INFORMATION TO USERS

While the most advanced technology has been used to photograph and reproduce this manuscript, the quality of the reproduction is heavily dependent upon the quality of the material submitted. For example:

- Manuscript pages may have indistinct print. In such cases, the best available copy has been filmed.

- Manuscripts may not always be complete. In such cases, a note will indicate that it is not possible to obtain missing pages.

- Copyrighted material may have been removed from the manuscript. In such cases, a note will indicate the deletion.

Oversize materials (e.g., maps, drawings, and charts) are photographed by sectioning the original, beginning at the upper left-hand corner and continuing from left to right in equal sections with small overlaps. Each oversize page is also filmed as one exposure and is available, for an additional charge, as a standard 35mm slide or as a 17"x 23" black and white photographic print.

Most photographs reproduce acceptably on positive microfilm or microfiche but lack the clarity on xerographic copies made from the microfilm. For an additional charge, 35mm slides of 6"x 9" black and white photographic prints are available for any photographs or illustrations that cannot be reproduced satisfactorily by xerography.
Ali, Sayel Ali Ahmad

UPPER BOUND FOR THE DEGREE OF AN APPROXIMATING MONOMIAL

The Ohio State University

Ph.D. 1987

University Microfilms International 300 N. Zeeb Road, Ann Arbor, MI 48106
UPPER BOUND FOR THE DEGREE OF AN APPROXIMATING MONOMIAL

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

by

Sayel Ali Ahmad Ali, B.Sc., M.Sc.

The Ohio State University
1987

Dissertation Committee:
B. M. Baishanski
R. Bojanic
W. Davis
P. Nevai

Approved by

Adviser
Department of Mathematics
ACKNOWLEDGEMENTS

I would like to express my deep gratitude to my advisor Professor Bogdan Baishanski for his guidance and his generous giving of both time and insight throughout this research.

Thanks are due to Professor William Davis for his remarks in the beginning of my research and for taking the time to read this dissertation. I would like to thank the other members of my dissertation committee, Professors Ranko Bojanic and Paul Nevai for their suggestions and remarks.

It is also a pleasure to acknowledge Mrs. Terry England for her skillful typing of this dissertation.

Finally, I thank my family for their continuous support and concern throughout my graduate study.
VITA

February 18, 1954 ................................................ born - Nawaimeh, Jordan

1976 ................................................................. B.Sc., The University of Jordan

1977-1978 .......................................................... Teacher, Zillah Primary School,
.............................................................................. Zillah, Libya

1980 ................................................................. M.Sc., University of Dundee,
................................................................................... Dundee, Scotland, U.K.

1980 ................................................................. Lecturer, Department of
.............................................................................. Mathematics, The University
.............................................................................. of Jordan, Jordan

1980-1981 ........................................................ Teaching Associate,
.............................................................................. Department of Mathematics,
.............................................................................. Illinois Institute of Technology,
.............................................................................. Chicago, Illinois

1981-1987 ........................................................ Teaching Associate,
.............................................................................. Department of Mathematics,
.............................................................................. The Ohio State University
.............................................................................. Columbus, Ohio

Publications

"Strict Decrease of the approximation Error", joint work with Professor Bogdan
Baishanski, Proceedings of the 1985 Alfred Haar Memorial Conference, Colloquia
Mathematica Societatis János Bolyai, Budapest, Hungary.

Fields of Study

Major Field: Mathematics
ABSTRACT

If $P$ is any polynomial of degree $\leq n$, and $m(x) = cx^k$ is a monomial of best approximation to $P$ in $L_p[a,b]$ among all monomials of degree $\geq n$, then

i) if $p = \infty$, no upper bound for $k$ exists, and

ii) if $1 \leq p < \infty$, there is $K_n = K_n(a, b; p)$ (independent of the polynomial $P$) such that

\begin{equation}
(*) \quad k \leq K_n(a, b; p).
\end{equation}

These facts are known [2,3], but the proof of the existence of the upper bound $K_n$ is not constructive. In particular, with $M_n$ denoting the best bound $K_n$ (i.e., $M_n$ is the infimum of all $K_n$ for which (*)& is true), no estimate for $M_n$ was available.

In this dissertation we have considered approximation by all quasi-monomials $cx^k$ ($k$ real and $\geq n$). We have obtained estimates for $M_n$ for the case of $L_2$-norm on the interval $[0, 1]$; our main result is that

\begin{equation}
(**) \quad \frac{1}{4} (n + 1)^3 \leq M_n \leq 6(n + 1)^3.
\end{equation}

The proof consists of three parts:
i) the $L_2$-problem under consideration is shown to be equivalent to an extremal problem on polynomials of degree $\leq n$ in $L_\infty(W_n(x)dx)$ where $W_n$ is a particular weight function;

ii) the upper bound for $M_n$ is obtained by adapting the method of G. G. Lorentz in [7], i.e., by using the Rahman-Schmeisser Lemma [8] (Section 4 of Chapter II in this dissertation), and also the formula

\[
(*** \quad \lim_{r \to 1^-} \frac{P_r(f; t) - f(t)}{1 - r} = -(\tilde{f})'(t),
\]

valid for smooth periodic function $f$ (here $P_r(f; t)$ is the Poisson transform of $f$, and $\tilde{f}$ is the conjugate function of $f$);

iii) the lower bound for $M_n$ is obtained by explicitly finding the Chebyshev polynomial for the weight $x$ on $[0, 1]$, and then by replacing the weight function $x$ by the weight $W_n(x)$.
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>ACKNOWLEDGEMENTS</th>
<th>ii</th>
</tr>
</thead>
<tbody>
<tr>
<td>VITA</td>
<td>iii</td>
</tr>
<tr>
<td>ABSTRACT</td>
<td>iv</td>
</tr>
<tr>
<td>LIST OF TABLES</td>
<td>vii</td>
</tr>
<tr>
<td>CHAPTER</td>
<td></td>
</tr>
<tr>
<td>I. INTRODUCTION</td>
<td></td>
</tr>
<tr>
<td>1. Notation</td>
<td>1</td>
</tr>
<tr>
<td>2. Origin of the problem</td>
<td>2</td>
</tr>
<tr>
<td>a) The conjecture of G. G. Lorentz</td>
<td>2</td>
</tr>
<tr>
<td>b) The inversion of the Lorentz Problem</td>
<td>2</td>
</tr>
<tr>
<td>c) The existence of the upper bound in $L_p$-Norms</td>
<td>3</td>
</tr>
<tr>
<td>d) The Statement of our Problem</td>
<td>4</td>
</tr>
<tr>
<td>3. Incomplete Polynomials</td>
<td>5</td>
</tr>
<tr>
<td>a) Uniform convergence of a sequence of incomplete polynomials to zero on an interval $[0, \delta], \delta &gt; 0$</td>
<td>5</td>
</tr>
<tr>
<td>b) Location of the maximum of an incomplete polynomial</td>
<td>6</td>
</tr>
<tr>
<td>c) A general version of the problem</td>
<td>7</td>
</tr>
<tr>
<td>d) The Saff-Varga Lemma</td>
<td>8</td>
</tr>
<tr>
<td>4. Correction to a proof of G. G. Lorentz</td>
<td>9</td>
</tr>
<tr>
<td>5. Comments on the proof of the Main Theorem</td>
<td>10</td>
</tr>
<tr>
<td>6. On the numerical results</td>
<td>11</td>
</tr>
<tr>
<td>II. THE MAIN THEOREM</td>
<td>12</td>
</tr>
<tr>
<td>1. Statement of the Main Theorem</td>
<td>12</td>
</tr>
<tr>
<td>2. Equivalent Form of the Main Theorem</td>
<td>13</td>
</tr>
<tr>
<td>3. The Poisson Transform and the Conjugate of Smooth Functions</td>
<td>18</td>
</tr>
<tr>
<td>4. The Rahman-Schmeisser Lemma</td>
<td>25</td>
</tr>
<tr>
<td>5. The Upper Bound</td>
<td>26</td>
</tr>
<tr>
<td>6. The Lower Bound</td>
<td>32</td>
</tr>
<tr>
<td>III. NUMERICAL VALUES OF $M_{n,0}$ AND POLYNOMIALS FOR WHICH THE DEGREE OF THE BEST APPROXIMATING QUASI-MONOMIAL IS LARGEST</td>
<td>40</td>
</tr>
<tr>
<td>LIST OF REFERENCES</td>
<td>44</td>
</tr>
</tbody>
</table>
LIST OF TABLES

Table | Page
--- | ---
1) The Upper Bound | 42
2) The Coefficients of the Extremal Polynomials | 43
1. Notation:

The norm $\| \cdot \|_K$ will denote the uniform norm on $K$.

The norm $\| \cdot \|_p$ will denote the $L_p$-norm on $[0, 1]$.

For any non-negative integer $n$, $\pi_n$ will denote the set of all polynomials of degree $\leq n$.

For $t > -\frac{1}{2}$ and a polynomial $P$, we define $E(P; t)$ by

$$E(P; t) = \inf_{c} \| P(x) - cx^t \|_2.$$

For any $S$, $S \subseteq \{ t : t > -\frac{1}{2} \}$, we define $E_S(P)$ by

$$E_S(P) = \inf\{ E(P; t) : t \in S \}.$$

We will use $E_\gamma(P)$ in place of $E_S(P)$ whenever $S$ is of the form $[\gamma, \infty)$ if $\gamma > -\frac{1}{2}$ or $(\gamma, \infty)$ if $\gamma = -\frac{1}{2}$. 
2. Origin of the problem:

a) The conjecture of G. G. Lorentz:

Conjecture: Among all polynomials of the form \( P(x) = \sum_{i=1}^{t} a_i x^{k_i} \), where \( t \) is
a fixed integer \( < N \), the polynomial of best uniform approximation to \( x^N \) of degree
\( < N \) has powers \( k_1 = N - t \), \( k_2 = N - t + 1 \), \( \ldots \), \( k_t = N - 1 \) ([7], pp 293).

This conjecture of Lorentz was proved by Borosh, Chui, and Smith [4]. They proved
the following more general theorem:

Theorem [4]: Let \( N, t, \) and \( k \) be fixed positive integers such that \( \ell < N \) and
\( \ell \leq k \). Let \( \lambda_1, \ldots, \lambda_k \) be integers such that

\[
0 \leq \lambda_1 < \ldots < \lambda_\ell < N < \lambda_{\ell+1} < \ldots < \lambda_k.
\]

Among all polynomials \( P(x) = \sum_{i=1}^{k} a_i x^{\lambda_i} \), the polynomial of best uniform approxi-
mation to \( x^N \) has powers

\[
N - \ell, \ldots, N - 1, N + 1, \ldots, N + k - \ell.
\]

In [15], P. W. Smith gave a proof (based on an observation by A. Pinkus) of the
above result in any \( L_p \) norm, \( 1 \leq p \leq \infty \).

b) The inversion of the Lorentz problem:

Suppose \( x^n \) is replaced by a polynomial \( P \in \pi_n \) and \( P \) is approximated by
monomials \( m(x) = ax^k \), \( k \geq 1 \). What is the degree of \( m(x) \) which best approximates
\( P \)?
An analogue of the above result will not hold, even in simple cases; for example, if \( P(x) = x^n - \frac{n}{n+1} x^{n-1} \), then among all monomials, the monomial of best \( L_2 \)-approximation to \( P \) on \([0,1]\) has power \( 3n + 1 \), [2].

This led to the question about the existence of an upper bound for the powers of the best approximating monomials if \( P \) runs over the set \( \pi_n \), [2].

c) The existence of the upper bound in \( L_p \)-norms:

Let \( \ell \) be a fixed positive integer.

If \( P \in \pi_n \) and \( Q_P(x) = \sum_{k=1}^{i} C_k(P)x^{\lambda_k(P)} \) is a polynomial of length \( \leq \ell \) (the length of a polynomial is the number of its non-zero coefficients) of best approximation to \( P \) in \( L_p[a,b] \) among all polynomials of length \( \leq \ell \), then

i) if \( p = \infty \) and \( 2\ell \leq n + 1 \), no upper bound for \( \lambda_{\ell}(P) \) (we assume \( \lambda_1(P) < \ldots < \lambda_{\ell}(P) \)) exists, and

ii) if \( 1 \leq p < \infty \), there is \( K_n = K_n(a,b;\ell,p) \) such that

\[
\lambda_{\ell}(P) \leq K_n.
\]

A proof of (i) and the special case when \( p = 2 \), \([a,b] = [0,1] \), and \( \ell = 1 \) is given in [2], and a proof of (ii) is given in [3]. In fact, stronger results were obtained in [3]; for example,

**Theorem [3]:** Let \( S \) be a set of non-negative integers, and denote by \( \pi_{\ell-1}(S) \) the collection of all polynomials of length \( \leq \ell - 1 \).
Let $K$ be a compact set in $L_p(a, b)$, $1 \leq p < \infty$, such that

$$K \cap \pi_{\ell-1}(S) = \phi.$$

i) If $\sum_{s \in S} \frac{1}{s+1} = \infty$ and, in case $p = 1$, $m \{x : f(x) = g(x)\} = 0$ for every $f \in K$, $g \in \pi_{\ell-1}(S)$, or

ii) if every function in $K$ is analytic on $[a, b]$ and, in case $a = -b$, $S$ contains infinitely many odd and infinitely many even integers, then there exists $d = d(K, S, \ell)$ such that if $f \in K$ and $P$ is a best approximation to $f$ in $\pi_{\ell}(S)$, then $\deg P \leq d$.

This is a pure existence theorem. The proof is not constructive.

d) The statement of our problem:

The question arises of obtaining an upper bound for the degree of the best approximating polynomials of length $\leq \ell$, when a polynomial of degree $\leq n$ is being approximated. It is natural to restrict ourselves to the simplest case, first, and we do this in this dissertation. Namely, we consider only the $L_2$-norm on the interval $[0, 1]$, we consider only the length $\ell = 1$, and instead of approximating by monomials $cx^k$, $k$ non-negative integer, we approximate by quasi-monomials $cx^t$, $t$ is real and $\geq n$.

If $E(P; t)$ and $E_n(P)$ are defined as in Section (1) of this chapter, it is easy to show (Lemma (1) in Chapter II) that there exists a constant $K_n$ such that if $P \in \pi_n$ then

$$(*) \quad E(P; t) > E_n(P) \quad \text{if} \quad t > K_n.$$

Now, the problem of this dissertation can be stated as follows:
Let \( M_n \) be the infimum of numbers \( K_n \) which satisfy formula (*) above. Give an estimate for \( M_n \) in terms of \( n \) only.

3. Incomplete polynomials:

To attack our problem, we have borrowed some powerful techniques developed in the theory of incomplete polynomials.

In (a) and (b) we present two aspects of this theory, in (c) we see how our problem can be related to the problems in incomplete polynomials, in (d) we state the Saff-Varga Lemma on weighted polynomials, which we used in the proof of the lower bound for \( M_n \), and which is the basis for our computation of \( M_n \).

a) Uniform convergence of a sequence of incomplete polynomials to zero on an interval \([0, \delta], \delta > 0\):

The name Incomplete Polynomials was initiated by G. G. Lorentz to denote polynomials of the form,

\[
(I.1) \quad \sum_{k=s}^{n} a_k x^k, \quad s > 0.
\]

In [7], G. G. Lorentz presented some problems and results in this new field which he calls approximation by incomplete or Lacunary Polynomials. One of the main results appeared in [7], is the following:

Theorem [7]: For each \( 0 < \theta < 1 \), there is \( 0 < \delta < 1 \) with the following property. If polynomials
(1.2) \[ P_n(x) = \sum_{k=s}^{n} a_k x^k, \quad s \geq n\theta, \]

defined for infinitely many \( n \), satisfying \( |P_n(x)| \leq M \), \( 0 \leq x \leq 1 \), then \( P_n(x) \to 0 \) uniformly on \([0, \delta] \).

The set of all polynomials of the form (1.2) will be denoted by \( I_\theta \) (this notation was used by Saff in [11]).

Lorentz defined \( \Delta(\theta) \) to be the supremum of numbers \( \delta \), for which the above theorem is true.

In [7], Lorentz proved that \( \Delta(\theta) \geq \theta^2 \), and the reverse inequality, \( \theta^2 \geq \Delta(\theta) \), was a consequence of the following result which was obtained independently by Saff and Varga [14] and M. W. Golitschek [6].

**Theorem [6, 14]:** For any \( 0 < \theta < 1 \) and any function \( f \in C[0,1] \) with \( f(x) = 0 \), \( 0 \leq x \leq \theta^2 \), there exists a sequence \( \{P_n(x)\}_{n=1}^{\infty} \) of polynomials in \( I_\theta \) such that \( \lim_{n \to \infty} P_n(x) = f(x) \) uniformly on \([0, 1]\).

Thus, it was established that \( \Delta(\theta) = \theta^2 \).

**b) Location of the maximum of an incomplete polynomial:**

Let \( P \) be any polynomial in \( I_\theta \), then the sequence \( \{Q_n\}_{n=1}^{\infty} \), \( Q_n(x) = \left( \frac{P(x)}{\|P\|_{[0,1]}} \right)^n \), satisfies the hypothesis of Lorentz theorem, which is stated in part (a) above. Since \( \Delta(\theta) \geq \theta^2 \), we have

\[ \lim_{n \to \infty} Q_n(x) = 0 \quad \text{for} \quad x \in [0, \theta^2), \]
but \( \| Q_n \|_{[0,1]} = 1 \), so if \( |Q_n(\xi)| = 1 \), then \( \xi \geq \theta^2 \). This argument of Saff [9], using Lorentz theorem, proves that if \( \xi(P) = \min\{\xi \in (0,1] : |P(\xi)| = \|P\|_{[0,1]} \} \), then

\[
\inf\{\xi(P) : P \not= 0, P \in I_\theta\} \geq \theta^2,
\]

and by taking \( \theta = \frac{s}{n} \), Saff deduced that, for \( P(x) = \sum_{k=s}^{n} a_k x^k \) and \( P \not= 0 \),

\[
\min\{\xi \in (0,1] : |P(\xi)| = \|P\|_{[0,1]} \} \geq (\frac{s}{n})^2.
\]

Also, using the above theorem of Saff, Varga, and Golitschek (stated in (a) above), Saff showed that there are polynomials in \( I_\theta \) which attains their maximum value at points arbitrary close to \( \theta^2 \). This proves the following theorem:

**Theorem [9]:** If \( 0 < \theta < 1 \), and \( \xi(P) \) is defined as above, then

\[
(1.3) \quad \inf\{\xi(P) : P \not= 0, P \in I_\theta\} = \theta^2
\]

Nothing seems to be known about the degree of a polynomial in \( I_\theta \) which attains its maximum close to \( \theta^2 \). The problem can be phrased as follows: given \( c > 0 \), what is the smallest degree \( n \) of a polynomial \( P_n \in I_\theta \) such that \( \|P_n\|_{[0,1]} = \|P_n\|_{[0,c\theta^2]} \)?

We obtained an answer to this question, in case \( c > \frac{3\pi^2}{16} \) and \( \theta \) sufficiently small; and in case \( c = 3 \) and arbitrary \( \theta \) (Lemma (8)). These results were essential in obtaining the lower bound for \( M_n \).

(c) A general version of the problem:

The problem of determining the location of the maximum of an incomplete polynomial can be viewed as a special case of the following problem:
Let $W_n(x) \in C[0,1]$ be such that $W_n(0) = 0$, $W_n(x) > 0$, for $x \in (0,1]$. Find an estimate for the number $\xi^*_n$.

$$\xi^*_n = \inf \{ \xi(P, W_n) : P \neq 0, P \in \pi_n \},$$

where $\xi(P, W_n) = \min \{ \xi \in (0,1] : |W_n(\xi)P(\xi)| = ||W_nP||_{[0,1]} \}$.

Special cases of this general problem are, for example,

i) as mentioned, the problem of incomplete polynomials, with $W_n(x) = x^\alpha$;

ii) the problem treated by Saff and Mhaskar in [9] for the weight $W_n(x) = \exp(-|x|^\alpha)$, $\alpha > 0$, on the interval $(-\infty, \infty)$ (which is easily reduces to a particular weight on $[0,1]$);

iii) our problem (as will be seen in Section (5) of this chapter), reduces to the case $W_n(x) = \prod_{k=0}^{\infty} (2k+1)x^{2k}$ on $[0,\alpha]$, for some $\alpha > 0$.

d) The Saff-Varga Lemma:

For most weights it is not possible to explicitly find the number $\xi^*_n$ defined in (c) above. Therefore, the following Lemma of Saff and Varga [12] is of great computational value, since, using this lemma, $\xi^*_n$ can be obtained with high accuracy by means of Remes Algorithm.

Lemma [10]: Suppose the weight function $W(x) \in C[0,1]$ satisfies $W(0) = 0$ and $W(x) > 0$ for $x \in [0,1]$. For each $n$, let

$$P^*_n(x) = P^*_n(W; x) = x^n - \sum_{i=0}^{n-1} c_i^* x^i$$

be the unique extremal polynomial for the Chebyshev problem
\[
\inf\{\| W(x)x^n - \sum_{i=0}^{n-1} c_ix^i \|_{[0,1]} : (c_0, \ldots, c_{n-1}) \in \mathbb{R}^n \},
\]

and set

\[
\xi_n^* = \min\{x \in (0, 1] : |W(x)P_n^*(x)| = \|WP_n^*\|_{[0,1]} \}.
\]

If \(P(x)\) is any real Lacunary polynomial of the form

\[
P(x) = \sum_{i=0}^{n} b_ix^{\mu_i}
\]

then

\[
|P(x)| \leq \frac{\|WP\|_{[0,1]}}{\|WP_n^*\|_{[0,1]}} |P_n^*(x)|, \quad \text{for all} \quad 0 \leq x \leq \xi_n^*.
\]

Consequently, if \(\xi \in (0, 1]\) satisfies \(W(\xi)P(\xi)| = \|WP\|_{[0,1]}\), where \(P \neq 0\) and of the above form, then

\[
\xi_n^* \leq \xi.
\]

4. Correction to a proof of G. G. Lorentz:

As mentioned before, one part of the proof of our Main Theorem is an adaptation of the Lorentz's proof of the inequality \(\Delta(\theta) \geq \theta^2\) in [7]. However, that proof, as presented in [7] contains, in its final part, a serious gap (or error). Namely, Lorentz shows that an estimate of the type

\[
(*) \quad \lim_{n \to \infty} \frac{1}{n} \log |P_n(x)| \leq A(r, a) + o(1 - r), \quad r \to 1
\]
holds for a sequence of polynomials $P_n$, for each $0 < a < \theta^2$, where $0 < r < 1$, $A(r) < 0$ and $r \to 1$ as $x \to a$. He concludes "It follows that for each $a < \theta^2$ and some $\varepsilon > 0$, $P_n(x) \to 0$ uniformly on $[a - \varepsilon, a]$. By "induction in the continuum" we obtain $P_n(x) \to 0$ on $[0, \theta^2]$." (There is a mistake in this which is easy to correct. Polynomials $P_n$ converge uniformly on the interval $[a - \varepsilon, a - \frac{\varepsilon}{2}]$, but not necessarily on $[a - \varepsilon, a]$).

The serious gap (or the error) is in the implicit claim that $\varepsilon$ can be chosen independently of $a$. Analyzing the derivation of the formula (*), we see that $o(1 - r)$ term in (8) comes from estimates of derivatives of functions $\log(\frac{1+a}{2} + \frac{1-a}{2} \cos t)$, and so it is not even plausible that the $o$-term is uniform for $a$ in a neighborhood of zero.

We can, however, salvage this proof of Lorentz in the following way: first, we apply another theorem of Lorentz (Theorem 5 in the same article, [7]) to show there exists $\delta > 0$ such that $P_n(x)$ converges uniformly to zero on $[0, \delta]$; then we show that for each $a$, $\delta \leq a < \theta^2$, there is $\varepsilon > 0$ (independent of $a$ and dependent only on $\delta$) such that $P_n$ converges uniformly to zero on each interval $[a - \varepsilon, a - \frac{\varepsilon}{2}]$.

5. Comments on the proof of the Main Theorem:

There are two crucial steps in the proof of the Main Theorem. The first crucial step is the transcription of our original problem (as stated in (d) of Section 2 above) into the following form:

Let $W_n(x) = \prod_{k=0}^{n} \frac{\sqrt{2x+1}}{(x+k+1)}$,

$$\xi_n = \sup\{\xi(P, W_n) : P \neq 0, \ P \in \pi_n\},$$

where $\xi(P, W_n) = \max\{\xi \geq n : |W_n(\xi)P(\xi)| = \|W_nP\|_{[n, \infty)}\}$. 
Give an estimate for $\xi_n$.

This step makes possible the use of some techniques developed in the studies on incomplete polynomials. For example, with some modification and adaptation (including the correction we mentioned in Section 4. above), it is possible to follow the method of Lorentz [7].

The second crucial step resides in Lemma (7), which we have derived by explicitly finding the Chebyshev polynomials for the weight $\omega$ on $[0, 1]$ (Lemma 6).

Lemma (7) makes it possible to replace the weight $\omega$ by the weight $W_n(\omega)$, and so to construct a counterexample, which gives a lower bound for $M_n$.

Finally, Lemma (8) can be viewed as a complement to Saff's Theorem, which we stated in part (b) of Section 3 above.

6. On the Numerical results:

In Chapter III we give numerical values for $M_n$ (of course $M_n$ depends on the set of approximating monomials). We will calculate $M_n$ when the set of approximating monomials is $\{ax^t : a \in \mathbb{R} ; t \in \mathbb{R} \text{ and } t \geq 0\}$. We will use $M_{n,0}$ instead of $M_n$.

To determine $M_{n,0}$, we need only to determine, by Saff-Varga Lemma, the Chebyshev polynomial for a certain weight and the first point at which this Chebyshev polynomial attains its maximum.

The computation of the Chebyshev polynomials is done by using the Remes Algorithm.
CHAPTER II

THE MAIN THEOREM

Statement of the Main Theorem:

For any polynomial $P$ and for any real $t$, $t > -\frac{1}{2}$, we let

$$E(P; t) = \inf \{ \| P(x) - c x^t \|_2 : c \in \mathbb{R} \}.$$ 

For $\gamma > -\frac{1}{2}$, we let

$$E_\gamma(P) = \inf \{ E(P; t) : t \in \mathbb{R}, t \geq \gamma \},$$

$$M_\gamma(P) = \sup \{ t : t \in \mathbb{R}, t \geq \gamma ; E(P; t) = E_\gamma(P) \},$$

and for any positive integer $n$,

$$(*) \quad M_{n, \gamma} = \sup \{ M_\gamma(P) : P \in \pi_n \}.$$ 

Then for all $n > 1$ we have,

$$\frac{1}{4}(n + 1)^3 \leq M_{n, n} \leq 6(n + 1)^3.$$ 

Remarks:

1) $M_\gamma(P)$ is well defined, since the set $\{ t : t \in \mathbb{R}, t \geq \gamma ; E(P; t) = E_\gamma(P) \}$ is non-empty, this follows from Lemma (1) below because $E(P; t)$ attains its infimum $E_\gamma(P)$. 

12
It also follows from Lemma (1) that the supremum in the definition of \( M_\gamma(P) \) is attained.

2) \( M_{n,\gamma} \) is finite for every \( \gamma \), this follows from Lemma (1), and our proof of the main theorem depends essentially on this fact.

3) It would be natural to consider also \( M_\gamma(P) \) and \( M_{n,\gamma} \), for \( \gamma = -\frac{1}{2} \), provided one defines \( E_{-\frac{1}{2}}(P) \) and \( M_{n,-\frac{1}{2}} \) by

\[
E_{-\frac{1}{2}}(P) = \inf\{E(P; t) : t \in R, \ t > -\frac{1}{2}\},
\]

\[
M_{-\frac{1}{2}}(P) = \sup\{t : t \in R, \ t > -\frac{1}{2}, \ E(P; t) = E_{-\frac{1}{2}}(P)\}, \quad \text{and}
\]

\[
M_{n,-\frac{1}{2}} = \sup\{M_{-\frac{1}{2}}(P) : P \in \pi_n\}.
\]

4) Since \( M_{n,\gamma} \) is an increasing function of \( \gamma \), the inequality

\[
M_{n,\gamma} \leq 6(n + 1)^3
\]

holds for all \( \gamma, -\frac{1}{2} < \gamma < n \), in particular for \( \gamma = 0 \) and \( \gamma = -\frac{1}{2} \). However, it is an open problem whether \( M_{n,0} \) or \( M_{n,-\frac{1}{2}} \) are still bounded below by a constant multiple of \((n + 1)^3\).

2. Equivalent Form of The Main Theorem:

Our main problem is to give an estimate for \( M_{n,n} \). In this section (in particular, in Corollary (2) which is a consequence of Theorem (1) and Lemma (1) below), we show that this is equivalent to finding an estimate for \( \xi_{n,n} \), where \( \xi_{n,n} \) is defined as in Corollary (2).
In addition, we establish in this section several simple facts that we shall need both for the proof of the Main Theorem and for the computational part.

**Theorem (1):** Let $P \in \pi_n$. There exists a polynomial $Q \in \pi_n$ such that, for $t > -\frac{1}{2}$

\[(II.1) \quad E(P; t) = \| P \|^2 - \left\{ U_n(t)Q(t) \right\}^2,\]

where $U_n(t) = \frac{\sqrt{2t+1}}{\prod_{k=0}^{n} (t+k+1)}$.

Moreover,

i) The mapping $P \rightarrow Q$ is a bijection on $\pi_n$.

ii) $Q(x) = \sum_{k=0}^{n} \frac{P^{(k)}(0)}{k!} \prod_{i \neq k}^{n} (x + i + 1)$.

iii) $P(x) = \sum_{k=0}^{n} (-1)^k \frac{Q(-k-1)}{k!(n-k)!} x^k$.

**Corollary (1):** There are bijections $P \rightarrow R$ and $P \rightarrow S$ on $\pi_n$ such that

\[(II.2) \quad E(P; \frac{x-1}{2}) = \| P \|^2 - 4^{n+1} \left\{ W_n(x)R(x) \right\}^2, \quad x > 0\]

where $W_n(x) = \frac{\sqrt{x}}{\prod_{k=0}^{n} (x+2k+1)}$ and $R(x) = Q(\frac{x-1}{2})$.

\[(II.3) \quad E(P; \frac{y-1}{2}) = \| P \|^2 - 4^{n+1} \left\{ V_n(y)S(y) \right\}^2, \quad y > 0\]

where $V_n(y) = \frac{\sqrt{y}}{\prod_{k=0}^{n} ((2k+1)y+1)}$, and $S(y) = y^n R(\frac{1}{y})$. 

Proof of Theorem (1): Let $P(x) = \sum_{k=0}^{n} a_k x^k$, and $t > -\frac{1}{2}$. By definition, $E(P; t)$ is the $L_2$-distance from $P$ to the subspace spanned by $x^t$, so by the well-known distance formula in inner product spaces, we have

$$E(P; t) = \frac{G(x^t, P)}{G(x^t)}$$

where $G(f_1, \ldots, f_m)$ is the Gram determinant on $\{f_1, \ldots, f_m\}$. This gives

$$(II.4) \quad E(P; t) = \frac{\| P \|_2^2 \| x^t \|_2^2 - (x^t, P)^2}{\| x^t \|_2^2} = \| P \|_2^2 - (2t + 1) \left( \sum_{k=0}^{n} \frac{a_k}{t + k + 1} \right)^2.$$

We write

$$(II.5) \quad \sum_{k=0}^{n} \frac{a_k}{t + k + 1} = \frac{1}{\prod_{k=0}^{n}(t + k + 1)} \sum_{k=0}^{n} a_k \prod_{i=0}^{n} (t + i + 1).$$

Then from (II.4) and (II.5), since $a_k = \frac{P^{(k)}(0)}{k!}$, the formulas in (II.1) and (ii) follow.

In (ii), if we let $x = -j - 1$, we get

$$Q(-j - 1) = \frac{P^{(j)}(0)}{j!} \prod_{i=0}^{n} (i - j) = (-1)^j P^{(j)}(0)(n - j)!,$$

from which (iii) follows.

Finally, the bijection follows from (ii) and (iii).

Lemma (1): Let $E_S(P) = \inf\{E(P; t) : t \in S\}$. 

i) For any set \( S \), \( S \subseteq \{ t : t > -\frac{1}{2} \} \), there are constants \( c_n = c_n(S) \) and \( T_n = T_n(S) \) such that if \( P \in \pi_n \), then

\[
E(P; t) > E_S(P), \text{ for } t \in S \cap \left[ \{ x : x < c_n \} \cup \{ x : x > T_n \} \right].
\]

ii) If \( S \) is relatively closed with respect to \( \{ t : t > -\frac{1}{2} \} \), then for every \( P \in \pi_n \), there is \( t \in S \) such that

\[
E(P; t) = E_S(P),
\]

i.e., the best approximation exists.

**Proof:** Let \( P \in \pi_n \), and \( P(x) = \sum_{i=0}^{n} a_i x^i \). We may assume that \( \| P \|_2 = 1 \), so there is a constant \( K(n) \) such that

\[
| a_k | \leq K(n) \text{ for } k = 0, \ldots, n.
\]

If \( F_P(t) = \sqrt{2t + 1} \left| \sum_{i=0}^{n} \frac{a_i}{i+i+1} \right| \), then

\[
| F_P(t) | \leq (n + 1)K(n) \frac{\sqrt{2t + 1}}{t + 1} \text{ for } t > -\frac{1}{2}.
\]

The last inequality implies that \( \lim_{t \to -\frac{1}{2}} E_P(t) = 0 \) uniformly for \( P \in \pi_n \) and \( \| P \|_2 = 1 \), and \( \lim_{t \to \infty} F_P(t) = 0 \) uniformly for \( P \in \pi_n \) and \( \| P \|_2 = 1 \). Therefore, there exist \( c_n = c_n(S) \) and \( T_n = T_n(S) \) such that
(II.6) \[ |F_P(t)| < \|F_P\|_S \quad \text{if} \quad t \in S \cap \{x : x < c_n\} \]
\[ \cup \{x : x > T_n\}, \]
(recall that, \( \|F_P\|_S = \sup_{t \in S} |F_P(t)| \)).

Since, \( E_S(P) = \inf \{E(P; t) : t \in S\} \), then by (II.4) we have,

(II.7) \[ E_S(P) = \|P\|_2^2 - \|F_P\|_2^2 = 1 - \|F_P\|_2^2. \]

Thus, (i) follows from (II.6) and (II.7).

If \( S \) is relatively closed with respect to \( \{t : t > -\frac{1}{2}\} \), then since \( F_P \neq 0 \) and \( \lim_{t \to -\frac{1}{2}} F_P(t) = \lim_{t \to \infty} F_P(t) = 0 \), (iii) follows by the continuity of \( F_P \).

Corollary (2): For \( \gamma > -\frac{1}{2} \) and \( P \in \pi_n \), we let

\[ \lambda_\gamma(P) = \max\{\xi : |U_n(\xi)P(\xi)| = \|U_nP\|_{[\gamma, \infty)}, \xi \geq \gamma\}, \]
\[ \xi_\gamma(P) = \max\{\xi : |W_n(\xi)P(\xi)| = \|W_nP\|_{[2\gamma + 1, \infty)}, \xi \geq 2\gamma + 1\}, \]
\[ \mu_\gamma(P) = \min\{\xi : |V_n(\xi)P(\xi)| = \|V_nP\|_{[0, \frac{1}{2\gamma + 1}]}, 0 \leq \xi \leq \frac{1}{2\gamma + 1}\}, \]
\[ \lambda_{n,\gamma} = \max\{\lambda_\gamma(P) : p \in \pi_n\} \]
\[ \xi_{n,\gamma} = \max\{\xi_\gamma(P) : P \in \pi_n\}, \quad \text{and} \]
\[ \mu_{n,\gamma} = \min\{\mu_\gamma(P) : P \in \pi_n\}, \]

where \( U_n, W_n, \) and \( V_n \) are as defined in Theorem (1) and Corollary (1).
If $M_{n,\gamma}$ is defined as in Theorem (1), then

$$(II.8) \quad M_{n,\gamma} = \lambda_{n,\gamma} = \frac{\xi_{n,\gamma} - 1}{2} = \frac{1}{\mu_{n,\gamma}} - 1.$$

**Proof:** We show the first equality in (II.8), the rest is obvious.

By (II.1),

$$E(P; t) = \| P \|_2^2 - \{U_n(t)Q(t)\}^2,$$

so, for $\gamma > -\frac{1}{2}$, we have

$$E_{\gamma}(P) = E(P; t) \text{ if and only if } |U_n(t)Q_n(t)| = \| U_nQ \|_{(\gamma, \infty)}.$$

Since the mapping $P \rightarrow Q$ is a bijection by Theorem (1), the first equality in (II.8) follows.

3. The Poisson Transform and The Conjugate of Smooth Functions

In the introduction (Part 3-a), we have mentioned a very important result of G.G. Lorentz on incomplete polynomials: $\Delta(\theta) \geq \theta^2$, proved in [7]. One step of the proof is showing that if

$$(II.9) \quad f(t) = \log(\frac{1 - a}{2} \cos t + \frac{1 + a}{2}), \quad a > 0$$

then
(II.10) \[ P_r(f; t) = f(t) - (1 - r)(\tilde{f})'(t) + o(1 - r), \quad r \to 1^- \]

and

(II.11) \[ (\tilde{f})'(x) = \frac{1}{2\pi} \int_0^\pi [f'(x - t) - f'(x + t)] \cot \frac{t}{2} dt, \]

where \( P_r(f; t) \) is the Poisson transform of \( f \) at \( t \),

\[ P_r(f; t) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(1 - r^2)f(\theta)}{1 - 2r \cos(\theta - t) + r^2} d\theta \]

and \( \tilde{f} \) is the conjugate function of \( f \),

\[ \tilde{f}(x) = \frac{1}{2\pi} \int_0^\pi [f(x - t) - f(x + t)] \cot \frac{t}{2} dt. \]

Lorentz's proof of (II.10) and (II.11) works, not only for the particular function (II.9), but for a wide class of functions. However, for the proof of the Main Theorem we need more precise results, including an estimate of the remainder term in the following Lemma.

**Lemma (2):** If \( f \) is periodic function of period \( 2\pi \) and has a bounded fourth derivative, then for all \( t \) and all \( r \in [0, 1) \) we have

(II.12) \[ P_r(f; t) = f(t) - (1 - r)(\tilde{f})'(t) + (1 - r)^2 H(f; r, t) \]

where \( |H(f; t, r)| \leq M = \max_t |f^{(4)}(t)|.\)
Proof: Let \( c_k = \frac{1}{2\pi} \int_0^{2\pi} f(t)e^{-ikt} dt \); \( k = 0, \pm 1, \pm 2, \ldots \). Integrating by parts four times we obtain,

\[
|c_k| \leq \frac{1}{2\pi |k|^4} \int_0^{2\pi} |f^{(4)}(t)| \, dt \leq \frac{M}{|k|^4}; \quad k = \pm 1, \pm 2, \ldots.
\]

So if we let \( C_k(t) \) be the general term of the Fourier series of \( f \) \( (C_k(t) = c_0 \) if \( k = 0 \), \( C_k(t) = c_k e^{ikt} + c_{-k} e^{-ikt}, k > 0 \)), then

\[
|C_k(t)| \leq \frac{2M}{k^4}; \quad k = 1, 2, \ldots
\]

and

\[
f(t) = \sum_{k=0}^{\infty} C_k(t).
\]

Since \( P_r(f; t) = \sum_{k=0}^{\infty} r^k C_k(t) \) for \( r \in [0, 1) \), then for \( r \in [0, 1) \) we have

\[
(II.13) \quad \frac{P_r(f; t) - f(t)}{1 - r} = - \sum_{k=1}^{\infty} C_k(t) \frac{r^k - 1}{r - 1} = - \sum_{k=1}^{\infty} (r^{k-1} + \cdots + r + 1) C_k(t).
\]

Since \( f \) is differentiable, \( \tilde{f} \) is bounded, and so it is integrable. Therefore, (see [6], page 156), the Fourier Series of \( \tilde{f} \) is

\[
-i \sum_{-\infty}^{\infty} (\text{sgn } k)c_k e^{ikt},
\]

and since \( |c_k| \leq \frac{M}{|k|^4} \), this Fourier Series converges uniformly, and so we have
\[
\bar{f}(t) = -i \sum_{-\infty}^{\infty} (\text{sgn } k) c_k e^{ikt} \quad \text{at every } t.
\]

Also, the differentiated series is uniformly convergent, thus

\[(\bar{f})'(t) = \sum_{-\infty}^{\infty} |k| c_k e^{ikt} = \sum_{k=1}^{\infty} k C_k(t). \tag{I.I.4} \]

From (I.I.3) and (I.I.4), and for \( r \in [0, 1) \), we have

\[
\frac{P_r(f; t) - f(t)}{1 - r} + (\bar{f})'(t) = -\sum_{k=2}^{\infty} [r^{k-1} + \ldots + r - (k - 1)] C_k(t).
\]

We write

\[
r^{k-1} + \ldots + r - (k - 1) = (r^{k-1} - 1) + \ldots + (r - 1)
\]

\[
= (r - 1)\{(r^{k-2} + \ldots + r + 1) + (r^{k-3} + \ldots + r + 1) + \ldots + (r + 1) + 1\}
\]

\[
= (r - 1)\{r^{k-2} + 2r^{k-3} + \ldots + (k - 2)r + k - 1\}
\]

\[
= (r - 1)a_k(r).
\]

Therefore, for \( r \in [0, 1) \) we have

\[(\text{II.15}) \quad \frac{P_r(f; t) - f(t)}{1 - r} + (\bar{f})'(t) = (1 - r) \sum_{k=2}^{\infty} a_k(r) C_k(t). \]

Since \(|a_k(r)C_k(t)| \leq \frac{k-1}{k^3} M\) for \( r \in [0, 1) \), the series in (II.15) converges uniformly in \( t \) and \( r \).

Let \( H(f; r, t) = \sum_{k=2}^{\infty} a_k(r) C_k(t) \).
Lemma (3): If \( f \) is periodic of period \( 2\pi \) and has a bounded second derivative, then (II.11) holds

i.e. \( (\bar{f})'(x) = \frac{1}{2\pi} \int_0^\pi [f'(x - t) - f'(x + t)] \cot \frac{t}{2} \, dt \).

Proof: From the definition of \( \bar{f} \), we have

\[
\bar{f}(x) = \frac{1}{2\pi} \int_0^\pi [f(x - t) - f(x + t)] \cot \frac{t}{2} \, dt \\
= -\frac{1}{2\pi} \lim_{\epsilon \to 0} \left( \int_{-\pi}^{-\epsilon} + \int_{\epsilon}^{\pi} \right) f(x + t) \cot \frac{t}{2} \, dt.
\]

So integrating by parts and noting that \([f(x+\epsilon) - f(x-\epsilon)] \log \sin \frac{\epsilon}{2} \to 0\), as \( \epsilon \to 0 \)

we get

\[
\bar{f}(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f'(x + t) \log | \sin \frac{t}{2} | \, dt.
\]

It follows that

\[
\frac{\bar{f}(x + h) - \bar{f}(x)}{h} = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{f'(x + h + t) - f'(x + t)}{h} \log | \sin \frac{t}{2} | \, dt
\]

Since \( \left| \frac{f'(x + h + t) - f'(x + t)}{h} \right| \leq \max | f''(x) | = M \), and \( \log | \sin \frac{t}{2} | \) is integrable, then by Lebesgue Dominated Convergence Theorem, we have
\[
\lim_{h \to 0} \frac{\tilde{f}(x + h) - \tilde{f}(x)}{h} = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{f'(x + t + h) - f'(x + t)}{h} \log |\sin \frac{t}{2}| \, dt - \frac{1}{\pi} \int_{-\pi}^{\pi} f''(x + t) \log |\sin \frac{t}{2}| \, dt
\]

\[
= \lim_{\varepsilon \to 0} \frac{1}{\pi} \left( \int_{-\pi}^{-\varepsilon} + \int_{\varepsilon}^{\pi} \right) f''(x + t) \log |\sin \frac{t}{2}| \, dt
\]

\[
= \lim_{\varepsilon \to 0} -\frac{1}{2\pi} \left( \int_{-\pi}^{-\varepsilon} + \int_{\varepsilon}^{\pi} \right) f'(x + t) \cot \frac{t}{2} \, dt
\]

\[
= \frac{1}{2\pi} \int_{0}^{\pi} [f'(x - t) - f'(x + t)] \cot \frac{t}{2} \, dt.
\]

**Remark:** Lemma (3) is just saying that, under certain conditions, the derivative of the conjugate function is the conjugate of the derivative of the function,

i.e. \( \tilde{f}'(t) = (\tilde{f})'(t) \) at every \( t \).

**Lemma (4):** Let \( f(t) = \log(A - B \cos t) \), \( A > |B| \). Then for \( r \in [0, 1) \) we have

\[
(II.16) \quad \rho_r(f; \pi) = \log(A + B) + (1 - r) \left( \sqrt{\frac{A - B}{A + B}} - 1 \right) + (1 - r)^2 H(A, B; r)
\]

where \( |H(A, B; r)| \leq C\delta^{-4} \) (\( C \) is an absolute constant) provided \( 1 - \frac{|B|}{A} \geq \delta \), \( \delta > 0 \).

**Proof:** Since \( f \) satisfies the hypothesis of Lemma (2), then by (II.12) we have

\[
\rho_r(f; \pi) = f(\pi) - (1 - r)(\tilde{f})'(\pi) + (1 - r)^2 H(f; r, \pi),
\]
and

\[ |H(f; r, \pi)| \leq \max_t |f^{(4)}(t)|. \]

We will use \( H(A, B; r) \) in place of \( H(f; r, \pi) \) since \( H(f; r, \pi) \) depends only on \( A, B \) and \( r \).

Since \( f(\pi) = \log(A + B) \), (II.16) will follow if we show that \( (\tilde{f})'(\pi) = 1 - \sqrt{\frac{A - B}{A + B}} \), and

\[ \max_t |f^{(4)}(t)| \leq C\delta^{-4} \text{ whenever } 1 - \frac{|B|}{A} \geq \delta. \]

By Lemma (3),

\[ (\tilde{f})'(\pi) = \frac{1}{2\pi} \int_0^\pi [f'(\pi - t) - f'(\pi + t)] \cot \frac{t}{2} dt \]

so

\[ (\tilde{f})'(\pi) = \frac{1}{2\pi} \int_0^\pi \left[ \frac{B \sin(\pi - t)}{A - B \cos(\pi - t)} - \frac{B \sin(\pi + t)}{A - B \cos(\pi + t)} \right] \cot \frac{t}{2} dt \]

\[ = \frac{B}{\pi} \int_0^\pi \frac{\sin t}{A + B \cos t} \cot \frac{t}{2} dt. \]

Letting \( u = \tan \frac{t}{2} \), we get

\[ (\tilde{f})'(\pi) = \frac{4B}{\pi} \int_0^\infty \left( \frac{1}{B + A + (A - B)u^2} \right) \left( \frac{1}{1 + u^2} \right) du \]

\[ = \frac{2}{\pi} \int_0^\infty \left( \frac{1}{1 + u^2} - \frac{1}{A + B + u^2} \right) du \]

\[ = 1 - \sqrt{\frac{A - B}{A + B}} \]
Since \( f'(t) = \frac{\lambda \sin t}{1 - \lambda \cos t}, \lambda = \frac{B}{A} \), we get

\[
f^{(4)}(t) = \frac{P(\lambda)}{(1 - \lambda \cos t)^4},
\]

where \( P(\lambda) \) is a polynomial in \( \lambda \) (with coefficients trigonometric polynomials in \( t \)).

Then since \( |\lambda| < 1 \), we have \( \max |f^{(4)}(t)| \leq C(1 - |\lambda|)^{-4} \).

So if \( 1 - \frac{|B|}{A} \geq \delta \), then \( |H(A, B; r)| \leq C\delta^{-4} \).

4. The Rahman-Schmeisser Lemma:

Lemma [8]: Let \( P \in \pi_n \) and \( M(x) \) a continuous positive function on some interval \([a, b]\) such that

\[
|P(x)| \leq M(x) \quad \text{for all } x \in [a, b].
\]

Then,

i) for \( c < b \),

\[
|P(c)| \leq \frac{1}{r^n} \exp \left\{ \frac{1}{2\pi} \int_0^{2\pi} \frac{(1 - r^2) \log M \left( \frac{b+c}{2} + \frac{b-c}{2} \cos t \right)}{1 + 2r \cos t + r^2} \, dt \right\},
\]

where \( r = \delta - \sqrt{\delta^2 - 1}, \delta = \frac{b+c-2a}{b-a} \).

ii) For \( c > a \),

\[
|P(c)| \leq \frac{1}{r^n} \exp \left\{ \frac{1}{2\pi} \int_0^{2\pi} \frac{(1 - r^2) \log M \left( \frac{a+c}{2} - \frac{b-a}{2} \cos t \right)}{1 + 2r \cos t + r^2} \, dt \right\},
\]

where \( r = \delta - \sqrt{\delta^2 - 1}, \delta = \frac{2c-b-a}{b-a} \).
5. The Upper Bound:

Lemma (5): Let $a = 2n + 1$ and $b \geq 12(n + 1)^3$.

i) There exists $\mu_n > 0$ such that,

\[(\text{II.17}) \quad \frac{1}{2} - \sum_{k=0}^{n} \sqrt{\frac{a + 2k + 1}{b + 2k + 1}} \geq \mu_n.\]

ii) For $c > b$, $\delta = \frac{2c - b - a}{b - a}$, and $\tau = \delta - \sqrt{\delta^2 - 1}$, we have

\[(\text{II.18}) \quad 1 - \tau \leq \sqrt{c - b}.\]

Proof:

i) Let $b_n = 12(n + 1)^3$, and

\[
\mu_n = \frac{1}{2} - \sum_{k=0}^{n} \sqrt{\frac{a + 2k + 1}{b_n + 2k + 1}}.
\]

We have

\[
\sum_{k=0}^{n} \sqrt{\frac{a + 2k + 1}{b_n + 2k + 1}} \leq \sqrt{\frac{2}{b_n}} \sum_{k=0}^{n} \sqrt{n + k + 1}
\]

\[
\leq \sqrt{\frac{2}{b_n}} \int_{0}^{n+1} \sqrt{x + n + 1} \, dx = \frac{8 - \sqrt{8}}{3\sqrt{b_n}} \sqrt{(n + 1)^3}
\]

\[
= \frac{8 - \sqrt{8}}{6 \sqrt{3}} \leq .498
\]

so $\mu_n \geq .002$. Since $\sum_{k=0}^{n} \sqrt{\frac{a + 2k + 1}{b + 2k + 1}}$ is decreasing in $b$, (II.17) follows.
\[ i ) \quad 1 - r = 1 - \delta + \sqrt{\delta^2 - 1} = \sqrt{\delta - 1(\sqrt{\delta + 1} - \sqrt{\delta - 1})} = \sqrt{\delta - 1} \frac{2}{\sqrt{\delta + 1} + \sqrt{\delta - 1}}. \]

Since \( \delta - 1 = 2 \frac{c-a}{b-a} \) and \( \delta + 1 = \frac{c-a}{b-a} \), we have

\[ 1 - r = \frac{2}{\sqrt{c-a} + \sqrt{c-b}} \sqrt{c-b}. \]

But \( c - a \geq 12(n+1)^3 - 2n - 1 > 4 \), so \( 1 - r \leq \sqrt{c-b} \).

**Proof of The Upper Bound:**

We will prove that

\[(II.19) \quad M_{n,n} \leq 6(n + 1)^3, \]

where \( M_{n,n} \) as defined by (*) in the statement of the main theorem.

By Corollary (2), \( M_{n,n} = \frac{\xi_{n,n} - 1}{2} \). So we need to show that

\[ \xi_{n,n} \leq 12(n + 1)^3 + 1 \]

where \( \xi_{n,n} = \max\{\xi(P) : P \in \pi_n\} \),

\[ \xi_n(P) = \max\{\xi : |W_n(\xi)P(\xi)| = \|W_nP\|_{2n+1,\infty}, \xi \geq 2n + 1\}, \quad \text{and} \]

\[ W_n(x) = \frac{\sqrt{x}}{\prod_{k=0}^{n} (x + 2k + 1)}. \]
Thus, it is clear that, to prove (II.19), it is enough to prove the following:

\[ \text{Let } P \in \pi_n, \quad F_n(x) = W_n(x)P(x) \quad \text{and } \parallel F_n \parallel_{2n+1,\infty} = 1. \]

Show that if \( \xi > 12(n+1)^3 \), then \( |F_n(\xi)| < 1. \)

The proof of (***) will follow from (1) and (2) below.

1) By Lemma (1), there is \( T_n \) such that if \( \xi > T_n \), then \( |F_n(\xi)| < 1. \) Since if \( T_n \leq 12(n+1)^3 \), our theorem is proved, so we assume that \( T_n > 12(n+1)^3 \).

2) There is \( \epsilon_n > 0 \) dependent only on \( n \) such that if \( 12(n+1)^3 < b \leq T_n \) and \( c \in (b, b+\epsilon_n) \), then \( |F_n(c)| < 1. \) (Observe that \( \{c : 12(n+1)^3 < c \leq T_n\} \subseteq U \{\{c : b < c < b+\epsilon_n\} : 12(n+1)^3 \leq b \leq T_n\}. \))

So we let \( F_n \) as in (**), and \( T_n \) as in (2), then we have

\[ |F_n(x)| \leq 1, \quad x \geq 2n+1 \]

i.e., \( |P(x)| \leq M(x), \quad x \geq 2n+1 \)

where \( M(x) = \frac{1}{W_n(x)} = x^{-\frac{1}{2}} \prod_{k=0}^{n} (x+2k+1) \). In particular for \( b, \) \( 12(n+1)^3 \leq b \leq T_n \),

\[ |P(x)| \leq M(x), \quad x \in [2n+1,b]. \]

By the Rahman-Schmeisser Lemma (Section 4 of this chapter), we have for \( c > b \),

\[ (II.20) \quad |P(c)| \leq \frac{1}{r^n} \exp\{P_c(\log M(\frac{b+a}{2} - \frac{b-a}{2} \cos(\cdot)); \pi)\}, \]

where \( r = \delta - \sqrt{\delta^2 - 1}, \delta = \frac{2c-a-b}{b-a}, \) and \( a = 2n+1. \) Since
\[
\log M\left(\frac{b+a}{2} - \frac{b-a}{2} \cos t\right) = -\frac{1}{2} \log\left(\frac{b+a}{2} - \frac{b-a}{2} \cos t\right) + \sum_{k=0}^{n} \log\left(\frac{b+a}{2} + 2k + 1 - \frac{b-a}{2} \cos t\right),
\]

then if we set,

\[
f_{0,b}(t) = -\frac{1}{2} \log\left(\frac{b+a}{2} - \frac{b-a}{2} \cos t\right), \quad \text{and} \quad f_{j+1,b}(t) = \log\left(\frac{b+a}{2} + 2j + 1 - \frac{b-a}{2} \cos t\right) \quad \text{for} \quad j = 0, \ldots, n;
\]

we have

\[
\log M\left(\frac{b+a}{2} - \frac{b-a}{2} \cos t\right) = \sum_{k=0}^{n+1} f_{k,b}(t),
\]

so we have

\[
(II.21) \quad P_r\left(\log M\left(\frac{b+a}{2} - \frac{b-a}{2} \cos t\right); \pi\right) = \sum_{k=0}^{n+1} P_r(f_{k,b}; \pi).
\]

Each \( f_{k,b} \) is of the form

\[
f_{k,b}(t) = C_k \log(A_k - B_k \cos t),
\]

for some \( A_k, B_k \) and \( C_k \), where \( A_k \) and \( B_k \) depend on \( b \). In particular,

\[
(II.22) \quad \begin{cases} 
C_0 = -\frac{1}{2}, & A_0 = \frac{b+a}{2}, & B_0 = \frac{b-a}{2} \\
C_k = 1, & A_k = \frac{b+a}{2} + 2k - 1, & B_k = \frac{b-a}{2} & \text{if} \ k = 1, \ldots, n+1 
\end{cases}
\]

since \( B_k = \frac{b-a}{2} \) and \( A_k \geq \frac{b+a}{2} \) for \( k = 0, 1, \ldots, n+1 \); then
(II.23) \[ 1 - \frac{B_k}{A_k} \geq 1 - \frac{b-a}{b+a} = \frac{2a}{b+a} \quad \text{for} \quad k = 0, \ldots, n+1. \]

But \( b \leq T_n \), so if we let \( \delta_n = \frac{2a}{T_n+a} \), then

\[ 1 - \frac{B_k}{A_k} \geq \delta_n > 0. \]

Thus, by (II.16) we have

\[
P_r(f_{k,b}; \pi) = C_k[\log(A_k + B_k) + (1-r) \left( \frac{A_k - B_k}{A_k + B_k} - 1 \right)] + \delta_n^2 H(A_k, B_k; \pi),
\]

(II.24) where \( |H(A_k, B_k; \pi)| \leq C\delta_n^{-4}. \)

So if we let \( K_n = (n+2)C\delta_n^{-4} \), then substituting (II.24) in (II.21) gives (notice that \( K_n \) is a constant depends only on \( n \)),

\[
P_r(\log M(\frac{b+a}{2} - \frac{b-a}{2} \cos(\cdot)); \pi)
\leq \sum_{k=0}^{n+1} C_k \left[ \log(A_k + B_k) + (1-r) \left( \frac{A_k - B_k}{A_k + B_k} - 1 \right) \right] + (1-r)^2 K_n.
\]

Substituting the values of \( A_k, B_k \) and \( C_k \) from (II.22) in the last inequality gives

(II.25) \[
P_r(\log M(\frac{b+a}{2} - \frac{b-a}{2} \cos(\cdot)); \pi)
\leq -\log \sqrt{b} + \sum_{k=0}^{n} \log(b + 2k + 1) + (1-r)^2 K_n
\]
\[ + (1-r) \left[ -\sqrt{\frac{a}{b}} - \frac{1}{2} + \sum_{k=0}^{n} \sqrt{\frac{a+2k+1}{b+2k+1}} \right].\]
Since $F_n(c) = P(c) \prod_{k=0}^{n} \frac{\sqrt{c}}{(c+2k+1)}$, then

$$\log | F_n(c) | \leq \log | P(c) | + \log \sqrt{c} - \sum_{k=0}^{n} \log | c + 2k + 1 | .$$

Therefore, by (II.20) and (II.25), we have for $c > b$,

$$\log | F_n(c) | \leq -n \log r + \log \sqrt{\frac{c}{b}} - \log \frac{c + 2k + 1}{b + 2k + 1}$$

$$+ (1 - r) \left[ -\frac{a}{b} - n - \frac{1}{2} + \sum_{k=0}^{n} \frac{a + 2k + 1}{b + 2k + 1} \right] + (1 - r)^2 K_n$$

Using formula (II.17) in Lemma (5), and removing the negative terms $-(1 - r)\sqrt{\frac{a}{b}}$

and $-\log \frac{c+2k+1}{b+2k+1}$ for $k = 1, 2, \ldots, n$; then for $c > b$, the last inequality gives

$$\log | F_n(c) | \leq -n \log r + \log \sqrt{\frac{c}{b}} - \log \frac{c + 1}{b + 1} + (1 - r)(-n - \mu_n) + (1 - r)^2 K_n .$$

Since $\sqrt{\frac{c}{b}} \leq \frac{c+1}{b+1}$ for $c \geq b \geq 1$, the last inequality gives, for $c > b$,

$$\text{(II.26)} \quad \log | F_n(c) | \leq -n \log r + (1 - r)(-n - \mu_n) + (1 - r)^2 K_n .$$

Now by (II.18) in Lemma (5), for $c > b \leq 12(n+1)^2$, we have $1-r \leq \sqrt{c-b}$. 

So, if $\varepsilon < \frac{1}{4}$ and $c \in (b, b + \varepsilon)$, then $1 - r \leq \sqrt{\varepsilon} < \frac{1}{2}$, and so $-\log r \leq (1-r) + (1-r)^2 K$ where $K$ is a constant independent of $\varepsilon$ if $\varepsilon < \frac{1}{4}$.

Using this estimate for $-\log r$ in (II.26) gives the following: For every $\varepsilon < \frac{1}{4}$

and $b \in [12(n+1)^3, T_n]$, we have for $c \in (b, b + \varepsilon)$,
\[
\log | F_n(c) | \leq \sqrt{\varepsilon[-\mu_n + \sqrt{\varepsilon(K_n + nK)}]}. \]

From the last inequality, it follows that there is \( \varepsilon_n < \frac{1}{4} \) such that if \( 12(n+1)^3 \leq b \leq T_n \) and \( c \in (b, b + \varepsilon_n) \), then \( | F_n(c) | < 1 \).

This completes the proof of the upper bound.

6. The Lower Bound:

Lemma 6 (Explicit form of the Chebyshev polynomial for the weight \( x \) on \( [0, 1] \)):

Let \( T_n(x) = x^n - \sum_{k=0}^{n-1} c_k x^k \) be the unique polynomial such that

\[
\| xT_n(x) \|_{[0,1]} = \inf \{ \| x(x^n - \sum_{k=0}^{n-1} c_k x^k) \|_{[0,1]} : (c_0, \ldots, c_{n-1}) \in \mathbb{R}^n \}. 
\]

Then

\[
T_n(x) = \prod_{i=1}^{n} (x - x_i),
\]

where \( x_i = \frac{1}{2} \left\{ 1 - \xi_n + (1 + \xi_n) \cos \frac{2\xi^2}{2(n+1)} \pi \right\} \) and

\[
\xi_n = \frac{1 - \cos \frac{\pi}{2(n+1)}}{1 + \cos \frac{\pi}{2(n+1)}}.
\]

Remark: The system \( \{x, x^2, \ldots, x^n\} \) is not a Haar System on the interval \([0, 1]\); however, by a generalization of the Chebyshev's theorem, the polynomial \( xT_n(x) \) in the statement of the lemma is unique and has the alternating property ([15], footnote in page 56).
Proof: Let $e_n = ||xT_n(x)||_{[0,1]}$. By the alternating property, $|xT_n(x)|$ attains its maximum $e_n$ at $n + 1$ points in $(0,1]$. So $xT_n(x)$ has $n + 1$ distinct zeros in $[0,1]$. Since $xT_n(x)$ has at most $n + 1$ zeros, then the $n + 1$ distinct zeros of $xT_n(x)$ are contained in $[0,1)$. It follows that $|xT_n(x)|$ is decreasing on $(-\infty,0)$, and if $c$ is the largest zero of $xT_n(x)$ then $|xT_n(x)|$ is increasing on $(c,\infty)$. Therefore, $|T_n(1)| = e_n$ and there is a unique point $-\xi_n \in (-\infty,0)$ such that

$$\xi_n T_n(-\xi_n) = e_n.$$ (II.27)

Thus, the polynomials $(\xi_n + x)(1 - x)[T_n(x) + xT'_n(x)]^2$ and $e_n^2 - x^2T_n^2(x)$ have exactly the same zeros.

Since the leading coefficient of $(\xi_n + x)(1 - x)[T_n(x) + xT'_n(x)]^2$ is $-(n + 1)^2$, then the polynomial $y = xT_n(x)$ satisfies the differential equation

$$(\xi + x)(1 - x)y'' = (n + 1)^2(e_n^2 - y^2), \quad y(0) = 0$$ (II.28)

The general solution of (II.28) is of the form

$$y = \pm e_n \cos[(n + 1)\arccos \frac{2x + \xi_n - 1}{1 + \xi_n} + c],$$

but the right hand side of the last equation is a polynomial if and only if $c = m\pi$, $m$ is an integer, so

$$y = \pm e_n \cos[(n + 1)\arccos \frac{2x + \xi_n - 1}{1 + \xi_n}].$$ (II.29)
Since \( y(0) = 0 \), we have
\[
\arccos \left( \frac{x_n - 1}{x_n + 1} \right) = \frac{2k + 1}{2(n+1)} \pi \quad \text{for some } k \in \{0, 1, \ldots, n\}.
\]

So
\[
\xi_n = \frac{1 + \cos \frac{2k + 1}{2(n+1)} \pi}{1 - \cos \frac{2k + 1}{2(n+1)} \pi} \quad \text{and} \quad \frac{x_n - 1}{x_n + 1} = \cos \frac{2k + 1}{2(n+1)} \pi.
\]

The \( n+1 \) zeros of \( y, \{x_0, \ldots, x_n\} \) are given by
\[
\frac{2x_i + \xi_n - 1}{\xi_n + 1} = \cos \frac{2i + 1}{2(n+1)} \pi \quad \text{for } i = 0, \ldots, n.
\]

Therefore, \( x_i = \frac{1}{2} (1 + \xi_n) \{\cos \frac{2i + 1}{2(n+1)} \pi + \frac{1 - \xi_n}{1 + \xi_n}\} \), and so by (II.30) we have,
\[
x_i = \frac{1}{2} (1 + \xi_n) \{\cos \frac{2i + 1}{2(n+1)} \pi - \cos \frac{2k + 1}{2(n+1)} \pi\}.
\]

Since \( x_i > 0 \) for \( i \neq k \), then for \( i \in \{0, \ldots, n\}\setminus\{k\} \) we have
\[
\cos \frac{2i + 1}{2(n+1)} \pi > \cos \frac{2k + 1}{2(n+1)} \pi,
\]
so \( k = n \), and by (II.30) we get
\[
(II.31) \quad \xi_n = \frac{1 - \cos \frac{\pi}{2(n+1)}}{1 + \cos \frac{\pi}{2(n+1)}} \quad \text{and} \quad \frac{x_n - 1}{x_n + 1} = -\cos \frac{\pi}{2(n+1)}.
\]
Lemma 7: If $T_n(x)$ is defined as in Lemma 6 and
\[ t_n = \min\{t \in (0, 1] : |tT_n(t)| = \|x T_n(x)\|_{[0,1]} \}, \]
then $t_n \leq \frac{3}{(n+1)^2}$.

Proof: Let $y(x) = x T_n(x)$, and $\xi_n$ be as in Lemma 6.

By Lemma 6, $x = 0$ is a simple zero of $y$. So the zeros of the second derivative $y''$ of $y$ are contained in the interval $[t_n, 1]$, and so $y''$ is of constant sign on $(-\infty, t_n)$. Since $y$ is a polynomial, $|y|$ is convex on $(-\infty, t_n)$. In particular it is convex on $[-\xi_n, 0]$, so we have
\[ \frac{|y(-\xi_n)|}{\xi_n} \geq |y'(0)|. \]

Since by (II.27) we have $|y(-\xi_n)| = e_n$, then the last inequality gives
\[ \frac{e_n}{\xi_n} \geq |y'(0)|. \]

But by (II.28) we have $\xi_n(y'(0))^2 = (n + 1)^2 e_n^2$, so the last inequality gives
\[ \frac{e_n}{\xi_n} \geq \frac{(n + 1)e_n}{\sqrt{\xi_n}} \]
which implies that
\[ (II.32) \quad \xi_n \leq \frac{1}{(n + 1)^2}. \]
From (II.29) we obtain that \( y'(t) = 0 \) if and only if

\[
\frac{2t + \xi_n - 1}{1 + \xi_n} = \cos \frac{k\pi}{n + 1} \quad \text{for some } k, \quad k = 0, 1, \ldots, n.
\]

It follows, since \( t_n = \min\{t : |y'(t)| = 0\} \) that

\[
\frac{2t_n + \xi_n - 1}{1 + \xi_n} = - \cos \frac{\pi}{n + 1} = - \left(2\cos^2 \frac{\pi}{2(n + 1)} - 1\right).
\]

So by (II.31) we have

\[
\frac{2t_n + \xi_n - 1}{1 + \xi_n} = 1 - 2\left(\frac{1 - \xi_n}{1 + \xi_n}\right)^2,
\]

which implies that

\[
t_n = \frac{3 - \xi_n}{1 + \xi_n} \xi_n \leq 3\xi_n.
\]

Finally from (II.32) we get

\[
t_n \leq \frac{3}{(n + 1)^2}.
\]

Remark: It is easy, of course, from \( t_n = \frac{3 - \xi_n}{1 + \xi_n} \xi_n \) and the expression for \( \xi_n \) in Lemma 6 to deduce an exact expression for \( t_n \), from which it follows that

\[
t_n \sim \frac{3\pi^2}{16n^2}, \quad n \to \infty.
\]
Proof of the Lower Bound:

We will prove that for all \( n > 1 \),

\[
M_{n,n} \geq \frac{1}{4} (n + 1)^3
\]

where \( M_{n,n} \) is defined as in the Main Theorem.

Let \( P_n(x) = T_n((2n + 1)x) \) and \( \lambda_n = \frac{t_n}{2n+1} \), where \( T_n \) and \( t_n \) are as in Lemmas 6 and 7.

Let \( G_n(x) = \prod_{k=0}^{n} \frac{\sqrt{x} P_n(x)}{((2k+1)x+1)} \) so,

\[
G_n(x) = \frac{x P_n(x)}{\sqrt{x} \prod_{k=0}^{n} ((2k+1)x+1)}, \quad x > 0.
\]

Since \( \sqrt{x} \prod_{k=0}^{n} ((2k+1)x+1) \) is increasing on \((0, \infty)\), then

\[(II.33) \quad |G_n(x)| < |G_n(\lambda_n)| \quad \text{for } x \in (\lambda_n, \frac{1}{2n+1}].\]

Therefore,

\[
\|G_n\|_{[0, \frac{1}{2n+1}]} = \|G_n\|_{[0, \lambda_n]},
\]

and from this it follows that \( \mu_n(P_n) \leq \lambda_n \), where \( \mu_n(P_n) \) is defined as in Corollary (2) on page 17.

By Lemma 7, \( t_n \leq \frac{3}{(n+1)^2} \). So

\[(II.34) \quad \lambda_n \leq \frac{3}{(2n+1)(n+1)^2}.\]
Since \( \mu_{n,n} \leq \lambda_n \), where \( \mu_{n,n} \) is defined as in Corollary (2), then by (II.34) we get

\[
\mu_{n,n} \leq \frac{3}{(2n + 1)(n + 1)^2},
\]

and so by (II.8) in Corollary 2, we have

\[
M_{n,n} \geq \frac{(2n + 1)(n + 1)^2}{6} - \frac{1}{2} \geq \frac{1}{4}(n + 1)^3 \quad \text{for all } n > 1.
\]

This completes the proof.

Remark: By the remark following Lemma 7, it is possible to improve the constant \( \frac{1}{4} \) in the lower bound for \( M_{n,n} \) for \( n \) large. Namely

\[
\lim \inf_{n \to \infty} \frac{M_{n,n}}{n^3} \geq \frac{16}{3\pi^2}.
\]

From Lemma (7) we easily deduce the following result which may be of interest as a complement to Saff’s Theorem (Part (b) of Section 3, Chapter I).

Lemma 8: Let \( 0 < \theta < 1 \), and \( I_\theta = \{ P_n(x) : P_n(x) = \sum_{k=0}^{n} a_k x^k, s \geq n\theta \} \). There is \( P_n \in I_\theta \) of degree \( n = \lceil \frac{1}{\theta} \rceil + 1 \) such that

\[
\| P_n \|_{[0,1]} = \| P_n \|_{[0,3\theta^2]}.
\]

i.e., \( t_n = \min\{t \in (0,1] : |P_n(t)| = \|P_n\|_{[0,1]} \} \leq 3\theta^2 \).

Proof: It is sufficient to let \( P_n(x) = x T_{n-1}(x) \), where \( T_{n-1}(x) \) is the Chebyshev polynomial of degree \( n - 1 \) as defined in Lemma 6. then, the conclusion follows from Lemma 7.
Remark: Let $\epsilon > \frac{3\pi^2}{16}$ and $n = \lceil \frac{1}{\theta} \rceil + 1$. By the remark following Lemma 7, if $n$ is sufficiently large (i.e., $\theta$ is sufficiently small), then $t_n \leq \frac{\epsilon}{n^2}$ (where $t_n$ is as in the statement of Lemma (8)).
CHAPTER III

NUMERICAL VALUES OF $M_{n,0}$ AND POLYNOMIALS FOR WHICH
THE DEGREE OF THE BEST APPROXIMATING QUASI-MONOMIAL IS LARGEST

Numerical Values of $M_{n,0}$

In our Main Theorem we showed that, if $P \in \pi_n$, and $m(x) = cx^t$ is a best approximation to $P$ in $L_2$-norm among all functions $ax^s$, $s \in \mathbb{R}$ and $s \geq n$, then

$$t \leq 6(n + 1)^3.$$ 

Also, we showed that, there is a polynomial $P \in \pi_n$ for which the above $t$ satisfies the inequality

$$t \geq \frac{1}{4}(n + 1)^3.$$ 

In an earlier remark we mentioned that it is still an open problem whether $M_{n,0}$ or $M_{n,-\frac{1}{2}}$ are still bounded below by a constant multiple of $(n + 1)^3$. For example, we don't know whether there is an absolute constant $K$ such that for everh $n$ there is $P_n \in \pi_n$ for which a best approximation $m(x) = cx^t$ in $L_2[0, 1]$ among all functions $ax^s$ ($s \in \mathbb{R}$ and $s \geq 0$) exists with $t \geq K(n + 1)^3$.

In this section we will give numerical values for $M_{n,0}$ for $1 \leq n \leq 16$, where $M_{n,0}$ is defined as in the statement of the Main Theorem. For this we will use the Remes Algorithm to find the unique monic polynomial $T_n(x)$ which satisfies
\[ \| V_n T_n \|_{[0,1]} = \inf \{ \| V_n(x)(x^n - \sum_{i=0}^{n-1} c_i x^i) \|_{[0,1]} : (c_0, \ldots, c_{n-1}) \in \mathbb{R}^n \} \]

where \( V_n(x) = \frac{\sqrt{n}}{\prod_{k=0}^{n-1} ((2k+1)x+1)} \). If \( \xi_n = \min \{ \xi : |V_n(\xi) T_n(\xi)| = \| V_n T_n \|_{[0,1]} \} \), then by Corollary (2)

\[ \mu_{n,0} \leq \xi_n \]

where \( \mu_{n,0} \) is as defined in Corollary (2). But by the Saff-Varga Lemma (Part (d) of Section (4), Chapter (i)) we have \( \xi_n \leq \mu_{n,0} \). So, \( \mu_{n,0} = \xi_n \).

Therefore by (II.8) in Corollary (2) we have

\[ M_{n,0} = \frac{1}{\xi_n} - 1 \]

Using these facts we obtain the following table:
Polynomials for which the degree of the best approximating quasi-monomial is largest (i.e. the degree is $M_{n,0}$):

Using the bijection formulas (in Theorem (1) and Corollary (1)) and the Chebyshev polynomial $T_n(x)$, mentioned in Section (1) above, we find the extremal polynomials for which the degree of the best approximations quasi-monomial among all quasi-monomials $ax^s$ ($s \in R$ and $x \geq 0$) is largest (i.e., $M_{n,0}$).

The following table shows the coefficients of the extremal polynomials ($P_n(x) = x^n + \sum_{k=0}^{n-1} a_k x^k$), $1 \leq n \leq 8$:

<table>
<thead>
<tr>
<th>$n$</th>
<th>$M_{n,0}$</th>
<th>$\frac{M_{n,0}}{(n+1)^3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5.82</td>
<td>0.7276</td>
</tr>
<tr>
<td>2</td>
<td>21.81</td>
<td>0.8076</td>
</tr>
<tr>
<td>3</td>
<td>52.63</td>
<td>0.8223</td>
</tr>
<tr>
<td>4</td>
<td>102.09</td>
<td>0.8167</td>
</tr>
<tr>
<td>5</td>
<td>175.15</td>
<td>0.8109</td>
</tr>
<tr>
<td>6</td>
<td>276.04</td>
<td>0.8048</td>
</tr>
<tr>
<td>7</td>
<td>409.06</td>
<td>0.7989</td>
</tr>
<tr>
<td>8</td>
<td>578.45</td>
<td>0.7935</td>
</tr>
<tr>
<td>9</td>
<td>788.51</td>
<td>0.7885</td>
</tr>
<tr>
<td>10</td>
<td>1043.80</td>
<td>0.7842</td>
</tr>
<tr>
<td>11</td>
<td>1348.48</td>
<td>0.7804</td>
</tr>
<tr>
<td>12</td>
<td>1706.95</td>
<td>0.7769</td>
</tr>
<tr>
<td>13</td>
<td>2123.54</td>
<td>0.7739</td>
</tr>
<tr>
<td>14</td>
<td>2602.57</td>
<td>0.7711</td>
</tr>
<tr>
<td>15</td>
<td>3148.37</td>
<td>0.7686</td>
</tr>
<tr>
<td>16</td>
<td>3765.28</td>
<td>0.7664</td>
</tr>
</tbody>
</table>
TABLE 2: The Coefficients of the Extremal Polynomials

<table>
<thead>
<tr>
<th>n</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>a₀</td>
<td>-0.6268</td>
<td>0.2811</td>
<td>-0.1063</td>
<td>0.0359</td>
<td>-0.0111</td>
<td>0.0032</td>
<td>-0.0009</td>
<td>0.0002</td>
</tr>
<tr>
<td>a₁</td>
<td>-1.1978</td>
<td>0.8812</td>
<td>-0.5046</td>
<td>0.2458</td>
<td>-0.1065</td>
<td>0.0423</td>
<td>-0.0156</td>
<td></td>
</tr>
<tr>
<td>a₂</td>
<td>-1.7588</td>
<td>1.7854</td>
<td>-1.3581</td>
<td>0.8558</td>
<td>-0.4707</td>
<td>0.2331</td>
<td></td>
<td></td>
</tr>
<tr>
<td>a₃</td>
<td>-2.3138</td>
<td>2.9892</td>
<td>-2.8284</td>
<td>2.1791</td>
<td>-1.4465</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>a₄</td>
<td>-2.8653</td>
<td>4.4905</td>
<td>-5.0763</td>
<td>4.6181</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>a₅</td>
<td>-3.4145</td>
<td>6.2881</td>
<td>-8.2619</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>a₆</td>
<td>-3.9622</td>
<td>8.3814</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>a₇</td>
<td>-4.5089</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>a₈</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
REFERENCES


44