} r^*_i = kb \). Notice,

\[ m_*/^* = r^*_0 - (1/k) \sum_{j=1}^{b} r^*_{0j} \]
$$= kb - r_0 - (1/k) \sum_{j=1,b} (k - r_{0j})^2$$
$$= r_0 - (1/k) \sum_{j=1,b} r_{0j}^2$$
$$= \ldots$$

and
$$= (1/p-1) \sum_{i=1,p} m_{II} r_i^k - m_{III}/p(p-1)$$
$$= (1/p-1) \sum_{i=1,p} m_{II} r_i^k - m_{III}/p(p-1)$$
$$= (1/p-1) \sum_{i=1,p} (r_i^k - (1/k) \sum_{j=1,b} r_{ij}^k) - m_{III}/p(p-1)$$
$$= (1/p-1) \sum_{i=1,p} m_{II}(r_i^k) - m_{III}/p(p-1)$$

Let us look at the relationship between $m_{II}(r_1)$ and $r_1$. By definition
$$A_0(r_1) = m_{II}(r_1 + 1) - m_{II}(r_1)$$
$$= (1/k)(k - 1 - 2[r_1/b])$$

Let $r_1 = (a_1)b + a_2$, where $0 \leq a_2 \leq b - 1$, then
$$A_0(r_1) = (1/k)(k - 1 - 2a_1) \geq 0$$

for $a_1 \leq (k-1)/2$. Hence one has $A_0(r_1) \geq 0$ for

$$r_1 \leq (k-1)b/2 + a_2 \leq (k-1)b/2 + b - 1 = (b(k+1)-2)/2$$

$m_{II}(r_1)$ is nondecreasing in $r_1$ when $r_1 \leq N_1$, where $N_1$ is the smallest element in $N$ which is $\geq (b(k+1)-2)/2$, since by (4.2) $m_{II}(r_1)$ is linear in $r_1$ when $a_1 b \leq r_1 < (a_1+1)b$. 
Now \( r_1^* = r_1 \) or \( r_1 < r_1^* \leq b_0^* \leq N_1 \), thus \( m_{11}(r_1^*) = m_{11}(r_1) \). Therefore

\[
\sum_{i=1,p} m_{11} = \sum_{i=1,p} m_{11}(r_1) \leq \sum_{i=1,p} m_{11}(r_1^*) = \sum_{i=1,p} m_{11}
\]

Since \( f' < 0 \) it follows that

\[
f((1/p-1) \sum_{i=1,p} m_{11} - m_{11}/p(p-1)) \leq f((1/p-1) \sum_{i=1,p} m_{11} - m_{11}/p(p-1)).
\]

Finally if \( d^* \) is not of the form given in Corollary 4.2 (or Lemma 4.3), then there exists \( d^{**} \) with \( r_0^{**} = r_0^* \leq kb/2 \) in the form of Corollary 4.2 (or Lemma 4.3) which yields a value of \( f(\cdot) \) less than or equal to that produced by \( d^* \). The lemma is now proved.

**Lemma 4.5.** For fixed values of \( r_0 \leq kb/2 \), suppose \( \Phi \), the \( r_i \), and the \( r_{ij} \) are as in Lemma 4.1 and Corollary 4.2 (or Lemma 4.3). Then

\[
(p-1) f((1/p-1) \sum_{i=1,p} m_{11} - m_{11}/p(p-1)) + f(m_{11}/p)
\]

is minimized when the \( r_{ij} \) are as equal as possible, i.e. \( r_{ij} = \lfloor r_0/b \rfloor \) or \( \lfloor r_0/b \rfloor + 1 \) for \( 1 \leq j \leq b \).

**Proof.** Since \( f \) is a real valued possibly infinite function on the set of all nonnegative numbers which is continuous on the set of all positive numbers, and has \( f' < 0 \) and \( f'' > 0 \), then \( f \) has the property that if \( \mu_1 \leq \mu_2 = \mu_3 = \ldots = \mu_p \), \( v_1 \leq v_2 = v_3 = \ldots = v_p \), \( \mu_1 \geq v_1 \) and if
Let us examine \((p-1)((1/p-1) \Sigma_l=1,p m_{ll} - m../p(p-1)) + m../p\) to see if it is constant. Now

\[
(p-1)((1/p-1) \Sigma_l=1,p m_{ll} - m../p(p-1)) + m../p
\]

\[
= \Sigma_l=1,p m_{ll} - m../p + m../p
\]

\[
= \Sigma_l=1,p m_{ll}
\]

is constant, since \(r_0\), the \(r_i\), and the \(\Sigma_j=1,b r_{ij}^2\) are fixed.

Hence, if one can show that \((1/p-1) \Sigma_l=1,p m_{ll} \geq m../p\) then by the property of \(f\) as just stated above, finding conditions under which \((p-1) f ((1/p-1) \Sigma_l=1,p m_{ll} - m../p(p-1)) + f (m../p)\) is a minimum is the same as finding the conditions under which \(m..\) is a maximum.

Now let us look at

\[
(1/p-1) \Sigma_l=1,p m_{ll} - m../p(p-1) - m../p
\]

\[
= (1/p-1)( \Sigma_l=1,p m_{ll} - m.. )
\]

\[
= (1/p-1) \left[ \Sigma_l=1,p \left( r_i - (1/k) \Sigma_j=1,b r_{ij}^2 \right) 
\]

\[
= \left( \Sigma_l=1,p r_i - (1/k) \Sigma_l=1,p \Sigma_j=1,b r_{ij}^2 - (kb - \Sigma_l=1,p r_i) \right)
\]
\[ + \frac{1}{k} \sum_{j=1}^{b} (k - \sum_{l=1}^{p} r_{lj})^2 \]

\[ = \frac{1}{p-1} \left\{ \sum_{l=1}^{p} r_{l1} - \left( \frac{1}{k} \sum_{j=1}^{b} \sum_{l=1}^{p} r_{lj} \right)^2 - k b \sum_{l=1}^{p} r_{li} \right\} \]

\[ + \frac{1}{k} (kb - 2k \sum_{l=1}^{p} r_{l1} + \sum_{j=1}^{b} (\sum_{l=1}^{p} r_{lj})^2) \]

\[ = \frac{1}{p-1} \left\{ \left( \frac{1}{k} \sum_{j=1}^{b} (\sum_{l=1}^{p} r_{lj})^2 - \sum_{l=1}^{p} \sum_{j=1}^{b} r_{lj}^2 \right) \right\} \]

\[ \geq 0 \]

\[ \Rightarrow m_{..} \text{ is the smallest one among} \]

\[ \left( \frac{1}{p-1} \sum_{l=1}^{p} m_{l1} - m_{..}/p(p-1), \ldots, \frac{1}{p-1} \sum_{l=1}^{p} m_{l1} - m_{..}/p(p-1), m_{..}/p \right). \]

The final step is to find conditions under which

\[ m_{..} = r_0 - \left( \frac{1}{k} \sum_{j=1}^{b} r_{0j} \right)^2 \]

is a maximum. However, for a fixed value of \( r_0 \), one can easily see that \( m_{..} = r_0 - \left( \frac{1}{k} \sum_{j=1}^{b} r_{0j} \right)^2 \) is maximized when the \( r_{0j} \) are as equal as possible, i.e. \( r_{0j} = 1 \) whenever \( j \neq b \), are either \([ r_0/b ] \) or \([ r_0/b ] + 1 \). The lemma follows.

**Lemma 4.6.** Suppose the \( r_{ij} \) and the \( r_{0j} \) are as in lemma 4.1 and 4.5. Then \( g(r_{ij}) = p^2 m_{l1} - 2p m_{l} + m_{..} \) is concave in each elementary interval.
Proof. In order to show the concavity of $g(r_i)$ in each elementary interval, one has to show the following. For $r_i - 1$, $r_i$, and $r_i + 1$ in the same elementary interval,

$$g(r_i + 1) + g(r_i - 1) \leq 2g(r_i),$$

i.e. $$g(r_i + 1) - g(r_i) - (g(r_i) - g(r_i - 1)) \leq 0,$$

or equivalently to show that

$$\Delta(r_i) - \Delta(r_i - 1) \leq 0$$

for $ab \leq r_i \leq (a+1)b$, where $a$ is integer.

Before we show that $\Delta(r_i) - \Delta(r_i - 1) \leq 0$, let us construct a reasonable strategy for increasing $r_i$. The increment from $r_i - 1$ to $r_i$ takes place in the $m^{th}$ block where it is assumed that $r_{im} = \lfloor r_i / b \rfloor$ and $r_{om}$ is the smallest of the $r_{ij}$'s whose corresponding $r_{ij}$'s are $\lfloor r_i / b \rfloor$. Then through some easy calculation, one can show that

$$\Delta(r_i - 1) = g(r_i) - g(r_i - 1)$$

$$= (p^2/k)(k - 1 - 2\lfloor r_i / b \rfloor) - (2p/k) r_{om}.$$ 

Also

$$\Delta(r_i) = g(r_i + 1) - g(r_i)$$

$$= (p^2/k)(k - 1 - 2\lfloor r_i / b \rfloor) - (2p/k) r_{on}.$$
where the increment from \( r_i \) to \( r_i+1 \) takes place at one of the remaining blocks, say the \( n^{th} \) block, for which \( r_{in} = \lfloor r_i/b \rfloor \), and the increment also follows the strategy just stated. Notice \( r_{om} \leq r_{on} \), so

\[
\Delta(r_i) - \Delta(r_{i-1}) = \frac{2p}{k}(r_{om} - r_{on}) \\
\leq 0, \text{ for abs } r_i \leq (a+1)b
\]

\[\Rightarrow g(r_i) \text{ is concave in each elementary interval.}\]

**Lemma 4.7.** For fixed value of \( r_0 \leq kb/2 \). Suppose \( \Phi \), the \( r_i \), the \( r_{ij} \), and the \( r_{ij} \) are as in Lemma 4.1, 4.5, and Corollary 4.2. Then

\[
\Sigma_{i=1,p} f \left( \frac{p}{p-1} m_{ii} - \frac{2}{p-1} m_{i+} + m_+/p(p-1) \right)
\]

is minimized when the \( r_i \) are as equal as possible, i.e. \( r_i = \lfloor (kb-r_0)/p \rfloor \) or \( \lfloor (kb-r_0)/p + 1 \rfloor \) for \( 1 \leq p \).

**Proof.** This lemma follows immediately from Lemma 4.2 and 4.6. In Lemma 4.6, it was shown that \( g(r_i) \) is concave in the basic interval (since \( g(r_i) \) is concave in each elementary interval), and combined with the fact that \( f \) is convex and decreasing, one can show that \( f \cdot g \) is convex in the basic interval. Since in Lemma 4.2 it was shown that the \( r_i \) are all in the basic interval, and since \( f \cdot g \) is convex in the basic interval, one can conclude that

\[
\min \Sigma_{i=1,p} f \left( \frac{p}{p-1} m_{ii} - \frac{2}{p-1} m_{i+} + m_+/p(p-1) \right) \text{ over all } r_i
\]
occurs when the \( r_i \) are as equal as possible, i.e. \( r_i \) is either \( \lfloor \frac{kb-r_0}{p} \rfloor \) or \( \lfloor \frac{kb-r_0}{p} \rfloor + 1 \) for \( 1 \leq kp \).

**Remark:** It can be shown that from Lemma 4.1 - 4.7,

\[
(p-1) f \left\{ (1/p(p-1)) \sum_{i=1}^{p} m_{ij} - m_{..}/p(p-1) \right\} + f \left( m_{..}/p \right)
\]

\[
= (p-1) f \left\{ (1/p(p-1)) \sum_{i=1}^{p} \left( r_i - \left( \frac{1}{k} \right) \sum_{j=1}^{k} r_{ij}^2 \right) - \left( \frac{1}{p(p-1)} \right) \left( r_0 - \left( \frac{1}{k} \right) \sum_{j=1}^{k} r_{0j}^2 \right) \right\}
\]

\[
+ f \left\{ (1/p) \left( r_0 - \left( \frac{1}{k} \right) \sum_{j=1}^{k} r_{0j}^2 \right) \right\}
\]

\[
= (p-1) f \left\{ (1/p(p-1)) \left( kb - r_0 - \left( \frac{1}{k} \right) \sum_{i=1}^{p} \sum_{j=1}^{k} r_{ij}^2 \right) - \left( \frac{1}{p(p-1)} \right) \left( r_0 - \left( \frac{1}{k} \right) \sum_{j=1}^{k} r_{0j}^2 \right) \right\}
\]

\[
+ f \left\{ (1/p) \left( r_0 - \left( \frac{1}{k} \right) \sum_{j=1}^{k} r_{0j}^2 \right) \right\}
\]

where \( \sum_{j=1}^{k} r_{0j}^2 = r_0 + \left( 2 \frac{r_0 - b}{b} \right) \left[ \frac{r_0}{b} \right] - b \left[ \frac{r_0}{b} \right]^2 \)

\( \sum_{i=1}^{p} \sum_{j=1}^{k} r_{ij}^2 = r_1 + \left( 2 \frac{r_1 - b}{b} \right) \left[ \frac{r_1}{b} \right] - b \left[ \frac{r_1}{b} \right]^2 \)

\( \sum_{i=1}^{p} \sum_{j=1}^{k} r_{ij}^2 \)

\[
= \left( kb - r_0 + \sum_{i=1}^{p} \left[ \frac{r_i}{b} \right] \left( 2 \frac{r_1 - b}{b} - b \left[ \frac{r_1}{b} \right] \right) \right)
\]

\[
= \left( kb - r_0 + \left( p - kb + r_0 + p \left[ \frac{kb - r_0}{p} \right] \right) \right)
\]

\[
\left[ \left( \frac{kb - r_0}{p} \right) / b \right]
\]
\[(2\left(\left(\frac{kb - r_0}{p}\right) - b - b^2\left(\left(\frac{kb - r_0}{p}\right)/b\right)\right) + \left(\frac{kb - r_0}{p}\right) - p\left(\left(\frac{kb - r_0}{p}\right)/p\right)\right)\]

\[\left(1/b\right)\left(\left(\frac{kb - r_0}{p}\right)/p\right) + 1\]  

\[\left(2\left(\left(\frac{kb - r_0}{p}\right)/p\right) + 1\right) - b - b^2\left(\left(1/b\right)\left(\left(\frac{kb - r_0}{p}\right)/p\right) + 1\right)\]\n
is a function of \(r_0\) only, say \(F(r_0)\).

**Theorem 4.1.** Suppose \(\Phi\) is of form (2.1). If \(d \in \mathcal{C}(p,k,b)\) such that \(M(d)\) is completely symmetric and

1. The \(r_i\) are as equal as possible, i.e.
   \[r_i = \left(\frac{kb - r_0}{p}\right)\text{ or } \left(\frac{kb - r_0}{p}\right)/p + 1\text{ for } 1 \leq i \leq p,\]

2. The \(r_{ij}\) are as equal as possible, i.e.
   \[r_{ij} = \left(\frac{r_i}{b}\right)\text{ or } \left(\frac{r_i}{b}\right)/b + 1\text{ for } 1 \leq i \leq p, 1 \leq j \leq b,\]

3. The \(r_{ij}\) are as equal as possible, i.e.
   \[r_{ij} = \left(\frac{r_0}{b}\right)\text{ or } \left(\frac{r_0}{b}\right)/b + 1\text{ for } 1 \leq j \leq b,\]

4. \(r_0\) is the nonnegative integer, \(0 \leq r_0 \leq kb/2\), which minimizes
   \[F(r)\text{ where } F(r)\text{ is given in (4.3).}\]

Then \(d\) is \(\Phi\)-optimal over \(\mathcal{C}(p,k,b)\).
As mentioned in the introduction, A-optimal designs are very appealing statistically, so it is worthwhile to examine such designs in detail. A design $d \in C(p,k,b)$ is A-optimal if it minimized $\text{tr} \, M^{-1}(d)$ over all $d$, or equivalently, it minimizes $\Phi(M(d)) = \Sigma_{i=1,p} f(\lambda_i)$, where $f(x) = 1/x$, over $C(p,k,b)$. From equation (4.3), $F(r_0)$ then becomes

$$F(r_0) = 1/\left( \Sigma_{i=1,p} m_{ij}/(p-1) - m../(p-1) \right) + 1/(m../p)$$

$$= \left\{ \left( 1/(p-1) \right) \Sigma_{i=1,p} \left( r_{ij} - (1/k) \Sigma_{j=1,b} r_{ij}^2 \right) \right\}^{-1}$$

$$- \left( 1/(p-1) \right) \left( r_0 - (1/k) \Sigma_{j=1,b} r_{0j}^2 \right)$$

$$+ p/\left( r_0 - (1/k) \Sigma_{j=1,b} r_{0j}^2 \right)$$

$$= p/(p-1) \left\{ p \left( k b - r_0 - (1/k) \Sigma_{i=1,p} \Sigma_{j=1,b} r_{ij}^2 \right) \right\}^{-1}$$

$$- \left( r_0 - (1/k) \Sigma_{j=1,b} r_{0j}^2 \right) + p/\left( r_0 - (1/k) \Sigma_{j=1,b} r_{0j}^2 \right)$$

$$= F_A(r_0), \text{ say.} \quad (4.5)$$

where $\Sigma_{i=1,p} \Sigma_{j=1,b} r_{ij}^2$, $\Sigma_{j=1,b} r_{0j}^2$ are as in (4.4)

Before we go into the next theorem, let us examine the necessary and sufficient conditions for a design $d$ whose information matrix $M(d)$ to be completely symmetric.

From sec. 1.2, the information matrix of $(\alpha_0 - \alpha_1, \alpha_0 - \alpha_2, \ldots, \alpha_0 - \alpha_p)'$ is the $p \times p$ matrix $M = (m_{ij})$ and

$$m_{ij} = r_{ij} - (1/k) \Sigma_{h=1,b} r_{ih}^2 \quad \text{if} \quad i = j$$

$$- (1/k) \Sigma_{h=1,b} r_{ih} r_{jh} \quad \text{if} \quad i = j$$
for $1 \leq i, j \leq p$. Let $\lambda_{ij} = \sum_{h=1}^{b} r_{ih} r_{jh}$, $0 \leq i, j \leq p$. $\lambda_{ij}$ represents the total number of times treatment $i$ and $j$ appear together in the $b$ blocks. For $M$ to be completely symmetric, the entries of $M$ must satisfy

(i) $(-1/k) \sum_{h=1}^{b} r_{ih} r_{jh} = (-1/k) \lambda_{ij}$, $1 \leq i = j \leq p$, are all equal, i.e.

$$= (-1/k) \lambda_1, \text{ say},$$

(ii) $\sum_{j=1}^{p} m_{ij} = m_i$, $1 \leq i \leq p$, are all equal, i.e.

$$\sum_{j=1}^{p} m_{ij} = m_i = (1/k) \sum_{h=1}^{b} r_{ih} r_{oh} = (1/k) \lambda_{io} = (1/k) \lambda_0, \text{ say}.$$ 

From (i) and (ii),

$$m_{ii} = r_1 - (1/k) \sum_{h=1}^{b} r_{ih}^2 = (1/k) ((p-1) \lambda_1 + \lambda_0)$$

Therefore the necessary and sufficient conditions for a design to be completely symmetric are

$$\lambda_{12} = \lambda_{13} = \cdots = \lambda_{p-1,p} = \lambda_1$$

$$\lambda_{10} = \lambda_{20} = \cdots = \lambda_{p,0} = \lambda_0$$
For a design $d \in \mathcal{C}(p,k,b)$ that satisfies the above conditions is called a Balanced Treatment Complete Block (abbreviated as BTCB) design (we use a similar terminology as in Bechhofer and Tamhane (1981)), since it is 'balanced' with respect to test treatments. The following theorem is a consequence of Theorem 4.1.

**Theorem 4.2.** Suppose $d \in \mathcal{C}(p,k,b)$ is a BTCB design such that

1. $r_i = \lceil (kb - r_0)/p \rceil$ or $\lceil (kb - r_0)/p \rceil + 1$ for $1 \leq i \leq p$,
2. $r_{ij} = \lceil r_i/b \rceil$ or $\lceil r_i/b \rceil + 1$ for $1 \leq i \leq p, 1 \leq j \leq b$,
3. $r_{ij} = \lceil r_0/b \rceil$ or $\lceil r_0/b \rceil + 1$ for $1 \leq j \leq b$,
4. $r_0$ is the nonnegative integer, $0 \leq r_0 \leq kb/2$, which minimizes $F_A(r)$ where $F_A(r)$ is as given in (4.5),

then $d$ is $A$-optimal over $\mathcal{C}(p,k,b)$. 

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Table 1

Values of $r_0$ which minimize

$$(p-1)f((1/p-1)\sum_{i,j=1}^{p} m_{ij} - m_{..} / p(p-1)) + f(m_{..} / p)$$

in the row-column design setting

for $6 \leq p \leq 10$, $6 \leq R, C \leq 20$.

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*Note: The value of \( r_0 \) will remain the same when R and C are interchanged.*
Table 2

Values of $r_0$ which minimize

$\frac{(p-1)}{f((1/p-1)\sum_{i=1,p} m_{1i} - m_{1i}/p(p-1)) + f(m_{1i}/p)}$

in the block design setting

for $2 \leq p \leq 10$, $2 \leq b \leq 15$, $2 \leq k \leq 20$.

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BIBLIOGRAPHY


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