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Lang, Mong-Lung

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DISSERTAION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

BY

Mong-Lung Lang, B.S.

*****

The Ohio State University

1987

Dissertation Committee:                       Approved by

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To My Parents
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It is my privilege to have Professor Koichiro Harada as my academic advisor, without whose guidance and encouragement, the completion of my thesis will be impossible. I thank him with my greatest respect.

I am also grateful to my graduate committee members, Professor Eiichi Bannai, Professor Ronald Solomon and Professor Sia Wong for their advice and discussion on the thesis.
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**ACKNOWLEDGEMENT**  
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v
INTRODUCTION

Let $G$ be a finite group and $F$ be the collection of all modular functions $f(z)$ satisfying:

1. $f(z)$ is a modular function with respect to a discrete subgroup $\Gamma$ of $SL_2(\mathbb{R})$ that contains $\Gamma_0(N)$ for some $N$. (i.e. $f(z)$ is meromorphic on $H^* = H \cup \text{cusps of } \Gamma$ where $H$ is the upper half plane)

2. The genus of $\Gamma$ is zero and $f(z)$ is a generator of a function field of $\Gamma$ (i.e. the genus of $\Gamma \backslash H^*$ is zero and $f(z)$ is a generator of a function field of $\Gamma \backslash H^*$).

3. At $z = \infty$, $f(z)$ has a Fourier expansion of the form:

$$q^{-1} + a_0 + \sum_{n=1}^{\infty} a_n q^n, \quad q = e^{2\pi i z}, \quad z \in H, \quad a_n \in \mathbb{C}$$

We say a pair $(G, \phi)$ is a moonshine for a finite group $G$, if $\phi$ is a function from $G$ to $F$ and if for $\sigma \in G$.

$$\phi_\sigma(z) = q^{-1} + a_0(\sigma) + \sum_{n=1}^{\infty} a_n(\sigma) q^n \quad (0.1)$$

then for $n \geq 1$, the mapping $\sigma \mapsto a_n(\sigma)$ from $G$ to $\mathbb{C}$ is a generalized character of $G$. In particular, $\phi_\sigma$ is a class function of $G$. 
In [2], Conway and Norton have assigned a "Thompson Series" of the form:

\[ t_\sigma = q^{-1} + H_1(\sigma) q + H_2(\sigma) q^2 + \ldots \in F \quad (0.2) \]

to each element \( \sigma \) of the Fischer-Griess "Monster" group \( M \) and conjectured that \( H_n \) are characters of \( M \) for all \( n \). This remarkable connection between the "Monster" \( M \) and modular functions is called \textbf{Monstrous Moonshine}.

One of the problem which arose from Conway-Norton paper is the Conway-Norton Problem:

\( (*) \) For an element \( \sigma \) in \( \ast O \), is there a class of elements \( \sigma_1 \) in \( M \) whose Thompson series \( t_{\sigma_1} \) has a form \( \eta_\sigma(z) + \text{constant} \) \quad (For the definition of \( \eta_\sigma(z) \) and \( \Theta_\sigma(z) \), see (1.2.2) and (1.2.3)).

The problem has been studied by many people. In [2], Conway and Norton studied elements in \( \ast O \) of weight 0 and proved that \( (*) \) is true for elements of weight 0 (i.e. if \( \sigma \) is of weight 0, then there is a class of elements \( \sigma_1 \) in \( M \) whose Thompson series \( t_{\sigma_1} \) has a form \( \Theta_\sigma(z)/\eta_\sigma(z) + \text{constant} \). In [7], Kondo and Tasaka studied elements in \( M_{24} \) (\( M_{24} \) can be naturally embedded in \( \ast O \)) and proved that \( (*) \) is true for elements in \( M_{24} \). Recently, Kondo [9] calculated \( \Theta_\sigma(z) \) for \( \sigma \) in...
and proved that (*) is false for \( \sigma \in 2^{12}\mathbb{M}_{24}\backslash\mathbb{M}_{24} \) (i.e. there exist some elements \( \sigma \in 2^{12}\mathbb{M}_{24}\backslash\mathbb{M}_{24} \) such that \( \Theta_\sigma(z)/\eta_\sigma(z) + \text{constant} \) is not a Thompson series \( t_{\sigma_1} \) for any \( \sigma_1 \) in \( \mathbb{M} \)).

The main purpose of this paper is to show that (*) is false for exactly 15 conjugacy classes of \( \cdot^0 \), and to find an obstruction to (*). In particular, we will show that if \( f_\sigma(z) = \Theta_\sigma(z)/\eta_\sigma(z) \) does not possess a corresponding class in \( \mathbb{M} \), then the Riemann surface whose function field is \( \mathbb{C}(f_\sigma(z)) \) cannot be realized as \( \Gamma_\sigma \backslash \mathbb{H} \) where \( \Gamma_\sigma \) is the fixing group of \( f_\sigma(z) \) in \( \text{SL}_2(\mathbb{R}) \).

We now give a brief description of the contents of this paper.

In Chapter I, we recall several facts on Frame shapes and classify every conjugacy class of \( \cdot^0 \) into (1) \( \sigma \), such that \( \text{wt} \sigma = 0 \); (2) \( \sigma \), such that \( \sigma \in \mathbb{M}_{24} \), a subgroup naturally embedded in \( \cdot^0 \); (3) \( \sigma \), such that \( \sigma \in 2^{12}\mathbb{M}_{24}\backslash\mathbb{M}_{24} \); (4) the remaining 21 conjugacy classes.

In Chapter II, we state the results obtained by Conway-Norton [2], Kondo-Tasaka [7] and Kondo [9]. As we mentioned above, [2], [7] and [9] provided an answer to (*) for elements of weight 0 and elements in \( 2^{12}\mathbb{M}_{24} \).
In Chapter III, we study (*) for the remaining elements in \( \cdot 0 \) (In Chapter I, Theorem I.2.4, we will show that studying (*) for the elements of \( \cdot 0 \) is equivalent to studying (*) for the conjugacy classes of \( \cdot 0 \)). First, we find a matrix representation for each conjugacy class. Second, we find the theta function \( \Theta_\sigma(z) \). Finally, we study \( \Theta_\sigma(z)/\eta_\sigma(z) \). The main results will be stated and proved in Theorem III.3.4, Theorem III.4.1 and III.4.2.
CHAPTER I
FRAME SHAPES OF CONWAY GROUP 0

In this chapter, we will recall several facts on Frame shapes.

I.1 Generalized permutation and Frame shapes

Definition I.1.1

(1) A symbol \( \sigma = \prod t^r_t = r_1 r_2 \ldots \), \( r_t \in \mathbb{Z} \) is called a generalized permutation if \( r_t > 0 \) for only a finite number of positive integer \( t \).

(2) \( \text{deg } \sigma = \sum_t \text{tr}_t \). \hspace{1cm} (I.1)

(3) \( \text{wt } \sigma = 1/2 \sum_t r_t \). \hspace{1cm} (I.2)

Definition I.1.2 Let \( G \) be a finite group and \( \rho \) be a d-dimensional representation of \( G \) over the rational number field \( \mathbb{Q} \), then we can assign to every element (or every conjugacy class) of \( G \) a generalized permutation of degree \( d \) as follows:

The characteristic polynomial \( \det(xI - \rho(\sigma)) \) of \( \rho(\sigma) \) can be written

in the form \( \prod (x^t - 1)^{r_t} \) where \( t \) ranges over all positive integers
dividing the order of \( G \). The generalized permutation \( \Pi_t^t \) of degree \( d_t \)

is called the Frame shape of \( \sigma \) with respect to the representation \( \rho \).

We also refer to Frame shape of a conjugacy class of \( G \), as two conjugate elements of \( G \) yield the same Frame shape.

1.2 Frame shape of Conway group \( \cdot 0 \)

As usual, we denote by \( \cdot 0 \) the automorphism group of Leech lattice. So \( \cdot 0 \) has a natural 24-dimensional representation \( \rho_o \) over \( \mathbb{Q} \) induced by its action on Leech lattice. According to Definition 1.1.2, we can assign to every conjugacy class of \( \cdot 0 \) a Frame shape of degree 24.

The Frame shape of every conjugacy class of \( \cdot 0 \) is listed in Appendix D (They are taken from [8]. See also [10]. These Frame shapes may have been known to other mathematicians too).

**Theorem 1.2.1** Let \( \sigma = \Pi_t^t \) be a Frame shape of a \( \cdot 0 \) element, then

\[
(1) \quad \text{deg } \sigma = 24 \\
(2) \quad \text{wt } \sigma \geq 0
\]

**Proof** (1) By definition, \( \text{deg } \sigma = \text{deg } \det(xI - \rho_o(\sigma)) = 24 \).

(2) Since the characteristic polynomial of \( \sigma \) is \( \pi(x^t - 1)^{t} \), we have

\[
\Sigma t \text{ number of times that } 1 \text{ appears as an eigen value.}
\]
We classify every conjugacy class of \( \cdot 0 \) into the following: (by an abuse of notation, the elements of \( \cdot 0 \) and their Frame shapes are often identified.)

1. \( \sigma \), such that \( \text{wt} \sigma = 0 \). (90 of them).
2. \( \sigma \), such that \( \sigma \in M_{24} \), a subgroup naturally embedded in \( \cdot 0 \). (21 of them).
3. \( \sigma \), such that \( \sigma \in 2^{12} M_{24} \setminus H_{24} \). (28 of them).
4. The remaining 21 conjugacy classes.

**Definition 1.2.2** Let \( \eta(z) \) be Dedekind eta-function

\[
\eta(z) = q^{1/24} \prod_{n=1}^{\infty} (1-q^n), \quad q = e^{2 \pi i z}, \quad z \in \mathbb{H}
\]  

where \( \mathbb{H} \) is the upper half plane. For a Frame shape \( \sigma = \prod \eta(t)^{x} \), we put

\[
\eta_\sigma(z) = \prod \eta(tz)^{x}.
\]

**Definition 1.2.3** Let \( \{e_1, \ldots, e_{24}\} \) be a natural basis of \( \mathbb{R}^{24} \), \( \sigma = \prod \eta(t)^{x} \)

a Frame shape. We define the following:

1. \( v(x,y) \); the inner product on \( \mathbb{R}^{24} \) with \( v(e_i,e_j) = 2 \delta_{ij} \).
2. \( L_\sigma = \{ x \in L \mid \sigma(x) = x \} \).
Theorem 1.2.4 \( \sigma \rightarrow \theta_{\sigma}(z) \) is a class function (i.e., if \( \sigma \) and \( \gamma \) are conjugate to each other in \( \mathcal{O} \), then \( \theta_{\sigma}(z) = \theta_{\gamma}(z) \)).

Proof Let \( \gamma = g \sigma g^{-1} \) for some \( g \in \mathcal{O} \), then

\[
L_{\gamma} = \{ g(x) \mid x \in L_{\sigma} \}
\]

\[
\theta_{\gamma}(z) = \sum_{y \in L_{\gamma}} q^{1/2} v(y, y)
\]

\[
= \sum_{x \in L_{\sigma}} q^{1/2} v(g(x), g(x))
\]

\[
= \sum_{x \in L_{\sigma}} q^{1/2} v(x, x)
\]

\[
= \theta_{\sigma}(z)
\]

From the above theorem, we conclude that \( \sigma \rightarrow \theta_{\sigma}(z) / \eta_{\sigma}(z) \) is also a class function, this implies that studying (*) for the elements in \( \mathcal{O} \) is equivalent to studying (*) for the conjugacy classes in \( \mathcal{O} \).
CHAPTER II

CONWAY-NORTON PROBLEM

THE KNOWN RESULTS

In this chapter we shall state the results obtained by Conway and Norton [2], Kondo and Tasaka [7], and Kondo [9]. The following notation is used.

\[
\Gamma_0(N) = \{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z}) \mid c = 0 \pmod{N} \} \quad (II.1)
\]

\( W_{N,e} \) - an Atkin-Lehner involution of \( \Gamma_0(N) \)

\[
W_{e} = \begin{bmatrix} ae & b \\ cN & de \end{bmatrix}, \ a, b, c, d \in \mathbb{Z},
\]

where \( e \parallel N \), i.e. \( e > 0 \) is a divisor of \( N \) (II.2)

with \( (e, N/e) = 1 \) and \( \det W_{e} = e \)

\[
\Gamma_0(n|h) = \begin{bmatrix} h & 0 \\ 0 & 1 \end{bmatrix}^{-1} \Gamma_0(-\frac{n}{h}) \begin{bmatrix} h & 0 \\ 0 & 1 \end{bmatrix}
\]

\[ = \{ \begin{bmatrix} a & b/h \\ cn & d \end{bmatrix} \in SL_2(\mathbb{R}) \mid a, b, c, d \in \mathbb{Z} \} \quad (II.3)\]

for example, \( \Gamma_0(8|2) = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \Gamma_0(4) \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \)

Let \( W_{e} \) be an Atkin Lehner involution of \( \Gamma_0(-\frac{N}{R}) \), then

\[ \Gamma_0(n|h) + w_{e}, \ldots = < \Gamma_0(n|h), \begin{bmatrix} h & 0 \\ 0 & 1 \end{bmatrix}^{-1} W_{e} \begin{bmatrix} h & 0 \\ 0 & 1 \end{bmatrix}, \ldots > \quad (II.4) \]
We can simplify our notations as follows:

\[
\begin{align*}
N - N^- &= \Gamma^0_1(N) \\
N + e, \ldots &= < \Gamma^0_1(N), \omega > \\
N^+ &= < \Gamma^0_1(N), \text{all its Atkin-Lehner involutions} > \\
n|h - n|h^- &= \Gamma^0_1(n|h) \\
n|h + e, \ldots &= \Gamma^0_1(n|h) + \omega, \ldots
\end{align*}
\]

Theorem II.1. (Conway-Norton [2])

If \( \sigma = \Pi_t r_t \) is a Frame shape of weight 0, then (*) is true. Namely,

there exists an element \( \sigma_1 \) in the Monster simple group \( M \), such that

\[
\Theta_\sigma(z)/\eta_\sigma(z) = t_{\sigma_1} + \text{constant, where } t_{\sigma_1} \text{ is a Thompson series for } \sigma_1.
\]

Proof: Given wt \( \sigma = 1/2 \sum r_t = 0 \), we have that 1 is not an eigenvalue,

this implies that \( \sigma \) acts fixed-point-freely on \( L \), i. e. \( L_\sigma = \{0\} \).

Hence, \( \Theta_\sigma(z) = 1 \). This gives \( \Theta_\sigma(z)/\eta_\sigma(z) = \eta_\sigma(z)^{-1} \).

A case by case inspection shows that \( \eta_\sigma(z)^{-1} \) appears in Table 3 of [2]. This means that (*) is true.
In [7], Kondo and Tasaka have determined $\theta_\sigma(z)$ explicitly in terms of the classical Jacobi theta functions and the Dedekind eta functions for every $\sigma \in M_{24}$. Furthermore, by using these expressions of $\theta_\sigma(z)$, they have shown:

**Theorem II.2 (Kondo-Tasaka [7])**

If $\sigma \in M_{24}$, then (*) is true. Furthermore, let $\sigma$, $\theta_\sigma(z)$, $\sigma_1$, and $\Gamma_\sigma$ be elements of $M_{24}$, their theta functions, elements of $M$ and corresponding discrete subgroups of $SL_2(R)$ given in the following table, respectively. Then,

1. $\theta_\sigma(z)/\eta_\sigma(z) = t_\sigma + \text{constant} = \text{Thompson series for } \sigma_1 + \text{constant},$

2. $\theta_\sigma(z)/\eta_\sigma(z)$ is a generator of a function field corresponding to $\Gamma_\sigma$, which is of genus 0. In particular, $\Gamma_\sigma$ is the fixing group of $\theta_\sigma(z)/\eta_\sigma(z)$.

<table>
<thead>
<tr>
<th>Frame shape</th>
<th>$\theta_\sigma(z)$</th>
<th>$\sigma_1$</th>
<th>$\Gamma_\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1_A$</td>
<td>$E_4(z)^3 - 45/16 \theta_1'(z)^8$</td>
<td>1A</td>
<td>1+</td>
</tr>
<tr>
<td>$2_A$</td>
<td>$E_4(2z)^2 - (15/256) \theta_2(z)^{16}$</td>
<td>2A</td>
<td>2+</td>
</tr>
<tr>
<td>$3_B$</td>
<td>$\theta(3)(2z)^6 - (9/4)(\theta_1'(z)\theta_1'(3z))^2$</td>
<td>3A</td>
<td>3+</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>$4C$</td>
<td>$1^42^44^4$</td>
<td>$\theta_3(2z)^{10} \cdot 5/4\theta_2(2z)^6\theta_4(2z)^2\theta_4(4z)^4$</td>
<td>$4A$</td>
</tr>
<tr>
<td>$5B$</td>
<td>$1^45^4$</td>
<td>$1/2(\hat{\theta}_2^4 + \theta_2^4 + \theta_3^4 + \theta_4^4) + 3\theta_2\theta_2\theta_3\theta_3$</td>
<td>$5A$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(2\theta_2^2\theta_3^2 + \theta_2\theta_2\theta_3^2 + 2\theta_3^2\theta_2)$</td>
<td></td>
</tr>
<tr>
<td>$6E$</td>
<td>$1^22^35^2$</td>
<td>$(\theta(3)(2z)\theta(3)(4z))^2$</td>
<td>$6A$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$3/4(\theta_2(2z)\theta_2(3z)\theta_4(2z)\theta_4(6z))^2$</td>
<td></td>
</tr>
<tr>
<td>$7B$</td>
<td>$1^37^3$</td>
<td>$\theta(7)(2z)^3 \cdot 3/2\theta_1'(x)\theta_1'(7z)$</td>
<td>$7A$</td>
</tr>
<tr>
<td>$8E$</td>
<td>$1^22.4.8^2$</td>
<td>$\theta_3(2z)^3\theta_3(4z)^3$</td>
<td>$8A$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$3/4\theta_2(2z)^2\theta_2(4z)\theta_4(2z)\theta_4(4z)^2$</td>
<td></td>
</tr>
<tr>
<td>$11A$</td>
<td>$1^211^2$</td>
<td>$\theta(11)(2z)^2$</td>
<td>$11A$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$1/4(\hat{\theta}_2^3 - \theta_3^3 + \theta_4^3)^2$</td>
<td></td>
</tr>
<tr>
<td>$14B$</td>
<td>$1.2.7.14$</td>
<td>$\theta(7)(2z)\theta(7)(4z)$</td>
<td>$14A$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$1/2\theta_2(2z)\theta_2(7z)\theta_4(2z)\theta_4(14z)$</td>
<td></td>
</tr>
<tr>
<td>$15D$</td>
<td>$1.3.5.15$</td>
<td>$\theta(3)(2z)\theta(3)(10z)$</td>
<td>$15A$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$3/2\theta(2z)\theta(6z)$</td>
<td></td>
</tr>
<tr>
<td>$23A$</td>
<td>$1.23$</td>
<td>$\theta(23)(2z) - 2\eta_9(x)$</td>
<td>$23A$</td>
</tr>
<tr>
<td>$2C$</td>
<td>$2^12$</td>
<td>$\theta_3(2z)^{12} \cdot 3/2\theta_1'(2z)^4$</td>
<td>$4A$</td>
</tr>
</tbody>
</table>
\begin{equation*}
\begin{array}{cccc}
\text{3}_D & 3^8 & \text{E}_4(3z) & 3C & 3|3 \\
\text{4}_D & 2^4 4^4 & 1/4(\theta_3(2z)^4+\theta_4(2z)^4)^2 & 4B & 4|2 \\
\text{4}_F & 4^6 & \theta_3(4z)^6 & 8B & 8|2^+ \\
\text{6}_I & 6^4 & \theta_3(6z)^4 & 12D & 12|3^+ \\
\text{10}_F & 2^2 10^2 & 1/4(\theta_3(z)\theta_3(5z)+\theta_4(z)\theta_4(5z))^2 & 20A & 20^+ \\
\text{12}_J & 2.4.6.12 & \theta^3(3z)\theta^3(8z) & 12C & 12|2^+ \\
\text{12}_N & 12^2 & \theta_3(12z)^2 & 24E & 24|6^+ \\
\text{21}_G & 3.21 & \theta^7(6z) & 21C & 21|3+4 \\
\end{array}
\end{equation*}

In [9] Kondo informed us, with proof in preparation, that \( \theta_{\sigma}(z) \) has been calculated for \( \sigma \in 2^{12}M_{24} \setminus M_{24} \). Furthermore (*) is true for 20 Frame shapes. Let \( \sigma, \theta_{\sigma}(z), \sigma_1, \) and \( \Gamma_\sigma \) be elements of \( 2^{12}M_{24} \setminus M_{24} \), their theta functions, elements of \( M \) and corresponding discrete subgroup of \( SL_2(\mathbb{R}) \), given in the following table, respectively. Then

1. If \( \sigma_1 \) appears, then \( \theta_{\sigma}(z)/\eta_{\sigma}(z) = t_{\sigma_1} + \text{constant} \) and

\( \theta_{\sigma}(z)/\eta_{\sigma}(z) \) is a generator of a function field corresponding to \( \Gamma_\sigma \).
which is of genus 0. In particular, $\Gamma_\sigma$ is the fixing group of
$\theta_\sigma(z)/\eta_\sigma(z)$.

(2) If $x$ appears, then (*) is false.

<table>
<thead>
<tr>
<th>Frame shape</th>
<th>$\theta_\sigma(z)$</th>
<th>$\sigma_1$</th>
<th>$\Gamma_\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-2_A$</td>
<td>$2^{16}/1^8$</td>
<td>2B</td>
<td>2-</td>
</tr>
<tr>
<td>$-4_A$</td>
<td>$1^84^8/2^8$</td>
<td>4A</td>
<td>4+</td>
</tr>
<tr>
<td>$4_B$</td>
<td>$4^8/2^4$</td>
<td>4C</td>
<td>4-</td>
</tr>
<tr>
<td>$-4_C$</td>
<td>$2^64^4/1^4$</td>
<td>4C</td>
<td>4-</td>
</tr>
<tr>
<td>$6_C$</td>
<td>$1^42^26^5/3^4$</td>
<td>x</td>
<td>x</td>
</tr>
<tr>
<td>$-6_C$</td>
<td>$2^53^46/1^4$</td>
<td>x</td>
<td>x</td>
</tr>
<tr>
<td>$-6_E$</td>
<td>$2^46^4/1^23^2$</td>
<td>6C</td>
<td>6+3</td>
</tr>
<tr>
<td>$6_G$</td>
<td>$2^36^3$</td>
<td>12A</td>
<td>12+</td>
</tr>
<tr>
<td>$8_B$</td>
<td>$2^4^8/4^4$</td>
<td>8B</td>
<td>8</td>
</tr>
<tr>
<td>$-8_C$</td>
<td>$1^84^4/2^24^2$</td>
<td>8A</td>
<td>8+</td>
</tr>
<tr>
<td>$8_D$</td>
<td>$2^4/4^2$</td>
<td>8D</td>
<td>8</td>
</tr>
<tr>
<td>$-8_E$</td>
<td>$2^3 4.8^2/1^2$</td>
<td>$\theta_3(4z)\theta_3(2z)^3 - 6\eta_\sigma(z)$</td>
<td>8E</td>
</tr>
<tr>
<td>$8_F$</td>
<td>$4^2 6^2$</td>
<td>$\theta(4z), D_4$</td>
<td>8C</td>
</tr>
<tr>
<td>$10_D$</td>
<td>$1^2 2.10^3/5^2$</td>
<td>$1/2\Sigma_{i=2}^4 \theta_1(2z)^3 \theta_1(10z)$</td>
<td>x</td>
</tr>
<tr>
<td>$-10_D$</td>
<td>$2^3 5^2/1^2$</td>
<td>$1/2\Sigma_{i=2}^4 \theta_1(2z)\theta_1(10z)^3$</td>
<td>x</td>
</tr>
<tr>
<td>$-12_E$</td>
<td>$1^2 3^2 4^2 12^2/2^2 6^2$</td>
<td>$\theta_3(2z)^2 \theta_3(6z)^2 - 4\eta_\sigma(z)$</td>
<td>12A</td>
</tr>
<tr>
<td>$12_G$</td>
<td>$4^2 12^2/2.6$</td>
<td>$\theta_4(2z)\theta_4(6z) + 2\eta_\sigma(z)$</td>
<td>12D</td>
</tr>
<tr>
<td>$12_I$</td>
<td>$1^2 4.6^2 12^3/2^2$</td>
<td>$3 \Sigma_{i=2}^{12z} \theta_1(4z)^3 \theta_1(12z)$</td>
<td>x</td>
</tr>
<tr>
<td>$-12_I$</td>
<td>$2^2 3^2 4.12/1^2$</td>
<td>$3 \Sigma_{i=2}^{12z} \theta_1(4z)\theta_1(12z)^3$</td>
<td>x</td>
</tr>
<tr>
<td>$-14_B$</td>
<td>$2^2 14^2/1.7$</td>
<td>$\theta(7)(4z)$</td>
<td>14B</td>
</tr>
<tr>
<td>$16_A$</td>
<td>$2^2 16^2/4.8$</td>
<td>$\theta_3(4z)\theta_3(8z)$</td>
<td>16A</td>
</tr>
<tr>
<td>$-16_B$</td>
<td>$1^2 16^2/2.8$</td>
<td>$\theta_3(4z)^2$</td>
<td>16C</td>
</tr>
<tr>
<td>$20_B$</td>
<td>$4.20$</td>
<td>$\theta_3(4z)\theta_3(20z)$</td>
<td>40B</td>
</tr>
<tr>
<td>$22_A$</td>
<td>$2.22$</td>
<td>$\theta_3(2z)\theta_3(22z) - 2\eta_\sigma(z)$</td>
<td>44AB</td>
</tr>
<tr>
<td>$24_E$</td>
<td>$2.4.8.24/4.12$</td>
<td>$\theta_3(4z)\theta_3(12z)$</td>
<td>24A</td>
</tr>
<tr>
<td>$-28_A$</td>
<td>$4.7.28/2.14$</td>
<td>$1/2\Sigma_{i=2}^4 \theta_1(2z)\theta_1(14z)$</td>
<td>28A</td>
</tr>
</tbody>
</table>
Concerning the Frame shapes $\sigma$ such that (*) is false, the following problem is very important.

**Problem II.3** Given $\sigma \in \mathfrak{S}_0$ such that (*) is false, what is the fixing group of $\Theta_{\sigma}(z)/\eta_{\sigma}(z)$ in $\text{SL}_2(\mathbb{R})$?

To answer this question, we need the following 7 lemmas.

**Lemma II.4** (M. Koike [4])

Let $\sigma$ be elements listed in the following table, then there exist some elements $\sigma$ and $\sigma'$ in $\mathfrak{S}_0$ such that $\Theta_{\sigma}(z)/\eta_{\sigma}(z) = t_{\sigma} + c/t_{\sigma'} + \text{constant}$ where $c$ is a constant.

<table>
<thead>
<tr>
<th>Frame shape</th>
<th>$g$</th>
<th>$g'$</th>
<th>$c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$6_C$</td>
<td>$1^4 2^5 3^4$</td>
<td>$6E$</td>
<td>$6D$, $-4$</td>
</tr>
<tr>
<td>$-6_C$</td>
<td>$2^5 3^4 6^{11}$</td>
<td>$6E$</td>
<td>$6B$, $12$</td>
</tr>
</tbody>
</table>
\begin{align*}
10_D & \quad 1^22.10^3/5^2 & 10E & \quad 10C, -2 & 25 \\
-10_D & \quad 2^35^210/1^2 & 10E & \quad 10D, 6 & 1 \\
12_I & \quad 1^24.62^212/3^2 & 12I & \quad 12B, -4 & 9 \\
-12_I & \quad 2^234.12/1^2 & 12I & \quad 12H, 4 & 1 \\
30_D & \quad 1.610.15/3.5 & 30G & \quad 30A, -3 & 1 \\
-30_D & \quad 2.35.30/1.15 & 30G & \quad 30F, 1 & 1
\end{align*}

Each line of the above table reads as
\[ \theta_\sigma(z)/\eta_\sigma(z) = t_g + c/t_g, + \text{constant} \]  
\hspace{10cm} (II.10)

For example, the first line shows that:
\[ \theta_\sigma(z)/\eta_\sigma(z) = t_{6E} + 81/(t_{6D} - 4) + 0 \]  
\hspace{10cm} (II.11)

**Lemma II.5** (See [5])

Let \( S_{\Gamma_0(N)} \) be the set of pairs \((c,a)\) satisfying

1. \((1,0) \in S_{\Gamma_0(N)}\)

2. \(c > 1, c \in \mathbb{N}, 1 \leq a < c, (c,a) = 1 \) and;

3. If \((c,a), (c,a_1) \in S_{\Gamma_0(N)}\), and \(a_1 \equiv a \mod (c,N/c)\), then \(a = a_1\).

Then, the set \( \{a/c \mid (c,a) \in S_{\Gamma_0(N)}\} \) is a complete set of representatives of all inequivalent cusps of \( \Gamma_0(N) \).
Lemma II.6 ([6], see [5] also)

Let \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z}) \), and \( t \in \mathbb{N} \), then there exists \( A_t = \begin{bmatrix} a_t & b_t \\ c_t & d_t \end{bmatrix} \in SL_2(\mathbb{Z}) \) and \( \alpha_t, \beta_t \in \mathbb{N}, \delta_t \in \mathbb{Z} \) such that

\[
\begin{bmatrix} a_t & b_t \\ c_t & d_t \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & \alpha_t \\ \beta_t & 0 \end{bmatrix} (II.12)
\]

(2) Let \( \sigma = \Pi_t \tau_t \) be a Frame shape, then

\[
\eta_\sigma(Az) = C (cz + d)^{1/2 \sum_{t} r_t \exp[\pi i z/12(\sum_{t} r_t/c - (t,c)^2)]} g(z)
\]

where \( g(z) = \prod_{n=1}^{\infty} [1 - \exp(2\pi i (\alpha_n z + \beta_n)/\delta_n)]^{r_t} \) and

\[
C = \Pi_t \eta_t(\begin{bmatrix} a_t & b_t \\ c_t & d_t \end{bmatrix})^{r_t} \Pi_t \delta_t^{-1/2} \exp(\pi i /12 \sum_{t} \beta_t/\delta_t r_t).
\]

Lemma II.7 Given \( \sigma = \Pi_t \tau_t \), \( c + 0 \), such that \( \sum_{t} r_t(c,r) = 0 \), the following hold.

\[
(1) \text{If } \sum_{t} r_t(c,r)^2/t > 0 \text{ then } \eta_\sigma(a/c) = 0 \quad (II.14)
\]

\[
(2) \text{If } \sum_{t} r_t(c,r)^2/t < 0 \text{ then } \eta_\sigma(a/c) = \infty \quad (II.15)
\]

\[
(3) \text{If } \sum_{t} r_t(c,r)^2/t = 0 \text{ then } \eta_\sigma(a/c) + 0, = \infty \quad (II.16)
\]
Proof: \( \sum r_t = 0, \) then

\[
\eta_a(Az) = C \exp[\pi i z/12 \left( \sum r_t(t,c)^2/t \right)] g(z) \tag{II.17}
\]

Since \( Az \to a/c \) and \( g(z) \to 1 \) as \( z \to \infty \) the lemma follows.

Remark Since any cusp of a given congruence subgroup in \( SL_2(R) \) has a form \( a/c \in \mathbb{Q}, c \neq 0, \) the above lemma can be used to obtain the behavior of \( \eta_a(z) \) at any cusp.

Lemma II.8 Let \( \Gamma \) be a discrete subgroup of \( SL_2(R) \) of the first kind, and let \( D \) be its fundamental domain, then

the area of \( D = \Lambda_\Gamma = 2\pi [2g - 2 + t + \sum (1 - 1/n_i)] \) \( \tag{II.18} \)

where

- \( g \) = genus of \( \Gamma \)
- \( t \) = number of inequivalent cusps of \( \Gamma \)
- \( n_i \) = order of inequivalent elliptic element \( \sigma_i \in \Gamma \)

Lemma II.9 Let \( \Gamma \) be a discrete subgroup of \( SL_2(R) \) that contains \( \Gamma_0(N) \) for some \( N \) and \( f(z) \) be a generator of \( C(\Gamma \backslash \mathbb{H}). \)

If \( F(z) = \frac{a_n f(z)^n + a_{n-1} f(z)^{n-1} + \ldots + a_0}{b_m f(z)^m + b_{m-1} f(z)^{m-1} + \ldots + b_0} \) \( \tag{II.19} \)

where \( a_i, b_j \in \mathbb{C} \)

then the fixing group \( \Gamma' \) of \( F(z) \) contains \( \Gamma. \) Furthermore,
\[ [ \Gamma' : \Gamma ] \leq M! \text{ where } M = \max\{ n, m \}. \]

**Proof.** Since \( \Gamma \) fixes \( f(z) \), it also fixes \( F(z) \), thus \( \Gamma' \Gamma' \).

On the other hand, from (II.19), we know that \( f(z) \) is a solution of \[ F(z) \left( b_n x^n + b_{n-1} x^{n-1} + \ldots + b_0 \right) - \]
\[ [ a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0 ] = 0 \quad (\text{II.20}) \]

Let \( f = f_1, f_2, \ldots, f_M \) be solutions of (II.20). Since \( \Gamma' \) fixes \( F(z) \), \( \Gamma' \) also fixes (II.20). This implies that \( \Gamma' \) permutes \( \{ f_1, \ldots, f_M \} \).

Therefore, \( \Gamma' \) can be embedded into \( S_M \). Let \( \Gamma'' \) be the kernel of the embedding, then \( \Gamma'/\Gamma'' \) is isomorphic to a subgroup of \( S_M \).

So \([ \Gamma' : \Gamma'' ] \leq M! \). Since \( \Gamma'' \subseteq \Gamma \), we have \([ \Gamma' : \Gamma ] \leq M! \).

**Lemma II.10** Let \( \Gamma = \Gamma_0 (N) \), then

\[ \Gamma_{1/n} = \{ \sigma \in \Gamma | \sigma(1/n) = (1/n) \} \]
\[ = \left< \begin{bmatrix} 1 & 0 \\ n & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -n & 1 \end{bmatrix} \right>, \quad x \in \mathbb{Z} \setminus \{0\} \quad (\text{II.21}) \]

such that \( n^2x/N \) is an integer of least absolute value.

**Proof:** Given \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma \) such that \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} (1/n) = (1/n) \), then

\[ \begin{bmatrix} 1 & 0 \\ -n & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ n & 1 \end{bmatrix} = -\infty, \text{ where } \Gamma_* = \left< \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right> \not\in \mathbb{Z}. \]

So \( \begin{bmatrix} 1 & 0 \\ -n & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ n & 1 \end{bmatrix} = [1, x], \) for some \( x \in \mathbb{Z} \), which implies
\[
\begin{bmatrix}
c & d \\ n & 1
\end{bmatrix} \begin{bmatrix}
1 & 0 \\ 0 & 1
\end{bmatrix} \begin{bmatrix}
1 & n \\ 0 & 1
\end{bmatrix} = \begin{bmatrix}
1 & -nx \\ n & 1
\end{bmatrix} \in \Gamma
\]

The lemma follows immediately.

Now the following theorem answers Problem II.3

**Theorem II.11.** Let \( \sigma \) and \( \Gamma_\sigma \) be elements in \( 2^{12} \text{M}_{24} \setminus \text{M}_{24} \) such that (*) is false and discrete subgroups in \( \text{SL}_2(\mathbb{R}) \) given in the following table, then \( \Gamma_\sigma \) is the fixing group of \( \theta_\sigma(z)/\eta_\sigma(z) \). Moreover, \( \Gamma_\sigma \) is of genus zero.

<table>
<thead>
<tr>
<th>Frame shape</th>
<th>( \Gamma_\sigma )</th>
</tr>
</thead>
<tbody>
<tr>
<td>6( _C )</td>
<td>( 1^42.6^5/3^4 )</td>
</tr>
<tr>
<td>-6( _C )</td>
<td>( 2^53^46/1^4 )</td>
</tr>
<tr>
<td>10( _D )</td>
<td>( 1^22.10^3/5^2 )</td>
</tr>
<tr>
<td>-10( _D )</td>
<td>( 2^35^210/1^2 )</td>
</tr>
<tr>
<td>12( _I )</td>
<td>( 1^24.6^212/3^2 )</td>
</tr>
<tr>
<td>-12( _I )</td>
<td>( 2^34.12/1^2 )</td>
</tr>
<tr>
<td>30( _D )</td>
<td>( 1.6.10.15/3.5 )</td>
</tr>
<tr>
<td>-30( _D )</td>
<td>( 2.3.5.30/1.5 )</td>
</tr>
</tbody>
</table>

**Proof:**
(1) \( \sigma = 1^{42}.65/3^4 \):

Let \( \sigma_1 = 2^{83}.4/1^{46} \), then the fixing group of \( t_{\sigma_1} \) is \( \Gamma_0(6) \).

Applying Lemma II.4, we have

\[
\Theta_\sigma(z)/\eta_\sigma(z) = t_{\sigma_1} + \frac{81}{t_{\sigma_1} - \frac{1}{t_{\sigma_1}}} + 0 \quad \text{(II.22)}
\]

\[
= (t_{\sigma_1}^3 - 13t_{\sigma_1}^2 + 85t_{\sigma_1} + 81) / (t_{\sigma_1}^2 - 13t_{\sigma_1} + 4)
\]

Let \( \Gamma \) be the fixing group of \( \Theta_\sigma(z)/\eta_\sigma(z) \).

Applying Lemma II.9, we have \([ \Gamma : \Gamma_0(6) ] \leq 6 \)

Applying Lemma II.5, II.6 and II.7, we have the following cusps \( \langle c_1 \rangle \) of \( \Gamma_0(6) \):

<table>
<thead>
<tr>
<th>( t_{\sigma_1}(c_1) )</th>
<th>0</th>
<th>1/2</th>
<th>1/3</th>
<th>1/6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t_{\sigma_1}(c_1) )</td>
<td>9</td>
<td>0</td>
<td>1</td>
<td>∞</td>
</tr>
<tr>
<td>( \Theta_\sigma/\eta_\sigma(c_1) )</td>
<td>∞</td>
<td>∞</td>
<td>1</td>
<td>∞</td>
</tr>
</tbody>
</table>

The above table shows that if we pick \( A \in \Gamma \), then \( A(1/3) \) is equivalent to 1/3. Without lose of generality, we may assume \( A(1/3) = 1/3 \).

Case 1. \([ \Gamma : \Gamma_0(6) ] = 2 \)

\([ \Gamma_{1/3} : (\Gamma_0(6))_{1/3} ] = 2 \)

Applying Lemma II.10, we have,

\[
\Gamma_{1/3} = <A | A = [ \begin{array}{cc} 1 & 0 \\ 3 & 1 \end{array} ][ \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} ][ \begin{array}{cc} 1 & 0 \\ -3 & 1 \end{array} ] > \quad \text{(II.23)}
\]

\( A(1/6) = 4/15 \) equivalent to 1/3.

This is a contradiction.
Case 2. \([ \Gamma : \Gamma_0(6) ] = 3 \).

\([ \Gamma_{1/3} : ( \Gamma_0(6) )_{1/3} ] = 3 \)

Applying Lemma II.10, we have,

\[
\Gamma_{1/3} = \langle A | A - \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} [ \begin{bmatrix} 1 & 2/3 \\ 0 & 1 \end{bmatrix} [ \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} ] >
\]

\(A(0) = 2/9\) equivalent to 1/3.

This is a contradiction.

Case 3. \([ \Gamma : \Gamma_0(6) ] = 6 \)

\([ \Gamma_{1/3} : ( \Gamma_0(6) )_{1/3} ] = 6 \)

Applying Lemma II.10, we have,

\[
\Gamma_{1/3} = \langle A | A - \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} [ \begin{bmatrix} 1 & 1/3 \\ 0 & 1 \end{bmatrix} [ \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} ] >
\]

\(A^2(0) = 2/9\) equivalent to 1/3.

This is a contradiction.

Summing up the above, we have \(\Gamma = \Gamma_0(6)\).

(2) \(\sigma = 2^{5}3^{6}6/1^{4}\):

Let \(\sigma_1 = 2^{8}3^{4}/1^{4}6^{8}\).

Applying Lemma II.4, we have

\[
\Theta_\sigma(z)/\eta_\sigma(z) = t_{\sigma_1} + \frac{1}{t_{\sigma_1}} + \frac{1}{t_{\sigma_1}} - 8
\]

Let \(\Gamma\) be the fixing group of \(\Theta_\sigma(z)/\eta_\sigma(z)\).

Applying Lemma II.9, we have \([ \Gamma : \Gamma_0(6) ] \leq 6 \).
Applying Lemma II.5, II.6, and II.7, we have

cusps \( c_1 \) of \( \Gamma_0(6) \)

<table>
<thead>
<tr>
<th>( c_1 )</th>
<th>0</th>
<th>1/2</th>
<th>1/3</th>
<th>1/6</th>
</tr>
</thead>
</table>

\[
\begin{align*}
t_{c_1}(c_1) & = 9 \\
\theta_{\sigma}/\eta_{\sigma}(c_1) & = 1
\end{align*}
\]

Case 1. \( [\Gamma : \Gamma_0(6) ] = 2 \).

Then \( [\Gamma_0 : (\Gamma_0(6))_0 ] = 2 \), and \( \Gamma_0 = < A | A = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} > \).

\( A(1/2) = 1/5 \) equivalent to 0. This is a contradiction.

Case 2. \( [\Gamma : \Gamma_0(6) ] = 3 \).

Then \( [\Gamma_0 : (\Gamma_0(6))_0 ] = 3 \), and \( \Gamma_0 = < A | A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} > \).

\( A(1/3) = 1/5 \) equivalent to 0. This is a contradiction.

Case 3. \( [\Gamma : \Gamma_0(6) ] = 6 \).

Then \( [\Gamma_0 : (\Gamma_0(6))_0 ] = 6 \), and \( \Gamma_0 = < A | A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} > \).

\( A(1/6) = 1/7 \) equivalent to 0. This is a contradiction.

Summing up the above, we have \( \Gamma = \Gamma_0(6) \).

(3) \( \sigma = 1^2 2.10^3/5^2 \):

Let \( \sigma_1 = 2.5^5/1.10^5 \).

Applying Lemma II.4, we have,

\[
\frac{\theta_{\sigma}(z)/\eta_{\sigma}(z)}{\theta_{\sigma}/\eta_{\sigma}(z)} = \frac{t_{c_1}}{t_{c_1}} + \frac{25}{t_{c_1}} + 1
\]

(II.27)

Let \( \Gamma \) be the fixing group of \( \theta_{\sigma}(z)/\eta_{\sigma}(z) \).
Applying Lemma II.9, we have \( [ \Gamma : \Gamma_0(10) ] \leq 6 \).

Applying Lemma II.5, II.6 and II.7, we have the following,

cusps \( (c_i) \) of \( \Gamma_0(10) \)

\[
\begin{array}{cccc}
0 & 1/2 & 1/5 & 1/10 \\
\end{array}
\]

\( e_{c_1}(c_i) \)

\[
\begin{array}{cccc}
4 & 1 & 0 & \infty \\
\end{array}
\]

\( \theta_{c_1}/\eta(c_i) \)

\[
\begin{array}{cccc}
\infty & \infty & 1 & \infty \\
\end{array}
\]

Case 1. \( [ \Gamma : \Gamma_0(10) ] = 2 \).

\( [ \Gamma_{1/5} : (\Gamma_0(10))_{1/5} ] = 2 \).

Applying Lemma II.10, we have,

\[
\Gamma_{1/5} = \langle A | A = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -5 & 1 \end{bmatrix} \rangle
\]

(II.28)

\( A(1/10) = 6/35 \) equivalent to \( 1/5 \).

This is a contradiction.

Case 2. \( [ \Gamma : \Gamma_0(10) ] = 3 \).

\( [ \Gamma_{1/5} : (\Gamma_0(10))_{1/5} ] = 3 \).

Applying Lemma II.10, we have,

\[
\Gamma_{1/5} = \langle A | A = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2/3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -5 & 1 \end{bmatrix} \rangle
\]

(II.29)

\( A(1/10) = 13/80 \) equivalent to \( 1/10 \).

\( A(1/5) = 1/5 \)

\( A(1/2) = 1/8 \) equivalent to \( 1/2 \).

\( A(0) = 2/13 \) equivalent to \( 0 \).

This implies that \( \Gamma \) has 4 cusps.

Applying Lemma II.8, we have \( \Lambda_{\Gamma_0(10)} = 6\pi \) and
2\pi - \Lambda_\Gamma = 2\pi(4 + 2g - 2 + \Sigma(1 - 1/n_i)) \geq 4\pi

This is a contradiction.

Case 3. \([ \Gamma : \Gamma_0(10) ] = 6.\)

\([ \Gamma_{1/5} : (\Gamma_0(10)^{1/5} ] = 6.\)

Applying Lemma II.10, we have,

\[\Gamma_{1/5} = < A | A = \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1/3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -5 & 1 \end{pmatrix}> \quad \text{(II.30)}\]

\[A^3(1/10) = 6/35 \text{ equivalent to } 1/5.\]

This is a contradiction.

Summing up the above, we have \(\Gamma = \Gamma_0(10).\)

(4) \(\sigma = 2^{3} 5^{2} 10^{1/2}.\)

Let \(\sigma_1 = 2.5^{5}/1.10^{5}.\)

Applying Lemma II.4, we have

\[\Theta_\sigma(z)/\eta_\sigma(z) = \tau_{\sigma_1} + \frac{1}{20} - 3 \quad \text{(II.31)}\]

Let \(\Gamma\) be the fixing group of \(\Theta_\sigma(z)/\eta_\sigma(z).\)

Applying Lemma II.9, we have \([ \Gamma : \Gamma_0(10) ] \leq 6.\)

Applying Lemma II.5, II.6, and II.7, we have the following,

<table>
<thead>
<tr>
<th>Cusps ((c_1)) of (\Gamma_0(10))</th>
<th>0</th>
<th>1/2</th>
<th>1/5</th>
<th>1/10</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\tau_{\sigma_1}(c_1))</td>
<td>4</td>
<td>1</td>
<td>0</td>
<td>-</td>
</tr>
<tr>
<td>(\Theta_\sigma/\eta_\sigma(c_1))</td>
<td>1</td>
<td>∞</td>
<td>∞</td>
<td>∞</td>
</tr>
</tbody>
</table>
Case 1. [ $\Gamma : \Gamma_0(10)$ ] = 2.

Then [ $\Gamma_0 : ( \Gamma_0(10) \ 0 )$ ] = 2, and $\Gamma_0 = < A \mid A = [ \frac{1}{5} \ 0 \ 1 ] >$.

$\Lambda(1/2) = 1/7$ equivalent to 0. This is a contradiction.

Case 2. [ $\Gamma : \Gamma_0(10)$ ] = 3.

Then [ $\Gamma_0 : ( \Gamma_0(10) \ 0 )$ ] = 3, and $\Gamma_0 = < A \mid A = [ \frac{1}{10/3} \ 0 \ 1 ] >$.

$\Lambda(1/10) = 3/40$ equivalent to 1/10.

$\Lambda(1/5) = 3/25$ equivalent to 1/5.

$\Lambda(1/2) = 3/16$ equivalent to 1/2.

$\Lambda(0) = 0$.

Similar to case 2 of (3), this is a contradiction.

Case 3. [ $\Gamma : \Gamma_0(10)$ ] = 6.

Then [ $\Gamma_0 : ( \Gamma_0(10) \ 0 )$ ] = 6, and $\Gamma_0 = < A \mid A = [ \frac{1}{5/3} \ 0 \ 1 ] >$.

$\Lambda^3(1/2) = 1/7$ equivalent to 0. This is a contradiction.

Summing up the above, we have $\Gamma = \Gamma_0(10)$.

(5) $\sigma = 1^{2} 4^{2} 6^{2} 12^{3} 2^{1}$.

Let $\sigma_1 = 4^{3} 6^{2} / 2^{2} 12^{4}$.

Applying Lemma II.4, we have,

\[ \Theta_\sigma(z)/\eta_\sigma(z) = \frac{\tau\sigma_1}{\tau\sigma_1} - \frac{\tau\sigma_1}{\tau\sigma_1} + 2 \]

Let $\Gamma$ be the fixing group of $\Theta_\sigma(z)/\eta_\sigma(z)$. 
Applying Lemma II.9, we have \( \Gamma : \Gamma_0(12) \leq 6 \).

Applying Lemma II.5, II.6 and II.7, we have the following,

<table>
<thead>
<tr>
<th>Cusps (c_i) of (\Gamma_0(12))</th>
<th>0</th>
<th>1/2</th>
<th>1/3</th>
<th>1/4</th>
<th>1/6</th>
<th>1/12</th>
</tr>
</thead>
<tbody>
<tr>
<td>(t_{\sigma_1}(c_i))</td>
<td>3</td>
<td>-3</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>\infty</td>
</tr>
<tr>
<td>(\theta_\sigma/\eta_\sigma(c_i))</td>
<td>\infty</td>
<td>-2</td>
<td>1</td>
<td>\infty</td>
<td>-6</td>
<td>\infty</td>
</tr>
</tbody>
</table>

From the above table, we know \(\Gamma\) has at least 4 cusps.

Applying Lemma II.8,

\[ 8\pi = \Lambda_{\Gamma_0}(12) \geq \Lambda_{\Gamma} \geq 2\pi(4 + 2g - 2 + \Sigma(1-1/n_1)) \geq 4\pi \]

So \(\Gamma : \Gamma_0(12) = 1\) or 2.

If \(\Gamma : \Gamma_0(12) = 2\), then \(\Gamma_{1/2} : (\Gamma_0(12))_{1/2} = 2\).

Applying Lemma II.10, we have,

\[ \Gamma_{1/2} = \langle A | A = [ 1 0 ] [ 1 0 \ 3/2 \ 1 \ -2 \ 1 ] \rangle \]

This is a contradiction.

So \(\Gamma = \Gamma_0(12)\).

\(5) \sigma = 2^3 3^2 4.12/1^2 \)

Let \(\sigma_1 = 4^3 6^3 2^3 12^4 \)

Applying Lemma II.4, we have,

\[ \theta_\sigma(z)/\eta_\sigma(z) = t_{\sigma_1} + \frac{1}{t_{\sigma_1}} - \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \]

\[ \text{(II.34)} \]
Let $\Gamma$ be the fixing group of $\Theta_\sigma(z)/\eta_\sigma(z)$.

Applying Lemma II.9, we have $[\Gamma : \Gamma_0(12)] \leq 6$.

Applying Lemma II.5 II.6 and II.7, we have the following,
cusps $(c_1)$ of $\Gamma_0(12)$

<table>
<thead>
<tr>
<th>$c_1$</th>
<th>0</th>
<th>1/2</th>
<th>1/3</th>
<th>1/4</th>
<th>1/6</th>
<th>1/12</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t_{\sigma_1}(c_1)$</td>
<td>3</td>
<td>-3</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

$\Theta_\sigma/\eta_\sigma(c_1)$

1 -6 0 -1 2 -2

Similar to (5), we have $[\Gamma : \Gamma_0(12)] = 1$ or 2.

If $[\Gamma : \Gamma_0(12)] = 2$, then $[\Gamma_{1/2} : (\Gamma_0(12))_{1/2}] = 2$.

Applying Lemma II.10, we have,

$\Gamma_{1/2} = \langle A | A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 3/2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \rangle$ (II.35)

$A(0) = 3/8$ equivalent to 1/4.

This is a contradiction.

So $\Gamma = \Gamma_0(12)$.

(7) $\sigma = 1.6.10.15/3.5$:

Let $\sigma_1 = 3.5/2.30$.

Applying Lemma II.4, we have,

$\Theta_\sigma(z)/\eta_\sigma(z) = t_{\sigma_1} + \frac{1}{t_{\sigma_1}} + \frac{1}{t_{\sigma_1}} + \frac{1}{t_{\sigma_1}} + 1$ (II.36)

Let $\Gamma$ be the fixing group of $\Theta_\sigma(z)/\eta_\sigma(z)$.

Applying Lemma II.9, we have $[\Gamma : \Gamma_0(30)+W_{15}] \leq 6$

Applying Lemma II.5, II.6 and II.7, we have the following.
cusps \( (c_1) \) of \( \Gamma_0(30)+\mathbb{H}_{15} \)

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1/2</th>
<th>1/3</th>
<th>1/6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \tau \sigma_1(c_1) )</td>
<td>2</td>
<td>( \infty )</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>( \Theta_{\sigma}/\eta_{\sigma}(c_1) )</td>
<td>( \infty )</td>
<td>( \infty )</td>
<td>1</td>
<td>( \infty )</td>
</tr>
</tbody>
</table>

Case 1. \( \Gamma : \Gamma_0(30)+\mathbb{H}_{15} \) = 2.
\[ \Gamma_{1/3} : (\Gamma_0(30)+\mathbb{H}_{15})_{1/3} \] = 2.

Applying Lemma II.10, we have
\[ \Gamma_{1/3} = < A | A - \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} > \] (II.37)
\[ A(1/6) = 16/51 \text{ equivalent to } 1/3. \]
This is a contradiction.

Case 2. \( \Gamma : \Gamma_0(30)+\mathbb{H}_{15} \) = 3.
\[ \Gamma_{1/3} : (\Gamma_0(30)+\mathbb{H}_{15})_{1/3} \] = 3

Applying Lemma II.10, we have
\[ \Gamma_{1/3} = < A | A - \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 10/3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} > \] (II.38)
\[ A(0) = 10/33 \text{ equivalent to } 1/3. \]
This is a contradiction.

Case 3. \( \Gamma : \Gamma_0(30)+\mathbb{H}_{15} \) = 6.
\[ \Gamma_{1/3} : (\Gamma_0(30)+\mathbb{H}_{15})_{1/3} \] = 6

Applying Lemma II.10, we have
\[ \Gamma_{1/3} = < A | A - \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 5/3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} > \] (II.39)
\[ A^2(0) = 10/33 \text{ equivalent to } 1/3. \]
This is a contradiction.

Summing up the above, we have $\Gamma = \Gamma_0(30)+\mathbb{W}_{15}$.

(8) $\sigma = 2.3.5.30/1.5$:

Let $\sigma_1 = 3.5./2/30$.

Applying Lemma II.4, we have,

$$\Theta_\sigma(z)/\eta_\sigma(z) = t_{\sigma_1} + \frac{1}{t_{\sigma_1} + \frac{2}{t_{\sigma_1} + \frac{3}{t_{\sigma_1} + \frac{30}{1.5}}}} - 1 \quad (\text{II.40})$$

Let $\Gamma$ be the fixing group of $\Theta_\sigma(z)/\eta_\sigma(z)$.

Applying Lemma II.9, we have $[\Gamma : \Gamma_0(30)+\mathbb{W}_{15}] \leq 6$.

Applying Lemma II.5, II.6, and II.7, we have the following,

<table>
<thead>
<tr>
<th>c_{i_1}</th>
<th>0</th>
<th>1/2</th>
<th>1/3</th>
<th>1/6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t_{\sigma_1}(c_{i_1})$</td>
<td>2</td>
<td>$\infty$</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$\Theta_\sigma/\eta_\sigma(c_{i_1})$</td>
<td>1</td>
<td>$\infty$</td>
<td>$\infty$</td>
<td>$\infty$</td>
</tr>
</tbody>
</table>

Case 1. $[Q : \Gamma_0(30)+\mathbb{W}_{15}] = 2$

Then $[\Gamma_0 : (\Gamma_0(30)+\mathbb{W}_{15})_0 ] = 2$, and $\Gamma_0 = <A \mid A = \begin{bmatrix} 1 & 0 \\ 15 & 1 \end{bmatrix}>$

$A(1/2) = 1/17$ equivalent to 0. This is a contradiction.

Case 2. $[\Gamma : \Gamma_0(30)+\mathbb{W}_{15}] = 3$.

Then $[\Gamma_0 : (\Gamma_0(30)+\mathbb{W}_{15})_0 ] = 3$, and $\Gamma_0 = <A \mid A = \begin{bmatrix} 1 & 0 \\ 10 & 1 \end{bmatrix}>$

$A(1/3) = 1/13$ equivalent to 0. This is a contradiction.

Case 3. $[\Gamma : \Gamma_0(30)+\mathbb{W}_{15}] = 6$. 
Then \[ \Gamma_0 : (\Gamma_0(30)+U_{15})_0 \] = 6, and \( \Gamma_0 = \langle A | A = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix} \rangle \)

\( A^2(1/3) = 1/13 \) equivalent to 0. This is a contradiction.

Summing up the above, we have \( \Gamma = \Gamma_0(30)+U_{15} \).
 CHAPTER III

CONWAY-NORTON PROBLEM

THE REMAINING 21 FRAME SHAPES OF *0

In this chapter, we will study (*) for the remaining elements (conjugacy classes) in *0. The main results will be stated and proved in Theorem III.3.4. Theorem III.3.4, together with Theorem II.1, II.2 will answer (*) (i.e. for an element σ in *0, we have decided whether there exists an element σ₁ in N such that θ_σ(z)/η_σ(z) = tσ₁ + constant or not). Furthermore, when (*) is false, an obstruction will be stated in Theorem III.4.2.

III.1 Matrix representation of *0 elements.

To give a complete study of Conway-Norton Problem for the remaining 21 conjugacy classes (as listed in Theorem III.1.2), a matrix representation of each conjugacy classes is necessary. To achieve this, we first state a theorem.

Theorem III.1.1 (See [1])

*0 is generated by α, β, γ, δ, ε and T, where

α = (∞)(0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22)
β = (∞)(0)(1,2,4,8,16,9,18,13,3,6,12)(5,10,20,17,11,22,21,19,15,7,14)

33
\[(12,21)(14,18)\]
\[\delta = (-)(0)(3)(15)(1,18,4,2,6)(5,21,20,10,7)(8,16,13,9,12)\]
\[(11,19,22,14,17)\]
\[\sigma \text{ and } T \text{ are listed in Appendix C.}\]

Next, we give a brief explanation of how to obtain matrix representations for the remaining 21 conjugacy classes.

1. Find a set of representatives of all conjugacy classes of of $M_{24}$ expressed as permutations of 24 letters explicitly.

2. Determine the matrix representation $A_4$'s for each representative.

(See Appendix B.)

3. Compute products of $T$ and $A_4$'s.

4. Study the Frame shape of each matrix computed in (3).

Finally, we state our results in Theorem III.1.2

**Theorem III.1.2** The matrix representation of the remaining 21 conjugacy classes are listed below:

<table>
<thead>
<tr>
<th>Frame shape</th>
<th>matrix representation of $\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$3_A$</td>
<td>$3^9/1^3$</td>
</tr>
<tr>
<td></td>
<td>$\left( A_2TA_{20}T \right)^3$</td>
</tr>
<tr>
<td>$5_C$</td>
<td>$5^5/1$</td>
</tr>
<tr>
<td></td>
<td>$\left( A_7TA_4 \right)^6$</td>
</tr>
</tbody>
</table>
\begin{align*}
-6_D & \quad 1^{5.3.6^4/2^4} & \quad (A_6^T)^2 \\
6_F & \quad 3^{3.6^3/1.2} & \quad (A_3^TA_{11})^3 \\
-6_F & \quad 1.6^6/2^2.3^3 & \quad (A_7^T)^3 \\
9_B & \quad 9^3/3 & \quad (A_2^TA_{20}^T) \\
9_C & \quad 1^3.9^3/3^2 & \quad (A_2^TA_{18}^T) \\
-10_E & \quad 1^{3.5.10^2/2^2} & \quad (A_7^TA_4)^3 \\
-12_D & \quad 2.3^{3.12^3}/1.4.6^3 & \quad (A_3^TA_{10})^3 \\
12_H & \quad 2^3.6.12^2/1.3.4^2 & \quad (A_3^TA_8) \\
-12_H & \quad 1.2^2.3.12^2/4^2 & \quad (A_6^T) \\
-12_K & \quad 1^3.12^3/2.3.4.6 & \quad * \\
15_E & \quad 1^{2.15^2}/3.5 & \quad (A_5^T) \\
-18_B & \quad 1^2.9.18/2.3 & \quad (A_2^TA_9^T) \\
18_G & \quad 1.2.18^2/6.9 & \quad (A_7^T) \\
-18_G & \quad 2^2.9.18/1.6 & \quad (A_3^TA_{11}) \\
20_G & \quad 1.2.10.20/4.5 & \quad (A_5^TA_{10}) \\
-20_G & \quad 2^2.5.20/1.4 & \quad (A_1^TA_{17}) 
\end{align*}
Proof: The proof is done by computer.

Remark: The main purpose of finding a matrix representation for each conjugacy classes in *0 is to use the matrix form of each element to determine their theta functions. Theta function of \( \sigma = 1^312^3/2.3.4.6 \) can be evaluated without its matrix representation. (see III.2.4).

III.2 Theta series of *0 elements.

To calculate the theta function of \( \sigma \), let us consider the matrix representation of each element: First, for simplicity, we still use \( \sigma \) to denote its matrix representation. Secondly, let \( V_\sigma \) be the eigen space corresponding to 1. Then \( L_\sigma = L \cap V_\sigma \). Practically, \( V_\sigma \) can be evaluated easily, but \( L \cap V_\sigma \) is not so easily determined. To achieve this, we introduce the following lemma.

Lemma III.2.1 (See [7])

The Leech lattice \( L \) in the Euclidean space \( \mathbb{R}^{24} \) can be described as disjoint sum in the following way:

\[
L = \bigcup_{x \in G} \left( \frac{1}{2}e_x + L_0 \right) \cup \left( \frac{1}{4}e_0 + \frac{1}{2}e_x + L_1 \right)
\]

where
(1) \( \Omega = \{ \ast, 0, 1, \ldots, 22 \} \) is a 24-point set and \( \mathcal{G} \subset \mathcal{F}(\Omega) \) is the (binary) Golay code on \( \Omega \). For codes and Golay code, see [1].

(2) Let \( \{ e_1, \ldots, e_{24} \} \) be a natural basis for \( \mathbb{R}^{24} \), then

\[
L_\delta = \{ x = \Sigma x_i e_i \in \mathbb{R}^{24} \mid \Sigma x_i = \delta \text{ (mod 2)} \} \text{ for } \delta = 0, 1.
\]

(3) For a subset \( X \) of \( \Omega \), we put \( e_X = \sum_{i \in X} e_i \).

Using the above lemma, we can prove the following theorem.

**Theorem III.2.2** Let \( \sigma \) and \( \theta_{\sigma}(z) \) be elements in \( \ast \) and functions defined in the following table, then \( \theta_{\sigma}(z) \) is the theta functions of \( \sigma \). (A,B,C,D,E are matrices listed in Appendix A).

<table>
<thead>
<tr>
<th>Frame shape</th>
<th>( \theta_{\sigma}(z) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3_A</td>
<td>( 3^9 / 1^3 )</td>
</tr>
<tr>
<td>5_C</td>
<td>( 5^5 / 1 )</td>
</tr>
<tr>
<td>-6_D</td>
<td>( 1^5 3.6^4 / 2^4 )</td>
</tr>
<tr>
<td>6_F</td>
<td>( 3^3 6^3 / 1.2 )</td>
</tr>
<tr>
<td>-6_F</td>
<td>( 1.6^6 / 2^2 3^3 )</td>
</tr>
<tr>
<td>9_B</td>
<td>( 9^3 / 3 )</td>
</tr>
<tr>
<td>9_C</td>
<td>( 1^3 9^3 / 3^2 )</td>
</tr>
</tbody>
</table>
-10_{E} \quad 135.10^2/2^2 \quad \theta(z, B)

12_{D} \quad 2.3^3.12^3/1.4.6^3 \quad \theta(z, \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix})

12_{H} \quad 2^3.6.12^2/1.3.4^2 \quad \theta(z, \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix})

-12_{H} \quad 1.2^2.3.12^2/4^2 \quad \theta(z, E)

-12_{K} \quad 13^3.12^3/2.3.4.6 \quad \theta(z, \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix})

15_{E} \quad 1^2.15^2/3.5 \quad \theta(z, \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix})

-18_{B} \quad 1^2.9.18/2.3 \quad \theta(z, \begin{bmatrix} 6 & 3 \\ 3 & 6 \end{bmatrix})

18_{C} \quad 1.2.18^2/6.9 \quad \theta(z, \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix})

-18_{C} \quad 2^2.9.18/1.6 \quad \theta(z, \begin{bmatrix} 12 & 6 \\ 6 & 12 \end{bmatrix})

20_{G} \quad 1.2.10.20/4.5 \quad \theta(z, \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix})

-20_{C} \quad 2^2.5.20/1.4 \quad \theta(z, \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix})

24_{F} \quad 1.4.6.24/3.8 \quad \theta(z, \begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix})

-24_{F} \quad 2.3.4.24/1.8 \quad \theta(z, \begin{bmatrix} 6 & 0 \\ 0 & 12 \end{bmatrix})

-30_{E} \quad 2.3.5.30/6.10 \quad \theta(z, \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix})

\textbf{Proof:} The proof is done by direct computation.
The proof for $\sigma = 3^9/1^3$ will be given below. (This is surely one of the more difficult cases.) Since Theorem III.1.2 provides us a matrix representation for $\sigma$, $V_\sigma$ can be evaluated easily. A basis for $V_\sigma$ is listed below, $(\dim V_\sigma = 6)$.

Choose $e_1, e_2, \ldots, e_6$ listed as below:

- $b+c-3d+e-f$
- $-d$
- $a$
- $-b+c+2d+f$
- $a+c+f$
- $d-e+f$
- $-a+b-2d+e-2f$
- $-a+b-c-d-f$
- $-a-c-d$
- $b-2d+e-f$
- $d$
- $d-e+f$
- $-a+b-2d+e-2f$
- $a-b+c+2f$
- $-d$

We can show easily the followings:

1. $(e_1, \ldots, e_6)$ are linearly independent
2. Let $c_1, \ldots, c_6 \in \mathbb{R}$ such that $c_1e_1 + \ldots + c_6e_6 \in L$ then $c_1, \ldots, c_6 \in \mathbb{Z}$.

This implies that $L_\sigma = L \cap V_\sigma$ is
We did not find a matrix representation for $\sigma = 1^{3}2^{3}/2.3.4.6$, but it is not necessary, as shown by the following lemma and corollary.

**Lemma III.2.3** Let $\gamma, \sigma$ be two elements in $*0$ such that

1. $\gamma = \sigma^n, \ n \in \mathbb{N}$
2. $\dim V_\gamma = \dim V_\sigma$

then $\theta_\gamma(z) = \theta_\sigma(z)$.

**Proof:** By (1) $V_\sigma \subseteq V_\gamma$; by (2) $V_\sigma = V_\gamma$. So $L_\sigma = L_\gamma$. So $R_\gamma(z) = \theta_\sigma(z)$.

**Corollary III.2.4** For $\sigma = 1^{3}2^{3}/2.3.4.6$, $\theta_\sigma(z) = \theta(z, \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix})$.

**Proof:** Let $\sigma = 1^{3}2^{3}/2.3.4.6, \ \gamma = 1.6^6/2.23^3$, then corollary follows immediately.

The next two sections give the main results of this paper.
III.3 The Modular Functions of \( \ast \cdot 0 \) elements

The main purpose of this section is to study the modular functions of the remaining elements (as listed in Theorem III.2.2). A main theorem will be stated and proved in III.3.4. This proves the conjecture of Koike [3].

**Lemma III.3.1** Let \( M(N) = M(\Gamma_0(N)) \) be the set of all functions invariant under \( \Gamma_0(N) \), holomorphic on \( \mathcal{H} \) and meromorphic at the cusps of \( \Gamma_0(N) \). Given \( f(z) \), \( g(z) \in M(N) \) such that \( f(z) - g(z) \) is holomorphic at all cusps of \( \Gamma_0(N) \). Then \( f(z) - g(z) = \text{a constant function} \).

**Proof:** Let \( F(z) = f(z) - g(z) \), then \( F(z) \in M(N) \). So \( F(z) \) is holomorphic on \( \mathcal{H} \), the upper half plane. By assumption, \( F(z) \) is also holomorphic at all cusps. So \( F(z) \) is holomorphic on \( \mathcal{H}^* = \mathcal{H} \cup \{ \text{cusps of } \Gamma_0(N) \} \), hence, holomorphic on the compact Riemann surface \( \Gamma_0(N) \backslash \mathcal{H}^* \). Therefore, \( F(z) = \text{constant} \).

**Lemma III.3.2** (See [5])

Let \( \sigma = \Pi t \) be a Frame shape in \( \ast \cdot 0 \). Assume

\[
(1) \quad \Sigma t r_t = 24
\]
(2) $\eta_\sigma(z)$ is invariant under the action of a discrete subgroup $\Gamma$ of $\text{SL}_2(\mathbb{R})$ containing $\Gamma_0(N)$ for some $N$.

(3) $\Gamma_{\infty} = \{\sigma \in \Gamma \mid \sigma(\infty) = \infty\}$ is equal to $< \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} >$.

(4) $z = \infty$ is the unique pole of $\eta_\sigma(z)$ among all inequivalent cusps of $\Gamma$.

Then $\eta_\sigma(z)$ is a generator of a function field corresponding to $\Gamma$.

Moreover, $\Gamma$ is of genus zero.

Lemma III.3.3 (see [5]) Let $A$ be a positive definite integral even symmetric $r \times r$ matrix with $r$ even, then

$$\theta(z, A) = i^{r/2} z^{-r/2} D^{-1/2} \theta(-1/Nz, A^*)$$

where

$A^*$ = the adjoint matrix of $A$

$D$ = determinant of $A$

$N$ = level of $A$

Now we are ready to prove the main theorem.

Theorem III.3.4 Let $\sigma, \theta_\sigma(z), \sigma_1$ and $\Gamma_\sigma$ be elements in $\cdot 0$, their theta functions, elements in $\mathcal{M}$ and corresponding discrete subgroups in $\text{SL}_2(\mathbb{R})$ given in the following table, respectively, then

(1) If $\sigma_1$ appears, then $\theta_\sigma(z)/\eta_\sigma(z) - \text{constant and } \theta_\sigma(z)/\eta_\sigma(z)$ is a generator of a function field corresponding to $\Gamma_\sigma$ which is of genus 0, in particular, $\Gamma_\sigma$ is the fixing group of $\theta_\sigma(z)/\eta_\sigma(z)$.

(2) If $x$ appears, then (*) is false.
<table>
<thead>
<tr>
<th>Frame shape</th>
<th>$\theta_\sigma(z)$</th>
<th>$\sigma_1$</th>
<th>$\Gamma_\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$3_C$</td>
<td>$3^9/1^3$</td>
<td>$\theta(z, A)$</td>
<td>3B</td>
</tr>
<tr>
<td>$5_B$</td>
<td>$5^5/1$</td>
<td>$\theta(z, B)$</td>
<td>5B</td>
</tr>
<tr>
<td>$-6_D$</td>
<td>$1^53.6^4/2^4$</td>
<td>$\theta(z, A)$</td>
<td>x</td>
</tr>
<tr>
<td>$6_F$</td>
<td>$3^6^3/1.2$</td>
<td>$\theta(z, C)$</td>
<td>6D</td>
</tr>
<tr>
<td>$-6_F$</td>
<td>$1.6^6/2^23^3$</td>
<td>$\theta(z, [\frac{4}{2} \frac{2}{4}] )$</td>
<td>6E</td>
</tr>
<tr>
<td>$9_B$</td>
<td>$9^3/3$</td>
<td>$\theta(z, \frac{6}{3} \frac{3}{6} )$</td>
<td>9B</td>
</tr>
<tr>
<td>$9_C$</td>
<td>$1^39^3/3^2$</td>
<td>$\theta(z, D)$</td>
<td>9A</td>
</tr>
<tr>
<td>$-10_E$</td>
<td>$1^35.10^2/2^2$</td>
<td>$\theta(z, B)$</td>
<td>x</td>
</tr>
<tr>
<td>$-12_D$</td>
<td>$2.3^312^3/1.4.6^3$</td>
<td>$\theta(z, [\frac{4}{2} \frac{2}{4}] )$</td>
<td>12B</td>
</tr>
<tr>
<td>$12_H$</td>
<td>$2^3.12^2/1.3.4^2$</td>
<td>$\theta(z, [\frac{6}{0} \frac{0}{6}] )$</td>
<td>12I</td>
</tr>
<tr>
<td>$-12_H$</td>
<td>$1.2^23.12^2/4^2$</td>
<td>$\theta(z, E)$</td>
<td>x</td>
</tr>
<tr>
<td>$-12_K$</td>
<td>$1^312^3/2.3.4.6$</td>
<td>$\theta(z, [\frac{4}{2} \frac{2}{4}] )$</td>
<td>12H</td>
</tr>
<tr>
<td>$15_E$</td>
<td>$1^215^2/3.5$</td>
<td>$\theta(z, [\frac{4}{1} \frac{1}{4}] )$</td>
<td>15C</td>
</tr>
<tr>
<td>$-18_B$</td>
<td>$1^29.18/2.3$</td>
<td>$\theta(z, [\frac{6}{3} \frac{3}{6}] )$</td>
<td>x</td>
</tr>
<tr>
<td>$18_G$</td>
<td>$1.2.18^2/6.9$</td>
<td>$\theta(z, [\frac{4}{2} \frac{2}{4}] )$</td>
<td>x</td>
</tr>
</tbody>
</table>
Proof: For simplicity, we set \( f_\sigma(z) = \theta_\sigma(z)/\eta_\sigma(z) \).

(1) \( \sigma = 3^9/1^3 \):

From Theorem III.3.1, we have \( \theta_\sigma(z) = \theta(z, A) \)

\[
\begin{array}{rrrrrr}
4 & -1 & 1 & 1 & -2 & 2 \\
-1 & 4 & -1 & 2 & 2 & 1 \\
A = & 1 & -1 & 4 & 1 & -2 & 2 \\
1 & 2 & 1 & 4 & 1 & 2 \\
-2 & 2 & -2 & 1 & 4 & -1 \\
2 & 1 & 2 & 2 & -1 & 4 \\
\end{array}
\]

Furthermore, we have the following,

<table>
<thead>
<tr>
<th>level weight character</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \theta_\sigma(z) ) 3 3 ((-3^-))</td>
</tr>
<tr>
<td>( \eta_\sigma(z) ) 3 3 ((-3^-))</td>
</tr>
</tbody>
</table>
From the above table, we have that $f_\sigma(z) \in M(3)$. To show $f_\sigma(z) - t_{3B} = 9$, where $3B = 1^{12/12}$, by applying Lemma III.3.1, it suffices to show $f_\sigma(z) - t_{3B}$ is holomorphic at 0 and 1/3 and equals to 9 at 1/3.

At $c_1 = 1/3$, we consider $M = [1 0; 3 1]$, since both $f_\sigma(z)$ and $t_{3B}$ are in $M(3)$, we have

$$(f_\sigma - t_{3B})(Mz) = (f_\sigma - t_{3B})(z) = f_\sigma(z) - t_{3B}(z)$$

$$= (q^{-1} - 3 + \sum_{n=1}^{\infty} a_n q^n) - (q^{-1} - 12 + \sum_{n=1}^{\infty} b_n q^n)$$

$$(III.1)$$

$$= 9 + \sum_{n=1}^{\infty} (a_n - b_n)q^n$$

as $z \to \infty$, we have $(f_\sigma - t_{3B})(1/3) = 9$

At $c_1 = 0$, we consider $M = [0 1; 1 0]$, applying Lemma II.6 and III.3.3, we have $f_\sigma(Mz) = \frac{9\theta(z/3, A^*)}{g(z)}$, where $A^*$ is the adjoint matrix of $A$ and $g(z)$ is a holomorphic function. As $z \to \infty$, we have $f_\sigma(0) = 9$.

On the other hand, applying Lemma II.7, we have that $t_{3B}(0) = 0$.

This implies that $f_\sigma(0) - t_{3B}(0) = 9$.

Applying Lemma III.3.1, we have $f_\sigma(z) - t_{3B} = 9$. Applying Lemma III.3.2, $f_\sigma(z)$ is a generator of a function field of $\Gamma_0(3)$.

$(2) \sigma = 5^{1/2}$:

From Theorem III.2.2, we have $\theta_\sigma(z) = \theta(z, B)$ where
4 1 1 1
B = 1 4 -1 -1
1 -1 4 -1
1 -1 -1 4

Furthermore, we have the following,

<table>
<thead>
<tr>
<th>level</th>
<th>weight</th>
<th>character</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \theta_\sigma(z) )</td>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td>( \eta_\sigma(z) )</td>
<td>5</td>
<td>2</td>
</tr>
</tbody>
</table>

From the above table, we have that \( f_\sigma(z) \in M(5) \). To show \( f_\sigma(z) - t_{5B}(z) = -5 \), where \( 5B = 1^6/5^6 \), applying Lemma III.3.1, it suffices to show \( f_\sigma(z) - t_{5B}(z) \) is holomorphic at 0, 1/5 and equals to 5 at 1/5.

At \( c_1 = 1/5 \), we consider \( A = \frac{1}{5} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \); Since both \( f_\sigma(z) \) and \( t_{5B}(z) \) are in \( M(5) \), we have,

\[
(f_\sigma - t_{5B})(Az) = f_\sigma(z) - t_{5B}(z)
\]

\[
- \left( q^{-1} - 1 + \sum_{n=1}^{\infty} a_n q^n \right) - \left( (q^{-1} - 6 + \sum_{n=1}^{\infty} b_n q^n) \right) = 5 + \sum_{n=1}^{\infty} (a_n - b_n) q^n
\]

As \( z \to \infty \), we have \( (f_\sigma - t_{5B})(1/5) = 5 \).

At \( c_1 = 0 \), we consider \( A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \); applying Lemma II.6 and III.3.3, we have that \( f_\sigma(Az) = \cdots \frac{58(z/5)}{g(z)} \cdots B^* \) where \( B^* \) is the adjoint matrix of
$B$ and $g(z)$ is a holomorphic function. As $z \to \infty$, we have $f_\sigma(0) = 5$. On the other hand, applying Lemma II.7, we have that $t_{5B}(0) = 0$. This implies that $f_\sigma(0) - t_{5B}(0) = 5$. Applying Lemma III.3.1, we have $f_\sigma(z) - t_{5B}(z) = 5$. Applying Lemma III.3.2, we have that $f_\sigma(z)$ is a generator of a function field of $\Gamma_\sigma(5)$.

(3) $\sigma = 1^{5}3.6^{4}/2^{4}$:

From Theorem III.2.2, we have $\theta_\sigma(z) = \theta(z, A)$ where $A$ is the $6 \times 6$ matrix in (1). Computation shows that the Fourier series of $f_\sigma(z)$ does not appear in Table 4 of [2], This implies (*) is false.

(4) $\sigma = 3^{3}6^{3}/1.2$:

From Theorem III.2.2, we have $\theta_\sigma(z) = \theta(z, C)$ where

\[
C = \begin{pmatrix}
6 & 0 & 3 & 3 \\
0 & 6 & 3 & 3 \\
3 & 3 & 6 & 3 \\
3 & 3 & 3 & 6
\end{pmatrix}
\]

Furthermore, we have the following,

<table>
<thead>
<tr>
<th>level</th>
<th>weight</th>
<th>character</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta_\sigma(z)$</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>$\eta_\sigma(z)$</td>
<td>6</td>
<td>2</td>
</tr>
</tbody>
</table>

From the above table, we have $f_\sigma(z) \in M(6)$. To show $f_\sigma(z) - t_{6D}(z) = 3$, where $6D = 1^{424}/3^{4}6^{4}$, by applying Lemma III.3.1, it suffices to
show that \( f_\sigma(z) - t_{6D}(z) \) is holomorphic at 0, 1/2, 1/3 and 1/6 and equals to 3 at 1/6.

To achieve this, we use the following identities [7]

\[
\theta(z, c) - \theta(z, [\frac{2}{1}, \frac{1}{2}]) \equiv \theta(z, [\frac{4}{2}, \frac{2}{4}]) - 6\eta_\sigma(z) \tag{III.3}
\]

\[
\theta(z, [\frac{(p+1)/2}{1}, \frac{1}{1}]) - \theta_2(2z)\theta_2(2pz) + \theta_3(2z)\theta_3(2pz) \tag{III.4}
\]

\[
\theta_2(z) = 2\eta(2z)^2/\eta(z) \tag{III.5}
\]

\[
\theta_3(2z) = \eta(2z)^5/\eta(z)^2\eta(4z)^2 \tag{III.6}
\]

Using the above identities, we have

\[
f_\sigma(z) = 16\eta_\sigma_1(z) + 4\eta_\sigma_2(z) + 4\eta_\sigma_3(z) + \eta_\sigma_4(z) - 6 \text{ where}
\]

\[
\sigma_1 = 1.4.8^22.12.24^2/3\cdot 6^4 \tag{III.7}
\]

\[
\sigma_2 = 1.4.7.12^7/2.3^66^22^4 \cdot 2^4 \cdot 8^22^4 / 1.3^54^3.12^3 \tag{III.8}
\]

\[
\sigma_3 = 2^6.28^22^4 / 1.3^58^22^4 \cdot 2^2 \cdot 12^3 \tag{III.9}
\]

\[
\sigma_4 = 2^6.2^8^22^4 / 1.3^58^22^4 \cdot 2^4 \cdot 2^2 \cdot 12^3 \tag{III.10}
\]

At \( c_1 = 1/6 \) and 0, similar to (1), we have \( f_\sigma(z) - t_{6D}(z) = 3 \)

At \( c_1 = 1/2 \), the following holds,

\[
A = \begin{bmatrix}
1 & 0 \\
2 & 1
\end{bmatrix}
\begin{bmatrix}
\eta_\sigma_1(Az) & \eta_\sigma_2(Az) & \eta_\sigma_3(Az) & \eta_\sigma_4(Az) & t_{6D}(Az)
\end{bmatrix}
\]

\[
\Sigma t_c(t,c)^2/t = 0 \quad 0 \quad + \quad + \quad +
\]
Applying Lemma II.7, we have that \( f_\sigma(z) - t_{6\sigma}(z) \) is holomorphic at \( 1/2 \).

At \( a_1 = 1/3 \), we consider \( A = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \), applying Lemma II.6, we have,

\[
16\eta_{\sigma_1}(Az) = -\frac{1}{8q^{1/2}}g_1(z)
\]

where \( g_1(z) \) is a holomorphic function.

\[
4\eta_{\sigma_2}(Az) = -\frac{1}{8q^{1/2}}g_2(z)
\]

where \( g_2(z) \) is a holomorphic function.

\[
4\eta_{\sigma_3}(Az) = -\frac{1}{2q^{1/2}}g_3(z)
\]

where \( g_3(z) \) is a holomorphic function.
\[ \eta_{\sigma}(A z) = \frac{v(\begin{smallmatrix} 2 & -1 \\ 3 & -1 \end{smallmatrix})^4 v(\begin{smallmatrix} 4 & -3 \\ 3 & -2 \end{smallmatrix})^3 v(\begin{smallmatrix} 4 & -1 \\ 1 & 0 \end{smallmatrix})^5}{v(\begin{smallmatrix} 3 & -2 \\ 1 & 0 \end{smallmatrix})^2 v(\begin{smallmatrix} 8 & -3 \\ 3 & -2 \end{smallmatrix})^2 v(\begin{smallmatrix} 4 & -1 \\ 1 & 0 \end{smallmatrix})^2 v(\begin{smallmatrix} 2 & -1 \\ 0 & 1 \end{smallmatrix})^2} \\
\quad \exp(\pi i (2 + 9/4 + 5/4)/12) \exp(\pi i (1 + 5 + 5/4 + 1/2 + 1/4)/12) - \frac{1}{4q^{1/2}} \ g_4(z) \\
= - \frac{1}{4q^{1/2}} \ g_4(z) \] (III.11)

where \( g_4(z) \) is a holomorphic function.

\[ t_{6D}(A z) = q^{-1/2} \frac{v(\begin{smallmatrix} 1 & -1 \\ 3 & -1 \end{smallmatrix})^4 v(\begin{smallmatrix} 2 & -1 \\ 3 & -1 \end{smallmatrix})^4 \exp(\pi i (4 + 2)/12)}{v(\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix})^4 v(\begin{smallmatrix} 2 & -1 \\ 1 & 0 \end{smallmatrix})^4 \exp(\pi i (4 + 2)/12)} \ g_5(z) \\
= - q^{-1/2} \ g_5(z) \] (III.12)

where \( g_5(z) \) is a holomorphic function.

So we have \( f_\sigma(z) - t_{6D}(A z) = \)

\[-q^{-1/2} (1/8 \ g_1(z) + 1/8 \ g_2(z) + 1/4 \ g_3(z) + 1/2 \ g_4(z) - g_5(z)) - 6 \]

As \( z \to \infty \), we have \( (f_\sigma - t_{6D})(1/2) = 3 \)

From the above argument, we have that \( f_\sigma(z) - t_{6D}(z) \) is holomorphic at all cusps of \( \Gamma_0(6) \). Applying Lemma III.3.1, we have

\[ f_\sigma(z) - t_{6D}(z) = 3. \] Applying Lemma III.3.2, we have that \( f_\sigma(z) \) is a generator of a function field of \( \Gamma_0(6) + W_2 \).

(5) \( \sigma = 1.6^{6}/2^{2}3^{3} \)
From Theorem III.2.2, we have \( \Theta_0(z) = \Theta(z, \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}) \)

Furthermore, we have the following,

<table>
<thead>
<tr>
<th>level</th>
<th>weight</th>
<th>character</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Theta_0(z) )</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>( \eta_0(z) )</td>
<td>6</td>
<td>1</td>
</tr>
</tbody>
</table>

where \(-3\) is Jacobi symbol.

From the above table, we conclude that \( f_0(z) = \Theta_0(z)/\eta_0(z) \in \mathbb{M}(6) \). To show \( f_0(z) - t_{6E}(z) = -3 \), where \( 6E = 2^83^4/1.6^8 \). By applying Lemma III.3.1, it suffices to show \( f_0(z) - t_{6E}(z) \) is holomorphic at 0, 1/2, 1/3 and equals to -3 at the cusp 1/6.

Using III.4, III.5 and III.6, we have,

\[
f_0(z) = 4\eta_{\sigma_1}(z) + \eta_{\sigma_2}(z) \quad \text{where}
\]

\[
\sigma_1 = 2^23^86^224^2/1.4.6^612 \quad \text{(III.13)}
\]

\[
\sigma_2 = 3^45^212^5/1.6^82^424^2
\]

At \( c_1 = 0, 1/2 \) and \( 1/3 \), we have the following,

\[
\begin{bmatrix}
\text{Sr}_{c}(t,c)^2/t \\
4\eta_{\sigma_1}(Az) \\
\eta_{\sigma_2}(Az) \\
t_{6E}(Az)
\end{bmatrix}
\]

\[
A = \begin{bmatrix}
0 & -1 \\
1 & 0
\end{bmatrix}
\]

\[
A = \begin{bmatrix}
1 & 0 \\
2 & 1
\end{bmatrix}
\]
From the above table and Lemma II.7, we have that $f_\sigma(z) - t_{6B}(z)$ is holomorphic at 0, 1/2 and 1/3.

At $c_1 = 1/6$, similar to (1), we have that $(f_\sigma - t_{6B})(1/6) = -3$

From the above argument, we have that $f_\sigma(z) - t_{6B}(z)$ is holomorphic at all the cusps of $\Gamma_0(6)$. Applying Lemma III.3.1, we have that $f_\sigma(z) - t_{6B}(z) = -3$. Applying Lemma III.3.2, we have that $f_\sigma(z)$ is a generator of a function field of $\Gamma_0(6)$.

(6) $\sigma = 9^3/3$:

From Theorem III.2.2, we have $\theta_\sigma(z) = \theta(z, [\begin{smallmatrix} 6 \\ 3 \\ 3 \end{smallmatrix}])$.

Furthermore, we have the following.

<table>
<thead>
<tr>
<th>level</th>
<th>weight</th>
<th>character</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta_\sigma(z)$</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>$\eta_\sigma(z)$</td>
<td>9</td>
<td>1</td>
</tr>
</tbody>
</table>

From the above table, we have $f_\sigma(z) \in M(9)$. To show $f_\sigma(z) - t_{9B}(z) = 3$, where $9B = 1^3/9^3$, by applying Lemma III.3.1, it suffices to show $f_\sigma(z) - t_{9B}(z)$ is holomorphic at 0, 1/3, 2/3 and 1/9 and equals to 3 at 1/9.

At $c_1 = 1/9$, similar to (1), we have $f_\sigma(z) - t_{9B}(z) = 3$.

At $c_1 = 0, 1/3, 2/3$. Using identity (III.4), we have
\[ f_\sigma(z) = 4\eta_{\sigma_1}(z) + \eta_{\sigma_2}(z), \] where

\[ \sigma_1 = 3.122362^2 / 6.9318 \]  

\[ \sigma_2 = 6.185^5 / 3.91236^2 \]  

Furthermore, we have the following,

\[ \text{E_r}(c,c^2/t) \quad 4\eta_{\sigma_1}(Az) \quad \eta_{\sigma_2}(Az) \quad t_{gB}(Az) \]

\[ A = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \quad 0 \quad 0 \quad + \]

\[ A = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \quad 0 \quad 0 \quad 0 \]

\[ A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \quad 0 \quad 0 \quad 0 \]

Applying Lemma II.7, we have that \( f_\sigma(z) - t_{gB}(z) \) is holomorphic at \( 0, 1/3, \) and \( 2/3 \). Applying Lemma III.3.1, we have that \( f_\sigma(z) - t_{gB}(z) = 3 \)

Applying Lemma III.3.2, we have \( f_\sigma(z) \) is a generator of a function field of \( \Gamma_0(9) \).

(7) \( \sigma = 13^3g^3/3^2 \):  

From Theorem III.2.2, we have \( \Theta_\sigma(z) = \Theta(z, D) \), where

\[ D = \begin{bmatrix} 4 & 1 & 1 & 2 \\ 1 & 4 & 1 & 2 \\ 1 & 1 & 4 & -1 \\ 2 & 2 & -1 & 4 \end{bmatrix} \]

Furthermore, we have the following,
<table>
<thead>
<tr>
<th>level</th>
<th>weight</th>
<th>character</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta_9(z)$</td>
<td>9</td>
<td>2</td>
</tr>
<tr>
<td>$\eta_9(z)$</td>
<td>9</td>
<td>2</td>
</tr>
</tbody>
</table>

From the above table, we have $f_9(z) \in M(9)$. To show $f_9(z) - t_{9A}(z)$

$=-3$, where $9A = 3^{12}/6^6$, applying Lemma III.3.1, it suffices to show

$f_9(z) - t_{9A}(z)$ is holomorphic at $0, 1/3, 2/3$ and $1/9$ and equals to $-3$

at $1/9$

At $c_0 = 0$, $1/9$, similar to (1), we have $f_9(z) - t_{9A}(z) = -3$

At $c_1 = 1/3, 2/3$, we use the following identities [11]:

$$\theta([a \ b]z, A) = i^{c^2} c^{-k} \det A^{-1/2} (cz + d)^k$$

$$\sum_{v \mod N} \xi(0, v) \theta(z, A, 1, v) \quad (III.15)$$

where

$A$ is a positive definite integral even symmetric matrix of dimension

$r = 2k$ and level $N$.

$$\xi(0, v) = \sum w \exp(\pi i (aw^T A w + v^T A w + dv^T A v)/cn^2)$$

$w = 0 \mod N$

$$\sum_{w \mod cN} \theta(z, A, 1, v) = \sum_{n \equiv v \mod N} \exp(\pi i A n z / N^2)$$

$Av = 0 \mod N$

Remark: $\theta(z, A, 1, v)$ is a modular form of weight $k$, level $N$.

$\theta(z, A, 1, v)$ is a cusp form if $v$ is a nonzero vector.
\[
\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{SL}_2(\mathbb{R})
\]

At \(c = 1/3\), we consider \(\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1/3 & 1 \end{bmatrix}\), applying (III.15), we have

\[
\Theta(z/(3z+1) , D) = -(3z+1)^2/81 \sum \xi(\psi, \nu) \Theta(z, D, 1, \psi) \\
\nu \equiv 0 \pmod{9}
\]

(III.16)

On the other hand, Applying Lemma II.6, we have

\[
\eta_0(z/(3z+1)) = v\left(\begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix}\right)^3 v\left(\begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix}\right)^3 \\
v\left(\begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix}\right)^2 \exp(\pi i (3 + 1 - 2)/12)
\]

\[
(3z+1)^2/(27)^{1/2} \eta_1(z)
\]

(III.17)

So \(f_\sigma(z/(3z+1)) = \sum \xi(\psi, \nu) \Theta(z, D, 1, \psi) \eta_1(z)\). (III.18)

Where \(\eta_1(z)\) is a holomorphic function.

On the other hand, \(t_{g_A}(z/(3z+1)) = 27^{1/2} \exp(\pi i (12 - 6 - 2)/12)\)

\[
v\left(\begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix}\right)^{12} v\left(\begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix}\right)^6 v\left(\begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix}\right)
\]

(III.19)

As \(z \to \infty\), we have \((f_\sigma - t_{g_A})(1/3) = -3\)

Similarly, we have \((f_\sigma - t_{g_A})(2/3) = -3\)

From the above argument, we have that \(f_\sigma(z) - t_{g_A}(z)\) is holomorphic at all the cusps of \(\Gamma_0(9)\). Applying Lemma III.3.1, we have \(f_\sigma(z) - t_{g_B}(z)\)
-3. Applying Lemma III.3.2, we have that \( f_\sigma(z) \) is a generator of a function field of \( \Gamma_\sigma(9)+ \).

(8) \( \sigma = 1^3 5.10^2/2^2 \):

From Theorem III.2.2, we have that \( \Theta_\sigma(z) = \Theta(z, B) \). Computation shows that the Fourier series of \( f_\sigma(z) \) does not appear in Table 4 of [2]. This implies that (*) is false.

(9) \( \sigma = 2.3^3 12^3 /1.4.6^3 \):

From Theorem III.2.2, we have \( \Theta_\sigma(z) = \Theta(z, \{\frac{4}{2}, \frac{2}{4}\}) \).

Furthermore, we have the following:

<table>
<thead>
<tr>
<th>level</th>
<th>weight</th>
<th>character</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Theta_\sigma(z) )</td>
<td>6</td>
<td>1</td>
</tr>
<tr>
<td>( \eta_\sigma(z) )</td>
<td>6</td>
<td>1</td>
</tr>
</tbody>
</table>

From the above table, we have \( f_\sigma(z) \in M(12) \). To show \( f_\sigma(z) - t_{12B}(z) = 5 \), where \( 12B = 1^4 4^4 6^4 / 2^4 3^4 12^4 \), applying Lemma III.3.1, it suffices to show that \( f_\sigma(z) - t_{12B}(z) \) is holomorphic at 0, 1/2, 1/3, 1/4, 1/6 and 1/12 and equals to 5 at 1/12.

At \( c_1 = 1/12 \), similar to (1), we have \( f_\sigma(z) - t_{12B}(z) = 5 \).

At \( c_1 = 0, 1/2, 1/4 \) and 1/6. Using (5), we have

\[
 f_\sigma(z) = \eta_{\sigma_1}(z) - 3\eta_{\sigma_2}(z),
\]

where

\[
 \sigma_1 = 2^5 4.6 / 1^2 3^2 12^3
\]

(III.20)
\[ \sigma_2 = 1^{34.6^9/2^{36.12^3}} \]

Furthermore, the followings hold

\[
\begin{array}{cccc}
\Sigma_T(t,c)^2/c & \eta_{\sigma_1}(Az) & \eta_{\sigma_2}(Az) & t_{12B}(Az) \\
A = [0 \ 1] & 0 & 0 & + \\
A = [1 \ 0] & + & 0 & 0 \\
A = [1 \ 0] & + & 0 & + \\
A = [1 \ 0] & 0 & + & 0 \\
\end{array}
\]

Applying Lemma II.7, we have that \( f_\sigma(z) - t_{12B}(z) \) is holomorphic at 0, 1/2, 1/4 and 1/6.

At \( c_1 = 1/3 \), we consider \( A = [\frac{1}{3} \ 0] \), applying Lemma II.6 we have

\[
\eta_{\sigma_1}(Az) = \frac{\nu([\frac{2}{3} \ -1])^5\nu([\frac{4}{3} \ -3])\nu([\frac{2}{1} \ -1])}{\nu([\frac{2}{3} \ -2])\nu([\frac{1}{1} \ -1])^3} \cdot \frac{1}{2q^{1/4}} g_1(z)
\]

\[
= \frac{1}{2q^{1/4}} g_1(z) \tag{III.21}
\]

where \( g_1(z) \) is a holomorphic function

\[
3\eta_{\sigma_2}(Az) = \frac{\nu([\frac{1}{3} \ -1])^2\nu([\frac{4}{3} \ -3])\nu([\frac{2}{1} \ -1])^9}{\nu([\frac{2}{3} \ -2])^8\nu([\frac{1}{1} \ -1])^3}\nu([\frac{2}{1} \ -1])^3
\]

\[
\frac{\exp(i(2 + 3/4 + 9/2)/12)}{\exp(i(3/2 + 6 + 3/4)/12)} = -\frac{3}{2q^{1/4}} g_2(z)
\]

where \(g_2(z)\) is a holomorphic function

So \(f_g(Az) = -iq^{1/4}(1/2g_1(z) - 3/2g_2(z))\)

On the other hand, we have

\[
t_{12B}(Az) = \frac{\exp(i(4 + 3 + 2)/12)}{\exp(i(2 + 4 + 1)/12)} = -\frac{1}{q^{1/4}} g_3(z)
\]

where \(g_3(z)\) is a holomorphic function

So we have \(f_g(Az) - t_{12B}(Az) = -iq^{1/4}(1/2g_1(z) - 3/2g_2(z) + g_3(z))\)

As \(z \to \infty\), we have \((f_g - t_{12B})(1/3) = 5\).

From the above argument, we have that \(f_g(z) - t_{12B}(z)\) is holomorphic at all cusps of \(\Gamma_0(12)\). Applying Lemma III.3.1, we have that \(f_g(z) - t_{12B}(z) = 5\). Applying Lemma III.3.2, we have that \(f_g(z)\) is a generator of a function field of \(\Gamma_0(12) + W_4\)

(10) \(\sigma = 2^3 \cdot 6 \cdot 12^2 / 1.3.4^2\):
From Theorem III.2.2, we have $\theta_\sigma(z) = \theta(z, [4 \ 2 \ 0 \ 1])$.

Applying identities (III.5) and (III.6), we have $\theta_\sigma(z) = \theta_3(6z)^2 - \eta_\sigma_1(z)$, where $\sigma_1 = 6^{10}/3^412^4$. This implies that $f_\sigma(z) = t_{12I}$.

(11) $\sigma = 1.2^23.12^2/4^2$:
From Theorem III.2.2, we have $\theta_\sigma(z) = \theta(z, E)$, where
\[ E = \begin{array}{cccc}
4 & 2 & 2 & 2 \\
2 & 4 & 1 & 1 \\
2 & 1 & 4 & 1 \\
2 & 1 & 1 & 4 \\
\end{array} \]

Computation shows that the Fourier series of $f_\sigma(z)$ does not appear in Table 4 of [2]. This implies (*) is false.

(12) $\sigma = 1^32^3/2.3.4.6$:
From Theorem III.2.2, we have $\theta_\sigma(z) = \theta(z, [4 \ 2 \ 2 \ 1])$.

Furthermore, we have the following,

<table>
<thead>
<tr>
<th>level</th>
<th>weight</th>
<th>character</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta_\sigma(z)$</td>
<td>6</td>
<td>1</td>
</tr>
<tr>
<td>$\eta_\sigma(z)$</td>
<td>12</td>
<td>1</td>
</tr>
</tbody>
</table>

From the above table, we have $f_\sigma(z) \in \mathcal{M}(12)$. To show $f_\sigma(z) = t_{12H}(z) = -1$, where $12H = 3^44^4/1^412^4$, applying Lemma III.3.1, it suffices to
show that $f_\sigma(z) - t_{12H}(z)$ is holomorphic at 0, 1/2, 1/3, 1/4, 1/6 and 1/12 and equals to -1 at 1/12.

At $c_1 = 0, 1/12$, similar to (1), we have $f_\sigma(z) - t_{12H}(z) = -1$.

At $c_1 = 1/2, 1/3, 1/4$ and 1/6, using III.4, III.5, and III.6, we have

$f_\sigma(z) = 4\eta_{\sigma_1}(z) + \eta_{\sigma_2}(z)$, where

$$\sigma_1 = 2.3.6.8^224^2/13^{12}$$

$$\sigma_2 = 3.4^6l_{2^2}/13^2.6^2.8^224^2$$

Furthermore, the followings hold.

<table>
<thead>
<tr>
<th>$\Sigma_{t,c}(t,c)^2/t$</th>
<th>$\eta_{\sigma_1}(Az)$</th>
<th>$\eta_{\sigma_2}(Az)$</th>
<th>$t_{12H}(Az)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A = [\begin{array}{c} 1 \ 2 \ 1 \end{array}]$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$A = [\begin{array}{c} 1 \ 3 \ 1 \end{array}]$</td>
<td>0</td>
<td>0</td>
<td>+</td>
</tr>
<tr>
<td>$A = [\begin{array}{c} 1 \ 4 \ 1 \end{array}]$</td>
<td>0</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>$A = [\begin{array}{c} 1 \ 6 \ 1 \end{array}]$</td>
<td>0</td>
<td>+</td>
<td>0</td>
</tr>
</tbody>
</table>

Applying Lemma II.7, we have $f_\sigma(z) - t_{12H}(z)$ is holomorphic at 0, 1/2, 1/3, 1/4, 1/6. Applying Lemma III.3.1, we have $f_\sigma(z) - t_{12H}(z) = -1$.

Applying Lemma III.3.2, we have that $f_\sigma(z)$ is a generator of a function field of $\Gamma_{\sigma}(12) + W_{12}$.

(13) $\sigma = l^215^2/3.5$: 
From Theorem III.2.2, we have $\theta_{\sigma}(z) = \theta(z, [4 1 31 4])$.

Furthermore, we have the following,

<table>
<thead>
<tr>
<th>level</th>
<th>weight</th>
<th>character</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta_{\sigma}(z)$</td>
<td>15</td>
<td>1</td>
</tr>
<tr>
<td>$\eta_{\sigma}(z)$</td>
<td>15</td>
<td>1</td>
</tr>
</tbody>
</table>

From the above table, we have $f_{\sigma}(z) \in M(15)$. To show $f_{\sigma}(z) - t_{15C}(z) = -1$, where $15C = 3^3 5^3 / 1^3 15^3$, it suffices to show that $f_{\sigma}(z) - t_{15C}(z)$ is holomorphic at $0, 1/3, 1/5$ and $1/15$ and equals to $-1$ at $1/15$.

At $c = 0$ and $1/15$, similar to (1), we have $f_{\sigma}(z) - t_{15C}(z) = -1$.

At $c = 1/3$ and $1/5$, we introduce the following identity,

$$\theta(z, [4 1 31 4]) = \theta_2(6z)\theta_2(10z) + \theta_3(6z)\theta_3(10z)$$

Applying (III.5) and (III.6), we have the following,

$$f_{\sigma}(z) = 4\eta_{\sigma_1}(z) + \eta_{\sigma_2}(z),$$

where

$$\sigma_1 = 3.5.12^2 20^2 / \omega^2 6.10.15^2$$

$$\sigma_2 = 3.5.6^5 10^5 / \omega^2 3^2 5^2 12^2 15^2 20^2$$

Furthermore, the followings hold.

<table>
<thead>
<tr>
<th>$t_{\tau}(t, c)^2$</th>
<th>$\eta_{\sigma_1}(Az)$</th>
<th>$\eta_{\sigma_2}(Az)$</th>
<th>$t_{15C}(Az)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A = $[\frac{1}{3} \quad 0]$</td>
<td>0</td>
<td>0</td>
<td>+</td>
</tr>
<tr>
<td>A = $[\frac{1}{5} \quad 0]$</td>
<td>0</td>
<td>0</td>
<td>+</td>
</tr>
</tbody>
</table>
Applying Lemma II.7, we have \( f_\sigma(z) - \tau_{156}(z) \) is holomorphic at 1/3 and 1/5. Applying Lemma III.3.1, we have \( f_\sigma(z) - \tau_{156}(z) = -1 \). Applying Lemma III.3.2, we have \( f_\sigma(z) \) is a generator of a function field of \( \Gamma_0(15) + W_{15} \).

(14) \( \sigma = 1^{2}9.18/2.3 \):

From Theorem III.2.2, we have \( \Theta(z) = \Theta(z, [\begin{array}{c} 6 \\ 3 \\ 6 \end{array}]) \). Computation shows that the Fourier series of \( f_\sigma(z) \) does not appear in Table 4 of [2], this implies that (*) is false.

(15) \( \sigma = 1.2.18^{2}/6.9 \):

From Theorem III.2.2, we have \( \Theta(z) = \Theta(z, [\begin{array}{c} 2 \\ 4 \end{array}]) \). Computation shows that the Fourier series of \( f_\sigma(z) \) does not appear in Table 4 of [2], this implies that (*) is false.

(16) \( \sigma = 2^{2}9.18/1.6 \):

From Theorem III.2.2, we have \( \Theta(z) = \Theta(z, [\begin{array}{c} 12 \\ 6 \\ 12 \end{array}]) \). Computation shows that the Fourier series of \( f_\sigma(z) \) does not appear in Table 4 of [2], this implies that (*) is false.

(17) \( \sigma = 1.2.10.20/4.5 \):

From Theorem III.2.2, we have \( \Theta(z) = \Theta(z, [\begin{array}{c} 2 \\ 6 \\ 2 \end{array}]) \).

Furthermore, we have the following:

level weight character
From the above table, we have $f_\sigma(z) \in M(20)$. To show $f_\sigma(z) - t_{20F}(z) = -1$, where $20F = 4^25^2/1^220^2$, by applying Lemma III.3.1, it suffices to show $f_\sigma(z) - t_{20F}(z)$ is holomorphic at $0, 1/2, 1/4, 1/5, 1/10$ and $1/20$ and equals to $-1$ at $1/20$.

At $c_4 = 0$ and $1/20$, similar to (1), we have $f_\sigma(z) - t_{20F}(z) = -1$.

At $c_4 = 1/4$ and $1/5$, using (III.4), (III.5) and (III.6), we have,

$$f_\sigma(z) = 4\eta_{\sigma_1}(z) + \eta_{\sigma_2}(z),$$

where

$$\sigma_1 = 5.8^240^2/1.2.10.20^2$$

$$\sigma_2 = 4^65.20^4/1.2^38^210^340^2$$

Furthermore, the followings hold,

$$\Sigma_{t,c}(t,c)^2/t \quad \eta_{\sigma_1}(Az) \quad \eta_{\sigma_2}(Az) \quad t_{20F}(Az)$$

$$A = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \quad 0 \quad + \quad +$$

$$A = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix} \quad 0 \quad 0 \quad +$$

Applying Lemma II.7, we have $f_\sigma(z) - t_{20F}(z)$ is holomorphic at $1/4$ and $1/5$.

At $c_4 = 1/2$, we consider $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$, applying Lemma II.6, we have,
\[ 4\eta_{1}(Az) = \frac{v\left(\begin{array}{cc} \frac{5}{2} & -1 \\ -1 & 0 \end{array}\right)v\left(\begin{array}{cc} 4 & -1 \\ -1 & 1 \end{array}\right)^{2}v\left(\begin{array}{cc} 20 & -1 \\ -1 & 0 \end{array}\right)^{2}}{v\left(\begin{array}{cc} 1 & -1 \\ -1 & 0 \end{array}\right)v\left(\begin{array}{cc} 4 & -1 \\ -1 & 1 \end{array}\right)v\left(\begin{array}{cc} 4 & -1 \\ -1 & 1 \end{array}\right)v\left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right)v\left(\begin{array}{cc} 10 & -1 \\ -1 & 2 \end{array}\right)^{2}} \]

\[ \exp(\pi i(3/5 + 1/2 + 1/10)/12) \cdot \exp(\pi i(1 + 1/5 + 1/12)/12) = \frac{1}{2q^{1/20}} \cdot g_{1}(z) \quad (\text{III.28}) \]

\[ - \frac{\exp(-\pi i/10)}{2q^{1/20}} \cdot g_{1}(z) \]

where \( g_{1}(z) \) is a holomorphic function.

\[ \eta_{2}(Az) = \frac{v\left(\begin{array}{cc} 2 & -1 \\ 1 & 0 \end{array}\right)^{6}v\left(\begin{array}{cc} 5 & -3 \\ 1 & 2 \end{array}\right)v\left(\begin{array}{cc} 10 & -1 \\ 0 & 1 \end{array}\right)^{4}}{v\left(\begin{array}{cc} 1 & -1 \\ 0 & 0 \end{array}\right)v\left(\begin{array}{cc} 1 & -1 \\ 0 & 0 \end{array}\right)^{3}v\left(\begin{array}{cc} 4 & -1 \\ 1 & 0 \end{array}\right)^{2}v\left(\begin{array}{cc} 6 & -1 \\ 1 & 0 \end{array}\right)^{3}v\left(\begin{array}{cc} 20 & -1 \\ 1 & 2 \end{array}\right)^{2}} \]

\[ \exp(\pi i(3 + 3/5 + 2/3)/12) \cdot \exp(\pi i(1 + 3 + 1/2 + 3/5 + 1/10)/12) = \frac{1}{2q^{1/20}} \cdot g_{2}(z) \quad (\text{III.29}) \]

\[ - \frac{\exp(-\pi i/10)}{2q^{1/20}} \cdot g_{2}(z) \]

So \( f_{\sigma}(Az) = \frac{\exp(\pi i/10)}{2q^{1/20}} \cdot (g_{1}(z) - g_{2}(z)) \) is holomorphic at \( 1/2 \).

On the other hand, since \( \Sigma_{t}(2,c)/t \) is holomorphic at \( 1/2 \). This implies that \( f_{\sigma}(z) - t_{20}f(z) \) is holomorphic at \( 1/2 \).

At \( c_{1} = 1/10 \), we consider \( \Lambda = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right] \), applying Lemma II.6, we have,

\[ 4\eta_{1}(Az) = \frac{v\left(\begin{array}{cc} 1/2 & -1 \\ -1 & 0 \end{array}\right)v\left(\begin{array}{cc} 4 & -1 \\ -1 & 1 \end{array}\right)^{2}v\left(\begin{array}{cc} 6 & -1 \\ -1 & 0 \end{array}\right)^{2}}{v\left(\begin{array}{cc} 1 & -1 \\ -1 & 0 \end{array}\right)v\left(\begin{array}{cc} 4 & -1 \\ -1 & 1 \end{array}\right)v\left(\begin{array}{cc} 4 & -1 \\ -1 & 1 \end{array}\right)v\left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right)v\left(\begin{array}{cc} 10 & -1 \\ -1 & 2 \end{array}\right)^{2}} \]

\[ \exp(\pi i(1 + 1/2 + 1/2)/12) \cdot \exp(\pi i(1 + 1 + 1 + 1)/12) = \frac{1}{2q^{1/20}} \cdot g_{1}(z) \quad (\text{III.30}) \]
where $g_1(z)$ is a holomorphic function.

$$\eta_{g_2}(Az) = \frac{1}{2q^{1/2}} \frac{\exp(\pi i (3 + 1 + 2)/12)}{\exp(\pi i (1 + 3 + 1/2 + 3 + 1/2)/12)}$$

$$= \frac{1}{2q^{1/2}} \frac{1}{g_2(z)}$$

(III.31)

where $g_2(z)$ is a holomorphic function.

So $f_\sigma(Az) = \frac{1}{2q^{1/2}} (g_1(z) - g_2(z))$ is holomorphic at $1/10$.

On the other hand, since $\Sigma_t(10, c)^2 / t = 0$, we have $t_{20F}(z)$ is holomorphic at $1/10$. This implies that $f_\sigma(z) - t_{20F}(z)$ is holomorphic at $1/10$. From the above argument, we have that $f_\sigma(z) - t_{20F}(z)$ is holomorphic at all cusps of $\Gamma_0(20)$. Applying Lemma III.3.1, we have $f_\sigma(z) - t_{20F}(z) = -1$. Applying Lemma III.3.2, we have that $f_\sigma(z)$ is a generator of a function field of $\Gamma_0(20) + W_{20}$.

(18) $\sigma = 2^{25} 20 / 1.4$.

From Theorem III.2.2, we have $\Theta_\sigma(z) = \Theta(z, [^{10}_{0}, ^{10}_{0}])$. 
Applying (III.5) and (III.6), we have \( \Theta_\sigma(z) = \Theta_3(10z)^2 = \eta_{\sigma_1}(z) \), where

\[ \sigma_1 = 10^{10}/5^420^4. \]

This implies that \( f_\sigma(z) = t_{20c}(z) \).

(19) \( \sigma = 1.4.6.24/3.8 \):

From Theorem III.2.2, we have \( \Theta_\sigma(z) = \Theta(z, [\frac{6}{0} \ 0]) \).

Applying (III.5) and (III.6), we have \( \Theta_\sigma(z) = \Theta_3(4z)\Theta_3(6z) = \eta_{\sigma_1}(z) \),

where \( \sigma_1 = 4^56^5/2^33^22^212^2 \). This implies that \( f_\sigma(z) = t_{24I}(z) \).

(20) \( \sigma = 2.3.4.24/1.8 \):

From Theorem III.2.2, we have \( \Theta_\sigma(z) = \Theta(z, [\frac{6}{0} \ 12]) \).

Applying (III.5) and (III.6), we have \( \Theta_\sigma(z) = \Theta_3(6z)\Theta_3(12z) = \eta_{\sigma_1}(z) \),

where \( \sigma_1 = 6^512^5/3^26^22^224^2 \). This implies that \( f_\sigma(z) = t_{24c}(z) \).

(21) \( \sigma = 2.3.5.30/6.10 \):

From Theorem III.2.2, we have \( \Theta_\sigma(z) = \Theta(z, [\frac{6}{1} \ 4]) \). Computation shows that the Fourier series of \( f_\sigma(z) \) does not appear in Table 4 of [2].

This implies that (*) is false.

III.4 Fixing groups of the Modular functions.

In this section, we will try to determine the fixing group of \( \Theta_\sigma(z)/\eta_\sigma(z) \) for every \( \sigma \in \sigma_0 \).
Case 1. If (*) is true for some $\sigma$, implying that $\theta_\sigma(z)/\eta_\sigma(z) = t_{\sigma_1} + c$ for some $\sigma_1 \in H$, then $\theta_\sigma(z)/\eta_\sigma(z)$ and $t_{\sigma_1}$ have the same fixing group and the fixing group of $t_{\sigma_1}$ is listed in Table 3 of [2].

Case 2. If (*) is false.

In this case, we have 15 conjugacy classes, the first 8 conjugacy classes are listed in Theorem II.11 with their fixing groups. The rest 7 conjugacy classes are $1^5 3.6^4/2^4$, $1^3 5.10^2/2^2$, $1.2^2 3.12^2/4^2$, $1^2 9.18/2.3$, $1.2 18^2/6.9$, $2^2 9.18/1.6$ and $2.3.5.30/6.10$.

Theorem III.4.1 will give the fixing group for every one of them. To prove Theorem III.4.1, Lemma II.5, II.6, II.7, II.8, II.9 and II.10 are needed.

**Theorem III.4.1** Let $\sigma$ and $\Gamma_\sigma$ be elements in $\Gamma_0$ and discrete subgroups in $SL_2(\mathbb{R})$, given in the following table, respectively, then $\Gamma_\sigma$ is the fixing group of $\sigma$. Moreover, $\Gamma_\sigma$ is of genus zero.

<table>
<thead>
<tr>
<th>Frame shape</th>
<th>$\Gamma_\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-6D</td>
<td>$1^5 3.6^4/2^4$</td>
</tr>
<tr>
<td>-10E</td>
<td>$1^3 5.10^2/2^2$</td>
</tr>
</tbody>
</table>
Proof: By an abuse of notation, we identify $\Pi_\eta(tz)^t$ and $\Pi_t^t$ together.

(1) $\sigma = 1^{53.64/24}$:

Using (1) of Theorem III.3.4, we have $\Theta_\sigma(z) = 1^{9/3^3} + 9(3^9/1^3)$, this implies

$f_\sigma(z) = 1^{24/3^66^4} + 9(3^{84}/1^{84})$

$$= \frac{1^{24}}{3^66^4} + 9\left(\frac{2^{84}}{3^66^4}\right)$$

(III.32)

$$= (f - 1) - \frac{8}{f - 1} - 7 + 9\left(\frac{8}{f - 1}\right)$$

where $f = t_{6E}(z)$, $6E = 2^{84}/1^{84}$.

Let $\Gamma$ be the fixing group of $f_\sigma(z)$, applying Lemma II.9, we have
Applying Lemma II.5, II.6 and II.7, we have

\begin{align*}
\text{cusps } (c_i) \text{ of } \Gamma_0(6) & \quad 0 \quad 1/2 \quad 1/3 \quad 1/6 \\
f(c_i) & \quad 9 \quad 0 \quad 1 \quad \infty \\
f_\sigma(c_i) & \quad 0 \quad \infty \quad \infty
\end{align*}

The above table shows that if we pick \( A \in \Gamma \), then \( A(1/2) \) is equivalent to \( 1/2 \). Without loss of generality, we may assume \( A(1/2) = 1/2 \).

Case 1. \( [\Gamma : \Gamma_0(6)] = 2 \). Then \( [\Gamma_{1/2} : (\Gamma_0(6))_{1/2}] = 2 \).

Applying Lemma II.10, we have,

\[ \Gamma_{1/2} = \langle A \mid A = \begin{bmatrix} 1 & 0 \\ 1/2 & 1 \end{bmatrix}, 3/2 \rangle, \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \rangle \quad (III.33) \]

\( A(0) = 3/8 \) equivalent to \( 1/2 \).

This is a contradiction.

Case 2. \( [\Gamma : \Gamma_0(6)] = 3 \). Similar to case 1, we have

\[ \Gamma_{1/2} = \langle A \mid A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, 1/2 \rangle, \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \rangle \quad (III.34) \]

\( A(1/6) = 5/14 \) equivalent to \( 1/2 \).

This is a contradiction.

Case 3. \( [\Gamma : \Gamma_0(6)] = 6 \). Similar to case 1, we have

\[ \Gamma_{1/2} = \langle A \mid A = \begin{bmatrix} 1 & 0 \\ 1/2 & 1 \end{bmatrix}, 1/2 \rangle, \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \rangle \quad (III.35) \]

\( A^2(1/6) = 5/14 \). This is a contradiction.

Summing up the above, we have \( \Gamma = \Gamma_0(6) \).

(2) \( \sigma = 1^{35}.10^2/2^2 \):
Using (2) of Theorem III.3.4, we have $\theta_\circ(z) = 1^5/5 + 5(5^5/1)$.

This implies,

$$f_\circ(z) = \frac{1^2z^2}{5^2} 10^2 + 5(2^2\frac{1^4}{5^4}1^4 10^2)$$

$$= \frac{1^2z^2}{5^2} 10^2 + 5(-\frac{2^2}{1^2} + \frac{2^2}{5^2} 10^2)$$

$$= f - \frac{4}{5^2} - 3 + 5(-\frac{2}{5^2} + \frac{2}{10^2})$$

(III.36)

where $f(z) = t_{10E}(z), 10E = 2.5^5/1.10^5$.

Let $\Gamma$ be the fixing group of $f_\circ(z)$, applying Lemma II.9, we have

$$[\Gamma : \Gamma_0(10)] \leq 6.\text{ Applying Lemma II.5,II.6, and II.7, we have}$$

**cusp** $(c_1)$ of $\Gamma_0(10)$

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1/2</th>
<th>1/5</th>
<th>1/10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(c_1)$</td>
<td>4</td>
<td>1</td>
<td>0</td>
<td>$\infty$</td>
</tr>
<tr>
<td>$f_\circ(c_1)$</td>
<td>$-23/3$</td>
<td>$\infty$</td>
<td>$\infty$</td>
<td></td>
</tr>
</tbody>
</table>

**Case 1.** $[\Gamma : \Gamma_0(10)] = 2$.

Applying Lemma II.10, we have

$$\Gamma_{1/2} = \langle A | A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} 10^{5/2} \begin{bmatrix} .1 & .0 \\ 1 & .2 \end{bmatrix} >$$

(III.37)

$A(1/2) = 5/12$ equivalent to $1/2$.

This is a contradiction.

**Case 2.** $[\Gamma : \Gamma_0(10)] = 3$.

Applying Lemma II.10, we have

$$\Gamma_{1/2} = \langle A | A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} 10^{5/3} \begin{bmatrix} .1 & .0 \\ 1 & .2 \end{bmatrix} >$$

(III.38)
$A(0) = 5/13, A(1/2) = -1/2, A(1/5) = 2/5$ and $A(1/10) = 43/110$. This implies $A$ fixes all equivalence classes of cusps. Since $\Gamma = \Gamma_0(10) + \Lambda \Gamma_0(10) + \Lambda^2 \Gamma_0(10)$, we conclude that $\Gamma$ still has 4 cusps. Applying Lemma II.8, we have $\Lambda \Gamma_0(10) = 6\pi$ and $2\pi = \Lambda \Gamma - 2\pi (2g - 2 + 4 + \sum (1 - 1/n^2)) \geq 4\pi$.

This is a contradiction.

Case 3. $[\Gamma : \Gamma_0(10)] = 6$.

Applying Lemma II.10, we have

$$\Gamma_{1/2} = \langle A | A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 5/6 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \rangle \quad (III.39)$$

$\Lambda^3(0) = 5/12$ equivalent to $1/2$.

This is a contradiction.

Summing up the above, we have $\Gamma = \Gamma_0(10)$.

(3) $\sigma = 1.2^2.3.12^2/4^2$:

Using Theorem 3.2 of [3], we have

$$f_\sigma(z) = f + \frac{16}{3} - \frac{3}{2} \cdot 2$$

where $f(z) = t_{121}(z)$, $12I = 4^4.6^2/2^2.12^4$.

Let $\Gamma$ be the fixing group of $f_\sigma(z)$, applying Lemma II.9, we have

$[\Gamma : \Gamma_0(12)] \leq 6$. Applying Lemma II.5, II.6 and II.7, we have

cusps $\{c_i\}$ of $\Gamma_0(12)$: 0, 1/2, 1/3, 1/4, 1/6, 1/12
From the above table, we have that \( \Gamma \) has at least 4 cusps.

Applying Lemma II.8, we have
\[
8\pi = \Lambda_{\Gamma_0(12)} \geq \Lambda_{\Gamma} = 2\pi(2g - 2 + 4 + \Sigma(1 + 1/n)) \geq 4\pi.
\]

This implies that \( \{ \Gamma : \Gamma_0(12) \} \leq 2 \).

If \( \{ \Gamma : \Gamma_0(12) \} = 2 \), applying Lemma II.10, we have
\[
\Gamma_{1/2} = \langle A | A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, 3/2 | \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} > \quad \text{(III.41)}
\]

A(0) = 3/8 equivalent to 1/4.

This is a contradiction.

Summing up the above, we have \( \Gamma = \Gamma_0(12) \).

\( (4) \sigma = 1.8, 1.8/2.3: \)

Using (6) of Theorem III.3.4, we have \( \Theta_\sigma(z) = 1^{3/3} + 3(9^{3/3}) \).

This implies that
\[
f_\sigma(z) = \frac{1.2}{9.18} + 3(-\frac{2.9^{2}}{1.18})
\]
\[
= f - \frac{2}{f} - 1 + 3(-\frac{f + 1}{f - \frac{1}{f}}) \quad \text{(III.42)}
\]

where \( f(z) = t_{18D}(z), 18D = 6.9^{3}/3.18^{3}. \)

Let \( \Gamma \) be the fixing group of \( f_\sigma(z) \), applying Lemma II.9, we have
[ \Gamma : \Gamma_0(18) ] \leq 6. Applying Lemma II.5, II.6 and II.7, we have

cusps \( c_1 \) of \( \Gamma_0(18) \)  
0 1/2 1/3 2/3 1/6 5/6 1/9 1/18

\( f(c_1) \)  
2 -1 2a^2 2a^4 a^5 a 0 =

\( f_\sigma(c_1) \)  
-5 -3 -6 b c =

where \( a = (1-3^{1/2}i)/2 \), \( b = -(3+3^{1/2}i)/2 \), \( c = -(3-3^{1/2}i)/2 \)

From the above table, we conclude that \( \Gamma \) has at least 6 cusps.

Applying Lemma II.8, we have

\[ 12\pi = \Lambda_\Gamma(18) \geq \Lambda_\Gamma \geq 2\pi(2\pi - 2 + 6 + \Sigma(1 - 1/n) \geq 8\pi. \text{ So } \Gamma = \Gamma_0(18). \]

(5) \( \sigma = 1.2.18^26.9; \)

Using (5) of Theorem III.3.4, we have \( \Theta_\sigma(z) = 1^6.6/2^3.3^2 + 6(1.6^3/2.3^3). \)

This implies that

\[ f_\sigma(z) = 1^6.6^3/1.2^4.3^2.18^2 = 6(1.6^3/1.2^3.3^2.18^2) \]

\[ = \frac{1.2}{9.18} \frac{1.18^2}{2.9} \frac{1.6^2}{2.3^3.18^3} + 6(\frac{1.6^2}{2.3^3.18^3} \frac{1.3^2.18^3}{2.3^3.18^3}) \]

\[ = \left(\frac{f - 2}{f^2 + 1}\right)^2 + 6\left(\frac{f^2}{f + 1} - \frac{1}{f^2 + 1}\right) \]

(III.43)

where \( f(z) = t_{18D}(z), 18D = 6.9^3/3.18^3 \)

Let \( \Gamma \) be the fixing group of \( f_\sigma(z) \), similar to (4), we conclude that \( \Gamma \)

has at least 6 cusps, hence \( \Gamma = \Gamma_0(18). \)

(6) \( \sigma = 2^29.18/1.6: \)
Using (6) of Theorem III.3.4, we have $\Theta_\sigma(z) = 2^3/6 + 3(18^3/6)$.

This implies that

$$f_\sigma(z) = 1.2/9.18 + 3(1.18^2/2^29)$$

$$= (f - \frac{2}{x} - 1) + \frac{3}{x^3} - 1$$

where $f(z) = t_{18D}(z)$, $18D = 6.9^3/3.18^3$

Let $\Gamma$ be the fixing group of $f_\sigma(z)$, similar to (4), we conclude that $\Gamma$ has at least 6 cusps, hence $\Gamma = \Gamma_0(18)$

(7) $\sigma = 2.3.5.30/6.10:

Using (13) of Theorem III.3.4, we have $\Theta_\sigma(z) = 3^25^2/1.15 - 1^215^2/3.5$.

This implies that

$$f_\sigma(z) = 3.5.6.10/1.2.15.30 - 1^26.10.15^2/2.3^25^230$$

$$= \frac{3.5.6.10}{1.2.15.30} - \frac{2.30}{3.5} - \frac{1^26.10.15^2}{2^23.5.30^2}$$

$$= f - 2 + \frac{2}{x^2} - \frac{2}{x^2} + 3 - \frac{2}{x^2}$$

where $f(z) = t_{30G}(z)$, $30G = 3.5/2.30$

Let $\Gamma$ be the fixing group of $f_\sigma(z)$, applying Lemma II.9, we have that

$[ \Gamma : \Gamma_0(30) + W_{15} ] \leq 6$. Applying Lemma II.5, II.6 and II.7, we have

cusps $(c_1)$ of $\Gamma_0(30)+W_{15}$

<table>
<thead>
<tr>
<th>$c_1$</th>
<th>0</th>
<th>1/2</th>
<th>1/3</th>
<th>1/6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(c_1)$</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>
Case 1. \( [\Gamma : \Gamma_0(30)+W_{15}] = 2 \). Applying Lemma II.10, we have

\[ \Gamma_{1/6} = \langle A | A = \begin{bmatrix} 1 & 0 \\ 5/2 & 1 \end{bmatrix} \rangle \]

\[ A(1/3) = 13/84 \text{ equivalent to } 1/6. \]

This is a contradiction.

Case 2. \( [\Gamma : \Gamma_0(30)+W_{15}] = 3 \). Applying Lemma II.10, we have

\[ \Gamma_{1/6} = \langle A | A = \begin{bmatrix} 1 & 0 \\ 5/3 & 1 \end{bmatrix} \rangle \]

\[ A(1/2) = 8/114 \text{ equivalent to } 1/6. \]

This is a contradiction.

Case 3. \( [\Gamma : \Gamma_0(30)+W_{15}] = 6 \). Applying Lemma II.10, we have

\[ \Gamma_{1/6} = \langle A | A = \begin{bmatrix} 1 & 0 \\ 5/6 & 1 \end{bmatrix} \rangle \]

\[ A^3(1/3) = 13/84. \] This is a contradiction.

Summing up the above, we have \( \Gamma = \Gamma_0(30) + W_{15} \).

Theorem III.4.2 Let \( \sigma \) be a \( \cdot 0 \) element such that (*) is false; then the Riemann surface whose function field is \( C(\theta_\sigma(z)/\eta_\sigma(z)) \) cannot be realized as \( \Gamma_\sigma \setminus \mathbb{H}^* \) where \( \Gamma_\sigma \) is the fixing group of \( \theta_\sigma(z)/\eta_\sigma(z) \) in \( SL_2(\mathbb{R}) \).

Proof: Let \( \sigma \) be a \( \cdot 0 \) element such that (*) is false. By Theorem II.2 and Theorem III.4.1, \( [C(\sigma_1) : C(\theta_\sigma(z)/\eta_\sigma(z))] = 3 \) for some \( \sigma_1 \in \mathcal{M} \).

Suppose \( C(\theta_\sigma(z)/\eta_\sigma(z)) \) is the function field of the Riemann surface
$\Gamma_1 \backslash \mathbb{H}^*$, this implies that $[\Gamma_\sigma : \Gamma_\sigma_1] = 3$, which contradicts with the fact that $\Gamma_\sigma = \Gamma_\sigma_1$. Thus the Riemann surface whose function field is $C(\Theta_\sigma(z) / \eta_\sigma(z))$ cannot be realized as $\Gamma_\sigma \backslash \mathbb{H}^*$. 
### APPENDIX A

\[
\begin{align*}
\mathbf{A} & = \begin{bmatrix}
4 & -1 & 1 & 1 & -2 & 2 \\
-1 & 4 & -1 & 2 & 2 & 1 \\
1 & -1 & 4 & 1 & -2 & 2 \\
-2 & 2 & -2 & 1 & 4 & -1 \\
2 & 1 & 2 & 2 & -1 & 4
\end{bmatrix} & \quad \mathbf{D} & = \begin{bmatrix}
4 & 1 & 1 & 2 \\
1 & 1 & 4 & -1 \\
2 & 2 & -1 & 4
\end{bmatrix} \\
\mathbf{B} & = \begin{bmatrix}
4 & 1 & 1 & 1 \\
1 & 4 & -1 & -1 \\
1 & -1 & 4 & -1 \\
1 & -1 & -1 & 4
\end{bmatrix} & \quad \mathbf{E} & = \begin{bmatrix}
4 & 2 & 2 & 2 \\
2 & 4 & 1 & 1 \\
2 & 1 & 4 & 1 \\
2 & 1 & 1 & 4
\end{bmatrix} \\
\mathbf{C} & = \begin{bmatrix}
6 & 0 & 3 & 3 \\
3 & 3 & 6 & 3 \\
3 & 3 & 3 & 6
\end{bmatrix}
\end{align*}
\]
APPENDIX B

The matrix representation of a permutation $A_i$ listed below can be derived in the following manner:

Let $A = \ldots \ldots (\ldots, i, j, \ldots)$ be a permutation and $[A]$ be its corresponding matrix, then the $(i+1)$th row of $[A]$ consists of zero except for an entry of 1 in the $(j+1)$th column. The 24th row (column) corresponds to $\infty$.

$A_1 = 2^81^8 = (4,22)(6,7)(8,18)(9,10)(11,12)(13,16)(15,20)(19,21)$

$A_2 = 1^63^6 = (0,1,2)(5,14,17)(6,21,19)(8,11,18)(9,20,15)(13,16,22)$

$A_3 = 1^64^4 = (1,18,4,2,6)(5,21,20,10,7)(8,16,13,9,12)(11,19,22,14,17)$

$A_4 = 1^64^22^2 = (4,6,22,7)(8,9,18,10)(11,15,12,20)(13,19,16,21)(3,14)(5,17)$

$A_5 = 1^37^3 = (8,3,0,2,4,9)(14,6,12,17,20,5,1)(7,10,22,19,11,15,18)$

$A_6 = 1^22.4.8^2 = (4,8,6,9,22,18,7,10)(13,12,19,20,16,11,21,15)(3,17,14,5)(\infty,0)$

$A_7 = 1^22.3^26^2 = (0,5,1,14,2,17)(6,16,21,22,19,13)(8,18,11)(9,15,20)(\infty,3)(4,7)$

$A_8 = 1^211^2 = (1,2,4,8,16,9,18,13,3,6,12)(5,10,20,17,11,22,21,19,15,7,14)$

$A_9 = 12^2 = (\infty,20,8,21,14,16,0,9,18,6,17,13)(1,10,11,7,3,4,2,15,12,19,5,22)$

$A_{10} = 6^4 = (\infty,8,14,0,18,17)(1,11,3,2,12,5)(4,15,19,22,10,7)(6,13,20,21,16,9)$

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<table>
<thead>
<tr>
<th>$A_{11}$</th>
<th>$A_{12}$</th>
<th>$A_{13}$</th>
<th>$A_{14}$</th>
<th>$A_{15}$</th>
<th>$A_{16}$</th>
<th>$A_{17}$</th>
<th>$A_{18}$</th>
<th>$A_{19}$</th>
<th>$A_{20}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$4^6$</td>
<td>$3^8$</td>
<td>$2^{12}$</td>
<td>$2^{4^2}$</td>
<td>$2.4.6.12$</td>
<td>$2.2.7.14$</td>
<td>$1.3.5.15$</td>
<td>$1.2.3$</td>
<td>$2^2.10^2$</td>
<td>$3.21$</td>
</tr>
<tr>
<td>$(\infty, 21, 0, 6)(1, 7, 2, 19)(3, 15, 5, 10)(4, 12, 22, 11)(8, 16, 18, 13)$</td>
<td>$(\infty, 14, 18)(0, 17, 8)(1, 3, 12)(2, 5, 11)(4, 19, 10)(6, 20, 16)(7, 15, 22)$</td>
<td>$(\infty, 0)(1, 22)(2, 11)(3, 15)(4, 17)(5, 9)(6, 19)(7, 13)(8, 20)(10, 16)$</td>
<td>$(\infty, 7, 3, 4)(0, 6, 14, 22)(1, 21, 17, 13)(2, 19, 5, 16)(8, 15)(9, 11)$</td>
<td>$(\infty, 0, 13, 5, 6, 1, 16, 14, 21, 2, 22, 17, 19)(8, 9, 18, 15, 11, 20)$</td>
<td>$(\infty, 14, 8, 6, 3, 12, 0, 17, 2, 20, 4, 5, 9, 1)(7, 11, 10, 15, 22, 18, 19)$</td>
<td>$(\infty, 14, 9, 19, 22, 16, 1, 20, 2, 5, 15, 11, 18, 13, 12)(4, 6, 0, 17, 10)$</td>
<td>$(0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22)$</td>
<td>$(0, 3, 1, 15, 19, 12, 18, 8, 4, 10)(6, 13, 14, 7, 11, 16, 17, 21, 22, 20)$</td>
<td>$(\infty, 7, 16, 22, 8, 13, 14, 4, 2, 6, 1, 9, 15, 20, 10, 19, 11, 0, 21, 17, 5)(3, 18, 12)$</td>
</tr>
</tbody>
</table>
### APPENDIX C

**Matrix T (a = 0.5, b = -0.5)**

\[
\begin{bmatrix}
0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a \\
0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b & 0 & 0 & b b \\
0 & 0 & a & 0 & 0 & 0 & 0 & b & 0 & b & 0 & 0 & 0 & 0 & 0 \\
a & 0 & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 & 0 \\
0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & b & b & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & a & b & 0 & 0 & b & 0 & 0 & 0 & 0 & b & 0 \\
0 & 0 & 0 & 0 & b & a & 0 & 0 & b & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & b & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & b b \\
0 & 0 & 0 & b & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & b b \\
0 & 0 & b & 0 & 0 & 0 & b & 0 & 0 & 0 & b & 0 & a & 0 & 0 \\
0 & 0 & 0 & 0 & b & 0 & 0 & 0 & 0 & b & 0 & b & 0 & 0 & 0 \\
a & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b & b & 0 \\
0 & 0 & 0 & 0 & b & 0 & 0 & 0 & 0 & b & b & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & b & 0 & 0 & 0 & 0 & b & 0 & 0 & 0 & a \\
0 & 0 & 0 & 0 & 0 & 0 & b & 0 & 0 & 0 & 0 & b & b & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & b & b & 0 & 0 & b & 0 & 0 & 0 & 0 & a \\
0 & 0 & 0 & 0 & 0 & 0 & b & 0 & 0 & 0 & 0 & b & 0 & 0 & a \\
b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b & a & 0 \\
a & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b
\end{bmatrix}
\]
Matrix $a$ (a = -1)

\[
\begin{array}{cccccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0

a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0

0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0

0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0

0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0

0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0

0 & 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0

0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 & 0

0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0

0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a

\end{array}
\]
APPENDIX D

The first column gives the Frame shapes of -0 elements.
The second column gives the theta functions.
The third column gives the Monster elements.
The last column gives the discrete subgroups in SL_2(R).

Each line of the table reads as
(1) If \( \sigma_1 \) appears, then \( \theta_\sigma(z)/\eta_\sigma(z) = t_{\sigma_1} \) + constant
(2) If x appears, then (*) is false.
(3) \( \Gamma_\sigma \) is the fixing group of \( \theta_\sigma(z)/\eta_\sigma(z) \).
(4) \( -n_x = n_x \) if \( -n_x \) does not appear.

The following notations are used:
(1) A, B, C, D, E are matrices listed in Appendix A.
(2) \( E_4(z) = 1/2 (\theta_2(z)^8 + \theta_3(z)^8 + \theta_4(z)^8) \)
(3) \( \theta_1(z) = \theta_2(z)\theta_3(z)\theta_4(z) \)
(4) \( \theta_4(p)(z) = \theta_2(z)\theta_2(pz) + \theta_3(z)\theta_3(pz) \)
(5) \( \theta(z,D_4) = -1/2 (\theta_3(z)^4 + \theta_4(z)^4) \)
(6) \( \theta_1(z) = \theta_1(2z) \)
(7) \( \theta_1(z) = \theta_1(10z) \)
(8) \( \hat{\Theta}_1 - \Theta_1(z) \)

(9) \( \Theta(z) = \Theta_2(z)\Theta_3(z) - \Theta_3(z)\Theta_2(z) \)

<table>
<thead>
<tr>
<th>Frame shape</th>
<th>( \Theta_\sigma(z) )</th>
<th>( \sigma )</th>
<th>( \Gamma_\sigma )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1_A</td>
<td>( 1^{124} )</td>
<td>( E_4(z)^3 \cdot 45/16\theta_4'(z)^8 )</td>
<td>1A</td>
</tr>
<tr>
<td>-1_A</td>
<td>( 2^{24}/1^{24} )</td>
<td>1</td>
<td>2B</td>
</tr>
<tr>
<td>2_A</td>
<td>( 1^{16}2^8 )</td>
<td>( E_4(2z)^2 \cdot 15/256\theta_2(z)^{16} )</td>
<td>2A</td>
</tr>
<tr>
<td>-2_A</td>
<td>( 2^{16}/1^8 )</td>
<td>( E_4(2z) )</td>
<td>2B</td>
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<td>$3/2\theta(2z)\Phi(6z)$</td>
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<td>$18^-$</td>
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<td>1^{2}20^{2}/4^{2}5^{2}</td>
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<td>20\text{F}</td>
<td>20+20</td>
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<td>\Theta_{3}(4z)\Theta_{3}(20z)</td>
<td>40\text{B}</td>
<td>40\text{J}2+</td>
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<td>1.2.10.20/4.5</td>
<td>\Theta(z, [\frac{4}{2}, \frac{2}{6}])</td>
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<td>42\text{B}</td>
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<td>42\text{D}</td>
<td>42+3,14</td>
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<td>21\text{C}</td>
<td>3.21</td>
<td>\Theta(7)(6z)</td>
<td>21\text{C}</td>
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<td>2.22</td>
<td>\Theta_{3}(2z)\Theta_{3}(22z)-2\eta_{6}(z)</td>
<td>44\text{AB}</td>
<td>44+</td>
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<td>1.23</td>
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<td>$\theta_1(2z)\theta_1(14z)$</td>
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<td>30_G</td>
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<td>$1/2^{\delta_2} \theta_1(6z)\theta_1(10z)$</td>
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<td>$30</td>
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LIST OF REFERENCES


[3] M. Koike, Moonshine of PSL$_2$(F$_q$) and the automorphism group of Leech lattice. To appear in Japanese J.


