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Relative risk aversion and stochastic dominance in multiattribute decision making

Lee, Dae Joo, PH.D.
The Ohio State University, 1987

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RELATIVE RISK AVERSION AND STOCHASTIC DOMINANCE
IN MULTIATTRIBUTE DECISION MAKING

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By

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* * * * *

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To My Parents
ACKNOWLEDGEMENTS

I would like to express my deep appreciation to my academic adviser, Dr. Richard A. Miller, for his invaluable guidance and constructive criticism throughout my study. I learned from him not only how to think but also what to think to be a scholar.

I am particularly indebted to my co-adviser, Dr. Jane M. Fraser, for her enthusiastic support and vigorous suggestions in preparing my dissertation. She taught me how to do research and helped almost all aspects in and out of my dissertation.

I also would like to acknowledge the important contributions of my Dissertation Committee and General Examination Committee members: Dr. Gordon M. Clark, Dr. John B. Neuhardt, and Dr. Richard J. Jagacinski.

I want to express my warm thanks to my colleagues in the Department of Industrial and Systems Engineering for their friendly advice and care. Without them, my life would have been boring and dreary.
My special thank goes to Ms. Lee Ann Brentlinger. She provided me the opportunity to have a financial support when it was most needed. Without it, I must have been at home three years ago.

My deepest gratitude goes to my beloved wife, Jae Yoon for her patience and understanding during the days and nights of my graduate study. Her faith in me has been the most driving force in the course of my graduate career.

Last, but not the least, I would like to express my deepest appreciation to my parents. My father not only gave me devoted support but also showed himself the way to be a scholar. My mother provided me endless encouragement whenever I needed it.
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CHAPTER I
INTRODUCTION

Human decision making has been one of the main subjects of many disciplines for a long time and it is easy to find literature in the areas of economics, psychology, business, and (industrial) engineering which deal with human decision making. This research aims at helping a single decision maker in making better decisions important to him. To do so, it is important not only to understand but also to improve the decision making process of the human.

Human decision making can be divided into two categories depending on whether the decision maker has full knowledge of the consequences\(^1\) of his choices: decision making under certainty and decision making under uncertainty. If the decision maker has full knowledge of the consequences, then we call this decision making under certainty. In other words, the decision maker knows for sure what the consequence will

---

1. The words "consequences" and "outcomes" are used interchangeably.
be when he chooses an alternative or an action\(^2\). In this case we can use a real-valued function on the set of all possible consequences to assign a ranking to all possible consequences (or alternatives). This function, which we will call a value function, represents the decision maker's preference by assuming that the preference relation is transitive and the consequences are comparable.

If we are interested in more than ordering of consequences, then it is necessary to develop a function, which we will call a cardinal value function, that not only represents a decision maker's preference structure under certainty but also provides an interval scale of measurements. Thus, a cardinal value function represents the decision maker's strength of preference. Note that a cardinal value function is unique up to a positive linear transformation while a value function is unique up to a positive monotone transformation.

But the decision maker does not always have full knowledge of the consequences. Often he may not know the outcome(s) of his action for sure because of uncertainty inherent in the nature of his choices. For instance, a lottery ticket may

\[^2\text{Actually, an action is the choice of an alternative which results in a consequence or some possible consequences. The words "action" and "alternative" are used interchangeably.}\]
result in either winning one million dollars or just losing the price of the ticket. That is, he may not know for sure what the consequence of his action (or alternative) will be even though he knows what the possible consequences are. Thus, we cannot use a (cardinal) value function in this case because the decision maker's preference under uncertainty is different from the one under certainty. In this case, we can use a function, which we will call a utility function, that can be used to represent his preference among lotteries on the consequences. Note that a utility function is unique up to a positive linear transformation.

In 1944, von Neumann and Morgenstern[40] formally introduced expected utility as the criterion in evaluating and selecting the most preferred alternative. They claimed that they "practically defined numerical utility as being that thing for which a calculus of expectations is legitimate"[34]. But it should be made clear at this point that we do not address the problem of what utility function is best for an organization to which a decision maker belongs. As Keeney and Raiffa[20] pointed out that we assume "the decision maker believes that in a specified decision context there is a particular preference structure that is appropriate for him." Here, a utility function represents a decision maker's preference among lotteries which are determined by two factors: strength of preference and attitude toward risk.
Since then, there have been many developments in assessing the decision maker's utility function and also in assessing his probabilities[34]. Also there have been claims from the area of psychology that the expected utility model does not describe well how a person makes decisions[11],[12],[18],[25],[26],[36],[37],[38],[39]. These assertions and claims are related mainly to the questions of assessing the decision maker's utility function and the descriptive power of the expected utility model. These questions implicitly suggest that results of assessment of the utility function are not always consistent. And this is particularly evident when an alternative results in multiattribute consequences. This research is an attempt to address such issues.

This research develops a method of aiding a decision maker to choose an alternative among available alternatives in a decision making situation under uncertainty with incomplete knowledge where possible consequences of an alternative are represented as multiple attributes. By uncertainty I mean the lack of knowledge, which is inherent in the states of nature concerning consequences. Uncertainty is usually represented by a probability distribution of consequences for each alternative. Incomplete knowledge concerns the assessment of the decision maker's utility function. That is, the decision is to be made when the utility function is not known exactly.
In this research, the following assumptions will be used. First, it is assumed that the feasible alternatives are fixed and that the attributes are fixed. Second, the decision is thought to be very important for the decision maker so that he wants his preference and attitude toward risk to be reflected as accurately as possible. Third, it is assumed that all possible consequences are, at least, on an ordinal scale and the consequences are numerically measurable. Finally, the decision maker prefers a consequence with a higher numerical value of utility to that with a lower one.

With these assumptions, an attempt will be made to describe the decision maker's utility function in multiattribute decision making under uncertainty. To do so, a multivariate cardinal value function will be used to represent the decision maker's strength of preference on the multiattribute consequences under certainty. Next, a utility function with respect to the cardinal value function will be defined to separate his risk attitude from the strength of preference. This utility function represents his relative risk attitude without the effect of his strength of preference.

A decision maker's utility function \( u \) can be accomplished in a composite functional form \( u_v(v) \) by first constructing a multivariate cardinal value function \( v \) and then constructing a utility function with respect to the cardinal value function \( u_v(\cdot) \). As will be shown in Chapter III, a decision
maker's utility function constructed from the two steps is invariant up to a positive linear transformation. Suppose a decision analyst assessed the decision maker's cardinal value function as \( v(x) \), his utility function given \( v(x) \) as \( u_v(v) \), thereby constructing his utility function as \( u(x) = u_v(v(x)) \). Again, another decision analyst assessed the decision maker's cardinal value function as \( \overline{v}(x) \), which is strategically equivalent to \( v(x) \), his utility function given \( \overline{v}(x) \) as \( w_v(v) \), thereby constructing his utility function as \( w(x) = w_v(\overline{v}(x)) \). Then, the utility functions \( u(x) \) and \( w(x) \) are strategically equivalent. Using this composite function, we are able to employ the expected utility model to find the most preferred alternative. It is important to note here that I do not intend to decompose a given utility function of a decision maker into a cardinal value function and a utility function with respect to the cardinal value function.

This representation of a utility function has been developed for a single attribute utility function. The contribution of this dissertation is to extend the representation to multiattribute utility functions by defining the notion of multiattribute relative risk aversion and by developing its measure.

Finally, stochastic dominance will be developed by acknowledging that assessment of a utility function is not easy. In
doing so, separation of the factors affecting the decision maker's preference will be used. Stochastic dominance is used to find the set of nondominated alternatives among available alternatives under the assumption that the decision maker's utility function is not exactly known\(^3\). This set is guaranteed to include the most preferred alternative if exact information on the decision maker's utility function is provided.

The contribution of this dissertation is to combine the representation of the multiattribute utility functions and the measure of multiattribute relative risk aversion with stochastic dominance in multiattribute decision making under uncertainty.

As a result, it is hoped to reduce the conflict between the desire to reflect the real world environment and the decision maker's perception of the problem and the desire to reduce the need to assess and aggregate large amounts of information when a multiattribute decision is to be made under uncertainty with incomplete knowledge. The organization of the dissertation is as follows.

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3. When we say that the function is not exactly known but its functional form is known, it implies that we only have information of its first and/or second partial derivatives.
In Chapter II, I address the problem of decision making under certainty using a multivariate cardinal value function to represent the decision maker's strength of preference in a multiattribute decision making problem. Also, I will present a set of equivalent criteria to compare different cardinal value functions in the form of a theorem.

In Chapter III, I address the problem of decision making under uncertainty using a utility function with respect to the multivariate cardinal value function used in Chapter II. The function captures the decision maker's attitude toward risk. In this chapter, I will define a measure of multiattribute relative risk aversion. This is a major step in extending the separation of the decision maker's attitude toward risk and strength of preference to multiattribute decision making so that we can assert that the decision maker's attitude toward risk may not always be relatively risk averse even when it is said to be risk averse.

Chapter IV focuses on stochastic dominance using the functional form of the decision maker's utility function in combination with the separation mentioned in Chapter II. I will define a stochastic dominance principle using the measure of decision maker's multiattribute relative risk aversion defined in Chapter III instead of using the usual measure of risk aversion.
The final chapter summarizes the results of this research and suggest further research topics stemming from what I accomplished in Chapters II, III, and IV.
If a decision is to be made under uncertainty, an alternative is usually described in the form of a lottery among outcomes each of which may be multi-attributed. Often the decision maker's preference among the outcomes and thus among lotteries can be represented by a multiattribute utility function $u$. Preferences among lotteries are determined by two factors; (1) strength of preference for the multiattribute consequences under certainty, and (2) attitude toward risk.

Let $v$ be a multiattribute cardinal value function\(^1\) which represents the decision maker's strength of preference. Also let $u_v$ be a utility function with respect to the cardinal value function $v$, which describes his attitude toward risk. Then we can model his multiattribute utility function $u$ as a composite function $u_v(v)$.

This separation of a multiattribute utility function $u$ has several advantages. First, we can help a decision maker

\footnote{1. More precisely, any cardinal value function which is strategically equivalent to $v$ can be used.}
think about a choice under uncertainty by isolating their preferences under certainty and then incorporating their risk attitude relative to the value assigned to the consequences. Second, we can assess the utility function \( u \) in an indirect manner when direct assessment of \( u \) is difficult. Furthermore, use of a multiattribute cardinal value function \( v \) is advantageous when the levels of attributes are not commensurable. Finally, we can focus on the decision maker's relative attitude toward risk without the effect of his strength of preference.

Once this separation has been established, we can apply the above findings concerning stochastic dominance to eliminate dominated alternatives even when the multiattribute utility function is not exactly known.

2.1 Human Decision Making

Decision making problems can be classified into two categories: decision making under certainty and decision making under uncertainty. The classification depends on the degree of knowledge the decision maker has about alternatives.

If a decision is to be made under certainty, an alternative is mapped into a unique consequence. That is, a
multiattribute consequence \( x \) is a numerical realization by a mapping from the alternative space (or empirical relational structure\(^2\)) to the consequence space (numerical relational structure\(^3\)). Then, the decision maker's strength of preference on the consequence can be represented by a multiattribute cardinal value function \( v \) and the value of a consequence \( x \) is represented as \( v(x) \). So we can effectively order a set of alternatives by comparing their corresponding values of \( v(x) \). That is, the cardinal value function \( v \) completely orders the set of consequences, thus, the set of alternatives.

If a decision is to be made under uncertainty, an alternative is usually described in the form of a lottery instead of a unique consequence with certainty. That is, a choice of an alternative does not result in a consequence for sure. So the decision maker's preference on the consequences can be represented by a multiattribute utility function \( u \) if the preferences among lotteries satisfy the following axioms [19], [31]. One of the well accepted utility functions which satisfy the assumptions is the von Neumann-Morgenstern (NM) utility function.

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2. See Krantz, et.al. [21]

3. See Krantz, et.al. [21]
The informal statements of axioms are:

1a. [Canonicity] The decision maker can imagine an experiment all of whose outcomes are "equally likely" in the sense that he would be indifferent between any two lotteries one of which entitles him to a certain valuable prize if some one, particular outcome occurs while the other entitles him to that same prize if some other one, particular outcome occurs.

1b. [Monotonicity] As regards any two lotteries one of which entitles him to a valuable prize if any one of \(n_1\) particular outcomes occurs while the other entitles him to that same prize if any one of \(n_2\) particular outcomes occurs, he will prefer the former lottery to the latter if and only if \(n_1\) is greater than \(n_2\).

2. The decision maker can select reference consequences \(c^*\) and \(c^*\) such that \(c^*\) is at least as attractive as any possible consequence of any of the available acts and \(c^*\) is at least as unattractive as any possible consequence of any of the available acts; and he can then:

a. [Evaluation of Consequences] Scale his preference for any possible consequence \(c\) by specifying a number \(\pi(c)\) such that he would be indifferent between (1) \(c\) for certain, and (2) a lottery giving a canonical chance \(\pi(c)\) at \(c^*\) and a complementary chance at \(c^*\);

b. [Evaluation of Events] Scale his judgment concerning any possible event \(E_0\) by specifying a number \(P(E_0)\) such that he would be indifferent between (1) a lottery with consequence \(c^*\) if \(E_0\) occurs, \(c^*\) if it does not, and (2) a lottery giving a canonical chance \(P(E_0)\) at \(c^*\) and a complementary chance at \(c^*\).
3. [Transitivity] As regards any set of lotteries among which the decision maker has evaluated his feelings of preference or indifference, these relations should be transitive. If, for example, he prefers lottery A to lottery B and is indifferent between lottery B and lottery C, then he should prefer lottery A to lottery C.

4. [Substitutability] If some of the prizes in a lottery are replaced by other prizes such that the decision maker is indifferent between each new prize and the corresponding original prize, then the decision maker should be indifferent between the original and the modified lotteries.

5. [Conditional and Posterior Preference] Lottery 1' should be preferred to lottery 1" given knowledge that E^ has occurred if and only if 1' would be preferred to 1" without this knowledge but with an agreement that both lotteries would be called off if E^ did not occur.

Preferences among lotteries are determined by two factors; (1) strength of preference for the consequences under certainty, and (2) attitude toward risk. Of these factors, the strength of preference is common to both decision making under certainty and under uncertainty. It can be represented by a cardinal value function v.

On the other hand, the attitude toward risk is unique to the case of decision making under uncertainty. The fact that preferences among lotteries can be described by a utility function u implies that the utility function is a mixture of both the strength of preference and the attitude toward risk of a decision maker.
2.2 Separation of Strength of Preference and Risk Attitude

When we analyze the decision maker's utility function, it is difficult to isolate the risk attitude from \( u \). Since \( u \) is a mixture of the two factors, we cannot state the decision maker's risk attitude independently of his strength of preference. Thus, the separation of the two factors becomes an important issue.

Let \( v \) be a cardinal value function on the set of all possible consequences which represents the decision maker's strength of preference. Also let \( u_v \) be a function with respect to the cardinal value function \( v \), which describes his attitude toward risk. Then we can model his utility function \( u \) as a composite function \( u_v(v) \).

Use of the composite function \( u_v(v) \) has several advantages. First of all, we conjecture that we can help a decision maker think about a choice under uncertainty. People may not feel comfortable at making decisions given a set of alternatives whose results are uncertain. Thus, they may deal with a problem as if it were a decision making under certainty and

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4. A function \( v \) provides an interval scale of measurement. The range of \( v \) is the set of real numbers and the domain is the set of all possible consequences.
then try to incorporate their knowledge of uncertainty into it. Suppose a decision maker is faced with a situation of choosing either Option A or Option B. Option A will result in 10 apples when the head of a coin appears and 3 apples when the tail appears. Similarly, Option B will result in 8 and 3 apples, respectively.

Now, we can think of the following scenario. Assuming that the head(tail) appears, the problem reduces either to choose Option A' (A") or B' (B") where each option will result in 10(3) or 8(3) apples, respectively, for sure. That is, the original problem under uncertainty reduces to several subproblems under certainty. If we assume that he prefers larger number of apples to smaller number, then he prefers Option A' to B' and he is indifferent between A" and B". Thus, he may conclude by thinking of both chances of appearing head and tail that he would choose Option A.

In other words, he first divides the given situation according to the chances. Now, he has several subproblems of decision making under certainty. Then, he assesses his value function to each consequence under certainty regardless of his risk attitude. In the next stage, he incorporates his knowledge of uncertainty to come up with a conclusion. Similarly, Tversky[36] argued that people modify the given situation as if they were problems under certainty; he called this the "isolation effect".
By acknowledging that people are prone to make inconsistent decisions given a choice problem under uncertainty, we argue that it would be better for them to think in two steps, first under certainty and and under uncertainty. This two-step method may not only lead them to make better decisions but also may give them better insight into the nature of the problem even though they may not naturally think that way.

The second advantage is that we can use the composite function \( u_v(v) \) when the attributes are not commensurable. If the attributes are represented in a fairly different fashion and thus not commensurable, then people sometimes find it difficult to aggregate information on these attributes in decision making under uncertainty. To ease this difficulty in a multiattribute decision making context, people often use heuristics such as lexicographic semi-order[10], elimination by aspects[36], the disjunctive rule, and the conjunctive rule. But these rules are not efficient because preference ordering from them is subject to the change of the order of the attributes being considered. Thus, it is better to aggregate attributes which are not commensurable using a cardinal value function and then assess the result under uncertainty to come up with his utility function.

The third advantage is that we can assess the utility function \( u \) indirectly when direct assessment of \( u \) sometimes results in inconsistent behavior. This can be done first by
assessing $v$, the cardinal value function under certainty. Next the assessment of $u_v$ is performed, which is a function with respect to the cardinal value function $v$. Then we can combine $v$ and $u_v$ to come up with the decision maker's utility function $u$. By doing so, we may have a better understanding of his decision behavior so we can describe his risk attitude more accurately in a decision making under uncertainty.

There is evidence that people are not good at assessing utility in making decisions under uncertainty. For instance, Fischhoff, et.al.[12] found that people sometimes find it difficult to evaluate or assign values to the possible consequences of alternatives in a decision making situation under uncertainty. That is, the direct assessment of the utility function is sometimes difficult in a decision making under uncertainty. Also, Peterson and Beach[28] argued that people show poor performance at inferential tasks as the degree of uncertainty increases.

Kahneman and Tversky[18] demonstrated that people are not good at making judgment on uncertain prospects. When decision making is done using extensive form of analysis [32,p129], we need to calculate conditional (and/or posterior) probabilities using Bayes theorem at chance nodes in a decision tree. But DuCharme[5], Edwards [9], and Phillips and Edwards[29] showed that people are, in general, conservative in assessing conditional (and/or posterior)
probabilities. Conservatism may be due to response bias[5], misperception[9], and misaggregation[29].

All this evidence illustrates that people are sometimes inconsistent in making decisions under uncertainty. But, it does not mean that people do not want to make a decision. Montgomery[27] emphasized "the fact that we have difficulties in evaluating the aspects of a decision situation cannot free us from the burden of making these evaluations." Thus, it may be better for a decision maker to use the separation of the two factors to assess his utility function in an indirect way than to assess it directly in order that his decision be consistent.

When we try to use the two steps, we have to have some means of estimating the decision maker's cardinal value function v and his utility function u_v given v. First, there are several procedures to estimate a multivariate additive cardinal value function such as direct rating, direct midpoint, and direct ordered metric[13]. These methods, however, can only be applied to additive multivariate cardinal value functions which is a rather strict assumption. So, in general, it is not always easy to estimate a multivariate cardinal value function.

Second, the assessment of u_v can be performed by setting up a simple lottery with prespecified values (like certainty
equivalent\textsuperscript{5}) similar to the estimation of a single attribute utility function. But it should be done with care because the attribute in this case may not be an intuitive one. Since the attribute is quite esoteric, the decision maker may feel difficulty in responding to the questions posed to him by a decision analyst.

The separation issue becomes even more important when the decision maker's utility function is not fully known. Even though we may know his strength of preference, we usually do not have a good grasp on the decision maker's preferences under uncertainty. Suppose that we do not know the decision maker's utility function, but that, we have information about the functional form of \( u_v \) and \( v \). We can employ stochastic dominance in order to determine the most preferred alternative or, at least, to determine the set of nondominated alternatives. This topic will be addressed in Chapter IV.

\textsuperscript{5} The certainty equivalent of a lottery \( L \) is an amount \( \hat{x} \) such that the decision maker is indifferent between the lottery itself and the amount \( \hat{x} \) for sure[20].
2.3 **Strength of Preference**

2.3.1 **Single Attribute Case**

Several researchers attempted to separate the two factors mentioned in the previous section so that we have a better understanding of what the decision maker has in mind on the problem. First, Debreu[4] and Krantz,et.al.[21] developed an axiomatic basis for a cardinal value function. They defined a so-called "positive difference structure" which requires the following axioms;

Definition 2. 16[21] Suppose that X is a nonempty set, X* a nonempty subset of X x X, and ≥* a binary relation on X*. The triple < X, X*, ≥* > is a positive difference structure if and only if , for all a, b, c, d, a', b', and c' ∈ X, and all sequences a₁, a₂, ..., aᵢ,.. ∈ X, the following six axioms are satisfied:

1. < X*, ≥* > is a weak order.
2. If ab, bc ∈ X*, then ac ∈ X*.

6. Whenever an important theorem or definition is stated, it will carry a certain number, e.g. 3.2. If it comes from other authors' work, then it will be followed by a bracketed reference number, e.g. [20].
3. If \( ab, bc \in X^* \), then \( ac >^* ab \) and \( ac >^* bc \).  
4. If \( ab, bc, a'b', b'c' \in X^* \), \( ab \geq^* a'b' \), and \( bc \geq^* b'c' \), then \( ac >^* a'c' \).
5. If \( ab, cd \in X^* \) and \( ab >^* cd \), then there exist \( d', d'' \in X \) such that \( ad', d'b, ad'' \), and \( d''b \in X^* \) and \( ad' \sim^* cd \sim^* d''b \).
6. If \( a_1, a_2, \ldots, a_i, \ldots \) is a strictly bounded standard sequence \( (a_{i+1}a_i \in X^* \), and \( a_{i+1}a_i \sim^* a_2a_1 \) for all \( a_i, a_{i+1} \) in the sequence; and for some \( d'd'' \in X^* \), \( d'd'' >^* a_1a_1 \) for all \( a_i \) in the sequence), then it is finite.

From the above definition of positive difference structure, it is possible to derive a function \( \Psi \) as follows[21]:

Theorem 2. 1[21] If \( < X, X^*, \geq^* > \) is a positive difference structure, then there exists \( \Psi : X^* \to \mathbb{R}^+ \) such that for all \( a, b, c, d \in X \), where \( \mathbb{R}^+ \) is the set of nonnegative real numbers

(i) if \( ab, cd \in X^* \), then \( ab \geq^* cd \) iff \( \Psi(ab) \geq \Psi(cd) \);
(ii) if \( ab, bc \in X^* \), then \( \Psi(ac) = \Psi(ab) + \Psi(bc) \).

If \( \Psi' \) also has these properties, then there exists \( \alpha > 0 \) such that \( \Psi' = \alpha \Psi \). If, in addition, for all \( a,

7. \( a >^* b \) if and only if \( a \geq^* b \) and not \( b \geq^* a \).
8. \( a \sim^* b \) if and only if \( a \geq^* b \) and \( b \geq^* a \).
Given Definition 2.1 and Theorem 2.1, Dyer and Sarin[8] proposed that a cardinal value function is a relevant representation of the decision maker's strength of preference. Now let $X$, in Definition 2.1 and Theorem 2.1, be the set of all possible consequences and $a$ and $b$ be elements of $X$. Then, $ab \in X^*$ represents the change of the consequence from $b$ to $a$ the decision maker realizes. Now, if we assume that the decision maker's preferences under certainty obey the axioms in Definition 2.1, we can define a binary relation $\succeq^*$ on $X^*$ where $X \times X \supseteq X^*$. Here, $ab \succeq^* cd$ means that the strength of preference for $a$ over $b$ is greater than or equal to the strength of preference for $c$ over $d$.

Then, by Theorem 2.1, there exists a function $\psi$ that represents the decision maker's preference differences. If the decision maker's preferences are complete, that is, either $ab \in X^*$ or $ba \in X^*$, then there exists a function $v : X \rightarrow \mathbb{R}^e$ such that $\psi(ab) = v(a) - v(b)$. That is, we can define a real-valued function $v$ from $X$ into $\mathbb{R}^e$ such that for all $a, b, c, d \in X$, the difference in the strength of preference between $a$ and $b$ is greater than the difference in the strength of preference between $c$ and $d$ if and only if
\[ v(a) - v(b) \geq v(c) - v(d) \quad (2.1) \]

and denoted as \( ab \geq^* cd \) for \( ab, cd \in X^* \). The function \( v \) is unique up to a positive linear transformation because

\[
\psi(ab) = v(a) - v(b) = [\alpha \cdot v'(a) + \beta] - [\alpha \cdot v'(b) + \beta] \\
= \alpha [v'(a) - v'(b)] \text{ for } \alpha > 0 \text{ and any } \beta.
\]

Thus, \( v \) is strategically equivalent to \( v' \).

As Theorem 2.1 implies it is easy to induce a binary relation \( \succ \) such that for all \( a, b, c \in X, ac \geq^* bc \) if and only if \( a \succ b \). In other words, if for all \( a, b, c \in X, ac \geq^* bc \), then \( \psi(ac) \geq \psi(bc) \) from Theorem 2.1 and \( v(a) - v(c) \geq v(b) - v(c) \) or \( v(a) \geq v(b) \) from (2.1). Thus, if \( ac \geq^* bc \) then \( a \succ b \). On the other hand, if \( a \succ b \), then \( v(a) \geq v(b) \) or \( v(a) - v(c) \geq v(b) - v(c) \), hence, \( ac \geq^* bc \). Therefore, a cardinal value function \( v \) represents the strength of preference of a decision maker. We can rank order a set of consequences (or alternatives) based on \( v(x) \) because under certainty every alternative maps into a consequence, \( x \), and every \( x \) is mapped into a \( v(x) \).

Dyer and Sarin\cite{8} introduced the concept of value satiation for a single-attribute cardinal value function \( v(x) \) under the assumption that \( v \) is increasing in \( x \). Value satiation is defined to be the decision maker's marginal increase in satisfaction scaled by \( v'(x) \) at consequence \( x \). As a measure for the cardinal value function \( v \), they defined \( m(x) \), called
coefficient of value satiation, as follows:

\[ m(x) = -\frac{v''(x)}{v'(x)} \]  \hspace{1cm} (2.2)

where \( v'(x) \) and \( v''(x) \) are the first and second derivatives of \( v(x) \), respectively. If \( m(x) \) is zero, then \( v''(x) \) is zero and \( v'(x) \) is a positive constant because \( v(x) \) is increasing in \( x \) by assumption. Thus, \( v(x) \) is a linear function with positive slope. Therefore, \( m(x) = 0 \) means constant marginal value. In a similar vein, \( m(x) > 0 \) (\( m(x) < 0 \)) corresponds to increasing (decreasing) marginal value.

Concerning the interpretation of \( m(x) \), Dyer and Sarin[8] considered the value of a small increase \( h \) in the consequence \( x \). Given regularity conditions\(^9\), there exists an increment \( \Delta \) such that

\[ v(x+h) = v(x) + \Delta \cdot v'(x) \]  \hspace{1cm} (2.3)

They defined \( s(x,h) = h - \Delta \) as the satiation sacrifice for the increase. It is interesting to compare (2.3) with the following:

\[ v(x+h) = v(x) + h \cdot q \]  \hspace{1cm} (2.4)

where \( q \) is the slope of the straight line connecting \( (x,v(x)) \)

\(^9\) \( v \) is increasing in \( x \) and continuously twice differentiable. Also, there exists a third derivative for \( v \) which is continuous.
and \((x+h, v(x+h))\). From (2.3) and (2.4), \(A \cdot v'(x) = h \cdot q\). If \(q = v'(x)\), then \(h = \Delta\) or \(s(x,h) = h - \Delta = 0\). That is, if \(v'(x)\) equals to the constant rate of change from \(x\) to \(x+h\), then the decision maker's preference difference is constant. If \(q < v'(x)\), then \(h > \Delta\) or \(s(x,h) = h - \Delta > 0\). That is, if \(v'(x)\), the rate of change of \(v(x)\), is larger than \(q\), i.e., \(v(x)\) is concave, then the decision maker's preference difference is decreasing. So, it can be thought of a local measure of departure from linearity.

The approximate relationship between the coefficient of value satiation \(m(x)\) and the satiation sacrifice \(s(x,h)\) was shown by Dyer and Sarin[8] as follows:

\[
s(x,h) \equiv (0.5)h^2 \cdot m(x) \tag{2.5}
\]

As they noticed, this approximation for \(s(x,h)\) is similar to Pratt's finding[30] that the risk premium for a small actuarially neutral risk is approximately \(r(x)\) times half the variance of the risk \(\sigma_x^2\), that is,

\[
\pi(x,z) \equiv (0.5)\sigma_x^2 \cdot r(x) \tag{2.6}
\]

where \(r(x) = - u''(x)/u'(x)\) and \(u(x)\) is a utility function.

### 2.3.2 Multiple Attribute Case

In general, we want a representation of the decision maker's preferences which can deal with multiattribute decision
making because an alternative may have more than one attribute. That is, it is hoped to extend the strength of preference notion to multiattribute decision making. Dyer and Sarin[7] extended their work on cardinal value functions to multiattribute decision making. However, they assumed that the attributes are preferentially independent10, which is a very strong assumption. Thus, their version of the multiattribute cardinal value function is the sum of single-attribute value functions. As far as I know nobody has attempted to analyze a multiattribute decision making problem under certainty without the assumption of preferential independence. The rest of this section accomplishes that task.

The extension of the strength of preference notion to multiattribute decision making is done by starting with a definition of positive difference structure in a multivariate situation. Since the main focus of this research is on multiattribute decision making, I present a straightforward re-statement of Definition 2.1 in which the objects \( a, b, c \) of \( X \) are now viewed as vectors \( \vec{a}, \vec{b}, \vec{c} \) of \( X \), to emphasize the multiattribute nature of the problem.

10. "The pair of attributes \( X \) and \( Y \) is preferentially independent of \( Z \) if the conditional preferences in the \( (x,y) \) space given \( z' \) do not depend on \( z' \)." ([20], p. 101)
Definition 2.2 Suppose that $\bar{X}$ is a nonempty set whose elements are vectors such that $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. Also, suppose $\bar{X}^*$ is a nonempty subset of $\bar{X} \times \bar{X}$ and $\geq^*$ a binary relation on $\bar{X}^*$. Thus, the binary relation $\geq^*$ is defined on $(\bar{X} \times \bar{X})$. The triple $< \bar{X}, \bar{X}^*, \geq^* >$ is a positive difference structure if and only if, for all $a, b, c, d, a', b'$, and $c' \in \bar{X}$, and all sequences $a^1, a^2, \ldots, a^i, \in \bar{X}$, the following six axioms are satisfied:

1. $< \bar{X}^*, \geq^* >$ is a weak order.

2. If $ab, bc \in \bar{X}^*$, then $ac \in \bar{X}^*$.

3. If $ab, bc \in \bar{X}^*$, then $ac \geq^* ab, bc$.

4. If $ab, bc, a'b', b'c' \in \bar{X}^*$, $ab \geq^* a'b'$, and $bc \geq^* b'c'$, then $ac \geq^* a'c'$.

5. If $ab, cd \in \bar{X}^*$ and $ab \geq^* cd$, then there exist $d', d'' \in \bar{X}$ such that $ad', d'b, ad''$, and $d''b \in \bar{X}^*$ and $ad' \geq^* cd \geq^* d''b$.

6. If $a^1, a^2, \ldots, a^i, \ldots$ is a strictly bounded standard sequence ($a_{i+1}a^i \in \bar{X}^*$, and $a_{i+1}a^i \geq^* a^2a^i$ for all $a^i, a_{i+1}$ in the sequence; and for some $d'd'' \in \bar{X}^*$, $d'd'' \geq^* a^i a^i$ for all $a^i$ in the sequence), then it is finite.

Similar to Definition 2.2, I present a theorem rewritten from Theorem 2.1 which is for single attribute case to emphasize that we are primarily concerned with multiattribute decision making.
Theorem 2. 2  If $< \overline{x}, \overline{x}^*, \geq^* >$ is a positive difference structure, then there exists $\psi : \overline{x}^* \rightarrow \mathbb{R}^+$ such that for all $a, b, c, d \in \overline{x}$

(i) if $ab, cd \in \overline{x}^*$, then $ab \geq^* cd$ iff $\psi(ab) \geq \psi(cd)$;

(ii) if $ab, bc \in \overline{x}^*$, then $\psi(ac) = \psi(ab) + \psi(bc)$.

If $\psi'$ also has these properties, then there exists $\alpha > 0$ such that $\psi' = \alpha \psi$. If, in addition, for all $a, b \in \overline{x}$, $a \neq b$, either $ab$ or $ba$ in $\overline{x}^*$, then there exists $v : \overline{x} \rightarrow \mathbb{R}$ such that for $ab$ in $\overline{x}^*$, $\psi(ab) = v(a) - v(b)$. If $v'$ has the same property, then there exists a constant $\beta$ such that $v' = v + \beta$.

Now let $\overline{x}^* = X_1 \times X_2 \times \cdots \times X_n$, in Definition 2. 2 and Theorem 2. 2, be the set of all possible multiattribute consequences and $a$ and $b$ be the elements of $\overline{x}$. Then, $ab \in \overline{x}^*$ represents how the decision maker feels about going from $b$ to $a$. Now, if we assume that the decision maker's preferences under certainty obey the axioms in Definition 2. 2, we can define a binary relation $\geq^*$ on $\overline{x}^*$ where $\overline{x} \times \overline{x} \supseteq \overline{x}^*$. Here, $ab \geq^* cd$ means that the strength of preference for $a$ over $b$ is greater than or equal to the strength of preference for $c$ over $d$.

Then, by Theorem 2. 2, there exists a function $\psi$ that represents decision maker's preference differences. If the decision maker's preferences are complete, that is, either
\( ab \in \Xi^* \) or \( ba \in \Xi^* \), then there exists a function \( v : \Xi \to \mathbb{R} \) such that \( \psi(ab) = v(a) - v(b) \). That is, we can define a real-valued function \( v \) from \( \Xi \) into \( \mathbb{R} \) such that for all \( a, b, c, d \in \Xi \), the difference in the strength of preference between \( a \) and \( b \) is greater than the difference in the strength of preference between \( c \) and \( d \) if and only if

\[
v(a) - v(b) \geq v(c) - v(d)
\]

and denoted as \( ab \preceq cd \) for \( ab, cd \in \Xi^* \). The function \( v \) is unique up to a positive linear transformation because

\[
\psi(ab) = v(a) - v(b) = [\alpha v'(a) + \beta] - [\alpha v'(b) + \beta] \\
= \alpha [v'(a) - v'(b)] \text{ for } \alpha > 0 \text{ and any } \beta.
\]

Thus, \( v \) is strategically equivalent to \( v' \).

Similar to the single-attribute case, Theorem 2.2 implies a binary relation \( \succ \) such that for all multiattribute consequences \( a, b, c \in \Xi \), \( ac \succeq bc \) if and only if \( a \succ b \). In other words, if for all \( a, b, c \in \Xi \), \( ac \succeq bc \), then \( \psi(ac) \geq \psi(bc) \) from Theorem 2.2 and \( v(a) - v(c) \geq v(b) - v(c) \) or \( v(a) \geq v(b) \) from (2.7). Thus, if \( ac \succeq bc \), then \( a \succ b \). On the other hand, if \( a \succ b \), then \( v(a) \geq v(b) \) or \( v(a) - v(c) \geq v(b) - v(c) \). Thus, \( a \succ b \) implies \( ab \succeq cd \). Therefore, a multiattribute cardinal value function \( v \) represents the strength of preference of the decision maker. We can rank order a set of alternatives based on \( v(x) \) because
under certainty every multiattribute alternative maps into a consequence, $\mathbf{x}$, and every $\mathbf{x}$ is mapped into a $v(\mathbf{x})$.

The rest of this section extends the work of Dyer and Sarin to the general multiattribute case with no assumption of preferential independence. We assume a preliminary process has identified alternatives and the relevant attributes for alternatives, and determined that there are $n$ attributes. We also assume $v(x_1, x_2, \ldots, x_n) = v(\mathbf{x})^{11}$ has been identified as a cardinal value function under certainty which measures the strength of preference of a decision maker on possible consequences. I assume that $v(\mathbf{x})$ is a nondecreasing function in each argument and continuously twice differentiable.

I define

$$M(\mathbf{x}) = \left[ -\frac{v_{ij}}{v_{1i}} \right] = \left[ \text{diag} (v') \right]^{-1} \nabla$$

where $\left[ \text{diag} (v') \right]$ is an $n \times n$ diagonal matrix with $(i, i)$-element $v_1 = \frac{\partial v}{\partial x_i}$, $i=1, 2, \ldots, n$, and $V$ is the $n \times n$ Hessian matrix $[v_{ij}(\mathbf{x})]$ where $v_{ij}(\mathbf{x}) = \frac{\partial^2 v}{\partial x_i \partial x_j}$, $i,j=1, 2, \ldots, n$. Here, I call $M$ the coefficient matrix of multivariate value satiation where the $(i, j)$ element can be interpreted as a local measure of value satiation of the decision maker when he realizes consequence $\mathbf{x}$.

---

11. To be consistent, I should use the notation $v$ instead of $v(\mathbf{x})$. But for the sake of convenience, I will use the notation $v(\mathbf{x})$. 
Similar to the definition by Dyer and Sarin[8], I define \( s(x,h) \) as the vector of satiation sacrifice for the increase. If a small increase \( h \) in the level of \( x \) occurs, there exists a \( \Delta \) such that

\[
v(x + h) = v(x) + \Delta^T v' \quad \text{where} \quad v' = \left[ \frac{\partial v}{\partial x_1}, \ldots, \frac{\partial v}{\partial x_n} \right]
\]

(2.9)

where \( T \) stands for transpose of a matrix (or a vector).

Now let \( s(x,h) \equiv h - \Delta \) be the vector of satiation sacrifice for the increase. If we expand \( v(x + h) \) and \( v(x + \Delta) \) using Taylor series,

\[
v(x + h) = v(x) + h^T v' + (0.5) h^T V h + o(h)
\]

(2.10)

\[
v(x + \Delta) = v(x) + \Delta^T v' + (0.5) \Delta^T V \Delta + o(\Delta)
\]

(2.11)

From (2.9) and (2.11)

\[
v(x + h) - v(x + \Delta) = -(0.5) \Delta^T V \Delta - o(\Delta).
\]

(2.12)

Also from (2.10) and (2.11)

\[
v(x + h) - v(x + \Delta) = (h^T - \Delta^T) v' + (0.5) h^T V h
\]

\[
- (0.5) \Delta^T V \Delta + o(h) - o(\Delta)
\]

(2.13)

Now combining (2.12) and (2.13) we have

\[
(h^T - \Delta^T) v' \equiv -(0.5) h^T V h.
\]

(2.14)

or \( s(x,h) = (h - \Delta) \equiv -(0.5) h^T V h \cdot ((v')^-)^T \)

where \( A^- \) is the generalized inverse of \( A \).
There are obvious similarities among (2.5), (2.6) and (2.14);

\[ s(x, h) \cdot v'(x) \equiv -(0.5)h^2 \cdot v''(x) \quad (2.5) \]

\[ \pi(x, z) \cdot u'(x) \equiv -(0.5)\sigma_z^2 \cdot u''(x) \quad (2.6) \]

\[ \varphi(x, h) \cdot v'(x) \equiv -(0.5)h^T \cdot V \cdot h \quad (2.14) \]

Next, let \( \lambda(x, h) \) be the ratio of the value increase in going from \( x - h \) to \( x \) to the value increase in going from \( x - h \) to \( x + h \) which is another measure of local departure of \( v \) from linearity. That is,

\[ \lambda(x, h) = \frac{v(x) - v(x-h)}{v(x+h) - v(x-h)} \quad (2.15) \]

\[ v(x) = \lambda(x, h) \cdot v(x+h) + (1-\lambda(x, h)) \cdot v(x-h) \quad (2.16) \]

\[ v(x + h) = v(x) + h^T \cdot v' + (0.5)h^T \cdot V \cdot h + o(h) \quad (2.17) \]

\[ v(x - h) = v(x) - h^T \cdot v' + (0.5)h^T \cdot V \cdot h + o(h) \quad (2.18) \]

Now if we substitute (2.17) and (2.18) to (2.16),

\[ 0 = -h^T \cdot v' + 2\lambda h^T \cdot v' + (0.5)h^T \cdot V \cdot h + o(h) \]

\[ (2\lambda-1)h^T \cdot v' \equiv -(0.5)h^T \cdot V \cdot h \]

\[ (2\lambda-1) \equiv -(0.5)(h^T \cdot V \cdot h)/(h^T \cdot v') \]

\[ \lambda(x, h) \equiv 0.5 - (1/4)(h^T \cdot V \cdot h)/(h^T \cdot v') \quad (2.19) \]
If $V$ is diagonal, i.e., $v_{ij} = 0$ for $i \neq j$, then

$$\lambda(x, h) = 0.5 + (1/4) \sum_{i=1}^{n} h_{i}(-\frac{v_{ii}}{v_{1}})$$

$$\equiv 0.5 + (1/4) \sum_{i=1}^{n} h_{i}M_{ii}$$

Thus, if $v(x)$ is an additive cardinal value function, then $\lambda(x, h)$ is one half plus the trace of $M$ each of which is multiplied by $(1/4)h_{i}$ and it is consistent with the result of Dyer and Sarin. Furthermore, if $v(x)$ is linear in each argument, then $\lambda(x, h) = 0.5$.

With these results, I next prove a theorem which is the multivariate generalization of a theorem of Dyer and Sarin [8].

**Theorem 2.3** Let $M^{a}(x)$ and $M^{b}(x)$ be the local coefficients to the cardinal value functions $\alpha(x)$ and $\beta(x)$. Also let $s^{i}(x, h)$ and $\lambda^{i}(x, h)$, $i = \alpha$, $\beta$, be the corresponding satisfaction sacrifices and ratios, respectively. Then the following conditions are equivalent:

(a) $M^{a}(x) \geq M^{b}(x)$

(b) $s^{a}(x, h) \geq s^{b}(x, h)$ for all $x$ and $h$

(c) $\lambda^{a}(x, h) \geq \lambda^{b}(x, h)$ for all $x$ and $h$

(d) $\frac{\alpha(t) - \alpha(v)}{\alpha(w) - \alpha(z)} \leq \frac{\beta(t) - \beta(v)}{\beta(w) - \beta(z)}$
<Proof>

(a) → (d)

By definition \( M_{11}^\alpha = -\alpha_{11}/\alpha_1 \) and \( M_{11}^\beta = -\beta_{11}/\beta_1 \).

Since \( M^\alpha - M^\beta \) is positive definite, \( M_{11}^\alpha - M_{11}^\beta \geq 0 \) or \(-\alpha_{11}/\alpha_1 \geq -\beta_{11}/\beta_1 \). If we integrate on both sides,

\[
\int_{z}^{v} (-\frac{\alpha_{11}}{\alpha_1}) \, dx_1 \geq \int_{z}^{v} (-\frac{\beta_{11}}{\beta_1}) \, dx_1
\]

\[-\log\frac{\alpha_1(v)}{\alpha_1(z)} \geq -\log\frac{\beta_1(v)}{\beta_1(z)} \quad \text{or} \quad \frac{\alpha_1(v)}{\alpha_1(z)} \leq \frac{\beta_1(v)}{\beta_1(z)} \quad \text{for} \quad z < v\]

By applying mean-value theorem,

\[
\frac{\alpha(t) - \alpha(v)}{\alpha_1(z)} \leq \frac{\beta(t) - \beta(v)}{\beta_1(z)} \quad \text{for} \quad z \leq v < t
\]

or

\[
\frac{\alpha_1(z)}{\alpha(t) - \alpha(v)} \geq \frac{\beta_1(z)}{\beta(t) - \beta(v)}
\]

Again by applying the mean-value theorem,

\[
\frac{\alpha(w) - \alpha(z)}{\alpha(t) - \alpha(v)} \geq \frac{\beta(w) - \beta(z)}{\beta(t) - \beta(v)} \quad \text{for} \quad z < w < t
\]

or

\[
\frac{\alpha(t) - \alpha(v)}{\alpha(w) - \alpha(z)} \leq \frac{\beta(t) - \beta(v)}{\beta(w) - \beta(z)} \quad \text{for} \quad z \leq x < w < t
\]

(d) → (c)

If we substitute \( x + h \) for \( t \), \( x \) for \( w \), \( x - h \) for \( v \), and \( x - h \) for \( z \) in (d), then the inequality becomes

\[
\frac{\alpha(x) - \alpha(x-h)}{\alpha(x+h) - \alpha(x-h)} \geq \frac{\beta(x) - \beta(x-h)}{\beta(x+h) - \beta(x-h)}
\]
But LHS of the above inequality is $\lambda^\alpha(x, h)$ and RHS is $\lambda^\beta(x, h)$ from (2.15). Thus, $\lambda^\alpha(x, h) \geq \lambda^\beta(x, h)$.

(c) $\leftrightarrow$ (b)

Let $A$ and $B$ be Hessian matrices for $\alpha(x)$ and $\beta(x)$. From (c),

$$\lambda^\alpha(x, h) \geq \lambda^\beta(x, h) \leftrightarrow \frac{-1/4}{h^\top A h} \geq \frac{-1/4}{h^\top B h} \geq \frac{-1/2}{h^\top \mathbf{g}'} \geq \frac{-1/2}{h^\top \mathbf{h}'} \leftrightarrow (s^\alpha)^\top \mathbf{g}' / (h^\top \mathbf{g}') \geq (s^\beta)^\top \mathbf{h}' / (h^\top \mathbf{h}')$$

Theorem 2.3 does not include Part (d) of the theorems in Dyer and Sarin[8] and Pratt[30] which is:

(d) $v_1(v_2^{-1}(t)) \ [u_1(u_2^{-1}(t))]$ is a concave function of $t$ where $v_1 [u_1]$ and $v_2 [u_2]$ are value [utility] functions for two different decision makers[8],[30].

This part is excluded here because (d) above is not suggestive of any intuition or insight in comparing value[utility] functions of two different decision makers. The only interpretation I could come up with is that $v_1[u_1]$ is more concave than $v_2[u_2]$ in an intuitive sense. But there is no formalism to compare the degree of concavity of two different functions. Also, the multivariate generalization of (d) is not obvious.

From Theorem 2.3, we can compare cardinal value functions for two different decision makers in terms of the measure of
value satiation or satiation sacrifice without the assumption that the cardinal value functions are concave.
3.1 Introduction

If a decision is to be made under uncertainty, one of the most important questions posed is the effect of a decision maker's attitude toward risk on the decision he makes. It is not too difficult to find evidence that the average human is not an EMV'er\(^1\) [2],[32]. One of the famous examples is the "St. Petersburg paradox" which lead Bernoulli[2] and Cramer\(^2\) to suggest that people use a utility function instead of monetary value to make decisions under uncertainty. As these examples show the attitude toward risk of a decision maker plays an important role in making the decision.

'When we ask whether the decision maker is risk averse, risk neutral, or risk prone, this is equivalent to asking for the

---

1. An EMV'er is the one whose "certainty monetary equivalent (CME)" is equal to the "expected monetary value (EMV)" of a lottery.[32]

functional form of the decision maker's utility function \( u \). It is important to note here that it is not necessary to know the exact form of the utility function itself. This idea will be used in Chapter IV to explore the principle of stochastic dominance.

Before proceeding further, I would like to make it clear the assumptions made about the underlying scales of the consequences. When a decision maker tries to rank a set of consequences, say \( x_1, x_2, \ldots, x_n \), in order of his preferences, it is immaterial whether the \( x_i \)'s are measured on an ordinal, interval, or ratio scale. For example, a decision maker is faced with two options; an option of trip to San Francisco for sure and an option of trip to Hawaii with probability of 0.5 and trip to New York with 0.5. To rank these options, it is necessary to assume a scale which transform the results of options, trips to San Francisco, Hawaii, and New York, into a consequence space. Here, he does not have to assume any particular scale, except nominal, of consequences to order them. Then and only then, he can rank these two options in order of his preference, on the condition that the consequences are, at least, on an ordinal scale.

When we are concerned about his risk attitude in terms of risk aversion, however, we need some sort of restriction on the scale of consequences due to the definition of risk aversion. That is, existence of the expected consequence of a
lottery assumes that the consequences are on an interval scale. Thus, from now on, I will assume that the consequences are on an interval scale.

Traditionally, a decision maker is called risk averse when he prefers a sure consequence which is equal to the expected consequence of a lottery to the lottery itself[20]. When the first derivative of $u$ is nonnegative, it is a nondecreasing function. If the second derivative of $u$ is negative, it is a concave function. A definition that can be shown to be equivalent is that a decision maker is risk averse if his utility function is concave.

This traditional definition of risk aversion is misleading. The functional form of the decision maker's utility function $u$ does not imply the decision maker's attitude toward risk in an absolute sense because the utility function is a mixture of two factors: strength of preference and attitude toward risk as mentioned in Chapter II. Thus, what has been traditionally named risk attitude reflects also the shape of the strength of preference function $v$.

By separating the two factors, Dyer and Sarin[8] defined the notion of relative risk aversion, which represents the decision maker's attitude toward risk only in terms of his
utility function with respect to the value function\(^3\). (I use the notation \(u_v(v)\) for this utility function.) Using relative risk aversion, we may have a better understanding of the decision maker's behavior and, therefore, we can describe his risk attitude more accurately in decision making under uncertainty.

The next section briefly reviews the definitions of the measure of risk aversion in both single attribute and multiple attribute decision making. And the following section will introduce the notion of relative risk aversion which exclusively deals with the decision maker's attitude toward risk.

### 3.2 Risk Aversion

#### 3.2.1 Single Attribute Case

Loosely speaking, we say a person is risk averse when he behaves conservatively. Formally, a decision maker is said to be risk averse when he prefers a sure consequence which is equal to the expected consequence of a lottery to the lottery.

---

3. From this point on, whenever the word "value function" appears, it means "cardinal value function".
itself[20]. Suppose that \( u \) is a monotonically increasing function and it is continuously twice differentiable. Then, as is well known, the decision maker's attitude toward risk can be stated using his utility function as follows[20]:

**Theorem 3.1**[20] A decision maker is risk averse if and only if his utility function \( u \) is concave and risk prone if and only if \( u \) is convex.

To take account of the effect of a decision maker's attitude toward risk on the choices he makes, Arrow[1] and Pratt[30] independently developed a measure of risk aversion. They were primarily concerned with decision making using a single attribute. Their measure of risk aversion is defined as follows:

**Definition 3.1**[30] Given a utility function \( u \) which is assumed to be monotonically increasing (that is, \( u'(x) > 0 \)), the function,

\[
r(x) = -\frac{u''(x)}{u'(x)}
\]

is interpreted as a measure of risk aversion where \( u'(x) \) and \( u''(x) \) are first and second derivatives of \( u(x) \), respectively.

From the Definition 3.1, it is easy to see that stating the decision maker's preference in terms of \( r(x) \) requires knowledge of the functional form of \( u \) but not necessarily of
the exact form. In other words, the measure \( r(x) \) is useful because it is invariant for a positive linear transformation of the utility function. Thus, we can represent a certain class of utility functions by use of the same measure \( r(x) \). But it is important to note that \( u''(x) \) itself is not a good measure of risk aversion. For example, as the utility function \( u(x) \) changes to \( cu(x) + b \), \( u''(x) \) changes to \( cu''(x) \) where \( c \) is a constant. But \( r(x) \) remains the same. It follows immediately from Theorem 3.1 that:

**Theorem 3.2** [20] If \( r(x) \) is positive [negative] for all \( x \), then the decision maker is risk averse [prone] and \( u \) is concave [convex].

This measure is also useful when we try to compare the attitude toward risk of two different people. In this case, we can determine whether one is more risk averse than the other by comparing their respective measures of risk aversion.

The measure \( r(x) \) is closely related to the risk premium \( \pi \) which is defined such that the decision maker is indifferent between receiving a sure amount \( E(Z) - \pi \) and receiving a risk

---

4. If we recall that the utility function is unique up to a positive monotone transformation, then \( u(x) \) and \( c \cdot u(x) \) \((c > 0)\) are equivalent. Keeney and Raiffa[20] call it "strategic equivalence".
That is, the risk premium $\pi$ is the solution of the following equation

$$u(x - \pi) = E[u(x+Z)]$$

(3.2)

where $z$ is an actuarially neutral risk[30]. By expanding both sides of (3.2), Pratt showed that

$$\pi = (0.5)\sigma_z^2 \cdot r(x) + o(\sigma_z^2) \equiv (-0.5) (u''/u')\sigma_z^2.$$  

(3.3)

3.2.2 Multiple Attribute Case

When the consequence of an alternative is represented as a multiattribute consequence, Pratt's measure of risk aversion is not enough to describe the decision maker's attitude toward risk. In a multiattribute decision making situation, Duncan[6] generalized the results of Pratt and introduced risk premiums in vector form $\pi(x; Z)$: For a given risk $Z$, a vector $\pi = (\pi_1, \pi_2, \ldots, \pi_n)^T \in \mathbb{E}^n$ is assigned to each vector $x$. If $E(Z) = 0$, the vector $\pi$ must satisfy

$$u(x - \pi) = E[u(x + Z)]$$

(3.4)

where (3.4) is a generalization of (3.2). Here we assume that $u$ is monotonically increasing in each component of $x$ and that $E[u(x + Z)]$ is finite. From the definition of the risk premium vector, Duncan stated the following theorem:
Theorem 3.3 [6] Let $\mathbf{Z}$ be an $n$-dimensional random vector with expectation $E[\mathbf{Z}] = 0$. If there exists a nonnegative risk premium vector $\mathbf{\pi}$ for all two-point gambles $\mathbf{Z}$, then $u$ is concave. Let $u$ be a concave utility function on $E^n$. Then there exists a nonnegative risk premium vector $\mathbf{\pi}$ such that (3.4) is satisfied.

Theorem 3.3 states that if the utility function $u$ is concave, then the risk premium vector is nonnegative. Since the decision maker is said to be risk averse if the utility function $u$ is concave, it also can be said that he is risk averse if the risk premium vector is nonnegative.

By expanding both sides of (3.4), we get an approximate equation

$$\mathbf{u}^T \cdot \mathbf{\pi} = \sum_{i=1}^{n} \pi_i u_i(x) \equiv (-0.5) \sum_{i=1}^{n} \sum_{j=1}^{n} u_{ij}(x) \sigma_{ij} = (-0.5) \operatorname{tr}(\mathbf{U} \cdot \mathbf{\Sigma})$$

(3.5)

where $\operatorname{tr}(\mathbf{A})$ is the trace of a matrix $\mathbf{A}$, $\mathbf{U}$ is the $nxn$ Hessian matrix $\mathbf{U} = [u_{ij}(x)]$, and $\mathbf{\Sigma} = \operatorname{var} (\mathbf{Z}) = (\sigma_{ij})$. Duncan called any solution $\mathbf{\pi}$ of (3.5) approximate risk premium vector similar to (3.3), which is of the form

$$\mathbf{\hat{\pi}} = (-0.5) (\mathbf{u}^T)^{-1} \cdot \operatorname{tr}(\mathbf{U} \cdot \mathbf{\Sigma})$$

(3.6)

and suggested that the natural solution of (3.5) is
\[ \hat{\mu} = \hat{\mu}^0 = (0.5)\text{dg}(R \cdot \Sigma) \] (3.7)

where
\[ R(x) \equiv \left[ -\frac{u_1'(x)}{u_1(x)} \right] = -[\text{diag}(u)]^{-1}u. \] (3.8)

Here \( R(x) \) is called the absolute risk aversion matrix where \( u_1(x) \) is the first partial derivative of \( u(x) \) with respect to \( x_1 \) and \( u_{ij}(x) \) is the second partial derivative with respect to \( x_i \) and \( x_j \). Also, \( \text{dg}(A) \) is the \( n \)-vector of main diagonal elements \( (a_{ii}) \) and \( \text{diag}(u) \) is the diagonal matrix with \((i,i)\)-element \( u_i \) where \( n \)-vector \( u = (u_1(x)) \). If we note the similarity between \( \hat{\pi}_i = \hat{\pi}_i^0 = (0.5) \sum_{j=1}^n (-u_{ij}/u_1)\sigma_{ij} \) and univariate risk premium \( \hat{\pi} = (0.5)(-u''/u')\sigma^2 \) from (3.3), then it seems to make sense to call \( R \) the absolute risk aversion matrix.

From the definitions of the approximate risk premium vector and the absolute risk aversion matrix, Duncan stated the following theorem:

**Theorem 3.4**[6] There exists an approximate risk premium vector which is nonnegative regardless of the risk if and only if \( u \) is concave. If \( u_1(x) = \cdots = u_n(x) \) and \( u \) is concave, i.e., \( R(x) \) is symmetric nonnegative definite, then the sum of the natural approximate risk premiums is nonnegative.

Theorem 3.4 implies that if the utility function is concave, then \( R(x) \) is nonnegative definite. Thus, operationally
speaking, if $R(\mathbf{x})$ is nonnegative definite, then the decision maker is risk averse.

A new type of risk aversion with respect to a utility function was proposed by Richard[33]. Consider a pair of attributes $X$ and $Y$ and let the levels of the attributes be $x$ and $y$, respectively. Also, let $u$ be a utility function defined on $X \times Y$. Now consider the following lotteries where $x_0, x_1 \in X$ and $y_0, y_1 \in Y$, $x_0 < x_1$ and $y_0 < y_1$:

i) $L_1$ gives a 0.5 probability of receiving $(x_0, y_0)$ and a 0.5 probability of receiving $(x_1, y_1)$.

ii) $L_2$ gives a 0.5 probability of receiving $(x_0, y_1)$ and a 0.5 probability of receiving $(x_1, y_0)$.

In lottery $L_1$, the decision maker receives "the best or the worst" with a 50-50 chance. In lottery $L_2$, he receives one of the more desirable and one of the less. From $L_1$ and $L_2$, Richard defined multivariate risk attitude of a decision maker as follows:

**Definition 3.2**[33] A decision maker is **multivariate risk averse** (MRA) if he prefers $L_2$ to $L_1$ ($L_2 \succeq L_1$) for all $x_0, x_1, y_0, and y_1$. He is **multivariate risk neutral** (MRN) if he is indifferent between $L_2$ and $L_1$ ($L_2 \sim L_1$) for all $x_0, x_1, y_0, and y_1$. He is **multivariate risk prone** (MRP) if he prefers $L_1$ to $L_2$ ($L_1 \succeq L_2$) for all $x_0, x_1, y_0, and y_1$. 


Given Definition 3.2, he showed a necessary and sufficient condition for MRA [MRN, MRP]:

**Theorem 3.5** [33] A decision maker is multivariate risk averse if and only if \( u_{xy}(x,y) \leq 0 \) for all \((x,y) \in X \times Y\); multivariate risk neutral if and only if \( u_{xy}(x,y) = 0 \) for all \((x,y) \in X \times Y\); multivariate risk prone if and only if \( u_{xy}(x,y) \geq 0 \) for all \((x,y) \in X \times Y\).

Recall that, in single attribute decision making, a decision maker is said to be risk averse if \( r(x) > 0 \) (equivalently \( u''(x) \) is negative), that is, if \( u \) is concave from Theorem 3.1. And if we extend Theorem 3.1 to the multiattribute case, then the decision maker is risk averse if the utility function \( u \) is concave. For multiattribute utility function \( u(x,y) \) to be concave, its Hessian matrix must be negative semidefinite. Thus, if a decision maker is risk averse in multiattribute decision making, then Hessian matrix of his utility function must be negative semidefinite.

According to Richard, however, that the utility function of a decision maker is concave does not necessarily mean that he is risk averse in a multiattribute sense. That is, Hessian matrix of the utility function is negative semidefinite does not imply that \( u_{xy}(x,y) \leq 0 \). Thus, it is possible that, even though the utility function is concave, it may not
necessarily imply that the decision maker is risk averse in the sense of Richard.

3.3 Relative Risk Aversion

3.3.1 Introduction

In the previous section we discussed the measure of risk aversion that has been used to characterize the decision maker's attitude toward risk. But as we know from Chapter II, $u$ is a mixture of two factors, strength of preference and attitude toward risk. Thus, it is very difficult to state the decision maker's risk attitude independent of his strength of preference. For example, it is possible that any two persons may agree on their values for certain consequences but they need not necessarily have the same risk attitude. Thus, we cannot accurately describe a person's attitude toward risk by using a measure based on his utility function defined on levels of attributes.

To circumvent the above difficulty, the measure of risk aversion needs to be redefined by use of the notion of relative risk aversion. The notion of relative risk aversion is a result of separation issue discussed in Chapter II. The decision maker's strength of preference can be described by a
value function $v$ and his attitude toward risk can be represented by a utility function $u_v$ with respect to a value function $v$. Then, we can focus on the decision maker's attitude toward risk by examining the properties of $u_v$ without confounding the effect of their strength of preference.

There are several advantages of using the notion of relative risk aversion. First, we can pay attention to a person's consistency in making decisions. That is, a decision maker may be said to be consistent if he shows a consistent relationship $u_v$ with respect to the value function $v$ whatever the value function may be. This implies that the decision maker may reveal his attitude toward risk in a consistent manner. Then, we can alleviate the burden of assessing his utility function in every circumstance. Second, in multiattribute decision making, the decision maker's attitude toward risk can be characterized as a function of only the level of $v$ instead of a function of all levels of attributes. That is, we only need to assess the functional form of $u_v$.

The notion of relative risk aversion was first introduced by Dyer and Sarin[8] for a single attribute decision making under uncertainty. Krzysztofowicz[22] did experiments and concluded that within a specific decision scenario the individual's relative risk attitude is constant. In the next section, I will review relative risk aversion in single attribute decision making and in the following section I will
propose a definition of relative risk aversion in multiple attribute decision making.

3.3.2 Single Attribute Case

If \( v \) is a value function which represents the decision maker's strength of preference and \( u_v(v) \) is a function which describes his attitude toward risk, then we can model his utility function \( u \) as a composite function \( u_v(v) \).

Suppose that \( u(x) \) is a monotonically increasing function and it is continuously twice differentiable. Let the utility function \( u(x) = u_v(v(x)) \) where \( v(x) \) is a value function defined in Chapter II. Now the decision maker's attitude toward risk can be stated in terms of \( u_v(v) \) by the following definition from Dyer and Sarin[8] where \( m(x) \) is defined by (2.2) and \( r(x) \) is defined by (3.1):

**Definition 3.3[8]** At \( x \in X' \), a decision maker is relatively risk averse if \( m(x) < r(x) \), relatively risk prone if \( m(x) > r(x) \), and relatively risk neutral if \( m(x) = r(x) \).

Since \( u(x) = u_v(v(x)) \), \( u'(x) \) and \( u''(x) \) are as follows:

\[
\begin{align*}
  u'(x) &= u'_v(v) v'(x) \\
  u''(x) &= u''_v(v) (v'(x))^2 + u'_v(v) v''(x)
\end{align*}
\]
Let \( r_v(v) \) be the measure of relative risk aversion where

\[
r_v(v) = - \frac{u''_v(v)}{u'_v(v)} \tag{3.11}
\]

From (2.2) and (3.1), the following theorem can be easily shown:

**Theorem 3.6** At \( x \in X' \), a decision maker is relatively risk averse if and only if \( r_v(v) > 0 \), relatively risk prone if and only if \( r_v(v) < 0 \), and relatively risk neutral if and only if \( r_v(v) = 0 \).

From Definition 3.3 and Theorem 3.6, the use of \( r_v(v) \) is advantageous because this expression is invariant for a family of value functions which are strategically equivalent. Even more, it is invariant for different decision makers whose utility functions are strategically equivalent. For example, let \( v(x) \) and \( \tilde{v}(x) \) be value functions which are strategically equivalent, say,

\[
\tilde{v}(x) = a\overline{v}(x) + b \tag{3.12}
\]

where \( a > 0 \) and \( b \) are constants. Also let \( u(x) \) and \( w(x) \) be utility functions for two different decision makers such that \( u(x) = u_\Psi(\overline{v}(x)) \) and \( w(x) = u_\Xi(\tilde{v}(x)) \) and they are strategically equivalent, that is, for \( c > 0 \) and \( d \),

\[
w(x) = c \cdot u(x) + d. \tag{3.13}
\]
If we take the derivatives of $u(x)$ and $w(x)$,

$$u'(x) = u'_v(v) \cdot \overline{v}'(x), \quad (3.14)$$

$$u''(x) = u''_v(v) \cdot (\overline{v}'(x))^2 + u'_v(v) \cdot \overline{v}''(x), \quad (3.15)$$

$$w'(x) = w'_v(v) \cdot \overline{v}'(x), \quad (3.16)$$

$$w''(x) = w''_v(v) \cdot (\overline{v}'(x))^2 + w'_v(v) \cdot \overline{v}''(x) \quad (3.17)$$

From (3.12) and (3.13)

$$\tilde{v}'(x) = a \cdot \overline{v}'(x) \text{ and } w'(x) = c \cdot u'(x) \quad (3.18)$$

So if we substitute (3.14) and (3.18) in (3.16),

$$c \cdot (u'_v(v) \cdot \overline{v}'(x)) = (w''_v(v)) \cdot (a \cdot \overline{v}'(x)) \quad (3.19)$$

or

$$w''_v(v) = \frac{c}{a} u'_v(v) \quad (3.20)$$

From (3.12) and (3.13)

$$\tilde{v}''(x) = a \cdot \overline{v}''(x) \text{ and } w''(x) = c \cdot u''(x) \quad (3.21)$$

So if we substitute (3.15), (3.20) and (3.21) in (3.17),

$$c \cdot u''(x) = w''_v(v) \cdot (a \cdot \overline{v}'(x))^2 + w'_v(v) \cdot (a \cdot \overline{v}''(x)) \quad (3.22)$$

or

$$w''_v(v) = \frac{c}{a^2} u'_v(v). \quad (3.23)$$

Now based on $u(x)$ and $\overline{v}(x)$,

$$r(x) - m(x) = - \frac{u''_v(v)}{u'_v(v)} \overline{v}'(x) = r_v(v) \cdot \overline{v}'(x) \quad (3.24)$$

and based on $w(x)$ and $\tilde{v}(x)$,

$$r(x) - m(x) = - \frac{w''_v(v)}{w'_v(v)} \tilde{v}'(x) = - \frac{(\frac{c}{a}) u''_v(v)}{(\frac{c}{a}) u'_v(v)} (a \cdot \overline{v}'(x)) \quad (3.25)$$

$$= - \frac{u''_v(v)}{u'_v(v)} \tilde{v}'(x) = r_v(v) \cdot \overline{v}'(x) \quad (3.25)$$
From (3.24) and (3.25), we can easily see that the measure of relative risk aversion is invariant for both cases. Therefore, if we use any of the value functions which are strategically equivalent, then we can effectively describe his relative risk attitude. Also if utility functions for two different decision makers are strategically equivalent, then their relative attitudes toward risk are the same by Definition 3.3.

3.3.3 Multiple Attribute Case

As mentioned in 3.3.1, the advantages of using relative risk aversion are more apparent in multiattribute decision making than in single-attribute decision making. The primary advantage is that, in multiattribute decision making, we can accurately describe the decision maker's attitude toward risk. Another advantage is due to the fact that $u_{v}(v)$ is a function of single variable. Thus, no matter how many attributes we have, we can use $u_{v}(v)$ to address the decision maker's attitude toward risk. Finally, if the decision maker were consistent in terms of his attitude toward risk, we just assess his value function without assessing his utility function in every situation.

Despite these advantages, however, there has been no formal attempts to extend the notion of relative risk aversion to
multiattribute decision making under uncertainty. We now present the definition of multiattribute relative risk aversion which generalizes Dyer and Sarin's definition for one attribute to the multiattribute case. Also we present a theorem which develops a measure of multiattribute relative risk aversion and the measure will be used to develop stochastic dominance in Chapter IV.

Let \( R(x) \) be the absolute risk aversion matrix defined by (3. 8) based on a utility function (under uncertainty) \( u(x) \) and \( M(x) \) be the matrix of the measure of the strength of preference defined in (2. 8) based on a value function (under certainty) \( v(x) \) where \( x \) is a multiattribute consequence denoted by \((x_1, x_2, \ldots, x_n)\). So \( R_{i,j}(x) = -\frac{u_{i,j}}{u_1} \) and \( M_{i,j}(x) = -\frac{v_{i,j}}{v_1} \).

Now let us define the following:

Definition 3. 4 An individual is multiattribute relatively risk averse [neutral, prone] if \( R(x) - M(x) \) is positive definite (semi-definite) [zero, negative definite (semi-definite)] in a strong (weak) form.

Theorem 3. 7 An individual is multiattribute relatively risk averse [neutral, prone] if and only if \( r_v(v) > [=, <] 0 \) where \( r_v(v) = -\frac{u_v''(v)}{u_v'(v)} \).
(a) Sufficiency

Let \( H(x) = R(x) - M(x) \) and \( H_{ij}(x) \) be the \((i,j)\)-element of \( H(x) \).

Then \( H_{ij}(x) = (- \frac{u_{ij}}{u_1}) - (- \frac{v_{ij}}{v_1}) \) \hspace{1cm} (3.26)

where \( u_i(x) = \frac{\partial u}{\partial x_i} = \left( \frac{\partial u}{\partial v} \right) \cdot \left( \frac{\partial v}{\partial x_i} \right) = u_v \cdot v_1 \) \hspace{1cm} (3.27)

and \( u_{ij}(x) = \frac{\partial^2 u}{\partial x_i \partial x_j} = \left( \frac{\partial^2 u}{\partial v^2} \right) \left( \frac{\partial v}{\partial x_i} \right)^2 + \left( \frac{\partial u}{\partial v} \right) \left( \frac{\partial^2 v}{\partial x_i \partial x_j} \right) \hspace{1cm} (3.28) \)

where \( u_{ij}(x) = \frac{\partial^2 u}{\partial x_i \partial x_j} = \left( \frac{\partial^2 u}{\partial v^2} \right) \left( \frac{\partial v}{\partial x_i} \right)^2 + \left( \frac{\partial u}{\partial v} \right) \left( \frac{\partial^2 v}{\partial x_i \partial x_j} \right) \)

Now, by plugging (3.27) and (3.28) into (3.26)

\[
H_{ij}(x) = - \frac{u_{ij}}{u_1} - (- \frac{v_{ij}}{v_1}) = - \frac{u_v \cdot v_1^2 + u_v \cdot v_1 + v_{ij}}{u_v \cdot v_1}
\]

\[
= - \frac{u_v \cdot v_1^2 - u_v \cdot v_1 + v_{ij}}{u_v \cdot v_1} = - \frac{u_v}{u_v} \cdot v_1 = r_v(v) \cdot v_1
\]

From the beginning we already know that \( v_1 \geq 0 \), i.e., the strength of preference function is a nondecreasing function.

---

5. The proof given here is for the case of relative risk aversion but other cases can be done similarly with ease.
So, the matrix $\mathbf{V}$ is positive semi-definite because its eigenvalues are zeros (with multiplicity of $n-1$) and $(v_1 + v_2 + \cdots + v_n)$. Therefore, if the decision maker is multiattribute relatively risk averse, that is, if $H$ is positive semi-definite, then $r_\nu(v)$ must be positive.

(b) Necessity

From part (a), we know that the matrix $\mathbf{V}$ is positive semi-definite. So if $r_\nu(v) > 0$, then $H(x)$ is positive semi-definite. And if $H(x)$ is positive semi-definite, then the decision maker is multiattribute relatively risk averse by the definition. Thus, the proof is complete.

Theorem 3.7 is very useful because if we assess a decision maker's utility function with respect to his value function, then his attitude toward risk can be stated in terms of $u_\nu$. Thus, we can focus on the decision maker's relative attitude toward risk without the effect of his strength of preference. In summary, in Chapter 3, we first defined the notion of relative risk aversion in multiattribute decision making and then developed a measure of multiattribute relative risk aversion which can be used to determine whether a decision maker is relatively risk averse or relatively risk prone.
4.1 Introduction

The first step of decision analysis is to identify all feasible alternatives and to determine the relevant attributes which characterize each alternative. The second step is to identify the decision maker's utility function, which was the topic of Chapter III. And the third step is to assess the probabilities, or the probability distribution, with which the consequences will occur. Once this information has been obtained, we are now in a position to evaluate each alternative in order to determine the best one or, at least, the set of nondominated alternatives.

Normatively, we use expected utility as the decision criterion to evaluate each alternative, that is, the decision maker should choose the alternative whose expected utility is the maximum. The expected utility of an alternative is the mathematical expectation of the utility function of the decision maker given the probabilities. Thus, we have to have
full knowledge of the decision maker's utility function and the probabilities to calculate the expected utility for each alternative to determine the best one.

If we know the utility function of the decision maker exactly, then we can determine "the" best alternative. But it is not always easy to assess the utility function exactly. If we do not know the utility function exactly, then we cannot calculate the expected utility for each alternative explicitly. In other words, we cannot compare alternatives directly.

We can use stochastic dominance to circumvent the above situation. The main purpose of stochastic dominance is to eliminate the dominated alternatives so as to reduce the number of nondominated alternatives given a set of feasible alternatives under consideration when the utility function is not known exactly. Thus, it is conjectured that the application of stochastic dominance will create a smaller set of nondominated alternatives. Finally, it is hoped that the decision maker can make a decision by himself from the resulting set of nondominated alternatives.

Traditionally, the decision maker's attitude toward risk has been captured in the form of a utility function. But, as mentioned in Chapters II and III, a utility function $u$ is a mixture of attitude toward risk and strength of preference so
that we cannot isolate the attitude toward risk from the strength of preference. As we will see in the next section stochastic dominance has been defined and developed in relation to the attitude toward risk of a decision maker based on a utility function \( u \). Therefore, stochastic dominance developed so far does not take account of the decision maker's risk attitude accurately.

To use information about the decision maker's risk attitude in stochastic dominance, it is better to use \( u_v \) instead of \( u \) because \( u_v \) describes his risk attitude exclusively without being confounded by a value function (or strength of preference). So when a utility function of the decision maker is not known exactly, we can state the stochastic dominance principle using \( u_v \). By doing so, we can provide him a set of nondominated alternatives which better reflects his attitude toward risk. The next section briefly reviews stochastic dominance based on a utility function \( u \). The following section will be devoted to the development of stochastic dominance based on \( u_v \).

4.2 Stochastic Dominance and Risk Aversion

Stochastic dominance is used to choose the most preferred alternative among the set of available alternatives or, at
least, to determine the smallest subset of alternatives which is guaranteed to include the most preferred alternative by considering a partial order of the alternatives.

One of the basic assumptions of stochastic dominance is that the decision maker's utility function is not known exactly\(^1\). Rather, only the functional form\(^2\) of the utility function is known where the form of a utility function is a way of representing the decision maker's attitude toward risk and his strength of preference. The lack of knowledge may be because either the decision maker is uncertain about his/her exact utility function or the assessment is incomplete. Another assumption is that the decision maker has exact knowledge about the probability distribution of consequences for each alternative.

Now, assume that there are \(m\) alternatives \(\{A_1, A_2, \ldots, A_m\}\) and \(n\) attributes \(\{X_1, X_2, \ldots, X_n\}\) for each consequence which are the representation of the decision maker's perception of each consequence.

---

1. When we say that the function is not exactly known but its functional form is known, it implies that we only have information of its first and/or second partial derivatives.

2. By "functional form" I mean the property the function has in terms of the signs of its first and second partial derivatives or sometimes the definiteness of its Hessian matrix.
A complete ordering relation on the set of alternatives is determined by the expected utility criterion which is stated as follows:

Let \( A_p \) and \( A_q \) be two alternatives under consideration, i.e., \( p, q \in IA \) where \( IA \) denotes the index set of all feasible alternatives, i.e., \( IA = \{1, 2, \ldots, m\} \). Then, \( A_p \) is preferred to \( A_q \) (\( A_p \succ A_q \)) iff \( E[u, F] \geq E[u, G] \) for \( p, q \in IA \) where \( E[u, \cdot] \) denotes the expected utility with respect to utility function \( u \) and specified probability distribution of consequences on \( X \), i.e., \( E[u, F] = \int_x u(x) dF(x) \). The distributions \( F \) and \( G \) describe the consequences of \( A_p \) and \( A_q \), respectively.

Utility functions can be classified according to their functional forms. The well known classes of utility functions are given below in a multiattribute setting[15]:

\[
S_1 = \{ u(x) \mid u \in C^1, \frac{\partial u}{\partial x_i} > 0, \text{ for all } i \in IX \}
\]

\[
S_2 = \{ u(x) \mid u \in C^2, u \in S_1, \frac{\partial^2 u}{\partial x_i^2} < 0, \text{ for all } i \in IX \}
\]

\[
S_3 = \{ u(x) \mid u \in C^3, u \in S_2, \frac{\partial^3 u}{\partial x_i^3} > 0, \text{ for all } i \in IX \}
\]

where \( C^1 \) represents the set of bounded \( i \)th differentiable functions, \( x_i \) is the \( i \)th element of an \( n \)-dimensional vector \( \mathbf{x} = (x_1, x_2, \ldots, x_n) \) and \( x_i \in (a_i, b_i)^3 \) and \( IX \) denotes the index set of attributes, i.e., \( IX = \{1, 2, \ldots, n\} \).

---

3. The domain of \( i \)th consequence is not necessarily restricted to finite upper and lower bounds.
If a decision maker's utility function belongs to the set $S_1$ (increasing utility functions), then the decision maker simply believes more of each attribute is better. So no specific attitude toward risk is revealed in this set. A utility function in the set $S_2$ (increasing concave utility functions) represents a risk averse attitude of the decision maker. And membership of $u$ in the set $S_3$ (increasing concave utility functions with positive third derivatives) means that the utility of a decision maker increases at a decreasing rate.

Now, let $u$ be an $n$-attribute utility function of a decision maker, i.e., $u = u(x_1, x_2, \ldots, x_n)$ and $E[u, F] = \int_S u(x) dF(x)$ where $S$ is a convex subset of $n$-dimensional Euclidean space. With the above classes of utility functions, stochastic dominance is defined as follows [3], [15], [16], [17], [41]:

Definition 4.1[15] Let $F$ and $G$ be the cumulative distribution functions of $n$-vector random variables for two alternatives $A_p$ and $A_q$, respectively. For $h = 1, 2, 3$, $F$ dominates $G$ (or alternative $A_p$ is preferred to alternative $A_q$) in the sense of $h$th-degree stochastic dominance, denoted as $F \geq_h G$ (or $A_p \geq_h A_q$) if $E[u, F] \geq E[u, G]$, for all $u \in S_h$ and for $p, q \in IA.$
For the single-attribute utility function \( n = 1 \), there are well-known theorems which provide necessary and sufficient conditions for stochastic dominance\([15],[16],[41]\):

**Theorem 4. 1\([15]\) For all \( x, y \in [a,b] \) and \( p, q \in IA \)

a) \( F \geq_1 G \iff F(x) \geq G(x) \) for all \( u \in S_1 \)

b) \( F \geq_2 G \iff \int_a^X F(t)dt \geq \int_a^X G(t)dt \) for all \( u \in S_2 \)

c) \( F \geq_3 G \iff \mu_F \geq \mu_G \) and \( \int_a^X \int_a^Y F(t)dt \cdot dy \geq \int_a^X \int_a^Y G(t)dt \cdot dy \)

where \( \mu_F = \int_a^b dF(x) \) for all \( u \in S_3 \)

There have been several efforts to develop stochastic dominance in multiattribute case. Levy\([23]\) developed sufficient rules for first and second degree stochastic dominance when the utility function is of the form \( u(x_1, x_2, \ldots, x_n) = u(y) \) where \( y = \prod_{i=1}^n x_i \) and outcomes are independent. Levy and Paroush\([24]\) developed first-degree stochastic dominance rules for multi-period additive utility functions and for univariate utility functions when outcomes are dependent. Huang, et.al.\([17]\) provided the theorem below for the multiattribute utility function, but it does not incorporate the decision maker's relative risk attitude nor does Theorem 4. 1.

**Theorem 4. 2\([17]\) For all \( x \in S \) and for \( p, q \in IA \)

if \( F_{i1} \geq G_{i1} \) for all \( i = 1, 2, \ldots, n \), then \( F \geq_1 G \)
if \( u \in S_1 \) and \( \frac{\partial g_{i|j}}{\partial x_j} \leq 0 \) for all \( i, j = 1, 2, \ldots, n \) and \( j < i \) where \( F_{i|j} \) and \( G_{i|j} \) are conditional cumulative distribution functions for alternatives \( A_p \) and \( A_q \), respectively, given \( (x_1, \ldots, x_{i-1}) \).

Theorem 4.2 implies that, for the decision maker who believes more of each attribute is better, his preference of \( A_p \) over \( A_q \) can be established by comparing the conditional cdf's \( F_{i|j} \) and \( G_{i|j} \) and by checking to see if the conditional cdf's \( G_{i|j} \) (\( i = 1, 2, \ldots, n \)) are nonincreasing with respect to \( x_1, x_2, \ldots, x_{i-1} \). The latter means that \( x_i \) and \( x_k \) (\( k = 1, 2, \ldots, i-1 \)) are nonnegatively correlated. That is, nonnegative correlation between \( x_i \) and \( x_k \) must be satisfied for \( A_p \) to be dominant over \( A_q \) in the sense of first degree stochastic dominance. Note that in Theorem 4.2, the ordering of the attributes is immaterial, thus, the nonnegative correlation must hold for every pair of attributes.

4.3 Stochastic Dominance and Relative Risk Aversion

When the decision is made under uncertainty, stochastic dominance has been traditionally defined and used with a decision maker's utility function \( u \) which is assumed not to be known exactly. In doing so, we usually classify decision
maker's utility function in terms of his attitude toward risk as we did in the previous section. But the classification in the previous section is not at all an accurate one because a utility function \( u \) represents not only the decision maker's attitude toward risk but also his strength of preference. For example, it is possible that the decision maker may be relatively risk neutral even though his utility function is classified as a risk averse utility function.

I propose to use the notion of relative risk aversion which was stated in Chapter III in developing stochastic dominance. On the one hand, it is an attempt to reflect the decision maker's attitude toward risk more accurately. By doing so we can come up with the set of nondominated alternatives which not only describes the decision maker's behavior in an appropriate manner but also prescribes what to choose.

On the other hand, we can state the decision maker's risk attitude based on \( u_v \) rather than the multivariate utility function \( u \). So, regardless of the number of attributes, we can effectively analyze the decision maker's risk attitude. Thus, we have a better understanding of how the decision maker does behave in multiattribute decision making under uncertainty.

In mathematical notation, the decision maker's utility function can be denoted as \( u(x) \) which has a multiattributed
domain. As before, this utility function can be represented using a value function \( v(x) \) to represent the strength of preference and \( u_v \) to represent the risk attitude. Then, the decision maker's utility function is denoted as \( u_v(v(x)) \) where \( u_v(v) \) is a univariate function which exclusively represents the decision maker's attitude toward risk.

4.3.1 Single Attribute Case

When possible outcomes of an alternative are represented as single attribute consequences, we can think of a class of utility functions \( U_1 \) such that

\[
U_1 = \{ u_v \mid \frac{du_v(v)}{dv} = u_v' \geq 0, \ x \in X \}
\]

\[
V_1 = \{ v \mid \frac{dv(x)}{dx} = v'(x) \geq 0, \ x \in X \} \text{ where } u(x) = u_v(v(x))
\]

Now, let \( A_p \) and \( A_q \) be two alternatives under consideration, i.e., \( p, q \in IA \) and \( F \) and \( G \) be the cumulative distribution functions for \( A_p \) and \( A_q \), respectively. Then, \( A_p \) is preferred to \( A_q \) (\( A_p \succ A_q \)) if \( E[u,F] \geq E[u,G] \) for \( p, q \in IA \). We can develop first-degree stochastic dominance based on this information as follows:

Theorem 4.3 For all \( x \in X \) and for \( p, q \in IA \), \( A_p \succeq_1 A_q \) if \( G(x) \geq F(x) \) for \( u_v \in U_1 \) and \( v \in V_1 \).
Given utility function \( u_v(v(x)) \) and cumulative distribution function (c.d.f.) \( F(x) \), the expected utility \( E(u,F) \) is
\[
E(u,F) = \int_{-\infty}^{\infty} u_v(v) dF(x) = \left[ u_v(v)F(x) \right]_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} u'_v(v)F(x)dx
= 1 - \int_{-\infty}^{+\infty} u'_v \cdot v'(x)F(x)dx
\]  \hspace{1cm} (4.1)

Similarly, \( E(u,G) \) given \( u_v(v(x)) \) and \( G(x) \) is
\[
E(u,G) = 1 - \int_{-\infty}^{+\infty} u'_v \cdot v'(x)G(x)dx
\]  \hspace{1cm} (4.2)

Now if two alternatives \( A_p \) and \( A_q \) are denoted by their c.d.f.'s \( F(x) \) and \( G(x) \), respectively, then by Definition 4.1 \( A_p \) is preferred to \( A_q \) if \( E(u,F) \geq E(u,G) \). So from (4.1) and (4.2),
\[
E(u,F) - E(u,G) = \int_{-\infty}^{+\infty} u'_v \cdot v'(x)G(x)dx - \int_{-\infty}^{+\infty} u'_v \cdot v'(x)F(x)dx
= \int_{-\infty}^{+\infty} u'_v \cdot v'(x) \left[ G(x) - F(x) \right] dx
= \int_{-\infty}^{+\infty} u'_v \cdot v'(x)H_1(x)dx
\]  \hspace{1cm} (4.3)

Since \( u_v(v) \in U_1 \) and \( v(x) \in V_1 \), \( u_v(v) \) and \( v(x) \) are nondecreasing. Therefore, \( E(u,F) \geq E(u,G) \) if \( G(x) \geq F(x) \). Thus, the proof is complete.

In Theorem 4.3, it was assumed that the utility function \( u_v \) and the value function \( v \) are nondecreasing. Thus, no specific attitude toward risk is inferred from the assumption.
Next, we can think of another class of utility functions $U_2$ which is a subset of $U_1$ such that

$$U_2 = \{ u_v | u_v \in U_1, \frac{d^2 u_v(v)}{dv^2} = u_v^\prime \leq 0, \, x \in X \}$$

or

$$U_2 = \{ u_v | u_v \in U_1, \, r_v(v) \geq 0, \, x \in X \}$$

and a subset of $V_1$ such that

$$V_{12} = \{ v | v \in V_1, \frac{d^2 v}{dx^2} \leq 0, \, x \in X \}.$$  

It is important to note that a utility function which belongs to $U_2$ represents relatively risk averse (RRA) behavior of the decision maker. That is a function $u_v$ that belongs to $U_2$ satisfies $u_v^{\prime \prime} \leq 0$ and $u_v^\prime > 0$ and thus, $r_v(v) = -\frac{u_v^{\prime \prime}}{u_v^\prime} \geq 0$. Then by Definition 3.1 the decision maker is relatively risk averse (RRA). Now we can state second-degree stochastic dominance in this case as follows:

**Theorem 4.4** Let $H_1(x) = G(x) - F(x)$ and $H_2(x) = \int_{-\infty}^{x} H_1(t) dt$.

For all $x \in X$ and for $p, q \in IA, A_p \geq A_q$ if $H_2(x) \geq 0$ for $u_v \in U_2$ (i.e., the decision maker is RRA) and $v \in V_{12}$.

<proof>

From (4.3),

$$E(u,F) - E(u,G) = \int_{-\infty}^{\infty} u_v^\prime \cdot v^\prime(x) H_1(x) dx$$

$$= \left[ u_v^\prime \cdot v^\prime(x) H_2(x) \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} (u_v^\prime \cdot (v^\prime(x)))^2 + u_v^\prime \cdot v^{\prime \prime}(x) \right) H_2(x) dx$$

$$= u_v^\prime \cdot v^\prime \cdot H_2(\infty) + \int_{-\infty}^{\infty} u_v^\prime \cdot \left[ r_v(v) \cdot (v^\prime)^2 + (-v^{\prime \prime}) \right] H_2(x) dx$$
Since \( v(x) \in V_{12}, u_v \in U_2, u_v' \geq 0, v' \geq 0, v'' \leq 0 \). So if the decision maker is RRA, i.e., \( r_v(v) > 0 \) and \( H_2(x) \geq 0 \) for all \( x \), then \( E(u,F) \geq E(u,G) \). Thus, the proof is complete.

### 4.3.2 Multiple Attribute Case

In the previous subsection, we developed theorems of stochastic dominance using \( u_v \) when the consequences of an alternative are represented as a single attribute. Sometimes the consequences can be represented as multiattribute consequences. First, suppose that possible consequences of an alternative are represented as two-attribute consequences \( x = (x_1, x_2) \). Then, we can imagine a utility function which belongs to \( U_2 \) where

\[
V_2 = \{ v \mid v(x), \frac{\partial v}{\partial x_1} = v_i \geq 0, i = 1, 2 \text{ and } \frac{\partial^2 v}{\partial x_1 \partial x_2} = v_{12} \leq 0 \}
\]

and \( u(x) = u_v(v(x)), x = (x_1, x_2), v \in V_2 \).

Let \( A_p \) and \( A_q \) be two alternatives and \( F \) and \( G \) be cumulative distribution functions (cdf's) [probability density functions (pdf's)] of \( x \) for \( A_p \) and \( A_q \), respectively, such that

\[
f(x_1, x_2) = f_{2|1}(x_2|x_1) \cdot f_1(x_1), \quad g(x_1, x_2) = g_{2|1}(x_2|x_1) \cdot g_1(x_1).
\]

Further, let \( f_i, g_i (F_i, G_i, i = 1, 2) \) be marginal pdf's (cdf's) and \( f_{2|1}, g_{2|1} (F_{2|1}, G_{2|1}) \) be conditional pdf's (cdf's). Then, dominance of \( A_p \) over \( A_q \) \( (A_p \succ A_q) \) can be established by the following theorem:
Theorem 4.5 - Stochastic Dominance with n = 2.

Define $H_1(x_1) = G_1(x_1) - F_1(x_1)$ and

$H_{2|1}(x_2|x_1) = G_{2|1}(x_2|x_1) - F_{2|1}(x_2|x_1)$.

For $p, q \in \mathbb{I}\mathbb{A}$, $A_p \geq A_q$ if $H_1(x_1) \geq 0$, $H_{2|1}(x_2|x_1) \geq 0$ for $v(x) \in V_2$, $u_v \in U_2$, and $\frac{\partial F_{2|1}(x_2|x_1)}{\partial x_1} \leq 0$.

<Proof>

$E(u, F) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u(x_1, x_2)f(x_1, x_2)dx_1dx_2$

$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u_v(v(x_1, x_2))f_{2|1}(x_2|x_1)f_1(x_1)dx_2dx_1$

$= \int_{-\infty}^{\infty} \left\{ \left[ u_v \cdot F_{2|1} \right]_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} u_v \cdot v_2 \cdot F_{2|1}dx_2 \right\} f_1dx_1$

$= \int_{-\infty}^{\infty} u_v(v(x_1, \infty)) \cdot f_1dx_1 - \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} u_v \cdot v_2 \cdot F_{2|1} \cdot f_1dx_1dx_2$

$= \left[ u_v \cdot F_1 \right]_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} u_v \cdot v_1 \cdot F_1dx_1 - \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} u_v \cdot v_2 \cdot F_{2|1} \cdot f_1dx_1dx_2$

$= 1 - \int_{-\infty}^{+\infty} u_v \cdot v_1 \cdot F_1dx_1 - \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} u_v \cdot v_2 \cdot F_{2|1} \cdot f_1dx_1dx_2$

$E(u, F) - E(u, G)$

$= \int_{-\infty}^{+\infty} u_v \cdot v_1(G_1 - F_1)dx_1 + \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} u_v \cdot v_2(G_{2|1}g_1 - F_{2|1}f_1)dx_1dx_2$

$= \int_{-\infty}^{+\infty} u_v \cdot v_1H_1dx_1 + \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} u_v \cdot v_2(G_{2|1}g_1 - F_{2|1}f_1)dx_1dx_2$

$= A + B$

A is nonnegative because $u_v \geq 0$, $v_1 \geq 0$, and $H_1 \geq 0$ from the assumptions. Now
\[ B = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} u^i v_2 (G_{2|1} g_1 - F_{2|1} f_1) dx_1 dx_2 \]
\[ \geq \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} u^i v_2 \cdot F_{2|1} (g_1 - f_1) dx_1 dx_2 \]
\[ = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left\{ u^i v_2 F_{2|1} (G_1 - F_1) \right\} \left( \frac{\partial}{\partial x_1} (u^i v_2 F_{2|1}) \right) (G_1 - F_1) dx_1 dx_2 \]
\[ = -\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left\{ u^i v_1 v_2 F_{2|1} + u^i v_1 v_{12} F_{2|1} + u^i v_2 \left( \frac{\partial F_{2|1}}{\partial x_1} \right) \right\} H_1 dx_1 dx_2 \]
\[ = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} u^i \left\{ r_v (v) \cdot v_1 v_2 F_{2|1} + (-v_{12}) F_{2|1} + v_2 \left( -\frac{\partial F_{2|1}}{\partial x_1} \right) \right\} H_1 dx_1 dx_2 \quad (4.6) \]

Since \( u^i \geq 0, v_1 \geq 0 \) (\( i = 1, 2 \)), \( F_{2|1} \geq 0 \), and \( H_1 \geq 0 \), (4.6) is nonnegative if \( v_{12} \leq 0, \frac{\partial F_{2|1}}{\partial x_1} \leq 0 \), and the DM is RRA.

In summary, \( E(u, F) - E(u, G) = A + B \) by the assumptions and \( B \) is nonnegative if \( v_{12} \leq 0, \frac{\partial F_{2|1}}{\partial x_1} \leq 0 \), and the DM is RRA. Thus, if the DM is RRA, \( \frac{\partial F_{2|1}}{\partial x_1} \leq 0 \), \( v \in V_2 \), and \( u^i \in U_2 \), then \( E(u, F) \geq E(u, G) \) provided \( H_1 \geq 0 \) and \( H_{2|1} \geq 0 \). Therefore, the proof is complete.

Let the value function be \( v(x_1, x_2) = v^1(x_1) + v^2(x_2) \) where \( f, g, f_1, g_1, f_{2|1}, g_{2|1}, F_1, G_1, F_{2|1}, \) and \( G_{2|1} \) are same as above in Theorem 4.5. Let \( V_2' \) be such that

\[ V_2' = \{ v \mid v(x_1, x_2) = v^1(x_1) + v^2(x_2), \frac{dv^i}{dx_1} \geq 0, i = 1, 2 \}. \]

Then the following corollary can be easily verified as a special case of Theorem 4.5.
Corollary 4.1 Additive Independence of Attributes

Suppose the decision maker is RRA. Then, for \( p, q \in IA \),
\[ A_p \geq_1 A_q \text{ if } G_1(x_1) \geq G_1(x_1), \quad G_{2|1}(x_2|x_1) \geq F_{2|1}(x_2|x_1) \text{ for } v \in V'_2, \quad u_v \in U_2, \quad \text{and } \frac{\partial G_{2|1}(x_2|x_1)}{\partial x_1} \leq 0.\]

Corollary 4.2 Probabilistic Independence

Let \( f(x_1,x_2) = \prod_{i=1}^{2} f_i(x_i) \), \( g(x_1,x_2) = \prod_{i=1}^{2} g_i(x_i) \).
Define \( H_i(x_i) = G_i(x_i) - F_i(x_i) \). Then for \( p, q \in IA \),
\[ A_p \geq_1 A_q \text{ if } H_i(x_i) \geq 0, \quad i = 1, 2, \quad \text{for } v \in V_2, \quad u_v \in U_2.\]

<Proof>

If the levels of attributes are probabilistically independent, then \( G_{2|1}(x_2|x_1) = G_1(x_1) \) and \( F_{2|1}(x_2|x_1) = F_1(x_1) \) in Theorem 4.5. Also, the correlation between \( x_i \) and \( x_j \) are zero because they are independent. Thus, \( \frac{\partial G_{2|1}(x_2|x_1)}{\partial x_1} = 0 \) and this condition is no longer necessary.

We now extend our discussion to the case when a consequence resulting from an alternative is represented as an \( n \)-attribute consequence \( x = (x_1,x_2,\ldots,x_n) \). Define a class of value functions \( V_n \) such that
\[ V_n = \{ v | v(x), \quad \frac{\partial v}{\partial x_i} = v_{i1} \geq 0, \quad i = 1,2,\ldots,n \}
\text{ and } \frac{\partial^2 v}{\partial x_i \partial x_j} = v_{ij} \leq 0 \text{ for } i < j, \quad i,j = 1,2,\ldots,n\).
Let $A_p$ and $A_q$ be two alternatives under consideration and $F$ and $G$ be the cdf's for alternatives $A_p$ and $A_q$, respectively. Also, let $f$ and $g$ be the pdf's for $A_p$ and $A_q$ such that

$$f(x) = \prod_{i=1}^{n} f_{1||}(x_i | x_{i-1})$$

where $f_{1||} = f_1(x_1) = f_1$

$$g(x) = \prod_{i=1}^{n} g_{1||}(x_i | x_{i-1})$$

where $g_{1||} = g_1(x_1) = g_1$

where $x_{i-1} = (x_1, x_2, \ldots, x_{i-1})$ and $f_{1||} = f_{1||}(x_i | x_{i-1})$.

Further, let $f_1, g_1 (F_1, G_1)$ be marginal pdf's (cdf's) and $f_{1||}, g_{1||} (F_{1||}, G_{1||}), i=2,3,\ldots,n$, be conditional pdf's (cdf's). Then, dominance of $A_p$ over $A_q (A_p \succeq A_q)$ can be established by the following theorem:

**Theorem 4.6 - stochastic dominance for n attributes**

Define $H_{1||} = G_{1||} - F_{1||}, i = 1, 2, \ldots, n$. For all $x \in X^n$ and for $p, q \in IA, A_p \succeq A_q$ if $v \in V_n, u \in U_2, H_{1||} \geq 0$ and $\frac{\partial F_{k||}}{\partial x_{k||}} \leq 0$ for $k > i, i, k = 1, 2, \ldots, n$.

**<Proof>**

Let $n = 2$. Then, from Theorem 4.5, the theorem holds true.

Now suppose that the theorem is true for $n = m$. Then we know if $H_{1||} \geq 0$ for $i = 1, 2, \ldots, m$, then $A_p \succeq A_q$ for $v \in V_m, u \in U_2$, and $\frac{\partial F_{k||}}{\partial x_{k||}} \leq 0$ for $k > i, i, k = 1, 2, \ldots, m$.

Now let's see if the theorem holds true when $n = m + 1$.

$$E(u, F) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} u(x_1, x_2, \ldots, x_{m+1}) \cdot f(x_1, x_2, \ldots, x_{m+1}) dx_1 \cdots dx_{m+1}$$

$$= \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} u(v(x)) \cdot \prod_{i=1}^{m+1} f_{i||} dx_1 \cdots dx_{m+1}$$
\[\begin{align*}
&= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} u_\nu(v(x)) \cdot \prod_{i=1}^{m} f_{i1} \cdot dx_1 \cdots dx_m \\
&- \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} u_\nu \cdot v_{m+1} \cdot F_{m+1} \cdot \prod_{i=1}^{m} f_{i1} \cdot dx_m d\bar{x}_m
\end{align*}\]

where \(d\bar{x}_1 = dx_1 \cdots dx_{i-1} dx_{i+1} \cdots dx_{m+1}\)

\[E(u, F) - E(u, G) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} u_\nu(v(x)) \cdot \left[ \prod_{i=1}^{m} f_{i1} \cdot \prod_{i=1}^{m} g_{i1} \right] dx_1 \cdots dx_m
\]

\[+ \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} u_\nu \cdot v_{m+1} \cdot \left[ G_{m+1} - F_{m+1} \cdot \prod_{i=1}^{m} f_{i1} \right] dx_1 \cdots dx_{m+1}
\]

\[= A + B\]

where \(A\) is the equivalent of \(E(u, F) - E(u, G)\) when \(n = m\) except \(u_\nu(v(x_1, x_2, \ldots, x_m, \infty))\) and thus we know that \(A \geq 0\). Now

\[B = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} u_\nu \cdot v_{m+1} \left[ G_{m+1} \cdot \prod_{i=1}^{m} g_{i1} - F_{m+1} \cdot \prod_{i=1}^{m} f_{i1} \right] dx_1 \cdots dx_{m+1}\]

\[\geq \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} u_\nu \cdot v_{m+1} \cdot F_{m+1} \cdot \left[ \prod_{i=1}^{m} g_{i1} - \prod_{i=1}^{m} f_{i1} \right] dx_1 \cdots dx_{m+1} = B'.\]

since \(H_{m+1} = G_{m+1} - F_{m+1} \geq 0\). And

\[\prod_{i=1}^{m} g_{i1} - \prod_{i=1}^{m} f_{i1} = g_{m1} \cdots g_1 - f_{m1} \cdots f_1 = (g_{m1} - f_{m1}) \prod_{i=1}^{m-1} g_{i1} + f_{m1} \left( \prod_{i=1}^{m-1} g_{i1} - \prod_{i=1}^{m-1} f_{i1} \right) = (g_{m1} - f_{m1}) \prod_{i=1}^{m-1} g_{i1} + f_{m1} \left[ (g_{m-11} - f_{m-11}) \prod_{i=1}^{m-2} g_{i1} + f_{m-11} \left( \prod_{i=1}^{m-2} g_{i1} - \prod_{i=1}^{m-2} f_{i1} \right) \right] = (g_{m1} - f_{m1}) \prod_{i=1}^{m-1} g_{i1} + f_{m1} \left( g_{m-11} - f_{m-11} \right) \prod_{i=1}^{m-2} g_{i1} + f_{m1} \cdot f_{m-11} \left( \prod_{i=1}^{m-2} g_{i1} - \prod_{i=1}^{m-2} f_{i1} \right).\]
\[ \begin{align*}
&= (g_{m1} \cdot f_{m1}) \prod_{i=1}^{m-1} g_{i1} + f_{m1} \cdot (g_{m1} \cdot f_{m1}) \prod_{i=1}^{m-2} g_{i1} \\
&+ f_{m1} \cdot f_{m-1} \cdot (g_{m-2} \cdot f_{m-2}) \prod_{i=1}^{m-3} g_{i1} \\
&+ f_{m1} \cdot f_{m-1} \cdot f_{m-2} \cdot (\prod_{i=1}^{m-3} g_{i1} - \prod_{i=1}^{m-3} f_{i1}) \\
&= \ldots.
\end{align*} \]

\[ \begin{align*}
&= (g_{m1} \cdot f_{m1}) \prod_{i=1}^{m-1} g_{i1} + f_{m1} \cdot (g_{m1} \cdot f_{m1}) \prod_{i=1}^{m-2} g_{i1} + \ldots \\
&+ (\prod_{i=1}^{m} f_{i1}) (g_{21} \cdot f_{21}) g_{11} + (\prod_{i=2}^{m} f_{i1}) (g_{11} - f_{11}) \\
&= \sum_{i=1}^{m} \left\{ (\prod_{i=k+1}^{m} f_{i1}) (g_{k1} \cdot f_{k1}) (\prod_{i=1}^{k-1} g_{i1}) \right\} \\
&= \sum_{i=1}^{m} K_i \text{ where } \prod_{i=m+1}^{m} f_{i1} = \prod_{i=1}^{0} g_{i1} = 1
\end{align*} \]

Thus,

\[ \begin{align*}
B' &= \int_{-\infty}^{+\infty} \ldots \int_{-\infty}^{+\infty} u_{\psi} \cdot v_{m+1} \cdot F_{m+1} \left[ \sum_{i=1}^{m} K_i \right] dx_1 \ldots dx_{m+1} \\
&= \int_{-\infty}^{+\infty} \ldots \int_{-\infty}^{+\infty} u_{\psi} \cdot v_{m+1} \cdot F_{m+1} \cdot K_1 dx_1 d\bar{x}_1 + \ldots \\
&\quad + \int_{-\infty}^{+\infty} \ldots \int_{-\infty}^{+\infty} u_{\psi} \cdot v_{m+1} \cdot F_{m+1} \cdot K_m dx_m d\bar{x}_m \\
&= B_1 + B_2 + \ldots + B_m \text{ where}
\end{align*} \]

\[ \begin{align*}
B_1 &= \int_{-\infty}^{+\infty} \ldots \int_{-\infty}^{+\infty} u_{\psi} \cdot v_{m+1} \cdot F_{m+1} \cdot (\prod_{i=2}^{m} f_{i1}) (g_{1} - f_{1}) dx_1 d\bar{x}_1 \\
&= \int_{-\infty}^{+\infty} \ldots \int_{-\infty}^{+\infty} \left\{ \left[ u_{\psi} \cdot v_{m+1} \cdot F_{m+1} \cdot (\prod_{i=2}^{m} f_{i1}) \right] G_{1} - F_{1} \right\} \left. \begin{array}{c}
\int_{-\infty}^{+\infty} \frac{d}{dx_1} (u_{\psi} \cdot v_{m+1} \cdot F_{m+1} \cdot \prod_{i=2}^{m} f_{i1}) (G_{1} - F_{1}) dx_1 \\
\end{array} \right\} d\bar{x}_1
\end{align*} \]
\[ \begin{align*}
&= \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} u^*_1 v_{m+1} F_{m+1} \cdot (\prod_{i=2}^{m} f_{i1}) H_{11} \cdot dx_1 \tilde{dx}_1 \\
&\quad - \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} u^*_1 v_{1,m+1} F_{m+1} \cdot (\prod_{i=2}^{m} f_{i1}) H_{11} \cdot dx_1 \tilde{dx}_1 \\
&\quad - \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} u^*_1 v_{m+1} \left( \frac{\partial}{\partial x_1} F_{m+1} \right) \cdot (\prod_{i=2}^{m} f_{i1}) H_{11} \cdot dx_1 \tilde{dx}_1 - \cdots \\
&\quad - \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} u^*_1 v_{m+1} F_{m+1} \cdot (\prod_{i=3}^{m} f_{i1}) \left( \frac{\partial}{\partial x_1} F_{21} \right) H_{11} \cdot dx_1 \tilde{dx}_1 \\
&= \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} r_v(v) u^*_1 v_{m+1} F_{m+1} \cdot (\prod_{i=2}^{m} f_{i1}) H_{11} \cdot dx_1 \tilde{dx}_1 \\
&\quad + \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} u^*_1 (-v_{1,m+1}) F_{m+1} \cdot (\prod_{i=2}^{m} f_{i1}) H_{11} \cdot dx_1 \tilde{dx}_1 \\
&\quad + \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} u^*_1 v_{m+1} \left( \frac{\partial}{\partial x_1} F_{m+1} \right) \cdot (\prod_{i=2}^{m} f_{i1}) H_{11} \cdot dx_1 \tilde{dx}_1 + \cdots \\
&\quad + \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} u^*_1 v_{m+1} F_{m+1} \cdot (\prod_{i=3}^{m} f_{i1}) \left( \frac{\partial}{\partial x_1} F_{21} \right) H_{11} \cdot dx_1 \tilde{dx}_1 \\
\end{align*} \]

The 1st term of \( B_1 \) is nonnegative because the decision maker is RRA (i.e., \( r_v(v) \geq 0 \)). The 2nd and the 3rd terms are nonnegative because \( v_{1,m+1} \leq 0 \) and \( \frac{\partial F_{m+1}}{\partial x_1} \leq 0 \) by assumption.

Also, each of the rest of the terms are nonnegative because \( \frac{\partial F_{i1}}{\partial x_1} \leq 0 \) (i.e., \( \frac{\partial F_{i1}}{\partial x_1} \leq 0 \)) for \( i = 2, 3, \ldots, m \). Thus, \( B_1 \geq 0 \).

\[ B_2 = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} u^*_1 v_{m+1} F_{m+1} \cdot (\prod_{i=3}^{m} f_{i1}) (g_{21} - f_{21}) g_1 dx_2 \tilde{dx}_2 \\
= \int_{-\infty}^{+\infty} \cdot \int_{-\infty}^{+\infty} \left\{ \left[ u^*_1 v_{m+1} F_{m+1} \cdot (\prod_{i=3}^{m} f_{i1}) \right] (G_{21} - F_{21}) \right\} \cdot \int_{-\infty}^{+\infty} \left( \frac{\partial}{\partial x_2} (u^*_1 v_{m+1} F_{m+1} \cdot (\prod_{i=3}^{m} f_{i1})) \right) (G_{21} - F_{21}) \cdot dx_2 \right\} g_1 dx_2 \]
Similar to $B_1$, the 1st term of $B_2$ is nonnegative because the
decision maker is RRA. Also the 2nd and the 3rd terms are
nonnegative because $v_{2,m+1} \leq 0$ and $\frac{\partial F_{m+1}}{\partial x_2} \leq 0$ by assumption.

And, each of the rest of the terms is nonnegative because\$\frac{\partial f_{i,1}}{\partial x_2} \leq 0$ (i.e., $\frac{\partial F_{m+1}}{\partial x_2} \leq 0$) for $i = 3, 4, \ldots, m$. Thus, $B_2 \geq 0$.

$$B_{m-1} = \int_{-\infty}^{+\infty} \ldots \int_{-\infty}^{+\infty} \left[ u_f v_{m+1} F_{m+1} \cdot f_m \cdot (G_{m-1} - F_{m-1}) \right] \left( \prod_{i=1}^{m-2} g_{i,1} \right) dx_{m-1} d\bar{x}_{m-1}$$

$$= \int_{-\infty}^{+\infty} \ldots \int_{-\infty}^{+\infty} \prod_{i=1}^{m-2} g_{i,1} \left( \frac{\partial}{\partial x_{m-1}} (u_f v_{m+1} F_{m+1} \cdot f_m) \right) (G_{m-1} - F_{m-1}) dx_{m-1}$$
= \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} u_\psi \cdot v_{m-1} \cdot v_{m+1} \cdot F_{m+1} \cdot f_m \cdot H_{m-1} \cdot (\prod_{i=1}^{m-2} g_{i+1}) \, dx_m \, dx_{m-1} \\
- \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} u_\psi \cdot v_{m-1,m+1} \cdot F_{m+1} \cdot f_m \cdot H_{m-1} \cdot (\prod_{i=1}^{m-2} g_{i+1}) \, dx_m \, dx_{m-1} \\
- \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} u_\psi \cdot v_{m+1} \cdot (\frac{\partial}{\partial x_m} F_{m+1} \cdot f_m \cdot H_{m-1} \cdot (\prod_{i=1}^{m-2} g_{i+1}) \, dx_m \, dx_{m-1} \\
- \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} u_\psi \cdot v_{m+1} \cdot F_{m+1} \cdot \left(\frac{\partial}{\partial x_m} f_{m+1} \right) \cdot H_{m-1} \cdot (\prod_{i=1}^{m-2} g_{i+1}) \, dx_m \, dx_{m-1} \\
= \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} r_v (v) \cdot u_\psi \cdot v_{m-1} \cdot v_{m+1} \cdot F_{m+1} \cdot f_m \cdot H_{m-1} \cdot (\prod_{i=1}^{m-2} g_{i+1}) \, dx_m \, dx_{m-1} \\
+ \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} u_\psi \cdot (-v_{m-1,m+1}) \cdot F_{m+1} \cdot f_m \cdot H_{m-1} \cdot (\prod_{i=1}^{m-2} g_{i+1}) \, dx_m \, dx_{m-1} \\
+ \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} u_\psi \cdot v_{m+1} \cdot (-\frac{\partial}{\partial x_m} F_{m+1} \cdot f_m \cdot H_{m-1} \cdot (\prod_{i=1}^{m-2} g_{i+1}) \, dx_m \, dx_{m-1} \\
+ \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} u_\psi \cdot v_{m+1} \cdot F_{m+1} \cdot (-\frac{\partial}{\partial x_m} f_{m+1} \cdot H_{m-1} \cdot (\prod_{i=1}^{m-2} g_{i+1}) \, dx_m \, dx_{m-1} \\
Since the decision maker is RRA (i.e., r_v (v) \geq 0), v_{m-1,m+1} \leq 0, \\
\frac{\partial F_{m+1}}{\partial x_m} \leq 0, and \frac{\partial f_m}{\partial x_m} \leq 0 (i.e., \frac{\partial F_{m+1}}{\partial x_m} \leq 0). Thus, B_m \geq 0.

B_m = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} u_\psi \cdot v_{m+1} \cdot F_{m+1} \cdot (g_{m1} - f_m) \cdot (\prod_{i=1}^{m-2} g_{i+1}) \, dx_m \, dx_{m-1} \\
= \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \left\{ [u_\psi \cdot v_{m+1} \cdot F_{m+1} \cdot (g_{m1} - f_m)] \right\} \, \int_{-\infty}^{+\infty} \left( \frac{\partial}{\partial x_m} (u_\psi \cdot v_{m+1} \cdot F_{m+1}) \right) (g_{m1} - f_m) \, dx_m \, (\prod_{i=1}^{m-2} g_{i+1}) \, dx_{m-1} \\
= -\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} u_\psi \cdot v_{m+1} \cdot F_{m+1} \cdot H_{m1} \cdot (\prod_{i=1}^{m-2} g_{i+1}) \, dx_m \, dx_{m-1} \\
- \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} u_\psi \cdot v_{m+1} \cdot F_{m+1} \cdot H_{m1} \cdot (\prod_{i=1}^{m-2} g_{i+1}) \, dx_m \, dx_{m-1} \\
- \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} u_\psi \cdot v_{m+1} \cdot (\frac{\partial}{\partial x_m} F_{m+1}) \cdot H_{m1} \cdot (\prod_{i=1}^{m-2} g_{i+1}) \, dx_m \, dx_{m-1}
Finally, $B_m \geq 0$ because the decision maker is RRA ($r_v(v) \geq 0$), $v_{m,m+1} \leq 0$, and $\frac{\partial F_m^{m+1}}{\partial x_m} \leq 0$.

From the above, $B_1 \geq 0$, $B_2 \geq 0$, ..., $B_{m-1} \geq 0$, and $B_m \geq 0$. So, $B' \geq 0$. Therefore, $A + B \geq 0$. That is, $E(u,F) - E(u,G) \geq 0$ when $n = m + 1$. Thus, the theorem holds true for $n \geq 3$ and the proof is complete.

Theorem 4. 6 deserves attention in several important aspects. First, a value function which belongs to the set $V_n$ satisfies the condition that $v_{i,j} \leq 0$ for $i, j = 1, 2, \ldots, n$ and $i < j$. This condition implies that when making a decision under certainty, the decision maker is conservative in the sense that he prefers a consequence in which the contribution of the level of each attribute to the value of the consequence does not differ greatly to a consequence in which the contribution differs greatly. It is interesting to note that this condition is similar to the one stated by Richard[33] in terms of the multivariate risk aversion of the decision maker when he makes a decision under uncertainty.
Next, the conditional cumulative distribution functions $F_{ij}$ for $i = 1, 2, \ldots, n$, satisfy $\frac{\partial F_{ij}}{\partial x_i} \leq 0$, $k > i$, $i, k = 1, 2, \ldots, n$. In other words, $F_{ij}$ is a nonincreasing function of $x_1, x_2, \ldots, x_i, \ldots, x_{k-1}$. Since $F_{ij}$ is a cumulative distribution function, it is nondecreasing in $x_k$ given any $x_{k-1} = (x_1, x_2, \ldots, x_{k-1})$. This condition implies a nonnegative correlation between $x_k$ and $x_i$ ($i < k$). Finally, a utility function $u_v$ which belongs to $U_n$ implies the relatively risk averse attitude of the decision maker.

In summary, Theorem 4.6 states that if a decision maker is multiattribute relatively risk averse, his strength of preference on consequences is conservative, and the levels of attributes are nonnegatively correlated, then we can screen out the dominated alternatives using the theorem.
CHAPTER V
SUMMARY AND CONCLUSION

5.1 Research Summary

This research has led to results to support a decision maker who faces a decision making problem in which an alternative is described in the form of a lottery among outcomes each of which may be multi-attributed. It is assumed that the feasible alternatives are fixed and that the attributes are fixed. The attributes are the representation of the decision maker's perception of each alternative.

First, a multivariate cardinal value function \( v \) is defined on a set of all possible consequences each of which is \( n \)-dimensional. A multivariate cardinal value function \( v \) represents the decision maker's strength of preference in multiattribute decision making under certainty. Then, a coefficient matrix \( M \) of a multivariate cardinal value function is developed which characterizes the value function of the decision maker. Theorem 2.3 showed that we can compare the strengths of preference for different decision makers using the coefficient matrix \( M \).
Based on a multivariate cardinal value function, the notion of multiattribute relative risk aversion has been defined in Definition 3.4 to represent the decision maker's attitude toward risk in multiattribute decision making under uncertainty. It was first developed by Dyer and Sarin[8] for a single attribute decision making.

Also, a measure $r_v(v)$ of multiattribute relative risk aversion is defined with respect to the multivariate cardinal value function $v$, which represents his multiattribute utility function in a composite functional form $u_v(v)$. And in Theorem 3.7, it was shown that a decision maker is multiattribute relatively risk averse [prone] if $r_v(v)$ is positive [negative]. Note that the preferences among lotteries are determined by two factors; (1) strength of preference for the consequences under certainty, and (2) attitude toward risk. Thus, use of multiattribute relative risk aversion enables us not only to separate the factors affecting the decision maker's preferences under uncertainty but also to exclusively focus on his attitude toward risk without the effect of his strength of preference.

This separation of the multiattribute utility function $u$ has several advantages. First, it may be better for decision makers to think in two steps, under certainty and under uncertainty. These two steps may not only lead them to make better decisions but also may give them a better insight
about the nature of the problem even though they may not naturally think in the two steps. Second, we can assess the utility function $u$ in an indirect manner when direct assessment of $u$ may result in inconsistent decisions. Furthermore, use of the multivariate cardinal value function $v$ is advantageous when the levels of attributes are not commensurable. Finally, we can focus on the decision maker's relative attitude toward risk without the effect of his strength of preference.

The expected utility is employed to evaluate an alternative given the utility function of the decision maker and the probability distribution of the consequences where the utility function $u_v(v)$ is used instead of $u$. By doing so, we can not only accurately describe the decision maker's risk attitude but also prescribe how they should behave in multiaattribute decision making under uncertainty. Therefore, the expected utility theory is sound in both descriptive and prescriptive sense.

The following assumptions are made in developing stochastic dominance. The multiaattribute utility function of the decision maker is not exactly known but the functional form is known. The probability distribution of the consequences is exactly known.
Given these assumptions, stochastic dominance is developed using multiattribute relative risk aversion. The purpose of stochastic dominance is to eliminate dominated alternatives from the set of feasible alternatives in order to come up with a set which includes a smaller number of nondominated alternatives. The stochastic dominance rule developed here (Theorem 4.6) can be used when the decision maker is conservative under certainty, his attitude toward risk is multiattribute relatively risk averse, and the levels of attributes are nonnegatively correlated.

5.2 Suggestions for Further Study

5.2.1 Measurability Issue

In Chapter II, a multivariate cardinal value function was defined and used to represent the strength of preference of a decision maker. Assessment of a cardinal value function may be done in several ways. One possibility is that the measurement could be done by revealed preference, but this method requires independence of preference differences which makes it hard to implement. Another possibility is to think of a cardinal value function as a transformation of the von Neumann-Morgenstern utility function. But the
transformation does not necessarily guarantee that ordering of alternatives (i.e., lotteries) using the utility function would be the same as that using the value function so they cannot be equivalent.

Now the only possibility is introspection of the decision maker. This implies that the measurement of the strength of preference is somewhat similar to psychologically direct scaling of consequences. As Fishburn[14] points out that "one must logically accept the ability to 'measure' preference differences introspectively much as one would go about measuring lengths with a measuring rod." Thus, it must be very carefully designed to measure the decision maker's strength of preference under certainty. Also it is important to note that people's judgments of consequences are usually on a relative scale rather than an absolute one. Therefore, it may sometimes be necessary to transform their judgments into a proper scale.

5.2.2 Stochastic-Statistical Dominance

The assumptions made in stochastic dominance are that the multiattribute utility function of the decision maker is not known exactly but the functional form is known and that the probability distribution of the consequences is exactly known. If the problem is to determine the most preferred
alternative when the probability distribution of the consequences is not exactly known, then it is useful to develop a stochastic-statistical dominance rule. Stochastic-statistical dominance was first introduced by Takeguchi and Akashi[35] which is to find a set of nondominated alternatives given the assumption that both the utility function and the probability distribution of the consequences are not known exactly. Expanding on Takeguchi and Akashi, we can employ the notion of multiattribute relative risk aversion along with the separation of the factors in developing stochastic-statistical dominance in multiattribute decision making under uncertainty.
BIBLIOGRAPHY


