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An options model of employee-firm contracts

Mahle, Stephen E., Ph.D.
The Ohio State University, 1987

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AN OPTIONS MODEL OF EMPLOYEE-FIRM CONTRACTS

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By

Stephen E. Mahle, B.A., M.A.

The Ohio State University

1987

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Advisor
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To Wendy and Jennifer, who know who they are and know why this work is dedicated to them. 100%.
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CHAPTER I

INTRODUCTION

Along with the specialization and division of labor that characterize a developed economy comes the need for contracts that specify (ex-ante) mutually agreed upon behaviors of economic agents. Labor-market contracts are among the most common of these, and as labor markets have become more sophisticated, so have compensation systems and the labor contracts that specify them. Virtually all individuals who do market work do so within the specifications of an (implicit or explicit) labor contract, and many of these contracts and attendant work settings are rather long-lived. Among American workers, seventy-five percent of workers aged 30-34 have jobs expected to last five years or more, and forty percent have jobs expected to last twenty-five years or more (Hall 1980, p.98). A similar point is made in other papers (Akerlof and Main 1980, Hall 1982).

The literature that has resulted from efforts to model long-term labor contracts has portrayed many different aspects of long-term contracts. Labor contracts have been dichotomized into implicit and explicit contracts (Azariadis 1975, Baily 1974, Gordon 1974). Either of these types of contracts has income-smoothing effects. The literature on firm-specific human capital explores the value of the
job-worker match in the context of the firm-specific skills that
workers attain by on-the-job learning (Becker 1975, Parsons 1972).
The model developed in the present paper is in the spirit of the firm-
specific human-capital literature and is easily conformable to being
either an implicit or explicit contract. Mincer and Jovanovic (1981)
incorporate human-capital considerations and search theory in a paper
that formalizes a widely held view into the observation that "The most
firmly established fact about labor mobility of all kinds is that it
declines with age." Lazear (1979) puts a cap on the length of the
labor contract by introducing mandatory retirement. One could go on
at great length detailing the different aspects of long-term labor
contracts that have been described in the literature. It seems rather
more interesting at this point, however, to consider an important
aspect of long-term labor contracts that has so far received little
attention in the literature, and is the major thrust of this paper.

In light of the extended duration of many job matches, it seems
clear that a very important aspect of the contracts that govern
many job matches is the seniority-based layoff criteria widely used by
firms. A common practice among employers is to lay off the least
senior worker first when the economic environment dictates a contrac-
tion of the size of the work force. This seniority-based layoff
structure is clearly a thing of value to the senior worker. The major
thrust of this paper is the construction of a model that determines
the value to the senior workers of the seniority benefits that specify
upon whom a layoff is to fall.
Once the model is developed in general, it is applied to pricing the seniority benefits of workers in a common work setting. In the work setting that we consider, we evaluate and price the system of seniority-based layoffs so predominant among union employees. The present analysis indicates that, when seniority-based layoffs are considered, we must rethink the notion that the compensation profile of union workers is flat with respect to age, even though the wage profile may be flat.

The model presented in this paper is developed in several iterations, beginning with a very simple model in the first iteration and proceeding to the model with increased complexity in subsequent iterations. In the first iteration of the model, we impose no requirements as to how or why seniority is granted, and make no assumptions about risk aversion or preferences (except that more of any good is preferred to less and that we are indifferent between identical items). We make no real attempt to construct any particular work situation, but do allude to one to be discussed in a subsequent iteration. While the model is ultimately constructed so as to be completely general as to the number of periods, the model presented in the first iteration is a two-period model.

We subsequently add structure and generality to the model in ways that bring the basic technique developed in the first iteration to bear on the task of pricing seniority benefits as they exist in familiar work settings. One subsequent iteration considers the value of seniority when the firm can renege on the right to work that it has issued to the senior worker. This makes the model more realistic,
since there are few cases where seniority rights provide an absolute guarantee of next-period employment. We then extend the model to three periods so that there is more than one period during which the worker is senior. Finally, we indicate how the model is affected by a variety of different assumptions regarding the stochastic process that determines the worker's spot wage.

Intermixed with these extensions of the mathematical structure of the model is the development of a specific type of work contract that awards seniority for length of service to the firm. This type of contract is of the sort widely used in both union and non-union manufacturing settings where seniority is awarded for length of service and the most senior worker is the worker that has the most experience with the current employer. These various iterations in the development of the model are the subject of Chapter Four. Chapter Five uses the model to analyze, in detail, the existing labor contract between Gould Defense Systems, Inc., Ocean Systems Division and The United Auto Workers. This contract illustrates the importance of several of the model's extensions contained in Chapter Four. Chapter Five uses the model of Chapter Four to determine analytically the value of the contract's seniority provisions for workers regardless of their individual level of seniority. The analysis indicates that the value of seniority is a function of five parameters and the stochastic process that drives the workers spot wage. Once we have found analytically the value of the seniority provisions in the contract for Gould workers with various levels of seniority, we extend the analysis to include a numerical evaluation of the value of these seniority
benefits. These numerical evaluations are done for a wide range of different settings of the five parameters of the value of seniority.

The analysis of Chapter Five is confined to valuing seniority when the workers labor has only one price. Chapter Six generalizes the analysis by allowing for two separate wage processes on the workers labor. This allows us to find the value of seniority for workers who have firm-specific human-capital.

To provide background for the analysis of Chapters Four, Five and Six, Chapter Two provides a simple intuitive characterization of the issues addressed in the paper and the assumed economic environment in which the contracts exist, while Chapter Three reviews one technique for deriving and solving the stochastic partial differential equations that are implied by the model.
CHAPTER II
A CHARACTERIZATION OF THE CONTRACT AND THE ENVIRONMENT

II.1 Introduction

Many firm-employee contracts spell out means by which seniority is attained and the privileges enjoyed by employees who have gained a specified level of seniority. In this paper, we consider situations that differ widely in the ways in which seniority is awarded, but we uniformly consider only the most important of the rights bestowed by seniority. The most important privilege that attends an award of seniority is the right to sell to an employer a unit of labor in the coming period or periods at some predetermined wage. No other seniority-based perquisites are considered in this paper.

The model presented in this paper is quite general with respect to the characteristics of the various work environments in which it can be applied. The model works well regardless of how seniority benefits are awarded, regardless of the relative wages of senior and non-senior workers and regardless of the relative productivity of senior and non-senior workers. Indeed, the formal development of the model proceeds most parsimoniously if we simply analyze the value of a seniority benefit that is awarded by lottery to certain members of a homogeneous workforce. This is, in fact, how the formal development
of the model contained in Chapter Four begins.

Clearly, there is nothing new in recognizing that senior workers have seniority rights that entitle them (under a wide range of circumstances) to a property right vis-a-vis non-senior workers to scarce employment opportunities. What is new is the recognition that these seniority rights are actually a special kind of an option and that modeling seniority rights as options allows us to draw on the well-established theory of options pricing to analyze the value of seniority benefits and thereby to establish the monetary value of the contracts that award seniority rights.

Having very briefly sketched the economic problem of valuing seniority rights and the options pricing solution of the problem that is developed in Chapter Four, in the remaining section of this chapter, we describe the economic environment in which these contracts are priced and then provide a more rigorous description of the seniority right to be modeled in Chapter Four.

II.2. The Economic Environment

The worker whose seniority benefit is to be priced is assumed to work in an environment where fluctuations in economic conditions cause his value of marginal product, both inside and outside the firm, to follow geometric Brownian motion. We denote this value of marginal product as \( W \). Formally, we posit that \( W \) follows the diffusion \( \frac{dw}{w} = \alpha_w dt + \sigma_w dz \), where \( \alpha_w \) is the expected increase in \( W \), \( \sigma_w \) is the instantaneous variance of \( W \) and \( dz \) is a Weiner process. A Weiner process is normally distributed with independently and identically distributed
increments. We assume that labor markets are competitive and free of impediments to factor mobility (mobility costs are zero) so that the worker whose seniority benefits are to be priced has as his best alternative to working next period for the contract wage in the present firm the alternative of costlessly moving into employment with an alternate firm at a wage equal to his VMP. The worker's alternative to exercising his option to work for the present firm next period then is to sell his labor in the spot market for VMP, his value of marginal product inside or outside the present firm. This assumption is relaxed in Chapter Six.

It is assumed that the contract under consideration specifies that workers are paid a contracted for fixed wage, $W_0$. In addition, the senior workers receive an option to sell, at the beginning of the next period, their next period's labor to the firm for the next period's contract wage $W_{t+1}$.

Having described a scenario with a senior worker whose VMP is a random variable that follows Geometric Brownian Motion and whose endowment provides an option-like seniority right that entitles him to work in the coming period for wage $W_{t+1}$ within a wide range of circumstances, we now turn our attention to describing a powerful tool for finding the dollar value of an option to sell a predetermined commodity to an established party at a fixed time for a certain price that has been developed in the finance literature over the last decade. This technique is referred to as the options pricing model and was introduced into the literature in 1973 by Black and Scholes.
Since the model developed in Chapter Four draws upon the same techniques as does Black-Scholes and several of the Black-Scholes extensions, it seems appropriate to provide a brief introduction to the options pricing model as developed by Black and Scholes before moving on to apply and extend the technique in Chapter Four.
CHAPTER III
PAST RESEARCH

Since the appearance of the seminal paper on options pricing of Black and Scholes (1973) many insightful papers have applied the ideas of Black and Scholes to pricing other securities, including stock, discount bonds, subordinated discount bonds, warrants, convertible bonds and callable bonds. Following in this tradition, this paper draws upon the options pricing model to develop a more complete model of firm-employee contracting behavior and use that model to price seniority benefits.

Two types of options concern us as we develop the seniority pricing model in Chapter Four. One is a call option and the other is a put option. Simple working definitions are:

(1) A "call" option: Is an option that entitles the bearer to purchase a designated commodity at a stipulated price on a predetermined day from the issuer of the "call" option.

(2) A "put" option: Is an option that entitles the bearer to sell a designated commodity at a stipulated price on a predetermined day to the issuer of the "put" option.

Put and call options that can be exercised at any time up to a fixed expiration date are referred to as American options, while
options that can be exercised only on a given date are known as European put and call options. The present paper deals only with European type options.

The stipulated price for which the designated commodity is sold is referred to as either the options exercise price or its striking price. The predetermined day on which the option can be exercised is referred to as the exercise date.

The notion of a hedge portfolio is important in the development of the analysis. A hedge portfolio is a portfolio of two or more assets that is riskless by construction. It consists of asset positions whose returns are negatively correlated so that if the value of the position in one asset falls, the value of the position in the other asset or assets rises. The position in the assets in the hedge portfolio can be either long or short positions. An investor with a long position in an asset accepts delivery of that asset. An investor with a short position in an asset must deliver the asset to the other party in the short position. A hedge portfolio can consist of two long positions or two short positions in negatively correlated assets. Alternatively, it can consist of a long position in one asset and a short position in another asset if the assets are positively correlated. The hedge portfolio can be imperfectly hedged, in which case the portfolio still has some risk but is less risky than the riskiest of the asset positions. The hedge portfolio considered here is perfectly hedged, so that the portfolio has no risk and hence pays the riskless rate of interest. The ratio of the two assets that render a portfolio riskless is referred to as the hedge ratio.
The Black and Scholes options pricing model largely uses arbitrage or perfect-substitute arguments to determine the competitive, efficient-market price of European call and put options on shares of stock.

Black and Scholes assume that

1) There are no penalties for short sales of stock
2) No transactions costs or taxes exist
3) The riskless rate of interest is constant
4) The stock pays no dividends and its price follows geometric Brownian Motion
5) The stock market operates continuously.

From these assumptions, Black and Scholes derive their well-known options pricing model. The set of assumptions used by Black and Scholes (hereafter B-S) in their 1973 paper are more severe than is necessary. The model has been shown to be robust to the relaxation of most of those assumptions. In an excellent review article, Smith (1976) cites several subsequent papers that relax, one at a time, most of those assumptions. Much of this section draws heavily on the treatment of Smith. Smith notes that Thorpe (1973) examines the effects of restrictions against the use of the proceeds of short sales. Ingersoll (1976) takes explicit consideration of the effect of differential taxes on capital gains and ordinary income. Merton (1976) argues that the continuous trading solution approximates the asymptotic limit of the discrete trading solution when the stock price movement is continuous. Merton (1973) generalizes the model to allow
for a stochastic interest rate. Merton (1976) and Cox and Ross (1976) use a Black-Scholes type of analysis to examine the case of discontinuous stock price movements. Merton (1973) and Thorpe (1973) modify the model to account for dividend payments on the underlying stock. Thus it appears that the model is not particularly sensitive to any one of the assumptions originally employed by Black and Scholes.

Black and Scholes show that it is possible to create a riskless hedge by forming a portfolio containing appropriate amounts of a stock and a European call option on this stock. The value of this portfolio is clearly the price of the stock times the number of shares in the portfolio plus the price of the call option times the number of calls in the portfolio, or

$$ V_H = S Q_S + C Q_C $$

where

- $V_H$ = Value of the hedge portfolio
- $S$ = Price of one share of the stock
- $Q_S$ = Quantity of shares in the portfolio
- $C$ = Price of one call option
- $Q_C$ = Quantity of call options in the portfolio.

The change in the value of the hedge portfolio is the total derivative of equation (3.1). Treating the quantities as fixed exogenously:

$$ dV_H = Q_S dS + Q_C dC. $$

The geometric Brownian motion mentioned in assumption (4) is a diffusion process that belongs to a class of stochastic processes that
are functions of Weiner processes and Itô's lemma allows differentia-
tion of these types of functions of Weiner processes. Because
rigorous derivation of Itô's lemma requires extensive use of advanced
mathematical techniques, it is omitted here. The interested reader is
referred to Friedman (1975) for a rigorous treatment of Itô's lemma.
I provide only a simple heuristic derivation that exploits the
familiar Taylor-series expansion.

Consider a Taylor-series expansion of the function

\[ C(S+dS, t+dt) \] where \( dS \) and \( dt \) are both stochastic functions.

\[ C(S+dS, t+dt) = C(S, t) + \frac{\partial C}{\partial S} dS + \frac{\partial C}{\partial t} dt + \frac{1}{2} \frac{\partial^2 C}{\partial S^2} dS^2 + \frac{1}{2} \frac{\partial^2 C}{\partial t^2} dt^2. \]

It can be shown that \( dt^2 = 0 \) and that \( dS^2 = \sigma^2 S^2 dt \), so that

\[ C(S+dS, t+dt) = C(S, t) + \frac{\partial C}{\partial S} dS + \frac{\partial C}{\partial t} dt + \frac{1}{2} \frac{\partial^2 C}{\partial S^2} \sigma^2 S^2 dt. \]

Noting that \( dC = C(S+dS, t+dt) - C(S, t) \)

\[ dC = \frac{\partial C}{\partial S} dS + \frac{\partial C}{\partial t} dt + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} dt \]

\[ = \frac{\partial C}{\partial S} dS + \left( \frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} \right) dt, \quad (3.3) \]

which is Itô's lemma.

If it is assumed that the stock return is a diffusion process of
the form

\[ \frac{dS}{S} = \alpha_S dt + \gamma_S dZ_s, \]

where \( \alpha_S \) is the instantaneous expected return on \( S \), \( \gamma_S \) is the
instantaneous variance of return on \( S \) and \( dZ_s \) is a Weiner process,
then Itô's lemma is applicable to equation 3.2 and substituting (3.3) into (3.2) for $dC$, 

$$
\quad dV_H = Q_S dS + Q_C \left[ \frac{\partial C}{\partial S} dS + \frac{\partial C}{\partial t} dt + \frac{1}{2} s^2 \sigma^2 \frac{\partial^2 C}{\partial S^2} dt \right]
$$

(3.4)

We next develop the so-called "hedge ratio", which is the appropriate ratio of stocks and options for the model's arbitrage arguments. We wish to have no change in the value of the hedge portfolio when the prices of stocks and options change. Without loss of generality, we may focus on one unit of stock. This means setting $Q_S = 1$. Then for the value of the hedge to be invariant to price changes, we select $Q_C = -\frac{1}{\partial C/\partial S}$. This yields

$$
Q_S dS + Q_C dC = dS + \frac{-1}{\partial C/\partial S} dC
$$


$$
= dS - \frac{dC}{\partial C/\partial S}
$$

$$
= dS - dS = 0,
$$

and the hedge is riskless. Substituting $Q_S = 1$ and $Q_C = -\frac{1}{\partial C/\partial S}$ into (3.4) yields:

$$
\quad dV_H = dS + \frac{-1}{\partial C/\partial S} \left[ \frac{\partial C}{\partial S} dS + \frac{\partial C}{\partial t} dt + \frac{1}{2} \frac{\partial^2 C}{\partial S^2} s^2 \sigma^2 dt \right]
$$

$$
= dS - 1 \left[ \frac{\partial C}{\partial t} + \frac{1}{2} \frac{\partial^2 C}{\partial S^2} s^2 \sigma^2 \right] dt
$$

$$
= \frac{-1}{\partial C/\partial S} \left[ \frac{\partial C}{\partial t} + \frac{1}{2} \frac{\partial^2 C}{\partial S^2} s^2 \sigma^2 \right] dt. \tag{3.5}
$$
Since the hedged portfolio is riskless, it must pay the riskless rate of interest. This means that, in equilibrium, to prevent arbitrage we must have:

\[
\frac{dV_H}{V_H} = rdt, \tag{3.6}
\]

so that \( \frac{dV_H}{V_H} \) becomes the rate of return earned by the hedged portfolio and \( rdt \) is the riskless rate times the change in time. Substituting the value of \( Q_S \) and \( Q_C \) into (3.1) and substituting this result along with (3.5) into (3.6), we obtain:

\[
\frac{-1}{\delta C/\delta S} \left[ \delta C + \frac{1}{\delta t} \frac{\partial ^2 C}{\partial S^2} S^2 \sigma^2 \right] dt = rdt.
\]

Multiplying through by the LHS denominator and dividing through by \( dt \), then multiplying through by \( \frac{\partial C}{\partial S} \),

\[
\frac{-1}{\delta C/\delta S} \left[ \delta C + \frac{1}{\delta t} \frac{\partial ^2 C}{\partial S^2} S^2 \sigma^2 \right] = rs + rc \frac{1}{\delta C/\delta S}.
\]

\[
\frac{\partial C + 1}{\delta t} \frac{\partial ^2 C}{\partial S^2} S^2 \sigma^2 = \frac{\partial C}{\partial S} rs + rc,
\]

and
This is a second-order partial differential equation for the value of the call option.

Along with the boundary condition that at its expiration date the call option has value \( C^* = \max(0, S^*-X) \), (3.7) may be transformed into the heat equation of classical physics and solved. It may also be solved using the technique of Cox and Ross (1975). The interested reader is referred to Black and Scholes (1973), p. 643 for the mathematics of the transformation to the heat equation. Smith (1976) contains additional citations for those interested in additional background on this derivation.

The solution of 3.7 subject to the boundary condition is

\[
\frac{\partial C}{\partial t} = rC - \frac{\partial C}{\partial S} rS - \frac{1}{2} \frac{\partial^2 C}{\partial S^2} S^2 \sigma^2. \tag{3.7}
\]

where \( N() \) is the standard normal CDF, \( \sigma^2 \) is the variance of the stock price, \( S \) is the stock price and \( X \) is the exercise price. Equation (3.8) is the Black-Scholes equation for the price of a call option.

The derivation of the put option price is perfectly symmetric to that of the call price and hence will be omitted. The put option pricing equation is

\[
C = S \cdot N \left( \frac{\ln(S/X) + (r + (\sigma^2/2)T)}{\sigma \sqrt{T}} \right) - e^{-rT} X \cdot N \left( \frac{\ln(S/X) + (r - (\sigma^2/2)T)}{\sigma \sqrt{T}} \right), \tag{3.8}
\]
\[
P = \mathcal{S} \cdot \exp\left\{ -\frac{\ln(S/X) - [r + (\sigma^2/2)]T}{\sigma\sqrt{T}} \right\} + X \mathcal{E}^{r T} \left\{ -\frac{\ln(S/X) - (r - \sigma^2/2)T}{\sigma\sqrt{T}} \right\}, \tag{3.9}
\]

which is the equilibrium price of an option to sell a stock valued at \( S \) for exercise price \( X \) at time \( T \).
CHAPTER IV

AN OPTIONS PRICING MODEL OF FIRM-EMPLOYEE CONTRACTS

IV.1 Introduction

The first three chapters of this paper describe a seniority right possessed by a worker and a technique for valuing that seniority right. Some of the more interesting uses of the model developed in this dissertation are applications to situations where specific assumptions about human-capital accumulation, wages, productivity and the manner in which seniority rights are obtained are appropriate. That notwithstanding, the model itself, in its simplest version, does not require any assumptions of this sort. To differentiate clearly between assumptions that are required for the seniority pricing technique and the assumptions made only for the purpose of establishing a realistic scenario to apply the model, it is useful to begin by developing the model in the simplest possible way. This is done in section 4.2 and we then add layers of complexity in subsequent sections. To that end, the present chapter is organized as follows. Section two presents the model in its most basic form. To establish a firm footing we make several assumptions, some of which are quite plausible and reasonable, and some of which are less reasonable. Very briefly, to develop the basic model in section 4.2
we assume that worker VMP follows geometric Brownian motion and that the seniority right to next-period work is always binding on the firm. Section three allows that the seniority right to work next period is not honored by the firm when it faces dire straits. Section four expands the model from two periods to three periods. Section five relaxes the assumption of geometric Brownian motion and considers the model under alternate specifications of the stochastic process that drives VMP. Section six applies the model to compensating wage differential analysis.

IV.2 The Basic Model

We begin with the simplest possible model structure and economic environment. This basic model concerns a worker to whom seniority has been granted. At the end of this section, we allude to the possibility that seniority is granted to the individual as a reward for length of service. Until then we need only allow that the individual has been granted seniority. The worker is employed in a productive activity where fluctuations in economic conditions (state variables) cause the shadow price of his labor, both in his best alternate employment and in the present firm, to follow geometric Brownian motion. A good working definition of geometric Brownian motion is that it is a continuous stochastic process whose increments are independent and log normally distributed. A more formal definition can be found in the section on alternative stochastic processes. Being granted seniority is posited to give the worker a guaranteed option to work in the subsequent period at some predetermined wage rate $W_{t+1}$. 
We refer to a guaranteed option to convey the meaning that the firm must honor this option regardless of the values of any other state variables. This is the type of option priced by the Black-Scholes option pricing model for stocks where the integrity of the option writer is guaranteed by the options exchange. Of course, certainty that the option will be honored when exercised is a much better assumption for stock options than it is for options to work next period. The model is first derived for the guaranteed option case to fix some key ideas before considering the more realistic case where the option is binding only under a wide range of circumstances.

Using the conceptual scenario developed in the preceding paragraph, we investigate the value to a worker of an option on employment with his present firm in the next period at a precontracted wage $W_{0,t+1}$. This type of option is referred to as a European put option on the worker's labor with exercise price (the price at which labor is sold) equal to $W_{0,t+1}$. Symmetrically, an option to buy an individual's labor at a prespecified date for a prespecified price $W_{0,t+1}$ is a European call option on the worker's labor with exercise price $W_{0,t+1}$.

Valuation equations for a put option and a call option on a commodity with the same price and parameters are nearly symmetric. Each value emerges from the other through the use of an important equation known as the put-call parity theorem. Several distinct ways of deriving the valuation equations for options exist. The method demonstrated in Chapter Three was the first method to appear in the literature and remains the most popular derivation. It brings to the
forestage an arbitrage or perfect-substitute assumption made in the present paper. A second solution method applies a theorem regarding the solution for a family of stochastic differential equations. This theorem plays a crucial role in the section in which we allow default on the option. Since both derivations supply important intuition, both are provided. The second derivation, which is contained in section 4.3, first derives the value of the put option and proceeds to value the call option using arbitrage arguments. The first derivation, provided here, parallels the derivation of Chapter Three.

IV.2.1 Derivation of Call and Put Option Valuation Equations

As is discussed in Chapter 3, Black and Scholes (1973) shows that it is possible to create a riskless hedge by forming a portfolio that contains a long position in a valuable commodity along with a short position in a call option on that commodity. For our purposes, the valuable commodity is the worker's labor which has spot price $W$.

Now, the value of the hedge portfolio is

$$V_h = WQ_1 + cQ_c$$  \hspace{1cm} (4.1)

where $Q_1$ = quantity of labor
$c$ = value of the call option
$Q_c$ = quantity of call option.

Assuming costless continuous stock and options market operation as Black-Scholes do assures that this hedge can be created and maintained. It is appropriate to consider what difficulties might
exist in taking the hedging positions required by equation 4.1. The hedge positions are ultimately arrived at by an arbitrage argument that plays upon the fact that if two portfolios of assets have the same payoff in each state of nature, then those two portfolios are perfect substitutes for each other. The same assumption leads to equation (3.7) in Chapter Three. It tells us that if there exists a portfolio of assets that has the same payoff in each possible state of nature as does the workers labor, then that portfolio and the workers labor are perfect substitutes.

Whenever we can demonstrate conditions under which the payoff vector for a unit of labor is spanned by the payoff vectors of a set of basis securities, then under those conditions, it matters little for our analysis whether or not the market for units of labor and options on units of labor are sufficiently complete to allow the hedge positions to be taken in labor and calls on labor. It seems reasonable that labor markets may not be complete in some dimensions. For example, a worker cannot typically buy full insurance against being laid off. However, a worker wishing to hedge against layoff might, for example, sell short some shares of stock in his current company, or buy put options on the same stock. If adverse times come that cause the firm to lay off the worker, these adverse times should also cause the price of the firm's stock to fall. This would enhance the value of the short position or put options position. Purchased in the correct quantities, put options on the stock issued by the worker's employer could be structured to provide a worker with a
riskless position insulated against losses from layoffs caused by the worker's employer falling on hard times.

We consider now, more formally, conditions under which the labor of the worker can be replicated by a portfolio of assets that yield payments identical to the payments to labor in all states of the world. Let $Y$ be a $k$-dimensional state variable that evolves as the diffusion process:

$$dY = \alpha dt + \gamma dz$$

where $z$ is an $N$-dimensional Weiner process.

Let $r$ be determined by:

$$\frac{dp_0}{p_0} = -rdt$$

where $p_0$ is the current price of the riskless asset.

Let the prices $p_1, \ldots, p_N$ of $N$ risky securities evolve as the diffusion

$$I^{-1}(p)dp = u dt + \sigma dz$$

where $I(p)$ is a diagonal matrix with diagonal elements $p_1, \ldots, p_N$ and $p_j$ is the price of the $j$th risky security; $dp$ is a $N \times 1$ vector of changes in prices of the $N$ risky securities; $u$ is an $N \times 1$ vector of expected increases in the prices of $N$ risky securities; $\sigma^2$ is a matrix of variances and covariances of the $N$ processes that drive the prices of the $N$ risky securities and $dz$ is an $N$-dimensional Weiner process. In short, $I^{-1}(p)dp$ is a multidimensional Brownian motion process. The $j$th element of $I^{-1}(p)dp$ is $\frac{dp_j}{p_j}$.

In what follows, a critical assumption is that $k < N$, i.e. we must have (at least) as many linearly independent securities as we have sources of uncertainty ($= \text{dimensions of the state variable}$). We next
introduce into this market asset $W(Y,t)$, which is the value of one unit of labor. $W$ depends only on current values of state variables and time, $t$. $W$ evolves as the diffusion process:

$$dW = MWdt + WS'dz$$
or equivalently,

$$\frac{dW}{W} = Mdt + S'dt.$$

Consider the portfolio with weights $w$, where $w$ is a non-null $N+1$ vector of percentage investments in the $N$ risky securities and $W$.

Let: (1) $e'w = 1$ where $e'$ is defined as $e'=(1,1,...,1)$

i.e., the weights sum to 1;

(2) $w' \begin{bmatrix} \sigma \\ s' \end{bmatrix} = 0$

i.e. the portfolio has no instantaneous risk;

(3) $w' \begin{bmatrix} u \\ m \end{bmatrix} = r$

i.e. the portfolio must earn the riskless rate.

But since $w' \neq 0$, by the theory of homogeneous systems we know that

$$\begin{vmatrix} \sigma & u-re \\ s' & m-r \end{vmatrix} = 0.$$
This means that there exists $k$ such that

$$w^*(s, u-re) = k(s', m-r).$$

By purchasing one unit of the portfolio $w^*$ (which is $w$ but without the investment in $W$) and $-k$ units of $W$, an investor assures himself no risk and no return. Essentially, he has taken a zero position in asset $W$. Alternatively, selling $1/k$ units of the portfolio $w^*$ leaves the investor who owns human capital with a zero position in his own human capital.

The result is that even though labor is not fully traded in markets, we can value it precisely as:

$$\frac{1}{k}w^*I(p).$$

This value reflects the requirement that the LHS and RHS both yield the same return in all states of the world. By buying and selling calls and puts on the $N$ traded assets, we can take a position identical to a position in calls and puts in the untraded asset.

The important assumption is that $\sigma$ be of rank $N$, so that each row of $\sigma$ maps the $dz$'s into an impact on the corresponding element of $I^{-1}(P)dp$.

If we have $k$ sources of uncertainty (the dimension of the state variable $Y$), then the number of linearly independent securities must equal $k$. Since we posit $N$ independent securities we require $k=N$. The alternatives are: (1) if $k > N$, we may not be able to price $W(Y,t)$ and (2) $k < N$, which assures us that some of the $N$ primary securities are
redundant. If securities are redundant, we may discard information on
them without any impact on the market completeness. We can discard
securities until \( N = k \) and the conditions of the assumptions are met.

Those familiar with this type of argument know that this analysis
boils down to assuming that state-dependent returns to asset \( W(Y,t) \)
are elements of a space spanned by the basis formed from the collec­
tion of state-dependent return vectors of the set of all assets.

Having determined a set of sufficient conditions under which the
hedge portfolio defined in (4.1) exists, we proceed to derive a valua­
tion equation for a call option on a unit of next-period labor with
exercise price \( W_{0t+1} \). Given the hedge portfolio defined by equation
(4.1), the change in the value of the hedge portfolio is the total
derivative of (4.1).

\[
dV_h = Q_1 dW + Q_c dc. \tag{4.2}
\]

If it is assumed that the spot wage, \( W \) follows geometric Brownian
motion, then using Itô's Lemma from Chapter Three,

\[
dc = \frac{\partial c}{\partial W} dW + \frac{\partial c}{\partial t} dt + \frac{1}{2} \frac{\partial^2 c}{\partial W^2} \sigma^2 dt. \tag{4.3}
\]

Substituting (4.3) into (4.2),

\[
dV_h = Q_1 dW + Q_c \left[ \frac{\partial c}{\partial W} dW + \frac{\partial c}{\partial t} dt + \frac{1}{2} \frac{\partial^2 c}{\partial W^2} \sigma^2 dt \right]. \tag{4.4}
\]
In general, \( dV^2 \) is stochastic, owing to the stochastic nature of the \( dW \) terms, but if we set
\[
Q_c = 1
\]
and \( Q_1 = -\partial c / \partial W \),
then the stochastic \( dW \) terms vanish and (4.4) becomes
\[
dV = \left\{ \frac{\partial c}{\partial t} + \frac{1}{2} \frac{\partial^2 c}{\partial W^2} W^2 \sigma^2 \right\} dt. \tag{4.5}
\]

With \( Q_c \) and \( Q_1 \) chosen in this manner, the hedge portfolio is riskless and, if perfect substitutes must pay the same return, its return must equal the riskless rate:
\[
\frac{dV_h}{V_h} = r dt. \tag{4.6}
\]

Paralleling Chapter Three, we substitute (4.4) and (4.5) into (4.6),
\[
\frac{\partial c}{\partial t} + \frac{1}{2} \frac{\partial^2 c}{\partial W^2} W^2 \sigma^2 = r,
\]
\[
\frac{\partial c}{\partial t} + \frac{1}{2} \frac{\partial^2 c}{\partial W^2} W^2 \sigma^2 = rc - rW(\partial c / \partial W),
\]
\[
\frac{\partial c}{\partial t} = rc - rW \frac{\partial c}{\partial W} - \frac{1}{2} \frac{\partial^2 c}{\partial W^2} W^2 \sigma^2. \tag{4.7}
\]
Equation (4.7) is a stochastic partial differential equation, which is part of a boundary-value problem whose solution is the valuation equation for an option to sell a unit of labor for $W_0$ in the next-period. The other necessary part of the boundary-value problem is the boundary condition that states that at the option's expiration date, the option has value equal to the maximum of zero or the difference between the spot price of labor and the contract wage. In symbols:

$$c^* = \text{Max} \left[ 0, W^* - W_0 \right], \quad (4.8)$$

where the starred variable indicates the value of the variable at the time of the option's expiration. Equation (4.7) along with the boundary condition (4.8) may be transformed into the already solved heat equation of classical physics (Black and Scholes, 1973), by making a straightforward but lengthy substitution. The interested reader is referred to pages 643-645 of Black and Scholes for the details of the transformation of equations (4.7) and (4.8) into the heat equation. The solution of the boundary-value problem posed by equations (4.7) and (4.8) is given by:

$$c = W_0 \cdot \exp \left\{ \ln \left( \frac{W}{W_0} \right) + \frac{r + \sigma^2/2}{\sigma \sqrt{T}} \right\} - W_0 e^{-rT} \cdot \exp \left\{ \ln \left( \frac{W}{W_0} \right) + \frac{r - \sigma^2/2}{\sigma \sqrt{T}} \right\} \quad (4.9)$$

where

$c$ = the value of a call option to buy one unit of labor at time $t$ for $W_0$

$W_0$ = contract wage rate

$W$ = spot price of labor
Derivation of the valuation equation for the put option exactly parallels the derivation of the valuation of the call option (equation 4.9) and is hence omitted. The resulting valuation equation for the put option is:

\[ p = -W \cdot N \left( \frac{-\ln(W/W_0) - (r+\sigma^2/2)T}{\sigma T} \right) + W_0 e^{-rT} \cdot N \left( \frac{-\ln(W/W_0) - (r-\sigma^2/2)T}{\sigma T} \right) \]  

Since equations (4.9) and (4.10) are expressions for call and put options on labor they correspond to equations (3.8) and (3.9).

This establishes that the value of a put option on a unit of labor with striking price \( W_0_{t+1} \) is given by equation 4.10. It also shows that the price of a call option on that labor is given by (4.9).

We may now distinguish between the wage of the worker with seniority and the wage of the worker without seniority. We denote the period-t wage of the worker with seniority by \( W_{t}^{sr} \) and the period-t wage of the worker without seniority as \( W_{t}^{ns} \). We first consider the total period-t compensation of the senior worker: this compensation consists of both the period-t wage, \( W_{t}^{sr} \) and the put option on the next period's labor whose value is given by (4.10). The total compensation of the senior worker is seen to be:

\[ c_{t}^{sr} = W_{t+1}^{sr} + p \]
where \( W_0 \) in the brackets is taken to be the contract price of period-\( t+1 \) senior labor.

The value of seniority, denoted \( V^s \) emerges quite naturally as the difference between the compensation of the senior worker and the compensation of the non-senior worker:

\[
V^s = c^s - W_0^ns
\]  
\[
= -W_0^{sr} - W_0^{ns} - W_0N \left\{ \ln\left( \frac{W}{W_0} \right) - \left( r + \sigma^2/2 \right)T \right\}
\]  
\[
+ W_0 e^{-rTN} \left\{ \ln\left( \frac{W}{W_0} \right) - \left( r - \sigma^2/2 \right)T \right\}
\]

In words, the value of differential seniority is the wage differential \( W_0 \) plus the right to sell one's labor next period for an agreed upon price. Indeed, as is argued at greater length in a later section of this chapter, situations where \( W_0 = 0 \) for all \( t \) are extremely common. For example, in union shops, it is the rule rather than the exception that all workers in a certain job class receive the same wage regardless of seniority. In this very common situation, the value of
seniority is simply the value of being able to sell one's labor next period.

IV.2.2 A Word About Assumptions

To this point, we have illustrated the first of what will be several iterations of a technique for pricing seniority rights. The development makes several assumptions. Some of these assumptions appear more or less reasonable in a wide range of settings. For example, even though it is a strong assumption, the assumption that the spot shadow price of labor follows geometric Brownian motion is a frequently made assumption. Many of the assumptions, however, seem reasonable in some situations and not terribly reasonable in others. For example, in many situations, the contract wage rate for senior workers and non-senior workers is the same, but there are many other situations where these wage rates are different. Other situations suggest that the certain honoring of a put option on labor is a rather wishful assumption, but circumstances exist where it reasonably approximates reality, since few twenty-year veterans of IBM are ever laid off. Relaxing of some of these restrictive assumptions is the subject of most of the subsequent iterations of the model. Before turning to the task of making the model more general, it seems appropriate to ask if there are any major classes of labor contracts that accord well with the assumptions of the basic model, since it is typical to be able to make less definitive statements about the characteristics of a model's solution as the level of the model's generality is raised.
We focus on the class of employment contracts that by some mechanism selectively bestow seniority and that also specify $W_{t}^{SR} = W_{t}^{NS}$ for all $t$. While this is certainly not a completely general class of contracts, it is nevertheless a broad class. It includes many union labor contracts and many contracts for non-union, blue-collar workers in large corporations. For example, Gould Corporation employs three classes of product inspectors; inspector A, inspector B and inspector C. Class C inspectors are paid $x_1$, regardless of seniority or productivity. All inspector B class employees are paid $x_2$ regardless of seniority or productivity and inspector A class employees are paid $x_3$. Inspector C class employees receive wage increases only when the entire class of inspector C employees receive wage increases or when they are promoted to inspector B. Clearly the result is that within a grade, workers in this firm have a horizontal wage profile with respect to experience. This sort of horizontal wage profile is very common, especially in large firms.

This scenario is developed much more extensively in Chapter Five; here we wish only to consider what can be said about the seniority-based employment rights. Gould is a major defense contractor with fairly stable prospects. As a result their most senior class A inspectors have virtually no chance of being denied employment in their lifetimes. This renders the seniority rights possessed by these senior inspector A's similar to those valued using equation 17 with $WD_t = 0$. In this class, the value of seniority is
Several illuminating comparative-static results concern the value of seniority. First,

1) \( \frac{\partial p}{\partial \sigma^2} > 0 \).

This indicates that the value of seniority is greater in employment situations where the variance of the spot price of labor is high. This accords well with our intuition. Intuition suggests that, ceteris paribus, a guarantee of employment is more valuable in industries that experience great variation in employment than in industries that show little or no such variation.

Second,

2) \( \frac{\partial p}{\partial W_0} > 0 \).

This shows that seniority is more valuable the higher the contract wage, as clearly it should be, since the promise of employment at a high wage is more valuable than the promise of employment at a low wage.

Third,

3) \( \frac{\partial p}{\partial W} < 0 \).

This says that the higher is the spot price of labor in alternative employments, the less valuable is an option to sell at a fixed labor price \( W_0 \).
The signs of these three derivatives accord well with intuition. They are standard results in the options literature and need not be derived here. It is seen in Chapter five that when we relax some of our restrictive assumptions these comparative statics become much more complicated.

IV.3 Relaxing the Assumption of Certain Options Honoring

The purpose of this section is to highlight and then relax the assumption that the put option held by the senior worker is always honored by the firm. We argue that a more complex options structure better suits the situation where firms, under some circumstances, may not honor the option held by the senior worker.

Consider the firm's position concerning the senior worker. The firm would not desire to maintain in its employ any worker for whom

\[ R_f = \sum_{t=1}^{T} \frac{1}{1+\text{r}}^t (\text{VMP}_t - \text{W}_t) < 0. \]  \hspace{1cm} (4.13)

In this case, hiring the worker offers negative net present value to the firm.

Consider the individual terms of this sum. Most of the \text{VMP}'s and \text{W}_o's are unknown. \text{VMP} is presumed known for the current period, while \text{W}_o is known for the current period and possibly for one or two periods into the future. The rest of these terms must be estimated by the firm. Given the firm's best estimate of \text{VMP} for \( t=2,\ldots,T \) and \text{W}_o for \( t=1,\ldots,T \), some value of \text{VMP}_t (\text{VMP} in the coming period) represents the minimal \text{VMP}_t that allows 4.13 to remain non-negative. Let
this value of $VMP_1$ be defined as $VMP_1^{**}$. If $VMP_1 < VMP_1^{**}$, the firm wishes to dismiss the employee.

One additional complication is that if the firm expects to be able to rehire senior workers in a subsequent period, the worker may be dismissed when $VMP_1$ exceeds $VMP_1^{**}$. This is because if it is able to rehire the senior worker, the firm is able to employ the worker in future periods where $VMP_t > WO_t$ without having to employ the worker in the current period when $VMP < WO$. This whole process of evaluation on the part of the firm is likely to be represented as a highly convoluted dynamic programming problem. The specifics of the firm's solution method are not of interest here. Our interest is to establish that some value of $VMP_1$ solves the firm's programming problem. We call that value $VMP_1^*$ and posit that for all values of $VMP_1$ such that $VMP_1 < VMP_1^*$ the firm becomes unwilling to honor the senior worker's put option when it is exercised. Since the possibility of the firm's reneging on the option clearly makes the option less valuable, it is interesting to consider the value of the option when the option is exercisable only for $VMP_1 > VMP_1^*$.

An alternate solution method for the partial differential equations of Chapter III proves particularly illuminating in the arguments that follow. The solution adapts a solution method used by several authors. The present development draws most directly on Smith (1979). The argument turns on the following theorem. Since the model makes no assumption regarding preferences of agents, if the valuation problem can be solved for one set of risk assumptions, then the solution must hold for any set of assumptions regarding risk.
The simplest set of assumptions regarding risk is that of risk neutrality of all agents. In such a world, all rates of return on assets would be equal, including the rates of return on put and call options. Therefore, the present price of a put option would be the discounted value of its terminal price, or

\[ p = e^{-rT}E(p^*), \]  

(4.14)

where \( p \) = value of the put option

\( p^* \) = terminal value of the put option

\( T \) = time to options expiration.

Given that \( VMP_t = \bar{W} \) (the spot wage) is distributed lognormally, we may write (4.14) as

\[ p = e^{-rT} \int_{0}^{W_0} (W_0 - W^*)L'(W^*)dW^*, \]  

(4.15)

where \( L'(W) \) is a lognormal density. A solution technique for such equations is provided by the following theorem. It has appeared in several options pricing papers and is usually attributed to Friedman (1975).

Theorem: Suppose \( L'(W^*) \) is a lognormal density function with

\[ p = \begin{cases} 
0 & \text{if } W*>\bar{W}_0 \\
e^{-rT}(W_0-W^*) & \text{if } \bar{W}_0>W*>\bar{W}_0 \\
0 & \text{if } W*<\bar{W}_0 
\end{cases} \]
Then \( E(p) = e^{-rT} \int_{\psi W_0}^{W_0} (W - W^*) L(L^*) dW^* \)

\[
-W \left\{ N \left\{ \frac{-\ln(W/W_0) - (r + \sigma^2/2)T}{\sigma \sqrt{T}} \right\} - N \left\{ \frac{-\ln(W/W_0) - (r - \sigma^2/2)T}{\sigma \sqrt{T}} \right\} \right\}
\]

\[
-e^{-rT} W_0 \left\{ N \left\{ \frac{-\ln(W/W_0) - (r - \sigma^2/2)T}{\sigma \sqrt{T}} \right\} - N \left\{ \frac{-\ln(W/W_0) - (r + \sigma^2/2)T}{\sigma \sqrt{T}} \right\} \right\}
\]

where \( \phi \) and \( \psi \) are arbitrary parameters.

The usefulness of this familiar theorem in the present case becomes apparent once two points are recognized. First, the value of a put option with exercise price \( W_0 \) that is sure to be honored when exercised is

\[
p = e^{-rT} \int_{W_0}^{W_0} (W - W^*) L(L^*) dW^*. \]

Second, a put option that is invalid when \( W^* (=VMP) \) is less than \( VMP^* \) has value

\[
p_0 = e^{-rT} \int_{VMP^*}^{W_0} (W - W^*) L(L^*) dW. \quad (4.16)
\]

Applying the above theorem to 4.15 with \( \psi = 1 \) and \( \phi = 1 \) yields the familiar Black-Scholes equation for a put option with exercise price \( W_0 \). Applying the theorem to equation 4.16 with \( \psi = VMP^*/W_0 \) and \( \phi = 1 \) yields equation (4.17):
When there is a possibility that the firm may default on the put option extended to the senior worker, the situation is more precisely modeled as the senior worker's receiving a put option with exercise price equal to \( W_0 \) from the firm but gives back to the firm a residual position similar to a put option on the worker's labor with the exercise price \( VMP^* \). This residual position is not exactly a put option. This point is discussed at greater length in Chapter Five.

**IV.4 Extending the Model to Three Periods:**

**The Case of the Stochastic Exercise Price**

The model as developed to this point has been a two-period model. During period one, seniority is conferred upon the employee and the employee works for \( W_0 \), the contract wage. At the beginning of period two the senior worker exercises his put option and works for \( W_0 \) (or \( W_0 \) and some pension). Since we have not defined the length of a period, no great loss of generality is involved with the two-period model.

It is interesting at this point to show the conceptual ease with which the model can be developed as a three-period model. We close this section by indicating how one would conceptually expand the model to more than three periods.
The modification of the model to accommodate a three-period time horizon is conceptually simple however it can be fairly demanding mathematically. Conceptually, in the three-period model seniority is conferred upon the employee at the beginning of period one. During period one, the employee works for WO and is endowed with a seniority right (with value p) to work in the second period. If the worker chooses to work in the second period he is paid \( WO_2 \) and option to work again in the third-period. He may or may not work during period three, at the end of which he retires. We first consider the option awarded in the second period. It is awarded if and only if the worker exercises the option awarded in the first period. Also, the second-period wage is paid only if the option awarded in the first period is exercised. As a result, the option awarded in period one is an option on period-two labor for the period-two wage and a period-three put option. This contrasts with the option awarded in period one in the two-period model, since exercising that option provides \( WO_2 \) in exchange for period-two labor. In both the two-period and the three-period models, seniority is awarded in the first period and confers a put option to sell a unit of labor in period two. Denoting a put option to sell a unit of labor in period \( t \) as \( PO_t \), figures 1 and 2 outline the two-period and three-period models in simple diagrammatic form.
Period-1

Seniority is awarded, conferring $PO_2$

$PO_2$ exercised

$PO_2$ not exercised

Period-2

one unit of period-two labor is sold for $WO_2$. Retire at the end of period-two.

exit the model

Figure 1: The Two-Period Model
Period-1
Seniority is awarded, conferring PO₂

exercise PO₂
non exercise

Period-2
one unit of period-two labor is sold for WO₂. PO₃ is conferred

exercise PO₃
non exercise

Period-3
one unit of period-three labor is sold for WO₃. Retire at the end of period-3.

exit the model

Figure 2: The Three-Period Model

From figures 1 and 2, it is clear that while PO₃ from the three-period model and PO₂ from the two-period model are simple put options, PO₂ from the three-period model is an option to sell a unit of labor for WO₂ + PO₃. PO₂ is therefore a more complex option than we have considered before. Its exercise price is no longer the predetermined WO₂, but is rather the stochastic sum WO₂ + PO₃.

We recall from Chapter Two that the spot wage W, follows the diffusion process:
where \( \alpha_w \) is the instantaneous expected increase in the wage per unit of time, \( \sigma_w \) is the instantaneous variance of the wage per unit of time and \( dZ_w \) is a standard Weiner process.

Since the put option \( P_0 \) is a function of the spot wage, its return follows the diffusion

\[
\frac{dP_0}{P_0} = \alpha_{P_0} dt + \sigma_{P_0} dZ,
\]

(4.19)

where \( \alpha_{P_0} \) is the instantaneous expected rate of return per unit time on the put and \( \sigma_{P_0}^2 \) is the instantaneous variance of the return (Geske, 1979).

The exercise price has two components. One of them is deterministic and the other is stochastic, which makes the total exercise price stochastic.

The exercise price, which is now \( W_0 + P_0 \) obeys the following diffusion process:

\[
\frac{d(W_0 + P_0)}{W_0 + P_0} = \alpha_{W_0+P_0} dt + \sigma_{W_0+P_0} dZ
\]

\[
= \alpha_{W_0+P_0} dt + \sigma_{P_0} dZ,
\]

(4.20)

since \( W_0 \) is deterministic. This diffusion differs from that given in (4.19) only in that its initial value exceeds the initial value of the diffusion in (4.19) by \( W_0 \).
Fisher (1978) has developed a model for pricing an option whose exercise price follows a diffusion process and equation (4.20) is such a process. We let the stochastic exercise price and the spot price of labor have correlation coefficient $\rho_{W,WO+PO}$. While with the constant exercise price a riskless hedge could be formed using only stocks and calls, when we let the exercise price be random, the riskless portfolio requires stock, calls and a security to hedge against movements in the exercise price. If such a hedging security does not exist, it can be created using the capital asset pricing model or the simple arbitrage argument used in section 4.2. Let this hedging security have return $r_h$ where

$$r_h = r + b$$

$b$ = risk premium on the hedge security

$r$ = riskless rate.

Paralleling the steps used to derive the B-S put pricing equation leads to a similar put pricing formula when the exercise price is uncertain.

$$-W \cdot N \left\{ \frac{1\ln(W/(WO+PO)) - [r_h - \alpha_p + (\sigma/2)]T}{\sigma T^{1/2}} \right\}$$

$$+ (WO+PO) e^{-\alpha_p} N \left\{ \frac{-1\ln(W/(WO+PO)) - [r_h - \alpha_p - (\sigma^2/2)]T}{\sigma T^{1/2}} \right\},$$

where $\sigma^2 = \sigma_W^2 + \sigma_{WO+PO}^2 - 2\rho_{W,PO}\sigma_W\sigma_{PO}$. 
Note that $\sigma_{WOP} = \sigma_{PO}$ since $W_0$ is fixed. Note also that if the exercise price is constant, then $\sigma_{PO} = 0$, $\alpha_{PO} = 0$, $r_h = r$ and equation (4.21) reduces to equation (4.10), the original Black-Scholes put option valuation equation.

Clearly the valuation expression, (4.21), for the put option with uncertain exercise price varies from the valuation expression (4.10), for options with certain exercise price only in two ways:

1) It replaces the riskless rate by the difference between the return on the asset that hedges the exercise price, $r_h$, and expected growth in the exercise price, $\alpha_x$; and

2) It replaces $\sigma^2$, the variance of the stock price, by $\sigma^2$ the variance of the product of the stock price and the exercise price.

To conclude, extending the model from a two-period model to a three-period model provides a more general model by allowing consideration of options with stochastic exercise price. When the model is extended from a two-period model to a three-period model, the most significant difference in the derivation is that in the three-period model, the option to work in the second period has a stochastic exercise price.

IV.5 Model Specification Under Alternative Assumptions Regarding the Stochastic Process that Drives $W$

Throughout the preceding development, it is assumed that the spot price of labor, $W$, is determined by a lognormal diffusion process:
\[ \frac{dW}{W} = \alpha_w dt + \sigma_w dZ_w. \]

This lognormal diffusion has many properties to recommend it. Not the least of these is that the percentage change in the spot wage that occurs in time \( dt \) is normally distributed with mean \( \alpha_w dt \) and variance \( \sigma^2 dt \). It is conventional to construct standard arguments about the percentage change in the wages being the result of the interplay of a large number of economic variables, which lets one invoke a central limit theorem to justify normality. It is, however, more satisfying to show that the model is quite robust to assumptions regarding the process that drives \( W \).

We begin the discussion of alternative stochastic processes by distinguishing between the two general classes of continuous-time stochastic processes. The first class is diffusion processes. These are characterized by continuous sample paths. A graph of the history of a diffusion looks very much like a random walk about a trend line. While we are unlikely to be able to guess the next value in a diffusion process, we are also relatively certain not to be too surprised by its next value. The second class of continuous-time stochastic processes is the jump process. A graph of a jump process looks like a predictable, deterministic trend line that occasionally has a discontinuity resulting from a large displacement of the process. In contrast to the diffusion process, one can almost always guess the next value of a jump process, but when one misses, one misses by a
large amount. Cox and Ross (1975) develop put option valuation equations for several jump, diffusion and combined jump-diffusion processes. Other authors consider wider ranges of processes. While little purpose is served by developing valuation equations here for various stochastic processes, it is useful to indicate specifically some of the other processes for which solutions exist.

The prototype diffusion process is the lognormal process we have considered all along:

\[
\frac{dW}{W} = \alpha \, dt + \sigma \, dz. \tag{4.22}
\]

The prototype jump process is given by

\[
\frac{dW}{W} = (-\lambda \varepsilon (k-1)) \, dt + (k-1) \, d\pi \tag{4.23}
\]

\[= (-\lambda \varepsilon (k-1)) \, dt + \left( \frac{\lambda dt}{k-1} \right) + \left( \frac{1 - \lambda dt}{1-\lambda dt} \right) . \]

where \( \pi \) is a continuous-time Poisson process, \( k-1 \) is the size of the jump when a jump occurs and \( \lambda dt \) is the small probability that a jump occurs.

In an unpublished paper, Cox and Ross (1975) show that the process given in 4.22 is the limit as \( \lambda dt \to 0 \) of the process given in (4.23). Taking limits of other more-complicated jump processes yield diffusion processes that have variances proportional to the level of the wage.
\[ dW_t = \alpha_w W_t dt + \sigma_w W_t dW_z, \]

and diffusions that have instantaneous variances of wages independent of the level of wages. The proofs that these diffusions are limiting cases of jump processes are quite sophisticated and are omitted. The interested reader is referred to the heuristic proof in Cox and Ross (1975) and to the more detailed treatment by Feller therein cited. For all but the most mathematically sophisticated, the proof of Cox and Ross should prove more than adequate.

As pointed out repeatedly in the course of this dissertation, the diffusion given in equation 4.22 is for intuitive purposes a process whose increments are independently and identically lognormally distributed random variables. More recently, McCulloch (1985) has derived an options pricing formula for the more general class of log-stable distributions, a class which contains the lognormal as a special case. The options pricing technique is valid for a wide range of distributional assumptions without changing its basic conclusions. This means that readers who take exception to our distributional assumptions can do so without taking exception to our techniques.

This section is not intended as a rigorous exposition of alternative stochastic processes. Its purpose is to give the reader a feel for the broad range of stochastic processes for which options pricing models have been developed. The point made here is that whether or not the assumption of geometric Brownian motion is reasonable, if there are objections to this assumption, we need not discard the model. Rather, we need only rely on one of the other formulations of the options pricing model.
IV.6 COMPENSATING DIFFERENTIALS

A major goal of this dissertation is to derive an expression for the value of seniority and the value of seniority is precisely the expression in equation (4.12), which indicates that seniority is valuable because it provides job security. Several papers in the literature have dealt with the added value of a job that has low variability of employment as compared to an otherwise identical job with high variability of employment. These papers comprise a part of what might be called the compensating wage differential literature and argue that jobs with high variability of employment must offer a higher wage than otherwise identical jobs that have a low variability of employment. This wage differential is defined as $r$, where $r$ serves to bring

$$(1-\pi_1)U(L_w, W) + \pi_1 U(L_{NW}, 0)$$
to equality with

$$(1-\pi_2)U(L_w, W+r) + \pi_2 U(L_{NW}, 0),$$
where $\pi_1 = P(\text{Layoff, low employment variability job})$

$\pi_2 = P(\text{Layoff, high employment variability job})$

$L_w = \text{Leisure given not laid off}$

$L_{NW} = \text{Leisure given laid off}$

$W = \text{Wage}$

$U(\ )$ is a utility function.

Hence the wage is higher in employment situations where employment variability is high. The variability of worker VMP is presumably the force that drives employment variability, since given a well-behaved production function, workers are retained so long as their VMP exceeds
some level, usually the wage. Abowd and Ashenfelter (1981) allow probability of employment to be stochastic with known mean and decompose the total wage differential into two components, the first of which is a certainty-equivalent compensation for expected layoff. The second component is a compensation for the uncertainty of employment and is based on the Friedman-Savage notion that for risk-averse workers the expected stochastic utility is exceeded by the utility of the expected value of the stochastic argument of the utility function. Hutchens (1983) and Adams (1985) test empirically for wage differentials using models based on the expected-utility model. The argument of Adams is that instead of employment insurance being offered by the firm as it is in the implicit contracts literature, the firm raises the wage paid during employment enough to provide the worker with a level of utility equal to that of a worker with lower wage and lower employment variance.

The model developed in earlier sections of this paper is well suited to analyzing the size of the compensating wage differential between an employment situation with a low propensity for layoff and an employment situation with a high propensity for layoff. The analysis that determines the size of the compensating wage differential is, in fact, a simple application of the same analysis that determines the value of seniority. This should not be surprising in any way, since we have argued all along that the value of seniority is the increased job security that it provides the worker in terms of the reduced layoffs that result from the worker exercising his option on next-period employment. The compensating differential is that extra
amount of money that must be paid to a worker so that a worker will accept a job with high layoff probability. Simply restating the compensating wage differential problem in terms of the options model indicates that the value of the compensating differential is given by a carefully constructed option.

To make this point, I consider three workers. Worker one is a non-senior worker in a low-layoff job. Worker two is a non-senior worker in a high-layoff job. Worker three is a worker in the same high-layoff job held by worker one, but whose seniority is such that his probability of layoff is exactly equal to that of the non-senior worker in the low-layoff-probability job. Assuming that the third worker exists and does have probability of layoff equal to the first worker, we now determine the compensating wage differential between the low-layoff job and the high-layoff job using simple arbitrage arguments.

In order that the wage differential that we derive reflects only the difference in layoff probability, we posit that the three workers do exactly the same work under exactly the same circumstances, except that two are employed in a high-layoff job and the third is employed in a low-layoff job. We define the probability that worker $i$ is laid off as $P(\text{layoff}_i)$, $i=1,2,3$ and note that

$$P(\text{layoff}_2) > P(\text{layoff}_1) = P(\text{layoff}_3).$$

We are interested in the compensating wage differential between workers one and two. This is, of course, exactly the wage differential between workers three and two. This is because workers one and
three have the same probability of layoff and do identical work so that, by the law of one price, they must be paid the same wage.

As we argued in Section 4.1, the technique for valuing seniority developed in this paper is applicable regardless of why or by what method seniority is awarded and regardless of the wage profile of workers with respect to length of service. Considering an application of the law of one price to the wages of senior and non-senior workers employed in the high-layoff job, it is clear that the total compensation of these two workers must be the same. Recalling that the total compensation of the senior worker in the high layoff-job is

\[ c_{SR} = w_{SR} + p \]

while the compensation of the non-senior worker in the high-layoff job is

\[ c_{NS} = w_{NS} \]

and therefore the compensation differential between the senior worker in the high-layoff job and the non-senior worker in the high-layoff job is \( w_{SR} - w_{NS} + p \). Since the compensation of the non-senior worker in the low-layoff job must be equal to the compensation of the senior worker in the high-layoff job, \( w_{SR} \), by the law of one price the compensating wage differential between the two non-senior workers, one in the low-layoff job and the other in the high-layoff job is also \( w_{SR} - w_{NS} + p \).

Thus, simple arbitrage arguments used in conjunction with the options model allow the options model to price the compensating wage differential that must exist between identical jobs, one with a high propensity for layoff and one with a low propensity for layoff.
Once the options model is established, intuition insists that the compensating differential be the price of the appropriate put option (or package of put options), since adding the put option to a position that contains the high probability of layoff job results in a portfolio that yields, in all states of nature, returns identical to portfolio that contains only the low probability of layoff job.
CHAPTER FIVE

AN APPLICATION OF THE MODEL: THE VALUE OF SENIORITY TO WORKERS AT

GOULD DEFENSE SYSTEMS, INC., OCEAN SYSTEMS DIVISION

In this section we extend the model developed in Sections 4.1 through 4.3 and use the extended model to analyze an existing labor contract in detail. The contract we analyze is between Gould Defense Systems, Inc., Ocean Systems Division and Local 1631 of the United Auto Workers. Gould is a major defense contractor headquartered in Cleveland, Ohio.

V.1 A Description of the Contract

The labor contract between Gould and the UAW specifies twelve different grades of labor, grades 12 through 23. There is a separate wage schedule for each labor grade. These wage schedules are upward sloping with respect to worker seniority for the first 18 months of each worker's service to the firm and are horizontal thereafter. Ceteris paribus, the higher the job grade, the higher is the wage schedule. There are 15 different departments in which employees work and most departments employ several different grades of labor. Some departments have as few as one job title and employ workers in only one labor grade. At the other extreme, the metal working department...
has eight different job titles and employs workers in five different labor grades. There is typically a progression upward through the different labor grades as a worker attains greater seniority. Some of the progressions are automatic with added seniority. We consider a department that has five wage grades of employment.

Seniority is defined by this contract as "the recognized length of service of each employee in the (collective) bargaining unit since the date of last hire in the Ocean Systems Division" (contract paragraph 81), except that "an employee shall be considered to be on probation and shall not be entitled to any seniority rights until after sixty (60) days after the date of last hire" (paragraph 82). We consider in detail the job progression of employees in the Testing Assurance-Electronic Test Department.

Job Progression

Testing Assurance-Electronic Test

<table>
<thead>
<tr>
<th>Labor Grade</th>
<th>Title</th>
</tr>
</thead>
<tbody>
<tr>
<td>23</td>
<td>General Electronic Tester</td>
</tr>
<tr>
<td>22</td>
<td>Senior Electronic Tester</td>
</tr>
<tr>
<td>20</td>
<td>Electronic Tester</td>
</tr>
<tr>
<td>19</td>
<td>Electronic Tester Learner</td>
</tr>
<tr>
<td>18</td>
<td>Package Tester</td>
</tr>
</tbody>
</table>

De jure, promotions through the ranks from Package Tester up to General Electronic Tester are made on the basis of seniority and
skill. De facto, it is typical that the skills that qualify an employee for a more senior job are obtained by working a less senior job. As a result, the de facto rule for promotions is that when an opening arises in a given labor grade, the most senior worker in the next lower job grade is promoted. For example, when an opening as a General Electronic Tester arises, the Senior Electronic Tester with the most seniority is promoted to General Electronics Tester. If this creates an opening for a Senior Electronic Tester, it is filled by the most senior of the grade 20 Electronic Testers, who becomes the least senior of the grade 22 Senior Electronic Testers. This continues to trickle down, until the most senior Package Tester is promoted to Electronic Tester Learner.

Wage increases are granted for one of three reasons. When a worker is promoted into a higher labor grade, or has a 3, 6, 9, 12, 15 or 18 month anniversary with the firm, the worker receives a small increase in wage that ranges between 6 and 22 cents an hour. In addition, each year the entire work force receives what is more-or-less a cost of living increase. More detail is provided in Table 1.
TABLE 1
Ocean Systems Division
Gould Inc.
Hourly Wage Structure

<table>
<thead>
<tr>
<th>Grade</th>
<th>Start</th>
<th>90 Days</th>
<th>6 Mos.</th>
<th>9 Mos.</th>
<th>12 Mos.</th>
<th>15 Mos.</th>
<th>18 Mos.</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>$7.89</td>
<td>$7.95</td>
<td>$8.01</td>
<td>$8.07</td>
<td>$8.13</td>
<td></td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>7.99</td>
<td>8.05</td>
<td>8.11</td>
<td>8.17</td>
<td>8.23</td>
<td></td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>8.09</td>
<td>8.16</td>
<td>8.23</td>
<td>8.30</td>
<td>8.38</td>
<td></td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>8.20</td>
<td>8.28</td>
<td>8.36</td>
<td>8.44</td>
<td>8.53</td>
<td></td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>8.34</td>
<td>8.42</td>
<td>8.50</td>
<td>8.58</td>
<td>8.67</td>
<td></td>
<td></td>
</tr>
<tr>
<td>17</td>
<td>8.47</td>
<td>8.56</td>
<td>8.65</td>
<td>8.74</td>
<td>8.84</td>
<td></td>
<td></td>
</tr>
<tr>
<td>18</td>
<td>8.64</td>
<td>8.73</td>
<td>8.82</td>
<td>8.91</td>
<td>9.01</td>
<td></td>
<td></td>
</tr>
<tr>
<td>19</td>
<td>8.73</td>
<td>8.82</td>
<td>8.91</td>
<td>9.00</td>
<td>9.09</td>
<td>$9.14</td>
<td>$9.19</td>
</tr>
</tbody>
</table>

Rates effective September 20, 1982 through September 18, 1983.

Note: For the following classifications a 25 cent per hour increase effective 3/21/83.

23ETG 23MHG 23IMG 22MAS 22MRS
22ETS 22BMO 22IMS 21MMA 21HRM
20ETR 22LTO 21DMO

As a result of this pay schedule, once an employee reaches his 18-month anniversary, he has a horizontal wage profile except for promotions and cost-of-living increases.

V.1.1 Layoffs

In the event of a change in business conditions that requires a reduction in the labor force, layoffs from each job classification are by seniority. However, if an employee is displaced from his job by layoff, he has the right to "bump" the least senior worker of the
next-lower job classification and take that job if he is able to do the work and has more seniority than the "bumped" employee. "Bumping" therefore is defined as a less senior worker being laid off from a lower-level job that the firm still intends to staff so as to provide a job opening for a more senior worker who would otherwise be laid off. The result of this "bumping" is increased employment security for the more senior worker and less employment security for the less senior worker. "Bumping" makes the application of the model to this contract substantially more complicated.

V.2 The Value of Seniority at Gould Ocean Systems, Inc.

We begin by considering a worker employed as a General Electronic Tester (labor grade 23) who has more seniority than any of the grade-18, 19, 20 and 22 testers employed in the department. We assume that all of the grade-22 testers have greater seniority than any of the grade-20 testers, all of whom have greater seniority than any of the grade-19 testers, all of whom have greater seniority than any of the grade-18 testers.

The seniority rights to employment that this contract provides are well-suited to being valued in the framework of the options model of labor contracts developed in this paper. The more senior labor grade 23 General Electronic Tester has an option to work in the next period that is valid so long as his VMP remains above VMP* as discussed in Section 4.3. In addition to this option, if the workers VMP falls below VMP* he has an option to occupy someone else's job, but at a slightly lower wage. We define:
\[ VMP23^* - \text{the minimal level to which the worker's VMP can fall and still have his option to work at labor grade 23 be valid} \]

\[ VMP22^* - \text{the minimal level to which the worker's VMP can fall and still have his option to work at labor grade 22 be valid} \]

\[ VMP20^* - \text{the minimal level to which the worker's VMP can fall and still have his option to work at labor grade 20 be valid} \]

\[ VMP19^* - \text{the minimal level to which the worker's VMP can fall and still have his option to work at labor grade 19 be valid} \]

\[ VMP18^* - \text{the minimal level to which the worker's VMP can fall and still have his option to work at labor grade 18 be valid} \]

\[ W023 - \text{the wage specified by the contract for an employee in labor grade 23} \]

\[ W022 - \text{the wage specified by the contract for an employee in labor grade 22} \]

\[ W020 - \text{the wage specified by the contract for an employee in labor grade 20} \]

\[ W019 - \text{the wage specified by the contract for an employee in labor grade 19} \]

\[ W018 - \text{the wage specified by the contract for an employee in labor grade 18} \]

The \( W_i, i=18, 19, 20, 22, 23 \) are data with \( W023 > W022 > W020 > W019 > W018 \). Since \( VMP_i^*, i=18, 19, 20, 22, 23 \) are determined by the zero's of the following variant of equation (4.13):

\[
R_f = \sum_{t=1}^{T} \left( \frac{1}{1+r} \right)^t (VMP_t - W01_t) = 0, \quad (5.1)
\]

we see that \( VMP23^* > VMP22^* > VMP20^* > VMP19^* > VMP18^* \). Only when the senior worker's VMP falls below \( VMP18^* \) does he not have a valid option to work in the next period. If the worker's VMP is such that \( VMP > VMP18^* \), he has an option to work next period for wage \( W018 \). If the
worker's VMP > VMP19*, he also has an option to work next period for wage W019. VMP greater than VMP20*, VMP22* and VMP23* results in the General Electronic Tester having valid options on employment at W020, W022 and W023 respectively. These are, of course, mutually exclusive opportunities: The worker can exercise only one of his valid options. When we consider the case where, for example, the worker's VMP is above VMP19* but below VMP20*, we assume that the rational worker would choose to work for W019, since W019 > W018 and more is preferred to less. Of course if the worker has VMP > VMP20*, he also has an option to work next period for W020 and so forth. The worker's options are summarized in Table 2.
Table 2: Options Possessed and Exercised by Labor Grade 23 Workers

<table>
<thead>
<tr>
<th>Worker’s VMP</th>
<th>Options Possessed by worker</th>
<th>Option Exercised</th>
</tr>
</thead>
<tbody>
<tr>
<td>VMP &lt; VMP18*</td>
<td>none</td>
<td>none</td>
</tr>
<tr>
<td>VMP18* &lt; VMP &lt; VMP19*</td>
<td>to work for W018</td>
<td>work for W018</td>
</tr>
<tr>
<td>VMP19* &lt; VMP &lt; VMP20*</td>
<td>to work for W018</td>
<td>work for W019</td>
</tr>
<tr>
<td></td>
<td>to work for W019</td>
<td>work for W019</td>
</tr>
<tr>
<td>VMP20* &lt; VMP &lt; VMP22*</td>
<td>to work for W018</td>
<td>work for W020</td>
</tr>
<tr>
<td></td>
<td>to work for W019</td>
<td>work for W020</td>
</tr>
<tr>
<td></td>
<td>to work for W020</td>
<td>work for W020</td>
</tr>
<tr>
<td>VMP22* &lt; VMP &lt; VMP23*</td>
<td>to work for W018</td>
<td>work for W022</td>
</tr>
<tr>
<td></td>
<td>to work for W019</td>
<td>work for W022</td>
</tr>
<tr>
<td></td>
<td>to work for W020</td>
<td>work for W022</td>
</tr>
<tr>
<td></td>
<td>to work for W022</td>
<td>work for W022</td>
</tr>
<tr>
<td>VMP &gt; VMP23*</td>
<td>to work for W018</td>
<td>work for W023</td>
</tr>
<tr>
<td></td>
<td>to work for W019</td>
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The value of seniority to a General Electronic Tester who is more senior than any lower labor grade workers in his department is derived in Section V.2.2 where we specify the value of these options assuming that the worker exercises the option on the job in the labor grade that pays him the highest wage.
V.2.1 A Digression on Equation (5.1)

Our basic argument is that while the senior worker has an option on next-period employment at the contract wage, the firm has the right not to honor that option if it is exercised during a period of sufficiently depressed economic conditions. Generalizing the analysis of Section 4.3 that led to equation (4.13) leads us to equation (5.1). Define next period to be Period 2. The argument that is summarized by equation (5.1) is that, for the \(i^{th}\) wage level, the firm uses next period's contract wage, \(W_{Oi}\), along with its best estimate of the workers future VMP, \(\mu_t\), \(t=3,\ldots, N\), and its best estimate of the contract wages, \(W_{Oi_t}\), \(t=3,\ldots, N\), to find the next-period VMP that is the lowest value of next-period worker VMP that allows hiring the worker to be a non-negative net present value (hereafter, NPV) project to the firm.

This means that for each wage level, VMP\(i^*\) is a function of next-period's contract wage for the \(i^{th}\) labor grade, expected future contract wages for salary grade \(i\) and expected future worker VMP's. This carries with it several key implications. First, assuming the distribution of a worker's VMP is invariant to changes in his salary grade, \(W_{O23} > W_{O22} > W_{O20} > W_{O19} > W_{O18}\) implies VMP\(23^*\) > VMP\(22^*\) > VMP\(20^*\) > VMP\(19^*\) > VMP\(18^*\). This means that in a downturn, a senior worker's VMP may be insufficient for his employment to be a non-negative NPV project in wage class 23, but still be sufficient for his employment to be a positive NPV project in a lower wage class. Second, assuming that workers know that this method of determining the VMP's exists, workers wage demands will take into account the fact that the probability of layoff increases monotonically with increases
in the contract wage. We make no attempt to specify rigorously the
dynamic programing problem that yields the VMP*'s, we simply posit
them as the lower bounds on worker VMP for the worker's options to be
binding on the firm.

V.2.2 The Value of the Seniority-Based Options on Employment to a
Senior General Electronic Tester

The position of General Electronic Tester is the highest labor
grade in the Testing Assurance-Electronic Test department at Gould
Ocean Systems. When economic conditions dictate a reduction in the
size of the labor force in that position, layoffs are made in reverse
order of seniority, with the least senior worker being the first
displaced from employment as a General Electronic Tester. This means
that more-senior General Electronic Testers have an option on employ­
ment as General Electronic Testers that less-senior General Electronic
Testers do not have. In addition, if the economic downturn that
precipitated the layoffs is severe enough that the VMP of the senior
General Electronic Tester falls below VMP23* so that his option on
employment as a General Electronic Tester at W023 is no longer bind­
ing, he still has an option to work next period in labor grade 22 as a
Senior Electronic Tester, so long as any worker is employed as a
Senior Electronic Tester. If the economic downturn is so severe that
the senior General Electronic Tester's VMP falls below VMP22*, then
his option on next-period employment as a Senior Electronic Tester at
W022 is not binding. Nevertheless, he still has an option on employ­
ment next period as an Electronic Tester at wage W020, so long as
anyone works as an Electronic Tester. If his VMP is below VMP20*, his
option to work at wage W020 is not binding on the firm. At this point
the worker can exercise his option to work for W019 as an Electronic
Tester Learner, so long as his VMP exceeds VMP19*. If VMP<VMP19*, the
worker can exercise his option on employment at W018 as a Package
Tester if his VMP exceeds VMP18*. If the worker's VMP falls short of
VMP18*, he has no binding option. The value of these options is given
in equation (5.2) as VSBO, the value of seniority based options on
employment.

\[ VSBO = e^{\int_{VMP23^*}^{VMP23^*}} (W023 - W*) L'(W*) dW* \] (5.2)

\[ + e^{\int_{VMP22^*}^{VMP22^*}} (W022 - W*) L'(W*) dW* \]

\[ + e^{\int_{VMP20^*}^{VMP20^*}} (W020 - W*) L'(W*) dW* \]

\[ + e^{\int_{VMP19^*}^{VMP19^*}} (W019 - W*) L'(W*) dW* \]

\[ + e^{\int_{VMP18^*}^{VMP18^*}} (W018 - W*) L'(W*) dW* \]

The first line of equation (5.2) is self explanatory: It is the
present value of (W023-W*) integrated across all values of W* above
VMP23* and below W023 inclusive of VMP23* and W023. This first line is the same as equation (4.16) except that W023 is substituted for W0 and VMP23* is substituted for VMP*.

The second line of (5.2) requires some additional interpretation. The second line represents the value of an option to work for W022 that is binding when VMP exceeds VMP22*. Hence the lower limit of integration is VMP22*. The upper limit of integration, \( \min \{VMP23*, W022\} \), requires interpretation. Before proceeding it is important to establish that these limits of integration in equation (5.2) (the W0i and VMPi*), are all assumed to be determined at the time that the contract wage levels (the W0i) are set. The VMPi* may be updated by the firm from time to time but are taken as being predetermined parameters for the purpose of valuing seniority. If VMP23* < W022, the upper bound of the integral in the second line of (5.2) is VMP23* which has a natural interpretation. In this case, the option to work for W022 begins to be valid at the same level of VMP that the option to work for W023 ceases to be valid.

We now consider the outcome when VMP23*>W022 and the terminal value of W is between VMP23* and W022. Equation (5.1) states that when VMP is below VMP23*, the worker does not have an option to work for wage W023. The worker does however have an option to work for W022 because when W is above W022, W is above VMP22*. Since the worker's VMP is the same in the present firm and in his best alternative employment opportunity and there are competitive labor markets with zero mobility costs, the worker can sell his labor outside of the firm for VMP. This means that his option to sell his labor for W022
has an exercise price that is less than the market value of the asset (the worker's labor) upon which the option is written. An option is a unilateral choice instrument: it is exercised if and only if the owner of the option finds exercise profitable. It is profitable to exercise a European put option if and only if, on the option's expiration date, the exercise price exceeds the market value of the commodity on which the option is written. Otherwise the option is of zero value and will not be exercised. The conditions posited in this paragraph are summarized in the fundamental boundary condition of put option pricing, that at the option's expiration date:

\[ p^* = \text{MAX}[0, W_{Oi} - W^*], \]

where \( p^* \) is the value of the put option at the option's expiration, \( W_{Oi} \) is the option's exercise price and \( W^* \) is the market value, at the option's expiration, of the commodity upon which the option is written. In our case, this commodity is the worker's labor. Hence, the value of any put option is zero when the exercise price (here \( W_{Oi} \)) is less than the terminal value of the commodity upon which the option is written. A put option with an exercise price below the market price of the asset upon which it is written is called an out-of-the-money option. Exercising an option results in the option holder receiving the exercise price and giving up the market value of the asset. Since a put option that is out-of-the-money at the option's expiration date has an exercise price below the market price of the asset, it is never exercised and has zero value according to the fundamental boundary condition.
When VMP at the option's exercise date is above W022, the option to work for W022 has no value and the option to work for W023 is not binding on the firm. The worker has no option of value when VMP23* > W* > W022 and this corresponds to a gap in the value of the seniority based options given in equation (5.2) and equation (5.3). The interpretations of the third, fourth and fifth integrals in equation (5.2) are analogous to the interpretation of the second integral.

For the sake of clarity, let us assume VMP23* > W022, VMP22* > W020 and VMP20* > W019, so that gaps exist and partition the density between VMP20* and +∞. The same analysis holds if we partition the space from zero to +∞. The partition of the density and the results of W* falling in each range are given below.

\[ W^* > W023 \rightarrow \text{Option to work for } W023 \text{ is valid and has value equal to } \text{MAX}[0, W023-W^*] = 0 \text{ by the fundamental boundary condition.} \]

\[ W023 > W^* > VMP23^* \rightarrow \text{Option to work for } W023 \text{ is valid and has value equal to } \text{MAX}[0, W023-W^*] = W023-W^* > 0 \text{ by the fundamental boundary condition.} \]

\[ \text{VMP23^* > W^* > W022} \rightarrow \text{Option to work for } W023 \text{ is not valid because } W^* < \text{VMP23^*}. \text{ Option to work for } W022 \text{ is valid and has value equal to } \text{MAX}[0, W022-W^*] = 0. \text{ The option to work for } W022 \text{ has no value for } W^* \text{ in this range and there is a "gap" in VSBO.} \]
WO22 > \( w^* \) > VMP22 \( ^* \) \( \Rightarrow \) Option to work for W022 is valid and has value equal to \( \text{MAX}[0, \text{VMP22} - w^*] - \text{VMP22} - w^* > 0 \).

VMP22 \( ^* \) > \( w^* \) > W020 \( \Rightarrow \) Option to work for W022 is not valid. Option to work for W020 is valid and has value equal to \( \text{MAX}[0, \text{W020} - w^*] = 0 \). There is a gap in VSBO.

W020 > \( w^* \) > VMP20 \( ^* \) \( \Rightarrow \) Option to work for W020 is valid and has value equal to \( \text{MAX}[0, \text{W020} - w^*] = \text{W020} - w^* \).

For these gaps in the value of VSBO to occur, it is necessary that either the present firm or an alternative firm (where the worker has the same VMP) always offer employment to the worker at or above wage W023, W022 and W020 when the workers VMP exceeds W023, W022 and W020 respectively.

If it is not true that the worker is always offered employment inside or outside the firm at or above W023, W022 and W020 when his VMP exceeds W023, W022 and W020, then these gaps cease to exist. For example, if the worker has firm-specific human-capital so that his value of marginal product in an alternate employment, \( \text{VMP}_{\text{alt}} \), is below his present firm VMP, then the competitive market pressures that led the worker's present employer to offer employment are absent. But it was exactly these market pressures that led to the existence of the gaps in the first place. In short, whether or not the gaps exist depends upon the worker's alternative opportunities. If the worker is, in the absence of the option, assured employment at or above W0i
when VMP exceeds WOi, then the gaps exist. Otherwise the gaps do not exist and equation (5.2) becomes (5.2'):

\[
\text{VSBO} = e^{\int_{\text{VMP23}}^{\text{VMP23}} (W_{023} - W^*) L'(W^*) dW^*} + e^{\int_{\text{VMP22}}^{\text{VMP22}} (W_{022} - W^*) L'(W^*) dW^*} + e^{\int_{\text{VMP20}}^{\text{VMP20}} (W_{020} - W^*) L'(W^*) dW^*}
\]

The same theorem that is used to evaluate equations (4.15) and (4.16) of section 4.3 can be used to evaluate equations (5.2) and (5.2'). Generalizing the notation and rearranging the theorem yields:

**Theorem:** Suppose \( L'(W^*) \) is a lognormal density function with

\[
p = \begin{cases} 
0 & \text{if } W^* > \psi WOi \\
1 - e^{\gamma T (W_{Oi} - W^*)} & \text{if } \psi WOi > W^* > \psi WOi \\
0 & \text{if } W^* < \psi WOi
\end{cases}
\]
\[ E(p) = e^{-rT} \int_{W_{01}}^{\phi W_{01}} (W_{01} - W^*) L'(W^*) dW^* \]

Applying this theorem to equation (5.2) one integral at a time yields the solution to the problem of valuing the seniority-based right to next-period employment. The first integral is evaluated by setting \( W_{01} = W_{023}, \phi = 1 \) and \( \psi = VMP_{23*}/W_{023} \). Defining \( \eta_{22} = MIN[VMP_{23*}, W_{022}] \), the second integral is evaluated by setting \( W_{01} = W_{022}, \phi = \eta_{22}/W_{022} \) and \( \psi = VMP_{22*}/W_{022} \). Defining \( \eta_{20} = MIN[VMP_{22*}, W_{020}] \), the third integral is evaluated by setting \( W_{01} = W_{020}, \phi = \eta_{20}/W_{020} \) and \( \psi = VMP_{20*}/W_{020} \). Defining \( \eta_{19} = MIN[VMP_{20*}, W_{019}] \), the fourth integral is evaluated by setting \( W_{01} = W_{019}, \phi = \eta_{19}/W_{019} \) and \( \psi = VMP_{19*}/W_{019} \). Defining \( \eta_{18} = MIN[VMP_{19*}, W_{018}] \), the fifth integral is evaluated using the theorem with \( W_{01} = W_{018}, \phi = \eta_{18}/W_{018} \) and \( \psi = VMP_{18*}/W_{018} \). The solution to equation (5.2) is the value, in a competitive labor market, to a senior General Electronic Tester, of the seniority-based right to next-period employment awarded to him by the labor contract between Gould and the UAW. This solution is given in equation (5.3)
\[ \text{VSBO} = -W N \left\{ \frac{-\ln(W/W023)-(r+\sigma^2/2)T}{\sigma/T} \right\} + e^{-rT_{WO23}} N \left\{ \frac{-\ln(W/W023)-(r-\sigma^2/2)T}{\sigma/T} \right\} \]

\[ + W N \left\{ \frac{-\ln(W/VMP23*)-(r+\sigma^2/2)T}{\sigma/T} \right\} - e^{-rT_{WO23}} N \left\{ \frac{-\ln(W/VMP23*)-(r-\sigma^2/2)T}{\sigma/T} \right\} \]

\[ -W N \left\{ \frac{-\ln(W/\eta22)-(r+\sigma^2/2)T}{\sigma/T} \right\} + e^{-rT_{WO22}} N \left\{ \frac{-\ln(W/\eta22)-(r-\sigma^2/2)T}{\sigma/T} \right\} \]

\[ + W N \left\{ \frac{-\ln(W/VMP22*)-(r+\sigma^2/2)T}{\sigma/T} \right\} - e^{-rT_{WO22}} N \left\{ \frac{-\ln(W/VMP22*)-(r-\sigma^2/2)T}{\sigma/T} \right\} \]

\[ -W N \left\{ \frac{-\ln(W/\eta20)-(r+\sigma^2/2)T}{\sigma/T} \right\} + e^{-rT_{WO20}} N \left\{ \frac{-\ln(W/\eta20)-(r-\sigma^2/2)T}{\sigma/T} \right\} \]

\[ + W N \left\{ \frac{-\ln(W/VMP20*)-(r+\sigma^2/2)T}{\sigma/T} \right\} - e^{-rT_{WO20}} N \left\{ \frac{-\ln(W/VMP20*)-(r-\sigma^2/2)T}{\sigma/T} \right\} \]

\[ -W N \left\{ \frac{-\ln(W/\eta19)-(r+\sigma^2/2)T}{\sigma/T} \right\} + e^{-rT_{WO19}} N \left\{ \frac{-\ln(W/\eta19)-(r-\sigma^2/2)T}{\sigma/T} \right\} \]

\[ + W N \left\{ \frac{-\ln(W/VMP19*)-(r+\sigma^2/2)T}{\sigma/T} \right\} - e^{-rT_{WO19}} N \left\{ \frac{-\ln(W/VMP19*)-(r-\sigma^2/2)T}{\sigma/T} \right\} \]
The interpretation of equation (5.3) proceeds most easily if we refer to the first line of the equation as line A, the second line as line B and so forth, so that the final line is referred to as line J.

Line A is a simple put option to sell one unit of labor to the firm in the next period for exercise price $W_{023}$. This is exactly equation (4.10) that gave the value of seniority in the most basic case where the senior worker's put option was sure to be honored.

Line B is the (negative) value to the worker of the contract's clause that specifies that the firm has the right to cancel the option on employment at $W_{023}$ in those adverse states of nature that force the senior workers $VMP$ below $VMP_{23*}$. This option possessed by the firm does not fit into the standard options framework, but it is identical to a put option with exercise price $VMP_{23*}$ except for the replacement of the exercise price where it multiplies $e^{-rT}$ by $W_{023}$.

Line C is somewhat more complicated due to the presence of $\eta_{22}$. $\eta_{22} = \text{MIN}[W_{022}, VMP_{23*}]$ is a predetermined parameter, but it can take either of two values. The reason that the upper limit of integration of the second integral in equation (5.2) takes this form is that in the competitive case the senior worker has an option on employment at $W_{022}$ that has value for observations of $VMP$ up to the minimum of $W_{022}$.
and VMP23*. The option to work for W022 is of no value if VMP exceeds W022, because in this case the worker will always be offered employment at W022 since his VMP exceeds the wage. Further, the option to work for W022 is of no value if VMP exceeds VMP23*, since in that case the worker has an option on employment at wage W023. W023 exceeds W022 and the rational worker is assumed always to exercise the option on the employment that pays the highest wage. If VMP23* exceeds W022 there is a gap in the density where the worker has no option of value, but this is only because an option to do something that is certain to be done in the absence of the option is of no value because it adds nothing to the opportunity set. In cases where W022 exceeds VMP23*, the option to work for W022 has no value in the range of VMP between VMP23* and W022 since the then binding option to work for W023 dominates the option to work for W022. Specifying the upper limit of the second integral as $\eta_{22} = \min\{\text{VMP23*}, \text{W022}\}$ summarizes these conditions. The use of $\eta_{20}$, $\eta_{19}$ and $\eta_{18}$ in the third, fourth and fifth integrals of (5.2) are substantiated in an analogous manner.

Lines A and B of (5.2) represent the value of the senior General Electronic Tester's option to work as a General Electronic Tester at contract wage W023. Lines C and D of (5.2) represent the value of the senior General Electronic Tester's option to work as a Senior Electronic Tester (a lower level position) at wage W022. Lines E and F, G and H and I and J of (5.2) represent the value of the senior General Electronic Testers options to work as an Electronic Tester for wage W020, as an Electronic Tester Learner for wage W019 and as a Package Tester for wage W018 respectively.
Before moving on, we consider once more lines C and D of (5.2). When \( W_{022} < VMP_{23*} \), line C is a simple put option with exercise price \( W_{022} \) to sell one unit of next-period labor. When \( W_{022} > VMP_{23*} \), line C no longer has an interpretation as a standard put option with exercise price \( W_{022} \). This is neither surprising nor alarming, since the option's bounds for exercise no longer coincide with its exercise price. Lines C and D remain the correct value of the contingent claim to next-period employment at wage \( W_{022} \) that is valid for VMP between \( VMP_{22*} \) and \( VMP_{23*} \). The analogous interpretations hold for lines E and F, G and H, and I and J.

Applying the theorem to equation (5.2') yields the value of the seniority-based options on the next-period employment in the case where the gaps do not exist. This value is given by equation (5.3').

\[
VSBO = -W N \left\{ \frac{-\ln(W/W_{023})-(r+\sigma^2/2)T}{\sigma \sqrt{T}} \right\} + e^{-rT_{W_{023}}} N \left\{ \frac{-\ln(W/W_{023})-(r-\sigma^2/2)T}{\sigma \sqrt{T}} \right\} \\
+ e^{-rT_{(W_{022}-W_{023})}} N \left\{ \frac{-\ln(W/VMP_{23*})-(r-\sigma^2/2)T}{\sigma \sqrt{T}} \right\} \\
+ e^{-rT_{(W_{020}-W_{022})}} N \left\{ \frac{-\ln(W/VMP_{22*})-(r-\sigma^2/2)T}{\sigma \sqrt{T}} \right\} \\
+ e^{-rT_{(W_{019}-W_{020})}} N \left\{ \frac{-\ln(W/VMP_{20*})-(r-\sigma^2/2)T}{\sigma \sqrt{T}} \right\}
\]
Eliminating the gaps in the valuation expression causes many of the terms in equation (5.3) to cancel, leaving a simpler expression.

The value of the seniority-based options on employment for a senior General Electronic Tester is given by equation (5.3) when we assume competitive behavior by firms and no mobility costs and no firm-specific human-capital. If these assumptions are not met we cannot assert that the worker will be offered, in the absence of an option, employment at or above wage \( W_{0i} \) when his VMP exceeds \( W_{0i} \). In this case the gaps do not exist and \( VSBO \) is given by equation (5.3').

The question of the existence of gaps in the valuation equation for \( VSBO \) is, in general, a theoretical question that turns on the existence or absence of competitive behavior, mobility costs and firm-specific human-capital. That notwithstanding, the Gould contract wage structure displays characteristics that make the theoretical question more or less moot for the specific contract that we are considering in this chapter. The reasoning behind this statement is as follows. The theoretical question revolves around the form of the upper limits of integration of the last four integrals of the valuation equations 5.2 and 5.2'. In equation 5.2 (the case of competition) the second integral has as its upper limit \( MIN[\text{VMP23}, W_{022}] \), while the
corresponding upper limit of 5.2' is simply VMP23*. Referring to Table 1, we see that, with 18 months of seniority, W023 = $10.03 while W022 = $9.81, so that W022 is 97.8 percent of W023. As a result, unless VMP23* exceeds 97.8 percent of W023, \( \min\{VMP23*, W022\} = VMP23* \). Otherwise, when the workers VMP fell to 97.8 percent of his contract wage, he would have no option to work at his job. The analysis of sections 5.3 and 5.4 suggest that a seniority benefit that is valid in so few states of the world would have little value indeed. Furthermore, while Gould was very closed-mouthed about layoff policy, the author knows personally several Gould employees with ten years tenure who have never been laid-off. Gould's unwillingness to provide information on layoff incidence makes it impossible to determine with certainty whether workers are laid off whenever their VMP falls below 97.8 percent of their contract wage, but the observable evidence, scant as it may be, leads one to discount the possibility that this is the case. The upshot of this discussion is simply that, in the case of Gould, equations 5.2 and 5.2' are almost certainly numerically identical, even though the question can only be answered by Gould itself.

V.2.2.1 Comparative Statics

Before moving on to valuing seniority provisions of the Gould contract for other workers, it is of interest to investigate the possibility of regularities in how the value of seniority is affected by changes in the parameters of VSBO. We concentrate on equation 5.2' and equation 5.3', since it is likely that equation 5.2 and equation
5.3 reduce to equation 5.2' and equation 5.3'. Equation 5.3' is a function of layoff points, VMP, contract wages, W0, the spot wage W, the interest rate r, and the variance of the spot wage, $\sigma^2$.

Beginning with the layoff points, VMP;:

$$\frac{\partial V_{SBO}}{\partial \text{VMP}^{23*}} = e^{-rT} \left( \frac{W_{022} - W_{023}N'}{\text{VMP}^{23*}} \right) \left\{ \frac{-\ln(W/\text{VMP}^{23*}) - (r - \sigma^2/2)T}{\sigma/T} \right\} < 0. \tag{5.4}$$

The negative sign is a result of all terms in equation 5.4 being positive except W022-W023, which is negative.

The partial derivatives of VSBO with respect to VMP22*, VMP20* and VMP19* are perfectly analogous to $\frac{\partial V_{SBO}}{\partial \text{VMP}^{23*}}$ and are omitted. The partial derivative of VSBO with respect to VMP18* is not quite analogous to the others, so we consider it next.

$$\frac{\partial V_{SBO}}{\partial \text{VMP}^{18*}} = W \cdot N' \left\{ \frac{-\ln(W/\text{VMP}^{18*}) - (r + \sigma^2/2)T}{\sigma/T} \right\} \frac{1}{\text{VMP}^{18*}\sigma/T} \tag{5.5}$$

$$-e^{-rT} \cdot W018 \cdot N' \left\{ \frac{-\ln(W/\text{VMP}^{18*}) - (r - \sigma^2/2)T}{\sigma/T} \right\} \frac{1}{\text{VMP}^{18*}\sigma/T}. \tag{5.5}$$

It is not immediately apparent that equation 5.5 is negative. However, if we rewrite the expression in a carefully chosen manner, we can simplify the expression and establish that it is negative.
Rewriting equation 5.5,

\[
\frac{\partial \text{VSBO}}{\partial \text{VMP18*}} = \left\{ W \cdot N' \left\{ -\frac{\ln(W/\text{VMP18*})-(r+\sigma^2/2)T}{\sigma/\sqrt{T}} \right\} \right. \\
\left. -e^{-rT} \text{VMP18*} \cdot N' \left\{ -\frac{\ln(W/\text{VMP18*})-(r-\sigma^2/2)T}{\sigma/\sqrt{T}} \right\} \right\} \frac{1}{\text{VMP18*} \sigma/\sqrt{T}} \]

In Appendix A, it is shown that

\[
W \cdot N' \left\{ -\frac{\ln(W/\text{VMP18*})-(r+\sigma^2/2)T}{\sigma/\sqrt{T}} \right\} \\
-e^{-rT} \text{VMP18*} \cdot N' \left\{ -\frac{\ln(W/\text{VMP18*})-(r-\sigma^2/2)T}{\sigma/\sqrt{T}} \right\} = 0,
\]

and as a result,

\[
\frac{\partial \text{VSBO}}{\partial \text{WO18*}} = -\frac{e^{-rT}(\text{WO18-\text{VMP18*})}}{\text{VMP18*} \sigma/\sqrt{T}} \cdot N' \left\{ -\frac{\ln(W/\text{VMP18*})-(r-\sigma^2/2)T}{\sigma/\sqrt{T}} \right\} < 0. \hspace{1cm} (5.6)
\]

The derivatives in equations 5.4 and 5.5 show that whenever the level of worker VMP required for the workers' option to be valid increases, the value of the option on next period employment decreases. This
result is consistent with intuition, since increasing $VMP_i^*$ reduces the number of states of the world in which the workers' option is valid.

Since most of the remaining comparative statics contain fairly large numbers of complicated yet often identical terms it seems prudent to simplify notation by defining:

\[
\text{ARGWO23}^+ = \left\{ \frac{-\ln(W/W_023) - (r+\sigma^2/2)T}{\sigma T} \right\}
\]

\[
\text{ARGWO23} = \left\{ \frac{-\ln(W/W_023) - (r-\sigma^2/2)T}{\sigma T} \right\}
\]

\[
\text{ARG23}^* = \left\{ \frac{-\ln(W/VMP23^*) - (r-\sigma^2/2)T}{\sigma T} \right\}
\]

\[
\text{ARG22}^* = \left\{ \frac{-\ln(W/VMP22^*) - (r-\sigma^2/2)T}{\sigma T} \right\}
\]

\[
\text{ARG20}^* = \left\{ \frac{-\ln(W/VMP20^*) - (r-\sigma^2/2)T}{\sigma T} \right\}
\]

\[
\text{ARG19}^* = \left\{ \frac{-\ln(W/VMP19^*) - (r-\sigma^2/2)T}{\sigma T} \right\}
\]

\[
\text{ARG18}^* = \left\{ \frac{-\ln(W/VMP18^*) - (r-\sigma^2/2)T}{\sigma T} \right\}
\]

\[
\text{ARG18}^{**} = \left\{ \frac{-\ln(W/VMP18^*) - (r+\sigma^2/2)T}{\sigma T} \right\}
\]
The impacts upon VSBO of changes in contract wage rates occur through two separate mechanisms, necessitating the use of a partial total derivative instead of a simple partial derivative. Changes in contract wage rates affect VSBO directly because, ceteris paribus, increasing the contract wage makes the option to receive that contract wage more valuable. However, in the present context it would seem rather unrealistic to expect that there would be an increase in the contract wage without an attendant increase in the minimum level of worker VMP required for the worker's option to be valid. To put it differently, VMP\_i\* is almost certainly influenced by WO\_i and as a result, the partial total derivative in equation (5.7) is indicated.

\[
\frac{\partial V_{\text{VSBO}}}{\partial WO_i^\text{\_i}} \frac{\partial V_{\text{VMP}}}{\partial V_{\text{MP}}^\text{\_i}} + \frac{\partial V_{\text{VSBO}}}{\partial WO_i^\text{\_i}} = 0
\]  

(5.7)

We have established that \( \frac{\partial V_{\text{VSBO}}}{\partial V_{\text{VMP}}^\text{\_i}} \) is negative. It seems clear that \( \frac{\partial V_{\text{MP}}^\text{\_i}}{\partial WO_i^\text{\_i}} \) is some positive function (in sections 5.3 and 5.4 we assume that \( \frac{\partial V_{\text{MP}}^\text{\_i}}{\partial WO_i^\text{\_i}} \) is a positive constant) since it would be highly counterintuitive to increase the contract wage without increasing the minimum level of VMP required to claim that wage. We then need only find \( \frac{\partial V_{\text{VSBO}}}{\partial WO_i^\text{\_i}} \) and assemble the components of equation (5.7).

From equation 5.3', we see that
Using arguments identical to those of appendix A, it can be shown that 
\[-WN'(ARGW023+) + e^{-rT} WO23 N'(ARGW023) = 0\] and as a result

\[\frac{\partial VSBO}{\partial WO23} = e^{-rT} [N(ARGW023) - N(ARG23*]) > 0. \quad (5.8)\]

This expression is positive since \(WO23 > VMP23*\) and \(N(\cdot)\) is a normal c.d.f.

When equations 5.4, 5.8 and 5.7 are combined, it is clear that \(*\frac{\partial VSBO}{\partial WO23}\) is the sum of a negative first term and a positive second term. In particular,

\[\frac{*\partial VSBO}{*\partial WO23} = e^{-rT} (WO22-WO23) N\left\{-\frac{\ln(W/VMP23*)}{\sigma_T} - \frac{(r-\sigma^2/2)}{T}\right\} \left\{\frac{1}{VMP23*\sigma_T} \cdot \frac{\partial VMP23*}{\partial WO23}\right\} + e^{-rT} N(ARGW023) - N(ARG23*) \leq 0. \quad (5.9)\]
Even if \( \frac{\partial \text{VMP}}{\partial \text{W}023} \) is set equal to a constant, this partial total derivative is of ambiguous sign. It is, of course, a trivial exercise analytically to set \( \frac{\partial \text{VMP}}{\partial \text{W}023} \) in such a way as to make this equation positive, negative or zero. However, the resulting expression would be rather useless because it would require a continuously and instantaneously adjusted layoff policy, since the expression would contain a continuous stochastic process.

The total partial derivatives of VSBO with respect to the other contract wage rates do not differ significantly from these and are hence omitted.

We now turn to the impact upon VSBO of the change in the variance of VMP:

\[
\frac{\partial \text{VSBO}}{\partial \sigma^2} = -\text{W} \cdot \text{N}'(\text{ARGW}023+) \cdot \frac{1}{2\sigma^2}(\sigma / \sqrt{T} - \text{ARGW}023+)
\]

\[
+ e^{-rT} \cdot \text{W}023 \cdot \text{N}'(\text{ARGW}023) \cdot \frac{1}{2\sigma^2}(\sigma / \sqrt{T} - \text{ARGW}023)
\]

\[
+ e^{-rT} \cdot (\text{W}022 - \text{W}023) \cdot \text{N}'(\text{ARG}23*) \cdot \frac{1}{2\sigma^2}(\sigma / \sqrt{T} - \text{ARG}23*)
\]

\[
+ e^{-rT} \cdot (\text{W}020 - \text{W}022) \cdot \text{N}'(\text{ARG}22*) \cdot \frac{1}{2\sigma^2}(\sigma / \sqrt{T} - \text{ARG}22*)
\]

\[
+ e^{-rT} \cdot (\text{W}019 - \text{W}020) \cdot \text{N}'(\text{ARG}20*) \cdot \frac{1}{2\sigma^2}(\sigma / \sqrt{T} - \text{ARG}20*)
\]
This derivative is of ambiguous sign and there does not seem to be a tractable set of parameter restrictions that would allow it to be signed.

The impact of the interest rate upon VSBO is also of ambiguous sign:

\[
\delta VSBO = \frac{\sqrt{\sigma}}{\sigma} \cdot \frac{N'(W_23) - N'(W_18)}{N'(W_23) - N'(W_18)} \quad (5.11)
\]

\[
= -rT \left[ W_23 \cdot N'(W_23) - N'(W_23) \right] + e \left[ W_20 \cdot N'(W_20) - N'(W_20) \right]
\]

\[
+ e \left[ W_19 \cdot N'(W_19) - N'(W_19) \right] - \frac{1}{2} \frac{1}{\sigma} \left( \sigma - \sqrt{\sigma} - \sqrt{\sigma} \right) \geq 0.
\]
Finally, we consider the partial of \( V_{SB0} \) with respect to the spot wage.

\[
\frac{\partial V_{SB0}}{\partial W} = N'(ARG18*+) - N'(ARG20*+)
+ \frac{1}{\sigma \sqrt{T}} [N'(ARG20*+) - N'(ARG23*+)]
+ \frac{-r T}{W_0} \left[ W_0 [N'(ARG23*) - N'(ARG20*+)] + W_0 [N'(ARG19*) - N'(ARG20*+)] + W_0 [N'(ARG18*) - N'(ARG19*)] \right] \lesssim 0.
\]

This derivative is also of ambiguous sign.

The sign ambiguity of the derivatives in equations 5.8, 5.9, 5.10 and 5.11 is neither surprising nor alarming. If anything, once one goes beyond very simple options, it is as common to find sign ambiguity of the derivatives as it is to find a derivative that can be signed. In Section 5.3 and 5.4, where we evaluate equation 5.3' numerically, it is borne out that the only derivative that can be signed is \( \frac{\partial V_{SB0}}{\partial V_{MP1}*} \). Additionally, it is seen that no tractable set of parameter restrictions can be imposed that make any of these unsigned derivatives monotonic.
To this point, this section establishes the value of the seniority based option on next-period employment to a senior General Electronic Tester provided for by the labor contract between Gould Ocean Systems, Inc. and the United Auto Workers and then goes on to show that the partial of VSBO with respect to VMPi* is the only one of the model's comparative statics that can be signed.

We now consider the total compensation of the senior General Electronic Tester. This compensation consists of the contract wage, W023, plus the value of the options on next-period employment given as VSBO in either equation (5.3) or (5.3'), whichever is appropriate to the situation. The total compensation of the senior worker is seen to be

\[ C_{SR} = W023 + VSBO. \]  

(5.13)

As a standard of comparison we now consider the compensation of a worker with no seniority. The difference between (5.13) and the compensation of the worker with no seniority in the Testing Assurance-Electronic Test Department is the value of seniority. The compensation of a worker with no seniority is the workers wage, since this worker has no options on future employment. It is important to point out that the purpose of this analysis is the valuation of seniority. As a result, when we calculate the total compensation of the senior worker and the total compensation of the non-senior worker, we do so for the purpose of subtracting the latter from the former to find the value of seniority. This means that there is no harm done by
excluding from $C^{SR}$ in (5.13) and $C^{NS}$ in (5.14) the many terms that appear identically in $C^{SR}$ and $C^{NS}$. Therefore we do not consider tangibles or intangibles such as health insurance policies, union dues or any other compensation related cost or benefit that accrues equally to senior and non-senior workers.

The total compensation of the non-senior worker is simply the wage paid, in this case $W_{018}$:

$$C^{NS} = W_{018}.$$  \hspace{1cm} (5.14)

The value of seniority is given by subtracting (5.14) from (5.13),

$$V_S = C^{SR} - C^{NS}$$

$$= W_{023} + V_{SOB} - W_{018}$$

$$= WD + V_{SOB},$$

where $WD$ is the wage differential and $V_{SOB}$ is given by equation (5.3) or (5.3').

The Gould-UAW contract specifies a complicated set of options that are given to the senior General Electronic Tester. This notwithstanding, the conceptual similarity between the value of seniority in the most basic case (see equation (4.12) in section 4.2.1) and the value of seniority here is striking. This value is still given by a package consisting of a wage differential and a package of options. The package of options has become much more complicated, but is still properly valued using options pricing techniques.
V.2.3 The Value of Seniority to a Senior Electronic Tester

Having determined that the value of the seniority based options on employment to a General Electronic Tester is given by equations (5.3) or (5.3'), we now turn to the task of finding the value of the corresponding group of options held by Senior Electronic Tester. Like the senior General Electronic Tester, the senior Senior Electronic Tester (wage grade 22) not only has an option on his own job in the next period, but also has a series of options on lower-level jobs that are valid in the event that his VMP drops below VMP22*. Finding the value of these options to the Senior Electronic Tester is more complicated than finding the value of the options possessed by the General Electronic Tester because there exist states of nature in which the Senior Electronic Tester can be "bumped" from his job by a General Electronic Tester whose VMP falls below VMP23*. This is because line 2 of equation (5.2) represents the General Electronic Tester's option on some Senior Electronic Tester's job. So, while a General Electronic Tester can only be a "bumper", a Senior Electronic Tester can either be a "bumper" or a "bumpee". To value properly the options held by the Senior Electronic Tester, we must account for the General Electronic Tester's option to bump the Senior Electronic Tester from his job and the impact that this has on the value of the Senior Electronic Tester's options.

To this point, we have needed to concern ourselves with only one worker, the worker whose option was being valued. Since this is no longer the case, we must turn our attention to a more detailed characterization of the firm's production function and the implications of
the seniority relationship between the workers. We assume that the firm's production function exhibits diminishing marginal productivity of labor and hence, diminishing value of marginal product of labor. This means that even if the workers are homogeneous, as more workers are hired into a given classification, the VMP of latter workers is less than the VMP of the first workers hired. This diminishing VMP takes on a particular significance when workers in a classification are ordered by seniority for layoff.

In Chapter Two we assume that worker VMP follows geometric Brownian motion. We continue to assume that VMP follows geometric Brownian motion, but we now consider many workers, each of whose VMP is different from the other. The specific relationship between the workers' VMP's is not of concern here. We require only that $VMP^1 > VMP^2 > \ldots > VMP^N$, where $VMP^j$ is the VMP of the $j^{th}$ worker hired, who is also the $j^{th}$ most senior worker in the job. This is a standard assumption. Clearly, this means that for each different worker, there is a different process that determines VMP. Generalizing the notation of Chapter Two, we define $W_{i,j}^1$ as the VMP or shadow wage of the $j^{th}$ most senior worker in job grade $i$, where $j=1, \ldots, N$ and $i=18, 19, 20, 22, 23$. We consider now a worker from labor grade 23 who has an option to work as a Senior Electronic Tester in labor grade 22 and more importantly, the impact of the General Electronic Tester's option to work for $W_{22}^0$ upon the distribution of $W_{22,j}^2$.

We define $N_{23}^1$ as the number of workers employed as General Electronic Tester in period one. The options discussed are valid and exercised in period-two. We likewise define $N_{22}^1, N_{20}^1, N_{19}^1$ and $N_{18}^1$ as
the number of Senior Electronic Testers, Electronic Testers, Electronic Tester Learners and Package Testers employed in period-one. We further define $N_{23}^2$, $N_{22}^2$, $N_{20}^2$, $N_{19}^2$ and $N_{18}^2$ as the respective period-two magnitudes. The relative number of employees in the different wage grades turns out to be important in determining the value of the seniority-based options on employment for workers in wage grade 22 and below. That is because the particular contract that we are considering states that in the event of a reduction in work force, any affected worker can "bump" the least senior worker in a lower labor grade and occupy that job, providing that the "bumping" employee has more seniority than the worker he displaces. Labor grade 22 workers that are at no risk of being displaced by being "bumped" by labor grade 23 employees have options on next-period employment very much like those possessed by the grade 23 workers. In contrast, grade 22 (and lower) employees who are at risk of being "bumped" have much more complicated options.

We first consider the $N_{22}^2 - N_{23}^1$ most senior grade 22 employees. These workers are not at the risk of being "bumped", because the least senior worker in a grade is always the first "bumped". If, for example, there are eight grade 22 workers in period-two and five grade 23 workers in period-one, the three most senior grade 22 workers cannot be bumped. They are at risk of grade 22 job loss because of the chance of a fall in their own VMP, but not because of a fall in the VMP of a grade 23 worker who then "bumps" them. As a result, the options held by the $N_{22}^2 - N_{23}^1$ most senior grade 22 workers are very
similar to the options held by the grade 23 workers. The value of these options is given by equation (5.15).

\[
\begin{align*}
\text{VSBO}^{22} & = e^{-rT} \int_{\text{VMP}^{22}*} (W^{22} - W^{22}, J^*) L'(W^{22}, J^*) dW^{22}, J^* \\
& + e^{-rT} \int_{\text{MIN}[\text{VMP}^{22*}, W^{20}]} (W^{20} - W^{22}, J^*) L'(W^{22}, J^*) dW^{22}, J^* \\
& + e^{-rT} \int_{\text{MIN}[\text{VMP}^{20*}, W^{19}]} (W^{19} - W^{22}, J^*) L'(W^{22}, J^*) dW^{22}, J^* \\
& + e^{-rT} \int_{\text{MIN}[\text{VMP}^{19*}, W^{18}]} (W^{18} - W^{22}, J^*) L'(W^{22}, J^*) dW^{22}, J^*. 
\end{align*}
\]

Equation (5.15) is identical to equation (5.2) except that the first line of equation (5.2) does not appear in equation (5.15). If the competitive, no mobility costs assumptions of Chapter Two do not hold, equations (5.15) and (5.16) may be modified as (5.2) and (5.3) were to eliminate the valuation gaps. Except where otherwise noted, we will assume the assumptions of Chapter Two hold for the remainder of this paper. Also the notation in equation (5.15) has been generalized as indicated earlier in this section. The first line of equation (5.2) is the value of a General Electronic Tester's option to work as a General Electronic Tester. The Senior Electronic Tester has no option to work as a General Electronic Tester. If laid off from his job, a worker can chose to move down a grade to remain employed, but
has no option to move up a grade. This means that while the General Electronic Tester has options on work in labor grades 23, 22, 20, 19 and 18, the Senior Electronic Tester has options on work only in labor grades 22, 20, 19 and 18.

The first line in equation (5.15) gives the value of the Senior Electronic Tester’s option to work for W022. This option is valid only for \( W_{22}^{1} > V_{M}^{22} \) and has value only for \( V_{M} \) below W022. The second line gives the value of the Senior Electronic Tester’s option to work for W020 as an Electronic Tester. This option is valid for \( W_{22}^{1} > V_{M}^{20} \) and has value for \( W_{22}^{1} \) up to \( \text{MIN}[V_{M}^{22}, W_{020}] \). The interpretation of this upper limit is identical to its interpretation in equation (5.2). The third and fourth lines of equation (5.15) have analogous interpretations as options to work for W019 and W018, respectively.

Equation (5.15) is solved using the same theorem and solution techniques as were used to solve equation (5.2). Suppressing the superscripts on \( W \), we state the solution as;
VSBO = \[-W \left\{ \frac{-\ln(W/W_{022})-(r+\sigma^2/2)T}{\sigma/T} \right\} + e^{-r_{T \cdot W022}} N \left\{ \frac{-\ln(W/W_{022})-(r-\sigma^2/2)T}{\sigma/T} \right\} \]

\[ + W \left\{ \frac{-\ln(W/VMP_{22})-(r+\sigma^2/2)T}{\sigma/T} \right\} + e^{-r_{T \cdot W022}} N \left\{ \frac{-\ln(W/VMP_{22})-(r-\sigma^2/2)T}{\sigma/T} \right\} \]

\[ -W \left\{ \frac{-\ln(W/\eta_{20})-(r+\sigma^2/2)T}{\sigma/T} \right\} + e^{-r_{T \cdot W020}} N \left\{ \frac{-\ln(W/\eta_{20})-(r-\sigma^2/2)T}{\sigma/T} \right\} \]

\[ + W \left\{ \frac{-\ln(W/VMP_{20})-(r+\sigma^2/2)T}{\sigma/T} \right\} + e^{-r_{T \cdot W020}} N \left\{ \frac{-\ln(W/VMP_{20})-(r-\sigma^2/2)T}{\sigma/T} \right\} \]

\[ -W \left\{ \frac{-\ln(W/\eta_{19})-(r+\sigma^2/2)T}{\sigma/T} \right\} + e^{-r_{T \cdot W019}} N \left\{ \frac{-\ln(W/\eta_{19})-(r-\sigma^2/2)T}{\sigma/T} \right\} \]

\[ + W \left\{ \frac{-\ln(W/VMP_{19})-(r+\sigma^2/2)T}{\sigma/T} \right\} + e^{-r_{T \cdot W019}} N \left\{ \frac{-\ln(W/VMP_{19})-(r-\sigma^2/2)T}{\sigma/T} \right\} \]

\[ -W \left\{ \frac{-\ln(W/\eta_{18})-(r+\sigma^2/2)T}{\sigma/T} \right\} + e^{-r_{T \cdot W018}} N \left\{ \frac{-\ln(W/\eta_{18})-(r-\sigma^2/2)T}{\sigma/T} \right\} \]

\[ + W \left\{ \frac{-\ln(W/VMP_{18})-(r+\sigma^2/2)T}{\sigma/T} \right\} + e^{-r_{T \cdot W018}} N \left\{ \frac{-\ln(W/VMP_{18})-(r-\sigma^2/2)T}{\sigma/T} \right\} \]
Equation (5.16) gives a closed-form solution to the value of the seniority-based options on employment to the \( N_{22}^2 N_{23}^1 \) most senior Senior Electronic Testers. It is perfectly analogous in both form and interpretation to the VSBO of the General Electronic Testers that is given by equation (5.3). Before we turn to the problem of finding the value of seniority to the less senior labor grade 22 worker, it is prudent to lay the groundwork by which to solve this problem.

V.2.4 A Digression on Mixed Stochastic Processes

We have so far assumed that the worker's VMP follows geometric Brownian motion. We are about to introduce a phenomena that from time to time causes the worker's VMP to jump by a substantial amount. This jump can be either positive or negative. Both before and after the jump, the worker's VMP follows geometric Brownian motion, but when the jump occurs, it displaces the process by a (possibly) random amount. A downward jump in VMP\(^j\) is caused by worker j being "bumped" from his job by a more senior worker. The Poisson process is the prototype jump process and we assume that the jump follows a Poisson process. Superimposing these jumps onto a geometric Brownian motion process results in a mixed geometric Brownian motion-Poisson process.

The mixed geometric Brownian motion-Poisson process can be described formally as:

\[
\frac{dW}{W} = (\alpha - \lambda E(k-1)) dt + \sigma W dZ + (k-1) d\omega
\]
\[
\begin{cases}
(\alpha_w - \lambda E(k-1)) dt + \sigma_w dz & \text{if the Poisson event does not occur} \\
(\alpha_w - \lambda E(k-1)) dt + \sigma_w dZ + (k-1) & \text{if the Poisson event does occur}
\end{cases}
\]

Here, \( \lambda \) is the number of Poisson events per unit of time, \( \sigma_w \) is the instantaneous expected growth in the shadow wage and \( (k-1) \) is the percentage change in the shadow wage if the Poisson event occurs. \( (k-1) \) is a random variable and \( d \lambda \) is an indicator function that takes the value of unity if the Poisson event occurs but is zero otherwise. \( \lambda E(k-1) \) is the expectation of change in the shadow wage resulting from Poisson events. The interested reader is referred to Merton (1976) for more discussion of this combined process.

V.2.5 The Value of Seniority to a Less-Senior Senior Electronic Tester

The purpose of this chapter is the application of the model developed in Chapter Four to the problem of finding the value of seniority to employees in the Testing Assurance-Electronic Test Department of Gould Ocean Systems, Inc. To this point, we have found closed form solutions that give (1) the value of seniority to a labor grade 23 worker with an arbitrary amount of seniority and (2) the value of seniority to any of the \( N_{22}^2 - N_{23}^1 \) most senior labor grade 22 workers. We now turn our attention to finding the value of seniority to other labor grade 22 workers.

These \( N_{23}^1 \) labor grade workers differ from the more-senior workers in that these less-senior labor grade 22 workers can be "bumped" from labor grade 22 employment by a labor grade 23 worker. We assume, for
the moment, that $N_{20} > N_{22} + N_{23}$. This means that there are enough labor grade 20 jobs that so that even the least senior grade 22 workers can have a grade 20 job, even if all grade 23 workers exercise their options on grade 20 employment. In these circumstances, finding the value of the seniority-based options on employment to that $j^{th}$ most senior labor grade 22 worker (for $j$ between $N_{22}^2 - N_{23}^1$ and $N_{22}^2$) is a more complicated problem than we have encountered before. The difference between the options possessed by these less senior labor grade 22 employees and the $N_{22}^2 - N_{23}^1$ most senior labor grade 22 employees is that the less senior grade 22 employees can be "bumped" from their jobs by laid off grade 23 workers. Being "bumped" is modeled as the arrival of a Poisson event that causes a downward shift in the worker's VMP. It is appropriate then that the difference between equation (5.15) and equation (5.17) is that the distribution of $w_p^{i,j}$ in equation (5.17) follows a mixed geometric Brownian motion-Poisson process, while the distribution of $w_p^{i,j}$ in equation (5.15) follows a simple geometric Brownian motion process. The value of the seniority-based options on next-period employment held by the $j^{th}$ most senior labor grade 22 worker (for $j$ between $N_{22}^2$ and $N_{22}^2 - N_{23}^1$) is given by equation (5.17) as $VSBO_{22}^j$.

$$VSBO_{22}^j = e^{-rT} \int_{VMP_{22}^*}^{W_{022}} (W_{022} - W_{22}^{22}, j^*) L_p (W_{22}^{22}, j^*) dW_{22}^{22}, j^*$$

$$+ e^{-rT} \int_{VMP_{20}^*}^{\text{MIN}[VMP_{22}^*, W_{020}]} (W_{020} - W_{22}^{22}, j^*) L_p (W_{22}^{22}, j^*) dW_{22}^{22}, j^*$$
Here it is understood that $W_{i,j}^{i,j}$ is the same variable as $W_{i,j}^{i,j}$ except that the former is governed by the distribution of worker $VMP$ when it is possible for a Poisson event to impact upon the distribution. Since we are considering situations where the grade 22 worker is not senior enough to guarantee that no Poisson event displaces him from employment at $W_{022}$, but enough jobs exist at $W_{020}$ that no Poisson event can displace the grade 22 worker from $W_{020}$ employment, only the distribution of $VMP$ in the first line of equation (5.17) is affected. If $N_{20}<N_{22}+N_{23}$, so that some labor grade 22 worker could be "bumped" from labor grade 20 employment, then the $W_{i,j}^{i,j}$ in the second line become $W_{i,j}^{i,j}$ terms because then the Poisson event can occur such that it affects the labor grade 22 worker's option on wage grade 20 employment. Equation (5.17) differs from equation (5.15) only in that the notation of equation (5.17) indicates the presence of the mixed distribution in the first line of equation (5.17). Unfortunately, the first line of equation (5.17) is not known to have a solution beyond the analytic solution to the problem given in equation (5.17).
V.2.6 The Value of Seniority-Based Options on Employment for Less Senior Workers

We have now considered explicitly the value of seniority for enough classes of workers that we have dealt with all of the difficulties that will arise. We can generalize the analysis to cover all other workers in labor classes 20, 19 and 18.

The value of the seniority-based options on next-period labor for workers in class 20 who are senior enough never to experience the Poisson event's impact upon any option is identical to equation (5.17), except that the first line of equation (5.17) is missing and the upper limit of the second integral becomes \( W_{020} \) instead of \( \min \{ W_{020}, VMP_{22}^* \} \). Also, \( \omega_{22,j^*} \) is replaced with \( \omega_{20,j} \), the VMP of the \( j \)th most senior labor grade 20 worker. The solution to this analytic equation will be the same as equation (5.16) except that the first two lines become zeros and \( \eta_{20} \) is replaced throughout by \( W_{020} \). For those workers who have insufficient seniority to be protected against the Poisson event affecting their own option on next-period employment at some wage grade \( i \), the \( \omega_{i,j^*} \) in the appropriate integral is replaced with \( \omega_{i,j}^{*p} \) and that integral no longer has a closed-form solution. By making the analogous modifications to the labor grade 20 result, we arrive at the labor grade 19 and labor grade 18 results.

V.3 The Numerical Evaluation of the Value of the Seniority Based Options on Employment

Having determined analytically the value of the seniority based options in Section 5.2, we now turn to the final task of the numerical
evaluation of VSBO. By carrying out this numerical analysis we can assess how important these seniority benefits are to the workers who possess them. In a sense, the purpose of this section is to ask whether VSBO is ten percent of the worker's compensation, or is it as high as twenty percent, or is it as low as one percent. The answer to this question is, of course, that it depends: it depends on the values of the parameters $\sigma^2, r, W_0, VMP^*$; and it depends on the current value of the worker's VMP.

We find the value of the seniority based options to a senior General Electronics Tester by evaluating equation 5.3'. It is important to indicate at the outset that it is not the goal of this section to find the value of the seniority based option to the senior General Electronics Tester, since the nature of VSBO assures us that the proverbial ink will not be dry on the page before the stochastic nature of $W$ has changed VSBO. Furthermore, any change in any of the parameters also changes VSBO. As a result, the more meaningful exercise is to determine the numerical value of VSBO for a range of different parameter settings and different current values of the random variable $W$ and then to investigate those parameters to which the value of VSBO is particularly sensitive. Evaluating equation 5.3' requires a current value of the stochastic process $W$ and also requires values for thirteen parameters; $r, \sigma^2, T$ and $W_0, VMP^*$ for $i = 18, 19, 20, 22, 23$. Carrying out this evaluation of equation 5.3' for even three different settings of each of those parameters results in $3^{14} = 4,782,969$ different estimates of VSBO. Further, while many of the parameters are set in advance by firm-worker agreement (such as
WO_i and VMP_i), or are arguably fairly stable over time, the assumption that worker VMP follows a stochastic process is essential to our arguments and indicates that W varies continually across time. As a result, while three different settings of each of the parameters may well give an adequate range of variation, it is desirable to evaluate VSBO for a wider range of values of W. Increasing the number of settings for W from three to twenty two (as we do) results in nearly thirty six million different parameter settings for which we must find the value of equation 5.3'. Even the most basic analysis of these results would result in a pile of computer output nearly a half a mile high. The unfortunate fact is that such a large volume of information is tantamount to no information at all, because even the most enthusiastic reader can no longer discern the pertinent information from the extraneous.

Fortunately, several simplifications are indicated here, most of which concern W0_i and VMP_i. In what follows, the parameter settings selected for use in the numerical evaluation of VSBO are listed and explained.

V.3.1 Parameter Values

Since VSBO as given in equation 5.3' is the value of seniority based options on employment at Gould Corporation, it is natural that the contract wage rate information provided by Gould is used. Hence initial values of W023, W022, W020, W019 and W018 are taken from the rightmost column of Table 1. We wish to investigate the sensitivity of VSBO to wage changes and since it is typical that when W023 changes
the other contract wage rates also change, we also evaluate VSBO with $W_{0_i}^{1} = 1.1 W_0$ and $W_{0_i}^{2} = 0.9 W_0$, for $i = 18, 19, 20, 22, 23$. By doing this we are able to investigate the impact on the value of VSBO of workers (1) receiving raises such that they receive contract wages of 110 percent of the wages specified in Table 1 and (2) receiving pay cuts such that they receive contract wages of 90 percent of the wages specified in Table 1. Assuming that contract wage rates move together in such a manner reduces the number of estimates of VSBO from 36 million to just over 400,000.

Gould was unwilling to provide any information whatsoever about their layoff policy or incidence of layoff other than what is contained in their union contract, so that there is no way of knowing the level of VMP below which a worker is laid off. As a result, we use the following procedure to establish the values of $VMP_i^*$ to be used in the analysis. For each contract wage, we use three settings of $VMP_i^*$. We first evaluate $5.3'$ with $VMP_i^*$ equal to sixty percent of the $i$th contract wage, then with $VMP_i^*$ equal to eighty percent of the $i$th contract wage, then with $VMP_i^*$ equal to ninety percent of the $i$th contract wage. For example, when the contract wage, $W_{023}$, enters $5.3'$ as $10.00$ an hour, $VMP_{23}^*$ enters $5.3'$ once as $9.00$ an hour, once as $8.00$ an hour and once as $6.00$ an hour. Since we have adopted the convention of defining $VMP_i^*$ as a percentage of $W_0$, it is useful to define the variable $VMP_i^*/W_0$ and to name that variable STAR. This simplification reduces the number of different parameter settings for which $5.3'$ is evaluated to 5,346. The parameter $T$ is normalized to one and all parameters are entered as annual magnitudes. Wages and
VMPs enter 5.3' in annual form (we assume 2080 hours per year), \( r \) is an annual interest rate and \( \sigma^2 \) is the instantaneous annual variance of the spot wage.

Using data from the National Longitudinal Survey, Randy Olsen has estimated that the annualized instantaneous variance of worker VMP is approximately 18 percent. We evaluate 5.3' using variances of 16, 18 and 20 percent. The range of values of the interest rate entered into 5.3' is six percent, eight percent and ten percent.

While the exercise of numerically evaluating equation 5.3' is clearly more interesting the more reasonable are the parameter settings, the most interesting information to be gleaned from this exercise are the responses of VSBO to changes in the parameter settings. Those empirical derivatives are of particular importance due to the sign ambiguity of the partial derivatives of VSBO with respect to \( W_1 \), \( \sigma \), \( r \) and \( W \) that we demonstrated in Section V.2.2.1.

To summarize the three settings chosen for each parameter in the evaluation of VSBO, we have

\[
\begin{align*}
W_0^{23}: & \quad 18,776.2, 20862.4, 22948.6 \\
W_0^{22}: & \quad 18,364.32, 20404.8, 22445.28 \\
W_0^{20}: & \quad 17,596, 19,556, 21507.2 \\
W_0^{19}: & \quad 17,203.68, 19,115.2, 21026.72 \\
W_0^{18}: & \quad 16,866.72, 18,740.8, 20,614.88 \\
VMP_23^*: & \quad 0.6W_0^{23}, 0.8W_0^{23}, 0.9W_0^{23} \\
VMP_22^*: & \quad 0.6W_0^{22}, 0.8W_0^{22}, 0.9W_0^{22} \\
VMP_20^*: & \quad 0.6W_0^{20}, 0.8W_0^{20}, 0.9W_0^{20}
\end{align*}
\]
Finally, we use twenty-two different values of the random variable $W$. We first consider $W = 20,862.4, 20,404.8, 19,552, 19,115.2$ and $18,740.8$. These are the annualized contract wages $W_{023}, W_{022}, W_{020}, W_{019}$ and $W_{018}$ respectively. In addition, we consider a range of other values of $W$. Those considered are $8,000, 10,000, 12,000, 14,000, 15,000, 16,000, 17,000, 18,000, 20,000, 22,000, 23,000, 24,000, 25,000, 26,000, 28,000, 30,000, 32,000 and 36,000.

To make notation simpler, we adopt the convention of referring to the 5-tuple of contract wages $W_{023}, W_{022}, W_{020}, W_{019}$ and $W_{018}$ as the contract wage vector. The multiplier of the contract wage vector is the value taken by the variable $G$, which takes values of 0.9, 1.0 and 1.1. Multiplying the contract wages given in the rightmost column of Table 1 by the three values of $G$ result in the three values of the contract wage vector given in this section. We also adopt the convention of referring to the 5-tuple of $VMP_{23}^*, VMP_{22}^*, VMP_{20}^*, VMP_{19}^*$ and $VMP_{18}^*$ as the required minimum VMP vector.

V.3.2 The Numerical Analysis of Equation 5.3'

We are interested in the response of the value of the seniority based options on employment to changes in parameter values for two reasons. First, some of the parameters of 5.3' are under the control
of the principals to labor contracts. As a result, it is of interest to contract negotiators to know how changing the parameter settings of 5.3' alters the value of seniority. Second, even the parameters that are outside the control of contract negotiators have an effect upon the value of seniority. By evaluating 5.3' for different settings of those parameters we can learn more about the impacts of these parameters on the value of seniority.

When 5.3' is evaluated for the 1,782 various combinations of the parameter values and W, values of 5.3' range from a low of $16.85 to a high of $2875.42. When considered in the context of the wide range of parameter values for which 5.3' is evaluated, this range seems entirely appropriate. VSBO takes its minimum value of $16.85 for a worker with current VMP equal to $8,000, W023 equal to $22948.6, W018 equal to $20614.88 and VMP18* equal to $18,553.39. With those fairly extreme parameter settings, his option on employment at W023 is valid only if his VMP rises by 158 percent and even his option on employment at W018 is valid only if his VMP rises by 132 percent. Since his variance of VMP is 16 percent, VMP of $18,553.39 (which is required for him to have a valid option on even the lowest wage employment) lies more than three standard deviations above his present VMP. The seniority based options on employment possessed by this worker have little value, because there is extremely little chance that his VMP will be high enough for any of the options to be valid on the exercise date.

At the other extreme is a worker with current VMP equal to $16,000, W023 equal to $22948.6 and VMP23* equal to $13,769.16. This
worker also has variance of VMP equal to 16 percent. His VMP can fall by 13.9 percent and still leave the worker with a valid option to work for W023 and his VMP can fall by 29.4 percent (to $12,368.93) and still leave him a valid option to work next period for W018.

While there is clearly information to be gathered by considering a very wide range of values of W, this analysis of the sensitivity of VSBO to changes in the parameters produces its most striking result when confined to values of W between $12,000 and $24,000. When we restrict W to this range and rank the 1134 evaluations of equation 5.3' by VSBO, we get very nearly the same rankings as if we rank them by VMP* when VMP* is expressed as a percentage of the wage. 375 of the 378 highest values of VSBO are those evaluations where VMP* is set at sixty percent of the contract wage (the lowest percentage at which we set VMP*), and 378 of the 378 lowest values of VSBO are those evaluations where VMP* is set at 90 percent of the contract wage (the highest percentage at which we set VMP*). Even though $\frac{\partial VSBO}{\partial VMP_1}$ was the only derivative of equation 5.3' that we have been able to sign, the strength of the impact of changes in VMP* on the value of VSBO was only partly anticipated. Not only does the sensitivity analysis confirm the negative sign of $\frac{\partial VSBO}{\partial VMP_1}$, but when we use a narrower (and probably more realistic) range of values of W the effect of VMP* is so strong that it virtually swamps all the other effects. In fact, with the exception of three situations where the present VMP is set at $24,000 (its highest value) and the contract wage W023 is set at
$18,776 (its lowest value), arranging the 1134 evaluations of 5.3' by VSBO also results in a perfect ordering with respect to VMP*.

At the other extreme, the impacts of changes in the interest rate and changes in the variance of VMP seem to be very small. To get a clearer picture of the relative impacts of the five factors upon the numerical magnitude of VSBO, we consider the elasticity of VSBO with respect to each of the five factors.

V.3.2.1 The Elasticities of VSBO With Respect to W and the Parameters

In carrying out the analysis of the sensitivity of VSBO to the wage process and to changes in the underlying parameters, we have generated a tremendous volume of information. Some of this information is interesting because it shows that changes in certain parameters cause large changes in VSBO and some is interesting because it shows that VSBO is rather insulated from changes in other parameters. The primary purpose of this section is to show that the parameters dichotomize nicely into one group that has small effects on VSBO and into another group that has large effects on VSBO. In the next section, we see that the parameters that affect VSBO very little are also parameters that are more or less outside the control of principals to labor contracts. We take that combination of facts to be sufficient reason to streamline the exposition by confining most of the graphical exposition of the impacts of those parameters to Appendix B.
For each of the five factors that affect VSBO (W, r, variance of VMP, the contract wage and STAR), we calculate, for each combination of settings of the other four factors, the arc elasticity of VSBO with respect to the factor, between the factor's highest value and its lowest value. For example, when $W = $18,000, STAR = .06, W023 = $20,862.4, W022 = $20,404.8, W020 = $19,556, W019 = $19,115.2, W018 = $18,740.8, $\sigma = .4$ and $r = .10$, the value of VSBO is $2,083.78$. When we change $r$ to .06, the value of VSBO changes to $2,286.24$. The resulting elasticity is

$$\frac{2083.78 - 2286.24}{\overline{(2083.78 + 2286.24)/2}} = \frac{-0.09266}{0.5} = -0.18532.$$ 

The result is that for each of the four parameters, we have the elasticity of VSBO with respect to that parameter for each of 594 (3·3·3·22) different settings of the parameters. While one can gain some insight into the relative impacts of the factors upon VSBO by direct observation of these individual elasticities, summary measures of the elasticities are far more readily digested and are provided in Table 3. Several interesting characteristics of VSBO are apparent from Table 3. First, the last two columns corroborate our finding that the partial of VSBO with respect to STAR is negative, while the other partials are all of ambiguous sign. Second, the column containing the mean of the absolute value of elasticity seems to show that the impacts upon VSBO of $r$ and $\sigma^2$ are much smaller than the impacts of the other parameters, STAR and the contract wage vector.
TABLE 3
SUMMARY ELASTICITY MEASURES

<table>
<thead>
<tr>
<th>ELAST. OF VSBO W.R.T.</th>
<th>MEAN OF ABS. VAL. OF ELAST.</th>
<th>MEAN ELAST.</th>
<th>S.D. OF ELAST.</th>
<th>MIN. ELAST.</th>
<th>MAX. ELAST.</th>
</tr>
</thead>
<tbody>
<tr>
<td>W</td>
<td>0.6166</td>
<td>-0.0890</td>
<td>0.7154</td>
<td>-1.3356</td>
<td>1.1361</td>
</tr>
<tr>
<td>r</td>
<td>0.1881</td>
<td>-0.1456</td>
<td>0.1639</td>
<td>-0.5297</td>
<td>0.3755</td>
</tr>
<tr>
<td>$\sigma^2$</td>
<td>0.4438</td>
<td>0.0331</td>
<td>0.5734</td>
<td>-0.5156</td>
<td>2.3773</td>
</tr>
<tr>
<td>W023</td>
<td>2.2645</td>
<td>1.7642</td>
<td>1.9125</td>
<td>-3.9928</td>
<td>5.5397</td>
</tr>
<tr>
<td>VMP23*</td>
<td>3.9006</td>
<td>-3.9006</td>
<td>0.4180</td>
<td>-4.7895</td>
<td>-2.7757</td>
</tr>
</tbody>
</table>

V.4 The Sensitivity of VSBO to Parameter Settings

Our concern in performing this sensitivity analysis is more than academic. Knowledge of how various parameters affect the value of seniority should be of value to those charged with negotiating the parameters of labor contracts. The parameters of the value of seniority are the interest rate, the variance of the spot VMP, the contract wage and the level of VMP below which the worker’s options cease to be valid. Of these, the last two, the contract wage and the level of VMP below which the worker’s option ceases to be valid are clearly under the control of contract negotiators. By contrast, it is not at all clear that the interest rate or the variance of worker VMP are under the control of contract negotiators. On the contrary, it seems quite clear that the interest rate is outside the influence of contract negotiators and barring malfeasance, the variance of worker VMP seems to be beyond the control of contract negotiators.
To the extent that control of the interest rate and the variance of worker VMP lies somewhere other than with contract negotiators, we have the fairly important result that the parameters of 5.3' least controllable by principals to labor contracts are also the parameters whose changes least affect the value of seniority. While the parameters most controllable by principals to labor contracts are those which most affect the value of seniority.

Before we move on to consider the specific changes in VSBO that result from changes in each of the parameters of 5.3', it is important to recall that VSBO is a function of several parameters and of a random variable W which is driven by a stochastic process. Thus, it is of little use to consider a single VSBO associated with a parameter setting. Rather, we should consider VSBO as a function of W for each parameter setting. This is most succinctly done in a series of graphs. Figure 3 gives the value of VSBO as a function of W for nine different parameter settings. These nine settings are the nine possible combinations of STAR and the multiplier of the contract wage vector when r takes the value .08 and the variance of spot VMP takes the value 0.18. Figures 3 through 16 have actual points at the twenty-two values of W specified on page 101. We interpolate between those points to make the figures more readable. Each curve graphed on figure 3 is labeled with a pair of numbers. The first of these numbers is the value of STAR for that curve and the second identifies the value of the contract wage for that curve. When the second number is 0.9, the contract wage vector is W023 = $18,776.2, ..., W018 = $18,866.72 (the low setting). When the second number is 1.0, the
contract wage vector is \$20,862.4, ..., \$18,740.8 (the median setting) and when the second number is 1.1, the contract wage vector is \$22,948.6, ..., \$20,614.88 (the high setting). Figures 8 through 16 are labeled in the same way. Figure 3 along with eight similar diagrams (one for each of the other possible combinations of \( r \) and \( \sigma^2 \)) summarizes all of the values that equation 5.4 takes as the parameters and \( W \) take on their various values. The similarity between figure 3 and the graphs for the other eight settings of \((r,\sigma^2)\) is so marked that we confine the other eight graphs to Appendix B.

V.4.1 AVSBO/\( \Delta \)STAR

Observation of the results graphed in figure 3 confirms that VSBO is monotonically decreasing in STAR. All other things equal, we see that low values of STAR are associated with high values of VSBO. It is not at all surprising that the value of the seniority based options should increase as the minimum level of worker VMP required for the option to be valid falls. Indeed, since this decrease in \( \text{VMP}_i^* \) means that there are more states of the world in which the worker’s options are valid, we would be quite surprised if this result did not obtain.

We also notice that as the value of STAR increases, so does the value of \( W \) at which VSBO has its maximum. This occurs because as STAR rises, so does the minimum level of the current value of \( W \) that is associated with any given probability of \( W \) rising to \( \text{VMP}_i^* \) by the option’s exercise date. In other words, the higher is the minimum VMP at which the firm will honor the option held by the senior worker, the
FIGURE 3
closer to the contract wage is that level of current VMP at which the value of the worker's options on next-period employment is at its maximum. When the minimum VMP at which the firm will honor the worker's options is relatively low, the worker's VMP can be relatively low and still have a relatively high probability of rising to the minimum VMP level required by the firm by the option's exercise date. But if the minimum VMP at which the firm will honor the option is relatively high, the worker's current VMP must be relatively high to have the same probability of rising to the minimum required level by the expiration date.

V.4.2 \( \Delta V_{SBO}/\Delta W_{O} \)

The relationship between VSBO and the contract wage structure is not so clear cut as the relationship between VSBO and \( V_{MP*} \). The partial derivative of VSBO with respect to \( V_{MP*} \) is negative, whereas the partial of VSBO with respect to \( W_{O} \) cannot be signed. The sensitivity analysis, however, does provide useful information on the behavior of \( \frac{\Delta V_{SBO}}{\Delta W_{O}} \). First, the sensitivity analysis confirms that \( \frac{\Delta V_{SBO}}{\Delta W_{O}} \) cannot be signed; for some parameter settings the derivative is positive and for others it is negative. In particular, \( \frac{\Delta V_{SBO}}{\Delta W_{O}} \) is positive for higher values of \( W \), although the value of \( W \) above which \( \frac{\Delta V_{SBO}}{\Delta W_{O}} \) becomes positive is a function of the other parameters, most notably (but not exclusively) \( V_{MP*} \). For low current values of \( W \), a high contract wage means that there is a small chance of the worker's VMP rising enough to make the options binding by the options' exercise
date. As a result, for low values of W, $\frac{\Delta V S B O}{\Delta W_{023}}$ is negative. In contrast, for high values of W, higher values of W023 increase the value of seniority, because high values of current W mean that there is a large chance that the worker's VMP will be sufficient to make the option valid at its exercise date and a high W023 means that valid options are options on high paying positions.

The bottom line is that there are two primary factors operating on $\frac{\Delta V S B O}{\Delta W_{023}}$. One is that the higher is the value of W023, the lower is the probability of the option being valid. The other is that the higher is the value of W023, the more valuable is the option when it is valid. For values of W substantially above VMP$_1^*$ there is little concern about the first factor and increasing W023 increases VSBO. On the other hand, for values of W substantially below VMP$_1^*$, there is little chance of the option being valid unless something changes dramatically. A sufficiently dramatic change would be a large enough decrease in W023 to lower VMP$_1^*$ to a point where the worker's option package would contain a valid option. In this case, $\frac{\Delta V S B O}{\Delta W_{023}}$ is negative.

Since the sign of the partial of VSBO with respect to the contract wage varies across different values of W and since the values of W at which the sign of $\frac{\Delta V S B O}{\Delta W_{023}}$ changes depends, in turn, on the value of STAR, we now consider the impact of STAR on $\frac{\Delta V S B O}{\Delta W_{023}}$. We begin by recalling that $\frac{\Delta V S B O}{\Delta W_{023}}$ is negative for low values of W because, ceteris paribus, the lower the value of W, the lower is the probability that the worker's options will be valid. Intuition tells us that this
problem is confounded by high values of STAR (since a higher value of
STAR corresponds to a higher value of W being required to yield a
given probability of VMP greater than VMP^*) and relieved by low
values of STAR. The sensitivity analysis bears out this reasoning:
the higher is the value of STAR, the higher is the value of W at which
\( \frac{\Delta V_{SBO}}{\Delta W_{023}} \) becomes positive. Indeed, the elasticity of VSBO with respect
to the contract wage rises as W rises for all values of all other
parameters.

V.4.3 The Effect of the Variance of W on VSBO

The variance would also be expected to influence the values of W
that are associated with a positive \( \frac{\Delta V_{SBO}}{\Delta W_{023}} \), since, ceteris paribus, a
high variance of W increases the chance of a low current value of W
rising enough by the option's exercise date for the option to be
valid. However, the same high variance also increases the chance of a
high current value of W falling enough by the option's exercise date
for the option not to be valid. As a result, while the variance of W
plays a role in determining the range of values of W that correspond
to a positive (or negative) value of \( \frac{\Delta V_{SBO}}{\Delta W_{023}} \), its effects are not sys-
tematic as are the effects of STAR and the wage multiplier.

V.4.4 The Components of VSBO

The value of the seniority based options to a senior General
Electronics Tester is the sum of the values of five separate options:
The value of the option to work in labor grade 23 and the value of the
options to work in labor grade 22, 20, 19 and 18. We conclude this
section with a discussion of these components of VSBO. Figure 4 is the graph of VOPT23, VOPT22, VOPT20, VOPT19 and VOPT18, each as a function of W, where VOPT1 is the value of the options to work in labor grade i. In Figure 4, we set $\sigma^2 = .18, r = .08,\ STAR = .8$ and set the wage in accordance with Table 1.

While Figure 3 graphs VSBO as a function of W for the nine different settings of STAR and $\sigma^2$, it is not desirable to consider a range of parameter settings for STAR and the contract wage when we disaggregate VSBO into VOPT23,..., VOPT18 in Figure 4 because the figure then becomes excessively cluttered.

The most striking attribute of the values of the components of VSBO is that for any parameter setting VOPT23 is substantially larger than any of the other VOPTi. In fact it is rare that any value of VOPTi exceeds one third of VOPT23 and extremely rare that any exceeds half of VOPT23. Moreover, those cases are all for parameter values that might well be considered so extreme as to be of no interest (for example, values of W at or below $10,000 and STAR = .9$). That VOPT23 exceeds any of the other VOPTi was to be expected, since from equation 5.2' and Table 1 it is clear that the density over which we integrate for VOPT23, $(W_{023} - VMP_{23\ast})$, is a great deal larger than the density over which we integrate for any other VOPTi. For example, to find VOPT19, we integrate from VMP_{19\ast} to VMP_{20\ast}. But this difference $(VMP_{20\ast} - VMP_{19\ast}) = (W_{020} \cdot STAR - W_{019} \cdot STAR) = [STAR(19552 - 19115.2)] - (STAR-436.8)$. This value ranges from 262.08 to 383.12, depending on the value of STAR. By contrast we find the value of VOPT23 by integrating an only very slightly different integrand over the range
VMP23* to W023 and W023 - VMP23* - W023 - STAR·W023 = (1·STAR)·(W023) = (1·STAR)·20862.4. This ranges from 2086.24 to 8344.96, depending on the value of STAR. As examples, the values of the upper and lower limits of integration of the first and fourth integrals in 5.2' are given in Table 4 for the contract wages in Table 1 and various values of the variable STAR.

TABLE 4

<table>
<thead>
<tr>
<th>INTEGRAL</th>
<th>VALUE OF UPPER LIMIT FOR STAR</th>
<th>VALUE OF LOWER LIMIT</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>W023</td>
<td>0.6 0.8 0.9</td>
</tr>
<tr>
<td></td>
<td>VMP23*</td>
<td>20862.4 20862.4 20862.4</td>
</tr>
<tr>
<td></td>
<td></td>
<td>12517.44 16689.92 18776.16</td>
</tr>
<tr>
<td>4</td>
<td>VMP20*</td>
<td></td>
</tr>
<tr>
<td></td>
<td>VMP19*</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>11980.8 15974.4 17971.2</td>
</tr>
<tr>
<td></td>
<td></td>
<td>11743.68 15658.24 17615.52</td>
</tr>
</tbody>
</table>

Given the extreme similarity of the integrands in the first and fourth integrals of 5.2' and the differences in the range of values over which the integration is performed, it is of little surprise that VOPT23 is several times as great as VOPT19. These same comments hold for VOPT23 in relation to VOPT22, VOPT20 and VOPT18.

To bring this observation into the context of the senior worker's right to bump a less senior worker, we see that the right to bump a worker from any single labor grade is worth much less than is the option to work in one's own labor grade. However, the privilege to bump workers from all lower wage grades accounts for a substantial
portion of VSBO. As figures 5, 6 and 7 show, the bumping privilege taken as a whole is, for certain parameter settings and for certain ranges of values of W, more valuable than the option to work in labor grade 23. Without exception, the values of W where VOPT23 is greater than the value of the bumping privilege are larger than the values of W where VOPT23 is less than the value of the bumping privilege.

Defining V.NO.23 as the value of the privilege to bump any lower grade worker, figure 5 shows that for the evaluations of VSBO with STAR equal to .6, VOPT23 exceeds V.NO.23 throughout. Figure 6 shows that for evaluations with STAR = .8, VOPT23 exceeds V.NO.23 for all but the lowest values of W and figure 7 shows that with STAR = .9, VOPT23 exceeds V.NO.23 for only the highest values of W. To summarize, as STAR rises, the value of W at which VOPT23 first exceeds V.NO.23 rises. Figures 5, 6 and 7 are for σ = 0.18, r = .08 and the median values of the contract wages. While the functions that generate these values are quite complicated it seems that the cause of this effect is that as STAR rises, the part of the density of 5.2' over which we integrate for VOPT23 falls and the area over which we integrate to obtain each of VOPT22, VOPT20, VOPT19 and VOPT18 rises.

It is also no surprise that the value of the options to work in lower wage grades is highest relative to the option to work in the highest wage grade when W is very low and is lowest when W is very high. This is because when W is very high, options to work at low wage rates are of relatively little value and the option to work at the high wage is of relatively great value because of the high probability that the option to work at a higher wage will be valid.
FIGURE 5
FIGURE 7
Conversely, when $W$ is very low, the value of options to work at high wage rates is relatively low while the value of options to work for low wage rates are high, because of the same low probability of a better option being valid. Of course, these relationships are complicated by other factors: in this case, the higher wage received when the option to work for $W_{023}$ is valid complicates the above explanation of the effects of STAR and $W$ on $\frac{\text{VOPT}_{23}}{V_{\text{NO}23}}$. This and other possible contributing complications notwithstanding, the reasons outlined above for the comparative statics of $\frac{\text{VOPT}_{23}}{V_{\text{NO}23}}$ seem sound and seem to capture the most important aspects of the relationships.
CHAPTER SIX

THE MODEL WITH FIRM SPECIFIC HUMAN CAPITAL

In the first five chapters of this paper we have constructed a model that shows that the option to sell one's labor at a pre-contracted price is valuable to the owners of labor. In showing this, we have assumed that at any point in time the worker's labor has one price, $W$, both inside and outside the firm. In the first five chapters, the closest that we come to diverging from this assumption is in section V.2.2, where we show how the model changes if a lack of competitive pressure allows the firm to diverge from the rule of hiring all workers whose VMP exceeds their wage. In section V.2.2, we still consider only one wage process for the worker, but investigate how the model can be adapted to situations where a lack of competition prevents the firm from voluntarily offering the competitive wage. At that point, we note that the existence of FSHC could lead the firm to display this non-competitive behavior. In the present chapter we generalize the model in a manner that allows us to consider explicitly two separate wage processes on the workers labor. One of these processes is $W$, the worker's VMP in the present firm (the process that we have considered all along), and the other process is WALT, the worker's wage process in the best alternate employment.
As is demonstrated in section V.3, in the simplest version of the model with no FSHC, the value of the option to sell labor that has spot price \( W \) for a contract wage \( W_0 \) is given by equation (4.15) as

\[
p = e^{-r(T-t)} \int_{0}^{W_0} (W_0 - W^*)L'(W^*)dW^* \tag{4.15}
\]

The intuition here is that the value of this option is the difference between \( W_0 \), the contract wage and \( W^* \), the spot price of labor at the options exercise date, integrated across all values of \( W^* \) for which \( W_0 \) exceeds \( W^* \). This is the appropriate formulation because at the beginning of the second period of this two-period model the worker will choose to work for \( W_0 \) instead of \( W^* \) if and only if \( W_0 \) exceeds \( W^* \). The worker has two alternatives: he can exercise his option to work for \( W_0 \), or he can work in the spot market for \( W^* \).

VI.1 Firm Specific Human Capital in the Basic Model

As the starting point in generalizing the model to allow for firm specific human capital, we begin with the basic model of sections IV.2 and IV.2.1, where we assumed that (1) options possessed by workers are sure to be honored regardless of the worker's VMP, (2) options are received in period one and are valid for period two and (3) workers have an option to work in only one position. These are the assumptions that are relaxed for the no-FSHC case in sections IV.3, IV.4 and chapter V.
In generalizing the model to consider firm-specific human capital, we consider a worker whose value in the present firm is $W$ and follows geometric Brownian motion, but whose value in the alternate employment is $WALT$ and follows another geometric Brownian motion process. In developing the FSHC arguments, we assume that the worker who possesses the options still has only two alternatives: he can exercise his option to work in the current firm for the contract wage $WO$, or he can allow the option to expire unexercised and accept employment at the alternative wage $WALT^*$. In the simplest version of the model, allowing for FSHC requires only that we carefully consider the worker's alternative of exercising his option and working for the contract wage. Without FSHC, the worker's alternative is to work for $W^*$. With FSHC, the worker's alternative is to work for $WALT^*$, which is the value of $WALT$ at the options exercise date. In the two-period model with certain honoring of the option, the worker simply selects the greater of $WO$ and $WALT^*$. Conceptually, nothing has changed except that the worker's alternative to exercising his option in the no-FSHC case is to work for $W^*$, while his alternative to exercising his option in the FSHC case is to work for $WALT^*$.

The analog of equation (4.15) when we allow for the presence of FSHC is given by

$$P_{FSHC} = e^{-rT} \int_{0}^{WO} (WO-WALT^*)L'(WALT^*)dWALT^*$$

(6.1)
where $L'(WALT^*)$ is a lognormal density function and $P_{FSHC}$ is the value of an option to work for $WO$ when the alternative is to work for $WALT$. We can solve equation 6.1 by applying the theorem on P34-35 with $W^*$ replaced by $WALT^*$ and $\psi=1$. According to the theorem, the solution to equation 6.1 is

$$P_{FSHC} = -WALT^* N\left\{ \frac{-\ln(WALT/WO)-(r+\sigma^2/2)T}{\sigma/\sqrt{T}} \right\} + W0 e^{-\rho T} N\left\{ \frac{-\ln(WALT/WO)-(r-\sigma^2/2)T}{\sigma/\sqrt{T}} \right\}. \quad (6.2)$$

In the simplest version of the model, we encounter little difficulty in extending the model to consider firm specific human capital.

VI.2 Relaxing the Assumption of Certainty that Options are Honored in the Presence of Firm Specific Human Capital

Generalizing the model to accommodate firm-specific human-capital does inhibit our ability to generalize the model in other directions. The most notable manifestation of this inhibition comes when we relax the assumption that the worker's option on next period employment is always valid. When we relax this assumption in the no-FSHC case, we use the theorem on page 34-35 to evaluate equation (4.16), which is the integral of $WO-W^*$ integrated across all values of $W^*$ between $VMP^*$.
and $W_O$. These limits of integration are appropriate, since the firm is not bound to honor the option when $W^*$ is below $VMP^*$ and since the worker will not exercise the option when $W^*$ exceeds $W_O$. In the FSHC case, the worker's alternative to exercising the option is to work for $WALT^*$; in the no-FSHC case, his alternative was to work for $W^*$. As a result, in the FSHC case the worker will exercise his option if $W_O > WALT^*$. However, the firm's right to refuse to honor the option is based upon the worker's value inside the firm and the firm will still refuse to honor the option when $W^* < VMP^*$. The problem that this presents is clear: The firm's decision is based upon $W^*$ while the worker's decision is based upon $WALT^*$ and the two are generated by different processes. Conceptually, this is no problem: the value of the option is now represented by the double integral

$$P_{FSHC} = \int_{VMP^*}^{\infty} \left\{ \int_0^{W_O} (W_O - WALT^*)L'(WALT^*)dWALT \right\} L'(W^*|WALT^*)dW^*.$$

Conceptually this is straightforward. We integrate $W_O - WALT^*$ across all states where the worker is both entitled to $W_O - WALT^*$, (those states where $W^* > VMP^*$) and also wishes to receive $W_O - WALT^*$ (those states where $W_O - WALT^*$ is positive). Solving the inner integral of 6.3 using the theorem of P34 yields
The solution to this integral is known not to exist. Of course, equation 6.4 could be solved numerically. However, just as it is of little interest to evaluate VSBO in Chapter five for any single set of parameters, it is of little interest to evaluate 6.4 for a single set of parameters. The reason is again that the integrand is stochastic and as a result the value of equation 6.4 changes continually. In Chapter five, we were able to deal with the stochastic nature of equation 5.3' by evaluating VSBO for a wide range of parameter values. Unfortunately, solving 6.4 is a much more complicated problem, requiring vastly more computer resources. As a result, budget constraints preclude numerically solving equation 6.4 for a range of different parameter values as we did with equation 5.3' in chapter five. Although the general solution of equation 6.4 must remain as a topic for future consideration, there is one case where this problem does possess a closed form solution.
VI.3 The Value of Seniority in the Presence of FSHC When WALT is Proportional to W.

An analytic solution to equation 7.3 is known not to exist. The inner integration can be completed but the outer cannot. At the root of the problem is the fact that the worker's decision to exercise the option is based upon the value of one variable (his wage outside the firm), while the firm's decision to honor the option is based upon another variable (the worker's value inside the firm). This requires that we consider a bivariate density to express the problem and the appropriate double integral across the bivariate lognormal density is known not to possess an analytic solution. There is, however, a case where the model can be solved analytically in the presence of FSHC. In the special case where the alternate wage is a constant percentage of the spot wage in the present firm (i.e., the case of $WALT - K \cdot W$, where $K$ is any constant), the firm's decision and the worker's decision can both be based upon the same random variable, $W$.

We begin the description of the solution by recalling that in the FSHC case the senior worker's endowment at the beginning of period two is his labor that has market value $WALT$ and an option to sell his labor for the contract wage $WO$. Furthermore, (1) the option is exchanged for an additional amount equal to $WO - WALT$ (bringing the worker's wealth to $WO$) if the option is both exercised and valid. (2) The option is valid if and only if $W > VMP*$ and (3) the option is exercised if and only if $WO > WALT*$. Of course, if $WALT = K \cdot W$, the condition for the option to be valid ($W > VMP*$) is equivalent to $WALT*$
Conditions 1, 2 and 3 imply that the worker receives the additional amount \( W_0 - W_{ALT} \) when \( W_0 > W_{ALT} > K \cdot VMP^* \), which is just another way of saying that the worker's option to sell his labor (which has spot price \( W_{ALT} \)) for \( W_0 \) is both valid and exercised if and only if \( W_0 > W_{ALT} > K \cdot VMP^* \).

The value of this option is given by

\[
W_0 \int_{K \cdot VMP^*}^{W_0} (W_0 - W_{ALT}) L'(W_{ALT}) dW_{ALT^*},
\]

where \( L'(W_{ALT}) \) is a lognormal density. Applying the theorem on page 35 with \( \phi = 1 \) and \( \psi = (K \cdot VMP)/W_0 \), we see that equation (6.5) has the solution

\[
P = -WALT \cdot N\left\{-\frac{\ln(WALT/W_0) - (r+\sigma^2/2)T}{\sigma \sqrt{T}}\right\}
+ W_0 e^{-rT} \cdot N\left\{-\frac{\ln(WALT/W_0) - (r-\sigma^2/2)T}{\sigma \sqrt{T}}\right\}
+WALT \cdot N\left\{-\frac{\ln(WALT/K \cdot VMP) - (r+\sigma^2/2)T}{\sigma \sqrt{T}}\right\}
- W_0 e^{-rT} \cdot N\left\{-\frac{\ln(WALT/K \cdot VMP) - (r-\sigma^2/2)T}{\sigma \sqrt{T}}\right\}.
\]

While the arguments above are useful for developing intuition, it is instructive to demonstrate that equation 6.5 can be derived more rigorously. Rewriting equation 6.3 as
we set \( WALT = K \cdot W \) and consider the bivariate density \( L(WALT*, W*) \).

This density can be factored into \( L_2(WALT*) \cdot L_3(W*|WALT*) \), where \( L_2(\cdot) \) is the marginal density of \( WALT* \) and \( L_3(\cdot) \) is the density of \( W* \) conditional on \( WALT* \). Since for any outcome of the random variable \( WALT* \) the outcome of the R.V. \( W* \) is \( W* = \frac{WALT*}{K} \), the conditional density becomes

\[
L_3(W*|WALT*) = 1 \text{ if } W* = \frac{WALT*}{K} \\
= 0 \text{ elsewhere.}
\]

Therefore, in this case, the bivariate density \( L(WALT*, W*) \) reduces to

\[
L(WALT*, W*) = L_2(WALT*) \cdot L_3(W*|WALT*) \\
= L_2(WALT*) \cdot 1 \\
= L_2(WALT*),
\]

for all values of \( W* \) and \( WALT* \) such that \( WALT* = K \cdot W* \). This, along with the fact that if \( W* = VMP* \) then \( WALT* = K \cdot VMP* \), allows us to write equation 6.3 as
We conclude this section by showing how equations 5.2, 5.2', 5.3
and 5.3' for the value of the seniority based options on next-period
employment for a senior General Electronic Tester at Gould Inc. are
modified to allow explicitly for the existence of firm-specific human
capital. We restrict our analysis to FSHC such that the alternate
wage is a constant fraction of the spot wage at Gould. In this case,
VSBO with FSHC is given in the competitive behavior case by equations
5.2 and 5.3 while the non-competitive behavior case is given by
equations 5.2' and 5.3', except that VMP_i* is replaced by K·VMP_i*, W
and W* are replaced by WALT and WALT* and the density L(·) is replaced
by the density L_2(·) throughout. For example, the FSHC analogue of
equation 5.3' is

\[ VSBO = WALT N \left\{ -\ln \left( \frac{WALT}{W_0} \right) - \left( r + \frac{\sigma^2}{2} \right) T \right\} \]

\[ + e^{-rt} W_0 N \left\{ -\ln \left( \frac{WALT}{W_0} \right) - \left( r - \frac{\sigma^2}{2} \right) T \right\} \]

\[ + e^{-rt} (W_0 - W_023) N \left\{ -\ln \left( \frac{WALT}{{K\cdot VMP_23^*}} \right) - \left( r - \frac{\sigma^2}{2} \right) T \right\} \]
\[ +e^{-rt}(W_{20}-W_{22}) N \left\{ \frac{-\ln(WALT/(K\cdot VMP22^*))-(r-\sigma^2/2)T}{\sigma\sqrt{T}} \right\} \]

\[ +e^{-rt}(W_{19}-W_{20}) N \left\{ \frac{-\ln(WALT/(K\cdot VMP20^*))-(r-\sigma^2/2)T}{\sigma\sqrt{T}} \right\} \]

\[ +e^{-rt}(W_{18}-W_{19}) N \left\{ \frac{-\ln(WALT/(K\cdot VMP19^*))-(r-\sigma^2/2)T}{\sigma\sqrt{T}} \right\} \]

\[ -e^{-rt}W_{18} N \left\{ \frac{-\ln(WALT/(K\cdot VMP18^*))-(r-\sigma^2/2)T}{\sigma\sqrt{T}} \right\} \]

\[ +\text{WALT} N \left\{ \frac{-\ln(WALT/(K\cdot VMP18^*))-(r+\sigma^2/2)T}{\sigma\sqrt{T}} \right\} . \]

It is possible to evaluate equation 6.7 numerically, just as we evaluated equation 5.3' numerically. To evaluate equation 6.7, we require values of the stochastic process WALT instead of values of the stochastic process \( W \), but since \( WALT = \text{K} \cdot W \), this provides no real difficulty. If we begin with the parameter settings we used in sections 5.3 and 5.4, we need only to multiply appropriate terms as indicated in equation 6.7 by the constant \( \text{K} \) and rerun the simulation programs to find the FSHC analogues of the numerical evaluations of VSBO contained in section 5.3 and 5.4.

Figure 8 graphs the value of the seniority right to next-period employment under the nine different specifications of the pair
FIGURE 3'}
Figure 3' is the exact analogue of figure 8 except that the worker whose VSBO is graphed in figure 3' has no FSHC, while the worker in figure 8 has FSHC such that $WALT = 0.9W$. Comparing figure 8 to figure 3', it is clear that for all $W$, $W_{0,i}$ and $V_MP_{i*}$, the value of the seniority based options on employment is greater for the worker with FSHC than for the worker without FSHC. In addition, further analysis shows that this result holds for all values of $\sigma^2$ and the interest rate. This is no surprise, since ceteris paribus, the worker is worse off without the option in the FSHC case than he is without the option in the no-FSHC case.

VI.4 Conclusion

This paper develops a model of labor contracts that makes use of techniques developed in the options pricing literature and uses that model to find the value of seniority to workers. In Chapter 5, the model is applied to a specific contract between the United Auto Workers and Gould Defense Systems, Inc., Ocean Systems Division. This contract has fairly complicated seniority systems that protect the interests of senior workers. In Chapter 5, the value of these seniority rights is found for Gould workers with various levels of seniority. This seniority value is shown to vary widely as the underlying parameters change.

A model that establishes the value of seniority is of more than just academic interest because of the large number of long-term work relationships that exist in the United States. Evidence is cited that
shows that most workers at some time in their working lives have substantial amounts of seniority. The ability to assign a monetary value to this seniority should be of interest to academics, firms and workers.
APPENDIX A

In this appendix, we show that

$$w \cdot N' \left\{ -\ln \left( \frac{W}{VMP18*} \right) - \left( r + \sigma^2 / 2 \right) T \right\} \frac{1}{\sigma \sqrt{T}} \quad (A.1)$$

$$-e^{-rtVMP18*} \cdot N' \left\{ -\ln \left( \frac{W}{VMP18*} \right) - \left( r - \sigma^2 / 2 \right) T \right\} \frac{1}{\sigma \sqrt{T}} = 0.$$  

Equation (A.1) holds if and only if

$$e^{-rtVMP18*} = \frac{N' \left\{ -\ln \left( \frac{W}{VMP18*} \right) - \left( r + \sigma^2 / 2 \right) T \right\} \frac{1}{\sigma \sqrt{T}}}{N' \left\{ -\ln \left( \frac{W}{VMP18*} \right) - \left( r - \sigma^2 / 2 \right) T \right\} \frac{1}{\sigma \sqrt{T}}}.$$  

(A.2)

Since $N(\cdot)$ is a normal p.d.f., equation (A.2) is equivalent to

$$e^{-rtVMP18*} = \frac{(1/\sqrt{2\pi})e^{-1/2} \left\{ -\ln \left( \frac{W}{VMP18*} \right) - \left( r + \sigma^2 / 2 \right) T \right\}^2 \sigma / \sqrt{T}}{(1/\sqrt{2\pi})e^{-1/2} \left\{ -\ln \left( \frac{W}{VMP18*} \right) - \left( r - \sigma^2 / 2 \right) T \right\}^2 \sigma / \sqrt{T}}.$$  

(A.3)
Taking natural logs of both sides of equation (A.3):

\[-rT \cdot \ln\left(\frac{W}{VMP18^*}\right) = \frac{1}{2\sigma^2 T} \left\{ \left[ -\ln(W/VMP18^*) - (r - \sigma^2/2)T \right]^2 \\
- \left[ -\ln(W/VMP18^*) - (r + \sigma^2/2)T \right]^2 \right\} \]

\[= \frac{1}{2\sigma^2 T} \left[ (-\ln(W/VMP18^*)^2 + 2\ln(W/VMP18^*) (r - \sigma^2/2)T \\
+ (r - \sigma^2/2)T^2 - (\ln(W/VMP18^*)^2 \\
+ 2\ln(W/VMP18^*) (r - \sigma^2/2)T + (r + \sigma^2/2)T^2) \right] \]

\[= \frac{1}{2\sigma^2 T} \left[ 2\ln(W/VMP18^*)T (-\sigma^2) + (r^2 + \sigma^4/4 - \sigma^2)T^2 \\
- (r^2 + \sigma^4/4 + \sigma^2)T^2 \right] \]

\[= \frac{1}{2\sigma^2 T} \left[ -2\ln(W/VMP18^*) \sigma^2 T - 2r\sigma^2 T^2 \right]. \]

Finally,

\[-\ln\left(\frac{W}{VMP18^*}\right) - rT = -rT \cdot \ln\left(\frac{W}{VMP18^*}\right). \quad (A.4)\]

Equation (A.1) holds if and only if equation (A.4) holds. Since equation (A.4) holds, we have shown that equation (A.1) holds.
APPENDIX B

This appendix contains the eight graphs referred to on page 109. These are graphs of VSBO as a function of W, VMPi* and WOi for various settings of \( r \) and \( \sigma^2 \). As is indicated in Chapter Five, changing the values of \( r \) and \( \sigma^2 \) has very little impact upon the value of VSBO, and these graphs make that very clear.

There are a total of nine combinations of the pair \((r, \sigma^2)\), when \( r \) take values of .06, .08 and .10 while \( \sigma^2 \) takes value of .16, .18 and .20. Figure 3 on page 110 gives VSBO as a function of WOi and VMPi* for \( r = .08 \) and \( \sigma^2 = .18 \). Figures 9 through 16 are graphs of VSBO as a function of WOi and VMPi* for the eight other combinations of \( r \) and \( \sigma^2 \). Each graph is labeled to show which combination of \( r \) and \( \sigma^2 \) applies to that graph. Additionally, as in figure 3, each curve on each graph is labeled to show the value of the contract wage multiplier and STAR that are in effect for the evaluation of equation 5.3' that lead to that curve. These labels take the form (value of STAR, value of the contract wage multiplier).
FIGURE 10. $r=.06, \sigma^2=.18$
FIGURE 11. $r = .06$, $\sigma^2 = .20$
FIGURE 12. $r = .08, \sigma^2 = .16$
FIGURE 13. $r = 0.08$, $\sigma^2 = 0.20$
FIGURE 15. \( \sigma = 10, \sigma^2 = .18 \)
FIGURE 16. $r=10$, $\sigma^2=.20$. 


Agreement Between Gould Defense Systems, Inc., Ocean Systems Division and United Auto Workers (AFL-CIO), Local Number 1631.


