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The Ohio State University, 1987
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APPLICATION OF RANDOM FIELD THEORY IN MAPPING PROBLEMS

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of the Ohio State University

By

Petros Georgios Patias, Dipl. Eng., M.Sc.

* * * * *

The Ohio State University
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CHAPTER I
INTRODUCTION

1.1 Terrain modelling and cartographic problems

We call terrain model a set of assumptions describing the spatial behavior of the terrain. Such a behavior is usually quite local, and very often changes rapidly (breaklines, ridges, etc.). In order to model this behavior we need to develop a flexible analytical apparatus, which adapts itself to terrain changes.

The correctness, consistency, efficiency, and flexibility of the terrain model is directly reflected in its implementation in actual problems. The way the terrain model is developed will suggest, to a certain extent, the complexity and accuracy of the solution of related problems. Moreover, it will suggest solutions to a broader class of problems.

So far, the problem of terrain modelling has not been sufficiently and consistently studied. In most of the cases, it has been assumed that, at least locally, the terrain elevation is characterized by a covariance
function of some exponential form (usually Gaussian). Such a covariance function has been either a-priori modelled, or fitted to some empirical data. In other instances, the terrain behavior is assumed to be similar to that of a spline, or some polynomial. In all cases the detection and proper treatment of the breaklines has remained a problem. All this piecemeal, and usually inconsistent practice in modelling terrain reflects in the solution of terrain related problems. The result was that problems like, interpolation and prediction of terrain elevations, breakline detection, drainage pattern recognition, generalization, computation of optimum sampling interval, on-line quality control of measurements, compression and storage of terrain data, etc. have been studied as if there was no connection between them.

It is clear, therefore, that what is needed is the development of a unified theoretical frame for the solution of terrain related cartographic problems. In this study such a theoretical frame is investigated and an efficient terrain modelling scheme, based on the theory of random fields is developed. The theory of random fields has been used--in one form or another--in the solution of problems in Geodesy, Photogrammetry, and Cartography. Using covariance functions, or power spectra in describing physical processes, assuming
randomness in measurement errors, or adopting Gauss-Markov models in estimation procedures, are all part of the theory of random fields. In this study we deal with a particular class of random fields: random fields with stationary increments.

Using the developed theoretical background, the problem of terrain generalization is solved. This is a problem which concerns the cartographic community for long, and it is believed, that the approach followed here gives a new insight to its solution. It is moreover anticipated, that the developed terrain model can help in the solution of other terrain related problems (like those mentioned above), as well. Therefore further research on the subject is recommended.

1.2 Objectives of the study

The principal objective of this research is twofold: The development of the theory of fractional random fields as related to cartography, and the application of the developed theory to the problem of terrain generalization.

Firstly, we review the theory of random fields and fractional calculus, and introduce the relevancy of the application to the cartographic field. The developed theory is then used to describe and simulate terrain
heights. This leads to a development of a unified theoretical frame for the solution of terrains related cartographic problems.

Secondly, since the terrain continuum is one of the main features appearing in topographic maps, its proper generalization is essential in the process of reducing the map scale. Usually terrain elevations are represented as contour lines, profiles, or grid data. Therefore the problem of generalization of terrain elevation reduces to the problem of finding a way to properly generalize one of its representation forms. In this study I will apply our terrain modelling technique, which is based on the Random Field theory, in solving the terrain generalization problem. I assume that the terrain is represented by profiles and I will try to prove that the proposed terrain model is very flexible in dealing with terrain generalization.

1.3 Organization of the report

The second chapter is devoted to a short review of the related theory, the description of stationary Random Fields, and the representation and description of the properties of Random Fields with stationary increments. I choose to work with spectral representations of Random
the subsequent discussion. Also useful is the formulation of the differential and integral operations on Random Fields.

The third chapter deals with the rather unfamiliar subject of Fractional Calculus. The notions of fractional integration and differentiation are introduced and their Laplace Transforms are studied. The purpose of this chapter is to introduce definitions and basic properties of linear operations of fractional order.

The purpose of the fourth chapter is twofold: Firstly, it expands the definitions of linear operations on random fields to fractional order. Using fractional integration on white noise, we are able to develop a random field called fractional white noise. Such a field has stationary increments and is later on used to model terrain. Secondly, the Weierstrass function is studied. It turns out that such a function can also give rise to random fields with stationary increments. An approximation to this function has been suggested, in order to develop to a simulation algorithm.

The simulation of fractional white noise is the subject of chapter five. Two simulation techniques have been suggested. Examples and computational considerations are also presented. These examples show surprising resemblance to actual terrain, suggesting that it is legitimate to model terrain as a fractional integral on
white noise. It is believed that such a terrain simulation method is very useful in instances of algorithm testing and accuracy evaluation, animation, and terrain classification. Besides, the simulation scheme is simple and easily invertible.

The inversion of the simulation scheme is dealt with at the sixth chapter, and shows the flexibility of the proposed terrain model. After introducing some robustification aspects of our estimation problem, we enter an estimation process in order to recover the underlying white noise. Mathematical and computational details are discussed and an example of the inversion is given. The inverse operation gives new insights in cartographic problems such as data compression, interpolation and prediction, terrain generalization, and breakline treatment.

In order to show the applicability of the developed theory to actual cartographic problems, we choose to tackle the problem of terrain generalization, which is a direct application of the inverse operation. The developed algorithm is proved to perform well in different terrain types, in the presence of breaklines, and in cases of non-homogeneous terrain.

Four actual terrain profiles are processed and are included as examples of the algorithm's performance. Finally concluding remarks are included in the last
chapter, whereas topics not directly relevant to the problems under study are left for the Appendices.
CHAPTER II
RANDOM FIELDS

2.1 Introduction

Given an n-dimensional, physical phenomenon, whose spatial behavior and development in time is controlled by probabilistic laws, we call its mathematical abstraction a random field. A mathematically rigorous definition will be given later, but for the time being the above definition is sufficient to explain the extended use of the random field theory in problems concerning physical phenomena.

Early investigations, which date back to the end of the 19th century, laid the foundations of the modern theory of random fields. Classical references like Gnedenko (1963), Doob (1960), Yaglom (1962), Gikhm and Skorokhod (1969), Loève (1963), and Prabhu (1965), study the subject in detail and create the mathematical apparatus for dealing with random fields.

Special classes of random fields, like Markov fields, Gaussian fields, stationary fields, or random fields with stationary increments, have drawn special attention, due to
their role in solving many physical problems. After the work of Kolmogorov (1941) and Wiener (1949), in estimation theory, the practical significance of the random fields has become particularly apparent. The resulting correlation theory is a well established mathematical tool with a great importance in the practical applications.

The term "random field" is normally used in the literature for the description of an n-dimensional phenomenon. The physical phenomenon we are dealing with, in this study, is the earth's terrain, described by profiles. Therefore our phenomenon is one dimensional and it is customary in this case to use the term "random function" instead.

Following the historically conditioned definition of a random field [Yaglom (1962)], let $T$ be an arbitrary set of elements $t, s, \ldots$. By a random field on $T$ we mean a function $\xi(t)$ whose values are random variables. We regard the function $\xi(t)$ as being specified if for any elements $t_1, t_2, \ldots t_n \in T$, we are given the distribution function

$$ F_{t_1, t_2, \ldots, t_n}(x_1, x_2, \ldots, x_n) = P(\xi(t_1) < x_1, \xi(t_2) < x_2, \ldots, \xi(t_n) < x_n) \quad (2.1) $$

of the $n$-dimensional random variable $(\xi(t_1), \xi(t_2), \ldots, \xi(t_n))$

The variable $t$ takes real values and in general is
interpreted as time, whereas the set $T$ normally consists of the set of all the real numbers $-\infty < t < \infty$.

Bearing in mind that a random function is nothing but a random field describing a one-dimensional physical phenomenon, I will use the more general term "random field" hereafter. The reason is that I believe that the techniques described here are readily applicable to terrain viewed as a two-dimensional phenomenon, as well as other three-dimensional phenomena like the earth's gravity field etc.

2.2 Basic notions in random field theory

As we have seen a random field is completely defined if all its finite-dimensional distribution functions (2.1) are known. Following the classical theory of Wiener (1949) and Yaglom (1962), we will restrict ourselves to the study of the first two moments of their distribution functions: the mean and the correlation function.

By definition the mean value of the random field $\xi(t)$ is

$$m(t) = E \{\xi(t)\} \quad (2.2)$$

and the correlation function is

$$B(t,s) = E \{\xi(t) \xi(s)\} \quad (2.3)$$
Of course we can always consider the "fluctuation" of a random field, ie. $\xi(t) - m(t)$ instead of the $(t)$ itself. In other words, we can always reduce our study to the case of random fields with zero mean value. The covariance function is defined as the centered function

$$B(t,s) - m(t) m(s)$$

and is very often encountered in the literature. In the case of $m(t) = \emptyset$, the notions of correlation and covariance function coincide, and we can thus refer to the function $B(t,s)$ either as the correlation or the covariance function.

In practice, many physical processes can be either considered to be Gaussian, due to the law of large numbers, or turn out to be Gaussian. In the case of normal or Gaussian random fields, the mean value $m(t)$ and the correlation function $B(t,s)$ determine all the distribution functions (2.1), since all these finite-dimensional distribution functions are multi-dimensional Gaussians [Yaglom (1962)]. Thus they completely specify $\xi(t)$. Therefore the study of the first two moments of Gaussian random fields is sufficient in the solution of problems related to those fields. Moreover, given a random field $y(t)$, a corresponding Gaussian field $x(t)$ exists, with the same means and covariances [Theorem
3.1, Doob (1960), p. 72]. This means that in a discussion involving only the mean and the correlation function of the \( y(t) \), it is no restriction to assume that the \( y(t) \) field is Gaussian.

Another important notion in the theory of random fields is "stationarity." Following the classical theory [Yaglom (1962)], the random field \( \xi(t) \) will be called stationary if all the finite-dimensional distribution functions (2.1) defining \( \xi(t) \) remain the same under translation:

\[
F_{t_1+r, t_2+r, \ldots, t_n+r}(x_1, x_2, \ldots, x_n) = F_{t_1, t_2, \ldots, t_n}(x_1, x_2, \ldots, x_n) \quad (2.4)
\]

for any \( t_1, t_2, \ldots, t_n, n \) and \( r \).

A stationary random field \( \xi(t) \) is characterized by a constant \( m \), which by the previous argument we consider to be zero, and by a function \( B(\tau) = B(t-s) \) of a single variable \( \tau \).

In general we may have different realizations of a random field in time. For the estimation of the mean value and the correlation function, we should consider a large number of realizations of the field \( \xi(t) \). However in practice, this is either not efficient, or not possible at all, since many physical processes cannot be realized in time more than once. However the ergodic theorem is applicable to stationary random fields (or at least to
those usually encountered in practice). The practical value of the notion of stationarity and ergodicity of random fields is evident by the fact, that we can compute the mean and the correlation function of a stationary random field by using just one realization.

2.3 Spectral representation of stationary random fields

It has been shown [Doob (1960), p.430,486, Yaglom (1962, p.36] that an arbitrary stationary field $\xi(t)$ can be represented by the following Fourier-Stieltjes integral:

$$\xi(t) = \int_{-\infty}^{\infty} e^{i\lambda t} dZ(\lambda) \quad (2.5)$$

where the function $Z(\lambda)$ is a random field satisfying the conditions.

$E[Z(\lambda)] = 0$ for all $\lambda$ \hspace{1cm} (2.6)

$E[(Z(\lambda_1 + \Delta\lambda_1) - Z(\lambda_1))(Z(\lambda_2 + \Delta\lambda_2) - Z(\lambda_2))] = 0$ \hspace{1cm} (2.7)

if the intervals $(\lambda_1, \lambda_1 + \Delta\lambda_1)$, and $(\lambda_2, \lambda_2 + \Delta\lambda_2)$ are disjoint. Such fields satisfying the conditions (2.6) and (2.7) are called random fields with uncorrelated increments. In other words, it is always possible to represent a stationary random field, as a sum of mutually
orthogonal stationary (in a wide sense) fields $Z(t)$, through the generalized Stieltjes integral (2.5).

The representation (2.5) has a very useful physical meaning. It explains the possibility of separating frequencies corresponding to different parts of the spectrum. That is why the representation (2.5) is often called the spectral decomposition of the stationary random field $\xi(t)$. In addition to that, it dictates a technique for simulating stationary random fields, as sums of periodic oscillations with given frequencies and random amplitudes and phases.

As we have seen, the correlation function $B(t,s)$ of a random field $\xi(t)$ is given by eq. (2.3) and, in the case of a stationary random field, it only depends on the difference $t-s = \tau$. Using the representation (2.5), we re-write (2.3) as

$$B(t-s) = E \left[ \left\{ \int_{-\infty}^{\infty} e^{i\lambda t} \, dZ(\lambda) \right\} \left\{ \int_{-\infty}^{\infty} e^{i\lambda s} \, dZ(\lambda) \right\} \right] =$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\lambda(t-s)} \, E(dZ(\lambda) \, dZ(\lambda)) =$$

$$= \int_{-\infty}^{\infty} e^{i\lambda(t-s)} \, E \{dZ(\lambda) \, dZ(\lambda)\} =$$

$$= \int_{-\infty}^{\infty} e^{i\lambda(t-s)} \, E \{dZ(\lambda)\}$$

(2.8)

Defining a non decreasing bounded function $F(\lambda)$ such that
$$F(\lambda+\Delta\lambda) - F(\lambda) = E \{Z(\lambda+\Delta\lambda) - Z(\lambda)\}$$
or

\[ dF(\lambda) = E \left| dZ(\lambda) \right|^2 > 0 \quad (2.9) \]

the correlation function is

\[ B(t-s) = \int e^{i\lambda(t-s)} dF(\lambda) \quad (2.9) \]

or

\[ B(\tau) = \int e^{i\lambda\tau} dF(\lambda) \quad (2.10) \]

The function \( F(\lambda) \) is called the spectral distribution function of the stationary random field \( \xi(t) \). Eq. (2.10) shows how the correlation function is determined from the spectral distribution function. For a given \( B(\tau) \) we can obtain \( F(\lambda) \) by using the inversion formula for characteristic functions [Gnedenko (1963), p.256, Prabhu (1965), p.32]:

\[ F(\lambda_2) - F(\lambda_1) = 4\text{Im} \lim_{T \to \infty} \frac{1}{2\pi} \int_{-T}^{T} \frac{e^{-i\lambda_1 t} - e^{-i\lambda_2 t}}{it} B(t) dt \quad (2.11) \]

If we define the function \( f(\lambda) \) as the spectral density function of the field \( \xi(t) \), such that:

\[ f(\lambda) = F'(\lambda) \quad (2.12) \]
or

\[ F(\lambda) = \int_{-\infty}^{\lambda} f(\lambda) d\lambda \]  \hspace{1cm} (2.13)

Then eq. (2.10) reads:

\[ B(\tau) = \int_{-\infty}^{\infty} e^{i\lambda \tau} f(\lambda) d\lambda \]  \hspace{1cm} (2.14)

provided that

\[ \int_{-\infty}^{\infty} |B(\tau)| d\tau < \infty \]  \hspace{1cm} (2.15)

Eq. (2.14) obviously shows that the correlation function \( B(\tau) \) and the spectral density function \( f(\lambda) \) are a Fourier pair. Therefore by the inverse Fourier Transform we obtain:

\[ f(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\lambda \tau} B(\tau) d\tau \]  \hspace{1cm} (2.16)

2.4 The differential and integral operators

Let \( \xi(t) \) be a random stationary field with spectral representation given by (2.5). By a linear operation on \( \xi(t) \) we mean a transformation transforming the field \( \xi(t) \)
into a $\hat{\xi}(t)$ field, where $\hat{\xi}(t)$ is either a finite sum of the form:

$$\hat{\xi}(t) = \int_{-\infty}^{\infty} e^{i\lambda t} \left[ \sum_j C_j e^{it j \lambda} \right] dZ(\lambda)$$  \hspace{1cm} (2.17)

or a limit in the mean of such finite sums [Doob (1960)].

The most general $\hat{\xi}(t)$ field is given by:

$$\hat{\xi}(t) = \int_{-\infty}^{\infty} e^{i\lambda t} C(\lambda) dZ(\lambda)$$  \hspace{1cm} (2.18)

The function $C(\lambda)$ is called the gain of the operator. Thus every linear operator has a gain, and every gain determines a linear operation, which defines a new (stationary) random field [Doob (1960)].

Suppose that:

$$\int_{-\infty}^{\infty} \lambda^2 dF(\lambda) < \infty$$  \hspace{1cm} (2.19)

and consider the gain

$$C(\lambda) = i\lambda$$  \hspace{1cm} (2.20)

since

$$\frac{\xi(t+h) - \xi(t)}{h} = \int_{-\infty}^{\infty} \frac{e^{i(t+h)\lambda} - e^{it\lambda}}{h} dZ(\lambda)$$  \hspace{1cm} (2.21)
and

$$\lim_{h \to 0} \frac{e^{i(t+h)} - e^{i t \lambda}}{h} = i \lambda \cdot e^{i t \lambda}$$  \hspace{1cm} (2.22)

then

$$\hat{\xi}(t) = \xi'(t) = \lim_{h \to 0} \int_{-\infty}^{\infty} \frac{e^{i(t+h)} - e^{i t \lambda}}{h} \ dZ(\lambda) =$$

$$= \int_{-\infty}^{\infty} e^{i t \lambda} \cdot i \lambda \cdot dZ(\lambda)$$  \hspace{1cm} (2.23)

In this way we have the spectral representation of the derivative $\xi'(t)$ of the stationary random process $\xi(t)$.

The correlation function of the new process $\xi'(t)$ is given as

$$B^{[1]}(\tau) = E \{ \xi'(t+\tau) \xi'(\tau) \} = \int_{-\infty}^{\infty} e^{i t \lambda} \cdot \lambda^2 \cdot dF(\lambda)$$  \hspace{1cm} (2.24)

and the spectral distribution function, as

$$F^{[1]}(\lambda) = \int_{-\infty}^{\lambda} \lambda^2 \ dF(\lambda)$$  \hspace{1cm} (2.25)

Considering eq. (2.16) and (2.24) we get the spectral density function of the derivative $\xi'(t)$ as

$$f^{[1]}(\lambda) = \lambda^2 \cdot f(\lambda)$$  \hspace{1cm} (2.26)
By further differentiation of the field $\xi'(t)$, we obtain the second derivative $\xi''(t)$, and so on for higher derivatives. Generalizing the above formulas, it can be shown [Yaglom, (1962)] that:

$$\xi^{[n]}(t) = \int_{-\infty}^{\infty} e^{it\lambda} (i\lambda)^n dZ(\lambda) ;$$  \hspace{1cm} (2.27)

with the corresponding correlation function

$$B^{[n]}(\tau) = \int_{-\infty}^{\infty} e^{it\lambda} (\lambda^2)^n dF(\lambda) ;$$  \hspace{1cm} (2.28)

the spectral distribution function

$$F^{[n]}(\lambda) = \int_{-\infty}^{\infty} (\lambda^2)^n dF(\lambda) ,$$  \hspace{1cm} (2.29)

and the spectral density function

$$f^{[n]}(\lambda) = \lambda^{2n} \cdot f(\lambda)$$  \hspace{1cm} (2.30)

provided that:

$$\int_{-\infty}^{\infty} \lambda^{2n} dF(\lambda) < \infty$$  \hspace{1cm} (2.31)

Suppose now that the gain function $C(\lambda)$ can be expressed as the Fourier Transform of an integrable function.
The corresponding linear operation can be identified with an integral averaging

\[ \hat{\xi}(t) = \int_{-\infty}^{\infty} e^{i\lambda t} \, C(\lambda) \, d\lambda = \int_{-\infty}^{\infty} e^{i\lambda t} \, d\lambda \int_{-\infty}^{\infty} e^{-i\lambda \tau} \, g(\tau) \, d\tau = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(\tau) \, \xi(t-\tau) \, d\lambda \, d\tau \]

\[ \int_{-\infty}^{\infty} |g(\lambda)|^2 \, d\lambda < \infty \]  

under the condition that:

Summarizing we show that the integral of a stationary random field \( \xi(t) \) is represented by:

\[ \hat{\xi}(t) = \int_{-\infty}^{\infty} g(\tau) \, \xi(t-\tau) \, d\tau \]  

(2.32)
We should also note that the gain function $C(\lambda)$ and the function $g(\tau)$ form a Fourier pair.

By analogy to the differential operator, the correlation function of the integral operator $\hat{\xi}(t)$ is

$$\hat{B}(\tau) = \int_{-\infty}^{\infty} e^{it\lambda} |C(\lambda)|^2 dF(\lambda)$$  \hspace{1cm} (2.36)$$

The spectral distribution function is

$$\hat{F}(\lambda) = \int_{-\infty}^{\infty} |C(\lambda)|^2 dF(\lambda)$$  \hspace{1cm} (2.37)$$

and the corresponding spectral density function is

$$\hat{f}(\lambda) = |C(\lambda)|^2 f(\lambda)$$  \hspace{1cm} (2.38)$$

For higher order iterated integration, the formulas (2.28)-(2.30) are still valid. In section 4.2 we will return to this subject to define the functions $C(\lambda)$ and $g(\tau)$ as they pertain to fractional integration.

2.5 Random fields with stationary increments

Consider a stationary random field $\xi(t)$. Let the indefinite integral of $\xi(t)$ be:

$$\xi(t) = \int_{0}^{t} \xi(s) \, ds$$  \hspace{1cm} (2.39)$$
The resulting random field $\xi(t)$ is not stationary. It belongs however to a special class of random fields: random fields with stationary increments [Doob (1953), p. 552, example 1). A field $\xi(t)$ for $t \in T$, is called a random field with stationary increments if the joint distribution function of the differences

$$\xi(t + \tau_2) - \xi(t + \tau_1)$$
$$\xi(t + \tau_3) - \xi(t + \tau_2)$$
$$\vdots$$
$$\xi(t + \tau_n) - \xi(t + \tau_{n-1})$$

is independent of $t$ for arbitrary $n$, $t$ and for $\tau_1, \tau_2, \ldots, \tau_n$ such that $t + \tau_i \in T$ (for $i = 1, 2, \ldots, n$) [Gikham & Skorokhod (1969), p. 4]. This is equivalent to the following: the process $\xi(t)$ is said to have stationary increments if the correlation function of the increments

$$B(t_1, t_2; t_3, t_4) = \mathbb{E} \left[ (\xi(t_2) - \xi(t_1)) (\xi(t_4) - \xi(t_3)) \right] \quad (2.40)$$

is independent of translations along the $t$-axis, i.e.

$$B(t_1+t, t_2+t; t_3+t, t_4+t) = B(t_1, t_2, t_3, t_4) \quad (2.41)$$

In the discrete parameter case the random field $\xi(t)$ has stationary increments if and only if the differences $\xi(t_1) - \xi(t_0), \xi(t_2) - \xi(t_1), \ldots$, constitute a stationary
field, in which extent we can write [Doob (1960), p.552, Yaglom (1962), p.92]:

$$\xi(t) - \xi(s) = \int_{-\infty}^{\infty} \frac{e^{it\lambda} - e^{is\lambda}}{i\lambda} \ (1 + \lambda^2)^{1/2} \ dZ(\lambda) \quad (2.42)$$

where the random field $Z(\lambda)$ has uncorrelated increments and

$$E \left| dZ(\lambda) \right|^2 = dF(\lambda) \quad (2.43)$$

the correlation function of the process $\xi(t)$ is

$$B(t,s) = E\{\xi(t) \xi(s)\} \quad (2.44)$$

Denoting by $A\xi(t)$ the stationary increment of $\xi(t)$ and having $\tau = t-s$, we can re-write (2.42) as:

$$A\xi(t) = \int_{-\infty}^{\infty} e^{it\lambda} \ (1 - e^{-i\tau\lambda}) \frac{(1+i\lambda)}{i\lambda} \ dZ(\lambda) \quad (2.45)$$

The function $F(\lambda)$ satisfying (2.43) is a bounded non-decreasing function, and is called the spectral distribution function of the stationary random increment $A\xi(t)$ [Yaglom (1955)]. If it is absolutely continuous its derivative

$$f(\lambda) = F'(\lambda) \quad (2.46)$$

will be called the spectral density function of $A\xi(t)$. 
Random fields with stationary increments can be further generalized by introducing fields with stationary increments of the second or higher order. The general theory of random fields with stationary increments of arbitrary order $n$, is a natural outgrowth of the above [Yaglom (1955), Yaglom (1962)]. In this case eq. (2.45) reads:

$$
\Delta^{[n]} \zeta(t) = \int_{-\infty}^{\infty} e^{it\lambda} (1-e^{i\tau \lambda})^n \frac{(1+i\lambda)^n}{(i\lambda)^n} dZ(\lambda) \quad (2.47)
$$

As in the case of stationary random fields, here we can also consider the "fluctuations" of $\zeta(t)$, that is the random field $\zeta(t) - E\{\zeta(t)\}$ instead of $\zeta(t)$. However it should be noted that the mean value $E\{\zeta(t)\}$ need not exist in general.

As far as the correlation function is concerned, in the case of random fields with stationary increments, its counterpart is the function:

$$
D(t,s; u,v) = E\{\{\zeta(u) - \zeta(t)\} \{\zeta(v) - \zeta(s)\}\} \quad (2.48)
$$

Such a function is called the structure function and has been extensively used in other works, e.g. Yaglom (1955), Yaglom (1962), Doob (1960), Von Neumann and Schoenberg (1941), Kolmogorov (1940).

As shown by Yaglom (1962), the structure function contains less information than the correlation function.
Knowing the correlation function we can determine the structure function as follows:
\[
D(t; u; v; s; v) = B(u; v) - B(u; s) - B(t; v) + B(t; s)
\]  \hspace{1cm} (2.49)

However, if we only know the structure function, the correlation function cannot be uniquely determined.

An important result of the definition of random fields with stationary increments is that the mathematical expectation of the increment \( \Delta \xi(t) \), during any time interval, is proportional to the length of the interval \((s-t)\), \cite{Yaglom1962}, i.e.
\[
E[\Delta \xi(t)] = \alpha (s-t)
\]  \hspace{1cm} (2.50)

where \( \alpha \) is a constant, and the function \( D(t; u; v) \) of the field \( \xi(t) \) depends only on the differences \((u-t)\) and \((v-t)\)
\[
D(t; u; v) = D(u-t, v-t)
\]  \hspace{1cm} (2.51)

We have seen that for the process \( \xi(t) \), its stationary increment of the first order is given by eq. (2.46). In this case the structure function can be easily proven to be \cite{Doob1960, p.552, Gikhm & Skorokhod1969, p.33, theorem 4}:
\[
D(t; \tau_1, \tau_2) = \int_{-\infty}^{\infty} e^{i\lambda t} (1-e^{-i\tau_1 \lambda}) (1-e^{i\tau_2 \lambda}) \frac{(1+i\lambda^2)}{\lambda^2} \, dF(\lambda)
\]  \hspace{1cm} (2.52)

where the function \( F(\lambda) \) has been previously defined, provided that
\[ \int_{-\infty}^{\infty} \lambda^2 \, dF(\lambda) < \infty \]  (2.55)

Similarly, the stationary increment \( \Delta[n] \zeta(t) \) of the \( n \)th order is given by eq. (2.47) and then the structure function is given as [Yaglom, (1955)]:

\[ D[n](t; \tau_1, \tau_2) = \int_{-\infty}^{\infty} e^{it\lambda} \left( 1 - e^{-i\tau_1 \lambda} \right)^n \left( 1 - e^{-i\tau_2 \lambda} \right)^n \frac{(1+\lambda^2)\lambda}{\lambda^{2n}} \, dF(\lambda) \]  (2.54)

It should be noted that as we have seen in the case of stationary fields, the correlation function \( B(\tau) \) is the Fourier transform of the spectral density function \( f(\lambda) \). This is not always true for the field \( \zeta(t) \). Whenever it exists, we can determine the spectral distribution function \( F(\lambda) \) (and therefore its derivative \( f(\lambda) \)) by inverting the integrals in eq. (2.52) and (2.54), according to the inversion formula for characteristic functions [see eq. (2.11)].

Returning back to the example given in the beginning of this section, it can be proven (Doob (1960), p.558) that a field with stationary increments has the form (2.39); that is, the random field \( \zeta(t) \) has a derived field \( \xi(t) \) such that \( \xi(t) = \zeta'(t) \), if and only if

\[ \int_{-\infty}^{\infty} \lambda^2 \, dF(\lambda) < \infty \]  (2.55)
CHAPTER III

FRACTIONAL OPERATORS

3.1 Introduction to Fractional Calculus

In mathematics, the concept of generalized differentiation and integration, that is the differential and integral operators of arbitrary (irrational, fractional, or complex) order is an old subject. Mathematicians, as early as, in 17th century, introduced the notion of fractional calculus, while attempts for formal definitions of derivatives of fractional order date back to 1815. Interesting historical details can be found in Davis (1936), and Oldham & Spanier (1974).

When dealing with the definition of fractional operators, Ross's (1975) explicit formulation of the problem is clear:

"For every function f(z), z = x + iy, of a sufficiently wide class, and every number v, irrational fractional, or complex, a function of \( \mathrm{cD}_z^v f(z) = g(z) \) (where \( c \) and \( z \) are the integration limits) could be assigned, subject to the following criteria:
1. If \( f(z) \) is an analytic function of the complex variable \( z \), the derivative \( cD_z^v f(z) \) is an analytic function of \( v \) and \( z \).

2. The operation \( cD_x^v f(x) \) must produce the same result as ordinary differentiation when \( v \) is a positive integer. If \( v \) is a negative integer, say \( v = -n \), then \( cD_x^{-n} \) must produce the same result as ordinary \( n \)-fold integration and \( cD_x^{-n} f(x) \) must vanish along with its \((n-1)\) derivatives at \( x = c \).

3. The operation of order zero leaves the function unchanged:

\[
cD_x^0 f(x) = f(x)
\]

4. The fractional operators must be linear:

\[
cD_x^{-v} (af(x) + bg(x)) = acD_x^{-v} f(x) + bcD_x^{-v} g(x)
\]

5. The law of exponents (or indices) for integration of arbitrary order should hold:

\[
cD_x^{-u} cD_x^{-v} f(x) = cD_x^{-(u+v)} f(x)
\]
Although at present, there exist many definitions of fractional operators we choose to work with the original Riemann-Liouville definition, which will be formally discussed in the following section.

Practical applications of the theory of fractional operators have not been fully studied so far, primarily due to unfamiliarity. Nevertheless, in the last several years increasing interest in the subject has been shown. In fact, fractional calculus can be used in many diverse areas, and its applications in diffusion problems are well known. We will see that fractional operators can be used advantageously in the solution of terrain related problems, as well.

3.2 Fractional integration

Among the various notations in existence today, we prefer to use the one used in Davis (1936), because of its compactness. We denote by

\[ cD_x^{-\nu} f(x) \]

integration of arbitrary order \( \nu > 0 \). The subscripts \( c \) and \( x \) denote the limits of integration of the definite integral.
Let us consider now the Cauchy integral of a function $f$. For $v = 1, 2, 3, \ldots$ we have:

$$\oint_{D^{-v}} f(x) = \frac{d^{-v}f(x)}{dx^{-v}} = \int_0^x f(y)dy \quad (3.1)$$

where the symbol $\oint$ stands for a $v$-fold integral. Eq. (3.1) can be also written as:

$$\int_0^x f(y)dy = \frac{1}{v!} \cdot \frac{dv}{dx} \int_0^x (x-y)^v f(y)dy \quad ; \quad v = 0, 1, 2, \ldots \quad (3.2)$$

This can be easily proven using the Leibnitz theorem for the $v$-fold integration:

$$\int_0^x (x-y)^v f(y)dy = \int_0^x \frac{dv}{dx} [(x-y)^v] f(y)dy + (x-y)^v f(y) \bigg|_{x, x}$$

$$= \int_0^x v! f(y)dy \quad (3.3)$$

Therefore:

$$\frac{1}{v!} \oint_{D^{-v}} (x-y)^v f(y)dy = \frac{1}{v!} \cdot \int_0^x f(y)dy = \int_0^x f(y)dy \quad (3.4)$$

using now the integral transform definition of the function:

$$\Gamma(x) = \int_0^x t^{x-1} e^{-t} dt \quad \text{for} \quad x > 0 \quad (3.5)$$
and its useful properties:

\[ \Gamma(x-1) = x! \]

\[ \Gamma(x-1) = x\Gamma(x) \tag{3.6} \]

\[ \frac{1}{\Gamma(x)} \sim \frac{x^{1/2-x}}{\sqrt{2\pi}} \exp(x) \quad ; \quad x \to \infty \]

we re-write eqs. (3.1) & (3.2) as:

\[ \int_0^x (x-y)^{v-1} f(y) \, dy \tag{3.7} \]

Further generalizing the above formula for arbitrary real \( v > 0 \), we are lead to Riemann-Liouville definition of the fractional integral operator.

It should be noted that the introduction of \( \Gamma(v) \) as denominator in eq. (3.7) ensures that for \( v \) integer, the fractional integral becomes an ordinary \( v \)-fold integral. Another noticeable fact is the appearance of forms analogous to (3.7), in the extension of the definition of the gamma function in the complex plane [Oldham & Spanier (1974), pp. 16-23]. This is not surprising, since the gamma function plays such a fundamental role in defining differentials and integrals of fractional order.

3.3 Fractional differentiation

Let \( v = r-k \) where \( r \) is an integer and \( r-1 < v < r \), and \( 0 < k < 1 \). Then using the law of exponents (see section 3.1, property 5) we have:
This is precisely the same definition given by Ramanujan's theorem V, in his third quarterly report [Berndt (1985), p.330], for the fractional derivative.

The question of extending the definition (3.7) of fractional integration to fractional differentiation is answered by letting \( v \) be real and larger than zero. Then

\[
_oD_x^{-v} f(x) = _oD_x^r _oD_x^{-k} f(x) = \frac{dr}{dx^r} \frac{1}{\Gamma(k)} \int_0^x (x-y)^{k-1} f(y)dy
\]  

(3.8)

where

\[
-v = r-k
\]  

(3.10)

and \( r \) is an integer. Letting \( r = 0 \) then \( v = k \) and the definitions of fractional integration (3.7) and differentiation (3.8) read:

\[
_oD_x^{-v} f(x) = \frac{1}{\Gamma(v)} \int_0^x (x-y)^{v-1} f(y)dy
\]  

(3.11)

\[
_oD_x^v f(x) = \frac{1}{\Gamma(-v)} \int_0^x (x-y)^{-v-1} f(y)dy
\]  

(3.12)
From the above discussion it is clear that both the fractional derivative and the fractional integral can be represented by the same formulation, and for their formal representation, we refer to Nishimoto's Unification Theorem [Nishimoto (1984), p.6], which for \( v \in \mathbb{R} \) reads (in our formulation):

If \( f(z) \) is an analytic function and \( _0D^v_x f(z) \) is given by eqs. (3.11) or (3.12), then

\[
_0D^v_x f(z) \begin{cases} 
\text{derivative for } v > 0 \\
\text{original for } v = 0 \\
\text{integral for } v < 0 
\end{cases}
\]

For every \( v \in \mathbb{R} \), if \( _0D^v_x f(z) \) exists. This is the reason why the expressions (3.11) and (3.12) are sometimes discussed in the literature as "differintegrals."

In the next section we are going to expose some aspects of the Laplace Transform of fractional differintegrals. These should be viewed as extensions of the differential and integral operators on random fields presented in the previous chapter, and as the first steps in formulating our terrain modelling problem.
3.4 The Laplace transform of fractional operators

The use of Laplace Transformation as a means for solving certain types of differential equations is probably too well known to require more than a passing reference here. In this study we will only try to relate the Laplace Transforms $L\{0D_x^V f(x)\}$ and $L\{0D_x^{-V} f(x)\}$ of the fractional differential and integral operators to the Laplace Transform of the differintegrable function $L\{f(x)\}$.

Defining the Laplace Transform of a function $f(x)$ as:

$$L\{f(x)\} = \int_0^\infty e^{-sx} f(t)\,dt \quad (3.13)$$

it is well known that:

$$L\{f'(x)\} = \int_0^\infty e^{-sx} f'(x)\,dx = e^{-sx} f(x) \bigg|_0^\infty + s \cdot \int_0^\infty e^{-sx} f(x)\,dx \quad (3.14)$$

Let $f(x)$ be of order $e^{ax}$ as $x$ approaches infinity.

By that we mean that the product $e^{-at} |f(t)|$ is bounded for all $t$ greater than some finite number $T$, and $a$ is some constant [Churchill (1972), p.6]. In other words the condition that $f(x)$ is $O(e^{at})$ is sufficient for the convergence of the Laplace integral (3.13), and therefore ensures the existence of the Laplace transform. Then whenever $s > a$, the first term becomes $-f(0)$ and so

$$L\{f'(x)\} = s \cdot L\{f(x)\} - f(0) \quad (3.15)$$
Generalizing for the \( n^{th} \) derivative of \( f(x) \) we have:

\[
L\{f^{[n]}(x)\} = s^n \cdot L\{f(x)\} - \sum_{k=0}^{n-1} s^{n-1-k} f^{(k)}(0) \quad (3.16)
\]

Further generalizing for \( v > 0 \) and real it can be proved that \( [\text{Oldham & Spanier (1974), p.133}] \):

\[
L\{\,_{0}D_{x}^{v} f(x)\} = s^v \cdot L\{f(x)\} - \sum_{k=0}^{r-1} s^{k} \cdot \,_{0}D_{x}^{-1-k} f(0) \quad (3.17)
\]

where \( r-1 < v < r \). It turns out that eq. (3.17) is also valid for \( v < 0 \). It thus gives the Laplace Transform of both the fractional integral and fractional differential operators.

For example, in the case of fractional integration, the summation in (3.17) vanishes and we are left with:

\[
L\{\,_{0}D_{x}^{-v} f(x)\} = s^v \cdot L\{f(x)\} \quad (3.18)
\]

(See also \( \text{Lavoie et.al. (1976), p.246-247.} \))

We recall that the difference between the Laplace transformation and the Fourier transformation is only that one of the axes is imaginary, in the latter. Thus eq. (3.18) extends the meaning of eq. (2.32), in the case of fractional differintegration. This extension of the meaning is necessary for our work, and we will return to eq. (3.18) in later sections.
3.5 Numerical evaluation of fractional integration and differentiation

For reasons of completeness we will discuss here the numerical aspects of fractional operators. For the numerical evaluation of eq. (3.11) and (3.12) many algorithms exist. The most efficient and relatively simple one is the R-L algorithm [Oldham & Spanier (1974), p. 136-148], which is based on the Riemann-Liouville definition:

\[ \mathcal{D}_x^{-\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_0^x (x-y)^{\nu-1} f(y) \, dy \]

The above equation can be written as:

\[ \mathcal{D}_x^{-\nu} f(x) = \frac{1}{\Gamma(\nu)} \sum_{i=0}^{N-1} \frac{(ix+x)/N}{y^{\nu+1}} \, dy \]  

(3.20)

from where it is clear that it is possible to produce a number of so-called R-L algorithms, each arising from a different approximation to the integral. Setting, for example:

\[ \left[ \frac{f(x-y)}{y^{\nu+1}} \right.] \, dy \approx \left[ \frac{(ix+x)/N}{[1+i - \frac{Ny}{x}] \cdot f(x - \frac{ix}{N}) + \left[ \frac{Ny}{x} - i \right] \cdot f(x - \frac{x}{N} - \frac{ix}{N})} \right] \, dy \]  

(3.21)
based on a linear interpolation between $f_i$ and $f_{i+1}$, we are directly led to our R-L algorithm:

$$D^{-v}_x f(x) = \frac{x^v N^{-v}}{\Gamma(v)} \sum_{i=0}^{N-1} \left[ f_{i+1} - f_i \right] \frac{[i+1]^v - i^v}{v} +$$

$$+ \frac{f_{i+1} - f_i}{1+v} \left[ [i-1]^{1+v} - i^{1+v} \right]$$

(3.22)

where $N+1$ are evenly spaced points in the range of $(0,x)$, of the independent variable, where the value of $f(x)$ is known. All the R-L algorithms are expressible as:

$$D^{-v}_x f(x) = \frac{x^v}{N^v} \sum_{i=1}^{N} W_i(v) f_i(x)$$

(3.23)

where $W_i(v)$ is a weighting factor whose value depends on $i$, $v$, and on the algorithm in question, but not on $N$ or $f$.

For our specific algorithm (3.22), we have:

$$W_i(v) = \begin{cases} 
\frac{(i+1)^{1+v} - 2i^{1+v} + (i-1)^{1+v}}{\Gamma(2+v)} & \text{for } 1 \leq i \leq N-1 \\
\frac{1}{\Gamma(2+v)} & \text{for } i = 0 \\
\frac{(N-1)^{1+v} - N^{1+v} + (1+v) N^v}{\Gamma(2+v)} & \text{for } i = N \\
0 & \text{for } i = -1
\end{cases}$$

(3.24)

Therefore eq. (3.23) together with eq. (3.24) present an alternative way in the computation of fractional derivative and integral of a function.
CHAPTER IV
FRACTIONAL WHITE NOISE AS A RANDOM FIELD

4.1 Introduction

As it has been discussed previously in section 2.5 a random field $\xi(t)$ with stationary increments can be always expressed in the form

$$\xi(t) = \int_0^t \xi(s)ds$$  (4.1)

where $\xi(t)$ is a stationary random field, as long as eq. (2.57) holds true. An interesting example of a random field with stationary increments is the one which is derived by an integral operation on a field $\xi(t)$ characterized by the property

$$dF(\lambda) = d\lambda \quad \text{or} \quad f(\lambda) = \text{constant}$$  (4.2)

Such a field is very often called white noise. Strictly speaking the spectral density (4.2) can never exist; otherwise the quantity

$$E \{(\xi(t))^2\} = \int_{-\infty}^{\infty} f(\lambda)d\lambda$$  (4.3)
would be infinite, or in other words eq. (2.57) does not hold anymore. However, the introduction of this fictitious stationary random field should be viewed only as a mathematical idealization similar to that of "the value of $\xi(t)$ at time $t$", or that of "point mass" etc.

This example of integrated white noise is very important in our application and we should clear up any possible complications. It is obvious that since the field $\xi(t)$ introduced by eq. (4.1) is not differentiable, although everywhere continuous, we cannot talk about its derivative $\xi'(t)$. However, in most of the circumstances when dealing with the derivative $\xi'(t)$ we are able to express the final result in terms of the integral $\xi(t)$ which is more physically meaningful than the field $\xi(t)$ itself. We will see, that the field $\xi(t)$ will be interpreted as a model for terrain elevation.

In the rest of this chapter, we will deal with fractional differintegration of white noise. It is a natural outgrowth of the theory discussed so far, and will prove to be a useful and effective tool in dealing with terrain related problems. Although similar random fields have been studied in the context of metric geometry [Kolmogorov (1940), Von Neumann & Schoenberg (1941), Hunt (1951), Yaglom (1955), and Lamperti (1962a)], such studies have never found much application; certainly none in the solution of cartographic problems.
It is the purpose of this chapter, to expand the classical theory of random fields using elements of Fractional Calculus, to derive useful characteristics of fields expressed as integrated white noise, and to show the importance of the Weierstrass function in the implementation of the random field theory.

4.2 Fractional operations on white noise

4.2.1 Fractional differintegration of white noise

Consider a stationary random field $\xi(t)$ whose spectral density is constant; for simplicity assume that $\xi(t)$ is real. Let us also assume that the field $\xi(t)$ is the derived field of a field $\zeta(t)$ such that:

$$\xi(t) = \frac{1}{\Gamma(v)} \int_0^t (t-s)^{v-1} \xi(s) \, ds \quad (4.4)$$

Denoting the integral kernel by:

$$g(t,s) = g(t-s) = \frac{1}{\Gamma(v)} (t-s)^{v-1} \quad (4.5)$$

we have

$$\zeta(t) = \int_0^t g(t,s) \xi(s) \, ds \quad (4.6)$$
As we have seen the process $\zeta(t)$ is not stationary but has
stationary increments, therefore we re-write eq. (4.6)
correctly using Stieltje's integral as:

$$\zeta(t) = \int_0^t g(t,s) \, d\zeta(s) \quad (4.7)$$

or the equivalent form:

$$\zeta(t) = \frac{1}{\Gamma(v)} \int_0^t (t-s)^{v-1} \, d\zeta(s) \quad (4.8)$$

Equation (4.7) or (4.8) is an important formula for
the rest of our work, and amazingly enough it is subjected
to different interpretations. One interpretation is that
the random field $\zeta(t)$ is the output of a filter, whose
input in the field $\xi(t)$, the integral kernel $g(t,s)$ being
the impulse response of the filter. In other words, the
field $\zeta(t)$ can be regarded as a weighted average of the
white noise $\xi(t)$ with $g(t,s)$ being the weight function.
Another and more useful interpretation is that according
to Fractional Calculus, eq. (4.8) is the definition of the
$v$-th fractional differintegral of the white noise $\xi(t)$.

Adopting this last interpretation, we will try, in the
following, to derive the probabilistic characteristics of
the $v$-th fractional integral (adopting $v$ as an arbitrary
positive real number) of the white noise $\xi(t)$. 
As we have discussed earlier, an arbitrary stationary random field $\xi(t)$ can be always represented in the form (4.9)

$$\xi(t) = \int_{-\infty}^{\infty} e^{i\lambda t} \, dZ(\lambda)$$  \hspace{1cm} (4.9)

where $Z(\lambda)$ is a random field with uncorrelated increments. The integral of such a field is represented as:

$$\hat{\xi}(t) = \int \, g(\tau) \xi(t-\tau) \, d\tau$$  \hspace{1cm} (4.10)

Consider now the gain function $C^{[v]}(\lambda)$ which is the Fourier (more generally Laplace) pair of the function $g(\tau) = g(t,s)$ given by eq. (4.5). Then the random field $\xi(t)$ can be written as

$$\xi(t) = \int_{-\infty}^{\infty} e^{i\lambda t} \cdot C^{[v]}(\lambda) \, dZ(\lambda)$$  \hspace{1cm} (4.11)

where

$$C^{[v]}(\lambda) = \int_{-\infty}^{\infty} e^{-i\lambda \tau} \left( \frac{\tau^{v-1}}{\Gamma(v)} \right) \, d\tau = (i\lambda)^v$$  \hspace{1cm} (4.12)

Recalling the discussion in section (3.4) the spectral distribution function of $\xi(t)$ is

$$F^{[v]}(\lambda) = \int_{-\infty}^{\lambda} |C^{[v]}(\lambda)|^2 \, dF(\lambda)$$  \hspace{1cm} (4.13)

and assuming existence of the spectral density function of $\xi(t)$, this will be given as before by

$$f^{[v]}(\lambda) = |C^{[v]}(\lambda)|^2 \, f(\lambda) = (i\lambda)^{2v} \, f(\lambda) = \lambda^{2v} \, f(\lambda)$$  \hspace{1cm} (4.14)
where \( f(\lambda) \) is the density function of the white noise \( \xi(t) \) and therefore constant.

The practical importance of eq. (4.14) could be realized if we view the density function of \( f^{[\nu]}(\lambda) \) plotted against the wavelength \( \lambda \) in a logarithmic-logarithmic scale. It is obvious then that it appears as a straight line

\[
\ln f^{[\nu]}(\lambda) = 2\nu \cdot \ln \lambda + \ln f(\lambda)
\]  
(4.15)

with slope

\[
\phi = -2\nu
\]  
(4.16)

### 4.2.3 The correlation and structure functions

Since the random field \( \xi(t) \) is not stationary it has no meaning to speak about its correlation function \( B(\tau) \). It makes however sense to speak about the correlation function of its increments, (structure function) which are stationary. Generally the \( n \)-th order stationary increments are given by:

\[
\Delta^{[n]} \xi(t) = \int_{-\infty}^{\infty} e^{it\lambda} (1-e^{-i\tau_1\lambda})^n \frac{(1+i\lambda)^n}{(i\lambda)^n} dF(\lambda)
\]  
(4.17)

and their structure function is given by:

\[
D^{[n]}(t; \tau_1, \tau_2) = \int_{-\infty}^{\infty} e^{it\lambda} (1-e^{-i\tau_1\lambda})^n(1-e^{-i\tau_2\lambda})^n \frac{(1-\lambda^2)^n}{\lambda^{2n}} dF(\lambda)
\]  
(4.18)
It turns out that such random fields $\zeta(t)$, form a special class of fields with stationary increments, termed random fields with stationary n-th increments. As it is mentioned in Yaglom (1955) (p.122) they are invariant with respect to similarity transformations and exhibit a spectral density function of the general form:

$$f^{[v]}(\lambda) = C \cdot \frac{|\lambda|^2|n-v|}{(1+\lambda^2)^n} ; \quad c > 0 , \quad \frac{1}{2} < v < n + \frac{1}{2} \quad (4.19)$$

Some further elaboration on the form of the structure function (4.18) is included in Appendix A.

Although the expression of the structure function in terms of its spectral components is very instructional, it is not handy in actual computations. Therefore another expression of the structure function should be derived. Using the laws of expectations, we obtain:

$$2E[(\zeta(t) - \zeta(s)) (\zeta(u) - \zeta(v))] = E[(\zeta(t) - \zeta(v))^2]$$

$$- E[(\zeta(t) - \zeta(u))^2] - E[(\zeta(s) - \zeta(v))^2] + E[(\zeta(s) - \zeta(u))^2] \quad (4.20)$$

We already know from the stationarity of the increments [see eq. (2.52)] that

$$E[(\zeta(i) - \zeta(j))^2] \propto (i-j)^{2v-1} \quad (4.21)$$
Therefore we have:

\[ D(t,s ; u,v) \propto \frac{1}{2} \left[ (t-v)^{2v-1} + (s-u)^{2v-1} - (t-u)^{2v-1} - (s-v)^{2v-1} \right] \] (4.22)

The proportionality constant is unknown but irrelevant, since we will always use the normalized structure function such that \( D(0) = 1 \).

Equation (4.22) provides a straightforward formula for the computation of the structure function, and it also reveals its special form. It is clear from (4.22) that the structure function depends only on the lengths of the increments of the random field \( \zeta(t) \). Such forms are termed Toeplitz forms and they are very important in practical applications. In estimation problems involving random fields of the \( \zeta(t) \) type, we are always faced with the inversion of variance-covariance matrices formed by functions of the type (4.22). Almost always the dimension of such matrices is quite large, making their inversion process problematic. By identifying these matrices as Toeplitz forms, their inversion becomes an easy matter, since only one row is needed for the computation of its inverse. Some results related to Toeplitz forms, which are useful in this study are given in Appendix B.

Another way to realize fractional operations on white noise is through the Weierstrass function. This function has the unusual characteristic of being everywhere continuous but nowhere differentiable. In this chapter, we
are going to study the Weierstrass function, and construct a random function out of it. Subsequently, it is going be proven that such a random function belongs to the family of random fields with stationary increments, like the fractional white noise.

4.3 The Weierstrass function

The function originated by Weierstrass (1872) is a periodic function of the form.

\[ W(t) = \sum_{n=0}^{\infty} a^n \cos b^n \pi t \]  

(4.23)

where \( b \) is an odd integer and

\[ 0 < a < 1 \]  

(4.24)

\[ ab > 1 + \frac{3}{2} \pi \]

It has been proven by Weierstrass that such a function is not differentiable at any \( t \). This result has been generalized by a number of authors [see references in Hardy (1916)] who have considered functions of the form

\[ C(t) = \sum_{n=0}^{\infty} a^n \cos b^n t \]  

(4.25)

or the form

\[ S(t) = \sum_{n=0}^{\infty} a^n \sin b^n t \]  

(4.26)
Different restrictions of the form (4.24) have been suggested, but most of them are artificial and arise merely in consequence of the limitations of the methods employed to prove non-differentiability of the function (4.23) and its versions. In fact if we are restricted to finite order derivatives of \( W(t) \), then it is sufficient that

\[ ab > 1 \]

(4.27)

Hardy (1916) with his celebrated theorem [Hardy (1916), pp. 303, Theorem 1.31] proved that neither of the functions (4.25) and (4.26) where

\[ b > 1 \text{ integer} \quad (4.28) \]

\[ 1 > a > 0 \]

possesses a derivative of finite order, under condition (4.27). Generalizing his result he also proved that the theorem holds for a real \( b > 1 \).

In our discussion we will refer to the Weierstrass function as having the more general form

\[ W(t) = \sum_{n=-\infty}^{\infty} a_n \exp (2\pi ib^n t) \]

(4.29)

which includes the two cases (4.25) and (4.26).

Let us consider now the construction of a random field out of the definition (4.29). If we allow the coefficients \( a \) and \( b \) to take values from a set of random numbers with mean 0 and variance 1, then \( W(t) \) is a sum of infinitely many contributions with random phases and amplitudes.
Such a technique has also been deployed previously in Berry and Lewis (1980) and Voss (1985). The investigation of the probabilistic characteristics of such a random field is the topic of the next section. This investigation will help us to connect the Weierstrass function to the theory of fractional white noise, which we have discussed before.

4.3.1 The correlation function and the spectrum

The Weierstrass function is not a stationary random function, since its variance

$$\text{Var} \{W(t)\} = E \{ W(t) \}$$

(4.30)

depends on $t$. However its increments are random and stationary. It can be proved [Berry & Lewis (1980), p.479] that the correlation function of the increments is of the form:

$$B(\tau) = E \{|W(t+\tau) - W(t)|^2\} = 2 \cdot \sum_{n=0}^{\infty} \frac{1-\cos b^n \tau}{b^{2n}}$$

(4.31)

and therefore independent of $t$. The fact that the Weierstrass function itself is not stationary can be easily deduced from (4.31) with $W(0) = 0$, and thus the assertion (4.30) follows.
A remarkable observation of Berry & Lewis (1980) is that the correlation function (4.31) is itself a Weierstrass function where $n$ is replaced by $2n$.

From the discussion in section (4.2.3) and in Appendix A [see eq. (A.6)], the form (4.31) of the correlation function is exactly the same as that of the structure function [see (A.6)] of a random field with stationary first order increments, with spectral density function of the form (4.19), as discussed in Yaglom (1955).

Regarding the computation of the spectral density function of the Weierstrass function, it is obvious that each frequency of the form $b^n$ has a spectral power (squared amplitude) equal to $a^{2n}$. [see eq.(4.29)]. Therefore the total power in frequencies $> b^n$ in the Weierstrass function is (remember $0 < a < 1$)

$$ S = a^{2n} + a^{2(n+1)} + a^{2(n+2)} + \ldots = \frac{a^{2n}}{1-a^2} \quad (4.32) $$

This means that the frequencies in the Weierstrass function span all the range from 0 to $\infty$ in an absolutely decreasing geometric progression.

Defining

$$ v = \frac{1}{2} - \frac{\log a}{\log b} \quad (4.33) $$

and using the fact that

$$ b^n = \frac{1}{\lambda} \quad (4.34) $$
We can easily prove that the total spectral power in eq. (4.32) is:

\[ S = \frac{\lambda^2 (\nu - \frac{1}{2})}{1 - a^2} \]  \hspace{1cm} (4.35)

which again matches eq. (4.19).

Subsequently using the restrictions (4.27) and (4.28) we can prove that

\[ 1/2 < \nu < 3/2 \]  \hspace{1cm} (4.36)

which is valid [see eq. (4.19)] for first order stationary increments, as has been confirmed in Yaglom (1955).

It is thus clear that the random field \( W(t) \) obtained from the Weierstrass function possesses all the characteristics of our previously defined fractional white noise, and it is as close to as allowed by the fact that one is discrete and the other continuous.

4.3.2 Approximation of the Weierstrass function

Although with the Weierstrass function approach we have already discretized the fractional white noise, a further approximation of the Weierstrass function is still essential from the efficiency of computations point of
view. Such an approximation although not dramatic, will help us later on to devise a recursive scheme for simulation of fractional white noise.

We observe that by adding new terms in the sum (4.29) we push the implied Nyquist frequency up by $b$, or equivalently we reduce the required spacing of $W(t)$ by $1/b$. In other words, we can view the development of $W(t)$ as a sequential addition of sine and cosine oscillations, with increasing frequency, and varying amplitudes. As an approximation to $W(t)$ we could substitute other periodic functions in the place of sine and cosine. In this way different approximations of the Weierstrass function could be achieved.

In this study we choose to substitute the triangular function

\[
T(t) = \begin{cases} 
1 - \frac{|t|}{T} & \text{for } |t| < T \\
0 & \text{for } |t| > T
\end{cases}
\]
In this case, the development of the approximated Weierstrass function, can be viewed as a sequential addition of triangular oscillations, with decreasing T (or increasing frequency) and varying amplitudes.

It should be noted that similar approximation techniques are also discussed in Voss (1985), Fournier et. al. (1982), and Carpenter (1986).
CHAPTER V

SIMULATION OF FRACTIONAL WHITE NOISE

5.1 Simulation based on the definition of fractional integral

Given the white noise \( \xi(t) \), a straight-forward simulation of its fractional \( v \)-th integral is a numerical evaluation of the definition (4.8). This evaluation can be accomplished by using either the numerical integration technique explained in section 3.5, or more crudely by just replacing the integral by a summation, which discretizes eq.(4.8) as follows:

\[
\xi(t) = \frac{1}{\Gamma(v)} \sum_{i=0}^{t} (t-j)^{v-1} \xi(i)
\]  

(5.1)

where \( v \) is any real number in the range \( 1/2 < v < 3/2 \). Such an algorithm uses \( 2N-1 \) white noise values, generated by a random number generator (fig. 1), in order to generate \( N \) values of \( \xi(t) \). Such random numbers are generated with zero mean and variance 1.
The random field $\zeta(t)$ possesses quite different characteristics, depending on the value of $v$. We can distinguish these cases, namely $1/2 < v < 1$, $v = 1$, and $1 < v < 3/2$. The correlation of the increments of $\zeta(t)$ is negative for $1/2 < v < 1$ and positive for $1 < v < 3/2$. This is clear from the definition (4.22) of the structure function. For instance, for $v < 1$ we have $D(t,s; t,s) < 0$ and for $v > 1$ $D(t,s; t,s) > 0$. For the exceptional case of $v = 1$ we get the classical Brownian motion, in which case it is known that the increments are uncorrelated. In all cases, however, the increments $\Delta\zeta(t)$ exhibit a long serial correlation. In fact, recent research [Fox et.al. (1985), Jacobi et.al. (1982), Mandelbrot et. al. (1968)] has shown that many natural phenomena, including the earth surface and the sea bottom, actually show such a long serial correlation.
Using the discrete form (5.1) different realizations of fractional white noise have been generated, corresponding to values of $v = 1.01$ to $v = 1.4$. In all examples presented in the next illustrations, the same seed value for the generation of the random numbers has been used. By varying this seed value different forms can be generated; such examples are given in Appendix C. The resemblance of these examples to actual terrain profiles is remarkable. A noticeable fact is that the roughness of the profiles increases progressively from $v = 1.4$ to $v = 1.01$. This fact has given rise to definitions of fractional dimension appearing in the literature \cite{Mandelbrot1982b, Berry1980}.\footnote{eq. Mandelbrot (1982b), Berry et.al. (1980)}.
Figure 2: Simulation examples for varying $v$. 
In order to exploit the characteristics of the simulated profiles the corresponding spectral density functions have been computed and given in the next figures. (Fig. 3-6).

For the computation of the spectrum, a linear trend \( y = a_0 + a_1 x \) has been firstly removed. Then the standard Fast Fourier Transform algorithm has been applied for the transformation of the data to the frequency domain and the subsequent computation of the spectral amplitudes. Aliasing effects have been reduced by applying a smoothing procedure using the standard Hanning window.

In the next figures the spectra of the simulated profiles are plotted in a log - log scale. As it is clear from eqs. (4.14) and (4.16), the order of integration \( v \) can be estimated from the slope of the spectrum in a log-log scale. For this purpose a least squares procedure has been used to estimate the best fitting straight line, to the spectral density function, and compute its slope. The computed \( v \)'s agree very well to the originally used ones and are also given below.
Figure 3: Power spectrum of simulated profile for \( v=1.1 \).
Estimated \( v=1.053 \)

Figure 4: Power spectrum of simulated profile for \( v=1.2 \).
Estimated \( v=1.208 \)
Figure 5: Power spectrum of simulated profile for $v=1.3$.
Estimated $v=1.365$

Figure 6: Power spectrum of simulated profile for $v=1.4$.
Estimated $v=1.423$
5.2 A recursive simulation scheme based on the approximated Weierstrass function

Alternatively a recursive algorithm is suggested for simulating fractional white noise. This algorithm differs from the previously presented one, in that it uses the approximated Weierstrass function, instead of the more generic definition of the fractional integral operator.

Voss (1985) has used the definition of the Weierstrass function as a starting point in simulating fractal profiles. His adaptation of the original formula (4.29) is

\[ W(t) = \sum_{n=-\infty}^{\infty} A_n r^{n(2\nu-1)} \sin(2\pi r^{-n} t + \phi_n) \]  

(5.2)

where \( A_n \) is a Gaussian random field, \( r \) is a scale factor, and \( \phi_n \) random phases uniformly distributed on the interval \((0, 2\pi)\). In terms of spectral density, although \( W(t) \) contains only discrete frequencies \( \lambda_n^{-1} = r^{-n} \), each component has an amplitude proportional to \( r^{n(2\nu-1)} \) in a bandwidth of \( \Delta \lambda^{-1} \propto \lambda^{-1} \) and thus the spectral density

\[ f(\lambda) \propto \frac{\text{amplitude}^2}{\Delta \lambda^{-1}} = \frac{1}{\lambda^{2\nu}} = \lambda^{2\nu} \]  

(5.3)

is in agreement to our previous discussion.

Our algorithm is based on such an interpretation, with some further modifications. First, as it is discussed in section 4.3.2, we use a triangular function, instead of the
sine function in eq. (5.2). It is obvious that in this case too, an addition of a new term to the sum $W(t)$ decreases the resolution (or the sampling interval) by a factor of $r$, and adds new fluctuations with variance $A^2 r^2 (2^{v-1})$. Such fluctuations are added to all previously computed elevations, and not only to the new points [in contrast to the algorithm implemented in Carpenter (1986)]. Second, we assume a value of $r = 2$, primarily for reasons of computational speed. Therefore the new points are always computed as mid-points of previously defined ones. Other computational aspects of the actual implementation of the algorithm are explained next.

5.2.1 Computational considerations

The algorithm works in a recursive densification manner. We begin with two given end-points. For these two points the elevations $Z_1$ and $Z_2$ are assumed known, and for the sake of argument, suppose that both are equal to zero. It is instructive to hereafter visualize the recursive densification as successive generation of different levels of detail, superimposed one on top of the other, as shown in Figure 7.
Figure 7: The recursive generation scheme

The lowest of these levels (level #1) consists of only the two known elevations $Z_1^{old} = Z_1^1$, and $Z_2^{old} = Z_2^2$. For the generation of the second level we use the triangular function (4.37) to interpolate an additional mid-point. That is:

$$Z_{1/2}^{old} = \frac{1}{2} (Z_1^{old} + Z_2^{old})$$  \hspace{1cm} (5.4)

Then, we add random fluctuations $\xi_i$; (white noise) of variance $\sigma_i^2$ given by eq.(5.6) for $n = 1$ (and zero mean) to all generated $Z$-values

$$Z_1^{new_1} = Z_1^{old} + \xi_1$$

$$Z_2^{new_1} = Z_2^{old} + \xi_2$$

$$Z_{1/2}^{new} = Z_{1/2}^{old} + \xi_3$$ \hspace{1cm} (5.5)

Subsequently, we repeat the process, recursively generating mid-points for the new intervals, and adding random fluctuations with zero mean and variance (5.6), to all the
fluctuations with zero mean and variance (5.6), to all the elevations;

\[ \sigma_n^2 = \frac{\sigma_0^2}{r^n(n^v - 1)} \]  

where \( \sigma_0^2 \) is the a-priori assume variance, associated with the assumed elevations \( Z_1 \) and \( Z_2 \). For simulation reasons we can regard \( \sigma_0^2 = 1 \) as being a unit, since the subsequent \( \sigma_n^2 \)’s are computed with respect to \( \sigma_0^2 \). Note that \( n \) is the order of the current level of simulation, \( v \) is the order of fractional integration, and the scale factor \( r \) is assumed equal to 2. Another possible version of the algorithm could use the known form (4.22) of the structure function to interpolate for the mid-point.

It is obvious that at level \( n \) the profile is described by

\[ N_n = r^{(n-1)} + 1 \]  

points, whereas the number \( M_n \) of the involved random fluctuations (random numbers, or white noise) is computed from the recursive formula
\[ M_n = M_{n-1} + N_n \quad ; \quad n \geq 2 \]  

(5.8)

and \( M_1 = 0 \)

5.2.2 Simulation examples

With the help of the described recursive algorithm, different examples have been simulated, with values of \( v \) ranging from \( v = 1.01 \) to \( v = 1.4 \). We chose to work in this range of \( v \), because, as we will see later on, actual terrain very rarely exceeds those limits. In order to demonstrate the recursive development of the random field, each simulation level has been plotted separately. An example of the simulation is given in Figure 8. A seed value of 123457.0 has been used for the generation of the random numbers, and \( v = 1.4 \). For level \#1 we have assumed zero elevations for the two end-points, and a total number of 8 levels have been generated. Other examples with varying seed value are given in Appendix D. In Figure 9, we keep the same seed value and vary the order of fractional integration from \( v = 1.01 \) to \( v = 1.4 \), in order to show the increase of the profile roughness with decreasing \( v \); something we have experienced in previous simulations, as well.
We should note that although we have assumed 8 simulation levels, this is by no means restrictive. As we recall from eq. (5.6) the higher is the level $n$, the smaller is the variance of the added fluctuations. Therefore the only restriction to the number of levels, is the drawing scale, since we should not continue generating features that cannot be graphically represented any more.
Figure 8: Recursive simulation of a profile for $v=1.4$. 
Figure 9: Recursive simulation examples for varying $v$. 
Spectral densities of the simulation examples in Fig. 9 have been estimated, the same way as described in section 5.1. The resulting spectra are shown in a double logarithmic scale in the next figures (Fig. 10-15). Also as in section 5.1, the slopes of the best fitting straight lines have been estimated and the computed values for $v$ agree with the originally used ones.
Figure 10: Power spectrum of simulated profile for $v=1.01$. Estimated $v=1.021$

Figure 11: Power spectrum of simulated profile for $v=1.1$. Estimated $v=1.062$
Figure 12: Power spectrum of simulated profile for $v = 1.2$.
Estimated $v = 1.212$

Figure 13: Power spectrum of simulated profile for $v = 1.3$.
Estimated $v = 1.324$
Figure 14: Power spectrum of simulated profile for $v=1.4$. Estimated $v=1.411$

Figure 15: Power spectrum of simulated profile for $v=1.49$. Estimated $v=1.501$
5.2.3 The structure function

In order to confirm our observations regarding the increase of the roughness as a function of the order \( v \) of integration, the structure function (4.22) has been plotted for different values of \( v \). As it has been discussed in section 5.1 the simulated process exhibits long serial correlation, a characteristic of many physical phenomena. In the range of \( 1 < v < 3/2 \) in which we are interested, the steepness of the structure function decreases and therefore the profile smoothness increases, with increasing \( v \).

At the limits \( v = 1 \) and \( v = 3/2 \) the random process becomes singular with completely uncorrelated increments (Brownian motion) in the first case, and completely correlated increments in the second. For values of \( 3/2 < v < 2 \) the structure function repeats itself with decreasing steepness as in the interval \( 1 < v < 3/2 \). For values of \( 1/2 < v < 1 \) the correlation becomes negative and the steepness of the structure function decreases until it becomes zero at value \( v = 1/2 \).
Figure 16: Form of the structure function for varying $v$. 

*Correlation vs. Correlation Length*
6.1 Introductory remarks

So far, I managed to develop algorithms to simulate a fractional integral operator on white noise, which closely resembles actual terrain. In most terrain related problems, however, we are given terrain elevations, which presumably can be modelled as fractional white noise, and we are asked to infer about the characteristics of the underlying process. Based then on such analysis, certain operations are performed in order to solve the specific problem. We thus have to invert the simulation procedure that was developed in previous chapters. Such an inversion is actually a way of coding the terrain elevations, in such a manner that problems like terrain generalization, breakline detection, data compression, etc., can be easily solved. Specifically, we will see in the next chapter how the terrain generalization problem can benefit from such an inversion procedure.
Since the simulated random process $\zeta(t)$ is a fractional integral of white noise $\xi(t)$, a first approach is that we can certainly recover $\xi(t)$ by just computing the corresponding fractional derivative of $\zeta(t)$. It is straightforward that the application of eq. (3.12) where $f(t) = \zeta(t)$ would provide us with the fractional derivative of $\zeta(t)$, that is the white noise $\xi(t)$. However this first thought was quickly abandoned for many reasons. First, the behavior of the gamma function $\Gamma(v)$ is completely different for $v > 0$ than for $v < 0$. For a positive value of the argument $v$ it is continuous, whereas for $v < 0$ it is discontinuous for all $v$ integers. Second, as discussed in section 5.1 we would need to estimate $2N-1$ values of the $\xi(t)$ function, given $N$ values of $\zeta(t)$ function. Such a problem does not have unique solution, and in order to obtain a unique solution some kind of constraint on the estimated $\xi(t)$ should be imposed, eg. a minimum norm constraint. However the addition of such a constraint is not clear. Third, such an inversion procedure has all the disadvantages of its simulation counterpart (see section 5.1), being similarly neither of local, nor of a recursive nature. For these reasons inversion of the simulation procedure described in section 5.1 has not been attempted.

The recursive simulation scheme based on the approximated Weierstrass function, however, is easily invertible. The inverse operation of such a simulation algorithm is the
topic of this chapter. Although it eliminates the disadvantages described above, it still involves estimation of more parameters than observations. In order to overcome this problem we have used a robust estimation technique based on re-weighted least squares, which is discussed next.

6.2 Robustification aspects of the estimation problem

The recursive simulation algorithm described in the last chapter can be formulated as follows:

\[ \zeta(t) = T(t) \{ \zeta(t_1) + \zeta(t_N) \} + B \cdot \xi(s) \]  

(6.1)

where \( \zeta(t_1) \) and \( \zeta(t_N) \) are the values of the function \( \zeta(t) \) at the two end points \( t = t_1 \) and \( t = t_N \). \( T(t) \) is the triangular function (4.37), \( B \) is a design matrix, and \( \xi(s) \) is white noise. We can re-write eq. (6.1) as:

\[ B \cdot \xi(s) + [T(t) \{ \zeta(t_1) + \zeta(t_N) \} - \zeta(t)] = 0 \]  

(6.2)

or as

\[ BV + W = 0 \]  

(6.3)

It is clear that in order to simulate a profile of \( N_n \) points, (corresponding to up to \( n \) levels), we need \( M_n \).
random numbers $\xi(s)$, where $M_n$ is given by eq. (5.8). For example for $n = 8$, we have $N_n = 129$ and $M_n = 261$. Therefore the involved matrices have dimensions of $B(N_n, M_n), V(M_n), W(N_n)$. The system of equations (6.3) has obviously infinitely many solutions (since $M_n > N_n$) and we can choose the one that minimizes the quadratic form

$$V^T P V = \text{min.} \tag{6.4}$$

Using the Lagrange multiplier method we minimize the function

$$\phi = V^T P V - K^T (B V + W) = 0 \tag{6.5}$$

and obtain the straightforward least square solution

$$V = - P^{-1} B^T (BP^{-1} B^T)^{-1} W \tag{6.6}$$

In general the vector $V$ containing the random sequence $\xi(t_i)$ is not the same with the random sequence used to simulate the values $\zeta(t_i)$. This, however, should not be a problem, since the random sequence $\xi(t_i)$ did not have any physical meaning and it was arbitrary in the first place. It is merely an auxiliary set of random numbers that fulfill the condition (6.3) and can therefore reconstruct the profile exactly.
Although, using least squares adjustment we are able to invert the simulation process and recover a set of random numbers, which can exactly regenerate the profile, still the solution is far from efficient. The point is that a method that replaces a set of \( N \) numbers with another set of twice as many numbers, which still contains the same amount of information, is clearly handicapped. Therefore it makes sense to seek a solution, which besides fulfilling the condition (6.3) also minimizes the number of \( V \)'s.

In order to find the minimum number of \( V \)'s, we use the least squares method repeatedly, and iteratively downweight large residuals, thus pressing as many \( V \)'s as possible to zero. In particular we use the weight function.

\[ p_i = \frac{1}{v_i^2} \]  

(6.7)

Thus we actually minimize the quantity

\[ v^T v = \sum_{i=1}^{M_n} v_i^2 \cdot \frac{1}{v_i^2} = M_n \rightarrow \text{min.} \]  

(6.8)

That is, we minimize the number \( M_n \) of the residuals \( V \), or in other words from all the possible solutions of (6.3) we pick up the one, which gives the minimum number of random numbers \( \xi(t_i) \) that can exactly reconstruct the given profile.
Beginning with a normal least squares adjustment, with $P = I$ we compute the $M_n$ non-zero (generally) residuals $V$. Then through an iterative process of re-weighted least squares, observations corresponding to the relatively bigger residuals are weighted down, whereas the rest of the observations are adjusted to fulfill eq. (6.3). The term "iteration" here is a little misleading, and it should not be confused with what we normally call "iteration" in least squares adjustments; it is clear that our model is linear and therefore no "iterations," with the regular meaning of the term, are needed for the solution (6.6) to converge.

6.3 Computational considerations

In inverting the recursive simulation algorithm, a least squares adjustment using robust estimation principles may take place. In order to compute the solution (6.6), the matrices $B, P, W$, and $M = BP^{-1}BT$ should be designed.

The matrix $B$ is formed by different $B_j$ sub-matrices, each one corresponding to a different simulation level of order $n$, i.e.

$$B = [B_2, B_3, \ldots, B_n] \quad (6.9)$$
The simulation procedure due to its recursive form gives rise to special structure for each $B_i$ sub-matrix. For example, for $i = n$

$$B_n = I$$ \hspace{1cm} (6.10)

for $i = n-1$

$$B_{n-1} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1/2 & 1/2 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1/2 & i/2 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$ \hspace{1cm} (6.11)

for $i = n-2$

$$B_{n-2} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
3/4 & 1/4 & 0 & 0 & 0 & 0 & 0 \\
1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 \\
1/4 & 3/4 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 3/4 & 1/4 & 0 & 0 & 0 \\
0 & 0 & 1/2 & 1/2 & 0 & 0 & 0 \\
0 & 0 & 1/4 & 3/4 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 3/4 & 1/4 \\
0 & 0 & 0 & 0 & 0 & 1/2 & 1/2 \\
0 & 0 & 0 & 0 & 0 & 1/4 & 3/4 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$ \hspace{1cm} (6.12)
and so on. For the general case for $i = k$, in the $k^{th}$ level we have $N_k$ new points and $B_k$ sub-matrix is given as:

$$B_k = \begin{bmatrix}
\frac{N}{N} & \frac{N-1}{N} & 0 & \cdots & 0 \\
\frac{N-1}{N} & \frac{N-1}{N} & 0 & \cdots & 0 \\
\frac{N-2}{N} & \frac{N-2}{N} & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
\frac{1}{N} & \frac{1}{N} & \cdots & \frac{1}{N} & 0 \\
\frac{0}{N} & \frac{0}{N} & \cdots & \frac{0}{N} & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
\frac{0}{N} & \frac{0}{N} & \cdots & \frac{0}{N} & 0 \\
\frac{0}{N} & \frac{0}{N} & \cdots & \frac{0}{N} & 0 \\
\cdots & \cdots & \ddots & \ddots & \ddots \\
\end{bmatrix}$$

(6.13)

The matrix $B$ entering the adjustment will have a dimension of $(N_n, M_n)$ and each individual $B_i$ sub-matrix will have $N_n$ rows and $N_i$ columns.
For example if \( n = 4 \) we have \( N_n = 9 \), \( M_n = 17 \) and the \( B_i \) sub-matrices have dimensions \( B_2(9,3) \), \( B_3(9,5) \), \( B_4(9,9) \). The final \( B \) matrix is given below.

\[
B = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3/4 & 1/4 & 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 \\
1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1/4 & 3/4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 3/4 & 1/4 & 1/2 & 1/2 & 0 & 0 & 0 & 0 \\
0 & 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1/4 & 3/4 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1/4 & 3/4 & 1/2 & 1/2 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

The \( W \) matrix is given by:

\[
W = [T(t) \{\zeta(t_1) + \zeta(t_w)\} - \zeta(t)]
\]  \hspace{1cm} (6.15)

and can be formed in a straightforward manner using the triangular function \( T(t) \), and the values of the function \( \zeta(t) \).

As far as the weight matrix \( P \) is concerned, for the first iteration it is the identity matrix, whereas for the subsequent solutions it is a diagonal matrix, with the diagonal elements given as
\[ p_{i+1} = \frac{1}{v_{i+1}^{2} + \varepsilon} \quad (6.16) \]

The constant \( \varepsilon \) has been added merely for numerical stability reasons. Suggested values for \( \varepsilon \) are \( \ll 10^{-6} \).

The programming can be further optimized by taking advantage of the very special structure of matrix \( B \). Because of this structure the normal matrix \( M \) has also a special form (Toeplitz form) and its inverse could be directly formed.

For the converge of the iterative procedure, a usual convergence criterion (e.g. \( |v_{i}^{k} - v_{i}^{k-1}| < \text{tolerance} \)) can be enforced. In all our testing examples the solution converged in 3-10 iterations.

6.4 An example on robust estimation

In this section an example of the inverse operation is given. Given the simulated profile \((t)\) of Fig. 8, the inversion algorithm is used to recover the \((t)\) random numbers. A maximum simulation level of \( n = 8 \) has been used, which corresponds to 129 profile points and 261 random numbers. Using robust estimation through an iterative procedure the number of random numbers has been compressed to 129. The rest have been assumed zero since
they are smaller than the tolerance value of $10^{-4}$. The table below gives the successive elimination of the random numbers through the iterations.

Table 1: Number of non-zero random numbers after each iteration.

<table>
<thead>
<tr>
<th>iteration #</th>
<th># of random numbers</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>261</td>
</tr>
<tr>
<td>2</td>
<td>244</td>
</tr>
<tr>
<td>3</td>
<td>204</td>
</tr>
<tr>
<td>4</td>
<td>172</td>
</tr>
<tr>
<td>5</td>
<td>154</td>
</tr>
<tr>
<td>6</td>
<td>142</td>
</tr>
<tr>
<td>7</td>
<td>139</td>
</tr>
<tr>
<td>8</td>
<td>135</td>
</tr>
<tr>
<td>9</td>
<td>130</td>
</tr>
<tr>
<td>10</td>
<td>129</td>
</tr>
</tbody>
</table>

The resulting reconstructed levels are shown in Fig. 17, and they should be compared to Fig. 8. The fidelity of the inverse operation can be easily realized with such a comparison. In particular we observe that the higher
levels (ie. levels 6, 7, 8) are almost identical, whereas characteristic points are still preserved in lower levels.

Figures 18, 19, and 20 show the random numbers resulted from the adjustment, after the 1st, 5th, and 10th iteration.

Figure 17: A level-by-level reconstruction of a profile, using the estimated random numbers.
Figure 18: Random numbers after 1st iteration.

Figure 19: Random numbers after 5th iteration.
Figure 20: Random numbers after 10th iteration.
As shown in Figures 18-20, not only about half of the initially used random numbers have been zeroed out, without any loss of information, but also the magnitude of the remaining ones is much smaller than the original \( \zeta(t) \) values. Therefore we can symbolically represent the profile by this sequence of random numbers instead of the original \( \zeta(t) \) values, reducing thus the storage requirements. Therefore, as a spin-off of this investigation, a data compression algorithm is possible. In order to further elaborate on the potential of such a compression technique, two examples of specific geometric features have been worked out and presented in Appendix E. In these examples the amount of information is known a-priori, but intentionally much more data points have been used to describe the profiles. The inversion algorithm in all cases determined correctly the optimum number of random numbers, conforming thus to the amount of information contained in the profile. This way the storage requirements have been drastically reduced, whereas not a bit of information has been lost.

Another important aspect of the inversion algorithm is the following: As we have seen the design matrix \( B \) can be structured as a series of \( B_i \) sub-matrices as in eq. (6.9). Each of these sub-matrices corresponds to a different simulation level. The recovered \( \xi(t) \) values after the adjustment can be inserted back in eq. (6.2) and
reconstruct the original $\zeta(t)$ values. Taking, however, advantage of the form (6.9), we can truncate the $B$ matrix to any desired level $k$, by simply ignoring the sub-matrices in (6.9) after the $B_k$ sub-matrix.

This way a level-by-level reconstruction of the original profile is possible (see Figure 17), the same way as the simulation has been performed. Comparing Fig. 17 with Fig. 8, which is the actual simulation of the same profile, we confirm the fidelity of the inversion also in a level-by-level basis. It is clear that each level contains different amounts of detail, while the overall characteristics of the profile are preserved. Based on this observation, a prototype generalization algorithm is developed in Chapter VII.
CHAPTER VII
TERRAIN GENERALIZATION

7.1 Introductory remarks

Map generalization is defined in the Glossary of Mapping, Charting, and Geodetic Terms (GMCG) as "Smoothing of the character of features without destroying their visible shape. Generalization increases as map scale decreases." [Steward (1974)]. In this case the feature to be generalized is the terrain elevations. Therefore, the purpose of terrain generalization is to smooth the terrain elevations to a degree appropriate to the scale reduction preserving at the same time the visible shape of the terrain.

Today a vast literature on terrain generalization exists. We can distinguish between two groups of generalization methods. One-dimensional methods, which deal with generalization of contours or profiles [e.g. Allan (1978)], and two-dimensional methods, which deal with generalization of grid points [e.g. Loon (1984)]. Generally, both groups use similar smoothing techniques,
applied in one or two dimensions, like weighted averaging, selective point elimination, Fourier spectrum truncation etc. However problems, like preservation of the character of the terrain, proper degree of generalization in connection to the scale reduction, and breakline definition and preservation, remain largely unanswered. The difficulty in dealing with such questions lies mostly on the fact that, although they are almost trivial matters in manual generalization, they cannot be described in a computer environment.

It is the author's belief, however that the actual problem lies somewhere else. I think that the problem of terrain generalization will continue to exist, as long as the problem of terrain simulation, or terrain classification exists. So long as we cannot model the terrain surface, with an efficient and well defined mathematical model, we cannot avoid using prohibitively sophisticated algorithms for generalization.

It is the purpose of this chapter to prove that the proposed terrain model is sophisticated enough to require only simple algorithms, in the solution of the generalization problems. It is also flexible enough to be used in terrain types with different characteristics. Such a modelling technique can relax the restriction of terrain homogeneity, which most of the generalization algorithms require, and correctly treat breaklines. Moreover, it is
easily expandable to two dimensions, easily adaptable to an interactive environment, and has an internal hierarchical nature, which is essential in retaining characteristic terrain features.

7.2. **A new approach to terrain generalization**

7.2.1 **Terrain generalization and the inverse operator**

The first step in developing a terrain generalization algorithm, is to clearly state assumptions concerning the behavior of the terrain. The set of such assumptions constitute a terrain model. Our assumption is that the terrain, as represented by profiles, can be formed by integrated white noise to a fractional order. By adapting such a model, we can estimate the fractional order of integration, and the underlying white noise, using the inverse operator. The way in which we have previously developed this inverse operation, guarantees the reconstruction of the profile by a set of random numbers, in a level-by-level basis. Such levels are formed with increasing degree of detail, or correspondingly decreasing degree of generalization. Although this technique can be classified as point elimination method, since with increasing generalization degree the number of profile points is decreasing, it grossly distinguishes itself from
other similar techniques. First, the reduction of the
detail information is controlled, so that the roughness of
the original terrain, at the original scale, remains
constant in all levels, at the derived scales. Thus the
character of the terrain is preserved. Second, the
elimination of the points is selective, in the sense that
characteristic features, like breaklines, are not smoothed
or eliminated during the process of generalization. Third,
the algorithm can be implemented in a way, that is very
efficient in a production environment. The only time
consuming task is the coding of the terrain elevations, in
terms of a set of random numbers. Once the inversion
operation is performed and the terrain code is stored, any
generalization level can be computed and displayed in a
matter of seconds, by a simple multiplication of two
matrices [see eq. (6.1)].

7.2.2 Heuristics and interactivity

Although the proposed algorithm is very selective in
the elimination of information, and the preservation of
characteristic points, like breakline points, there is
always the possibility that the cartographer judges (due to
the purpose of the map) that some information, which
otherwise would be eliminated, should be preserved. It is
obvious, that such problems can be tackled efficiently only in an interactive environment. Most of the generalization algorithms, however, do not facilitate such an option. Using, for example, a weighted average—which seems to be a popular method—one can interactively select and try different weight functions, but certainly there is no way to pin-point and preserve characteristic points, at least in an efficient manner. Therefore true interactivity does not exist.

On the other hand, generally all the algorithms are designed such that they successfully work in most of the cases generally encountered in practice. It is however, impossible to design an algorithm, which can predict all possible cases. In such instances, we should proceed heuristically and not algorithmically. It is essential, therefore, to pay attention to the aspects of heuristics and interactivity during the design of a terrain generalization algorithm.

The proposed algorithm facilitates both options. Suppose, for example, that in some instance the algorithm (wrongly) does not preserve a breakpoint, or that the user wants to preserve a small feature (which otherwise would be eliminated through the generalization). Both these categories of problems can be easily solved in an interactive environment. As we have mentioned, the proposed algorithm selects and stores terrain information on
hierarchically ordered levels. Levels high in the hierarchy contain finer detail than lower levels. Refer, for example, to Figure 8 and suppose that level #3 corresponds to the original scale, whereas level #7 corresponds to the derived scale. Also suppose that the point $Z_{1/4}^{new}$ is a feature we need to preserve at level #1. Obviously, the only thing we have to do is to "push" that point down hierarchically, from level #3 to level #1. Subsequently, the profile at level #1 will be described by the points $Z^{old}$, $Z_{1/4}^{new}$, $Z_{2}^{old}$. Cases in which we may wish to apply interactivity appear in Appendix E.

7.2.3 Remarks on practical implementation of the proposed method

Assuming $n$ simulation levels and a scale factor $r$, then obviously the number of points at the highest level (which corresponds to the original profile) is given by eq. (5.7) as $N = r^{(n-1)} + 1$. In our algorithm we have assumed $n = 8$, and $r = 2$; thus $N = 129$. However, in practice, we should assume that a profile can consist of any arbitrary number of points, not always a multiple of 129. Such a problem can be overcome easily, and we have to suggest several alternatives.
First, we could, of course, work only those points whose number is an integer multiple of 129, and disregard the rest. Although this is a practice followed in generalization algorithms, that are based on weighted averages, where the edge points are disregarded, it is not very attractive to consider throwing out "modulus (number of profile points, 129)" points.

A second alternative, is to extend with "zeroes" the profile, so that the total number of points is an integer multiple of 129. In this case, the whole profile is considered, without increasing the minimum number of random numbers, which are required to describe the terrain. As it is obvious from the examples in Appendix E, the "zeroes" do not introduce any new information to the profile.

Another implementation problem, which is connected to the number \( n \) of assumed levels, is the following. If \( N_n \) is the number of points, that correspond to the highest level \( n \), then the biggest matrix to be inverted in the adjustment process has dimension \( N_n \). Therefore a judgement, of whether we should use large \( n \) and therefore few fractions of the actual profile (thus, smaller matrix to be inverted), or smaller \( n \) and more fractions of the profile, should be based on on-site restrictions of computer memory and processing time.
7.3 Generalization examples

7.3.1 Data description

In order to demonstrate the performance of the proposed algorithm for terrain generalization, four actual terrain profiles have been chosen to work with. These are long profiles (6-8 km) digitized from aerial photography. They have a uniform sampling interval of 5m and 8m and cross areas of quite different land forms. The data has been acquired in the process of a pilot project in Queensland, Australia and has been supplied by Dr. K. Kubik.

As is expected in actual circumstances, inhomogeneity of the data is experienced in the first three profiles P1, P2, and P3. In order to avoid bias in the estimation of the spectral densities, these profiles have been treated as if consisting of two parts each. Thus P1 has been separated to parts P11 and P12, P2 in P21 and P22; and P3 in P31 and P32. This separation in parts has taken place only in the calculation of the spectrum. During the generalization process the profiles have been processed as a whole. The reason is that we wanted to test the algorithm in the presence of inhomogeneity and breaklines. The four profiles are referred to as P1, P2, P3, and P4, and their specifications are given below.
a. Profile P1

The profile P1 has been digitized from aerial photography of scale 1:19,000 with a terrain sampling interval of 5m. It consists of 1290 points measured over steep sided ridges, close to mountainous area. The average undulation of the ridges is about 100m., their extension being 100-300m. The characteristic land form is presented in Fig. 21 as a clip of the aerial photograph. In figures 22-24 the profile and its two parts are plotted together with their Fourier spectra in a double logarithmic scale; useful for the subsequent discussion.

b. Profile P2

This profile has been also digitized from aerial photography of photoscale 1:33,000 and represents a strongly undulating and forested land form (Fig. 25), with undulations of 120m on average and up to a maximum of 500m. It also consists of 1290 points with a sampling interval of 5m. The profile, its parts and their spectra in a double logarithmic scale are plotted in Figures 26-28.

c. Profile P3

It consists of 1161 points digitized with a sampling interval of 5m from aerial photography (Fig. 29) of 1:37,000 photoscale. It contains a number of characteristic features: an ancient outcrop, rivers,
marshland, and farmland. The undulations are in order of 1m, except for the outcropping rock with undulations of 2m. The actual profile, its parts and their spectra are given in Figures 30-32.

d. Profile P4

This profile has been digitized from existing aerial photography of higher photoscale (1:78,000). It consists of 903 points with a sampling interval of 8m. The characteristic land form (Fig. 33) is a very gently undulating smooth agricultural area, with undulations of 3m over 100m. Both the profile and its spectrum are plotted in Figures 34 and 35.
Figure 21: Aerial photography of area of profile Pl.
Figure 22: Profile P1.
Figure 23: Profile P11 and its power spectrum.
Figure 24: Profile P12 and its power spectrum.
Figure 25: Aerial photography of area of profile P2.
Figure 26: Profile P2.
Figure 27: Profile P21 and its power spectrum.
Figure 28: Profile P22 and its power spectrum.
Figure 29: Aerial photography of area of profile P3.
Figure 30: Profile P3.

Elevation (meters)

Distance (meters)

$10^{-1}$
Figure 31: Profile P31 and its power spectrum.
Figure 32: Profile P32 and its power spectrum.
Figure 33: Aerial photography of area of profile P4.
Figure 34: Profile P4.
Figure 35: Power spectrum of profile P4.
The spectra of the four profiles have been subsequently analyzed as before. From the slopes of their spectra in a double logarithmic scale (see Chapter 5) the order of integration, corresponding to our model of fractionally integrated white noise, has been estimated and tabulated below.

Table 2: Order of fractional integration $v$ of the test profiles P1, P2, P3, P4.

<table>
<thead>
<tr>
<th>Profile</th>
<th># of pts.</th>
<th>$v$</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>P11</td>
<td>695</td>
<td>1.35</td>
</tr>
<tr>
<td>P12</td>
<td>595</td>
<td>1.35</td>
</tr>
<tr>
<td>P2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>P21</td>
<td>668</td>
<td>1.44</td>
</tr>
<tr>
<td>P22</td>
<td>622</td>
<td>1.40</td>
</tr>
<tr>
<td>P3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>P31</td>
<td>515</td>
<td>1.23</td>
</tr>
<tr>
<td>P32</td>
<td>646</td>
<td>1.10</td>
</tr>
<tr>
<td>P4</td>
<td>903</td>
<td>1.20</td>
</tr>
</tbody>
</table>

In profiles P1 and P2 the $v$ values are in the order of 1.4, showing a remarkable agreement and indicating very smooth and old forms.
The profiles P3 and P4 yielded values of $v$ close to 1.2, indicating very small irregular undulations, which may be partly due to errors in data capture.

In all cases the order of fractional integration $v$ agrees with our perception of terrain roughness, which is independent of the plotting scale. Although research on this subject goes beyond the scope of this study, it is hoped that future insights could lead to the development of a terrain classification algorithm, which is much needed in Cartography.

7.3.2 Data processing

The data processing consists of two steps. In the first step, for each profile the inversion algorithm is applied and the underlying white noise is estimated. We have assumed that each profile can be decomposed in 8 levels, with increasing degree of detail, overlapping each other. This corresponds to $2^{(8-1)} + 1 = 129$ points processed at a time. In other words each profile has been decomposed in parts of 129 points processed sequentially. No overlap between the sections was applied. We should note that the maximum number of levels we have used has been chosen arbitrarily and in practice 2-3 levels will be normally sufficient. Remember that if the plotting scale
of the last \( n^{th} \) level is 1:M, the \((n-1)^{th}\) level corresponds to generalization at scale 1:2M, the \((n-2)^{th}\) level to scale 1:4M and so on. The point is that in practice we will probably never need a scale reduction of more than 4 times the original one. But in any case the algorithm is quite flexible in respect to this. It should also be noted that depending on the selected number of levels, we can choose the number of points, which are simultaneously processed in the adjustment [see eq. (5.7)]. The fact that we have chosen to work with 8 levels should not be interpreted as optimal number, if such an optimal number exists. In this study we are more interested in the functionality and correctness of the algorithm and less in its efficiency and optimality.

In the adjustment process, a value of \( \epsilon = 10^{-10} \) has been used in the definition of the weights [eq. (6.28)]. Test runs have proved that the value of \( \epsilon \) does not influence the results as long as it is small (in the range of \( 10^{-6} - 10^{-10} \)). The value \( \epsilon = 10^{-10} \) has been chosen only because somewhat faster convergence of the process has been experienced. On the other hand, instead of externally restricting the number of iterations by some convergence criterion, we let the solution converge in 10 iterations, which in most of the cases proved to be too many. Finally in all the cases we were able to represent the actual profiles with a set of random numbers in a compressed form.
In order to make clear such a compression, in the following table we tabulate the number of non-zero random numbers after each iteration for the four profiles.
Table 3: Number of non-zero random numbers after each iteration for the four test profiles.

<table>
<thead>
<tr>
<th>Iteration #</th>
<th>P1 1290 pts</th>
<th>P2 1290 pts</th>
<th>P3 1161 pts</th>
<th>P4 903 pts</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2610</td>
<td>2610</td>
<td>2349</td>
<td>1827</td>
</tr>
<tr>
<td>2</td>
<td>2577</td>
<td>2593</td>
<td>2290</td>
<td>1735</td>
</tr>
<tr>
<td>3</td>
<td>2347</td>
<td>2439</td>
<td>1995</td>
<td>1404</td>
</tr>
<tr>
<td>4</td>
<td>1943</td>
<td>2057</td>
<td>1638</td>
<td>1162</td>
</tr>
<tr>
<td>5</td>
<td>1634</td>
<td>1691</td>
<td>1412</td>
<td>1029</td>
</tr>
<tr>
<td>6</td>
<td>1462</td>
<td>1517</td>
<td>1276</td>
<td>962</td>
</tr>
<tr>
<td>7</td>
<td>1363</td>
<td>1400</td>
<td>1198</td>
<td>932</td>
</tr>
<tr>
<td>8</td>
<td>1319</td>
<td>1337</td>
<td>1144</td>
<td>890</td>
</tr>
<tr>
<td>9</td>
<td>1290</td>
<td>1315</td>
<td>1121</td>
<td>867</td>
</tr>
<tr>
<td>10</td>
<td>1270</td>
<td>1300</td>
<td>1096</td>
<td>833</td>
</tr>
</tbody>
</table>

The above number of random numbers include only the random numbers which are non-zero. A threshold value of $10^{-4}$ has been arbitrarily set to decide if a random number should be considered as non-zero; random number smaller than
this threshold were set to zero and only the non-zero ones have been counted.

After the estimation of the random numbers, the second step in the data processing follows. All the non-zero random numbers are inserted back into the mathematical model [eq. (6.13)], for the computation of the different levels for each profile. In the following figures (Fig. 36-39) these different levels have been plotted at the original scale, in order to present the successive elimination of detailed information. The chosen original scales are in conformity with those suggested by the International Society of Photogrammetry [Ghosh (1971)], with respect to the photoscales.

Although irrelevant to our discussion on terrain generalization, some interesting facts that emerged during the course of the data processing are addressed here. First, the convergence rate is not the same for all the profiles; for profile P4 7-8 iterations are enough, whereas for P2 may be 10 iterations are still not adequate. Second, the selected threshold value of $10^{-4}$ clearly affects the apparent convergence rate, and is obviously responsible for the "fluctuations" at the last iterations. Third, we should note that the convergence rate of the process, the selection of optimal values for $\epsilon$, and for the threshold, the required number of iterations, and other such issues are not raised here, simply because they are
problems connected to the data compression, which is clearly not our main topic. As far as the generalization technique itself is concerned, it would not matter if we would stop at the 5th iteration, or perform no iterations at all. We simply tried to develop a technique, that would not need to use more points than the original profile consists of. We have shown that by a robustification of our estimation problem, we are able to drastically reduce the number of random numbers required to exactly reconstruct the profile. This number is shown to be at most equal to the number of points of the original profile. If by continuing the iteration process we could further compress the included information to fewer parameters, it is clearly desirable, and for the benefit of a data compression algorithm, but not necessary as far as the generalization technique is concerned.
Figure 36: Different generalization levels for profile Pl plotted at original scale.  
Horizontal scale = 1:30,000  
Vertical scale = 1: 2,000
Figure 37: Different generalization levels for profile P2 plotted at original scale.
Horizontal scale = 1:30,000
Vertical scale = 1:5,000
Figure 38: Different generalization levels for profile P3 plotted at original scale. Horizontal scale = 1:30,000 Vertical scale = 1:2,000
Figure 39: Different generalization levels for profile P4 plotted at original scale.
Horizontal scale = 1:30,000
Vertical scale = 1: 2,000
However the generalization results cannot be correctly evaluated unless the different levels are plotted at their correct scales. Given the photoscales we choose the scales 1:20,000 horizontal and 1:2,000 (1:5,000 for P3) vertical as the plotting scales of the original four profiles. Then each subsequent level is plotted to twice the scale of the previous level, as shown in the next figures. (Figures 40 - 43)
Figure 40: Generalization of profile Pl at different scales.
Figure 41: Generalization of profile P2 at different scales.
Figure 42: Generalization of profile P3 at different scales.
Figure 43: Generalization of profile P4 at different scales.
7.3.3 Comments on generalization results

The generalization results seem to be quite satisfactory. Unfortunately in the related literature there exist no means, eg. a performance index, to express and evaluate the performance of a generalization algorithm, and we will therefore use only visual judgement of the results.

Although we cannot measure the performance of the algorithm using quantitative means, we can use nevertheless qualitative aspects. When we manually generalize terrain an intuitive requirement is to preserve the character of the terrain, that is its roughness and its characteristic points (i.e. points on breaklines). In computer assisted generalization although the same requirement is still valid, it is not easily achievable. Let us see how the proposed algorithm fulfills this requirement.

First, we have assumed that the terrain roughness is described by the order of fractional integration of the underlying white noise; or in other words, it is described by the slope of the spectrum in a double logarithmic scale. In the recursive simulation algorithm we have controlled the magnitude of the detail [see eq. (5.6)], which is added in each level, such that the slope of the spectrum remains constant. This means that at every level the terrain is described by the same fractional integration order. In
other words, the roughness of the terrain is the same for all levels. The inverse operation, therefore, also preserves the roughness of the terrain through the different levels. Therefore our generalization algorithm does not change the terrain roughness.

Second, it is obvious (Fig. 40) that our algorithm also preserves breaklines. Besides, it works successfully even if the terrain is not homogeneous (Figures 40-42). We will not try here to theoretically explain this characteristic of the algorithm. It can be proved, however, that the estimation method we have followed in the inverse operation is equivalent to estimation using the so-called \( L^b \) splines [eq. Inoue (1986)], where \( b \) is a real number. And the characteristic of the splines to preserve breaklines is well known. From the above we conclude that the character of the terrain is preserved during the generalization, and this is the reason why, at least visually, the results are satisfactory.
CHAPTER VIII
CONCLUSIONS

Terrain modelling using the theory of random fields and fractional calculus has been proven to be flexible and efficient. The resulting terrain model, as fractionally integrated white noise, has given rise to a recursive simulation algorithm, which is presumably useful in many cartographic applications. With the use of a robust estimation scheme the simulation process has been inverted, and a new insight in many cartographic problems has been obtained. A direct application of the inverse operation has led to a proposed terrain generalization algorithm, which demonstrates the applicability of the studied theory to cartographic problems.

The theory of random fields, although is well developed theoretically, and has found application in other fields, it has never been systematically studied in terms of its potential application in cartographic problems. In the course of this research we have proved that random fields is an excellent tool in modelling terrain. Beginning with a random process such as white noise we have
demonstrated that it is possible to generate features, which resemble the systematic character of actual terrain. The resulting random field is not stationary but has stationary increments.

The developed simulation algorithm, although an approximation, sufficiently demonstrates the terrain modelling scenario. The simulated profiles show remarkable resemblance to actual terrain, and can be effectively used in a future terrain classification. Subsequently, the applicability of the proposed terrain model to generalization has been proved by the developed inversion algorithm.

The evaluation of the generalization algorithm, through actual terrain profiles, has yielded desirable results. The applicability and efficiency of the algorithm are due mainly to two properties of the process. First, it preserves the character of the terrain; namely, it preserves the terrain roughness, and the characteristic points, unlike other generalization algorithms, where additional constraints on such points are required. Second, it facilitates the options of interactivity and heuristics, whenever special problems require such options. Finally, it is noted, that the generalization problem is not tackled as a problem remote from other cartographic problems, but in the context of terrain modelling, classification, and data compression. In fact, such an
approach, not only gives a solid theoretical background to terrain generalization, but also integrates the solution of many cartographic problems, as this is the case in a production environment.

Besides its theoretical beauty, the theory of random fields could be actually utilized in the solution of a variety of cartographic problems. The conducted research is neither exhaustive nor exclusive on the subject. It is only a first step on the topic and further research is therefore recommended.
BIBLIOGRAPHY


Lanczos, C., "Discourse on Fourier Series", Oliver and Boyd Co., Edinburg.


APPENDIX A

FURTHER ELABORATION ON THE STRUCTURE FUNCTION (4.18)
Here we will further elaborate on the form (4.18) of the structure function, developed in section 4.2.3. One of its derived forms [see eq. (A.6)] is very helpful in understanding the connection between the Weierstrass function, exposed in section (4.3.1) and random fields with stationary increments.

Beginning with the structure function of a random field $\xi(t)$ with stationary increments of order $n$:

$$D^{(n)}(t; \tau_1, \tau_2) = \int_{-\infty}^{\infty} e^{it\lambda} (1-e^{-i\tau_1\lambda})^n (1-e^{i\tau_2\lambda})^n \frac{(1+\lambda^2)^n}{\lambda^{2n}} dF(\lambda) \quad (A.1)$$

and substituting for the spectral density function

$$f(\lambda) = dF(\lambda) \quad (A.2)$$

we have:

$$D^{(n)}(t; \tau_1, \tau_2) = \int_{-\infty}^{\infty} e^{it\lambda} (1-e^{-i\tau_1\lambda})^n (1-e^{i\tau_2\lambda})^n \frac{(1+\lambda^2)^n}{\lambda^{2n}} f(\lambda) d\lambda \quad (A.3)$$

or:

$$D^{(n)}(0; \tau, \tau) = \int_{-\infty}^{\infty} (1-e^{-i\tau_1\lambda})^n (1-e^{i\tau_1\lambda})^n \frac{(1+\lambda^2)^n}{\lambda^{2n}} f(\lambda) d\lambda \quad (A.4)$$

But the expression

$$(1-e^{-i\tau_1\lambda})^n (1-e^{i\tau_2\lambda})^n = [(1-e^{-i\tau_1\lambda})(1-e^{i\tau_2\lambda})]^{n/2} [(1-e^{-i\tau_3\lambda})(1-e^{i\tau_4\lambda})]^{n/2}$$
and since
\[ (1-e^{-i\tau\lambda})(1-e^{i\tau\lambda}) = (1 - e^{-i\tau\lambda} - e^{i\tau\lambda} + 1 = 2 - (e^{i\tau\lambda} - e^{-i\tau\lambda}) = \]
\[ \frac{2}{1} \left[ 1 - \frac{e^{i\tau\lambda} - e^{-i\tau\lambda}}{2} \right] = 2(1 - \cos \tau \lambda) \]
therefore
\[ \left[ (1-e^{-i\tau\lambda})(1-e^{i\tau\lambda}) \right]^{n/2} = 2^{n/2} (1 - \cos \tau \lambda)^{n/2} \]
and
\[ (1-e^{-i\tau\lambda})^n (1-e^{i\tau\lambda})^n = 2^n (1 - \cos \tau \lambda)^n \]
Thus (A.4) becomes:
\[ \mathbf{D}[n](0; \tau, \tau) = 2 \int_{-\infty}^{\infty} (1 - \cos \tau \lambda) \left( \frac{1 + \lambda^2}{\lambda^{2n}} \right)^n f(\lambda) d\lambda \]  
(A.5)
Substituting for \( f(\lambda) \) the expression (4.19) we get:
\[ \mathbf{D}[n](0; \tau, \tau) = 2^n C \int_{-\infty}^{\infty} (1 - \cos \tau \lambda)^n \left( \frac{1 + \lambda^2}{\lambda^{2n}} \right)^n \cdot \frac{\lambda^{2(n-v)}}{(1+\lambda^2)^{n}} d\lambda = \]
\[ = 2^n C \int_{-\infty}^{\infty} \frac{(1 - \cos \tau \lambda)^n}{\lambda^{2n}} d\lambda \]  
(A.6)
\[ = C \int_{-\infty}^{\infty} \frac{\sin^2 n \tau \lambda}{\lambda^{2n}} d\lambda \]  
(A.7)
we can furthermore elaborate on (A.7) as follows:

\[ D^{[n]}(0; \tau, \tau) = C \int_{-\infty}^{\infty} \frac{\sin^2 \lambda \tau}{\lambda^2 v} \, d\lambda = \]

\[ = -C \cdot \frac{\Gamma(1-2v) \cos^n \left( \frac{1-2v}{2} \right) \pi}{2^n (1-v)} = \]

\[ = -C \cdot \frac{\Gamma(1-2v) \sin^n (v \pi)}{2^n (1-v)} = \]

\[ = -C \cdot \Gamma(1-2v) \sin^n (v \pi) \cdot 2^n (v-1) = \]

\[ = \frac{-2 \cdot C}{4} \cdot \Gamma(1-2v) \cdot \sin^n (v \pi) \quad (A.8) \]

Evaluating (A.8) for \( \tau=1 \) and \( n=1 \), i.e. for stationary increments of the first order we finally get

\[ D(0; 1, 1) = -\frac{2^v \cdot C}{4} \cdot \Gamma(1-2v) \sin(v \pi) \quad (A.9) \]
APPENDIX B

TOEPLITZ FORMS
It has been noticed that certain integral equations of the Volterra type [Froberg (1969)]:

\[ f(x) = \int_{a} K(x,y) g(y) \, dy \]  

(B.1)

involve kernels \( K(x,y) = K(x-y) \), which depend only on the difference \( x-y \) and not on the individual \( x \) and \( y \). Such forms are always involved [see theorem, p.170, Granander & Szego (1958)] in estimation problems of stationary random fields, and they are called Toeplitz forms. The resulted matrices are of special structure, which if exploited can considerably reduce expenditures of machine resources.

More formally [Arushanian et.al. (1983)], we call a Toeplitz matrix, any real or complex square matrix of the form

\[
T = \begin{bmatrix}
\alpha_0 & \alpha_1 & \alpha_2 & \ldots & \alpha_{m-1} \\
\alpha_{-1} & \alpha_0 & \alpha_1 & \ldots & \alpha_{m-2} \\
\alpha_{-2} & \alpha_{-1} & \alpha_0 & \ldots & \alpha_{m-3} \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
\alpha_{-m+1} & \alpha_{-m+2} & \alpha_{-m+3} & \ldots & \alpha_0
\end{bmatrix}
\]  

(B.2)

whose elements along the main diagonal and along each co-diagonal are equal. Such a matrix \( T \) is completely specified by its first row and column.
There are many classes of Toeplitz matrices, and we can distinguish between simple Toeplitz, block Toeplitz matrices etc. A special category of Toeplitz matrices which is of our interest are the column-circulant Toeplitz matrices defined as follows.

A column-circulant matrix, is a Toeplitz matrix, with the further property that

\[ a_{-i} = a_{m-i} , \quad i = 1, 2, \ldots, m-1 \]  

(B.3)

Thus, it has the representation

\[
CT = \begin{bmatrix}
  a_0 & a_{m-1} & a_{m-2} & \cdots & a_{m-l+1} \\
  a_1 & a_0 & a_{m-1} & \cdots & a_{m-l+2} \\
  a_2 & a_1 & a_0 & \cdots & a_{m-l+3} \\
  \vdots & \vdots & \vdots & \ddots & \ddots \\
  a_{m-1} & a_{m-2} & a_{m-3} & \cdots & a_{m-l}
\end{bmatrix}
\]

(B.4)

with row order \( m \), at least equal to its column order 1.

A column-circulant Toeplitz matrix is completely specified by its first column; each further column may be obtained from the previous one by a downward cyclic shift.

As it has been mentioned in section 4.2.3, the structure function of a random field with stationary increments has the form (4.22) and depends only on the lengths of the increments. Then according to the theorem
lengths of the increments. Then according to the theorem of Granander & Szego (1958) (pp.170) it is a Toeplitz form.

Variance-covariance matrices arising from such a structure function are therefore Toeplitz matrices. For demonstration reasons we have computed variance-covariance matrices based on the structure function (4.22) for varying degree v of fractional integration. Two examples are given below.

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It is clear that these matrices are actually column-circulant Toeplitz, and are therefore symmetric and persymmetric (ie. symmetric with respect to the other main diagonal as well).

The special structure of column-circulant Toeplitz matrices has been well studied in the literature and many
algorithms, for their inversion, have been suggested over the years [see eg. Kutikov (1967)].

For the inversion of such matrices we only need to store one column (or row) and not the whole matrix; thus drastically reducing the required computer memory. On the other hand, the suggested algorithms implement the characteristics of column-circulant Toeplitz, and the resulted inversion is much faster than the inversion of regular matrices.

The advantage, of implementing column-circulant Toeplitz matrices in terrain related problems, is very important. The point is that problems (like interpolation, contouring, prediction, and filtering), where inversion of variance-covariance matrices are required, can be solved in real time, using smaller computers.
APPENDIX C

SIMULATION EXAMPLES USING THE DEFINITION OF FRACTIONAL INTEGRAL
Figure 44: Simulation example 1. Seed value is 325.0, v=1.4.
Figure 45: Simulation example 2. Seed value is 92,705.0, $\nu=1.4$. 
Figure 46: Simulation example 3. Seed value is 100.0, \( v=1.4 \).
APPENDIX D

RECURSIVE SIMULATION EXAMPLES
Figure 47: Recursive simulation example 1. Seed value is 92,705.0, v=1.4. Simulation levels 2 through 7.
Figure 48: Recursive simulation example 1. Final (level-8) simulated profile and its power spectrum in a double logarithmic scale.
Figure 49: Recursive simulation example 2. Seed value is 10.0, v=1.4. Simulation levels 2 through 7.
Figure 50: Recursive simulation example 2. Final (level-8) simulated profile and its power spectrum in a double logarithmic scale.
Figure 51: Recursive simulation example 3. Seed value is 214,748,364.0, v=1.4. Simulation levels 2 through 7.
Figure 52: Recursive simulation example 3. Final (level-8) simulated profile and its power spectrum in a double logarithmic scale.
APPENDIX E

ROBUST ESTIMATION EXAMPLES OF GEOMETRIC FEATURES
Figure 53: Example #1: Level-by-level reconstruction and robust estimation results.
At Level #3 through Level #5 there appears no spike.

Figure 54: Example #2: Level-by-level reconstruction and robust estimation results.