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The character tables of certain association schemes

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The Ohio State University, 1987
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THE CHARACTER TABLES OF CERTAIN ASSOCIATION SCHEMES

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

by

Sung Yell Song, B.S., M.Ed., M.S.

****

The Ohio State University
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INTRODUCTION

Let \( X = (X, \{R_i\}_{0 \leq i \leq d}) \) be a (commutative) association scheme. Let \( A_i \) be the adjacency matrix with respect to the relation \( R_i \), and let \( \mathcal{A} = (A_0, A_1, \ldots, A_d) = \langle E_0, E_1, \ldots, E_d \rangle \) be the Bose-Mesner algebra (a subalgebra of the full matrix algebra of degree \( |X| \) over \( X \)) with \( E_0, E_1, \ldots, E_d \) the primitive idempotents (which are uniquely determined as a set). Let

\[
A_i = \sum_{j=0}^{d} p_i(j) E_j.
\]

Then the \((d+1) \times (d+1)\) matrix \( P = (p_{ij}) \), whose \((i,j)\)-entry \( p_{ij} \) is \( p_j(i) \), is called the first eigenmatrix of the association scheme \( X \). We also call it the character table of the association scheme \( X \). We use the notation: \( k_i = p_i(0) \) and \( m_i = \) the rank of matrix \( E_i \) \((0 \leq i \leq d)\).

Let \( G \) be a finite group, and let \( C_0 = \{1\}, C_1, \ldots, C_d \) be all the conjugacy classes of \( G \). For \( x, y \in G \), we define \((x, y) \in R_i \) if and only if \( yx^{-1} \in C_i \). Then \( \mathcal{G} = (G, \{R_i\}_{0 \leq i \leq d}) \) becomes a commutative association scheme. Let \( P \) be the first eigenmatrix of the association scheme \( \mathcal{G} \). Then the normalized matrix:

\[
T = \text{diag}(\sqrt{m_0}, \sqrt{m_1}, \ldots, \sqrt{m_d}).P.\text{diag}(1/k_0, 1/k_1, \ldots, 1/k_d)
\] (1)
is identical with the character table of the group $G$. More generally, let $Q$ be a finite loop or quasi-group (cf. [2,8]). For $x \in Q$, there are defined permutations of $Q$, $L(x): y \mapsto xy$, $R(x): y \mapsto yx$. Let $Gr(Q)$ be the group generated by all the $L(x)$ and $R(x)$ with $x \in Q$. Then taking the orbits of the group $Gr(Q)$ acting on the set $Q \times Q$ as the relations, we get a commutative association scheme (cf. [11]). The matrix $T$ defined above in (1) is called the character table of the loop (or quasi-group) $Q$. (If $Q$ is a group, then $T$ is the character table of the group $Q$.) But we think that (unnormalized) $P$ itself is easier to handle than (the normalized) $T$ for our purpose. So, in this dissertation we refer the first eigenmatrix $P$ as the character table of a loop $Q$ (or a group $G$).

A loop is called a Moufang loop if the identity $(xy)(zx) = x((yz)x)$ is satisfied. A loop $Q$ is called simple, if there is no nontrivial loop homomorphism from $Q$. If $Q$ is a finite simple loop, then the association scheme $(Q, \{R_i\}_{0 \leq i \leq d})$ is primitive, i.e., the graphs $(Q, R_i)$ are connected for all $i \geq 1$. (Actually, it is proved that $Q$ is a simple loop if and only if the loop association scheme $\chi(Q)$ is primitive.) Simple Moufang loops were studied extensively, because their structures are close to finite simple groups. See Doro [7] for recent results on simple Moufang loops. Recently, Liebeck [12] has shown, by using the classification of finite simple groups, that there is no other finite simple non-associative (non-group) Moufang loop than the following loops $M^*(q)$ defined by Paige [13] for any prime power $q$. Define

$$M(q) = \left\{ \begin{pmatrix} a & \alpha \\ \beta & b \end{pmatrix} \mid ab - \alpha \circ \beta = 1, \quad a, b \in GF(q), \quad \alpha, \beta \in (GF(q))^3 \right\}$$

with multiplication by
where $\circ$ and $\times$ denote the scalar and vector products in $(GF(q))^3$. Then we have $|M(q)| = q^3(q^4 - 1)$. For $q = 2^r$, define $M^*(q) = M(q)$, and for $q = p^r$ with an odd prime $p$, define $M^*(q) = M(q)/Z$ where $Z = \{(1,0), (-1,0,-1)\}$. So, $|M^*(q)| = q^3(q^4 - 1)/(q - 1,2)$.

In Chapter 2, we will determine the character tables of $M(q)$ and $M^*(q)$. Our first main result, which will be proved in Section 2.2, is stated as follows.

**Theorem 2.2.3** Let $q = 2^r$. Then $M(q)$ has $q + 1$ conjugacy classes (i.e., the association scheme is of class $q$), and the character table is given by

$$
\bar{P} = \begin{pmatrix}
1 & (q^3 + 1)(q^3 - 1) & q^3(q^3 - 1) & \ldots & q^3(q^3 - 1) & q^3(q^3 + 1) \\
1 & q^2 - 1 & -q^2(q - 1) & \ldots & -q^2(q - 1) & q^2(q + 1) \\
1 & -(q^3 + 1) & \vdots & \vdots & A = (a_{k\ell}) & 0 \\
1 & -(q^3 + 1) & \vdots & \vdots & & \vdots \\
1 & q^3 - 1 & \vdots & \vdots & 0 & B = (b_{mn}) \\
1 & q^3 - 1 & \vdots & \vdots & & 1
\end{pmatrix}
$$

where $a_{k\ell} = -q^2(\sigma^{k\ell} + \sigma^{-k\ell})$ for $1 \leq k \leq q/2$, $1 \leq \ell \leq q/2$, and $b_{mn} = q^3(\rho^{mn} + \rho^{-mn})$ for $1 \leq m \leq (q - 2)/2$, $1 \leq n \leq (q - 2)/2$ with $\sigma = \exp(2\pi\sqrt{-1}/(q + 1))$ and $\rho = \exp(2\pi\sqrt{-1}/(q - 1))$. 
Remark. Note that the character table (i.e., the $P$-matrix) of the group $PSL(2, q)$ is also of size $q + 1$ and is given by

$$
P = \begin{pmatrix}
1 & (q+1)(q-1) & q(q-1) & \cdots & q(q-1) & q(q+1) & \cdots & q(q+1) \\
1 & 0 & -(q-1) & \cdots & -(q-1) & q+1 & \cdots & q+1 \\
1 & -(q+1) \\
\vdots & \vdots & A = (a_{kt}) & 0 \\
1 & -(q+1) \\
1 & q-1 \\
\vdots & \vdots & 0 \\
1 & q-1
\end{pmatrix}
$$

where $a_{kt} = -q(\sigma^{kt} + \sigma^{-kt})$ and $b_{mn} = q(\rho^{mn} + \rho^{-mn})$ (cf. Section 1.1). So, it is self-evident how the $P$-matrix of $M(q)$, $q = 2^r$, is obtained by modifying that of $PSL(2, q)$, (i.e., of $\chi(PSL(2, 2^r))$).

The corresponding result for odd $q$, which is proved in Section 2.3, is summarized as follows.

**Theorem 2.3.3.** Let $q = p^r$ with $p$ an odd prime. Then $M^*(q)$ has $(q+3)/2$ conjugacy classes (i.e., the association scheme is of class $(q+1)/2$), and the character table is given by
\[
\begin{pmatrix}
1 & (q^3 + 1)(q^3 - 1) & q^3(q^3 - 1) & \ldots & q^3(q^3 - 1) & \frac{1}{2}q^2(q^3 + e) & q^3(q^3 + 1) & \ldots & q^3(q^3 + 1) \\
1 & q^3 - 1 & -q^2(q - 1) & \ldots & -q^2(q - 1) & \frac{1}{2}q^2(q + e) & q^2(q + 1) & \ldots & q^2(q + 1) \\
1 & -(q^3 + 1) & 0 & & & & & & \\
\vdots & & & & & A = (a_{kl}) & \vdots & & 0 \\
1 & -(q^3 + 1) & 0 & & & & & & \\
1 & (q^3 - e) & a_1, a_2, \ldots, a_s & e & & & b_1, b_2, \ldots, b_t & & \\
1 & q^3 - 1 & 0 & & & d_1 & & & B = (b_{mn}) \\
\vdots & & & & & \vdots & & & \\
1 & q^3 - 1 & 0 & & & d_t & & & \\
\end{pmatrix}
\]

where (i) \( e = 1 \), \( s = (q - 1)/4 \), and \( t = (q - 5)/4 \) if \( q \equiv 1 \pmod{4} \); \( e = -1 \), \( s = t = (q - 3)/4 \) if \( q \equiv -1 \pmod{4} \). (ii) \( a_{kl} = -q^3(a^{2k} + a^{-2k}) \) for \( 1 \leq k \leq s \), \( 1 \leq l \leq s \), and \( a = \exp(2\pi \sqrt{-1}/(q + 1)) \); \( b_{mn} = q^3(\rho^{2mn} + \rho^{-2mn}) \) for \( 1 \leq m \leq t \), \( 1 \leq n \leq t \) and \( \rho = \exp(2\pi \sqrt{-1}/(q - 1)) \). (iii) If \( e = 1 \), then \( a_i = c_i = 0 \) for \( 1 \leq i \leq s \), \( b_j = (-1)^j 2q^3 \) and \( d_j = (-1)^j q^3 \) for \( 1 \leq j \leq t \), and \( e = (-1)^{t+1}q^3 \); if \( e = -1 \), then \( a_i = (-1)^{i-1}2q^3 \), \( c_i = (-1)^{i-1}q^3 \) for \( 1 \leq i \leq s \) and \( b_j = d_j = 0 \) for \( 1 \leq j \leq t \), and \( e = (-1)^t q^3 \).

**Remark.** This table \( \tilde{P} \) is related to the character table of the association scheme \( \overline{X}(PSL(2, q)) \), i.e., the association scheme obtained from the conjugacy classes of \( PSL(2, q) \) by additionally combining two conjugacy classes (of order \( p \)) of \( PSL(2, q) \) which are conjugate in \( PGL(2, q) \) into a single class. Namely, if we replace all the \( q^3 \) by \( q \) and \( q^2 \) by 1 in the \( \tilde{P} \)-matrix in Theorem 2.3.3, then that is the \( P \)-matrix of the association scheme \( \overline{X}(PSL(2, q)) \) (cf. Table 3 and Table 4).
Our next purpose in this dissertation (which will be discussed in Chapter 3) is to show that the results obtained for the character tables of the Paige's simple Moufang loops are special cases in more general character tables of association schemes (permutation groups) which are coming from the action of orthogonal groups \( O_{2m}^+(q) \) acting on the set of non-singular (projective) points. These are discussed in Sections 3.1 (for \( q = 2^n \)) and 3.3 (for \( q = p^n, p \) an odd prime), respectively.

Let \( V \) be a \( 2m \)-dimensional vector space over \( GF(q) \), and let \( f(x) \) be a nonsingular quadratic form on \( V \) with Witt index \( m \). Let \( X \) be the set of non-isotropic (projective) points. Then \( |X| = q^{m-1}(q^m - 1) \). \( G = GO_{2m}^+(q) \) (or simple group \( O_{2m}^+(q) \)) acts on \( X \) transitively if \( q \) is even, and it acts transitively on each half of \( X \) if \( q \) is odd. Any of these transitive permutation groups gives a symmetric association scheme of class \( q \) if \( q \) is even, and class \( (q + 1)/2 \) if \( q \) is odd. (It is shown that this permutation group is isomorphic to the permutation group \( Gr(M^*(q)) \) on \( M^*(q) \) if \( m = 4 \).) Moreover, it is shown that for \( m \geq 2 \) the \( P \)-matrix of this association scheme is obtained from the \( P \)-matrix given in Theorem 2.2.3 or Theorem 2.3.3 by replacing all the expressions \( q^3 \) by \( q^{m-1} \) and \( q^2 \) by \( q^{m-2} \) (and fixing all others). (Compare Tables 9, 11, and 12 to Tables 5, 7, and 8, respectively.) Similarly, if we put \( m = 2 \) in this replacement, we get the \( P \)-matrix of the association scheme coming from \( PSL(2, q) \) (discussed in Theorem 2.2.3 and Theorem 2.3.3). The reason why the \( P \)-matrix for \( m = 2 \) is related to \( PSL(2, q) \) is explained from the fact that \( O_3^+(q) \cong PSL(2, q) \times PSL(2, q) \), while the stabilizer \( O_3(q) \cong PSL(2, q) \). In conclusion, the \( P \)-matrix of our association scheme (coming from \( G \)) is obtained from that of association scheme coming from \( PSL(2, q) \), and this observation includes the \( M^*(q) \)-case as the special case for \( m = 4 \).
The results for the association schemes coming from $O^{+}_{2m}(q)$ are also generalized for the association schemes coming from $O^{-}_{2m}(q)$. These are discussed in Sections 3.2 (for $q = 2^n$) and 3.4 (for $q$ odd), respectively.

Let $V$ be a $2m$-dimensional vector space over $GF(q)$, and let $f(x)$ be a non-singular quadratic form on $V$ of Witt index $m - 1$. Then $G = GO^{+}_{2m}(q)$ (or $O^{-}_{2m}(q)$) acts transitively on the non-isotropic points if $q$ is even; and transitively on each half of the non-isotropic points if $q$ is odd. We again get a symmetric association scheme of class $q$ if $q$ is even, and $(q + 1)/2$ if $q$ is odd. Moreover, the character table of this association scheme is obtained by replacing $q^m$ by $-q^m$ and $q^{m-1}$ by $-q^{m-1}$ in the $P$-matrix of the association scheme of $O^{+}_{2m}(q)$ (cf. Tables 10, 13 and 14).
CHAPTER I
PRELIMINARIES

The purpose of this chapter is to describe the basic notations which will be used repeatedly and is to present the fundamental theory of association schemes which will be referred to throughout. In Section 1.0, we will provide the well-known results of association schemes and will illustrate two important association schemes which will establish the background of the association schemes discussed in this dissertation. In Section 1.1 and Section 1.2, we will compute the structure numbers and character tables of the association schemes coming from $PSL(2, q)$ as a model for the association schemes discussed later. In the last section, we will prove two useful enumerative lemmas which will be used frequently.

1.0. Association schemes

Let $X$ be a nonempty finite set and $\{R_i\}_{0 \leq i \leq d}$ be a partition of $X \times X$ into $d + 1$ classes. Then a configuration $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq d})$, or briefly $\mathcal{X}$, is called an association scheme of class $d$ if it satisfies the following properties:

(A1) $R_0 = \{(x, x) | x \in X\}$.

(A2) For any $R_i$, $^{t}R_i = R_{i'}$ for some $i' \in \{0, 1, 2, \ldots, d\}$, where $^{t}R_i = \{(x, y) | (y, x) \in R_i\}$. 
(A3) For every pair \((x, y) \in R_h\), the number of \(z \in X\) such that \((x, z) \in R_i\), \((x, y) \in R_j\) is a constant \(p_{ij}^h\) depending only on \(h, i, j\). The constant \(p_{ij}^h\) is called the intersection number (or parameter) of \(X\).

If an association scheme satisfies the additional property

\[(A4) \ p_{ij}^h = p_{ji}^h \text{ for all } h, i, j,\]

then it is called a commutative association scheme.

If it satisfies the additional property

\[(A5) \ t' = i \text{ for all } i,\]

then it is called a symmetric association scheme.

For a given association scheme \(X = (X, \{R_i\}_{0 \leq i \leq d})\), the \(i^{th}\) adjacency matrix \(A_i\) is defined to be the matrix of degree \(|X|\) whose rows and columns are indexed by the elements of \(X\) and whose \((x, y)\) entries are

\[(A_i)_{xy} = \begin{cases} 1 & \text{if } (x, y) \in R_i, \\ 0 & \text{otherwise} \end{cases}\]

It is obvious that \(A_0 + A_1 + \cdots + A_d = J\), the all 1 matrix, and the conditions (A1), \(\cdots\), (A5) are equivalent to the following (A1)', \(\cdots\), (A5)', respectively.

\[(A1)' \ A_0 = I.\]

\[(A2)' \ t' A_i = A_{i'} \text{ for some } i' \in \{0, 1, 2, \ldots, d\}.\]

\[(A3)' \ A_i A_j = \sum_{k=0}^{d} p_{ij}^k A_k \text{ for all } i, j.\]

\[(A4)' \ A_i A_j = A_j A_i \text{ for all } i, j.\]

\[(A5)' \ t' A_i = A_{i'} \text{ for all } i,\]
where $^t A_i$ is the transpose of the matrix $A_i$. Note that the symmetricity (A5)$^t$ implies the commutativity (A4)$^t$.

In what follows, we assume that the given association scheme $X = (X, \{R_i\}_{0 \leq i \leq d})$ is commutative. Let $k_i$ denote the number of elements $y \in X$ such that $(x, y) \in R_i$ for a fixed $x \in X$. The positive integer $k_i$ is called the valency of the $i$th relation $R_i$. Let $B_i$ denote the matrix of degree $d + 1$ whose $(j, h)$-entry is $p^h_{ij}$. The matrix $B_i$ is called the $i$th intersection matrix of the scheme. The following equations are easily checked.

\begin{align*}
|X| &= \sum_{i=0}^{d} k_i \\
k_i &= p^0_{ii} = \sum_{j=0}^{d} p^h_{ij} \\
p^h_{i0} &= \delta_{ih} \\
p^0_{ij} &= k_i \cdot \delta_{ij} \\
k_h \cdot p^h_{ij} &= k_j \cdot p^j_{ih} \\
\sum_{k=0}^{d} p^k_{ij} \cdot p^h_{j\ell} &= \sum_{k=0}^{d} p^k_{ji} \cdot p^h_{kq}
\end{align*}
where \( \delta_{ij} \), the Kronecker delta, denotes 1 if \( i = j \), 0 otherwise.

Let \( \mathcal{A} \) be the subalgebra of \( M_n(\mathbb{C}) \) spanned by the adjacency matrices \( A_0, A_1, \ldots, A_d \), where \( M_n(\mathbb{C}) \) is the full matrix algebra of degree \( n = |X| \) over the complex field \( \mathbb{C} \). \( \mathcal{A} \) is a commutative algebra of dimension \( d + 1 \) by (A3) and (A4). This algebra \( \mathcal{A} \) is called the adjacency algebra of \( X \) or the Bose-Mesner algebra of \( X \). Let \( \mathcal{B} \) be the vector space spanned by \( B_0, B_1, \ldots, B_d \), then \( \mathcal{B} \) is a subalgebra of \( M_{d+1}(\mathbb{C}) \). Note that \( B_0 = I \) and \( B_i B_j = \sum_{h=0}^{d} p_{ij}^h B_h \) from (1.0.3) and (1.0.6). The algebra \( \mathcal{B} \) is called the intersection algebra of \( X \). In fact, the intersection algebra \( \mathcal{B} \) is isomorphic to the adjacency algebra \( \mathcal{A} \) by the correspondence \( B_i \) to \( A_i \). In particular, \( A_i \) and \( B_i \) have the same minimal polynomial.

The adjacency algebra \( \mathcal{A} \), which is known to be a semisimple algebra, admits a set of primitive idempotents \( E_0, E_1, \ldots, E_d \), ordered so \( |X| \cdot E_0 = J \), which are uniquely determined. Thus

\[
A_j = \sum_{i=0}^{d} p_j(i) E_i
\]

for certain complex numbers \( p_j(i) \). Namely, \( p_j(i) \) is the eigenvalue of \( A_j \) associated with the eigenspace spanned by the columns of \( E_i \). The complex number \( d + 1 \) by \( d + 1 \) matrix \( P \), whose \( (i,j) \)-entry is defined by \( p_j(i) \), is called the first eigenmatrix of the association scheme \( X \).

Let \( m_i \) be the rank of \( E_i \) and call it the \( i \)th multiplicity of \( X \). The following relations are well known.

\[
p_0(i) = 1. \quad (1.0.7)
\]
\[ k_i = p_i(0). \] (1.0.8)

\[ \sum_{\ell=0}^{d} p_{\ell}(i) = \begin{cases} |X| & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases} \] (1.0.9)

\[ \sum_{\ell=0}^{d} \frac{1}{k_{\ell}} \cdot p_{\ell}(i) \cdot \overline{p_{\ell}(j)} = \frac{|X|}{m_i} \cdot \delta_{ij}, \] (1.0.10)

where \( \overline{\alpha} \) denotes the complex conjugate of \( \alpha \).

\[ m_i = \frac{|X|}{\sum_{\ell=0}^{d} \frac{1}{k_{\ell}} |p_{\ell}(i)|^2} \] (1.0.11)

by solving (1.0.10) for \( m_i \).

\[ p_{ij}^{*} = \frac{1}{|X| \cdot k_{j}} \sum_{\ell=0}^{d} p_{i}(\ell) \cdot p_{j}(\ell) \cdot \overline{p_{k}(\ell)} \cdot m_{\ell}. \] (1.0.12)

Denoting the \( i \)th row (vector) of the first eigenmatrix \( P \) by \( v_i \), i.e., \( v_i = (p_0(i), p_1(i), \ldots, p_d(i)) \), the column vector \( ^t v_i \), \( i = 0, 1, 2, \ldots, d \), is characterized as the common right eigenvector of \( B_0, B_1, \ldots, B_d \). More precisely,

\[ B_j \cdot ^t v_i = p_j(i) \cdot ^t v_i \]

or
\[ B_j \cdot ^tP = ^tP \cdot P_j \]  \hspace{1cm} (1.0.13)

where \( P_j = \text{diag}[p_j(0), p_j(1), \ldots, p_j(d)] \), the diagonal matrix whose diagonal entries are \( p_j(0), p_j(1), \ldots, p_j(d) \).

In the rest of this section, we will discuss two important examples of association schemes.

Example 1. Let \( G \) be a finite group and let \( C_0 = \{1\}, C_1, C_2, \ldots, C_d \) be the conjugacy classes of \( G \). Let \( R_i \) be the relation on \( G \) defined by

\[(x, y) \in R_i \iff yx^{-1} \in C_i. \] \hspace{1cm} (1.0.14)

Then \( G \times G = R_0 \cup R_1 \cup \cdots \cup R_d \), and \( R_i \cap R_j = \emptyset \) if \( i \neq j \), and \( R_0 = \{(x, x) \mid x \in G\}\). \( R_i \in \{R_0, R_1, \ldots, R_d\} \) from the fact that \( C_i' = \{a \in G \mid a^{-1} \in C_i\} \) is also a conjugacy class of \( G \). Moreover, the number

\[ p_{ij}^h = |\{(a, b) \in C_i \times C_j \mid ab = c\}| \quad \text{for} \quad c \in C_h \] \hspace{1cm} (1.0.15)

does not depend on the choice of \( c \in C_h \). Hence

\[ X(G) = (G, \{R_i\}_{0 \leq i \leq d}) \]

becomes a commutative association scheme with \( k_i = |C_i| \). This association scheme \( X(G) \) is called the group association scheme of \( G \). Note that the group association scheme becomes a symmetric association scheme if \( C_i' = C_i \).
Let \( \{\chi_0, \chi_1, \ldots, \chi_d\} \) be the set of all irreducible characters of \( G \) and let \( f_i = \chi_i(1) \) be the degree of the character \( \chi_i \). Let \( a_i \) be a representative of the conjugacy class \( C_i \), and let \( C_i \) be the formal sum of the elements of \( C_i \), i.e.,

\[
C_i = \sum_{a \in C_i} a.
\]

Then we can write

\[
C_i C_j = \sum_{h=0}^{d} p_{ij}^h \cdot C_h. \tag{1.0.16}
\]

Furthermore, from the relation between the algebra generated by \( C_0, C_1, \ldots, C_d \), as a subalgebra of the group algebra of \( G \) over \( \mathbb{C} \), and the adjacency algebra \( A \) of the group association scheme \( X(G) \), we have

\[
p_{j}(i) = \frac{k_j \chi_i(a_j)}{f_i}, \tag{1.0.17}
\]

with a suitable rearrangement of the indices. This implies that the group character table \( T \) of \( G \) and the first eigenmatrix \( P \) of the group association scheme \( X(G) \) have the following relation (with a suitable permutation on rows and columns if it is necessary).

\[
P = F^{-1} \cdot T \cdot K, \tag{1.0.18}
\]

where \( F \) and \( K \) are the \( d + 1 \) by \( d + 1 \) diagonal matrices \( \text{diag}[f_0, f_1, \ldots, f_d] \) and \( \text{diag}[k_0, k_1, \ldots, k_d] \), respectively. Note that \( f_i = \sqrt{m_i} \) where \( m_i \) is the \( i \)th multiplicity of \( X(G) \).
Because of the above relation, the eigenmatrix $P$ can be regarded as the character table of the association scheme $\mathcal{X}$. Henceforth the first eigenmatrix $P$ of the association scheme will be called the character table of the association scheme.

Example 2. Let $G$ be a transitive permutation group on $\Omega$, $\Omega$ a $n$-set, and let $G$ act on $\Omega \times \Omega$ in such a way that $(x, y)^g = (x^g, y^g)$ for $x, y \in \Omega$, $g \in G$. Let $R_0, R_1, \ldots, R_d$ be the $G$-orbits on $\Omega \times \Omega$, ordered so $R_0 = \{(x, x) \mid x \in \Omega\}$. Then

$$\mathcal{X}(G, \Omega) = (\Omega, \{R_i\}_{0 \leq i \leq d})$$

forms an association scheme. For $x \in \Omega$, denote

$$R_i(x) = \{y \in \Omega \mid (x, y) \in R_i\},$$

then $R_0(x) = \{x\}$, $R_1(x), R_2(x), \ldots, R_d(x)$ are the orbits of $G_x$, the stabilizer of $x$ in $G$, on $\Omega$. It holds that $R_i(x)^g = R_i(x^g)$ for all $g \in G$, and all $i \in \{0, 1, 2, \ldots, d\}$. So $k_i = |R_i(x)|$ for all $i$, and the intersection numbers are given as

$$P_{ij}^k = |\{x \in R_i(x) \mid y \in R_j(x)\}| \quad \text{for} \quad y \in R_k(x). \quad (1.0.19)$$

Note that $\mathcal{X}(G, \Omega)$ is commutative if and only if the permutation character of $G$ on $\Omega$ is multiplicity free (cf. [1]). $\mathcal{X}(G, \Omega)$ is a symmetric association scheme if and only if for any $(x, y) \in R_i$, $i = 0, 1, \ldots, d$, there exists an element $g \in G$ such that $x^g = y$ and $y^g = x$.
1.1. Group association schemes of $SL(2, q)$ with $q = 2^n$ and their character tables.

In this section, let $F$ be the finite field of $q = 2^n$ elements, and let $\nu$ be a generator of the cyclic group $F^* = F - \{0\}$. Then $G = SL(2, q) = PSL(2, q)$ has exactly $q + 1$ conjugacy classes

\begin{align*}
C_0 &= \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}, \\
C_1 &= \left\{ \begin{pmatrix} a & c \\ d & b \end{pmatrix} \in G \mid a + b = 0 \right\} - \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}, \\
C_i &= \left\{ \begin{pmatrix} a & c \\ d & b \end{pmatrix} \in G \mid a + b = \nu^{i-1} \right\} \text{ for } i = 2, 3, \ldots, q.
\end{align*}

Let $X(G) = X(SL(2, q)) = (SL(2, q), \{R_i\}_{0 \leq i \leq q})$ be the group association scheme of class $q$ defined by

$$(x, y) \in R_i \iff yx^{-1} \in C_i \quad (0 \leq i \leq q).$$

Then $X(G)$ is a symmetric association scheme. The valencies $k_i$'s are given by

\begin{align*}
k_1 &= (q + 1)(q - 1) \\
k_i &= \begin{cases} 
q(q - 1) & \text{if } N[t^2 + \nu^{i-1}t + 1 = 0] = 0 \\
q(q + 1) & \text{if } N[t^2 + \nu^{i-1}t + 1 = 0] = 2
\end{cases}
\end{align*}

where $N[t^2 + \nu^{i-1}t + 1 = 0]$ denotes the number of solutions of $t^2 + \nu^{i-1}t + 1 = 0$ in $F$.

Lemma 1.1.1. Let $a_{ij}^k$ denote the parameters of the group association scheme $X(SL(2, 2^n))$, and let $N[f(x_1, x_2) = 0]$ denote the number of solutions of $f(x_1, x_2) = 0$ in $F^2$. Then
\[ a_{11}^1 = N[x_1^2 + x_2^2 + 1 = 0] - 2, \]
\[ a_{1j}^1 = N[x_1^2 + x_2^2 + \nu^{j-1}x_2 + 1 = 0], \quad \text{for } 2 \leq j \leq q, \]
\[ a_{ij}^1 = N[x_1^2 + \nu^{i-1}x_1 + x_2^2 + \nu^{j-1}x_2 + 1 = 0], \quad \text{for } 2 \leq i \leq j \leq q, \]
\[ a_{ij}^h = N[(x_1 + 1)^2 + \nu^{h-1}(x_1 + 1)x_2 + x_2^2 = 0] - 1, \quad \text{for } 2 \leq h \leq q, \]
\[ a_{ij}^h = N[x_1^2 + x_2^2 + \nu^{i-1}x_1 + \nu^{j-1}x_2 + \nu^{h-1}x_1x_2 + 1 = 0], \quad \text{for } 2 \leq h \leq i \leq j \leq q. \]

Note that the remaining parameters \( a_{ij}^h \) are easily computed from the above parameters by using the well-known equalities \( a_{ij}^h = a_{ji}^h \) and \( k_h \cdot a_{ij}^h = k_j \cdot a_{ih}^j \). (This is true for any symmetric association scheme, henceforth we will restrict ourselves to the case \( h \leq i \leq j \).)

**Proof of Lemma 1.1.1.** For \( h \geq 2 \), let \( z = \binom{0 \ 1}{1 \ \nu^{h-1}} \in C_h \). Since
\[ a_{ij}^h = |\{(x, y) \in C_i \times C_j | xy = z\}|, \]
we need to count the number of vectors \((x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4)\) in \( F^8 \) satisfying the system of equations
\[ xy = \begin{pmatrix} x_1 & x_2 \\ x_4 & x_3 \end{pmatrix} \begin{pmatrix} y_1 & y_2 \\ y_4 & y_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & \nu^{h-1} \end{pmatrix} = z; \]
\[ x_1x_3 + x_2x_4 = 1; \quad x_1 + x_3 = \nu^{i-1}; \]
\[ y_1y_3 + y_2y_4 = 1; \quad y_1 + y_3 = \nu^{j-1}. \]

By a straightforward computation we have
\[ a_{ij}^h = | \{(x_1, x_2) \in F^2 | x_1^2 + x_2^2 + x_1 x_2 + 1 = 0 \} | , \]

for all \( i, j \) such that \( 2 \leq h \leq i \leq j \leq q \), so that the last equality holds.

If \( i = 1 \) and \( h = j \geq 2 \), then, similarly, solving the system of equations obtained from \( xy = z; x \in C_1 \) (i.e., \( x_1 + x_3 = 0 \)); \( y \in C_h \), with the caution that \( x \neq (1 0 \ 0 1) \), we have

\[ a_{1h}^h = | \{(x_1, x_2) \in F^2 | x_1^2 + x_2^2 + x_1 x_2 + 1 = 0 \} | -1 . \]

In the same manner, the rest of equalities are easily checked by setting \( z = (0 1 0 1) \).

Although all parameters are computed explicitly in terms of one variable \( q \) as we know the number \( N[\cdots] \), it will be convenient to use the above expression in order to see the relation between the parameters \( a_{ij}^h \) of \( \mathcal{X}(SL(2, 2^n)) \) and those of other association schemes that will be discussed later.

To describe the character table (\( P \)-matrix) of the association scheme \( \mathcal{X}(SL(2, 2^n)) \), we use the well-known group character table of \( SL(2, 2^n) \).

Taking the elements

\[ 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad u = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad w = \begin{pmatrix} \nu & 0 \\ 0 & \nu^{-1} \end{pmatrix}, \]

and \( \nu \) an element of order \( q+1 \), the group character table of \( SL(2, q) \), \( q = 2^n \) is given as follows: (cf. [6], [14], etc.)
for $1 \leq s \leq \frac{q}{2}$, $1 \leq \ell \leq \frac{q}{2}$, $1 \leq t \leq \frac{q-2}{2}$, $1 \leq m \leq \frac{q-2}{2}$, where $\sigma$ and $\rho$ are primitive $(q + 1)$th root and $(q - 1)$th root of $1$, respectively.

Let $[x]$ denote the conjugacy class of a group containing $x$. Then as in the above character table, we can also arrange the conjugacy classes of $SL(2, q)$, $q = 2^n$, as the following order:

$$
C_0 = [1], \ C_1' = [u], \ C_2' = [v], \ C_3' = [v^2], \cdots, \ C_{c+1}' = [v^c],
$$

$$
C_{c+2}' = [\omega], \cdots, \ C_q' = [\omega^d],
$$

where $c = \frac{q}{2}$, $d = \frac{q-2}{2}$. Then the character table ($P$-matrix) of the association scheme $X'(SL(2, 2^n)) = (SL(2, 2^n), \{R_j\}_{0 \leq j \leq q})$, which is defined by $(x, y) \in R_j'$ if and only if $yx^{-1} \in C_j'$, is given by Table 1, where $\sigma = \exp(2\pi \sqrt{-1}/(q + 1))$, $\rho = \exp(2\pi \sqrt{-1}/(q - 1))$, $c = \frac{q}{2}$, $d = \frac{q-2}{2}$.

Now let $\pi$ be the bijection on the index set $\{0, 1, \ldots, q\}$ such that

$$
\pi(i) = j \iff C_i = C_j' \ (\iff R_i = R_j'),
$$

and let $\Pi$ be the permutation matrix whose $(i, j)$-entry $\Pi_{ij}$ is defined by

$$
\Pi_{ij} = \begin{cases} 
1 & \text{if } \pi(i) = j \\
0 & \text{otherwise}
\end{cases}.
$$
Then we have the following:

**Lemma 1.1.2** Let $P$ be the character table of $\mathcal{X}'(SL(2, 2^n))$ and $\Pi$ be the above permutation matrix. Then the character table of the association scheme $\mathcal{X}(SL(2, 2^n))$ is $P \cdot \Pi^{-1}$.

**Proof** Obvious. \qed

For later use, we make the following remark.

**Remark 1.1.3** Let $B_i$ denote the $i$-th intersection matrix of $\mathcal{X}(SL(2, 2^n))$ and let $P$ be the character table of $\mathcal{X}'(SL(2, 2^n))$. Then for given $i$ with $i(i) = j$, the following equality holds:

$$B_i \cdot (\Pi \cdot {}^t P) = (\Pi \cdot {}^t P) \cdot P_j \quad (1.1.1)$$

where $P_j$ is the diagonal matrix with $[p_j(0), p_j(1), \ldots, p_j(q)]$ on the diagonal and $^t P$ denotes the transpose of the matrix $P$.

1.2. **Group association schemes of $SL(2, q)$ and $PSL(2, q)$ with $q = p^n$ ($p$ odd) and their character tables.**

In this section let $F$ be the finite field of $q = p^n$ elements, $p$ an odd prime, and let $\nu$ be a generator of the cyclic group $F^*$. Denote

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad -1 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad u = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad u' = \begin{pmatrix} 1 & 0 \\ \nu & 1 \end{pmatrix}, \quad \omega = \begin{pmatrix} \nu & 0 \\ 0 & \nu^{-1} \end{pmatrix}$$
in \( G = SL(2, q) \). \( G \) contains an element \( v \) of order \( q + 1 \). For any \( x \in G \), let \([x]\) denote the conjugacy class of \( G \) containing \( x \). Then \( G \) has exactly \( q + 4 \) conjugacy classes

\[ [1], [-1], [u], [u'], [-u], [-u'], [v], \ldots [v^c], [\omega], \ldots [\omega^d], \]

satisfying

<table>
<thead>
<tr>
<th>( x )</th>
<th>1</th>
<th>-1</th>
<th>( u )</th>
<th>( u' )</th>
<th>-( u )</th>
<th>-( u' )</th>
<th>( v^e )</th>
<th>( \omega^m )</th>
</tr>
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<tbody>
<tr>
<td>([x])</td>
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<td>1</td>
<td>( \frac{1}{2}(q^2 - 1) )</td>
<td>( \frac{1}{2}(q^2 - 1) )</td>
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<td>( \frac{1}{2}(q^2 - 1) )</td>
<td>( q(q - 1) )</td>
<td>( q(q + 1) )</td>
</tr>
</tbody>
</table>

for \( 1 \leq \ell \leq c = \frac{q - 1}{2} \), \( 1 \leq m \leq d = \frac{q - 3}{2} \).

Denote \( \epsilon = (-1)^{\frac{q - 1}{2}} \). Let \( \sigma \in \mathbb{C} \) be a primitive \( (q + 1) \)th root of 1, \( \rho \in \mathbb{C} \) a primitive \( (q - 1) \)th root of 1. Then the complex character table of \( G = SL(2, p^n) \), \( p \) an odd prime, is given by

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<thead>
<tr>
<th>( i )</th>
<th>1</th>
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<th>( u )</th>
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<th>( v^e )</th>
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<td>1G</td>
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<tr>
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<tr>
<td>( \theta_2 )</td>
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for \( 1 \leq i \leq \frac{q - 1}{2} \), \( 1 \leq j \leq \frac{q - 3}{2} \), \( 1 \leq \ell \leq \frac{q - 1}{2} \), \( 1 \leq m \leq \frac{q - 3}{2} \).
Let $X(SL(2, q))$ be the group association scheme of class $q+3$, naturally defined by using these $q + 4$ conjugacy classes. Now we slightly modify this association scheme so that two pairs of its associate classes are to be combined pairwise into two single associate classes. The modified association scheme $\overline{X}(SL(2, q))$ of class $q + 1$, which becomes a symmetric association scheme, is now defined in detail.

Let $C_0, C_1, \ldots, C_{q+1}$ denote the following subsets of $G$:

\[
C_0 = \{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \}, \\
C_1 = \{ \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \}, \\
C_i = \{ \begin{pmatrix} a & c \\ d & b \end{pmatrix} \in G^* \mid a + b = \nu^{i-1} \} \quad \text{for } i = 2, 3, \ldots, q, \\
C_{q+1} = \{ \begin{pmatrix} a & c \\ d & b \end{pmatrix} \in G \mid a + b = 0 \},
\]

where $G^* = G - (C_0 \cup C_1)$. For the notational convenience, let the indices $i$ and $j$ for which $\nu^{i-1} = 1 + 1$ and $\nu^{j-1} = (-1) + (-1)$ in $F$ be denoted by $i_0$ and $j_0$, respectively. Then it is easily checked that

\[
C_{i_0} = [u] \cup [u'], \quad C_{j_0} = [-u] \cup [-u'],
\]

and each of the remaining $C_i$'s is identified with one of the remaining conjugacy classes of $SL(2, q)$.

Now define $(x, y) \in R_i$ if and only if $yx^{-1} \in C_i$, for $i = 0, 1, 2, \ldots, q+1$. Then $\overline{X}(SL(2, q)) = (SL(2, q), \{R_i\}_{0 \leq i \leq q+1})$ becomes a symmetric association scheme of class $q + 1$. The valencies of $\overline{X}(SL(2, q))$ are given by
\[ k_0 = k_1 = 1, \]

\[
k_i = \begin{cases} 
q(q-1) & \text{if } N[t^2 - \nu^{i-1}t + 1 = 0] = 0, \\
(q-1)(q+1) & \text{if } N[t^2 - \nu^{i-1}t + 1 = 0] = 1, \text{ for } i = 2, 3, \ldots, q, \\
q(q+1) & \text{if } N[t^2 - \nu^{i-1}t + 1 = 0] = 2, 
\end{cases}
\]

\[
k_{q+1} = \begin{cases} 
q(q-1) & \text{if } N[t^2 + 1 = 0], \text{ i.e. if } q \equiv -1 \pmod{4}, \\
q(q+1) & \text{if } N[t^2 + 1 = 0], \text{ i.e. if } q \equiv 1 \pmod{4}. 
\end{cases}
\]

(Note that \( k_{i_0} = k_{j_0} = (q - 1)(q + 1) \).)

**Lemma 1.2.1** Let \( b_{ij}^h \) denote the parameters of the association scheme \( \mathcal{X}(SL(2, q)) \) of class \( q + 1 \). Then

\[
b_{ij}^h = N[1 + x_1^2 + x_2^2 - \nu^{i-1}x_1 - \nu^{j-1}x_2 + \nu^{h-1}x_1x_2 = 0]
\]

for \( 2 \leq h \leq i \leq j \leq q \) except the following special cases:

\[
b^{i_0}_{i_0 j_0} = b^{j_0}_{j_0 i_0} = b^{j_0}_{j_0 i_0} = b^{j_0}_{j_0 i_0} = N[x_1 + x_2 - 1 = 0] - 2,
\]

\[
b^{h}_{i_0 h} = b^{h}_{i_0 h} = N[(x_1 - 1)^2 + \nu^{h-1}x_2(x_1 - 1) + x_2^2 = 0] - 1, \quad \text{for } 2 \leq h \leq q,
\]

\[
h \neq i_0, h \neq j_0,
\]

\[
b^{q+1}_{i_0 q+1} = b^{q+1}_{q+1 i_0} = N[(x_1 - 1)^2 + x_2^2 = 0] - 1,
\]

\[
b^{q+1}_{j_0 q+1} = b^{q+1}_{q+1 j_0} = N[(x_1 + 1)^2 + x_2^2 = 0] - 1,
\]

\[
b^{h}_{q+1 q+1} = N[1 + x_1^2 + x_2^2 - \nu^{i-1}x_1 + \nu^{h-1}x_1x_2 = 0], \quad \text{for } 2 \leq h \leq i \leq q,
\]

\[
b^{h}_{q+1 q+1} = N[1 + x_1^2 + x_2^2 + \nu^{h-1}x_1x_2 = 0], \quad \text{for } 2 \leq h \leq q,
\]

\[
b^{q+1}_{q+1 q+1} = N[1 + x_1^2 + x_2^2 = 0].
\]
Proof. For \(2 \leq h \leq i \leq j \leq q\), and for a given \(z \in C_h\), \(v_h^{ij}\) is equal to the number of all ordered pairs \((x, y) \in C_i \times C_j\) such that \(xy = z\). So setting \(z = \begin{pmatrix} 0 & 1 \\ -1 & \nu^{h-1} \end{pmatrix}\), \(x = \begin{pmatrix} x_1 & x_2 \\ x_4 & x_3 \end{pmatrix}\), and \(y = \begin{pmatrix} y_1 & y_2 \\ y_4 & y_3 \end{pmatrix}\), it suffices to count the number of solutions of the system of equations obtained from

\[
\begin{pmatrix} x_1 & x_2 \\ x_4 & x_3 \end{pmatrix} \begin{pmatrix} y_1 & y_2 \\ y_4 & y_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & \nu^{h-1} \end{pmatrix}
\]

under the additional conditions

\[x_1 + x_3 = \nu^{i-1},\quad x_1x_3 - x_2x_4 = 1;\]
\[y_1 + y_3 = \nu^{j-1},\quad y_1y_3 - y_2y_4 = 1,
\]
in the underlying field \(F\).

By a straightforward computation, we have

\[x_3 = \nu^{i-1} - x_1,\quad x_4 = x_2^{-1}(\nu^{i-1}x_1 - x_1^2 - 1);\]
\[y_1 = x_2,\quad y_2 = -\nu^{h-1}x_2 - x_1 + \nu^{i-1},\]
\[y_3 = \nu^{h-1}x_1 - x_2^{-1}(\nu^{i-1}x_1 - x_1^2 - 1),\quad y_4 = -x_1;
\]
\[1 + x_1^2 + x_2^2 + \nu^{h-1}x_1x_2 - \nu^{i-1}x_1 - \nu^{j-1}x_2 = 0.
\]

Hence
\[ b_{ij}^h = | \{ (x_1, x_2) \in V_2(F) \mid 1 + x_1^2 + x_2^2 + \nu^{h-1}x_1x_2 - \nu^{i-1}x_1 - \nu^{j-1}x_2 = 0 \} | \]

\[ = N[1 + x_1^2 + x_2^2 + \nu^{h-1}x_1x_2 - \nu^{i-1}x_1 - \nu^{j-1}x_2 = 0] \]

for \( 2 \leq h \leq i \leq j \leq q \) except for the special cases. If \( h = i = j = i_0 \), then the equation

\[ 1 + x_1^2 + x_2^2 + \nu^{h-1}x_1x_2 - \nu^{i-1}x_1 - \nu^{j-1}x_2 = 0 \]

should be written as

\[ (x_1 + x_2 - 1)^2 = 0, \]

and thus

\[ b_{i_0,i_0}^{i_0} = N[x_1 + x_2 - 1 = 0] - 2 \]

due to the fact that \( N[x_1 + x_2 - 1 = 0] \) counts two solutions \( \left( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) \)

and \( \left( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \) which do not belong to \( C_{i_0} \times C_{i_0} \).

The remaining cases are similarly checked. This completes the proof. \( \square \)

Let \( \overline{X}'(SL(2, q)) = (SL(2, q), \{ R_i' \}_{0 \leq i \leq q+1}) \) be the symmetric association scheme defined by

\[ (x, y) \in R_i' \iff yx^{-1} \in C_i' , \]
where

\[ C_0' = [1], \quad C_1' = [-1], \quad C_2' = [u] \cup [u'], \quad C_3' = [-u] \cup [-u'], \]

\[ C_4' = [v], \quad C_5' = [v^2], \ldots \]

\[ C_{e+1}' = [v^e], \quad C_{e+4}' = [\omega], \ldots \]

where \( e = \frac{q-1}{2}, \quad d = \frac{q-3}{2}. \)

It is easy to see, by using the group character table of \( SL(2, q) \), that the character table of the association scheme \( X'(SL(2, q)) \), \( q \) an odd, is given by Table 2, where

\[ \varepsilon = (-1)^{\frac{q-1}{2}}, \quad e = \frac{q-1}{2}, \quad d = \frac{q-3}{2}, \quad \sigma \in \mathbb{C} \text{ is a primitive } (q+1) \text{ th root of 1 and } \rho \in \mathbb{C} \text{ a primitive } (q-1) \text{ th root of 1.} \]

Note that the character table of the association scheme \( \overline{X}(SL(2, q)) \) is equivalent to the character table \( P \) of \( \overline{X}'(SL(2, q)) \) up to permutations of its columns and rows.

Now we will investigate the structure and the character tables of the association schemes coming from the finite groups \( PSL(2, q) \).

Let \( \overline{z} \) denote the element of \( G = PSL(2, q) \) viewing as the image of the element \( x \in SL(2, q) \) under the natural epimorphism from \( SL(2, q) \) to \( G \). (\( G \) is the factor group of \( SL(2, q) \) modulo the center \( Z = \{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \} \) of \( SL(2, q) \).) Let \( u, u', w, v \) be the elements of \( SL(2, q) \) defined as before. Then \( G \) has exactly

\[ \frac{q-1}{2} + 3 \]}

conjugacy classes

\[ [1], [\overline{u}], [\overline{u'}], [\overline{v}], [\overline{v^2}], \ldots, [\overline{w}], [\overline{w^2}], \ldots, [\overline{w^b}], \]

satisfying that
(i) if \( q \equiv 1 \pmod{4} \), then \( a = b = \frac{q-1}{4} \) and

\[
\begin{array}{ccccccc}
| \bar{x} | & 1 & \frac{1}{2}(q^2 - 1) & \frac{1}{2}(q^2 - 1) & q(q - 1) & q(q + 1) & \frac{1}{2}q(q + 1) \\
\end{array}
\]

for \( 1 \leq \ell \leq a = \frac{q-1}{4}, 1 \leq m < b = \frac{q-1}{4} \);

(ii) if \( q \equiv -1 \pmod{4} \), then \( a = \frac{q+1}{4} \) and \( b = \frac{q-3}{4} \), and

\[
\begin{array}{ccccccc}
| \bar{x} | & 1 & \frac{1}{2}(q^2 - 1) & \frac{1}{2}(q^2 - 1) & q(q - 1) & \frac{1}{2}q(q - 1) & q(q + 1) \\
\end{array}
\]

for \( 1 \leq \ell < a = \frac{q+1}{4}, 1 \leq m \leq b = \frac{q-3}{4} \).

Define

\[
\bar{C}_0 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\},
\]

\[
\bar{C}_i = \left\{ \begin{pmatrix} a & c \\ d & b \end{pmatrix} \in G - \bar{C}_0 \mid a + b = \nu^i \right\}, \quad \text{for } 1 \leq i \leq \frac{q-1}{2},
\]

\[
\bar{C}_{\frac{q+1}{2}} = \left\{ \begin{pmatrix} a & c \\ d & b \end{pmatrix} \in G \mid a + b = 0 \right\}.
\]

There is an index \( i_0 \ (1 \leq i_0 \leq \frac{q-1}{2}) \) such that \( \bar{C}_{i_0} = [\bar{u}] \cup [\bar{v}] \). Then the association scheme \( \bar{\mathcal{X}}(PSL(2, q)) = (PSL(2, q), \{R_i\}_{0 \leq i \leq \frac{q+1}{2}} \) is defined by

\[
(\bar{x}, \bar{y}) \in R_i \iff yx^{-1} \in \bar{C}_i \quad (0 \leq i \leq \frac{q+1}{2}),
\]
is a symmetric association scheme of class $\frac{q+1}{2}$. The valencies are given by

$$k_i = \begin{cases} 
q(q - 1) & \text{if } N[t^2 - \nu^i t + 1 = 0] = 0, \\
(q - 1)(q + 1) & \text{if } N[t^2 - \nu^i t + 1 = 0] = 1 \ (\text{i.e., if } i = i_0), \\
q(q + 1) & \text{if } N[t^2 - \nu^i t + 1 = 0] = 2,
\end{cases}$$

for $i = 1, 2, \ldots, \frac{q-1}{2}$,

$$k_{\frac{q+1}{2}} = \begin{cases} \frac{1}{2}q(q + 1) & \text{if } q \equiv 1 \ (mod \ 4), \\
\frac{1}{2}q(q - 1) & \text{if } q \equiv -1 \ (mod \ 4). 
\end{cases}$$

Notice that there are $\frac{q-1}{4}$ (resp. $\frac{q-5}{4}$) $i$'s for which $k_i = q(q - 1)$ (resp. $q(q + 1)$) if $q \equiv 1 \ (mod \ 4)$, while there are $\frac{q-3}{4}$ (resp. $\frac{q-3}{4}$) $i$'s for which $k_i = q(q - 1)$ (resp. $q(q + 1)$) if $q \equiv -1 \ (mod \ 4)$.

**Lemma 1.2.2** Let $d_{ij}^h$ denote the parameters of the association scheme $\Xi(PSL(2, q))$. Then

$$d_{ij}^h = N[1 + x_1^2 + x_2^2 + \nu^h x_1 x_2 - \nu^i x_1 \mp \nu^j x_2 = 0]$$

for $1 \leq h \leq i \leq j \leq \frac{q-1}{2}$, except the following cases:

$$d_{i_0i_0}^0 = N[(x - 1)^2 + 2(x_1 \mp 1)x_2 + x_2^2 = 0] - 2,$$

$$d_{i_i}^0 = N[(x_1 + x_2)^2 - \nu^i (x_1 \pm x_2) + 1 = 0], \text{ for } i_0 < i \leq \frac{q-1}{2},$$

$$d_{ij}^{i_0} = N[(x_1 + x_2)^2 - \nu^i x_1 \mp \nu^j x_2 + 1 = 0], \text{ for } i_0 < i < j \leq \frac{q-1}{2},$$

$$d_{i_0h}^h = N[1 + x_1^2 + x_2^2 - 2x_1 \mp \nu^h x_2 + \nu^h x_1 x_2 = 0] - 1, \text{ for } 1 \leq h \leq \frac{q-1}{2},$$
$h \neq i_0$, where $N[1 + x_1^2 + x_2^2 + \nu^h x_1 x_2 - \nu^i x_1 \mp \nu^j x_2 = 0]$ denotes the number of solutions $(x_1, x_2) \in F^2$ such that $1 + x_1^2 + x_2^2 + \nu^h x_1 x_2 - \nu^i x_1 \mp \nu^j x_2 = 0$.

**Proof** Similar to the proof of Lemma 1.2.1, so omitted. \(\square\)

It is possible to write all those parameters in terms of $q$, for instance,

\[
\begin{align*}
d_{i_0}^{d_0} &= 2q - 2, \\
d_{i_4}^{d_4} &= q, \text{ or } 3q \text{ depending on } N[1 - \nu^i t + t^2 = 0] = 0 \text{ or } 2, \\
d_{i_4}^{d_4} &= 2q, \text{ for } i_0 < i < j \leq \frac{q - 1}{2}, \\
d_{i_0}^{h} &= q + 1, \text{ or } 3q - 3 \text{ depends on } N[t^2 - \nu^h t + 1 = 0] = 0 \text{ or } 2 \text{ (} h \neq i_0). \\
\end{align*}
\]

As before, $X'(PSL(2, q))$ denotes the association scheme defined by $(\bar{x}, \bar{y}) \in R'_i$ if and only if $\bar{y}x^{-1} \in C'_i$, for $i = 0, 1, \ldots, \frac{q + 1}{2}$, where $C'_0 = [\overline{1}], C'_1 = [\overline{u}] \cup [\overline{u}]$, $C'_2 = [\overline{v}], C'_3 = [\overline{v^2}], \ldots, C'_{a+1} = [\overline{u^a}], C'_{a+2} = [\overline{u}], \ldots, C'_{2^{a+1}} = [\overline{u^b}]$. Then the character table ($P$-matrix) of $X'(PSL(2, q))$ is given as follows:

(i) If $q \equiv 1 \pmod{4}$, then see Table 3, where $a = \frac{q - 1}{4}$.

(ii) If $q \equiv -1 \pmod{4}$, then see Table 4, where $b = \frac{q - 3}{4}$.

Note that the character table of $X(PSL(2, q))$ is equivalent to the character table of $X'(PSL(2, q))$ up to a suitable permutation of its columns and rows.

**1.3. Two enumerative lemmas**

As a preparation for the later use, two enumeration lemmas for the numbers of solutions of some quadratic equations over a finite field together with some notation are introduced in this section.
Let $F$ be the finite field $GF(q)$ and $V_n(F)$ be an $n$-dimensional vector space over $F$. (It is often denoted by $V$ or $F^n$, too.) For a polynomial $f(x_1, x_2, \ldots, x_n) \in F[x_1, x_2, \ldots, x_n]$, and $b \in F$, let $N[f(x_1, x_2, \ldots, x_n) = b]$ denote the number of solutions of the equations $f(x_1, x_2, \ldots, x_n) = b$ on $V_n(F)$. In particular, denote

$$\lambda_b(2m) = N[x_1 x_{m+1} + x_2 x_{m+2} + \cdots + x_m x_{2m} = b].$$

**Lemma 1.3.1**

$$\lambda_0(2m) = q^m + q^{m-1}(q^m - 1),$$
$$\lambda_b(2m) = q^{m-1}(q^m - 1), \quad \text{for } b \in F^*. $$

**Proof** Straightforward. $\square$

Let $F$ be a field of an odd characteristic, and let $\alpha$ be a non-square element in $F^*$. Denote

$$\mu_b(2m) = N[2(x_1 x_m + x_2 x_{m+1} + \cdots + x_{m-1} x_{2(m-1)}) + x_{2m-1}^2 - \alpha x_{2m}^2 = b].$$

**Lemma 1.3.2**

$$\mu_0(2m) = -q^m + q^{m-1}(q^m + 1)$$
$$\mu_b(2m) = q^{m-1}(q^m + 1) \quad \text{for } b \in F^*.$$

**Comment** In the sequel, $N[\cdots]$ will be often used to denote the number of solutions of the equation between brackets in the underlying finite field, considering only the indeterminates actually written down. For instance, $N[a_1 x_2^2 + a_2 x_4^2 = b]$ refers to the number of solutions of the indicated equation in $F^2 (= V_2(F))$.
Proof of Lemma 1.3.2  Since

\[ N[x_{2m-1}^2 - \alpha x_{2m}^2 = 0] = 1, \]

and for \( b \neq 0 \)

\[ N[x_{2m-1}^2 - \alpha x_{2m}^2 = b] = q + 1, \]

so

\[
\mu_0(2m) = N[x_1x_m + x_2x_{m+1} + \cdots + x_{m-1}x_{2(m-1)} = -2^{-1}(x_{2m-1}^2 - \alpha x_{2m}^2)] \\
= \lambda_0(2(m - 1)) + (q^2 - 1) \cdot \lambda_1(2(m - 1)) \\
= -q^m + q^{m-1}(q^m + 1),
\]

and for \( b \neq 0 \)

\[
\mu_b(2m) = N[x_1x_m + x_2x_{m+1} + \cdots + x_{m-1}x_{2(m-1)} = -2^{-1}(b - (x_{2m-1}^2 - \alpha x_{2m}^2))] \\
= (q + 1)\lambda_0(2(m - 1)) + (q^2 - q - 1) \cdot \lambda_1(2(m - 1)) \\
= q^{m-1}(q^m + 1). \]
CHAPTER II

CHARACTER TABLES OF PAIGE'S SIMPLE MOUFANG LOOPS

In this chapter we determine the conjugacy classes and the character tables of Paige's Moufang loops $M(q)$ and $M^*(q)$. If we were interested only in the determination of the character table of the attached association scheme, this case is nothing but a special case of more general results which will be obtained in the next chapter. However, the existence of loop structure in this case makes the situation more directly related to the groups $SL(2, q)$ and $PSL(2, q)$ than other general cases. It also raises the question whether there exists any loop (or quasi-group) structure in other general permutation groups considered in the next chapter.

2.0. Paige's Moufang loops.

A quasigroup $Q$ is a system consisting of a nonempty set $Q$ and a binary operation on $Q$ such that, in the equation $xy = z$, knowledge of two of $x, y, z$ specifies the third uniquely in $Q$. A loop is a quasigroup with an identity $1$.

An element $e$ of a loop $Q$ is in the center $Z(Q)$ of $Q$ if and only if $ex = xe$, $c(xy) = (ex)y$, $(xe)y = x(ey)$, and $(xy)e = x(ye)$ for all $x, y \in Q$. A subloop $N$ of $L$ which satisfies $xN = Nx$, $(Nx)y = N(xy)$, and $y(xN) = (xy)N$ for all $x, y \in Q$ is called a normal subloop of $L$. 

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The permutations $L(x)$ and $R(x)$ for any loop $Q$ are defined by the equations

$$aL(x) = xa; \quad aR(x) = ax,$$

for all $a \in Q$. Let $Gr(Q)$ be the permutation group on $Q$ generated by $L(x)$ and $R(x)$ for all $x \in Q$. Let $I^*$ be the set of all elements $U$ in $Gr(Q)$ such that $1U = 1$. Then $I^*$ is the group generated by $T(x) = R(x)L(x)^{-1}$, $R(x, y) = R(x)R(y)R(xy)^{-1}$, $L(x, y) = L(x)L(y)L(yx)^{-1}$ for all $x, y \in Q$. Two elements $x$ and $y$ are conjugate (i.e., $x \sim y$) if there exists an element $U \in I^*$ such that $xU = y$. The relation $\sim$ is obviously an equivalent relation. For a given loop $Q$, $N$ is a normal subloop of $Q$ if and only if $NI^* \subseteq N$. It is also clear that if an element $x$ is contained in a normal subloop $N$, all conjugates of $x$ are in $N$. Note that $x \sim y$ if and only if $(1, x)$ and $(1, y)$ are in the same orbit of $Gr(Q)$ on $Q \times Q$.

A Moufang loop $M$ is a loop that satisfies one of the equivalent three Moufang identities $(xy)(zx) = x[(yz)x]$, $x[y(xx)] = [(xy)x]z$, and $[(xx)y]x = z[x(yx)]$, for all $x, y, z \in M$. In any Moufang loop, thus $xx^{-1} = x^{-1}x = 1$, $(xy)^{-1} = y^{-1}x^{-1}$, $L(x)^{-1} = L(x^{-1})$, and $R(x)^{-1} = R(x^{-1})$.

In the rest of this section, we will describe the infinite class of finite simple nonassociative Moufang loops constructed by Paige ([13]).

Let $F'$ be the finite field $GF(q)$ and let $R$ be the set of all matrices $\begin{pmatrix} a & \alpha \\ \beta & b \end{pmatrix}$ where $a, b \in F$ and $\alpha$ and $\beta$ are 3-dimensional coordinate vectors in $F^3$. An alternative ring $R$ (i.e., $x(xy) = (xx)y$, $(yx)x = y(xx)$ for all $x, y \in R$) is constructed from these matrices by first defining equality and addition to be ordinary matrix equality and addition with vector equality and addition for the vector elements of the matrices. For
\( \alpha, \beta \in F^3 \), denote \( \alpha \circ \beta \) and \( \alpha \times \beta \) the scalar and vector products. Now defining multiplication in \( R \) by

\[
\begin{pmatrix} a & \alpha \\ \beta & b \end{pmatrix} \begin{pmatrix} c & \gamma \\ \delta & d \end{pmatrix} = \begin{pmatrix} ac + \alpha \circ \delta & a\gamma + d\alpha - \beta \times \delta \\ c\beta + b\delta + \alpha \times \gamma & bd + \beta \circ \gamma \end{pmatrix},
\]

we get the simple alternative ring \( R \). By Paige, it is proved that

\[
L = \left\{ \begin{pmatrix} a & \alpha \\ \beta & b \end{pmatrix} \in R \mid ab - \alpha \circ \beta \neq 0 \right\}
\]

is a Moufang loop, and that

\[
M = \left\{ \begin{pmatrix} a & \alpha \\ \beta & b \end{pmatrix} \in R \mid ab - \alpha \circ \beta = 1 \right\}
\]

is a normal Moufang subloop of \( L \). That is, for a given \( q = p^n, p \) a prime, there is a Moufang loop \( M = M(q) \) of size \(| M(q) | = q^{3(q^4 - 1)} \). In this dissertation this Moufang loop is called Paige's Moufang loop. The following properties are easily proved.

1. \( M(q) \) is not a group.

2. The center \( Z \) of \( M(q) \) is the group \( \{ (1, 0), (-1, 0) \} \) of order 2 or the trivial group \( \{ (1, 0) \} \) depends on the parity of \( q \).

3. \( M^*(q) = M(q)/Z \) is a simple Moufang loop of size \( q^{3(q^4 - 1)/(q - 1, 2)} \).

2.1. Conjugacy classes of Paige's Moufang loops.

In this section we will determine the conjugacy classes of the Paige's Moufang loops \( M(q) \) and \( M^*(q) \).
Theorem 2.1.1. Let \( q = p^n \) with \( p \) an odd prime. Then the conjugacy classes of Paige's Moufang loops \( M = M(q) \) and \( M^* = M^*(q) \) are described as follows. Here, let us denote

\[
1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad -1 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad u = \begin{pmatrix} 1 & 0 \\ e_1 & 1 \end{pmatrix}, \quad w = \begin{pmatrix} \nu & 0 \\ 0 & \nu^{-1} \end{pmatrix}
\]

where \( \nu \) is a generator of the cyclic group \( F^* = F \setminus \{ 0 \} \) and \( e_1 = (1,0,0) \), \( 0 = (0,0,0) \) \( \in F^3 \); and let \( v \) be an element of order \( q + 1 \) in \( M \) (such element is shown to exist).

(1) \( M \) has \( q + 2 \) conjugacy classes \([1], [-1], [u], [-u], [v], [v^2], \ldots, [v^c], [w], [w^2], \ldots, [w^d]\), satisfying

\[
\begin{array}{c|c|c|c|c|c}
[\mathbb{Z}] & 1 & -1 & u & -u & v^\ell \\
\hline
| [\mathbb{Z}] | & 1 & 1 & (q^3 + 1)(q^3 - 1) & (q^3 + 1)(q^3 - 1) & q^3(q^3 - 1) \\
\end{array}
\]

for \( 1 \leq \ell \leq c = \frac{q-1}{2} \), \( 1 \leq m \leq d = \frac{q^3 - 1}{2} \).

(2) \( M^* \) has \( \frac{q-1}{2} + 2 \) conjugacy classes \([\overline{1}], [\overline{u}], [\overline{v}], [\overline{v^2}], \ldots, [\overline{v^d}], [\overline{w}], [\overline{w^2}], \ldots, [\overline{w^b}]\), satisfying that

(i) if \( q \equiv 1 \pmod{4} \), then \( a = b = \frac{q-1}{4} \) and

\[
\begin{array}{c|c|c|c|c|c}
[\mathbb{Z}] & \overline{1} & \overline{u} & \overline{v^\ell} & \overline{w^b} & \overline{w^m} \\
\hline
| [\mathbb{Z}] | & 1 & (q^3 + 1)(q^3 - 1) & q^3(q^3 - 1) & \frac{1}{2}q^3(q^3 + 1) & q^3(q^3 + 1) \\
\end{array}
\]

for \( 1 \leq \ell \leq a = \frac{q-1}{4} \), \( 1 \leq m < b = \frac{q-1}{4} \),
(ii) if $q \equiv -1 \pmod{4}$, then $a = \frac{q+1}{4}$ and $b = \frac{q-3}{4}$, and

<table>
<thead>
<tr>
<th>$\bar{x}$</th>
<th>$\bar{u}$</th>
<th>$\bar{v}^\ell$</th>
<th>$\bar{v}^a$</th>
<th>$\bar{v}^m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td>{ \bar{x} }</td>
<td>$</td>
<td>$1$</td>
<td>$(q^3 + 1)(q^3 - 1)$</td>
</tr>
</tbody>
</table>

for $1 \leq \ell < a = \frac{q+1}{4}$, $1 \leq m \leq b = \frac{q-3}{4}$.

**Proof** The proof is given in the following series of steps.

**Step 1.** $z \sim z^{-1}$ for all $z \in M$.

For given $z = \left( \begin{array}{c} a \\ \beta \\ b \end{array} \right) \in M$, choosing $x = \left( \begin{array}{c} 0 \\ \delta \\ 0 \end{array} \right) \in M$ such that $\alpha \circ \delta + \beta \circ \gamma = 0$, we get $z \cdot T(x) = \left( \begin{array}{c} b \\ -\beta \\ a \end{array} \right) = z^{-1}$, by applying the equality $\delta \times (\alpha \times \gamma) = (\delta \circ \gamma)\alpha - (\alpha \circ \delta)\gamma$.

Let $C_{ab}$ denote the set of all elements whose diagonal is $[a, b]$.

**Step 2.** For each $a \in F^*$, $x \sim y$ for any two elements $x$ and $y$ in $C_{aa^{-1}} = \left\{ \left( \begin{array}{c} a \\ \beta \\ a^{-1} \end{array} \right) \in M - \{1,-1\} \right\}$.

For given $y = \left( \begin{array}{c} a \\ 0 \\ a^{-1} \end{array} \right)$, $z = \left( \begin{array}{c} a \\ 0 \\ a^{-1} \end{array} \right)$ in $C_{aa^{-1}}$, assuming $\alpha$ and $\alpha_1$ are linearly independent, choose $\beta_1$ such that $\alpha \circ \beta_1 = \alpha_1 \circ \beta_1 = -1$. Then $y \cdot T \left( \left( \begin{array}{c} 0 \\ -\beta_1 \\ 0 \end{array} \right) \right) = \left( \begin{array}{c} a^{-1} \\ \beta_1 \\ a \end{array} \right)$ and $\left( \begin{array}{c} a^{-1} \\ \beta_1 \\ a \end{array} \right) \cdot T \left( \left( \begin{array}{c} 0 \\ -\beta_1 \\ 0 \end{array} \right) \right) = z$ so that $y \sim z$. If $\alpha$ and $\alpha_1$ are linearly dependent, then using a third vector $\alpha_2$ which is linearly independent to $\alpha$ we get $y \sim \left( \begin{array}{c} a \\ 0 \\ a^{-1} \end{array} \right) \sim z$. Similarly, we have $\left( \begin{array}{c} a \\ 0 \\ a^{-1} \end{array} \right) \sim \left( \begin{array}{c} a \\ 0 \\ a^{-1} \end{array} \right)$. Hence, in particular, we are done for the case when $a = \pm 1$. For $a \neq \pm 1$, in addition, the equalities $\left( \begin{array}{c} a \\ 0 \\ a^{-1} \end{array} \right) \cdot T \left( \left( \begin{array}{c} 1 \\ 0 \\ -\beta_1 \end{array} \right) \right) = \left( \begin{array}{c} a \alpha \\ (a - a^{-1})\beta \\ a^{-1} \end{array} \right)$ and $\left( \begin{array}{c} a \\ \alpha \\ a^{-1} \end{array} \right) \cdot T \left( \left( \begin{array}{c} 0 \\ -\beta_1 \\ 0 \end{array} \right) \right) = \left( \begin{array}{c} a^{-1} \\ 0 \\ a \end{array} \right)$ complete the proof of Step 2.
Step 3. For each \( b \in F \), \( x \sim y \) for any two elements \( x \) and \( y \) in \( C_{ab} = \left\{ \left( \begin{array}{c} 0 \\ \alpha \\ \beta \\ b \end{array} \right) \in M \right\} \).

Suppose \( \alpha \) and \( \gamma \) are linearly independent for given \( y = \left( \begin{array}{c} 0 \\ \alpha \\ \beta \\ b \end{array} \right) \), \( z = \left( \begin{array}{c} 0 \\ \gamma \\ \delta \\ b \end{array} \right) \) in \( C_{ab} \). Then choosing \( \varepsilon \) such that \( \alpha \circ \varepsilon = \varepsilon \circ \gamma = -1 \) and setting \( \beta_1 = \frac{1}{2}(\beta + \varepsilon) \), \( \alpha_1 = \frac{1}{2}(\alpha + \gamma) \), and \( \varepsilon_1 = \frac{1}{2}(\varepsilon + \delta) \), we have \( y \cdot T\left( \left( \begin{array}{c} 0 \\ -\beta_1 \\ 0 \\ 0 \end{array} \right) \right) = \left( \begin{array}{c} b \\ \alpha \\ \beta \\ b \end{array} \right) \), \( \left( \begin{array}{c} b \\ \alpha \\ \beta \\ b \end{array} \right) \cdot T\left( \left( \begin{array}{c} 0 \\ -\alpha_1 \\ 0 \\ 0 \end{array} \right) \right) = \left( \begin{array}{c} 0 \\ \gamma \\ \delta \\ b \end{array} \right) \), and \( \left( \begin{array}{c} b \\ \gamma \\ \delta \\ 0 \end{array} \right) \cdot T\left( \left( \begin{array}{c} 0 \\ -\gamma \\ 0 \\ 0 \end{array} \right) \right) = z \), so that \( y \sim z \). Suppose \( \alpha \) and \( \gamma \) are linearly dependent, then we can use a third vector \( \varphi \) which is linearly independent to \( \alpha \) to show \( y \sim z \).

Step 4. \( x \sim y \) for any two elements \( x \) and \( y \) in \( C_{ab} = \left\{ \left( \begin{array}{c} a \\ \alpha \\ \beta \\ b \end{array} \right) \in M \right\} \) if \( ab \neq 1 \) and \( ab \neq 0 \).

For given \( y = \left( \begin{array}{c} a \\ \alpha \\ \beta \\ b \end{array} \right) \), \( z = \left( \begin{array}{c} a \\ \gamma \\ \delta \\ b \end{array} \right) \) in \( C_{ab} \), without loss of generality, assuming \( \alpha \) and \( \gamma \) are linearly independent, choose \( \varepsilon \) such that \( \alpha \circ \varepsilon = \gamma \circ \varepsilon = ab - 1 \). If \( a \neq \pm 1 \), then setting \( x_1 = \left( \begin{array}{c} a^{-1} \\ 0 \\ 0 \\ a \end{array} \right) \), \( x_2 = \left( \begin{array}{c} 0 \\ \alpha_1 \\ \beta_1 \\ 0 \end{array} \right) \), \( x_3 = \left( \begin{array}{c} 0 \\ \alpha_1 \\ \beta_1 \\ 0 \end{array} \right) \), and \( x_4 = \left( \begin{array}{c} 0 \\ \varepsilon_1 \\ 0 \\ 0 \end{array} \right) \) where \( \beta_1 = (\varepsilon - a^{-2}\beta)(a^{-1} - 1)^{-1} \), \( \alpha_1 = (\gamma - a^2\alpha)(ab - 1)^{-1} \), \( \varepsilon_1 = (\delta - a^{-2}\varepsilon)(a - 1)^{-1} \), and \( \alpha_1 = (\gamma - a^2\alpha)(ab - 1)^{-1} \), we get \( y \cdot R(x_1, x_2) = \left( \begin{array}{c} a \\ \alpha \\ \beta \\ b \end{array} \right) \), \( \left( \begin{array}{c} a \\ \alpha \\ \beta \\ b \end{array} \right) \cdot R(x_1, x_3) = \left( \begin{array}{c} a \\ \gamma \\ \delta \\ b \end{array} \right) \), and \( \left( \begin{array}{c} a \\ \gamma \\ \delta \\ b \end{array} \right) \cdot R(x_1, x_4) = z \). Suppose \( a = \pm 1 \) but \( b \neq \pm 1 \), then interchanging the roles played by \( a \) and \( b \), \( \alpha \) and \( \beta \), we have again \( y \sim z \). Now, if \( a = -b = 1 \), then \( \left( \begin{array}{c} 1 \\ -1 \\ a \\ -b \end{array} \right) \cdot T\left( \left( \begin{array}{c} 1 \\ -\beta \\ 0 \end{array} \right) \right) = \left( \begin{array}{c} 3 \\ \alpha \\ 5\beta \\ -3 \end{array} \right) \) and \( \left( \begin{array}{c} 1 \\ \gamma \\ \delta \\ 1 \end{array} \right) \cdot T\left( \left( \begin{array}{c} 1 \\ -\gamma \\ 0 \end{array} \right) \right) = \left( \begin{array}{c} 3 \\ \gamma \\ 5\delta \\ -3 \end{array} \right) \) reveals that \( y \sim z \) again by the above argument except the cases when the characteristic of \( F \) is 3 or 5, which cases are already done by the Steps 3 and 2, respectively.
Step 5. For each \( t \in F \), \( S_{2t} = \left\{ \begin{pmatrix} a & \alpha \\ \beta & b \end{pmatrix} \in M - \{1,-1\} \mid a + b = 2t \right\} \) forms a conjugacy class in \( M \).

For any \( \left( \begin{array}{c} t \\ \beta \\ \alpha \\ t \end{array} \right) \in C_{tt} \), and for any \( a \in F \), since we can choose \( \beta_1 \) such that \( \alpha \circ \beta_1 = a \) and \( \left( \begin{array}{c} t \\ \beta \\ \alpha \\ t \end{array} \right) \cdot T \left( \begin{pmatrix} 1 & 0 \\ -\beta_1 & 1 \end{pmatrix} \right) = \left( \begin{array}{c} t - \alpha \circ \beta_1 \\ \beta - (\alpha \circ \beta_1)\beta_1 \\ t + \alpha \circ \beta_1 \end{array} \right) \), so any two elements in
\( S_{2t} = \bigcup_{a \in F} C_{t-a,t+a} \) are conjugate each other.

By a straightforward computation we can check that for any \( z = \left( \begin{array}{c} t - a \\ \beta \\ \alpha \\ t + a \end{array} \right) \in S_{2t} \), the traces of \( z \cdot T(x) \), \( z \cdot R(x,y) \), and \( z \cdot L(x,y) \) are always \( 2t \) for all \( x,y \) in \( M \).

Consequently, \( M \) has \( q+2 \) conjugacy classes \( \{1\}, \{-1\}, \) and \( S_{2t} \) for all \( t \in F \).

Step 6. (1) The conjugacy classes \( [w], [w^2], \ldots, [w^{q^2-3}] \) are all distinct. (2) \( M \) has an element \( v \) of order \( q + 1 \) and the conjugacy classes \( [v], [v^2], \ldots, [v^{q+1}] \), are all distinct. (3) Therefore, the \( q + 2 \) conjugacy classes are also described as \( [1], [-1], [u], [-u], [v], [v^2], \ldots, [v^{q+1}], [w], [w^2], \ldots, [w^{q^2-3}] \).

For (1), we see that \( w^k = \left( \begin{array}{cc} \nu^k & 0 \\ 0 & \nu^{-k} \end{array} \right) \) belongs to the conjugacy class \( S_{2t_k} \) with \( 2t_k = \nu^k + \nu^{-k} \) for \( k = 1, 2, \ldots, \frac{q^2-3}{2} \). Since \( t_i = t_j \) implies that \( \nu^{i+j} \equiv 1 \pmod{q} \), which is impossible for \( i, j \) (\( i \neq j \)) in \( \{1, 2, \ldots, \frac{q^2-3}{2} \} \), so part (1) of Step 6 follows.

For (2), let \( \left( \begin{array}{ccc} a & 0 & c \\ d & 0 & 0 \\ b & 0 & 0 \end{array} \right) \) be an element of order \( q + 1 \) in \( SL(2,q) \). Then \( v' = \left( \begin{array}{ccc} a & 0 & c \\ d & 0 & 0 \\ b & 0 & 0 \end{array} \right) \) is an element of order \( q + 1 \) in \( M(q) \). Furthermore, \( v' \) belongs to the conjugacy class \( S_{2t} \) with \( 2t = a + b \). Thus, in particular, choosing \( v = \left( \begin{array}{c} t \\ \beta \\ \alpha \\ t \end{array} \right) \in S_{2t} \), we can easily check that the traces of \( v, v^2, \ldots, v^{q+1} \) are all distinct. As an immediate consequence, we have part (3) of Step 6.

The size of each conjugacy class follows from the fact that \( |C_{ab}| = q^2(q^3 - 1) \) if \( ab \neq 1 \), \( |C_{aa^{-1}}| = (q^3 - 1)(q^2 + 1) + 1 \), \( |C_{11}| = |C_{-1,-1}| = (q^3 - 1)(q^2 + 1) \),
and further that \(|u| = |S_2| = |C_{11}| \) \(+ (q - 1)|C_{ab}| \), \(|v^t| = q \cdot |C_{ab}|\), and 
\(|w^m| = |C_{\nu m, \nu^{-m}}| \) \(+ |C_{\nu^{-m}, \nu^m}| \) \(+ (q - 2)|C_{ab}|\).

This completes the proof of part (1). It is now easy to see the part (2). ☐

**Theorem 2.1.2** Let \(q = 2^n\). Then the conjugacy classes of Paige's simple Moufang loop \(M^* (= M)\) are described as follows. Here we denote \(1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\), \(u = \begin{pmatrix} 1 & 0 \\ e_1 & 1 \end{pmatrix}\), \(w = \begin{pmatrix} \nu & 0 \\ 0 & \nu^{-1} \end{pmatrix}\) where \(\nu\) is a generator of the cyclic group \(F^* = F \setminus \{0\}\) and \(e_1 = (1, 0, 0), 0 = (0, 0, 0) \in F^3\). Let \(v\) be an element of order \(q + 1\) in \(M\) (such element is shown to exist).

\(M^* (= M)\) has exactly \(q + 1\) conjugacy classes \([1], [u], [v], [v^2], \ldots, [v^c], [w], [w^2], \ldots, [w^d]\) satisfying

<table>
<thead>
<tr>
<th>(x)</th>
<th>1</th>
<th>(u)</th>
<th>(v^t)</th>
<th>(w^m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(</td>
<td>x</td>
<td>) = 1</td>
<td>((q^3 + 1)(q^3 - 1))</td>
<td>(q^3(q^3 - 1))</td>
</tr>
</tbody>
</table>

for \(1 \leq \ell \leq c = \frac{q}{2}, 1 \leq m \leq d = \frac{q - 2}{2}\).

**Proof** The proof is given in three steps.

**Step 1.** For each \(t \in F^*\), \(x \sim y\) for every \(x, y\) in \(S_t = \left\{ \begin{pmatrix} a & \alpha \\ \beta & b \end{pmatrix} \in M \mid a + b = t \right\}\).

Suppose we set \(\beta_1, \alpha_1, \text{ and } \delta_1\) such that \(\beta_1 = t^{-1}(\beta + \epsilon)\), \(\alpha_1 = t^{-1}(\alpha + \gamma)\), and \(\delta_1 = t^{-1}(\delta + \epsilon)\), then
\[
\begin{align*}
(0 \quad \alpha) \cdot T\left(\begin{pmatrix} 1 \\ \beta_1 \\ 0 \end{pmatrix}\right) &= \begin{pmatrix} 0 \\ \alpha \end{pmatrix}, \\
(0 \quad \alpha) \cdot T\left(\begin{pmatrix} 1 \\ 0 \quad \alpha \end{pmatrix}\right) &= \begin{pmatrix} 0 \\ \gamma \end{pmatrix}, \\
(0 \quad \gamma) \cdot T\left(\begin{pmatrix} 1 \\ \delta_1 \\ 0 \end{pmatrix}\right) &= \begin{pmatrix} 0 \\ \gamma \end{pmatrix}.
\end{align*}
\]

This reveals that \( \begin{pmatrix} 0 \\ \alpha \end{pmatrix} \sim \begin{pmatrix} 0 \\ \gamma \end{pmatrix} \) whenever \( \alpha \) and \( \gamma \) are linearly independent (because then we can choose \( \epsilon \) such that \( \alpha \circ \epsilon = \gamma \circ \epsilon = 1 \)). Moreover, if we set \( \delta \) such that \( \alpha \circ \delta = a \), then

\[
\begin{pmatrix} a \\ \alpha \end{pmatrix} \cdot T\left(\begin{pmatrix} 1 \\ \delta_1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ \alpha \end{pmatrix}.
\]

Therefore, it is easy to see that every pair of elements whose traces are the same, are conjugate each other.

Step 2. \( x \sim y \) for every \( x,y \) in \( S_0 = \left\{ \begin{pmatrix} a \\ \alpha \end{pmatrix} \in M - \{1\} \right\} \).

From the equalities:

\[
\begin{align*}
(1 \quad \alpha) \cdot T\left(\begin{pmatrix} 0 \\ \beta \end{pmatrix}\right) &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\
(1 \quad 0) \cdot T\left(\begin{pmatrix} 0 \\ \beta \end{pmatrix}\right) &= \begin{pmatrix} 1 \\ \gamma \end{pmatrix}, \\
(1 \quad \alpha) \cdot T\left(\begin{pmatrix} 0 \\ \beta_1 \end{pmatrix}\right) &= \begin{pmatrix} 1 \\ \beta + \beta_1 \end{pmatrix}, \\
\begin{pmatrix} a \\ \alpha \end{pmatrix} \cdot T\left(\begin{pmatrix} 1 \\ \beta_1 \end{pmatrix}\right) &= \begin{pmatrix} a + \alpha \circ \beta_1 \\ \alpha \circ \beta_1 \end{pmatrix}.
\end{align*}
\]
we can see that \( x \sim y \) for any \( x, y \in S_0 \).

**Step 3.** For every \( t \in F \), \( S_t \cdot I^* \subseteq S_t \). For any \( Z = \begin{pmatrix} a & \alpha \\ \beta & t + a \end{pmatrix} \in S_t \), by a straightforward computation we can show that the traces of the matrices \( z \cdot T(x) \), \( z \cdot R(x,y) \), and \( z \cdot L(x,y) \) are \( t \). That is, \( S_t \) is a set which is invariant under inner mapping group \( I^* \).

As a consequence, we have exactly \( q + 1 \) conjugacy classes, \([1]\), \([\begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix}]\), \([\begin{pmatrix} 0 & \alpha \\ \beta & \nu \end{pmatrix}]\), \( \cdots \), \([\begin{pmatrix} 0 & \alpha \\ \beta & \nu^{q-2} \end{pmatrix}]\), \([\begin{pmatrix} 0 & \alpha \\ \beta & 1 \end{pmatrix}]\), where \( \alpha = \beta = (1, 0, 0) \in F^3 \).

Now the proof of the theorem is analogous to the proof of part (1) of Theorem 2.1.1. □

### 2.2. The character table of \( M(2^n) \).

In this section we compute the character table of (the association scheme of) \( M = M(2^n) \). Notation is the same as in the previous section (cf. Theorem 2.1.2), but we use the following indexing of the conjugacy classes of \( M(2^n) \):

\[
\begin{align*}
C_0 &= [1], \\
C_1 &= \begin{pmatrix} 0 & e_1 \\ e_1 & 0 \end{pmatrix}, \\
C_i &= \begin{pmatrix} 0 & e_1^{i-1} \\ e_1 & \nu^{i-1} \end{pmatrix} \text{ for } 2 \leq i \leq (q - 1), \\
C_q &= \begin{pmatrix} 0 & e_1 \\ e_1 & 1 \end{pmatrix} 
\end{align*}
\]

Let \( \mathcal{X}(M) = (M, \{R_i\}_{0 \leq i \leq q}) \) be the loop association scheme defined by

\[
(x, y) \in R_i \iff yx^{-1} \in C_i.
\]

Then \( \mathcal{X}(M) \) is a symmetric association scheme of class \( q \).

**Lemma 2.2.1** The valencies of the association scheme \( \mathcal{X}(M(2^n)) \) are given by
\[ k_1 = (q^3 + 1)(q^3 - 1), \]
\[ k_i = \begin{cases} q^3(q^3 - 1) & \text{if } N[t^2 + \nu^{i-1}t + 1 = 0] = 0, \\ q^3(q^3 + 1) & \text{if } N[t^2 + \nu^{i-1}t + 1 = 0] = 2, \end{cases} \]

for \( i = 2, 3, \ldots, q \).

**Proof** Since \( k_i = |C_i| \), it is straightforward. \(\square\)

Lemma 2.2.2 Let \( m_{ij}^h \) denote the parameters of \( X(M(2^n)) \). Then

\[ m_{ij}^h = q^3(q^2 - 1) + q^2 \cdot a_{ij}^h, \quad \text{for all } h, i, j \]

except the following cases:

\[ m_{11}^1 = q^3(q^2 - 1) + q^2(a_{11}^1 + 2) - 2 \]
\[ m_{1h}^h = m_{h1}^h = q^3(q^2 - 1) + q^2(a_{1h}^h + 1) - 1, \quad \text{for } 2 \leq h \leq q. \]

(Recall that \( a_{ij}^h \) are the parameters of the association scheme \( X(SL(2, 2^n)) \) (see Lemma 1.1.1).)

**Proof** For \( 1 \leq h \leq i \leq j \leq q \), \((h, i, j) \neq (1, 1, 1)\), put \( z = \begin{pmatrix} 0 & e_1 \\ e_1 & e^{h-1} \end{pmatrix} \in C_h \), and let \( x = \begin{pmatrix} x_1 \\ (x_5, x_6, x_7) \\ x_8 \end{pmatrix} \) and \( y = \begin{pmatrix} y_1 \\ (y_5, y_6, y_7) \\ y_8 \end{pmatrix} \), so that \( m_{ij}^h \) is equal to the number of vectors \((x_1, x_2, \ldots, x_8, y_1, \ldots, y_8) \in V_{16}(F) \) which satisfies the system of equations:
\[
\begin{align*}
    x_1y_1 + (x_2, x_3, x_4) \circ (y_5, y_6, y_7) &= 0, \\
x_1(y_2, y_3, y_4) + y_8(x_2, x_3, x_4) - (x_5, x_6, x_7) \times (y_5, y_6, y_7) &= (1, 0, 0), \\
y_1(x_5, x_6, x_7) + x_8(y_5, y_6, y_7) + (x_2, x_3, x_4) \times (y_2, y_3, y_4) &= (1, 0, 0), \\
x_8y_8 + (x_5, x_6, x_7) \circ (y_2, y_3, y_4) &= \nu^{h-1}, \\
x_1x_8 - (x_2, x_3, x_4) \circ (x_5, x_6, x_7) &= 1, \\
y_1y_8 - (y_2, y_3, y_4) \circ (y_5, y_6, y_7) &= 1, \\
x_1 + x_8 &= \nu^{i-1}, \\
y_1 + y_8 &= \nu^{j-1}.
\end{align*}
\]

By simplifying the system of equations, we have

\[
\begin{align*}
    x_5 &= x_2 + \nu^{j-1} + x_1\nu^{h-1}, \\
y_1 &= x_2, \\
y_2 &= x_1 + \nu^{i-1} + x_2\nu^{h-1}, \\
y_3 &= x_7 + x_3\nu^{h-1}, \\
y_4 &= x_6 + x_4\nu^{h-1}, \\
y_5 &= x_1, \\
y_6 &= x_4, \\
y_7 &= x_3, \\
y_8 &= x_2 + \nu^{j-1}; \\
x_1^2 + \nu^{i-1}x_1 + x_2^2 + \nu^{j-1}x_2 + \nu^{h-1}x_1x_2 + x_3x_6 + x_4x_7 &= 1.
\end{align*}
\]

This implies that

\[
\begin{align*}
m^h_{ij} &= \left| \{(x_1, x_2, x_3, x_4, x_5, x_6, x_7) \in F^6 \mid x_3x_6 + x_4x_7 \\
    &\quad = 1 + x_1^2 + \nu^{i-1}x_1 + x_2^2 + \nu^{j-1}x_2 + \nu^{h-1}x_1x_2 \} \right|
\end{align*}
\]

for \(1 \leq h \leq i \leq j \leq q\) except \((h, i, j) = (1, 1, 1)\) and \((h, i, j) = (h, 1, h)\). By Lemma 1.3.1, we have

\[
m^h_{ij} = \lambda_0(4) \cdot N[1 + x_1^2 + \nu^{i-1}x_1 + x_2^2 + \nu^{j-1}x_2 + \nu^{h-1}x_1x_2 = 0] \\
+ \lambda_1(4) \cdot (q^2 - N[1 + x_1^2 + \nu^{i-1}x_1 + x_2^2 + \nu^{j-1}x_2 + \nu^{h-1}x_1x_2 = 0]) \\
= q^3(q^2 - 1) + q^2 \cdot a^h_{ij}.
\]
For $m_{1h}^h = m_{h1}^h$, setting $\nu^{j-1} = 0$ in the above system of equations, we have

$$m_{1h}^h = \lambda_0(4) \cdot N[(1 + x_1)^2 + \nu^{h-1}(1 + x_1)x_2 + x_2^2 = 0]$$

$$+ \lambda_1(4) \cdot (q^2 - N[(1 + x_1)^2 + \nu^{h-1}(1 + x_1)x_2 + x_2^2 = 0]) - 1$$

$$= q^3(q^2 - 1) + q^2(a_{1h}^h + 1) - 1,$$

by avoiding the solution $(a, y) = (1, z)$ of the system of equations (because it does not belong to $C_1 \times C_h$).

For $m_{11}^h$, set $z = \left( \begin{array}{cc} 0 & e_1 \\ e_1 & 0 \end{array} \right)$, and repeat the same argument, so that we have

$$m_{11}^1 = \left\{ (x_1, x_2, x_3, x_4, x_6, x_7) \in F^6 \mid x_3x_6 + x_4x_7 = 1 + x_1^2 + x_2^2 \right\} - 2$$

$$= \lambda_0(4) \cdot N[1 + x_1^2 + x_2^2 = 0] + \lambda_1(4) \cdot (q^2 - N[1 + x_1^2 + x_2^2 = 0]) - 2$$

$$= q^3(q^2 - 1) + q^2(a_{11}^1 + 2) - 2,$$

by avoiding two pairs $(x, y) = (1, z)$ and $(x, y) = (z, 1)$ which are also solutions of $x_3x_6 + x_4x_7 = 1 + x_1^2 + x_2^2$. Because of the equality $k_h m_{ij}^h = k_j m_{ij}^h$, we complete the proof.

Now we construct the character table of the association scheme $\chi(M(2^n))$.

**Theorem 2.2.3.** The character table $\bar{P}$ of the association scheme $\chi(M(2^n))$ is given by Table 5, up to a suitable permutation on its columns and rows.

Note that comparing with $P$ given in Section 1.1, each entry of $\bar{P}$ which does not lie on the first row nor the first two columns is exactly equal to $q^2$ times the corresponding entry of $P$, i.e., the $(i, j)$-entry $\bar{p}_j(i) = q^2 \cdot p_j(i)$ for $i \geq 1, j \geq 2$. Each entry of the second column $\bar{p}_1(i)$ is equal to $q^2(p_1(i) + 1) - 1$ for $i > 1$. 
Proof of Theorem 2.2.3. Let \( \mathcal{M}_i \) denote the \( i \)th intersection matrix (whose \((j, h)\)-entry is \( m_{ij}^{kh} \)) of \( \mathcal{X}(M(2^n)) \), while \( B_i \) (whose \((j, h)\)-entry is \( a_{ij}^{kh} \)) denote that of \( \mathcal{X}(SL(2, 2^n)) \). Using the same notation as in Section 1.1, we will show that for each \( i = 0, 1, 2, \ldots, q \),

\[
\mathcal{M}_i \cdot (\Pi \cdot \tilde{P}) = (\Pi \cdot \tilde{P}) \cdot \tilde{P}_j 
\]

(2.2.1)

where \( \tilde{P}_j = \text{diag}(\tilde{p}_j(0), \tilde{p}_j(1), \ldots, \tilde{p}_j(q)) \), \( j = \pi(i) \). Once we prove this, it means that \( \tilde{P} \cdot \Pi^{-1} \) is the character table (\( P \)-matrix) of \( \mathcal{X}(M(2^n)) \). (It is easily shown that \( \tilde{P} \) is non-singular.)

To show the equality (2.2.1), we have only to show for any given \( i \), the \((k, \ell)\)-entry of the lefthand side, \( \sum_{\alpha=0}^{q} m_{ik}^{\alpha} \cdot \tilde{p}_{\pi(\alpha)}(\ell) \), and the \((k, \ell)\)-entry of the righthand side, \( \tilde{p}_{\pi(k)}(\ell) \cdot \tilde{p}_{\pi(i)}(\ell) \), are coincide for every fixed \((k, \ell)\).

First let \( i \geq 2 \). If \( k \geq 2 \) and \( k \neq i \), then for each \( \ell \geq 1 \),

\[
\sum_{\alpha=0}^{q} m_{ik}^{\alpha} \cdot \tilde{p}_{\pi(\alpha)}(\ell) = m_{ik}^{1} \cdot \tilde{p}_{1}(\ell) + \sum_{\alpha=2}^{q} m_{ik}^{\alpha} \cdot \tilde{p}_{\pi(\alpha)}(\ell) \\
= \{q^3(q^2 - 1) + q^2 \cdot a_{ik}^{1}\} \{q^2(p_{1}(\ell) + 1) - 1\} \\
+ \sum_{\alpha=2}^{q} \{q^3(q^2 - 1) + q^2 \cdot a_{ik}^{\alpha}\} q^2 \cdot p_{\pi(\alpha)}(\ell) \\
= (q^2 - 1)\{q^3(q^2 - 1) + q^2 a_{ik}^{1}\} \\
+ q^4 \sum_{\alpha=1}^{q} a_{ik}^{\alpha} \cdot p_{\pi(\alpha)}(\ell) + q^5(q^2 - 1) \sum_{\alpha=1}^{q} p_{\pi(\alpha)}(\ell) \\
= q^4 p_{\pi(k)}(\ell) \cdot p_{\pi(i)}(\ell) = \tilde{p}_{\pi(k)}(\ell) \cdot \tilde{p}_{\pi(i)}(\ell),
\]
by Lemma 2.2, the equality in (1.1.1), and the equalities \(a_{ik}^1 = q\) and \(\sum_{\alpha=1}^{q} p_{\pi(\alpha)}(\ell) = -1\). If \(k = i \geq 2\), then for each \(\ell \geq 1\),

\[
\sum_{\alpha=0}^{q} m_{ii}^\alpha \cdot \tilde{p}_{\pi(\alpha)}(\ell) = m_{ii}^0 + m_{ii}^1 \cdot \tilde{p}_1(\ell) + \sum_{\alpha=2}^{q} m_{ii}^\alpha \cdot \tilde{p}_{\pi(\alpha)}(\ell)
\]

\[
= m_{ii}^0 + q^2(q^2 - 1) \cdot a_{ii}^1 - q^3(q^2 - 1) + q^4 \sum_{\alpha=1}^{q} a_{ii}^\alpha \cdot p_{\pi(\alpha)}(\ell).
\]

From the fact that

\(m_{ii}^0 = q^3(q^3 - 1),\ a_{ii}^0 = q(q - 1),\) and \(a_{ii}^1 = 0\) if \(N[t^2 + \nu^{i-1} + 1 = 0] = 0\),

and

\(m_{ii}^0 = q^3(q^3 + 1),\ a_{ii}^0 = q(q + 1),\) and \(a_{ii}^1 = 2q\) if \(N[t^2 + \nu^{i-1} + 1 = 0] = 2\),

we have

\[
m_{ii}^0 + q^2(q^2 - 1) \cdot a_{ii}^1 - q^3(q^2 - 1) = q^4 \cdot a_{ii}^0,
\]

and thus

\[
\sum_{\alpha=0}^{q} m_{ii}^\alpha \cdot \tilde{p}_{\pi(\alpha)}(\ell) = q^4 \sum_{\alpha=0}^{q} a_{ii}^\alpha \cdot p_{\pi(\alpha)}(\ell) = [\tilde{p}_{\pi(i)}(\ell)]^2.
\]
Now suppose $k = 1$, $(i \geq 2)$ then for each $\ell \geq 1$,

$$
\sum_{\alpha=1}^{q} m_{i1}^\alpha \cdot \bar{p}_{\pi(\alpha)}(\ell) = \sum_{\alpha=1}^{q} m_{i1}^\alpha \cdot \bar{p}_{\pi(\alpha)}(\ell) \\
= m_{i1}^1 \cdot \bar{p}_1(\ell) + m_{i1}^i \cdot \bar{p}_{\pi(i)}(\ell) + \sum_{\alpha=2}^{q} m_{i1}^\alpha \cdot \bar{p}_{\pi(\alpha)}(\ell).
$$

Since

$$
m_{i1}^1 \cdot \bar{p}_1(\ell) = \{q^3(q^2 - 1) + q^2 \cdot a_{i1}^1\}\{q^2(p_1(\ell) + 1) - 1\} \\
= (q^2 - 1)\{q^3(q^2 - 1) + q^2 a_{i1}^1\} + q^5(q^2 - 1)p_1(\ell) + q^4 a_{i1}^1 \cdot p_1(\ell),
$$

and

$$
m_{i1}^i \cdot \bar{p}_{\pi(i)}(\ell) = \{q^3(q^2 - 1) + q^2(a_{i1}^i + 1) - 1\}q^2 \cdot p_{\pi(i)}(\ell) \\
= q^2(q^2 - 1) \cdot p_{\pi(i)}(\ell) + q^5(q^2 - 1) \cdot p_{\pi(i)}(\ell) + q^4 a_{i1}^i \cdot p_{\pi(i)}(\ell),
$$

so,

$$
\sum_{\alpha=0}^{q} m_{i1}^\alpha \cdot \bar{p}_{\pi(\alpha)}(\ell) = q^2(q^2 - 1)p_{\pi(i)}(\ell) + (q^2 - 1)\{q^3(q^2 - 1) + q^2 a_{i1}^1\} \\
+ q^5(q^2 - 1) \sum_{\alpha=1}^{q} p_{\pi(\alpha)}(\ell) + q^4 \sum_{\alpha=1}^{q} a_{i1}^\alpha \cdot p_{\pi(\alpha)}(\ell) \\
= q^2(q^2 - 1)p_{\pi(i)}(\ell) + q^4 \cdot p_{\pi(i)}(\ell) \cdot p_{\pi(i)}(\ell) \\
= q^2 \cdot p_{\pi(i)}(\ell)\{q^2(p_1(\ell) + 1) - 1\} \\
= \bar{p}_1(\ell) \cdot \bar{p}_{\pi(i)}(\ell).
$$
When $i = 1$, in the same manner but applying

$$m_{11}^1 = q^3(q^2 - 1) + q^2(a_{11}^1 + 2) - 2$$
$$m_{1h}^1 = q^3(q^2 - 1) + q^2(a_{1h}^1 + 1) - 1$$
as it is needed, we complete the proof. □

2.3. The character tables of $M(p^n)$ and $M^*(p^n)$ for an odd prime $p$.

In this section we compute the character tables of $M = M(p^n)$ and $M^* = M^*(p^n)$ for an odd prime $p$. Notation is the same as in the Section 2.1 (cf. Theorem 2.1.1). Let us fix the indexing of the conjugacy classes of $M(p^n)$ by

$$C_0 = [1],$$
$$C_1 = [-1],$$
$$C_i = \left[ \begin{array}{cc} 0 & e_{i-1} \\ -e_1 & 0 \end{array} \right] \quad (\text{for } 2 \leq i \leq q - 1),$$
$$C_q = \left[ \begin{array}{cc} 0 & e_{1} \\ -e_1 & 1 \end{array} \right],$$
$$C_{q+1} = \left[ \begin{array}{cc} 0 & e_{1} \\ -e_1 & 0 \end{array} \right].$$

Let $\mathcal{X}(M) = (M, \{R_i\}_{0 \leq i \leq q+1})$ be the naturally defined loop association scheme (whose ordering of the relation $R_i$'s corresponds to that of the $C_i$'s). Then $\mathcal{X}(M)$ is a symmetric association scheme of class $q + 1$.

Lemma 2.3.1. The valencies of the association scheme $\mathcal{X}(M(p^n))$ are given by

$$k_0 = k_1 = 1,$$
$$k_i = \begin{cases} (q^3 - 1)(q^3 + 1) & \text{if } N[t^2 - \nu^{i-1}t + 1 = 0] = 1, \\ q^3(q^3 - 1) & \text{if } N[t^2 - \nu^{i-1}t + 1 = 0] = 0, \\ q^3(q^3 + 1) & \text{if } N[t^2 - \nu^{i-1}t + 1 = 0] = 2, \end{cases}$$

for $i = 2, 3, \ldots, q$, for $i = 2, 3, \ldots, q$, for $i = 2, 3, \ldots, q$,
\[ k_{q+1} = \begin{cases} q^3(q^3 + 1) & \text{if } q \equiv 1 \mod 4, \\ q^3(q^3 - 1) & \text{if } q \equiv -1 \mod 4. \end{cases} \]

**Proof.** Omitted.

Let \( i_0, j_0 \) \((i_0 < j_0)\) denote the indices for which \( k_{i_0} = k_{j_0} = (q^3 - 1)(q^3 + 1) \).

(There are two such indices.)

**Lemma 2.3.2** Denote the parameters of \( \mathcal{X}(M(p^n)) \) by \( n^h_{ij} \) and let \( b^h_{ij} \) be the parameters of \( \mathcal{X}(SL(2,p^n)) \). Then

\[ n^h_{ij} = q^3(q^2 - 1) + q^2 \cdot b^h_{ij} \]

for all \( h, i, j \) except the following special cases:

\[
\begin{align*}
    n^i_{i_0i_0} &= n^j_{j_0j_0} = n^j_{i_0j_0} = n^i_{j_0i_0} = q^3(q^2 - 1) + q^2(b^i_{i_0i_0} + 2) - 2, \\
    n^h_{ih} &= n^h_{hi} = q^3(q^2 - 1) + q^2(b^h_{ih} + 1) - 1,
\end{align*}
\]

for \( h = 2, 3, \ldots, q + 1, \ h \neq i_0, \ h \neq j_0, \)

\[
    n^{q+1}_{j_{0q+1}} = n^{q+1}_{q+1j_0} = q^3(q^2 - 1) + q^2(b^{q+1}_{j_{0q+1}} + 1) - 1.
\]

**Proof.** Similar to the proof of Lemma 2.2.2, so it is omitted. \( \square \)

**Theorem 2.3.3.** The character table of the association scheme \( \mathcal{X}(M(p^n)) \) is equivalent to the matrix \( \hat{P} \) in Table 6, up to a suitable permutation on its columns and rows, where \( c = \frac{q-1}{2}, \ d = \frac{q^3-3}{2}, \ e = (-1)^{\frac{q-1}{2}}, \ \sigma = \exp(2\pi \sqrt{-1}/(q + 1)), \ \rho = \exp(2\pi \sqrt{-1}/(q - 1)). \)
Proof. Using the above lemmas and the results in Section 1.2, it is proved in the same manner as in the proof of Theorem 2.2.3. □

In the rest of this section, we consider the character table of the simple Moufang loop $M^*(p^n)$. As we proved in Theorem 2.1.1, $M^*(p^n)$ consists of $(q + 3)/2$ conjugacy classes. Let $X(M^*)$ be the loop association scheme determined by the following indexing of the conjugacy classes of $M^*$:

$$
C_0 = \{1\},
C_i = \left\{ \left( \begin{array}{cc} a & \alpha \\ \beta & b \end{array} \right) \in M^* \mid a + b = \nu^i \right\} \quad \text{(for } 1 \leq i \leq (q - 1)/2),
C_{\kappa+1}^{\kappa-1} = \left\{ \left( \begin{array}{cc} a & \alpha \\ \beta & b \end{array} \right) \in M^* \mid a + b = 0 \right\}.
$$

Lemma 2.3.4. The valencies of $X(M^*(q))$ are

$$
k_0 = 1
k_i = \left\{ \begin{array}{ll}
(q^3 - 1)(q^3 + 1) & \text{if } N[t^2 - \nu^{i-1}t + 1 = 0] = 1, \\
q^3(q^3 - 1) & \text{if } N[t^2 - \nu^{i-1}t + 1 = 0] = 0, \\
q^3(q^3 + 1) & \text{if } N[t^2 - \nu^{i-1}t + 1 = 0] = 2,
\end{array} \right.
$$

for $i = 1, 2, \ldots, \frac{q - 1}{2}$.

$$
k_{\kappa+1}^{\kappa-1} = \left\{ \begin{array}{ll}
\frac{1}{2}q^3(q^3 + 1) & \text{if } q \equiv 1 \text{ (mod } 4), \\
\frac{1}{2}q^3(q^3 - 1) & \text{if } q \equiv -1 \text{ (mod } 4).
\end{array} \right.
$$

Proof. Omitted. □

Lemma 2.3.5. Let $\ell_{ij}^h$ denote the parameters of $X(M^*(q))$ and let $d_{ij}^h$ denote those of $X(PSL(2, q))$. Then
\[ \ell_{i,j}^h = 2q^3(q^2 - 1) + q^2 \cdot d_{i,j}^h \]

for all \( h, i, j \in \{1, 2, \ldots, \frac{q-1}{2} \} \) except the following cases:

\[ \ell_{i,i_0}^{i_0} = 2q^3(q^2 - 1) + q^2(d_{i_0,i_0}^{i_0} + 2) - 2, \]
\[ \ell_{i_0^h}^{h} = \ell_{h,i_0}^{h} = 2q^3(q^2 - 1) + q^2(d_{h,i_0}^{h} + 1) - 1, \]

for \( h = 1, 2, \ldots, \frac{q-1}{2}, h \neq i_0 \).

**Proof.** It is straightforward by Lemma 1.2.2 and Lemma 2.3.2. \( \square \)

The character table of the association scheme \( X(M^*(q)) \) is now given by the following theorem which is proved by similar method as used in the proof of Theorem 2.2.3 (or Theorem 2.3.3).

**Theorem 2.3.6.** Let \( \sigma \) be a primitive \((q+1)\)th root of 1, \( \rho \) a primitive \((q-1)\)th root of 1, and denote \( a = \frac{q-1}{4}, b = \frac{q-3}{4} \). Then the character table of the association scheme \( X(M^*(p^n)) \) is equivalent, up to a suitable permutation on the columns and rows, to the following matrix \( \mathcal{P} \):

(i) If \( q \equiv 1 \pmod{4} \), then see Table 7;

(ii) If \( q \equiv -1 \pmod{4} \), then see Table 8.
CHAPTER III

CHARACTER TABLES OF THE ASSOCIATION SCHEMES
OBTAINED FROM THE ACTION OF ORTHOGONAL GROUPS

For a given non-singular quadratic form on a $2m$-dimensional vector space over $GF(q)$ with Witt index $m$ or $m - 1$, there is defined an orthogonal group. This group acts on the set of non-isotropic projective points (w.r.t. the given quadratic form) transitively if $q$ is even; acts transitively on each half of the set if $q$ is odd. The character tables of the group association scheme coming from these transitive permutation groups are constructed via investigation of the relation between these association schemes and those of $PSL(2, q)$. Through the construction it is observed that the character tables of the association schemes of $PSL(2, q)$ and the Paige’s Moufang loops are realized as special cases of the character tables of those association schemes discussed in this chapter.

3.0. Quadratic forms and orthogonal groups over $GF(q)$.

Let $V$ be a finite dimensional vector space over a finite field $F = GF(q)$. A quadratic form $f$ on $V$ is a function from $V$ to $F$ satisfying the condition that

$$f(\lambda x + \varphi y) = \lambda^2 f(x) + \varphi^2 f(y) + \lambda \varphi (x, y)$$
for all \( x, y \in V \) and \( \lambda, \varphi \in F \), where \( \langle x, y \rangle \) is a bilinear form on \( V \). (If the characteristic of \( F \) is odd, then \( \langle x, y \rangle \) is a symmetric form on \( V \) and there is a one to one correspondence between the quadratic form \( f \) and the symmetric bilinear form \( \langle x, y \rangle \) on \( V \). If the characteristic of \( F \) is even, then the bilinear form \( \langle x, y \rangle \) is an alternating bilinear form.)

For a quadratic form \( f \) on \( V \), let us define \( D = \{ x \in V \mid \langle x, y \rangle = 0, \text{ for all } y \in V \} \). The dimension of the space \( D \) over \( F \) is called the defect of \( f \). A quadratic form \( f \) is said to be non-degenerate if no non-zero vector \( x \in D \) satisfies \( f(x) = 0 \). For a quadratic form \( f \), the isometry group \( O(V, f) \) is defined by

\[
O(V, f) = \{ T \in GL(V) \mid f(Tx) = f(x) \text{ for all } x \in V \}
\]

and is called the orthogonal group of \( f \).

Let \( f \) be a non-degenerate quadratic form on \( V \). The Witt index \( \mu \) is defined as the dimension of maximal totally singular subspaces (which are all conjugate by the action of \( O(V, f) \)) of \( V \). (A subspace \( W \) of \( V \) is called singular if \( f(x) = 0 \) for all \( x \in W \).)

In what follows we always assume that the dimension of \( V \) is even \( 2m \geq 4 \). Then it is easily seen that \( \mu \leq m \).

The non-degenerate quadratic forms of \( 2m \)-dimensional vector spaces over \( F \) are classified as follows.

1. Let \( q = p^n \) with \( p \) an odd prime. Then there are exactly two non-equivalent quadratic forms \( f_1 \) and \( f_2 \):
\( f_1(x) = 2(x_1x_{m+1} + x_2x_{m+2} + \cdots + x_mx_{2m}) \) \quad (\mu = m),

\( f_2(x) = 2(x_1x_m + x_2x_{m+1} + \cdots + x_{m-1}x_{2m-2}) + x_{2m-1}^2 - \alpha x_{2m}^2 \) \quad (\mu = m - 1),

where \( \alpha \) is a non-square element in \( F \). We write \( GO_{2m}^+(p^n) = O(V, f_1) \) and \( GO_{2m}^-(p^n) = O(V, f_2) \).

(2) Let \( q = 2^n \). Then there are exactly two non-equivalent quadratic forms \( f_1 \) and \( f_2 \).

\( f_1(x) = x_1x_{m+1} + \cdots + x_mx_{2m} \) \quad (\mu = m),

\( f_2(x) = x_1x_{m+1} + \cdots + x_{m-1}x_{2m-1} + \alpha x_m^2 + x_mx_{2m} + \alpha x_{2m}^2 \) \quad (\mu = m - 1),

where \( \alpha t^2 + t + \alpha \) is an irreducible polynomial over \( F \). We write \( GO_{2m}^+(2^n) = O(V, f_1) \) and \( GO_{2m}^-(2^n) = O(V, f_2) \).

Let \( O_{2m}^+(q) \) and \( O_{2m}^-(q) \) be the (simple) group \( P\Omega_{2m}^+(q) \) and \( P\Omega_{2m}^-(q) \) attached to \( GO_{2m}^+(q) \) and \( GO_{2m}^-(q) \), respectively. (cf. Dieudonne [5], Atlas [3].)

3.1. The character table of \( X(O_{2m}^+(2^n), \Omega) \).

In this section let \( V \) be a \( 2m \)-dimensional vector space over \( F = GF(q) \), \( q = 2^n \), and let \( f \) be the non-singular quadratic form \( f_1 \) with index \( m \) (cf. Section 3.0), i.e.,

\[ f(x) = f_1(x) = x_1x_{m+1} + x_2x_{m+2} + \cdots + x_mx_{2m}. \]

Let \( \Omega \) be the set of all non-singular 1-dimensional vector spaces (projective points) of \( V \). Then we get \( |\Omega| = q^{m-1}(q^m - 1) \) by Lemma 1.3.1. Abusing the notation, let \( x \) denote the 1-dimensional subspace \( \langle x \rangle \), spanned by \( x \). Also we assume that \( f(x) = 1 \).
for all \( x \in \Omega \). Although \( x \) denotes a non-singular projective point in \( \Omega \), and \( x \) also denotes the representing vector of the point \( x \in \Omega \) satisfying that \( f(x) = 1 \), it should be clear from the context.

Since \( O_{2m}^+(q) \) acts on \( \Omega \) transitively, and since the orbits of \( O_{2m}^+(q) \) on \( \Omega \times \Omega \) are given by the following \( R_i \)'s,

\[
R_0 = \{ (x, x) \mid x \in \Omega \},
R_1 = \{ (x, y) \mid f(x + y) = 0, x \neq y, x, y \in \Omega \},
R_j = \{ (x, y) \mid f(x + y) = \nu^{j-1}, x, y \in \Omega \},
\]

for \( j = 2, 3, \ldots, q \), we have a symmetric association scheme, \( \mathcal{X}(O_{2m}^+(2^n), \Omega) = (\Omega, \{ R_i \}_{0 \leq i \leq q}) \), of class \( q \) defined by these \( q + 1 \) orbits.

**Lemma 3.1.1.** In the association scheme \( \mathcal{X}(O_{2m}^+(2^n), \Omega) \), we have

\[
k_1 = (q^{m-1} + 1)(q^{m-1} - 1),
k_i = \begin{cases} q^{m-1}(q^{m-1} - 1) & \text{if } N[t^2 + \nu^{i-1}t + 1 = 0] = 0, \\ q^{m-1}(q^{m-1} + 1) & \text{if } N[t^2 + \nu^{i-1}t + 1 = 0] = 2, \end{cases}
\]

for \( i = 2, 3, \ldots, q \).

**Proof** Let \( y = e_1 + e_{m+1} = (1, 0, \ldots, 0, 1, 0, \ldots, 0) \in V_{2m}(F) \). Then, using Lemma 1.3.1, we have
\[ k_1 = | \{ x \in \Omega \mid (x, y) \in R_1, x \neq y \} | \]
\[ = | \{ (x_1, x_2, \ldots, x_{2m}) \in V_{2m}(F) \mid x_1 + x_{m+1} = 0, x_1 x_{m+1} + \cdots + x_m x_{2m} = 1 \} | -1 \]
\[ = | \{ (x_2, x_3, \ldots, x_{2m}) \in V_{2m-1}(F) \mid x_2 x_{m+2} + x_3 x_{m+3} + \cdots + x_m x_{2m} = 1 + x_{m+1}^2 \} | -1 \]
\[ = \lambda_0(2(m - 1)) + (q - 1) \cdot \lambda_1(2(m - 1)) - 1 \]
\[ = (q^{m-1} + 1)(q^{m-1} - 1), \]
because \( N[1 + x_{m+1}^2 = 0] = 1. \) For \( 2 \leq i \leq q, \)

\[ k_i = | \{ x \in \Omega \mid f(x + y) = \nu^{i-1} \} | \]
\[ = | \{ (x_2, x_3, \ldots, x_{2m}) \in V_{2m-1}(F) \mid x_2 x_{m+2} + \cdots + x_m x_{2m} = 1 + x_{m+1}^2 + \nu^{i-1} x_{m+1} \} | \]
\[ = \begin{cases} q \cdot \lambda_1(2(m - 1)) = q^{m-1}(q^{m-1} - 1), \\ 2 \lambda_0(2(m - 1)) + (q - 2) \cdot \lambda_1(2(m - 1)) = q^{m-1}(q^{m-1} + 1) \end{cases} \]
according as \( N[x_{m+1}^2 + \nu^{i-1} x_{m+1} + 1 = 0] = 0 \) or \( 2. \)

The following lemma shows the relation between the two parameters of \( X(O_{2m}(2^n), \Omega) \) and \( X(SL(2, 2^n)). \)

**Lemma 3.1.2.** Let \( s_{ij}^h \) denote the parameters of \( X(O_{2m}^+(2^n), \Omega), \) while \( a_{ij}^h \) denote those of \( X(SL(2, 2^n)). \) Then

\[ s_{ij}^h = q^{m-2}(q^{m-2} - 1) + q^{m-2} \cdot a_{ij}^h, \]
for all \( h, i, j \) except the following cases:
\[ s_{11}^1 = q^{m-1}(q^{m-2} - 1) + q^{m-2}(a_{11}^1 + 2) - 2, \]
\[ s_{hh}^h = s_{h1}^h = q^{m-1}(q^{m-2} - 1) + q^{m-2}(a_{1h}^h + 1) - 1 \quad \text{for } 2 \leq h \leq q. \]

**Proof.** For \( h, i, j \) in \( \{2, 3, \ldots, q\} \), set \( y = e_1 + e_{m+1}, \ z = e_2 + \nu^{h-1}e_{m+1} + e_{m+2} \) so that \( f(y) = f(z) = 1 \) and \( (y, z) \in R_h \), and put \( x = (x_1, \ldots, x_{2m}) \). Then

\[
\begin{align*}
s_{ij}^h &= \left| \{ x \in \Omega \mid f(y + x) = \nu^{i-1}, \ f(x + x) = \nu^{j-1} \} \right| \\
&= \left| \{ x \in \mathbb{F}_{2m} \mid x_1 + x_{m+1} = \nu^{i-1}, \ x_2 + \nu^{h-1}x_1 + x_{m+2} = \nu^{j-1}, \ x_1x_{m+1} + \cdots + x_mx_{2m} = 1 \} \right| \\
&= \left| \{ (x_3, x_4, \ldots, x_{2m}) \in \mathbb{F}_{2m-2} \mid x_3x_{m+3} + \cdots + x_mx_{2m} = f(x_{m+1}, x_{m+2}) \} \right|
\end{align*}
\]

where \( f(x_{m+1}, x_{m+2}) = 1 + x_{m+1}^2 + \nu^{i-1}x_{m+1} + \nu^{j-1}x_{m+2} + \nu^{h-1}x_{m+1}x_{m+2} + x_{m+2}^2 \).

Since \( N[f(x_{m+1}, x_{m+2}) = 0] = a_{ij}^h \) by Lemma 1.1.1, we have

\[
\begin{align*}
s_{ij}^h &= \lambda_0(2(m - 2)) \cdot a_{ij}^h + \lambda_1(2(m - 2)) \cdot (q^2 - a_{ij}^h) \\
&= q^{m-1}(q^{m-2} - 1) + q^{m-2} \cdot a_{ij}^h
\end{align*}
\]

by Lemma 1.3.1.

For \( h = i = j = 1 \), by replacing \( \nu^{h-1}, \nu^{i-1}, \) and \( \nu^{j-1} \) by 0 in the above argument, we have \( 1 + x_{m+1}^2 + x_{m+2}^2 \) instead of \( f(x_{m+1}, x_{m+2}) \), so that

\[
\begin{align*}
s_{11}^1 &= \lambda_0(2(m - 2)) \cdot N[1 + x_{m+1}^2 + x_{m+2}^2 = 0] \\
&\quad + \lambda_1(2(m - 2))(q^2 - N[1 + x_{m+1}^2 + x_{m+2}^2 = 0]) - 2 \\
&= q^{m-1}(q^{m-2} - 1) + q^{m-2}(a_{11}^1 + 2) - 2
\end{align*}
\]
(by avoiding the possibility that \(N[1 + x^2_{m+1} + x^2_{m+2} = 0]\) counts the cases \(x = y\) and \(x = z\)).

For \(h = j \geq 2, i = 1\), again by replacing \(\nu^{j-1}\) by 0 in \(f(x_{m+1}, x_{m+2})\), we have

\[
\begin{align*}
    s_{1,h}^h &= \{(x_3, \ldots, x_{2m}) \in F^{2m-2} \mid x_3x_{m+3} + \cdots + x_mx_{2m} \\
    &= 1 + x_{m+1}^2 + \nu^{h-1}x_{m+2} + \nu^{h-1}x_{m+1}x_{m+2} + x_{m+2}^2 \} - 1 \\
    &= \lambda_0(2(m-2)) \cdot N[(1 + x_{m+1})^2 + \nu^{h-1}(1 + x_{m+1})x_{m+2} + x_{m+2}^2 = 0] \\
    &\quad + \lambda_1(2(m-2)) \cdot (q^2 - N[(1 + x_{m+1})^2 + \nu^{h-1}(1 + x_{m+1})x_{m+2} + x_{m+2}^2 = 0]) - 1 \\
    &= q^{m-1}(q^{m-2} - 1) + q^{m-2}(a_{1,h}^h + 1) - 1
\end{align*}
\]

because of the condition that \(x \neq y\).

All remaining cases are checked similarly. \(\square\)

**Theorem 3.1.3.** The character table of the association scheme \(\chi(O_{2m}^+(2^n), \Omega)\) is equivalent, up to a suitable permutation on its columns and rows, to the matrix \(\tilde{P}\) in Table 9, where \(q = 2^n\), \(c = \frac{q}{2}\), \(d = \frac{q-2}{2}\), \(\sigma = \exp(2\pi\sqrt{-1}/(q + 1)), \rho = \exp(2\pi\sqrt{-1}/(q - 1))\).

**Remarks.** Observe that, comparing this matrix \(\tilde{P}\) with the matrix \(P\) given in Section 1.1, we have

\[
\begin{align*}
    \tilde{p}_j(i) &= \begin{cases} 
        q^{m-2}(p_1(i) + 1) - 1 & \text{if } j = 1 \\
        q^{m-2} \cdot p_j(i) & \text{if } j \geq 2
    \end{cases} \quad \text{for all } i \geq 1.
\end{align*}
\]

Also, note that, if we plug 2 for all \(m\) in \(\tilde{P}\) then we have \(P\), and that if we plug 4 for all \(m\) in \(\tilde{P}\) then we have another \(\tilde{P}\) given in the Theorem 2.2.3.
Proof of Theorem 3.1.3. Let $S_i$ denote the $i$th intersection matrix, whose $(j, h)$-entry is $s_{ij}^h$, of $X(O_{2m}^+(2^n), \Omega)$, while $B_i$ denote that of $X(SL(2, 2^n))$. From the results of Lemma 1.1.2 and Lemma 3.1.2, and the above remarks, it is just an analogue of the proof of Theorem 2.2.3 to show that

$$S_i \cdot (\Pi \cdot t \tilde{P}) = (\Pi \cdot t \tilde{P}) \cdot \tilde{P}_j$$

where $\tilde{P}_j = \text{diag}[\tilde{p}_j(0), \tilde{p}_j(1), \ldots, \tilde{p}_j(q)]$ and $j = \pi(i)$. That is, the character table of $X(O_{2m}^+(2^n), \Omega)$ must be $\tilde{P} \cdot \Pi^{-1}$.

Remark. It is easy to see that the association scheme $X(M(2^n))$ is isomorphic to $X(O_{2m}^+(2^n), \Omega)$ and that the association scheme $X(SL(2, 2^n))$ is isomorphic to $X(O_{2m}^+(2^n), \Omega)$.

3.2. The character table of $X(O_{2m}^-(2^n), \Theta)$.

In this section, let $V$ be a $2m$-dimensional vector space over $F = GF(q)$, $q = 2^n$, and let $f$ be the nonsingular quadratic form $f_2$ with index $m - 1$ (cf. Section 3.1), i.e.,

$$f(x) = f_2(x) = x_1x_{m+1} + x_2x_{m+2} + \cdots + x_{m-1}x_{2m-1} + \alpha x_m^2 + x_mx_{2m} + \alpha x_{2m}^2.$$

Let $\Theta$ be the set of all non-singular projective points with respect to the quadratic form $f$. Then we have $|\Theta| = q^{m-1}(q^m + 1)$. In this section, again, we denote any projective point in $\Theta$, spanned by $x$, simply by $x$ and we assume that $f(x) = 1$ for all $x \in \Theta$.

$O_{2m}^-(2^n)$ acts on $\Theta$ transitively. We have a symmetric association scheme.
\[ \chi(O_{2m}(2^n), \Theta) = (\Theta, \{R_i\}_{0 \leq i \leq q}) \]

by using the \( q + 1 \) orbits of \( O_{2m}(2^n) \) on \( \Theta \times \Theta \) which are given by

\[
R_0 = \{(x, x) \mid x \in \Theta\}, \\
R_1 = \{(x, y) \mid f(x + y) = 0, \ x \neq y, \ x, y \in \Theta\}, \\
R_i = \{(x, y) \mid f(x + y) = \nu^{i-1}, \ x, y \in \Theta\}, \text{ for } i = 2, 3, \ldots, q.
\]

**Lemma 3.2.1.** In \( \chi(O_{2m}(2^n), \Theta) \), we have

\[
k_1 = (q^{-m-1} - 1)(q^{-m-1} + 1), \\
k_i = \begin{cases} 
q^{-m-1}(q^{-m-1} + 1) & \text{if } N[t^2 + \nu^{i-1}t + 1 + (\alpha
nu^{i-1})^2 = 0] = 2, \\
q^{-m-1}(q^{-m-1} - 1) & \text{if } N[t^2 + \nu^{i-1}t + 1 + (\alpha
nu^{i-1})^2 = 0] = 0,
\end{cases}
\]

for \( 2 \leq i \leq q \).

**Proof.** Similar to the proof of Lemma 3.1.1. \( \square \)

Notice that in \( \chi(O_{2m}(2^n), \Theta) \), there are \( \frac{q}{2} \) indices \( i \) for which \( k_i = q^{-m-1}(q^{-m-1} + 1) \), \( \frac{q-2}{2} \) indices \( i \) for which \( k_i = q^{-m-1}(q^{-m-1} - 1) \), while in \( \chi(O_{2m}(2^n), \Omega) \), there are \( \frac{q}{2} \) indices \( i \) for which \( k_i = q^{-m-1}(q^{-m-1} - 1) \) and \( \frac{q-2}{2} \) indices \( i \) for which \( k_i = q^{-m-1}(q^{-m-1} + 1) \).

**Lemma 3.2.2.** Let \( t_{ij}^h \) denote the parameters of \( \chi(O_{2m}(2^n), \Theta) \). Then

\[
t_{ij}^h = q^{-m-1}(q^{-m-2} + 1) - q^{-m-2} \cdot a_{ij}^h
\]

for all \( h, i, j \) except the following cases:
Theorem 3.2.3. The character table of the association scheme $X(O_{2m}^- (2^n), \Theta)$ is equivalent to the matrix $\tilde{P}^-$ in Table 10, up to a suitable permutation on its columns and rows.

Remark. The relation between the matrix $\tilde{P}^-$ and the matrix $P$ given in Section 1.1, is written as

$$\tilde{p}_j^-(i) = \begin{cases} -q^{m-2}(p_j(i) + 1) - 1 & \text{if } j = 1, \\ -q^{m-2} \cdot p_j(i) & \text{if } j \geq 2, \end{cases}$$

for all $i \geq 1$, where $\tilde{p}_j^-(i)$ and $p_j(i)$ are the $(i, j)$-entries of $\tilde{P}^-$ and $P$, respectively.

Proof of Theorem 3.2.3. It is analogous to the proof of Theorem 3.1.3 (hence to Theorem 2.2.3), so it is omitted. \qed

3.3. The character table of $X(O_{2m}^+ (p^n), \Omega), p$ an odd prime.

In this section let $V$ be a $2m$-dimensional vector space over $F = GF(q), q = p^n, p$ an odd prime. Let $\Omega$ be the set of all non-singular (i.e., non-isotropic) $1$-dimensional subspaces of $V$ with respect to the non-singular quadratic form

$$f(x) = f_1(x) = 2(x_1x_{m+1} + x_2x_{m+2} + \cdots + x_mx_{2m})$$

of index $m$ and let $\Omega_1$ and $\Omega_2$ be the set of all square-type and the set of all non-square-type elements of $\Omega$, respectively. Then we have $|\Omega_1| = |\Omega_2| = \frac{1}{2} q^{m-1}(q^m - 1)$. 
Abusing the notation, we let $x$ denote both the element of $\Omega_1$ and the representing vector with $f(x) = 1$. It is known that the orthogonal group $O_{2m}^+(q)$ with respect to the quadratic form $f$ over $GF(q)$ acts on both $\Omega_1$ and $\Omega_2$ transitively. The orbits of $O_{2m}^+(q)$ on $\Omega_1 \times \Omega_1$ can be given by $R_0, R_1, \ldots, R_{\frac{q-1}{2}}$, where

$$R_0 = \{(x, x) \mid x \in \Omega_1\}$$

$$R_i = \{(x, y) \mid \langle x, y \rangle = 2^{-i} \nu^i, x, y \in \Omega_1\},$$

for $i = 1, 2, \ldots, \frac{q-1}{2}$,

$$R_{\frac{q+1}{2}} = \{(x, y) \mid \langle x, y \rangle = 0, x, y \in \Omega_1\}.$$

From this we have a symmetric association scheme

$$\mathcal{X}(O_{2m}^+(q), \Omega_1) = (\Omega_1, \{R_i\}_{0 \leq i \leq \frac{q-1}{2}}).$$

We note that similarly we have an association scheme from the action of $O_{2m}^+(q)$ on $\Omega_2$ which is, in fact, isomorphic to the association scheme $\mathcal{X}(O_{2m}^+(q), \Omega_1)$.

**Lemma 3.3.1.** In $\mathcal{X}(O_{2m}^+(q), \Omega_1)$, we have

$$k_i = \begin{cases} q^{m-1}(q^{m-1} - 1) & \text{if } N\{t^2 - \nu^i t + 1 = 0\} = 0, \\ (q^{m-1} - 1)(q^{m-1} + 1) & \text{if } N\{t^2 - \nu^i t + 1 = 0\} = 1, \\ q^{m-1}(q^{m-1} + 1) & \text{if } N\{t^2 - \nu^i t + 1 = 0\} = 2, \end{cases}$$

for $i = 1, 2, \ldots, \frac{q-1}{2}$,

$$k_{\frac{q+1}{2}} = \begin{cases} \frac{1}{2} q^{m-1}(q^{m-1} + 1) & \text{if } q \equiv 1 (mod 4), \\ \frac{1}{2} q^{m-1}(q^{m-1} - 1) & \text{if } q \equiv -1 (mod 4). \end{cases}$$
Proof. Let \( y = 2^{-1}e_1 + e_{m+1} = (2^{-1}, 0, \ldots, 0, 1, 0, \ldots, 0) \in \Omega_1 \). Then

\[
\begin{align*}
    k_i &= | \{ x \in \Omega_1 - \{y\} \mid \langle x, y \rangle = 2^{-1} \nu^i \} | \\
    &= | \{(x_1, \ldots, x_{2m}) \in V_{2m}(F) - \{y, -y\} \mid x_1 = 2^{-1}(\nu^i - x_{m+1}) \}, \\
    2(x_2 x_{m+2} + \cdots + x_m x_{2m}) &= 1 - 2x_1 x_{m+1} | \\
    &= \begin{cases} 
        q^{m-1}(q^{m-1} - 1) & \text{if } N[\nu^{i} x_{m+1} + 1 = 0] = 0, \\
        q^{m-1}(q^{m-1} + 1) & \text{if } N[\nu^{i} x_{m+1} + 1 = 0] = 2, \\
        (q^{m-1} - 1)(q^{m-1} + 1) & \text{if } N[\nu^{i} x_{m+1} + 1 = 0] = 1, 
    \end{cases}
\end{align*}
\]

using the fact that \( N[2(x_2 x_{m+2} + \cdots + x_m x_{2m}) = 1 - \nu^i x_{m+1} + x_{m+1}^2] = \lambda_0(2(m - 1)) \cdot N[1 - \nu^i x_{m+1} + x_{m+1}^2 = 0] + \lambda_1(2(m - 1))(q - N[1 - \nu^i x_{m+1} + x_{m+1}^2 = 0]). \) So, the assertion is true for all \( i = 1, 2, \ldots, \frac{q-1}{2}. \)

\[
\begin{align*}
    k_{x_{m+1}} &= | \{ x \in \Omega_1 \mid \langle x, y \rangle = 0 \} | \\
    &= \frac{1}{2} | \{(x_1, \ldots, x_{2m}) \in V_{2m}(F) \mid x_1 2^{-1} x_{m+1} = 0 \}, \\
    2(x_2 x_{m+2} + \cdots x_m x_{2m}) &= 1 + x_{m+1}^2 | \\
    &= \begin{cases} 
        \frac{1}{2} q^{m-1}(q^{m-1} + 1) & \text{if } q \equiv 1 (\text{mod } 4), \\
        \frac{1}{2} q^{m-1}(q^{m-1} - 1) & \text{if } q \equiv -1 (\text{mod } 4), 
    \end{cases}
\end{align*}
\]

because \( N[1 + x_{m+1}^2 = 0] = 0 \) if \( q \equiv -1 (\text{mod } 4), \) \( N[1 + x_{m+1}^2 = 0] = 2 \) if \( q \equiv 1 (\text{mod } 4). \)

Lemma 3.3.2 Let \( u^h_{ij} \) denote the parameters of \( \mathcal{X}(O_{2m}^+(q), \Omega_1) \) while \( d^h_{ij} \) denote those of \( \mathcal{X}(PSL(2, q)) \) as in Section 1.2. Then

\[
u^h_{ij} = 2q^{m-1}(q^{m-2} - 1) + q^{m-2}d^h_{ij},
\]

for all \( h, i, j \in \{1, 2, \ldots, \frac{q-1}{2}\} \) except the following cases:
where $i_0$ is the index for which $k_{i_0} = (q^{m-1} - 1)(q^{m-1} + 1)$.

**Proof** Set $x = 2^{-1}e_1 + e_{m+1}$, $y = 2^{-1}v^h e_1 + 2^{-1}e_2 + e_{m+2}$, then

$$u_{i_0}^{i_0} = 2q^{m-1}(q^{m-2} - 1) + q^{m-2}(d_{i_0}^{i_0} + 2) - 2,$$

$$u_{i_0}^{h} = u_{h_i_0}^{h} = 2q^{m-1}(q^{m-2} - 1) + q^{m-2}(d_{i_0}^{h} + 1) - 1, \text{ for } 1 \leq h \leq \frac{q-1}{2}, \ h \neq i_0$$

for $1 \leq h \leq i \leq j \leq \frac{q-1}{2}$. That is, $u_{ij}^{h}$ is equal to the number of vectors $(z_1, z_2, \ldots, z_{2m})$ in $V_{2m}(F) - \{x, y, -x, -y\}$ which satisfy

$$z_1 = 2^{-1}(\nu^i - z_{m+1}),$$

$$z_2 = 2^{-1}(\nu^j - \nu^h z_{m+1} - z_{m+2}),$$

$$2(z_3 z_{m+3} + \cdots + z_{2m} z_{2m}) = 1 - z_1 z_{m+1} - 2 z_2 z_{m+2}$$

$$= 1 - \nu^i z_{m+1} + \nu^j z_{m+2} + \nu^h z_{m+1} z_{m+2} + z_{m+2}^2.$$

So, in particular, if $h = i = j = i_0$, then

$$u_{i_0}^{i_0} = N[2(z_3 z_{m+3} + \cdots + z_{2m} z_{2m})$$

$$= 1 - 2 z_{m+1} + z_{m+1}^2 + 2 z_{m+2} + 2 z_{m+1} z_{m+2} + z_{m+2}^2] - 2$$

$$= \lambda_0(2(m-2)) \cdot (d_{i_0}^{i_0} + 2) + \lambda_1(2(m-2)) \cdot (2q^2 - d_{i_0}^{i_0} - 2) - 2$$

$$= q^{m-2}(d_{i_0}^{i_0} + 2) + 2q^{m-1}(q^{m-2} - 1) - 2$$

by Lemma 1.2.5 and Lemma 1.3.1. If $1 \leq h \leq \frac{q-1}{2}$ and $h \neq i_0$ then

$$u_{i_0}^{h} = N[2(z_3 z_{m+3} + \cdots + z_{2m} z_{2m})$$

$$= 1 - 2 z_{m+1} + z_{m+1}^2 + \nu^h z_{m+2} + \nu^h z_{m+1} z_{m+2} + z_{m+2}^2] - 1$$

$$= q^{m-2}(d_{i_0}^{h} + 1) + 2q^{m-1}(q^{m-2} - 1) - 1.$$
Similarly, we can check all the remaining cases by Lemma 1.2.5 and Lemma 1.3.1.

Now we are ready to describe the character table of $\chi(O_{2m}^{\pm}(q), \Omega_1)$.

**Theorem 3.3.3.** Let $\sigma$ denote a primitive $(q + 1)$th root of 1 and $\rho$ denote a primitive $(q - 1)$th root of 1, and let $a = \frac{q-1}{4}$ and $b = \frac{q-3}{4}$. Then the character table of $\chi(O_{2m}^{\pm}(q), \Omega_1)$ is equivalent, up to a suitable permutation on its columns and rows, to one of the matrices in Tables 11 and 12 depending on $q$;

(i) $q \equiv 1 (\text{mod } 4)$, see Table 11;

(ii) $q \equiv -1 (\text{mod } 4)$, see Table 12.

**Remark.** Comparing the matrices $\tilde{P}$ with the character table $P$ of $\chi'(PSL(2, q))$, we observe that

$$\tilde{p}_j(i) = \begin{cases} q^{m-2} \cdot p_j(i) & \text{if } j \geq 2, \\ q^{m-2}(p_1(i) + 1) - 1 & \text{if } j = 1, \end{cases}$$

for all $i \geq 1$. Now Theorem 3.3.3 is proved in the same manner as before.

**3.4. The character table of $\chi(O_{2m}^{-}(p^n), \Theta)$, $p$ an odd prime.**

In this section let $V$ be a $2m$-dimensional vector space over a finite field $F = GF(q)$, $q = p^n$, $p$ an odd prime. Let $\Theta_1$ (resp. $\Theta_2$) be the set of all non-singular square-type (resp. non-square-type) 1-dimensional subspaces of $V$ with respect to the non-singular quadratic form

$$f(x) = f_2(x) = 2(x_2x_m + x_2x_{m+1} + \cdots + x_{m-1}x_{2(m-1)}) + x_{2m-1}^2 - \alpha x_{2m}^2.$$
Then \(|\Theta_1| = |\Theta_2| = \frac{1}{2}q^{m-1}(q^m + 1)\). Let \(O_{2m}^-(q)\) be the orthogonal group with respect to \(f\). Let \(x\) denote an element of \(\Theta_1\) as well as its representing vector \(x\) with \(f(x) = 1\).

\(O_{2m}^-(q)\) acts transitively on \(\Theta_1\) and the orbits of \(O_{2m}^-(q)\) on \(\Theta_1 \times \Theta_1\) are given by the \(R_i\)'s,

\[
R_0 = \{(x,x) \mid x \in \Theta_1\},
\]
\[
R_j = \{(x,y) \mid (x,y) = 2^{-1}v^j, \ x, y \in \Theta_1\},
\]
for \(j = 1, 2, \ldots, \frac{q-1}{2}\),

\[
R_{\frac{q+1}{2}} = \{(x,y) \mid (x,y) = 0, \ x, y \in \Theta_1\}.
\]

Then we have a symmetric association scheme

\[
\mathcal{X}(O_{2m}^-(q), \Theta_1) = (\Theta_1, \{R_i\}_{0 \leq i \leq \frac{q+1}{2}}).
\]

We note that similarly we have an association scheme from the transitive action of \(O_{2m}^-(q)\) on \(\Theta_2\) which is isomorphic to \(\mathcal{X}(O_{2m}^-(q), \Theta_1)\).

**Lemma 3.4.1.** In the association scheme \(\mathcal{X}(O_{2m}^-(q), \Theta_1)\), we have

\[
k_i = \begin{cases} 
q^{m-1}(q^{m-1} + 1) & \text{if } N[t^2 - v^j t + 1 = 0] = 0, \\
q^{m-1}(q^{m-1} - 1) & \text{if } N[t^2 - v^j t + 1 = 0] = 2, \\
(q^{m-1} + 1)(q^{m-1} - 1) & \text{if } N[t^2 - v^j t + 1 = 0] = 1,
\end{cases}
\]
for \(i = 1, 2, \ldots, \frac{q-1}{2}\),
Lemma 3.4.2. Let \( v_{ij}^h \) denote the parameters of the association scheme \( X(\mathcal{O}_{2m}(q), \Theta_1) \) while \( d_{ij}^h \) denote those of \( X(\text{PSL}(2, q)) \). Then

\[
v_{ij}^h = 2q^{m-1}(q^{m-2} + 1) - q^{m-2} \cdot d_{ij}^h,
\]

for all \( h, i, j \in \{1, 2, \ldots, \frac{q-1}{2}\} \) except the following cases:

\[
v_{i_0i_0}^h = 2q^{m-1}(q^{m-2} + 1) - q^{m-2}(d_{i_0i_0}^h + 2) - 2,
\]

\[
v_{i_0i_0}^h = v_{i_0i_0}^h = 2q^{m-1}(q^{m-2} + 1) - q^{m-2}(d_{i_0i_0}^h + 1) - 1,
\]

for \( 1 \leq h \leq \frac{q-1}{2}, h \neq i_0 \) where \( i_0 \) is the index for which \( k_{i_0} = (q^{m-1} + 1)(q^{m-1} - 1) \).

Proof. Omitted. \( \square \)

Through the exactly same procedure we have done for the Theorem 3.3.3, and performing similar computations we have done for the Theorem 2.2.3, we have the following result.

Theorem 3.4.3. Let \( \sigma \) and \( \rho \) denote a primitive \((q+1)\)th root and a primitive \((q-1)\)th root of 1, respectively, and denote \( a = \frac{q-1}{4}, b = \frac{q-3}{4} \). Then the character table of the association scheme \( X(\mathcal{O}_{2m}(q), \Theta_1) \) is equivalent to one of the matrices \( \bar{P}^- \) in Tables 13 and 14, up to a permutation on its columns and rows.

(i) \( q \equiv 1 (mod 4) \), see Table 13;
(ii) $q \equiv -1 \pmod{4}$, see Table 14.

Remark. Comparing $\bar{P}$ and $\bar{P}$ in Theorem 3.3.3, we can see that $\bar{P}$ can be obtained from $\bar{P}$ by replacing all $q^{m-1}$ and $q^{m-2}$ by $-q^{m-1}$ and $-q^{m-2}$ and vice versa.
LIST OF REFERENCES


TABLE 1

The character table of $\chi(SL(2, 2^n))$

$$
P = 
\begin{pmatrix}
1 & (q + 1)(q - 1) & q(q - 1) & q(q - 1) & \ldots & q(q - 1) & q(q + 1) & q(q + 1) & \ldots & q(q + 1) \\
1 & 0 & -(q - 1) & -(q - 1) & \ldots & -(q - 1) & q + 1 & q + 1 & \ldots & q + 1 \\
1 & -(g + 1) & -q(\sigma + \sigma^{-1}) & -q(\sigma^2 + \sigma^{-2}) & \ldots & -q(\sigma^e + \sigma^{-e}) & 0 & 0 & \ldots & 0 \\
1 & -(g + 1) & -q(\sigma^2 + \sigma^{-2}) & -q(\sigma^4 + \sigma^{-4}) & \ldots & -q(\sigma^{2e} + \sigma^{-2e}) & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & -(g + 1) & -q(\sigma^e + \sigma^{-e}) & -q(\sigma^{2e} + \sigma^{-2e}) & \ldots & -q(\sigma^{e^2} + \sigma^{-e^2}) & 0 & 0 & \ldots & 0 \\
1 & g - 1 & 0 & 0 & \ldots & 0 & q(\rho + \rho^{-1}) & q(\rho^2 + \rho^{-2}) & \ldots & q(\rho^d + \rho^{-d}) \\
1 & g - 1 & 0 & 0 & \ldots & 0 & q(\rho^2 + \rho^{-2}) & q(\rho^4 + \rho^{-4}) & \ldots & q(\rho^{2d} + \rho^{-2d}) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & g - 1 & 0 & 0 & \ldots & 0 & q(\rho^d + \rho^{-d}) & q(\rho^{2d} + \rho^{-2d}) & \ldots & q(\rho^{e^2} + \rho^{-e^2})
\end{pmatrix}
$$
TABLE 2

The character table of $\bar{X}(SL(2, p^n))$, $p$ an odd prime

$$

table

\begin{array}{cccccccccccc}
1 & 1 & q^2 - 1 & q^2 - 1 & q(q - 1) & q(q - 1) & \cdots & q(g - 1) & q(g + 1) & q(g + 1) & \cdots & q(g + 1) \\
1 & 1 & 0 & 0 & -(g - 1) & -(g - 1) & \cdots & -(g - 1) & g + 1 & g + 1 & \cdots & g + 1 \\
1 & -1 & -(g + 1) & (-1)^2(g + 1) & -(\sigma + \sigma^{-1}) & -(\sigma^2 + \sigma^{-2}) & \cdots & -(\sigma^e + \sigma^{-e}) & 0 & 0 & \cdots & 0 \\
1 & (-1)^2 & -(g + 1) & (-1)^3(g + 1) & -(\sigma^2 + \sigma^{-2}) & -(\sigma^4 + \sigma^{-4}) & \cdots & -(\sigma^{2e} + \sigma^{-2e}) & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
1 & (-1)^e & -(g + 1) & (-1)^{e+1}(g + 1) & -(\sigma^e + \sigma^{-e}) & -(\sigma^{2e} + \sigma^{-2e}) & \cdots & -(\sigma^{e^2} + \sigma^{-e^2}) & 0 & 0 & \cdots & 0 \\
1 & -\epsilon & -(g + 1) & (g + 1)\epsilon & (-1)^22q & (-1)^32q & \cdots & (-1)^{e+1}2q & 0 & 0 & \cdots & 0 \\
1 & \epsilon & g - 1 & (g - 1)\epsilon & 0 & 0 & \cdots & 0 & -2q & (-1)^2q & \cdots & (-1)^{2d}q \\
1 & -1 & g - 1 & -(g - 1) & 0 & 0 & \cdots & 0 & q(\rho + \rho^{-1}) & q(\rho^2 + \rho^{-2}) & \cdots & q(\rho^d + \rho^{-d}) \\
1 & (-1)^2 & g - 1 & (-1)^2(g - 1) & 0 & 0 & \cdots & 0 & q(\rho^2 + \rho^{-2}) & q(\rho^4 + \rho^{-4}) & \cdots & q(\rho^{2d} + \rho^{-2d}) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
1 & (-1)^d & g - 1 & (-1)^d(g - 1) & 0 & 0 & \cdots & 0 & q(\rho^d + \rho^{-d}) & q(\rho^{2d} + \rho^{-2d}) & \cdots & q(\rho^{2d} + \rho^{-2d}) \\
\end{array}
$$
**TABLE 3**

The character table of $\overline{X}(PSL(2, p^n))$, $p^n \equiv 1 \pmod{4}$

$$
\begin{align*}
\text{P} = &\begin{pmatrix}
1 & q^2 - 1 & q(q - 1) & q(q - 1) & \cdots & q(q - 1) & q(q + 1) & q(q + 1) & \cdots & q(q + 1) & \frac{1}{2} q(q + 1) \\
1 & 0 & -(q - 1) & -(q - 1) & \cdots & -(q - 1) & q + 1 & q + 1 & \cdots & q + 1 & \frac{1}{2} (q + 1) \\
1 & -(q + 1) & -q(a^2 + a^{-2}) & -q(a^4 + a^{-4}) & \cdots & -q(a^{2a} + a^{-2a}) & 0 & 0 & \cdots & 0 & 0 \\
1 & -(q + 1) & -q(a^4 + a^{-4}) & -q(a^8 + a^{-8}) & \cdots & -q(a^{4a} + a^{-4a}) & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & -(q + 1) & -q(a^{2a} + a^{-2a}) & -q(a^{4a} + a^{-4a}) & \cdots & -q(a^{2a^2} + a^{-2a^2}) & 0 & 0 & \cdots & 0 & 0 \\
1 & q - 1 & 0 & 0 & \cdots & 0 & q(p^2 + p^{-2}) & q(p^4 + p^{-4}) & \cdots & q(p^{2(a-1)} + p^{-2(a-1)}) & \frac{1}{2} q(p^{2a} + p^{-2a}) \\
1 & q - 1 & 0 & 0 & \cdots & 0 & q(p^4 + p^{-4}) & q(p^8 + p^{-8}) & \cdots & q(p^{4(a-1)} + p^{-4(a-1)}) & \frac{1}{2} q(p^{4a} + p^{-4a}) \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & q - 1 & 0 & 0 & \cdots & 0 & q(p^{2(a-1)} + p^{-2(a-1)}) & q(p^{2(a-1)} + p^{-2(a-1)}) & \cdots & q(p^{2(a-1)^2} + p^{-2(a-1)^2}) & \frac{1}{2} q(p^{2(a-1)^2} + p^{-2(a-1)^2}) \\
1 & q - 1 & 0 & 0 & \cdots & 0 & q(p^{2a} + p^{-2a}) & q(p^{2a} + p^{-2a}) & \cdots & q(p^{2a(a-1)} + p^{-2a(a-1)}) & \frac{1}{2} q(p^{2a(a-1)} + p^{-2a(a-1)})
\end{pmatrix}
\end{align*}
$$
### TABLE 4

The character table of $\overline{\chi}(PSL(2, p^n))$, $p^n \equiv -1 \pmod{4}$

$$
P = \begin{pmatrix}
1 & g^2 - 1 & g(g - 1) & g(g - 1) & \ldots & g(g - 1) & 1/2g(g - 1) & g(g + 1) & g(g + 1) & \ldots & g(g + 1) \\
1 & 0 & -(q - 1) & -(q - 1) & \ldots & -(q - 1) & -1/2(q - 1) & q + 1 & q + 1 & \ldots & q + 1 \\
1 & -(q + 1) & -q(a^2 + a^{-2}) & -q(a^4 + a^{-4}) & \ldots & -q(a^{2k} + a^{-2k}) & -1/2q(a^{2k+1} + a^{-2k+1}) & 0 & 0 & \ldots & 0 \\
1 & -(q + 1) & -q(a^2 + a^{-2}) & -q(a^4 + a^{-4}) & \ldots & -q(a^{2k} + a^{-2k}) & -1/2q(a^{2k+1} + a^{-2k+1}) & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & -(q + 1) & -q(a^2 + a^{-2}) & -q(a^4 + a^{-4}) & \ldots & -q(a^{2k} + a^{-2k}) & -1/2q(a^{2k+1} + a^{-2k+1}) & 0 & 0 & \ldots & 0 \\
1 & q - 1 & 0 & 0 & \ldots & 0 & 0 & q(a^2 + a^{-2}) & q(a^4 + a^{-4}) & \ldots & q(a^{2k} + a^{-2k}) \\
1 & q - 1 & 0 & 0 & \ldots & 0 & 0 & q(a^2 + a^{-2}) & q(a^4 + a^{-4}) & \ldots & q(a^{2k} + a^{-2k}) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & q - 1 & 0 & 0 & \ldots & 0 & 0 & q(a^2 + a^{-2}) & q(a^4 + a^{-4}) & \ldots & q(a^{2k} + a^{-2k})
\end{pmatrix}
$$
\begin{table}
\caption{The character table of $X(M(2^n))$}

\[
\begin{bmatrix}
1 & (q^3 + 1)(q^3 - 1) & q^3(q^3 - 1) & q^3(q^3 - 1) & \ldots & q^3(q^3 - 1) & q^3(q^3 + 1) & q^3(q^3 + 1) & \ldots & q^3(q^3 + 1) \\
1 & q^2 - 1 & -q^2(q - 1) & -q^2(q - 1) & \ldots & -q^2(q - 1) & q^2(q + 1) & q^2(q + 1) & \ldots & q^2(q + 1) \\
1 & -(q^3 + 1) & -q^3(\sigma + \sigma^{-1}) & -q^3(\sigma^2 + \sigma^{-2}) & \ldots & -q^3(\sigma^e + \sigma^{-e}) & 0 & 0 & \ldots & 0 \\
1 & -(q^3 + 1) & -q^3(\sigma^2 + \sigma^{-2}) & -q^3(\sigma^4 + \sigma^{-4}) & \ldots & -q^3(\sigma^{2e} + \sigma^{-2e}) & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
1 & -q^3 + 1 & -q^3(\sigma^e + \sigma^{-e}) & -q^3(\sigma^{2e} + \sigma^{-2e}) & \ldots & -q^3(\sigma^{e2} + \sigma^{-e2}) & 0 & 0 & \ldots & 0 \\
1 & q^3 - 1 & 0 & 0 & \ldots & 0 & q^3(\rho + \rho^{-1}) & q^3(\rho^2 + \rho^{-2}) & \ldots & q^3(\rho^{d} + \rho^{-d}) \\
1 & q^3 - 1 & 0 & 0 & \ldots & 0 & q^3(\rho^2 + \rho^{-2}) & q^3(\rho^4 + \rho^{-4}) & \ldots & q^3(\rho^{2d} + \rho^{-2d}) \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
1 & q^3 - 1 & 0 & 0 & \ldots & 0 & q^3(\rho^d + \rho^{-d}) & q^3(\rho^{2d} + \rho^{-2d}) & \ldots & q^3(\rho^{e2} + \rho^{-e2}) \\
\end{bmatrix}
\end{table}
TABLE 6

The character table of \( X(M(p^n)), p \) an odd prime

\[
\begin{bmatrix}
1 & 1 & q^6 - 1 & q^6 & q^2(q^3 - 1) & q^2(q^3 - 1) & \ldots & q^2(q^3 - 1) & q^2(q^3 + 1) & q^2(q^3 - 1) & \ldots & q^2(q^3 + 1) \\
1 & 1 & q^2 - 1 & q^2 & -q^2(q - 1) & -q^2(q - 1) & \ldots & -q^2(q - 1) & q^2(q - 1) & q^2(q - 1) & \ldots & q^2(q - 1) \\
1 & -1 & -(q^3 + 1) & -(q^3 + 1) & -q^3(q + 1) & -q^3(q + 1) & \ldots & -q^3(q + 1) & q^3(q + 1) & q^3(q + 1) & \ldots & q^3(q + 1) \\
1 & -(1)^2 & -(q^3 + 1) & -(q^3 + 1) & -q^3(q^2 + 1) & -q^3(q^2 + 1) & \ldots & -q^3(q^2 + 1) & q^3(q^2 + 1) & q^3(q^2 + 1) & \ldots & q^3(q^2 + 1) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\
1 & (-1)^e & -(q^3 + 1) & -(q^3 + 1) & -q^3(q^2 + 1) & -q^3(q^2 + 1) & \ldots & -q^3(q^2 + 1) & q^3(q^2 + 1) & q^3(q^2 + 1) & \ldots & q^3(q^2 + 1) \\
1 & -\varepsilon & -(q^3 + 1)\varepsilon & (q^3 + 1)\varepsilon & -(1)^32q^3 & -(1)^32q^3 & \ldots & -(1)^32q^3 & (1)^32q^3 & (1)^32q^3 & \ldots & (1)^32q^3 \\
1 & \varepsilon & q^3 - 1 & (q^3 - 1)\varepsilon & 0 & 0 & \ldots & 0 & -2q^3 & (1)^32q^3 & \ldots & (1)^32q^3 \\
1 & -1 & q^3 - 1 & -(q - 1) & 0 & 0 & \ldots & 0 & q^3(q + 1) & q^3(q + 1) & \ldots & q^3(q + 1) \\
1 & -(1)^2 & q^3 - 1 & -(1)^2(q^3 - 1) & 0 & 0 & \ldots & 0 & q^3(q^2 + 1) & q^3(q^2 + 1) & \ldots & q^3(q^2 + 1) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\
1 & (-1)^d & q^3 - 1 & -(1)^d(q^3 - 1) & 0 & 0 & \ldots & 0 & q^3(q^2 + 1) & q^3(q^2 + 1) & \ldots & q^3(q^2 + 1) \\
\end{bmatrix}
\]
TABLE 7

The character table of $X(M(p^n))$, $p^n \equiv 1 \pmod{4}$

\[
\begin{array}{cccccccc}
1 & q^2 - 1 & q^3(q^2 - 1) & q^3(q^2 + 1) & \ldots & q^3(q^2 + 1) & \ldots & q^3(q^2 + 1) & \frac{1}{2}q^3(q^2 + 1) \\
1 & q^2 - 1 & -q^2(q - 1) & q^2(q + 1) & \ldots & q^2(q + 1) & \ldots & q^2(q + 1) & q^2(q + 1) \\
1 & -(q^2 + 1) & -q^2(q^2 + q^{-3}) & -q^2(q^2 + q^{-3}) & \ldots & 0 & \ldots & 0 & 0 \\
1 & -(q^2 + 1) & -q^2(q^2 + q^{-3}) & -q^2(q^2 + q^{-3}) & \ldots & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
1 & -(q^2 + 1) & -q^2(q^2 + q^{-3}) & -q^2(q^2 + q^{-3}) & \ldots & 0 & \ldots & 0 & 0 \\
1 & q^2 - 1 & 0 & 0 & \ldots & 0 & q^2(q^2 + q^{-2}) & \ldots & q^2(q^2 + q^{-2}) \\
1 & q^2 - 1 & 0 & 0 & \ldots & 0 & q^2(q^2 + q^{-2}) & \ldots & q^2(q^2 + q^{-2}) \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
1 & q^2 - 1 & 0 & 0 & \ldots & 0 & q^2(q^2 + q^{-2}) & \ldots & q^2(q^2 + q^{-2}) \\
1 & q^2 - 1 & 0 & 0 & \ldots & 0 & q^2(q^2 + q^{-2}) & \ldots & q^2(q^2 + q^{-2}) \\
\end{array}
\]
TABLE 8

The character table of \( \mathcal{X}(M^*(p^n)) \), \( p^n \equiv -1 (\text{mod} 4) \)

\[
\hat{P} = \begin{pmatrix}
1 & g^0 - 1 & g^1(g^2 - 1) & g^2(g^2 - 1) & \cdots & g^l(g^2 - 1) & g^3(g^2 + 1) & g^4(g^2 + 1) & \cdots & g^l(g^2 + 1) \\
1 & g^2 - 1 & g^3(g - 1) & g^4(g - 1) & \cdots & g^l(g - 1) & g^3(g + 1) & g^4(g + 1) & \cdots & g^l(g + 1) \\
1 & -(g^3 + 1) & -g^2(a^2 + o^2) & -g(a^4 + o^2) & \cdots & -g^3(a^{2k+1} + o^{-2k-1}) & 0 & 0 & \cdots & 0 \\
1 & -(g^3 + 1) & -g^2(a^4 + o^3) & -g^4(a^4 + o^3) & \cdots & -g^3(a^{2k+1} + o^{-2k-1}) & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
1 & -(g^3 + 1) & -g^2(a^{2k+1} + o^{2k+1}) & -g^4(a^{2k+1} + o^{2k+1}) & \cdots & -g^3(a^{2k+1} + o^{-2k-1}) & 0 & 0 & \cdots & 0 \\
1 & g^1 - 1 & 0 & 0 & \cdots & 0 & g^1(p^2 + p^{-2}) & g^4(p^4 + p^{-4}) & \cdots & g^l(p^{2k} + p^{-2k}) \\
1 & g^2 - 1 & 0 & 0 & \cdots & 0 & g^1(p^2 + p^{-2}) & g^4(p^4 + p^{-4}) & \cdots & g^l(p^{2k} + p^{-2k}) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
1 & g^3 - 1 & 0 & 0 & \cdots & 0 & g^1(p^2 + p^{-2}) & g^4(p^4 + p^{-4}) & \cdots & g^l(p^{2k} + p^{-2k}) \\
\end{pmatrix}
\]
The character table of $\Gamma(0_2(2^n), G)$

\begin{table}[h]
\begin{tabular}{cccccccc}
\hline
& $1$ & $
u_1$ & $\cdots$ & $\nu_{n-1}$ & $\nu_n$ & $\cdots$ & $\nu_{2n-2}$ & $\nu_{2n-1}$ \\
\hline
$1$ & $1$ & $1$ & $\cdots$ & $1$ & $1$ & $\cdots$ & $1$ & $1$ \\
$\nu_1$ & $1$ & $1$ & $\cdots$ & $1$ & $1$ & $\cdots$ & $1$ & $1$ \\
$\nu_n$ & $1$ & $1$ & $\cdots$ & $1$ & $1$ & $\cdots$ & $1$ & $1$ \\
$\nu_{2n-1}$ & $1$ & $1$ & $\cdots$ & $1$ & $1$ & $\cdots$ & $1$ & $1$ \\
\hline
\end{tabular}
\end{table}
TABLE 10

The character table of $X(O_{2m}^*(2^n), \Theta)$

\[
\begin{pmatrix}
1 & (q^{m-1} - 1)(q^{m-1} + 1) & q^{m-1}(q^{m-1} + 1) & \cdots & q^{m-1}(q^{m-1} + 1) & q^{m-1}(q^{m-1} - 1) & q^{m-1}(q^{m-1} - 1) & \cdots & q^{m-1}(q^{m-1} - 1) \\
1 & -q^{m-2} - 1 & q^{m-2}(q^{m-1} - 1) & \cdots & q^{m-2}(q^{m-1} - 1) & -q^{m-2}(q^{m-1} + 1) & -q^{m-2}(q^{m-1} + 1) & \cdots & -q^{m-2}(q^{m-1} + 1) \\
1 & q^{m-1} - 1 & q^{m-1}(\sigma + \sigma^{-1}) & q^{m-1}(\sigma^2 + \sigma^{-2}) & \cdots & q^{m-1}(\sigma^2 + \sigma^{-2}) & 0 & 0 & \cdots & 0 \\
1 & q^{m-1} - 1 & q^{m-1}(\sigma^2 + \sigma^{-2}) & q^{m-1}(\sigma^4 + \sigma^{-4}) & \cdots & q^{m-1}(\sigma^2 + \sigma^{-2}) & 0 & 0 & \cdots & 0 \\
& \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & q^{m-1} - 1 & q^{m-1}(\sigma^2 + \sigma^{-2}) & q^{m-1}(\sigma^2 + \sigma^{-2}) & \cdots & q^{m-1}(\sigma^2 + \sigma^{-2}) & 0 & 0 & \cdots & 0 \\
1 & -(q^{m-1} + 1) & 0 & 0 & \cdots & 0 & -q^{m-1}(\rho + \rho^{-1}) & -q^{m-1}(\rho^2 + \rho^{-2}) & \cdots & -q^{m-1}(\rho^d + \rho^{-d}) \\
1 & -(q^{m-1} + 1) & 0 & 0 & \cdots & 0 & -q^{m-1}(\rho^2 + \rho^{-2}) & -q^{m-1}(\rho^4 + \rho^{-4}) & \cdots & -q^{m-1}(\rho^{2d} + \rho^{-2d}) \\
& \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & -(q^{m-1} + 1) & 0 & 0 & \cdots & 0 & -q^{m-1}(\rho^d + \rho^{-d}) & -q^{m-1}(\rho^{2d} + \rho^{-2d}) & \cdots & -q^{m-1}(\rho^{2d} + \rho^{-2d})
\end{pmatrix}
\]
TABLE II

The character table of $X(O_{2m}^+(p^n), \Omega_1)$, $p^n \equiv 1 \pmod{4}$

\[
\begin{pmatrix}
1 & (q^{m-1} + 1)(q^{m-1} - 1) & q^{m-1}(q^{m-1} - 1) & \ldots & q^{m-1}(q^{m-1} + 1) & q^{m-1}(q^{m-1} + 1) & \frac{1}{2} q^{m-1}(q^{m-1} + 1) \\
1 & q^{m-2} - 1 & -q^{m-2}(q - 1) & \ldots & -q^{m-2}(q + 1) & q^{m-2}(q + 1) & \frac{1}{2} q^{m-2}(q + 1) \\
1 & -(q^{m-1} + 1) & -q^{m-1}(q^2 + q^{-2}) & \ldots & -q^{m-1}(q^2 + q^{-2}) & 0 & 0 \\
1 & -(q^{m-1} + 1) & -q^{m-1}(q^4 + q^{-4}) & \ldots & -q^{m-1}(q^4 + q^{-4}) & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & -(q^{m-1} + 1) & -q^{m-1}(q^{2a} + q^{-2a}) & \ldots & -q^{m-1}(q^{2a} + q^{-2a}) & 0 & 0 \\
1 & q^{m-1} - 1 & 0 & \ldots & 0 & q^{m-1}(p^2 + p^{-2}) & \frac{1}{2} (p^2 + p^{-2}) \\
1 & q^{m-1} - 1 & 0 & \ldots & 0 & q^{m-1}(p^4 + p^{-4}) & \frac{1}{2} (p^4 + p^{-4}) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & q^{m-1} - 1 & 0 & \ldots & 0 & q^{m-1}(p^{2(a-1)} + p^{-2(a-1)}) & \frac{1}{2} (p^{2(a-1)} + p^{-2(a-1)}) \\
\end{pmatrix}
\]
TABLE 12

The character table of $\mathcal{X}(O_{2m}^+(p^n), \Omega_1)$, $p^n \equiv -1 \ (mod\ 4)$

$$
\bar{P} = \begin{pmatrix}
1 & (q^{m-1} - 1)(q^{m-1} + 1) & q^{m-1}(q^{m-1} - 1) & ... & q^{m-1}(q^{m-1} - 1) & \frac{q^{m-1}}{2}(q^{m-1} - 1) & q^{m-1}(q^{m-1} + 1) & \ldots & \frac{q^{m-1}}{2}(q^{m-1} + 1) & \frac{q^{m-1}}{2}(q^{m-1} + 1) & q^{m-1}(q^{m-1} + 1) \\
1 & q^{m-2} - 1 & -q^{m-2}(q - 1) & ... & -q^{m-2}(q - 1) & \frac{q^{m-2}}{2}(q - 1) & q^{m-2}(q + 1) & \ldots & q^{m-2}(q + 1) & q^{m-2}(q + 1) & q^{m-2}(q + 1) \\
1 & -(q^{m-1} + 1) & -q^{m-1}(\sigma^2 + \sigma^{-2}) & ... & -q^{m-1}(\sigma^2 + \sigma^{-2}) & -\frac{q^{m-1}}{2}(\sigma^{2k+1} + \sigma^{-2k+1}) & 0 & ... & 0 & 0 & 0 \\
1 & -(q^{m-1} + 1) & -q^{m-1}(\sigma^4 + \sigma^{-4}) & ... & -q^{m-1}(\sigma^4 + \sigma^{-4}) & -\frac{q^{m-1}}{2}(\sigma^{4k+1} + \sigma^{-4k+1}) & 0 & ... & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
1 & -(q^{m-1} + 1) & -q^{m-1}(\sigma^{2k+1} + \sigma^{-2k+1}) & ... & -q^{m-1}(\sigma^{2k+1} + \sigma^{-2k+1}) & -\frac{q^{m-1}}{2}(\sigma^{2k+1} + \sigma^{-2k+1}) & 0 & ... & 0 & \ldots & \ldots \\
1 & q^{m-1} - 1 & 0 & ... & 0 & 0 & q^{m-1}(\rho^2 + \rho^{-2}) & ... & q^{m-1}(\rho^{2k} + \rho^{-2k}) & q^{m-1}(\rho^{2k} + \rho^{-2k}) & q^{m-1}(\rho^{2k} + \rho^{-2k}) \\
1 & q^{m-1} - 1 & 0 & ... & 0 & 0 & q^{m-1}(\rho^4 + \rho^{-4}) & ... & q^{m-1}(\rho^{4k} + \rho^{-4k}) & q^{m-1}(\rho^{4k} + \rho^{-4k}) & q^{m-1}(\rho^{4k} + \rho^{-4k}) \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
1 & q^{m-1} - 1 & 0 & ... & 0 & 0 & q^{m-1}(\rho^{2k} + \rho^{-2k}) & ... & q^{m-1}(\rho^{2k} + \rho^{-2k}) & q^{m-1}(\rho^{2k} + \rho^{-2k}) & q^{m-1}(\rho^{2k} + \rho^{-2k}) \\
\end{pmatrix}
$$
TABLE 13

The character table of $X(O_{2m}^{-}(p^n), \Theta_1)$, $p^n \equiv 1 (\mod 4)$

$$
\begin{pmatrix}
1 & q^{m-1} - 1(q^{m-1} + 1) & q^{m-1}(q^{m-1} + 1) & \ldots & q^{m-1}(q^{m-1} + 1) & q^{m-1}(q^{m-1} - 1) & \ldots & q^{m-1}(q^{m-1} - 1) & \frac{1}{2}q^{m-1}(q^{m-1} - 1) \\
1 & -q^{m-2} - 1 & q^{m-2}(q - 1) & \ldots & q^{m-2}(q - 1) & -q^{m-2}(q + 1) & \ldots & -q^{m-2}(q + 1) & -\frac{1}{2}q^{m-2}(q + 1) \\
1 & q^{m-1} - 1 & q^{m-1}(q^2 + \sigma^{-2}) & \ldots & q^{m-1}(q^2 + \sigma^{-2}) & 0 & \ldots & 0 & 0 \\
1 & q^{m-1} - 1 & q^{m-1}(q^4 + \sigma^{-4}) & \ldots & q^{m-1}(q^4 + \sigma^{-4}) & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
1 & -(q^{m-1} + 1) & 0 & \ldots & 0 & -q^{m-1}(\rho^2 + \rho^{-2}) & \ldots & -q^{m-1}(\rho^2 + \rho^{-2}) & -\frac{1}{2}q^{m-1}(\rho^2 + \rho^{-2}) \\
1 & -(q^{m-1} + 1) & 0 & \ldots & 0 & -q^{m-1}(\rho^4 + \rho^{-4}) & \ldots & -q^{m-1}(\rho^4 + \rho^{-4}) & -\frac{1}{2}q^{m-1}(\rho^4 + \rho^{-4}) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & -(q^{m-1} + 1) & 0 & \ldots & 0 & -q^{m-1}(\rho^{2(a+1)} + \rho^{-2(a+1)}) & \ldots & -q^{m-1}(\rho^{2(a+1)} + \rho^{-2(a+1)}) & -\frac{1}{2}(\rho^{2(a+1)} + \rho^{-2(a+1)}) \\
1 & -(q^{m-1} + 1) & 0 & \ldots & 0 & -q^{m-1}(\rho^{2a} + \rho^{-2a}) & \ldots & -q^{m-1}(\rho^{2a} + \rho^{-2a}) & -\frac{1}{2}(\rho^{2a} + \rho^{-2a})
\end{pmatrix}
$$
TABLE 14

The character table of $X(O_{2m}(p^n), \Theta_1)$, $p^n \equiv -1 \pmod{4}$

$$
\begin{array}{cccccccc}
& 1 & (q^{m-1} - 1)(q^{m-1} + 1) & q^{m-1}(q^{m-1} + 1) & \ldots & q^{m-1}(q^{m-1} + 1) & \frac{q^{m-1}}{2}(q^{m-1} + 1) & q^{m-1}(q^{m-1} - 1) & q^{m-1}(q^{m-1} - 1) \\
1 & q^{m-2} - 1 & q^{m-2}(q - 1) & \ldots & q^{m-2}(q - 1) & \frac{q^{m-2}}{2}(q - 1) & -q^{m-2}(q + 1) & \ldots & -q^{m-2}(q + 1) \\
1 & q^{m-1} - 1 & q^{m-1}(a^2 + a^{-2}) & \ldots & q^{m-1}(a^2 + a^{-2}) & \frac{q^{m-1}}{2}(a^3(b+1) + a^{-3}(b+1)) & 0 & \ldots & 0 \\
1 & q^{m-1} - 1 & q^{m-1}(a^6 + a^{-6}) & \ldots & q^{m-1}(a^6 + a^{-6}) & \frac{q^{m-1}}{2}(a^4(b+1) + a^{-4}(b+1)) & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & q^{m-1} - 1 & q^{m-1}(a^2b^2 + a^{-2}b^{-2}) & \ldots & q^{m-1}(a^2b^2 + a^{-2}b^{-2}) & \frac{q^{m-1}}{2}(a^2b^2(b+1) + a^{-2}b^{-2}(b+1)) & 0 & \ldots & 0 \\
1 & q^{m-1} - 1 & q^{m-1}(a^3b^3 + a^{-3}b^{-3}) & \ldots & q^{m-1}(a^3b^3 + a^{-3}b^{-3}) & \frac{q^{m-1}}{2}(a^3b^3(b+1)^2 + a^{-3}b^{-3}(b+1)^2) & 0 & \ldots & 0 \\
1 & -(q^{m-1} + 1) & 0 & \ldots & 0 & 0 & -q^{m-1}(\rho^2 + \rho^{-2}) & \ldots & -q^{m-1}(\rho^2 + \rho^{-2}) \\
1 & -(q^{m-1} + 1) & 0 & \ldots & 0 & 0 & -q^{m-1}(\rho^4 + \rho^{-4}) & \ldots & -q^{m-1}(\rho^4 + \rho^{-4}) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & -(q^{m-1} + 1) & 0 & \ldots & 0 & 0 & -q^{m-1}(\rho^6 + \rho^{-6}) & \ldots & -q^{m-1}(\rho^6 + \rho^{-6})
\end{array}
$$