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Buchner, Helmut Josef

CONTROL OF ROBOT MANIPULATORS ON TASK ORIENTED SURFACES BY NONLINEAR DECOUPLING FEEDBACK AND COMPENSATION OF CERTAIN CLASSES OF DISTURBANCES

The Ohio State University
Ph.D. 1986

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Nonlinear Decoupling Feedback and Compensation of Certain
Classes of Disturbances

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the Degree Doctor of Philosophy in the
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by
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The Ohio State University
1986

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To whom it may concern
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CHAPTER I

Introduction

1.1 Scope of the Dissertation

Interest in the field of robot manipulators has proliferated since the 1960's. Robotics research has grown rapidly into a multi-disciplinary area comprising subjects like physics, mechanics, electro-technology, computer science, and last but not least, mathematics [1]. Primarily urged by economical problems about productivity, costs, and health hazards in certain industrial environments, private firms and government institutions have sought alternative solutions in employing robot manipulators. In the industrial environment robot manipulators shall improve reliability, cost, productivity, quality, and shall eliminate exposition of humans to hazardous environments. Despite the obvious advantages of robots in industrial environments, socio-economical impacts on workers as well as government agencies might be expected [2].

The steady increase of robot manipulators in industry and their diversified usage in a growing number of tasks demands an increase on the measures of performance. These performance measures are:

i) better path accuracies at higher speed,

ii) reduced energy consumption,

iii) design of software tools to allow more flexible path planning,
iv) better interaction in the work-space environment, e.g. with tools, work pieces, other robots, and humans,

v) more effective utilization of visual, tactile or other sensory information.

These demands require investigations in such areas as kinematics, dynamics, trajectory planning, visual and tactile sensing, vision systems, control robot languages, and more recently, machine intelligence.

Although most robot manipulators perform one and the same movement repeatedly, the enormous advantage of a robot over a simple machine is its flexibility to perform other tasks as well. However, this advantage requires a much greater sophistication concerning control strategies and computer algorithms. For instance, when a special one degree-of-freedom manipulator performs a particular task, it most likely possesses one simple control input. Contrarily, a six degree-of-freedom manipulator has six inputs which, in order to perform the same task, must receive input signals that assure proper coordination of all the links. In fact, generating these proper input signals that depend on a particular application integrates the problem of path planning and control of a robot manipulator.

A nonlinear control scheme that theoretically guarantees correct coordination of a manipulator gripper to follow some particular path while at the same time reduces the number of closed loop control inputs is developed in this dissertation. Suppose end-effector or tool movements of a manipulator in some n-dimensional coordinate frame can be expressed as a surface or portions of concatenated surfaces of dimension \(l\). Moreover, assume that the surface, which will be denoted as task oriented surface \(S_T\) can be expressed by \(n - l\) differentiable functions of the states of a manipulator only. Finally, it is assumed that points on \(S_T\) are given by \(l\) functions of the state vector as well. When the former functions are viewed
to be output channels, it is possible to apply nonlinear input-output decoupling feedback based on differential geometric system theory to control movements on $S_T$ by $l$ input channels only. It is immediately obvious that if $l = 1$ the manipulator can be controlled with one input only executing useful work and simplifying the design of a reference trajectory. When a suitable reference trajectory is defined, portions of different surfaces can be connected together by matching the surface boundaries. Computer simulations will demonstrate the merit of the nonlinear decoupling feedback on different surfaces $S_T$.

Digital computer simulations demand a mathematical manipulator model which is suitable as a demonstrative example throughout this dissertation. Such a model also serves illustrative purposes in the development of the control methodologies. A three link, three dimensional model is formulated using Newtonian dynamics. Two alternative methods are presented to describe coordinate rotation between coordinate frames. The two methods are based on Bryant angles and Euler parameters. Both formulations possess a problem which is discussed in this work. A possible solution related to Euler parameters is proposed and verified by computer simulations.

Controlling robot manipulators by nonlinear decoupling feedback alone may yield expected results in an ideal environment only. Unmodeled dynamics, parameter uncertainties, and external disturbances are commonly some of the reasons that the actual behaviour strays from the theoretically predicted one. In this dissertation studies are conducted for eliminating a class of disturbance signals entering in the dynamics of a manipulator. The class of disturbances discussed are solutions of linear differential equations of arbitrary order, but with known eigenvalues. Also, from a local point of view, gravitational effects can be viewed as a disturbance of constant magnitude. This suggests eliminating gravitation
by using a standard servocompensator. In order to show that the latter class of external disturbances can be asymptotically rejected by applying linear servocompensator theory as well, the Volterra series expansion of a manipulator example is introduced [3].

The nonlinear decoupling scheme presented in this section requires that manipulators be stabilizable by using output feedback. Occasionally it becomes necessary to estimate the state of a manipulator from its output variables, i.e. end-effector position and velocity. Therefore some basic issues regarding observability of nonlinear analytic systems applied to a particular class of manipulators are provided. The nonlinear observability analysis may also aid in recovering a particular state which becomes unmeasurable due to the failure of a sensor.

1.2 Organization of the Dissertation

This dissertation consists of seven chapters and two appendices. In Chapter 2 some pertinent literature in the field for modeling and control of general rigid body dynamics is briefly reviewed. The emphasis of the literature survey in this chapter, however, focuses on contributions in robotics research. A summary of some linear and nonlinear system theoretical concepts concludes this chapter. In Chapter 3 the equation of motion of a three-link, three-dimensional robot manipulator model is derived. This chapter contains a comparative study about Bryant angles and Euler parameters. A numerical stabilization method for Euler parameter based dynamic formulation is suggested.

The notion of task oriented surfaces is introduced in Chapter 4. A nonlinear input-output decoupling controller which integrates path planning and manipulator dynamics is developed and represents the main portion of this chapter. Digital computer simulations illustrate the devised control strategies.
In Chapter 5 linear servocompensator theory for disturbance rejection is briefly introduced. It is shown that this theory may be applied to eliminate the effects of gravitation upon manipulators. Moreover, servocompensator theory is applied to asymptotically reject some class of disturbances. Nonlinear observability theory based on differential geometry is introduced in Chapter 6. Among others, one of its application is associated with stabilizing dynamic systems using output feedback. For example, the end-effector position and velocity of a robot manipulator are considered to be the output functions when nonlinear input-output decoupling on a task oriented surface is used. Other applications may be related to studies of nonlinear observers. In Chapter 7 the dissertation is summarized and some remaining problems are pointed out. From these problems future research work is proposed.
CHAPTER II
Survey of Previous Work

2.1 Introduction

In this chapter some pertinent research in the area of manipulator control is surveyed. In Section 2.2 the basic literature on rigid body dynamics is presented for modeling rigid-body manipulators as studied in Chapter 3. Section 2.3 presents a brief survey of literature in the area of stability and control of general rigid body systems and other mechanical systems. Robotic systems are a special case of general mechanical systems and have attracted considerable attention lately. For this reason certain significant contributions in robotics research is presented separately in Section 2.4. Nonlinear control theories pertaining to this dissertation are introduced and related works are briefly summarized.

2.2 Modeling of Rigid Body Dynamics

In this section some pertinent literature on the dynamics of systems of rigid bodies is reviewed. Although interest in theoretical mechanics arose several hundred years ago, technological advances in this century have spurred significant research activities in this field. Mathematically speaking, the dynamical equations of mechanical systems can be established based on either Newtonian, Lagrangian, or Hamiltonian formalisms [4]-[7].

The Newtonian formulation has been frequently used to model multi-linkage
systems [8]-[14]. In the Newtonian or more precisely, in the Newton-Euler formulation, the notion of each body $i$ is expressed by three translational and three rotational second order differential equations. When several bodies are assembled in a linkage system, kinematical constraints are imposed upon the "free links" to reduce the number of degrees-of-freedom. In Chapter 3 the latter formalism is employed to arrive at the equation of motion of a manipulator model for studying a number of new control strategies.

The Lagrangian formulation is based on the Lagrangian $L$, which is the difference of the kinetic energy $E_k$ and the potential energy $E_p$ [15]-[18]. Define $\dot{X}_i$ to be the velocity of the center of mass of body $i$, $m_i$ to be the mass of body $i$, $\omega_i$ to be the angular velocity of body $i$ in body coordinates, $I_i$ to be the inertia matrix, and $k$ the unit vector in the opposite direction of gravity. The Lagrangian $L$ may be stated for a system of $n$ links as

$$L = \frac{1}{2} \sum_{i=1}^{n} m_i \dot{X}_i^T \dot{X}_i + W_i^T J_i W_i - m_i g k X_i$$

(2.1)

where $g$ is the gravitation constant. Let $X_i = X_i(\Theta)$, $W_i = W_i(\Theta, \dot{\Theta})$ then the equation of motion of the rigid body assemblage Eq. (2.1) becomes

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}_j} \right) - \frac{\partial L}{\partial \theta_j} = N_j$$

(2.2)

where $\Theta = (\theta_1, \theta_2, \ldots, \theta_n)^T$ is the generalized angular position vector of a linkage system.

The equation of motion of rigid body structures can also be expressed in the Hamiltonian form as mentioned earlier. The latter can be derived from the Lagrangian by means of the Legendre transform [19]. Let

$$p_i = \frac{\partial L}{\partial \dot{\theta}_i} \quad i = 1, \ldots, n$$

(2.3a)
Define the Hamiltonian as

\[ H = E_k + E_p \]

\[ H = \sum_{i=1}^{n} p_i \dot{q}_i - L \]  

(2.4)

Then the equation of motion of a mechanical system can be stated as \(2n\) first order differential equations

\[ \dot{q}_i = \frac{\partial H}{\partial p_i} \quad i = 1, \ldots, n \]  

(2.5a)

\[ \dot{p}_i = -\frac{\partial H}{\partial q_i} + N_i \]  

(2.5b)

The Hamiltonian form of the equation of motion is particularly suitable for optimal control, [19], [20], and for studying nonlinear controllability of mechanical systems [14], [21]. If the equation of motion of the system with several links need to be formulated, the complexity of deriving such equation becomes unmanageable. Therefore, systematic procedures have been formulated which are based on state space formulations and which also incorporate concepts from graph theory [22]-[28]. In addition to these systematic procedures, software packages for computer aided derivation of the equations of motion have been developed [29]-[32]. Moreover, formalisms are proposed that are suitable for symbolic algebraic manipulation [33].

The orientation of a rigid body with respect to any arbitrary coordinate frame can be expressed in terms of Euler or Bryant angles [24], [34]. However, Euler and Bryant angles possess an inherent algebraic problem inasmuch the mapping from angular velocities in some reference frame to the corresponding velocities in
some body coordinate system is globally not an isomorphism (see Section 3.3). To alliviate the above problem an alternative mathematical description is desirable. A possible alternative method is based on generalized quaternions which commonly are also denoted Euler or Euler-Rodriques parameters. Occasionally, they are also termed Cayley-Klein parameters [35].

In multi-articulate systems, as for example robots or musculo-skeletal systems, the number of degrees of freedom may vary depending on a particular task. If a robot touches an object with its gripper or an articulated hand picks up a tool, etc., the dynamics of such systems become restrained. Also in human locomotion the number of degrees-of-freedom varies depending upon the phase of a particular movement being executed [11]-[13], [36], [37]. Mathematically speaking, restraining movements of a manipulator or neuromusculoskeletal models is equivalent to imposing constraints upon the equation of motion of the dynamical system. Most commonly the constraints are of holonomic type, simple nonholonomic type, or of both types of constraints [13], [18],[38]-[40]. In mechanical systems holonomic constraints are analytic functions in the generalized position only and can be expressed as

\[ C_H(\Theta) = 0 \]  (2.6)

where \( \Theta = (\theta_1, \theta_2, \ldots, \theta_n) \).

On the other hand simple-nonholonomic constraints are analytic functions of the generalized position and velocities, however, such that the velocities enter linearly

\[ C_N(\Theta, \dot{\Theta}) = 0 \]  (2.7)

where \( \dot{\Theta} = (\dot{\theta}_1, \dot{\theta}_2, \ldots, \dot{\theta}_n) \).
The difference between holonomic and simple nonholonomic constraints is that for simple nonholonomic constraints Eq. (2.6) is not an exact one-form of some analytic function. Otherwise a simple nonholonomic constraint could be transformed to a holonomic constraint by simple integration. Imposing motion restraints upon mechanical systems can be done with constraint forces and constraint torques whose magnitudes can be computed as explicit function of the state, i.e. position and velocity, and the input torques [22], [39], [41], [42]. A comparison of the complexity of different algebraic methods to compute constrained forces has been discussed by Langer et al.[43].

2.3 Stability and Control of Rigid Body Dynamic Systems

Whenever a mathematical description of a multilinkage system has been obtained, a natural, secondary step in the analysis or synthesis of multibody dynamics is to investigate stability of a system and to discuss the implementation of a suitable controller. Historically, stability studies of rigid body dynamical systems were investigated on simple inverted or non-inverted pendulums, because of the relative ease of its mathematical representation [44]. Bavarian et al. [45] studied the stability and control of a "free-body" planar pendulum which is constrained to the ground. Using a linearized model Golliday [46] showed that linear gain feedback stabilizes a double inverted pendulum. For a general $n$-link planar multibody system Hemami et al. [47] showed that for linearized manipulator models pole-placement and decoupling can be achieved simultaneously. Moreover, stability of such systems can be proven by constructing a Lyapunov function. In fact, Lyapunov functions are often employed to verify global or local stability of equations of motion [48], [49]. Postural stability and control of a two link pendulum using nonlinear feedback gain was studied by Hemami and Camana [50]. They
also studied the effects of discretization noise in the same work. Vukobratović and Stokić [51] studied decentralized control and force feedback to reduce coupling among sub-systems of locomotion manipulators. Özgüner and Hemami [52] proved that the linearized approximation of a mechanical assemblage can be controlled by decentralized state feedback.

All the studies in this section so far have one common characteristic; namely, all the former mechanical systems are mathematically described by a minimal number of state variables. This means the solutions of their differential equations do not evolve on a submanifold of lower dimension than the dimension of the state space. But in contrast to this, there exist applications of rigid body system whose dynamical behaviour is restricted onto a sub-manifold of lower dimension than the dimension of states. This type of system is generally referred to as constrained dynamical system. Usually, constrained dynamical system exhibit an inherent instability problem when they are numerically simulated. A remedy of this problem applied to normalized quaternions is proposed by Wittenburg [24]. A feedback stabilization method to stabilize quaternions (i.e. Euler parameters) is presented in Chapter 3 of this dissertation. Numerical instabilities of constrained dynamical system are investigated in celestial mechanics [53]. Numerical stabilization methods are proposed to solve the problem of Keplerian motion [54]-[56]. In fact, Keplerian systems are holonomically constrained with \( v = \text{constant} \).

Baumgarte [57], [58] investigated stabilization of holonomic connection constraints using the Lagrangian formulation. Von Grünhagen [59] studied modified dynamic formulations to obtain a stable error behaviour during numerical simulation. Ceranowicz et al. [60] studied constrained mechanical systems by designing invariant constraint subspaces.

In this section robot manipulators are excluded. These special type of me-
chanical assemblage are considered in a separate section. However, most problems that are addressed in this section are intertwined with control of robot arms.

2.4 Kinematic, Dynamics and Control of Robot Manipulators

In this section a sample of the vast amount of literature in the field of robotics research is reviewed. The subject of robotic research may be subdivided into a number of topics as shown:

- kinematics
- dynamic formulation
- stability and control
- path planning
- obstacle avoidance
- computer real-time implementation
- CAD systems, artificial intelligence and robot control languages.

As a matter of fact, all the above topics are closely related in robotics such that it is not always possible to assign a particular problem to only one of the above listed items. However, the robotic related literature will be introduced according to the listed subjects above with the exception of CAD and AI systems.

2.4.1 Manipulator Kinematics

Studies in the field of manipulator kinematics are concepts that relate position, velocity, and acceleration of a reference point or coordinate frame on the robot to an arbitrary reference frame in the environment of that robot.
The derivation of accurate formulae for manipulator kinematics is of fundamental importance for a successful application of robots in assembly tasks. Wu and Lee [61] present a procedure for estimating the Cartesian kinematic error depending on the manipulator arm position. They found that the error estimate can be obtained from observational data without knowing the values of the kinematic errors. An exact kinematic model of an existing robot arm is established by Bazergi et al. [62]. They have found that approximate results are unacceptable in a simulation model. Besides an accurate kinematical description of a manipulator arm, efficient algorithms are needed to relate respectively the joint angles and joint velocities to the manipulator's end-effector position and velocity, and vice versa. Featherstone [63] proposes an inverse kinematic algorithm for robot manipulators possessing a spherical wrist joint; i.e., a joint in which the three distal revolute axes intersect at one point. Manipulators of the latter type allow the partitioning of the joint variables into two sets which can be independently computed from the orientation and the translational position/velocity of an end-effector. Hollerbach and Sahar [64] present an efficient algorithm for calculating the inverse kinematic acceleration of a manipulator with a spherical wrist. Their work shows that the algorithm works synergistically with the inverse dynamic calculations. A rather well-known issue in manipulator kinematics is that the inverse kinematic problem can be only solved in closed form for certain models of manipulators [65]. For robot manipulators whose inverse kinematic can not be solved in closed form, Lumelsky [66] and Angeles [67] proposed iterative methods.

Orin and Schrader [68] discuss and compare six different methods for computing the Jacobian matrix which relates joint velocities to end-effector velocities. The results show that the efficiency in some cases may strongly depend on the reference coordinate chosen. Recently, Stanišić and Pennock [69] introduce a kine-
matic manipulator model with seven joints. The novelty about this model is that it has seven joints possessing a nondegenerate solution which is given in closed form.

The issue of robot arm kinematics may be supplemented by studies of the kinematics of more or less sophisticated gripper for object handling. Salisbury and Craig [70] discuss kinematic as well as control concepts of a multifinger mechanical hand. They investigated conditions for object grasping with a three-finger hand and applied techniques for obtaining the kinematic parameters of that hand. Mechanisms of prehension and grasp of objects of generic shapes are studied by Rovetta [71]. He proposes a simple mechanical hand which is adaptable to function as a robot extremity. A multijointed three finger hand is constructed by Okada [72]. He discusses the computer control function for this hand and demonstrates its practicality by a number of experiments during which the hand has to precisely manipulate some objects. Similar studies were conducted by Kobayashi [73] who discusses grasping and manipulation of objects by a three finger 12-joint hand. He presents necessary conditions for grasping and ways to control the hand based on the former conditions.

2.4.2 Manipulator Dynamics, Stability, and Control

In addition to a precise kinematic description of manipulator arms, good dynamic formulations in terms of differential equations must be established. The differential equations are indispensable in designing an appropriate controller of any robot arm. Aside from classical mechanical methods already discussed, procedures which are particularly suited to robot manipulators were established to arrive at the equations of motion [74], [75]. Raibert [76] noted that fixed analog servo loops with constant gains around each degree-of-freedom are in no way suf-
ficient for precision control of manipulators particularly at higher speed. In order to account for the nonlinearities Raibert proposes to precompute coefficients of a robot arm and to store them into a memory. From the memory the coefficients can be retrieved in real time depending upon the present configuration, i.e. the vector $\Theta$.

Whitney [77] proposes that manipulators should be controlled by computing the angular velocities as function of the end-effector velocities. He stated that this method would coordinate motion of various actuators to allow end-effector movements along some world coordinates. In a later paper Whitney [78] presents the mathematics of coordinated velocity and position control together. He proposes that the endpoint position of a manipulator gripper can be specified in meaningful external coordinates. In fact, task oriented surfaces may be viewed as a special case of a meaningful external coordinate system. Takegaki and Arimoto [79] introduced the concept of task oriented coordinate control by implementing holonomic constraints. Under the assumption that some variables are bounded, they claim that linear state feedback stabilizes the manipulator. However, the boundedness assumption does not guarantee an adequate local performance, because fictitious holonomic constraints are unstable [58]. [59]. A recursive on-line computational method for calculating the manipulator input torques is developed by Luh and his colleagues [80]. Their method is based on the Newton-Euler equations of motion using $q$, $\dot{q}$ and $\ddot{q}$, where the latter three vectors are the generalized position, velocity and acceleration, respectively. The computation time of the proposed scheme was compared with alternative approaches.

Vukobratović and Stokić [81] saw a need to establish a dynamical description of manipulators about a specified nominal trajectory. They decouple the system into a number of sub-systems corresponding to actuators. Next, a decentralized
A control law is applied to stabilize the manipulator dynamics about the nominal trajectory. A numerical example intends to validate the proposed scheme. Khatib [82] investigates control of manipulators in operational space (i.e., task space) and avoids singularities by defining a domain from which the singularity points are excluded. Walker and Orin [83] introduce four different methods for obtaining the joint acceleration vector for motion simulation.

Achieving manipulator fine motion and controlling compliant motion of a manipulator may require some sort of force feedback strategies [84], [85]. Likewise, a force control method is shown to be beneficial for dynamically decoupling the motion trajectory of a robot arm [86]. Orin and Oh [87] discuss the need to control forces in robot mechanisms with closed kinematic chains. They recognized that the failure to do so could yield damage to a linkage element in this chain.

Nonlinear feedback principles were established for robot arms under consideration of force control [88] and errors in the feedback law [89]. Optimal or near optimal control strategies were proposed for robot arm control. Kahn and Roth [90] proposed a near minimum time control for open loop manipulators with input torque constraints. Using a linearized model Kahn et al. showed that the sub-optimal solution is reasonably close to the optimal one. An exact minimum time control was established by Shin and McKay [91]. They showed that an optimal solution can be obtained with relative ease despite bounded input constraints. A manipulator control algorithm with input bounds was formulated by Spong et al. [92]. The optimal control law was reduced to a solution of a quadratic programming problem. A near-minimum time control for point to point movements was considered by Shin and McKay [93] who found that a minimum cartesian end-effector path is not necessarily equivalent to minimum time. Geering and his co-workers [94] present a time optimal motion control for robots in assembled tasks and explain
its physical significance. Futura and colleagues [95] devise a trajectory tracking control based on a linearized manipulator model and quadratic regulator. Practically speaking, robot controllers are usually implemented on a digital computer performing real time computations. This fact suggests that discretized dynamic manipulator models may be computationally more efficient and accurate [96], [97].

In addition to the various control strategies presented in this section, model-referenced adaptive control and decentralized variable structure control algorithms were proposed [98], [99].

2.4.3 Path Planning and Obstacle Avoidance

The synthesis of manipulator trajectories has attracted considerable attention in the field of robotics. Typically, manipulator trajectory synthesis is guided by the premise to obtain cartesian movements of the end-effector. Paul [100] and Tayler [101] discuss interpolation methods which yields cartesian end-effector motion. Luh and Lin [102] proposed an iterative approximate programming algorithm for obtaining an optimum path being composed of straight line segments. Vukobratović and Kirčanski [103] investigate optimal robot trajectories based on minimum energy consumption and input inequality constraints. They found that the only suitable numerical optimization algorithm is based on dynamic programming. In a later paper, Vukobratović and Kirčanski [104] suggested a dynamic approach to nominal trajectory planning taking into account not only the kinematic but also the dynamic formula of a manipulator. They found that the energy consumption will be significantly lower than for purely kinematical approaches.

Luh and Lin [105] present an approximate joint trajectory method to perform cartesian path motion. They join cubic spline and quartic polynomials with continuous first and second derivatives together. Their results show that cubic spline
functions require the least computing time, while the quartic polynomials give best fit, but needs the longest computation time. A minimum time path planning scheme under consideration of the manipulator dynamics is presented by Kim and Shin [106]. The authors found that many manipulator tasks may be performed in joint space instead of cartesian gripper space. By specifying a set of corner-points in joint space, their algorithm computes a time minimum path connecting those corner-points.

In addition to path planning algorithms that are founded on optimization criteria such as minimum time, minimum fuel or any other optimization function, robot arm trajectory synthesis which avoids obstacles have also been considered. An incremental obstacle avoidance algorithm was developed by Loeff and Soni [107]. Using an influence function the influence of all obstacles on each link is determined and then used to guide the manipulator between the known obstacles. Barmish et al. [108] point out some difficulties that may arise in specifying a control which guarantees avoidance of a given set in the state space. Schmitendorf and his colleagues [109] give a necessary and sufficient condition for guaranteed avoidance control of a time varying linear system.

Luh and Campbell [110] developed an algorithm for synthesising a collision free path which is minimal in distance using the Stanford arm. They define polygonal obstacles which enclose real obstacles and use a linear programming algorithm to find a shortest path outside the forbidden regions.

2.4.4 Computer Implementation and Robot Control Languages

Although robot manipulators are commonly controlled by digital computer systems, little work has been reported on multiprocessor implementation of robot control systems. Grygier and Hemami [111] and Hemami et al. [112] introduce
distributed models for simulating the dynamic behaviour of multibody systems. Zheng and Hemami [113] conceptualized a multiprocessor scheme for computing the applied torques of multibody dynamic systems. They employed Newton-Euler state space formulation proposed in [22] eliminating the need of a recursive algorithm. A multiprocessor system for cartesian space control was suggested by Lin and Chen [114]. They compare computational complexity as well as different microprocessor based implementations.

The breath of life of digital computers are, of course, the computer programs. For industrial robot manipulators a number of programming languages have been developed, some of which are compared by Gruver et al. [115]. Volz and his colleagues [116] use the programming language ADA for defining robot-based manufacturing cells. The former cells include robot control programs, vision systems, NC-machines, and CAD systems.

In this section a selection of research works which has evolved over the last two decades has been presented in a brief form. These works provide a good overview of the state-of-the-art in robotics research for the present.

2.5 Survey of Relevant System Theory

In this section a number of linear and nonlinear system theoretic concepts, which are related to the topics of this dissertation are reviewed. In the following chapter a three link manipulator is formulated using Euler parameters. Because it is known that Euler parameter based dynamical formulations are numerically unstable, several linear and nonlinear stability tests are introduced. Stability of linear time-invariant systems is well understood and can easily be tested employing a number of stability criterions, e.g. eigenvalue test, Routh-Hurvitz criterion, Nyquist plots, Bode diagrams, Lyapunov functions, etc. [117]-[122]. Contrary to
linear systems, stability of nonlinear systems is far more difficult to prove. Besides testing for local stability of linearized nonlinear systems, global stability is usually verified using Lyapunov's direct method. The difficulty of this method, however, is the absence of general design rules for constructing a suitable Lyapunov function [123]-[125]. In the next chapter a Lyapunov function is constructed for proving stability of an Euler parameter based multibody dynamic model. For proving global asymptotic stability about an operating point $x_0 = 0$ a Lyapunov function $V(X(t))$ must satisfy

i) $V(X)e^{Ci}$ continuous functions with $X \in \mathbb{R}^n$, $V \in \mathbb{R}$, and all

$$\frac{\partial V(X)}{\partial X_i} e^{Ci}$$

must exist, $i = 1, \ldots, n$.

ii) $V(X) = 0$ if $X = 0$

$V(X) > 0$ if $X \neq 0$

iii) $\dot{V}(X) < 0$ if $X \neq 0$

Although stability of nonlinear systems is sometimes difficult or impossible to prove, even more intriguing is the analysis of system performance and its modification by state or output feedback. Most frequently, dynamic feedback systems are refined to obtain

i) insensitivity to or independence from some class of disturbances

ii) input-output decoupling

iii) ability to track a desired signal

iv) desired responsive behaviour by pole and eigenvector assignment.

In Chapter 4 differential geometric system theory is utilized for designing a robot controller. The applied nonlinear theoretical concepts stem from linear geo-
metric theory and are a generalization thereof. For linear time invariant dynamic systems Wonham [126] establishes a series of geometric criteria for achieving disturbance decoupling by state feedback and output feedback and for obtaining single input-single output decoupling. One of the basic ideas in the work of Wonham is the notion of invariant subspaces. Let $V \subset U$ be two linear spaces and define a mapping $A : U \rightarrow U$, then one says $V$ is $A$ invariant if

$$AV \subset V$$

(2.8)

A system $\dot{X} = AX + Bu$ is said to be $(A,B)$ invariant if there exists a mapping $K$ such that

$$(A + BK)V \subset V$$

(2.9)

An alternative way of writing the former relation of $(A,B)$ invariance is

$$AV \subset V + \text{im}(B)$$

(2.10)

where \text{im} (\cdot) means the image of a linear map. Wonham and Morse showed that the latter two statements are in fact equivalent [127]. Wonham [126] further defines controllable and observable subspaces in a geometric setting. The controllable subspace $\{A \mid B\}$ of the pair $(A,B)$ is defined as

$$\{A \mid B\} = \text{im}(B) + \text{im}(AB) + \cdots + \text{im}(A^{n-1}B)$$

(2.11)

The observability subspace can be defined similarly. Using the geometric approach according to Wonham and Morse a number of other control problems may be solved; e.g. dynamic compensator, pole and eigenvector assignment, observer design, etc. [128]. Wonham and Morse's geometric approach has been applied to control biped locomotion by Hemami and Wyman [39]. The latter authors design a controller for maintaining holonomic constraints using a linearized model.

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However, in recent years a number of differential geometric concepts have evolved for analyzing nonlinear systems. These concepts may be considered a generalization of the geometric approach for linear systems put forth by Wonham and Morse [127]. In the sequel a number of references on nonlinear geometric control theory are introduced. These references provide a quick overview of existing nonlinear concepts and, in addition, present some of these concepts which are applicable to manipulator control.

Among earlier works on nonlinear system theory, Brockett [129]-[131] uses differential geometry and Lie theory to point out their applicability to control systems. The mathematical foundation of differential geometry can be found in [132], [133].

Controllability of nonlinear systems is discussed in a differential geometric setting by Sussmann [134], Haynes et al. [135], Hermann et al. [136], and Brockett [131]. Let

$$\dot{X} = f(X) + \sum_{i=1}^{m} g_i(X)u_i$$

be a nonlinear system where $X \in M$, $u_i \in \mathbb{R}$, $M$ an $n$-dimensional $C^\infty$ manifold, and $f(X), g_i(X)$ are $C^\infty$ vector fields on $M$. The system Eq. (2.12) is said to be locally accessible at any point $X \in M$ if

$$\dim \{f(X), g_i(X)\}_{LA} = n.$$  

(2.13)

The set $\{f(X), g_i(X)\}_{LA}$ is the Lie algebra generated by the vector fields $f(X)$, $g_i(X)$, $i = 1, \ldots, m$ by taking Lie brackets. The Lie bracket for two vector fields $g_1(X)$ and $g_2(X)$ is defined as

$$[g_1, g_2](X) = \frac{\partial g_2(X)}{\partial X} g_1(X) - \frac{\partial g_1(X)}{\partial X} g_2(X).$$  

(2.14)
Dynamic systems which satisfy Eq. (2.12), however, may sometimes reach all points on the manifold $M$, only in negative time. Therefore, one says a system is strongly accessible if all points in a given set can be reached in positive time. The condition which a nonlinear system Eq. (2.11) must satisfy to be strongly accessible is ([137], p.28)

$$f(X) \notin \{f(X), g_i(X)\}_{LA} = \{f(X), g_i(X)\}_{LAS}$$

$$\dim \{f(X), g_i(X)\}_{LAS} = n$$

The set $\{f(X), g_i(X)\}_{LAS}$ is the accessibility Lie algebra generated by the vector fields of $f(X), g_i(X), i = 1, \ldots, m$, but which does not contain the vector field $f(X)$ explicitly.

Studying observability using real analysis was proposed by Griffith et al. [138] and Kou et al. [139] and differential geometry by Herman et al. [136]. In linear system theory controllability and observability are dual concepts, while in nonlinear system theory they are not. Given the system Eq. (2.11) and a set of $C^\infty$ output functions

$$y = h(X)$$

where $y \in \mathbb{R}^r$ and $h_i(X) \in C^\infty, i = 1, \ldots, r$, then the system represented by Eqs. (2.11) and (2.16) is locally observable if,

$$\dim \{dL_{T_j} h_j(X)\}_{LA} = n$$

for all $x \in M$ (2.18)

where $dL_{T_j} h_j(X)$ denotes the one-form i.e. gradient, of the functions $L_{T_j} h_j$ which are generated by repeatedly taking the Lie derivative of $h_j, j = 1, \ldots, r$ in the direction of the vector field $T_j$. The vector field $T_i$ is an element of the set of vector fields generated by the set $\{f(X), g_i(X)\}$. For a more detailed discussion refer to Chapter 6.
In the beginning of this section Wonham's concept on \((A,B)\) invariant subspaces for linear systems were introduced. More recently, Hirschorn \cite{140} and Isidori et al. \cite{141} generalized Wonham's \((A,B)\) invariance notion to smooth nonlinear systems. In nonlinear system theory the analogue of linear subspaces are distributions \(\Delta\) which are linear subspaces of the tangent space \(T_p(M)\) of a \(C^\infty\) manifold \(M\). A distribution \(\Delta(X)\) is invariant under a vector field \(f(X)\) if for every \(T(X)\in\Delta(X)\) the following holds

\[
[f,T](X) \subseteq \Delta(X) \quad \text{for all } x \in M
\]  

(2.19)

where \([\cdot,\cdot]\) denotes the usual Lie bracket. Given the system Eq. (2.11) a distribution \(\Delta(X)\) is \((f(X),g(X))\) invariant if there exists a feedback pair \((\alpha(X),\beta(X))\) with feedback

\[
U = \alpha(X) + \beta(X)V
\]  

(2.20)

where \(\alpha(X)\) and \(\beta(X)\) are smooth functions with values in \(R^m\) and \(R^{m \times m}\), respectively, such that

\[
[f + g\alpha,\Delta](X) \subseteq \Delta(X)
\]

\[
[g\beta,\Delta](X) \subseteq \Delta(X)
\]  

(2.21)

Isidori et al. \cite{141} showed that the former is equivalent to

\[
[f,\Delta](X) \subseteq \Delta(X) + \text{range } (g(X))
\]

\[
[g,\Delta](X) \subseteq \Delta(X) + \text{range } (g(X))
\]  

(2.22)

Hirschorn \cite{140} uses his results to obtain some type of disturbance decoupling by a nonlinear feedback law.
Isidori et al. [142] utilizing the results of \((f, g)\) invariant distributions to insulate the output \(y\) from some disturbance \(w\) by a nonlinear control law given in Eq. (2.19). The nonlinear system dynamics is given as

\[
\dot{X} = f(X) + g(X)u + p(X)w \tag{2.23}
\]
\[
y = h(X) \tag{2.24}
\]

Moreover, Isidori et al. show in the same work that it may be possible to obtain single input, single output decoupling such that the closed loop system under a local coordinate transformation satisfies

\[
\dot{z}_i = \tilde{f}_i(z_i) + \tilde{g}_i(z_i)u_i \tag{2.25}
\]
\[
y_i = h_i(z_i) \tag{2.26}
\]

The above noninteracting control law is one of the main tools being employed in Chapter 4 to yield simpler control of a manipulator arm on a task oriented surface \(S_T\). Claude [143] devises some useful algorithm to better utilize the results in [140], [142] and provides two examples for noninteracting control.

A number of works are reported that provide necessary and sufficient conditions for the existence of a nonlinear feedback law given in Eq. (2.19) such that the closed loop system behaves locally equivalent to a linear system [144], [145]. The latter results can be useful to utilize Davison's results on servocompensator locally. Nijmeijer [146] studies the invertibility of affine nonlinear control systems. His results may be applicable to the invertibility (singular) problem of robot manipulators. In Chapter 4 the concepts on invariant distributions is used to achieve manipulator control on task oriented surfaces.

The problem of rejecting certain types of disturbances is discussed in Chapter 5. Consequently, a few relevant papers which pertain to disturbance compen-
sation are pointed out.

Davison [147] provides necessary and sufficient conditions for the existence of a feedforward controller in a linear multivariable system, so that measurable disturbances are rejected and the outputs track some class of preassigned functions asymptotically. Davison and Goldenberg [148] and Davison [149] show that there exist necessary and sufficient conditions for the design of a robust controller of a general servomechanism problem. The feedback robust controller guarantees asymptotic disturbance rejection and tracking of some class of signals and also permits parameter variations in the plant. Given the plant

\[ \dot{X} = AX + Bu + \omega \]
\[ e = CX + Du \]
\[ y_m = C_mX + Dmu \]  

(2.27)

where \( X \in \mathbb{R}^n \) is the state, \( u \in \mathbb{R}^m \) the input, \( \omega \in \mathbb{R}^n \) the disturbance, \( e \in \mathbb{R}^v \) the error, \( y_m \) the measurable outputs with \( \dim(y_m) = \text{rank}(C_m) \), and \( \omega \) satisfies a linear differential equation. Davison [149] showed that the error \( e \to 0 \) when \( t \to \infty \) if

i) \((A, B)\) is stabilizable

ii) \((C_m, A)\) is detectable

iii) \( m \geq v \)

iv) The transmission zeros of \((C, A, B, D)\) are disjoint from the eigenvalues of the disturbance \( \omega \).

An identification scheme for the latter feedback-feedforward controller was proposed [150]. It was assumed that the plant is unknown, but linear, time-invariant, and stable. Moreover, it was assumed that the inputs can be excited and the outputs of
the plant be measured. A nonlinear optimization method was suggested to obtain
the gain parameters of a servocontroller based on certain constraint requirements
[151]. For stabilization or pole placement of linear time-invariant plants it may be
necessary to recover the state of the plants from observing their output signals.
The design of an observer for reconstructing the state $x$ of a system can be found
in [118], [119]. With the aid of Volterra theory one easily can show that linear
servocompensator theory suffices to asymptotically reject a class of disturbances
which robot manipulators could be exposed to.

In order to analyse the input-output behaviour of nonlinear analytic systems,
Volterra proposed a series expansion for nonlinear functions. This series, now
termed Volterra series, was studied by Norbert Wiener, who applied Volterra Series
to nonlinear systems [152]. Given a function

$$y(t) = T[u(t)]$$

with $y, u \in \mathbb{R}$ and $T[\cdot]$ is an operator acting on the input $u$. Volterra showed that
under certain conditions Eq. (2.25) can be expressed in terms of an infinite series
by

$$y(t) = \int_{-\infty}^{+\infty} h_1(T_1)u(t - T_1)dT_1$$
$$+ \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} h_2(T_1, T_2)u(t - T_1)u(t - T_2)dT_1dT_2$$
$$+ \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} h_3(T_1, T_2, T_3)u(t - T_1)u(t - T_2)u(t - T_3)dT_1dT_2dT_3$$
$$+ ...$$

(2.29)
in which, for causal systems, $h_i(T_1, \ldots, T_i) = 0$ for any $T_j < 0$, $0 < j \leq i$. The
functions $h_i(T_1, \ldots, T_i)$ are called the Volterra kernels. Because of the difficulty to
compute the kernel functions $h_i(T_1, \ldots, T_i)$, the Volterra theory has not attracted
much attention in the last two decades. However, recent progress in nonlinear system theory using differential geometry and Lie theory has again refocused attention to Volterra series. Brockett [153] showed that analytic controlled differential equations that are linear in the control have a Volterra series solution provided there is no finite escape time. Although, Volterra series are in general infinite, it was shown that certain analytic nonlinear systems have a finite Volterra series representation [154]. Given the Volterra series expansion of simple manipulator, it is shown in Chapter 5 that Davison's servocompensator theory may be applied to yield asymptotic disturbance rejection of a class of disturbances. Under certain conditions it may even be possible to track a desired bounded output signal [155], [156].

In an earlier part of this section it was pointed out that Davison's servocompensator theory requires that the linear system be observable, which, in turn, enables one to design an asymptotic observer. For linear time invariant systems, observers are relatively straightforward to design. However, the state variables of a nonlinear plant are not as easily observed, if at all, as in linear systems. Although much more difficult than for linear systems, a few studies on nonlinear observers with linear error dynamics are known [157], [158].

2.6 Summary

In this chapter some of the previous works which are related to this dissertation are surveyed. Since robotics research is related to a multitude of mathematical and physical subjects, this literature survey emphasizes on the subjects of modelling of multi-body system and their derivation of equations of motion, stability, and control. Furthermore, dynamics, control, and path planning of robot manipulators and linear as well as nonlinear system theories are presented.
CHAPTER III
A Comparison of a Manipulator Based on Euler or Bryant Angles
Versus Euler Parameters

3.1 Introduction

In this chapter the dynamics of the manipulator models, which serve as a test bed for the upcoming studies are formulated for a three-dimensional system. In Section 3.2, a three-dimensional three-link rigid body model is introduced. Its equations of motion are established in accordance with the state space formulation as presented by Hemami [22] with some modifications to reduce the computational burden. Of course, alternative state space representations may be used as well [24]-[26].

An alternative description of rigid body dynamics is presented in Section 3.3. This formulation incorporates Euler parameters, so that the computation of trigonometric functions are avoided. Section 3.3 underscores some inherent difficulties with Euler or Bryant angles, on the one hand and Euler parameters on the other hand. In Section 3.4 the numerically unstable, Euler parameter based dynamical description is stabilized. Lyapunov's direct method is employed to show that the proposed feedback methodology stabilizes the system globally.

Two digital computer simulations are presented in Section 3.5 contrasting the above two alternative dynamical formulations. The numerically stabilizing feedback is evaluated as well. A summary of this chapter is given in Section 3.6.
3.2 Dynamics of a Three-dimensional Three-link Manipulator

The derivation of equations of motion of a manipulator or anthropomorphic system in three-dimensional inertial space is more complex in comparison with planar systems. The dynamical equation will be established closely resembling the approach of [22]. The modifications will be pointed out in the sequel of this section.

Consider Fig. 1 which shows a general rigid body $i$. Rigidly attached to it is a body coordinate system $(x_i, y_i, z_i)$. Let us specify another coordinate system, which could be the inertial or reference coordinate system by $(X, Y, Z)$. Then there exists a coordinate transformation which specifies the angular orientation of a rigid body in the inertial reference frame. The transformation matrix $A_i$ can be computed by applying three successive rotations. Depending upon the order of rotation, the angles of rotation are named Euler or Bryant angles. Commonly, the Euler angles specify a sequence of rotation about the $Z, X,$ and $Z$ axis, or sometimes called 3-1-3 rotation. The Bryant angles, on the other hand, are obtained by rotating the coordinate system about the $X, Y,$ and $Z$ axes or 1-2-3 rotation. In this section Bryant angles will be employed to specify the orientation of a rigid body. Consider Fig. 2 which shows the three successive rotations which describe the coordinate rotation. Each transformation can be specified by a homogeneous rotation matrix as

$$A^1 = \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \phi_1 & \sin \phi_1 \\
0 & -\sin \phi_1 & \cos \phi_1
\end{bmatrix} ; A^2 = \begin{bmatrix}
\cos \phi_2 & 0 & -\sin \phi_2 \\
0 & 1 & 0 \\
\sin \phi_2 & 0 & \cos \phi_2
\end{bmatrix}$$
Figure 1: Body coordinate system and kinematic parameters of link i.
Figure 2: Coordinate rotation and Bryant angles.
Using Eq. (3.1) the transformation from the inertial or any other reference coordinate frame to a body coordinate frame is $A = A^3 \cdot A^2 \cdot A^1$. Using the abbreviated notation for $c_i = \cos \phi_i$ and $s_i = \sin \phi_i$ the directional cosine matrix $A$ yields

$$
A = \begin{bmatrix}
  \cos \phi_3 & \sin \phi_3 & 0 \\
 -\sin \phi_3 & \cos \phi_3 & 0 \\
  0 & 0 & 1
\end{bmatrix}
$$

(3.1)

Let a body be in arbitrary rotational motion with respect to some reference frame and let $w = (w_1 \ w_2 \ w_3)^T$ be the angular velocity of this body with respect to its body coordinate system. Moreover, define $\Phi = (\phi_1 \ \phi_2 \ \phi_3)^T$ be the time derivative of the Bryant angles $\Phi = (\phi_1 \ \phi_2 \ \phi_3)^T$. Then there exists a linear transformation such that

$$
w = B \dot{\Phi}
$$

(3.3)

where the elements of $B$ are obtained as follows.

$$
B = A_3 e_3 \dot{\phi}_3 + A^3 A^2 e_2 \dot{\phi}_2 + A^3 A^2 A^1 e_1 \dot{\phi}_1
$$

(3.4)

Where $e_i = (\delta_{1i} \ \delta_{2i} \ \delta_{3i})^T$ and $\delta_{ij}$ is the Kronecker delta. From Eq. (3.4) $B$ becomes

$$
B = \begin{bmatrix}
  c_2 c_3 & s_3 & 0 \\
 -c_2 s_3 & c_3 & 0 \\
  s_2 & 0 & 1
\end{bmatrix}
$$

(3.5)

The inverse of the matrix $B$ is
\[
B^{-1} = \begin{bmatrix}
\frac{c_3}{c_2} & s_3 & 0 \\
\frac{s_3}{c_2} & c_3 & 0 \\
-s_3 t_2 & s_3 t_2 & 1
\end{bmatrix}
\] (3.6)

where \(\tan \phi_2\) is abbreviated as \(t_2\). The transformation \(B^{-1}\) does not exist whenever the angle \(\phi_2\) assumes the critical values of \(\phi_2 = \pi/2 + n\pi, \ (n = \ldots, -2, -1, 0, 1, 2, \ldots)\). The choice of Euler angles will not eradicate this problem as is shown in [24]. The two linear transformations \(A\) and \(B\) are essential in deriving the dynamical equation of interconnected rigid bodies.

In what follows the dynamics of a three-link manipulator as shown in Fig. 3 is worked out in state space form, similar to the method presented in [22].

Resort back to Fig. 1 which shows the body coordinates of body \(i\). Let these coordinates coincide with the principal axis of this body. Further, let the inertia tensor of body \(i\) be \(J_i = \text{diag}(j_{i1} j_{i2} j_{i3})\). Moreover, all vector quantities bearing an index \(i\) are always referred to the \(i\)th body coordinates. The physical vector quantities mentioned before are, for example, the holonomic constraint forces, the angular velocity, and angular acceleration, etc. The orthogonal transformation matrices \(A_i\) as introduced above, transforms vectors from the \((i-1)\)th to the \(i\)th coordinate system of the manipulator. Let \(\Gamma_i\)'s be the holonomic constraint forces and \(\Delta_i\)'s be the simple nonholonomic constraint torques at joint \(i\), respectively. Define a vector cross product \(K \times \Gamma\) using matrix notation as \(\bar{K}\Gamma\) where \(\bar{K}\) is a skew symmetric matrix

\[
\bar{K} = \begin{bmatrix}
0 & -k_3 & k_2 \\
k_3 & 0 & -k_1 \\
-k_2 & k_1 & 0
\end{bmatrix}
\] (3.7)

Also define \(f_i(w) = -w_i \times (J_i w_i)\) where
Figure 3: Three-link manipulator with body and reference coordinate systems.
Because all three-links are rigidly connected and each joint has one degree-of-freedom the dynamic equations of the three-link manipulator are

\[ J_1 \dot{W}_1 = f_1(W_1) + \bar{K}_1 \Gamma_1 - \bar{L}_1 A_2^T \Gamma_2 + Q_1 \Lambda_1 - A_2^T Q_2 \Lambda_2 + R_1 u_1 - A_2^T R_2 u_2 \]
\[ J_2 \dot{W}_2 = f_2(W_2) + \bar{K}_2 \Gamma_2 - \bar{L}_2 A_3^T \Gamma_3 + Q_2 \Lambda_2 - A_3^T Q_3 \Lambda_3 + R_2 u_2 - A_3^T R_3 u_3 \]
\[ J_3 \dot{W}_3 = f_3(W_3) + \bar{K}_3 \Gamma_3 + Q_3 \Lambda_3 + R_3 u_3 \]

(3.9)

\[ m_1 \ddot{X}_1 = \Gamma_1 - A_2^T \Gamma_2 + m_1 A_1 G \]
\[ m_2 \ddot{X}_2 = \Gamma_2 - A_3^T \Gamma_3 + m_2 A_2 A_1 G \]
\[ m_3 \ddot{X}_3 = \Gamma_3 + m_3 A_3 A_2 A_1 g \]

(3.10)

where \( G = (0 \ 0 \ -g)^T \) and \( g \approx 9.81\text{m/s}^2 \). Eqs. (3.9) and (3.10) can be written in matrix form as

\[ N_{11} \dot{W} = N_{12} + N_{13} \Gamma + N_{14} A + N_{16} U \]
\[ N_{21} \dot{X} = N_{22} \Gamma + N_{22} \]

(3.11)

(3.12)

where the matrices and vectors are defined as

\[ N_{11} = \begin{bmatrix} J_1 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & J_3 \end{bmatrix}, \quad N_{12} = \begin{bmatrix} f_1(w_1) \\ f_2(w_2) \\ f_3(w_3) \end{bmatrix}, \quad N_{13} = \begin{bmatrix} \bar{K}_1 & -\bar{L}_1 A_2^T & \bar{K}_2 & -\bar{L}_2 A_3^T & \bar{K}_3 \end{bmatrix}, \quad N_{14} = \begin{bmatrix} Q_1 & -A_2^T Q_2 & 0 \\ 0 & Q_2 & -A_3^T Q_3 \end{bmatrix} \]

(3.13)
The vector $\Gamma$ is the vector of the holonomic constraint forces which act between the interconnection points of the manipulator. The simple nonholonomic constraint torques which limit rotation of two bodies relative to each other about certain axes are given by vector $\Lambda$. Since each link only has one rotational degree-of-freedom with respect to the previous link, let $R_i$ indicate the axis of rotation of that link relative to link $i - 1$ with $\|R_i\| = 1$. From this it immediately follows that the vectors $R_1, R_2,$ and $R_3$ are given by

\[
R_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad R_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}
\]

In order to restrict rotation of a body $i$ with respect to body $i - 1$ about its remaining two axes two vectors must be found for each $R_i$ which are orthogonal to $R_i$. These vectors are the column vectors in the matrices below.
For a more detailed derivation or proof respectively see [5], [24]. As a result of restricting rotational motion about the axes $R_i$ for each body $i$, the three homogeneous rotation matrices are

$$Q_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \quad Q_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \quad Q_3 = Q_2$$

The holonomic kinematical constraints can be established by expressing the position of the center of gravity of each link in terms of the transformation matrices $A_i$ and the body parameters $K_i$ and $L_i$. They are

$$X_1 + A_1^T K_1 = 0 \quad (3.14a)$$

$$X_2 + A_1^T A_2^T K_2 + A_1^T D_1 = 0 \quad (3.14b)$$

$$X_3 + A_1^T A_2^T A_3^T K_3 + A_1^T A_2^T D_2 + A_1^T D_1 = 0 \quad (3.14c)$$

where $D_i = L_i - K_i$. Taking the second derivative of Eqs. (3.13) and transforming the acceleration of the center of gravity with respect to body $i$ yields
\( \ddot{x}_1 = -\dot{w}_1 K_1 - W_1 W_1 K_1 \)  
(3.15a)

\( \ddot{x}_2 = -\dot{w}_2 K_2 - W_2 W_2 K_2 + A_2 [\dot{W}_1 D_1 + W_1 W_1 D_1] \)  
(3.15b)

\( \ddot{x}_3 = -\dot{w}_3 K_3 - W_3 W_3 K_3 + A_3 [\dot{W}_2 D_2 + W_2 W_2 D_2 + A_2 (\dot{W}_1 D_1 + W_1 W_1 D_1)] \)  
(3.15c)

where

\[
W_i = \begin{bmatrix}
0 & -w_{i3} & w_{i2} \\
-w_{i3} & 0 & -w_{i1} \\
-w_{i2} & w_{i1} & 0
\end{bmatrix}
\]

Equations (3.14) can be written in matrix form as

\[
\ddot{x} = M_1 \dot{w} + M_2(w)
\]  
(3.16)

where

\[
M_1 = \begin{bmatrix}
\dot{K}_1 & 0 & 0 \\
-A_2 \dot{K}_1 & \dot{K}_2 & 0 \\
-A_3 A_2 \dot{K}_1 & A_3 \dot{K}_2 & \dot{K}_3
\end{bmatrix}
\]

\[
M_2(w) = \begin{bmatrix}
-W_2^2 K_1 \\
-W_2^2 K_2 + A_2 \ddot{x}_1 \\
-W_3^2 K_3 + A_3 \ddot{x}_2
\end{bmatrix}
\]

(3.17)

The angular velocities \( w \) are symbolically written as

\[
w = E_1(\theta)\dot{}\dot{\theta}
\]  
(3.18)

where

\[
E_1(\theta) = \begin{bmatrix}
R_1 & 0 & 0 \\
A_2 R_1 & R_2 & 0 \\
A_3 A_2 R_1 & A_3 R_2 & R_3
\end{bmatrix}
\]

Differentiating Eq. (3.16) with respect to time yields the angular acceleration of each link. Thus
\[ \dot{w} = E_1(\theta) \ddot{\theta} + E_2(\theta, \dot{\theta}) \tag{3.19} \]

The vectors \( \dot{\theta} = (\dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3)^T \) and \( \ddot{\theta} = (\ddot{\theta}_1, \ddot{\theta}_2, \ddot{\theta}_3)^T \) are the angular velocity and angular acceleration at the joints of the manipulator.

The holonomic and simple nonholonomic constraints are assumed to be satisfied for all time, so that the constraint forces \( \Gamma \) and the constraint torques \( \Lambda \) may be eliminated by projection. Substituting Eq. (3.15) into Eq. (3.12) and combining the result with Eq. (3.11) gives

\[
\begin{bmatrix}
N_{11} & 0 \\
0 & N_{21}
\end{bmatrix}
\begin{bmatrix}
I \\
M_1
\end{bmatrix}
\dot{w} + \begin{bmatrix}
N_{12} \\
N_{21}M_2(w)
\end{bmatrix}
= \begin{bmatrix}
0 \\
N_{22}
\end{bmatrix} \Gamma + \begin{bmatrix}
N_{13} \\
N_{22}
\end{bmatrix} \Lambda + \begin{bmatrix}
N_{14} \\
0
\end{bmatrix} U
\tag{3.20}
\]

Eliminating \( w \) and \( \dot{w} \) by substituting Eqs. (3.16) and (3.17) into Eq. (3.18) yields

\[
\begin{bmatrix}
N_{11} & 0 \\
0 & N_{21}
\end{bmatrix}
\begin{bmatrix}
I \\
M_1
\end{bmatrix}
(E_1(\theta) \ddot{\theta} + E_2(\theta, \dot{\theta})) + \begin{bmatrix}
N_{12}(E_1(\theta) \dot{\theta}) \\
N_{21}M_2(E_1(\theta) \dot{\theta})
\end{bmatrix}
= \begin{bmatrix}
0 \\
N_{22}
\end{bmatrix} \Gamma + \begin{bmatrix}
N_{13} \\
N_{22}
\end{bmatrix} \Lambda + \begin{bmatrix}
N_{14} \\
0
\end{bmatrix} U
\tag{3.21}
\]

Define the matrix

\[
P = \begin{bmatrix}
I \\
M_1
\end{bmatrix}
\tag{3.22}
\]

After premultiplying the transpose of Eq. (3.19) by Eq. (3.20), the directional cosine matrix associated to the constraint forces \( \Gamma \) will be annihilated by \( P^T \).

Geometrically speaking the columns of \( P \) span the tangent space of the holonomic constraint surface, while the columns of \( N_3^T = (N_{13}^TN_{12}^T) \) are normal vectors to the
latter constraint surface. Therefore, \( P \) is orthogonal to \( N_3 \). It follows that Eq. (3.19) becomes

\[
P^T N_1 P \left( E_1 \dot{\theta} + E_2 (\theta, \dot{\theta}) \right) + P^T \begin{bmatrix} N_{12}(E_1(\dot{\theta})) \\ N_{21} M_2(E_1(\dot{\theta})) \end{bmatrix} = N_1 \begin{bmatrix} 0 \\ N_2 \end{bmatrix} \Lambda + N_1 \begin{bmatrix} 0 \\ 0 \end{bmatrix} U
\]

(3.23)

with

\[
N_1 = \begin{bmatrix} N_{11} & 0 \\ 0 & N_{21} \end{bmatrix}
\]

The simple nonholonomic constraint torques can be eliminated in a similar fashion by premultiplying Eq. (3.21) by \( E_1^T(\theta) \) as given in Eq. (3.16), because

\[
E_1^T N_{14} = 0
\]

However, when the principal axes do not coincide in the initial or zero position, the elimination of the constraint torques will become more difficult.

After defining a number of new matrices as shown below, one obtains the dynamical equation of the three-link manipulator in its minimal state space representation. Let

\[
\begin{align*}
H_1(\theta) &= E_1^T P^T N_1 P E_1 \\
H_2(\theta, \dot{\theta}) &= E_1^T P^T N_1 P E_2(\dot{\theta}) + E_1^T \left[ N_{12}(E_1 \dot{\theta} + M_1^T N_{21} M_2(E_1 \dot{\theta})) \right] \\
H_3(\theta) &= E_1^T M_1^T N_{22} \\
H_4 &= I_{n \times n}
\end{align*}
\]

(3.24-3.27)

the nonlinear differential equation is
\[ H_1(\theta)\ddot{\theta} + H_2(\theta, \dot{\theta}) = H_3(\theta) + H_4U \] (3.28)

with \( \theta, \dot{\theta}, \ddot{\theta} \in \mathbb{R}^3 \).

In the forthcoming chapters, Eq. (3.26) will be used to study trajectory planning algorithms and compensators for achieving asymptotic disturbance rejection. Also, the manipulator dynamics which is based on Bryant angles will be contrasted by establishing the dynamics of the same manipulator using Euler parameters in a redundant, non-minimal state space formulation. Fig. 3 shows a manipulator model whose equation of motion is given in Eq. (3.26).

### 3.3 Manipulator Dynamics Described by Euler Parameters

In order to motivate the use of Euler parameters in describing the equation of motion of manipulators in general, let us resort back to Eq. (3.3) in Section 3.2. Eq. (3.3) is once more given below for convenience.

\[
\begin{bmatrix}
w_1 \\
w_2 \\
w_3
\end{bmatrix} =
\begin{bmatrix}
\cos \phi_2 \cos \phi_3 & \sin \phi_3 & 0 \\
-\cos \phi_2 \sin \phi_3 & \cos \phi_3 & 0 \\
\sin \phi_2 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\dot{\phi}_1 \\
\dot{\phi}_2 \\
\dot{\phi}_3
\end{bmatrix}
\] (3.29)

As mentioned before, the inverse of the mapping above, i.e., \( B^{-1} : w \rightarrow \dot{\Phi} \) does not exist for some critical values of \( \phi_2 = \pi/2 + n\pi, \ n \in \{-2, -1, 0, 1, 2, \ldots\} \). A similar dilemma occurs if Euler angles are chosen instead of Bryant ones. The only difference is that the critical values of one of the Euler angles is \( n\pi, \ n \in \{-1, 0, 1, 2, \ldots\} \).

In order to circumvent the singularity problem associated to Bryant and Euler angles, let us revisit Eq. (3.2) and notice that there exist an eigenvector \( v \) and eigenvalue \( \lambda = 1 \) such that

\[ Av = v \] (3.30)
where $\|v\| = 1$. Euler’s Theorem [24] says that any vector which is transformed from some basis into another basis by a transformation $A$ can be rotated through an angle $\psi$ about the real eigenvector $v$. Define a quantity $q_0$ and a vector $\tilde{q}$ by

$$q_0 = \cos \frac{\psi}{2} \quad (3.31)$$
$$\tilde{q} = v \sin \frac{\psi}{2} \quad (3.32)$$

with

$$\tilde{q} = (q_1 \ q_2 \ q_3)^T$$

From Eqs. (3.29) and (3.30) it is easy to deduce that there exists a constraint such that

$$q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1 \quad (3.33)$$

Let $X$ be a vector in some arbitrary basis. Define a vector $\tilde{X}$ in the same basis which initially coincides with $X$ and is rotated about $v$ by an angle $\psi$. The vectors can be related by the equation

$$\tilde{X} = X + 2\tilde{q} \times (\tilde{q} \times X) + 2q_0\tilde{q} \times X \quad (3.34)$$

where $\times$ denotes the usual vector cross product. From Eq. (3.31) the transformation matrix $A$ may now be written in terms of the four Euler parameters as shown below

$$A = \begin{bmatrix} 2(q_0^2 + q_1^2) - 1 & 2(q_1 q_2 + q_0 q_3) & 2(q_1 q_3 - q_0 q_2) \\ 2(q_1 q_2 - q_0 q_3) & 2(q_0^2 + q_2^2) - 1 & 2(q_2 q_3 + q_0 q_1) \\ 2(q_1 q_3 + q_0 q_2) & 2(q_2 q_3 - q_0 q_1) & 2(q_0^2 + q_3^2) - 1 \end{bmatrix} \quad (3.35)$$

In order to determine an expression which relates the angular velocity $\omega$ in body coordinate system with the velocities $\dot{q}_i$, $i = 0, \ldots, 3$, differentiate Eq. (3.33) with respect to time and make use of Poisson’s equation. This yields
\[ \dot{A} = -AW \] 

(3.36)

where \( W \) is as shown in Eq. (3.14). After some manipulations one gets [24]

\[
\begin{bmatrix}
0 \\
w_1 \\
w_2 \\
w_3
\end{bmatrix} = 2
\begin{bmatrix}
q_0 & q_1 & q_2 & q_3 \\
-q_1 & q_0 & q_3 & -q_2 \\
-q_2 & -q_3 & q_0 & q_1 \\
-q_3 & q_2 & -q_1 & q_0
\end{bmatrix}
\begin{bmatrix}
\dot{q}_0 \\
\dot{q}_1 \\
\dot{q}_2 \\
\dot{q}_3
\end{bmatrix}
\] 

(3.37)

where the first row is obtained after taking the time derivative of the constraint Eq. (3.31). The matrix in Eq. (3.35) is orthonormal and hence, its inverse is its transpose. Solving for \( \dot{q}_i \) in terms of \( q_i, i = 0, \ldots, 4 \) yields

\[ \dot{q} = W_i \dot{q} \] 

(3.38)

where

\[ W_i = \frac{1}{2}
\begin{bmatrix}
0 & -w_1 & -w_2 & -w_3 \\
w_1 & 0 & -w_3 & -w_2 \\
w_2 & -w_3 & 0 & w_1 \\
w_3 & w_2 & -w_1 & 0
\end{bmatrix}
\]

It is obvious from the treatise so far that a singularity problem for Euler parameters does not exist. Furthermore, dynamical equations which are formulated based on the Euler parameter approach eliminate the need of evaluating trigonometric functions. Eqs. (3.33) and (3.36) show that the dynamics of rigid bodies can be expressed in terms of polynomials of the state of the system. One pertinent problem which is associated with the Euler parameter representation, however, shall not be ignored. As a matter of fact, embedding dynamical systems into a larger state space requires additional care whenever digital computer simulations are performed. Geometrically speaking, the trajectories of a rigid body dynamical
system move on a submanifold $M_s$, here $M_s \subset R^9$, having a velocity vector, which
is the time derivative of the vector $Q$ and which is always tangent to the solution-
submanifold and at any point $z_0$ on $M_s$. Because the velocity vector is tangential
to the submanifold, any usual finite integration method yield trajectories which
do not lie on the submanifold, except for linear vector spaces. For the case of rigid
body dynamical systems whose equations of motion are represented by Euler pa-
rameters, Eq. (3.31) cannot be satisfied for all time, because the state will diverge
from such constraint surface. In order to overcome this problem it is possible to
renormalize the Euler parameters after each integration step, or one introduces
artificial non-physically meaningful feedback, which keeps the trajectories on or in
an $\varepsilon$-neighborhood of the submanifold. The latter methodology will be employed
in the next section after introducing the Euler parameter representation of the
three-link manipulator as given in Section 3.3.

In order to express the equations of motion of the three-link rigid body manip-
ulator in the $X = (Q \dot{\theta})^T$ state space representation, the trigonometric functions
in Eq. (3.26) must be replaced by polynomials in terms of Euler parameters $q$.
This can be easily accomplished by defining the matrices $A_i$ in terms of $q_i$ instead
of $\cos \theta_i$ and $\sin \theta_i$. Because rotation of each link is limited to rotate about one
axis only, the three matrices $A_i, i = 1, \ldots, 3$ are given as

$$A_1 = \begin{bmatrix}
2q_{10}^2 - 1 & 2q_{10}q_{11} & 0 \\
-2q_{10}q_{11} & 2q_{10}^2 - 1 & 0 \\
0 & 0 & 1
\end{bmatrix} \quad (3.39)$$

$$A_2 = \begin{bmatrix}
2q_{20}^2 - 1 & 0 & -2q_{20}q_{21} \\
0 & 1 & 0 \\
2q_{20}q_{21} & 0 & 2q_{20}^2 - 1
\end{bmatrix} \quad (3.40)$$

45
\[
A_3 = \begin{bmatrix}
2q_{30}^2 - 1 & 0 & -2q_{30}q_{31} \\
0 & 1 & 0 \\
2q_{30}q_{31} & 0 & 2q_{50}^2 - 1
\end{bmatrix}
\]  \hspace{1cm} (3.41)

where

\[
\cos \theta_i = 2q_{i0}^2 - 1 \quad \sin \theta_i = 2q_{i0}q_{i1}
\]

Hence, Eq. (3.27) may be written in terms of \( Q, \dot{\theta}, \ddot{\theta} \) as

\[
H_1(Q)\ddot{\theta} + H_2(Q, \dot{\theta}) = H_3(Q) + H_4U
\]  \hspace{1cm} (3.42)

In addition to Eq. (3.40), six bilinear differential equations must be added to Eq. (3.40) to establish a relationship between \( \ddot{\theta} \) and \( Q \). These equations are in matrix form

\[
\begin{bmatrix}
\dot{q}_{10} \\
\dot{q}_{11} \\
\dot{q}_{20} \\
\dot{q}_{21} \\
\dot{q}_{30} \\
\dot{q}_{31}
\end{bmatrix} = \begin{bmatrix}
0 & -\dot{\theta}_1 \\
\dot{\theta}_1 & 0 \\
1/2 & 0 & -\dot{\theta}_2 \\
& \dot{\theta}_2 & 0 \\
& & \dot{\theta}_3 & 0
\end{bmatrix} \begin{bmatrix}
q_{10} \\
q_{11} \\
q_{20} \\
q_{21} \\
q_{30} \\
q_{31}
\end{bmatrix}
\]  \hspace{1cm} (3.43)

Eqs. (3.40) and (3.41) completely describe the motion of the manipulator in the \(( Q, \dot{\theta} )\) state space, whose domain is \( T^3 \times R^3 \), where \( T^3 \) is the triple cartesian product \( T^3 = S^1 \times S^1 \times S^1 \) and \( S^1 \) being the unit circle.

In this section the equations of motion are established based on Euler parameters. In the next section a feedback mechanism is introduced to stabilize \( Q \) in an \( \varepsilon \)-neighborhood of the domain \( T^3 \). The Euler angle and Euler parameter approach are then contrasted using digital computer simulations.
Figure 4: Domain of link i of the Euler parameters based manipulator and integration error $\varepsilon$. 

$\Omega_{n+1} = 1 + \varepsilon$

$q_{II}^2 + q_{ol}^2 = 1$
3.4 Numerical Stabilization of the Euler-Parameter Based Manipulator

In this section a mechanism is provided to stabilize the equation of motion of the three-link robot given by Eqs. (3.40) and (3.41). Let \( S \) be the dynamical system of the manipulator given by

\[
H_1(Q) \dot{\theta} + H_2(Q, \dot{\theta}) = H_3(Q) + H_4 U
\]

\[
S : \dot{Q} = T(\dot{\theta}) Q
\]  \hspace{1cm} (3.44)

with \( Q \in T^3, \theta \in \mathbb{R}^3, U \in \mathbb{R}^3 \) and the matrices \( H_1(\cdot), H_2(\cdot, \cdot), H_3(\cdot), H_4, \) and \( T(\cdot) \) are of appropriate sizes. The elements of the aforementioned matrices are analytic functions of the state variables \( Q \) and \( \theta \). Because one may assume that \( S \) is complete [136], all trajectories of \( S \) starting at some initial point \( X_o = (Q_o \ \dot{\theta}_o) \), satisfying \( X_o \in D \), where \( D = T^3 \times \mathbb{R}^3 \), and having a bounded input \( U \) satisfy \( X(t) \in D, \ te\left[t_0, \infty\right) \). Unfortunately, this is only true for analytic solutions of the differential equation (3.42). The numerically computed trajectories of the embedded, dynamical systems \( S \) will not satisfy \( X(t) \in D \) for all \( t \in \left[t_0, \infty\right) \), given that \( X_o = X(t_o) \in D \). This fact can easily be shown by observing that any vector \( \dot{Q} \), \( ||Q|| \neq 0 \) will be tangent to the surface \( T^3 \) at \( Q \), but will not be an element of \( D \), i.e., \( \dot{Q} \in D \) for all \( ||\dot{Q}|| \neq 0 \). Using a standard numerical integration technique, one computes \( \Delta Q_n \) such that \( Q_{n+1} = Q_n + \Delta Q_n \), and \( t_{n+1} = t_n + h \), where \( h \) is the integration step size, \( h > 0 \). The vector \( \Delta Q_n \) can be thought of having its origin at \( Q_n \), \( Q_n \in D \), and its endpoint outside the surface \( T^3 \) but limited by the tangent line at \( Q_n \).

Fig. 5 indicates the domain of a one-link of a manipulator which rotates around one axis. Without artificial, numerical stabilization the trajectory will diverge from the domain \( D \), eventually causing a collapse of the numerical simulation. In order to stabilize the system \( S \) numerically, the input \( U \) in Eq. (3.42)
Figure 5: Illustration of the integration error behaviour with and without feedback correction \( \delta_n \).
will not suffice. It is not difficult to show that the manipulator is controllable if it is given by its minimal realization. This means the state space is \((\theta, \dot{\theta}) \in \mathbb{R}^6\); see also [14] for a similar proof for a planar manipulator. Also, the system \(S\) will be controllable on the domain \(T^3 \times \mathbb{R}^3\) using the state space \((Q, \dot{\theta})\), because there exists an isomorphism that maps \(Q \rightarrow \theta\) and vice versa. However, in order to study numerical stability together with the regular stability of manipulator, the domain \(D = T^3 \times \mathbb{R}^3\) must be extended to include all the neighbour points of \(T^3\). Hence, the new extended domain \(D_e = \mathbb{R}^9\), which means that the Euler parameters could assume any real value.

Consider Fig. 5 together with Eq. (3.41) which shows that there exists an error \(\varepsilon_i\) for each link which is radially outward directed and orthogonal to the surface of \(S^1 \times \mathbb{R}\). This observation suggests the implementation of artificial feedback that acts radially to decrease the error \(\varepsilon_i\) for each link of the manipulator. Hence, the control can be separated to stabilise the numerical error and to control the manipulator motion via input \(U\). Geometrically the stabilization problem of the system can be divided to obtain stability in two subspaces. These two spaces can be identified as a normal space which is defined radially inward or outward of the domain \(T^3\) and the tangential space of \(T^3 \times \mathbb{R}^3\) to control motion on \(T^3 \times \mathbb{R}^3\). Define a positive definite matrix \(K \in \mathbb{R}^{n \times n}\), a diagonal positive matrix \(L = \text{diag}(L_1, L_1, L_2, L_2, L_3, L_3)\), a diagonal matrix \(R = \text{diag}(R_1, R_1, R_2, R_2, R_3, R_3)\), with \(R_i = \sqrt{q_{ii}^2 + q_{ii}^2} ; R_i \neq 0\), and a vector \(\tilde{Q} = (q_{10}(R_1 - 1), q_{11}(R_1 - 1), q_{20}(R_2 - 1), q_{21}(R_2 - 1), q_{30}(R_3 - 1), q_{31}(R_3 - 1))\).

**Lemma 3.1** The system \(S\) can be made stable within the extended domain \(D_e\) such that its equilibrium lies on \(D\) using a closed loop system as follows.

\[
H_1(Q)\ddot{\theta} + H_2(Q, \dot{\theta}) = H_3(Q) - K\dot{\theta}
\]
\[ \dot{Q} = T(\dot{\theta})Q - LR^{-1}\dot{Q} \]  \hspace{1cm} (3.45)

**Proof:** Define a Lyapunov function \( V \) as
\[ V = \frac{1}{2} \dot{R}^T \dot{R} + \frac{1}{2} \dot{\theta}^T H_1 \dot{\theta} - \int_0^t \dot{\theta}^T H_3 d\tau \]  \hspace{1cm} (3.46)

with
\[ \dot{R} = (R_1 - 1, R_2 - 1, R_3 - 1)^T \]

The second and third term of the Lyapunov function are physically meaningful and represent the kinetic and potential energy of the manipulator, respectively.

After differentiating Eq. (3.44) with respect to time one gets
\[ \dot{V} = \dot{R}^T \dot{R} + \dot{\theta}^T (H_1 \dot{\theta} + H_2^2) - \dot{\theta}^T H_3 \]  \hspace{1cm} (3.47)

Substituting the differential equation Eq. (3.43) of the closed loop system into Eq. (3.45) gives (see Appendix A for details).
\[ \dot{V} = \dot{R}^T \dot{R}(T(\dot{\theta})Q - LR^{-1}\dot{Q}) + \dot{\theta}^T (H_3 - K\dot{\theta}) - \dot{\theta}^T H_3 \]  \hspace{1cm} (3.48)

where
\[ \dot{R} = \dot{R} \cdot \dot{Q} \]  \hspace{1cm} (3.49)

and
\[ \dot{R} = \begin{bmatrix} \frac{1}{R_1} Q_1^T & 0 & 0 \\ 0 & \frac{1}{R_2} Q_2^T & 0 \\ 0 & 0 & \frac{1}{R_3} Q_3^T \end{bmatrix} \]

\[ Q_i = \begin{bmatrix} q_{i0} \\ q_{i1} \end{bmatrix} \quad i = 1 \ldots 3 \]
Noting that $\bar{R}T(\dot{\theta})Q = 0$, one gets

$$\dot{V} = -\bar{R}^T\bar{R}L^{-1}Q - \dot{\theta}^T K \dot{\theta}$$  \hspace{1cm} (3.50)

A simplification of the first term in Eq. (3.43) yields

$$\dot{V} = -\bar{R}^T L_a \bar{R} - \dot{\theta}^T k \dot{\theta}$$  \hspace{1cm} (3.51)

with $L_a = \text{diag}(L_{11}, L_{22}, L_{33})$.

Clearly, the function $\dot{V}$ is negative definite for all points in $D_e$ which are not on the equilibrium surface $T^3$ or if the system is not in equilibrium. Consequently, the system $S$ given by Eq. (3.43) is globally asymptotically stable.

Q.E.D.

The objective of the former stability discussion was to show that the system can be stabilized on the domain $D$ with its equilibrium on $T^3$. It was not desired to show that $S$ can be made stable at a particular point $Q \in T^3$, $\dot{\theta} = 0$. This can be done by adding position feedback to the system $S$ via the input $U$ as will be shown in the next section.

An alternative approach to avoid the numerical instability of Euler parameters has been stated by Wittenburg [24]. In his work he suggests to renormalize the Euler parameters to 1 after each integration step by dividing $q$ by its norm $q <= q/\|q\|$.

Before concluding this section some notes on the feedback gains $L_{ii}$ will provided. From Fig. 5 one may notice that the rate of divergence from the embedded submanifolds depends on $\dot{Q}$ and the integration step size $h$, if first order approximation for the integration method is used.

Define the error at integration step $n$ for link $i$ as $\varepsilon_n^i$.
\[ R_n^i = \|Q_n^i\|, \quad \text{and} \quad \dot{Q}_n^i = T_n^i Q_n^i \]

where
\[
T_n^i = \frac{1}{2} \begin{bmatrix}
0 & -\theta_n^i \\
\theta_n^i & 0
\end{bmatrix}, \quad Q_n^i = (q_{i0} q_{i1})_n^T
\]

Then from Fig. 6 one can easily show that
\[
Q_{n+1}^i = Q_n^i + h \cdot \dot{Q}_n^i
\]

(3.52)

where \( h \) is the integration step size, \( h > 0 \). If one notices that the two vectors \( Q_n^i \) and \( \dot{Q}_n^i \) are orthogonal, the error \( \epsilon_{n+1}^i \) can be computed as
\[
\epsilon_{n+1}^i = \sqrt{(1 + \epsilon_n^i)^2 + h^2 \|Q_n^i\|^2 - 1}
\]

(3.53)

The superscript for each body \( i \) has been dropped from Eq. (3.51) for better reading. It should be noted, however, that the equations in the present discussion are valid for each link of the three-link manipulator.

Substituting \( \|\dot{Q}_n^i\|^2 = \frac{1}{4} \dot{\theta}^2 (1 + \epsilon_n^i)^2 \) yields
\[
\epsilon_{n+1}^i = \sqrt{(1 + \epsilon_n^i)^2 + \frac{h^2}{4} \dot{\theta}^2 (1 + \epsilon_n^i)^2 - 1}
\]

(3.54)

The above may be rewritten as
\[
\epsilon_{n+1}^i = (1 + \epsilon_n^i) \sqrt{1 + \frac{h^2}{4} \dot{\theta}^2} - 1
\]

(3.55)

To achieve a small integration error, the integration step size should be chosen such that \( h^2 \cdot \dot{\theta}^2 / 4 \ll 1 \), which may require knowledge of an upper bound on \( \dot{\theta} \). If the latter condition is indeed true Eq. (3.52) may be approximated by
\[
\epsilon_{n+1}^i \approx \epsilon_n^i + \frac{h^2}{8} \dot{\theta}_n^2
\]

(3.56)
Figure 6: Reference trajectory of the manipulator tip versus time.
where \( \varepsilon_n \) satisfies \( \varepsilon_n \ll 1 \). In order to contrast the error estimate given in Eq. (3.53) after stabilizing feedback is introduced, let \( L_{ii} \) be \( k \) for link \( i \) and define \( \delta_n = -\frac{k\varepsilon_n}{1+\varepsilon_n}Q_n \cdot h \). The quantity \( \delta_n \) is a vector which is collinear to \( Q_n \). It will be directed outwards if \( \varepsilon_n < 1 \) and inwards if \( \varepsilon_n > 1 \). Using feedback the error \( \dot{\varepsilon}_n \) becomes

\[
\dot{\varepsilon}_{n+1} = \sqrt{(1 + \dot{\varepsilon}_n - h\kappa \varepsilon_n)^2 + \frac{h^2}{4} \dot{\vartheta}^2 (1 + \dot{\varepsilon}_n)^2} - 1
\]  

Eq. (3.55) may be expressed as

\[
\dot{\varepsilon}_{n+1} = (1 + \dot{\varepsilon}_n) \sqrt{(1 - h\frac{k\dot{\varepsilon}_n}{1 + \varepsilon_n})^2 + \frac{h^2}{4} \dot{\vartheta}^2} - 1
\]  

For small errors \( \dot{\varepsilon}_n \) it is reasonable to assume that

\[
(1 - h\frac{k\dot{\varepsilon}_n}{1 + \varepsilon_n}) \gg \frac{h^2}{4} \dot{\vartheta}^2
\]

This assumption leads to the following approximation.

\[
\dot{\varepsilon}_{n+1} \approx (1 + \dot{\varepsilon}_n - hK \dot{\varepsilon}_n) \left( 1 + \frac{h^2}{8(1 + \dot{\varepsilon}_n - hK \dot{\varepsilon}_n)^2} \right) - 1
\]

After simplification of Eq. (3.57) one gets

\[
\dot{\varepsilon}_{n+1} \approx \dot{\varepsilon}_n(1 - h\kappa) + \frac{h^2}{8(1 + (1 - h\kappa)\dot{\varepsilon}_n)^2} \dot{\vartheta}^2 (1 + \dot{\varepsilon}_n)^2
\]

A comparison of Eq. (3.51) and Eq. (3.58) clearly show that the feedback gain \( k \) can reduce the integration error.

In general there exists an upper bound \( \rho \) such that \( |\dot{\vartheta}| \leq \rho \). With this in mind Eq. (3.58) becomes

\[
\dot{\varepsilon}_{n+1} \approx \dot{\varepsilon}_n(1 - h\kappa) + \frac{h^2}{8(1 + (1 - h\kappa)\dot{\varepsilon}_n)^2} \rho^2 (1 + \dot{\varepsilon}_n)^2
\]
When the integration step size $h$ is selected such that $h \cdot \rho \ll 1$ the error $\hat{\xi}_{n+1}$ can be made to converge rapidly to zero.

In the next section the numerical stabilization technique as presented here is being verified by digital computer simulations. The results are contrasted with conventional, or Bryant angle based manipulator dynamics.

### 3.5 Comparative Simulation of the Contrasted Manipulator Dynamics

In this section the results of two digital computer simulations are discussed. The simulations intend to verify our stability discussion of the Euler parameter based manipulator as treated in Section 3.3 and 3.4. These simulation results are compared with a computer simulation of the Bryant angle based robot as derived in Section 3.2. Both simulations are conducted with identical kinematic and kinetic parameters. Also, as a matter of course, both robot models are controlled to perform an identical desired cartesian path within their workspace. Tables 1 and 2 exhibit the kinematic and kinetic parameters, respectively.

For the movement of the manipulator tip a straight line path is selected. The initial position of the tip is selected to be $Y_f = (1.4m, 0m, 2m)$ and the final position is $Y_p = (0.4m, 0.4m, 1m)$. The desired motion was performed within a time frame of $t_{ex} = 1s$, and the integration step size was $h = 10ms$. The cartesian path planning algorithm as well as the stability mechanism are not discussed in this section. These topics are deferred for a more in-depth treatment in the next chapter.

The reference trajectories of the manipulator are given in Fig. 6 and Fig. 8 respectively for the tip position and tip velocity as a function of time. Fig. 7 shows the desired path along which the manipulator tip shall move in its inertial, or world reference frame. The total length of the cartesian path is $l = 1.47m$.
Figure 7: Desired reference trajectory of the manipulator tip in reference coordinates.
Figure 8: Reference velocity of the manipulator tip versus time.
Figure 9: Trajectories of the Bryant angles of the manipulator versus time.
Figure 10: Trajectories of the angular velocities of the manipulator versus time. (Euler parameters based manipulator model)
Figure 11: Error of the desired tip position versus time. (Bryant angles based dynamics)
Figure 12: Input torques of the manipulator versus time. (Bryant angles based dynamics)
Figure 13: Euler parameters $q_{10}$ versus time.
Figure 14: Euler parameters $q_{11}$ versus time.
Figure 15: Trajectories of the angular velocities versus time. (Euler parameters based dynamics)
Figure 16: Error of the desired tip position versus time. (Euler parameters based dynamics)
Figure 17: Error of the Euler parameter constraint condition versus time.
along which the tip moves. The maximum reference tip velocities are $2m/s$ for the first and third link and $0.8m/s$ for the second link. The simulation results of the Bryant angle based dynamics are shown in Fig. 9-12, those of the Euler parameter based one are presented in Fig. 13-17. The motion trajectories of the Bryant angles $\theta_1 \ldots \theta_3$ are given in Fig. 9, and their first time derivative $\dot{\theta}_1 , \ldots \dot{\theta}_3$ in Fig. 10. The former figure shows that the maximum angular velocity is assumed by link three and is approximately $4rad/s$. As has been discussed in the previous section, the maximum angular velocity is a useful figure for estimating the radial error of the Euler parameters based dynamics; compare Eq. (3.58). An additional important measure to compare the two different dynamical representations is the error of the tip of the robot. They are likewise important for judging the path planning mechanism as well as the incorporated feedback scheme. Fig. 11 shows that the maximum error occurs approximately at the maximum tip speed of the manipulator. The maximum error $\|e_t\|$ from the desired path is less than $16.5mm$ and it vanishes if $\|\dot{\theta}\| \to 0$. The three input torques at the joint of the manipulator are exhibited in Fig. 12. The input torques are nicely bounded and not of excessive magnitude, although the input torque $u_2$ for link two reaches a value of about $280Nm$. A simulation diagram of the Euler parameter based manipulator with feedback loops is shown in Fig. 18. Figures 13 and 14 show the plots of the Euler parameters $q_{io}$ and $q_{ii}, i = 1 \ldots 3$, as a function of time, respectively. Since the latter plots cannot be directly compared with the Bryant angles, the angular velocities $\dot{\theta}_1 \ldots \dot{\theta}_3$ and the error of the desired tip position versus time are given. The angular velocities, as they are state variables, in the Euler parameterized dynamics, are shown in Fig. 15. A comparison of the latter figure with Fig. 10 indicates no discernible difference of either dynamical simulation. Likewise, the error of the desired tip position on Figs. 11 and 16 are indistinguishable.
Figure 18: Simulation diagram of the Euler parameters based manipulator with feedback.
The most important results of this analysis, however, are shown in Fig. 17. Fig. 17 exhibits the error of the constraint condition using stabilizing feedback as proposed in the previous section. From Eq. (3.58) in Section 3.5 the suggested feedback gain to stabilize the Euler parameters numerically is $L_{ii} = \frac{1}{k_i^2}$. This implies that the error $\varepsilon_n$ is of order $0((h\dot{\theta})^2)$ when the Euler integration algorithm is employed [159]. The implementation of a more accurate integration method will, of course, yield better results. In the present discussion the 4th order Runge-Kutta method was used to obtain the results as discussed. A closer look at the results of Fig. 17 indeed shows that the actual error is better than the estimated one for the Euler integration method. Knowing the integration step size $h$ and the angular velocity $\dot{\theta}$ suggests that the error $\varepsilon_n$ is in the order of $0((h\dot{\theta})^6)$ which certainly is a better result.

Hence, the simulation results clearly show that Euler parameters can be stabilized in a $\varepsilon$-neighborhood of the constraint surface, once a bound on $\dot{\theta}$ is known. From the latter bound the integration step size and the feedback gain matrices $L$ can be approximately chosen.

3.6 Summary

In this chapter the dynamics of a three-dimensional three-link manipulator, whose dynamics was derived from Newton-Euler equations using Bryant angles for coordinate rotations, was established. It will serve as a test environment in the following chapters. An alternative, Euler parameter based model, was established to be contrasted with the aforementioned one. The numerically unstable, redundant Euler parameter based model was stabilized by artificial, non-physically meaningful feedback. Stability was proved by Lyapunov's direct method. Two digital computer simulations are conducted to verify the feedback scheme and to
Table 1: Kinematic Manipulator Parameters

<table>
<thead>
<tr>
<th>Link i</th>
<th>$k_i[m]$</th>
<th>$l_i[m]$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$x$</td>
<td>$y$</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

shed more light into the qualitative and quantitative aspects of the two different dynamical descriptions.
Table 2: Kinetic Manipulator Parameters

<table>
<thead>
<tr>
<th>Link $i$</th>
<th>Inertia [kg m$^2$]</th>
<th>Mass [kg]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.0 1.0 6.0</td>
<td>25</td>
</tr>
<tr>
<td>2</td>
<td>0.5 0.5 5.0</td>
<td>20</td>
</tr>
<tr>
<td>3</td>
<td>0.5 0.5 25.0</td>
<td>10</td>
</tr>
</tbody>
</table>
CHAPTER IV

Trajectory Planning on Task Oriented Surfaces and Nonlinear Decoupling

4.1 Introduction

In an industrial environment robot manipulators are employed to accomplish particular tasks within complex manufacturing processes. The manipulator shall be able to improve cost, reliability, and other criteria of the manufactured product in order to be useful. Achieving this may not always be easy. In fact, the latter criteria may only be satisfied if the manipulator movements are properly planned and executed. This requires that attention be given to path planning methodologies that are task oriented and are most suitable for a particular application as part of a manufacturing process.

A task oriented surface or path $S_T$ can be defined to insert and tighten a bolt into a hole by superimposing rotational motion and axial, inwards directed motion of the end-effector of a manipulator. In fact, for this particular example the rotational and translational gripper movements are mechanically related by the slope of the thread. This indeed means that the degrees-of-freedom of the employed robot arm shall be limited to one during the latter task by a proper feedback controller.

It is shown in the forthcoming discussion that a task oriented surface $S_T$ can in general be specified by a number of output functions in some arbitrary coordi-
nate system which is most suitable for a particular application. Recent nonlinear system-theoretical results, that are based on differential geometric concepts are employed to prescribe the gripper behaviour as functions of one or two input parameters depending upon the dimension of the surface $S_T$. The output channels are decoupled from the inputs by means of invariant distributions. This methodology also comprises the concepts of invariant constraint subspaces as introduced by Hemami et al. [39] to maintain some outputs at a constant value. Beyond this it is shown that the nonlinear decoupling method may allow arbitrary pole placement whenever the system can be made linear locally. An alternative nonlinear control scheme is discussed by Freud [161].

A decision scheme is presented to switch sets of output channels sequentially in order to obtain more sophisticated end-effector movements. Digital computer simulations are conducted to verify the concepts developed in this chapter.

4.2 Output Invariance and Input-Output Decoupling

Consider the mechanical manipulator with three rotational degrees-of-freedom (DOF) as established in Chapter 3. The dynamics of this manipulator is described by three second order nonlinear differential equations. For this manipulator it is possible to define three analytic output functions which relate the joint angles to the position of the gripper in the workspace coordinate system. The manipulator dynamics $E$ together with the end-effector output functions is

$$\Sigma : \ H_1(\Theta)\dot{\Theta} = H_2(\Theta, \dot{\Theta}) + H_3 U$$

$$y_p = H_p(\Theta)$$ (4.1)

where $\Theta, \dot{\Theta}, \ddot{\Theta} \in R^3, y_p \epsilon R^3$, and $H_1, H_2, H_p$ are matrices and vectors whose entries are analytic functions, and $H_3$ is a constant $3 \times 3$ invertible matrix. The
vector \( \mathbf{y}_p = (y_{p1}, y_{p2}, y_{p3})^T \) represents the translational position of the gripper in the coordinate system \((Y_1Y_2Y_3)\) given in Fig. 20. For the aforementioned manipulator the output functions in the \((Y_1Y_2Y_3)\) coordinate system are

\[
\begin{align*}
    y_{p1} &= \cos \Theta_1 (d_2 \sin \Theta_2 + d_3 \sin(\Theta_2 + \Theta_3)) \\
    y_{p2} &= \sin \Theta_1 (d_2 \sin \Theta_2 + d_3 \sin(\Theta_2 + \Theta_3)) \\
    y_{p3} &= d_1 + d_2 \cos \Theta_2 + d_3 \cos(\Theta_2 + \Theta_3)
\end{align*}
\] (4.2a, 4.2b, 4.2c)

The latter equations define the set \( R_p, R_p \subset \mathbb{R}^3 \), which is the reachable set of the manipulator's end-effector. This is equivalent to \( R_p \) coinciding with the workspace of the manipulator. Let a task oriented smooth surface \( S_T \) be in the reachable space \( R_p \) which, in general, represents the desired path of the end-effector in some reference frame. A task oriented surface, for instance, could be a cartesian path or a circular path.

In general, manipulators possess more than three DOF's which renders a gripper to change also its orientation within the workspace of a manipulator. When a robot arm allows full freedom in its orientation and position, three additional output functions ought to be specified. This makes it possible to control rotational motion (i.e., orientation change) of the robot hand in some coordinated fashion together with translational movements of the gripper. The rotational velocity of a gripper with respect to the gripper's body coordinate system can symbolically be expressed as

\[
\dot{y}_o = H_o(\Theta, \dot{\Theta})
\] (4.3)

where \( \dot{y}_o \in \mathbb{R}^3 \) and \( H_o(\cdot, \cdot) \) is an analytic function of the state of the manipulator.
Define the state of the system $\Sigma$ to be $X = (\Theta, \dot{\Theta})^T \in \mathbb{R}^{2n}$ where henceforth a general robot arm is considered. However, the three link manipulator will be used as an example throughout this chapter.

Let the manipulator dynamics be rewritten in a more abstract setting as

$$\dot{X} = F(X) + G(X)U$$
$$y = H(X)$$

(4.4)

where

$$F(X) = (\dot{\Theta} - H_1^{-1}(\Theta)H_2(\Theta, \dot{\Theta}))^T$$
$$G(X) = (0 - H_1^{-1}(\Theta)H_3)^T$$
$$H(X) = (H_p(\Theta) - H_o(\Theta, \dot{\Theta}))^T$$

with $U \in \mathbb{R}^n, y \in \mathbb{R} \times \mathbb{R}^3, R_p \subset \mathbb{R}^3$. Let us further assume that the manipulator is non-redundant, i.e., it has six degrees of freedom.

Consider Fig. 7 which shows that the gripper moves along a cartesian path between the point $Y_f = (1.4, 0m, 2.0m)^T$ and $Y_f = (0.4m, 0.4, 1m)$. For this path it is possible to find three output functions in a coordinate system $\hat{Y}$ such that two output channels remain constant and one output channel specifies the location of the end-effector on the straight line connecting the points $Y_f$ and $Y_f$. Because only one output channel is varying it will be shown in the sequel that one can design a controller that allows motion control with just one reference input trajectory. In general terms, suppose that any task oriented surface $S_T$ in the work space of a manipulator can be expressed in terms of an output vector $H(X)$ such that at most two output channels may vary. Then these two channels uniquely specify a point on a two dimensional surface $S_T$. Most practically the two output channels
should be associated to position and orientation of the gripper on the surface $S_T$. If a well defined relationship exists between the position and the orientation of a gripper then this relation specifies an additional invariant output channel. Thus, one channel remains not invariant and in addition specifies any point on the task oriented surface $S_T$.

If it is possible to find sufficient invariant output functions that specify a task oriented surface $S_T$, then these output channels induce a subspace $M_S \subset R^{2n}$. Traditionally the subspace $M_S$ ($M_S$ may also be specified as a submanifold) is expressed in terms of holonomic and nonholonomic constraints. While holonomic constraints specify the curve for translational motion of the end-effector, the nonholonomic constraints limit the end-effectors rotation about some arbitrary axis of rotation. Symbolically, holonomic non-time-varying constraints of translational motion can be written as

$$C(\Theta) = 0$$  \hspace{1cm} (4.5)

and nonholonomic constraints may be expressed as

$$C(\Theta, \dot{\Theta}) = 0$$  \hspace{1cm} (4.6)

Eq. (4.6) will be linear in $\dot{\Theta}$ if the constraints are of simple non-holonomic type.

For a moment let us reconsider the three link manipulator. Suppose it is desired to drive the manipulator tip collinear with respect to the $Y_1$ coordinate of the workspace. In doing this, the output variables $y_{p2}$ and $y_{p3}$ in Eq. (4.2) must remain constant, i.e., invariant, during the manipulator movement. This means the two output equations Eq. (4.2b) and Eq. (4.2c) specify the task oriented surface $S_T$ in a similar fashion as a set of holonomic constraint equations would
To restrict rotational motion of an end-effector of a general manipulator, simple nonholonomic type constraints should be established as discussed below.

Reconsider Euler's Theorem [24] in Section 3.3. It states that two arbitrary linear basis can be aligned with each other by rotating one of the bases about a vector \( v_r \) through an angle \( \psi \). The vector \( v_r \) is an eigenvector of some homogeneous transformation matrix \( A \) with eigenvalue \( \lambda = \pm 1 \). Let two homogeneous (orthonormal) rotation matrices \( A_I \) and \( A_F \) define the orientation of a manipulator with respect to some inertial reference frame at some initial and some final time, respectively. Then, \( A_I \) and \( A_F \) can be related by

\[
A_F = A_r A_I
\]

Solving for \( A_r \) gives

\[
A_r = A_F A_I^{-1}
\]

For each rotation matrix \( A_r \) there exist a non-unique eigenvector \( v_r \) about which the coordinate system is being rotated. Let \( \omega_g = (\omega_{g1} \ \omega_{g2} \ \omega_{g3})^T \) be the angular velocity of a gripper in the gripper's body coordinates. The desired angular velocity of the gripper \( \omega_g \) about the vector \( v_r \) is

\[
\omega_g = v_r \cdot \dot{\psi}_r
\]

where it is assumed that the eigenvector \( v_r \) is normalized to one, i.e. \( \| v_r \| = 1 \). The scalar quantity \( \dot{\psi}_r \) is the angular velocity about the vector \( v_r \). In general, however, \( \omega_g \) can be stated as function of the generalized coordinates and velocities as given below

\[
\omega_g = B_g(\Theta)\hat{\Theta}
\]

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A manipulator which has full rotational degrees-of-freedom satisfies $\omega_p \in \mathbb{R}^3$ assuming no physical limitations are imposed upon the angular velocities. For, if a desired rotation is being executed, Eq. (4.10) must satisfy Eq. (4.9). This means that the vector space of $\omega_p$ may only be spanned by the vector $v_r$. Hence, equating Eqs. (4.9) and (4.10) gives

$$B_p(\Theta)\dot{\Theta} = v_r \psi$$

(4.11)

Define a matrix $Q_r \in \mathbb{R}^{3 \times 3}$ whose row vectors are orthogonal to $v_r$ and together with $v_r$ span the space $\mathbb{R}^3$.

Premultiplying Eq. (4.11) with $Q_r$ yields two simple nonholonomic constraint equations. They are

$$Q_r B_p(\Theta)\dot{\Theta} = 0$$

(4.12)

On the other hand if Eq. (4.11) is premultiplied by $v_r^T$ one gets

$$\dot{\psi}_r = v_r^T B_p(\Theta)\dot{\Theta}$$

(4.13)

Define a new output vector $y = (y_1 \ y_2)^T$ which is partitioned such that $y_1$ and $y_2$ contain the nonconstant and the constant output channels, respectively. The dimension of $y_1$ is either one or two depending upon whether gripper rotation may take place independently of translational motion. The dimension of the partition $y_2$ depends also on the latter criterion as well as on the number of degrees-of-freedom of the manipulator. For example, if the three link manipulator shall move in parallel with the $Y_1$ axis as mentioned previously, then the varying output channel is shown by Eq. (4.2a) and the invariant output partition $y_2$ is given by Eqs. (4.2b) and (4.2c).
Suppose the same cartesian type motion shall be performed with a six DOF robot arm with concurrent change of its gripper orientation. In this case the output partitions \( y_1 \) and \( y_2 \) are augmented by Eq. (4.13) and Eq. (4.12), respectively.

Having defined a partition on the output vector \( y \) the objective in the remaining part of this section is as follows. It is desired to find a closed loop system of some robot manipulator with nonlinear control

\[
U = \alpha(X) + \beta(X)V
\]

such that a number of control objectives are achieved:

i) The output channels in the partition \( y_1 \) are mutually input/output decoupled. This means that rotational and translational motion can be controlled independently on \( S_T \) via two inputs \( v_1 \) and \( v_2 \).

ii) The output partition \( y_2 \) can be made to remain constant. (This may also depend on the initial conditions).

iii) The closed loop system is asymptotically stable.

If the above requirements can be satisfied with smooth control as given in Eq. (4.14) and the end-effector satisfied \( y = H(X(t_0))eS_T \) at time \( t_0 \), then \( H(X(t))eS_T \) for all time \( t \in [t_0, t_1] \). Moreover, the controls \( v_1 \) and \( v_2 \) steer the robot arm to any point on the task oriented surface \( S_T \) such that one input controls translational motion and the other one rotational motion of the end-effector.

Let the closed loop dynamics of a nonredundant manipulator with static feedback pair \( (\alpha(X), \beta(X)) \) be

\[
\dot{X} = \tilde{F}(X) + \tilde{G}(X)V
\]

\[
y = H(X)
\]
where
\[
\tilde{F}(X) = F(X) + G(X)\alpha(X)
\]
\[
\tilde{G}(X) = G(X)\beta(X)
\]

Next, the notion of an invariant distribution is introduced. An invariant distribution can be seen to be the analogue to invariant linear subspaces.

**Definition 4.1** A distribution \(\Delta\) is said to be locally controlled invariant if for each \(x \in M\) there exist a neighbourhood \(U\) of \(x\) and a feedback pair \((\alpha(x), \beta(x))\) defined on \(U\) with the property that \(\Delta\) is invariant under the vector fields \(\tilde{F}(X), \tilde{G}_i(X), i = 1, \ldots, n \leq 6\) for all \(x \in U\); i.e.,
\[
[F, \Delta](X) \subset \Delta(X) \quad (4.16a)
\]
\[
[G_i, \Delta](X) \subset \Delta(X) \quad (4.16b)
\]

where \(\tilde{G}_i\) is the \(i\)-th column of the matrix \(\tilde{G}\). According to Frobenius' Theorem (Appendix B, Theorem B.1) a distribution \(\Delta\) is completely integrable if it is invariant under \(\tilde{F}\) and \(\tilde{G}\). Moreover, the flow of the latter vector fields evolve on an \(m\)-dimensional manifold with \(m = \dim \Delta\).

**Lemma 4.1** Suppose \(\Delta\) is an involutive and nonsingular distribution on \(M\). Moreover, let \(\tilde{G} = \text{span}\ \{G_1, \ldots, G_n\}\) and \(\Delta + \tilde{G}\) be nonsingular on \(M\) as well. Then \(\Delta\) is locally controlled invariant if, and only if,
\[
[F, \Delta](X) \subset \Delta(X) + \tilde{G}(X) \quad (4.17a)
\]
\[
[G_i, \Delta](X) \subset \Delta(X) + \tilde{G}(X) \quad 1 \leq i \leq 6 \quad (4.17b)
\]
A proof of this lemma is given in [141]. From the previous lemma it is interesting to observe that a locally controlled invariant distribution $A$ is independent of the feedback pair $(\alpha(X), \beta(X))$. To guarantee that an output is independent of some input as required in condition (iii) of the aforementioned problem statement, the following lemma applies.

**Lemma 4.2** The output $y_i = h_i(X)$ is independent of input $v_j$ if, and only if, for all $k \geq 1$ and for arbitrary choices of vector fields $\tau_1, \ldots, \tau_k \in \{\tilde{F}(X), \tilde{G}_1(X), \ldots, \tilde{G}_n(X)\}$

$$L_{\tilde{G}_j} h_i(X) = 0$$
$$L_{\tilde{G}_j} L_{\tau_1} \ldots L_{\tau_k} h_i(X) = 0$$  (4.18)

**Proof** [160] (p. 97).

Using Definition 4.1 and the latter lemma, one may state a more useful one.

**Lemma 4.3** The output $y_i$ is unaffected by input $v_j$ if, and only if, there exists a distribution $\Delta$ that fulfills

i) $\Delta$ is invariant under $\{\tilde{F}, \tilde{G}_1, \ldots, \tilde{G}_n\}$

ii) $\tilde{G}_j(X) \in \Delta \subset (\text{span}\{dh_i\})^\perp$

**Proof** [160].

The aforementioned two lemmas have important implications in the area of constraint dynamic systems [39]. For instance, if one defines a smooth function $h_i(X)$ to represent a holonomic or simple nonholonomic constraint which shall satisfy $h_i(X) = 0$ for all possible inputs $u_j, j = 1, \ldots, n$, then it is obvious that every (open loop) vector field $G_j$ should satisfy $G_j \in \Delta$. Whereas $\Omega^\perp = \Delta$ is the
annihilator of the codistribution spanned by the one-forms of $h_i(X)$, i.e., $dh_i(X)$, as defined in Definition B.8. Moreover, the distribution $\Delta$ must be invariant under the vector fields of the set $\{F(X), G_1(X), \ldots, G_n(X)\}$. If condition (i) and (ii) are satisfied in Lemma 4.3 for all input vector fields, $G_j$, one may apply Lemma B.4 to establish a local coordinate transformation that may guarantee that $h_i(X) = 0$ for all $t \in [t_0, t_1]$ provided that $h_i(X(t_0)) = 0$.

In general, however, the vector fields in the set $\{F(X), G_1(X), \ldots, G_n(X)\}$ do not satisfy condition (i) and (ii) in Lemma 4.3 unless some suitable feedback pair $(\alpha(X), \beta(X))$ can be found such that the closed loop system satisfies the latter conditions. In accordance with Hemami et al. [39] a distribution $\Delta$ which fulfills Lemma 4.3 for all $G_i, i = 1, \ldots, n$, may be called a constraint distribution $\Delta_w$.

In terms of the present discussion of introducing a task oriented surface it is presumed that the task oriented surface can always be expressed in terms of a maximal number of output invariant channels. This means that any task oriented translational path of a gripper can be defined by two constant output functions and one varying one which specifies the points on that path. For example, reconsider the three link manipulator whose tip shall move in parallel with the $Y_1$ coordinate of the reference frame as shown in Fig. 3. Because the tip coordinates in the directions of $Y_2$ and $Y_3$ remain constant, the two invariant output functions can be readily established.

The advantage of expressing the surface $S_T$ in terms of a maximal number of output invariant channels is that it minimizes the number of output channels to be actively controlled. Henceforth it also minimizes the number of control inputs. Recall that the output vector $y$ can be partitioned into $y = (y_1, y_2)^T$ such that $y_1$ are the varying output channels and $y_2$ the invariant ones. In terms of the state vector $x$, they are
\[(y_1, y_2)^T = (h_1(X) \ h_2(X))^T \] (4.19)

Since \(h_2(X)\) = constant and from Lemma 4.3 follows that the constraint distribution \(\Delta_w\) must satisfy

i) \(\Delta_w(X) \subset (\text{span}\{dh_2(X)\})^\perp\)

ii) \([F(X), \Delta_w(X)] \subset \Delta_w(X)\)

\([G_i(X), \Delta_w(X)] \subset \Delta_w(X) \quad i = 1, \ldots, n\)

iii) \(G_i(X) \in \Delta_w(X)\)

To achieve independent control of translational and rotational motion of a manipulator gripper, the \(i\)-th output of the partition \(y_1\) shall only be controlled by the \(i\)-th input. In terms of the three link robot arm cited before, the dimension \(\dim(h_1) = 1\). For a general manipulator the dimension is at most \(\dim(y_1) = 2\).

A feedback pair \((\alpha(X) \ \beta(X))\) with noninteracting control can be found according to Lemma B.4 and Lemma 4.3 if there can be found locally, a nonsingular distributions \(\Delta_1, \Delta_2, \Delta_3\) with \(\Delta_3 = \Delta_w\) satisfying

i) The \(\Delta_i\)'s are invariant under the vector fields of the closed loop system; i.e.,

\([\tilde{F}, \Delta_i](X) \subset \Delta_i(X)\)

\(\quad j = 1, 2, \ i = 1, 2, 3\) (4.20)

\([\tilde{G}_j, \Delta_i](X) \subset \Delta_i(X)\)

ii)

\(\Delta_1 \subset (\text{span}\{dh_{11}\})^\perp = \tilde{H}_1\) (4.21a)

\(\Delta_2 \subset (\text{span}\{dh_{12}\})^\perp = \tilde{H}_2\) (4.21b)

\(\Delta_3 \subset (\text{span}\{dh_2\})^\perp = \tilde{H}_3\) (4.21c)
iii) \[ \dot{G}_1 \in \Delta_2 \]
\[ \dot{G}_2 \in \Delta_1 \]

\[ \dot{G}_1, \dot{G}_2 \in \Delta_3 \]

(4.22a)  
(4.22b)  
(4.22c)

In terms of the three link manipulator moving its tip along the \( Y_1 \) coordinate, one gets \( \text{dim}(y_1) = 1 \) and

\[
dh_{11} = dh_1 = \begin{pmatrix}
-\sin \Theta_1 (d_2 \sin \Theta_2 + d_3 \sin(\Theta_3 + \Theta_8)) \\
\cos \Theta_1 (d_2 \cos \Theta_2 + d_3 \cos(\Theta_3 + \Theta_8)) \\
\cos \Theta_1 d_3 \cos(\Theta_3 + \Theta_8)
\end{pmatrix}^T
\]

(4.23)

\[
dh_2 = \begin{pmatrix}
\cos \Theta_1 (d_2 \sin \Theta_2 + d_3 \sin(\Theta_3 + \Theta_8)) \\
\sin \Theta_1 (d_2 \cos \Theta_2 + d_3 \cos(\Theta_3 + \Theta_8)) \\
\sin \Theta_1 d_3 \cos(\Theta_3 + \Theta_8)
\end{pmatrix}^T
\]

(4.24)

To find a feedback pair \((\alpha(X), \beta(X))\) such that the requirements (i) - (iii) are satisfied, an algorithm is outlined which yields a set of largest locally invariant distributions \( \Delta_i \) in \( \bar{H}_i \). For a detailed discussion of this algorithm see Claude [143] and Isidori [160].

Define a characteristic number \( \rho_i \) for each \( h_i(X) \) such that

\[
L_{G_j} L_F^k h_i(X) = 0
\]

(4.25a)

for all \( k < \rho_i, j = 1, \ldots, 6 \) and all \( x \in M \); and

\[
L_{G_j} L_F^{\rho_i} h_i(X) \neq 0
\]

(4.25b)

for some \( x \in M \).

Define a matrix \( A(X) \) with elements \( a_{ij}(X) \) and a vector \( b_i(X) \) with elements \( b_i(X) \) as follows

\[
a_{ij}(X) = L_{G_j} L_F^{\rho_i} h_i(X)
\]

(4.26)

\[
b_i(X) = L_F^{\rho_i+1} h_i(X)
\]

(4.27)
Isidori has shown that the local output noninteracting control problem is solvable with feedback pair \((\alpha(X), \beta(X))\) if and only if \(\text{rank}(A(X)) = \text{number of output channels for all } x\), i.e. \(\text{rank}(A(X)) = 6\) [160]. Define a matrix \(\tilde{A}(X) \in \mathbb{R}^{n \times q}\) and a vector \(\tilde{b}(X) \in \mathbb{R}^n\) with \(n = \text{number of DOF's and } q = \text{number of closed loop control inputs such that}\)

\[
\begin{align*}
\tilde{A}(X) &= A(X) \beta(X) \quad (4.28a) \\
\tilde{b}(X) &= A(X) \alpha(X) + b(X) \quad (4.28b)
\end{align*}
\]

The entries \(a_{ij}(X)\) of the matrix \(\tilde{A}(X)\) relate the \(i\)-th output to the \(j\)-th input.

This means, for example, if single input, single output decoupling is desired then \(\tilde{A}(X)\) will be diagonal provided the number of inputs is equal to the number of outputs. For the three link robot arm introduced in Chapter 3, \(\tilde{A}(X) \in \mathbb{R}^{3 \times 1}\) with \(\tilde{A}(X) = (\tilde{a}_{11}(X) \; O \; O)^T\).

In order to find the actual entries in \(\tilde{A}(X)\) and \(\tilde{b}(X)\), one may resort to a lemma of Claude [143] who has shown that every locally controlled invariant distribution contained in \(\tilde{H}_i\) is also contained in a distribution \(\tilde{\Delta}_i\) which is invariant under the closed loop system \(\tilde{F}(X)\) and \(\tilde{G}_i(X)\) \(i = 1, \ldots, q\). This means

\[
\begin{align*}
[\tilde{F}, \tilde{\Delta}_i] &\subset \tilde{\Delta}_i \\
[\tilde{G}_j, \tilde{\Delta}_i] &\subset \tilde{\Delta}_i \quad (4.29)
\end{align*}
\]

if and only if

\[
\begin{align*}
d\tilde{a}_{ij}(X) &\subset \tilde{\Delta}_i^\perp \\
d\tilde{b}_i(X) &\subset \tilde{\Delta}_i^\perp \quad (4.30)
\end{align*}
\]
In particular, if $A(X)$ is of full rank at some $x$ then in a neighbourhood $U$ of $x$ the
distribution $\Delta_i$ is equal to the largest locally controlled invariant distribution $\Delta_i$
contained in $\tilde{H}_i$ [143].

Suppose the matrix $A(X)$ is invertible. Then the statement from Claude
provides a simple way to obtain an input-output decoupled system. Let $\tilde{b}_i = 0,
\tilde{a}_{ij} = 0$ if $i \neq j$ and $\tilde{a}_{jj} = \text{const.} \neq 0$ if $i = j$, then

\begin{align*}
\beta(X) &= A^{-1}(X) \tilde{\alpha}(X) \\
\alpha(X) &= -A^{-1}(X) \tilde{b}(X)
\end{align*}

Clearly, the differentials of all $\tilde{b}_i$'s and all $\tilde{a}_{ij}$ belong to $\tilde{H}_i^{-1}$ and hence the feedback
pair will decouple the closed loop dynamics of the manipulator. Nevertheless a
feedback pair such as given by Eq. (4.31) does not guarantee stability of the
decoupled closed loop dynamics. To find a feedback pair $(\alpha(X), \beta(X))$ such that
the manipulator dynamics is stable and the control objectives are (i) - (iii) are
met, define a new set of coordinates in the following way. Let

\[ z_{ij} = L_{P_i} h_i(X) \]  \hspace{1cm} (4.32)

for all $0 \leq j \leq \rho_i$ and $1 \leq i \leq \ell$, where $\ell$ is equal to the number of output functions.
For our three link robot arm $\ell = 3$; for general manipulators $\ell = 6$.

If $j < \rho_i$ one can show that

\[ \dot{z}_{ij} = z_{i,j+1} \]  \hspace{1cm} (4.33)

To show the above, let

\[ z_{ij} = L_{P_i}^j h_i \]  \hspace{1cm} (4.34)
and differentiate with respect to time gives

\[ \dot{z}_{ij} = \frac{\partial z_{ij}}{\partial X} \dot{X} \]  \hfill (4.35a)

\[ = \frac{\partial z_{ij}}{\partial X} \left( \tilde{F}(X) + \sum_{k=1}^{q} \tilde{G}_k v_k \right) \]  \hfill (4.35b)

\[ = L_{\tilde{F}} z_{ij} + \sum_{k=1}^{q} L_{\tilde{G}_k} z_{ij} v_k \]  \hfill (4.35c)

\[ \dot{z}_{ij} = L_{\tilde{F}} I_F^j h_i + \sum_{k=1}^{q} L_{\tilde{G}_k} L_{\tilde{F}}^j h_i v_k \]  \hfill (4.35d)

But \( L_{\tilde{F}}^j h_i = L_{\tilde{F}}^j h_i \) for \( j \leq \rho_i \) and from Eq. (4.25) one gets

\[ \dot{z}_{ij} = L_{\tilde{F}}^{j+1} h_i \]  \hfill (4.36)

Hence, Eq. (4.33) is verified.

Whereas in the case of \( j = \rho_i \)

\[ \dot{z}_{ij} = \tilde{b}_i(X) + \tilde{a}_{ij}(X) v_j \]  \hfill (4.37)

with \( \tilde{a}_{ij} = 0 \) if \( i \neq j \) (compare Ref. [160] p. 156). The function \( \tilde{b}_i(X) \) and \( \tilde{a}_{ij}(X) \) are not unique and only must satisfy the condition given in Eq. (4.30). It was stated before that if \( A(X) \) is invertible at some \( x \) then \( \tilde{\Delta}_i \equiv \Delta_i \) in a neighbourhood \( U \) of \( x \). Hence, the differentials \( d\tilde{a}_{ij} \) and \( d\tilde{b}_i \) must be contained in \( \tilde{H}_i^\perp \). In fact, the largest locally invariant distribution contained in \( \tilde{H}_i \) is \( (U_{\Delta_i=0} \text{span}\{dL_{\tilde{F}}^k h_i\})^\perp \) ([160], p. 156).

Therefore, a valid choice for \( \tilde{b}_i(x) \) and \( \tilde{a}_{ij}(X) \) is

\[ \tilde{b}_i(X) = \sum_{k=0}^{\rho_i} c_{ik} L_{\tilde{F}}^k h_i(X) \]  \hfill (4.38)

\[ \tilde{a}_{ij}(X) = 0 \quad \text{if} \quad i \neq j \]

\[ = \text{const.} \neq 0 \quad \text{if} \quad i \neq j \]  \hfill (4.39)
It is now instructive to provide a concrete example using the familiar three link manipulator to demonstrate the above state transformation. For the three link robot arm $i = 3$, $\rho_i = 1$, $j = 1$. With these parameters the new dynamics of the robot arm becomes

$$
\begin{bmatrix}
\dot{z}_{10} \\
\dot{z}_{11} \\
\dot{z}_{20} \\
\dot{z}_{21} \\
\dot{z}_{30} \\
\dot{z}_{31}
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
z_{10} \\
z_{11} \\
z_{20} \\
z_{21} \\
z_{30} \\
z_{31}
\end{bmatrix}
+ \begin{bmatrix}
v_1 \\
v_2 \\
v_3
\end{bmatrix} (4.40a)
$$

$$
\begin{bmatrix}
y_1 \\
y_21 \\
y_22
\end{bmatrix} =
\begin{bmatrix}
z_{10} \\
z_{20} \\
z_{30}
\end{bmatrix} (4.40b)
$$

The internal structure of the locally linear robot arm dynamics distinctly shows that the three outputs are decoupled and that only the states $z_{10}$ and $z_{11}$ may be affected by the input $v_1$. More importantly, however, the locally linear dynamics can be stabilized by proper choice of the constants $c_{ik}$. A block diagram of the three-link manipulator with feedback loop is shown in Fig. 19. When a general robot manipulator is considered having additional degrees-of-freedom to change the gripper orientation some supplemental considerations should be recognized.

Resort back to Eqs. (4.12) and (4.13) and notice that these equations are not total differentials. This means, for instance, that Eq. (4.13) can not be expressed as

$$
\dot{\psi}_r = \frac{\partial \psi_r}{\partial \Theta} \dot{\Theta} (4.41)
$$

Consequently, there exists no unique functions $\psi_r = \psi_r(\Theta)$ and $\psi_e = \psi_e(\Theta)$. Where $\psi_e \in R^2$ is the integral of possible error of Eq. (4.12), i.e.

$$
\dot{\psi}_e = Q_r B_g(\Theta) \dot{\Theta} (4.42)
$$
Figure 19: Block diagram of the three-link manipulator with feedback loops.
Despite this problem it is desired to stabilize Eqs. (4.12) and (4.13) and to control Eq. (4.13) as well.

Define three states as

\[ \psi_r = \int_{t_0}^{t} v_r^T B_g(\Theta) \dot{\Theta} \, d\tau \]  

(4.43)

and

\[ \psi_e = \int_{t_0}^{t} Q_r B_g(\Theta) \dot{\Theta} \, d\tau \]  

(4.44)

where \( \psi_r \in \mathbb{R} \) and \( \psi_e \in \mathbb{R}^2 \), \( \psi = (\psi_r, \psi_e) \). The latter three states will be denoted quasi coordinates, because they can not be defined under the usual local coordinate transformation outlined earlier. The quasi coordinate vector \( \psi \) will be computed indirectly in a controller by integration as defined in Eqs. (4.33) and (4.34), because the states are not functionally related to the position \( \Theta \) of the manipulator. In order to control end-effector rotation the quasi coordinates \( \psi \) are fed back via the input vector \( V \).

In the following section two examples of gripper movements on task oriented surfaces are provided. It is shown that proper initial conditions and the nature of the controls \( v_i \) are also important to maintain gripper movements on a predefined surface \( S_T \). Moreover, a limitation of the presented control scheme for manipulators is pointed out and a possible remedy to this problem is discussed.

4.3 Implementation of Task Oriented Surfaces

The execution of desired trajectories can be conveniently accomplished using cartesian, cylindrical, spherical, or any other coordinate system, whichever is most suitable for a particular application. However, the output functions \( h_i(X) \) which are associated to translational gripper motion should be chosen such that two of
the $h_i(\cdot) = \text{constant}$, and the remaining one will then specify the motion along the task oriented path. Likewise, two output channels $h_i(X)$ which specify rotational motion should satisfy $h_i(X) = \text{constant} = 0$, because rotation only should take place about the angle $\psi_r$ (compare Eqs. (4.12) and (4.13)). Due to the nature of the dynamical system, the output functions may be chosen freely in order to determine any arbitrary task oriented surface. More importantly, it is possible to connect pieces of trajectories together which are defined in different coordinate systems. Connecting reference trajectories together essentially can be accomplished in a straightforward fashion when one assumes that the reference trajectories are only finitely many times differentiable. This assumption is required to satisfy the boundary conditions at the interconnection points of task oriented surfaces. As a matter of fact, to obtain continuous input torque trajectories the input functions $v_i$ shall be at least twice differentiable and continuous.

Two different feedback schemes can be easily implemented for translational motion of the gripper depending upon the choice of the coefficient $c_{10}$ and $c_{11}$; compare Eqs. (4.38) and (4.39). A feedforward compensator can be implemented if $v_i$ is chosen to be

$$v_i = \frac{-c_{10}}{\hat{a}_{11}} y_{1\text{ref}}(t) \quad (4.45)$$

where the coefficients $c_{10}$ and $c_{11}$ determine the locus of the poles of the local linear system.

On the other hand it is possible to construct a closed loop feedback servocompensator when in Eq. (4.38) the coefficient $c_{10}$ is chosen to be $c_{10} = 0$ resulting in a Type 1 system. For such a system a proper stabilizing feedback $v_i$ is

$$v_i = \hat{c}_{1}(y_{10} - y_{1\text{ref}}(t)) \quad (4.46)$$
where $y_1 = z_{10}$ and $\tilde{c}_1$ is some arbitrary constant such that the pair $(\tilde{c}_1 \cdot \tilde{a}_{11}, c_{11})$ stabilizes the locally linear system.

The implementation and control of a task oriented surface $S_T$ is considered next. A task oriented surface can be specified as a cartesian product of a curve along which a manipulator's end-effector will move and a vector $v_r$ about which rotational motion of the same end-effector will take place. In the previous section, it was pointed out that any point on a task oriented surface can be reached by controlling the output functions in the partition $y_1$ while the vector $y_2$ remains constant. In order to clarify this further let us consider a concrete example of Cartesian path motion.

### 4.3.1 Cartesian Task Oriented Surface

Suppose the end-effector of the manipulator as discussed in Chapter 3 shall move along the $Y_1$ axis in some inertial coordinate frame, while no motion shall take place along the $Y_2$ and $Y_3$ axes in the same reference frame. Define three output functions $y_1 = h_1(t, X_0, v_1)$, $y_2 = h_2(t, X_0) = y_{20} =$ constant, and $y_3 = h_3(t, X_0) = y_{30} =$ constant. The coefficients $y_{20}$ and $y_{30}$ define a line collinear to the $Y_1$ axis. Using the manipulator model as discussed in Chapter 3 the three output functions had been stated in Eq. (4.2) and are repeated here

\[
\begin{align*}
    h_1(X) &= \cos \theta_1(d_2 \sin \theta_2 + d_3 \sin(\theta_2 + \theta_3)) \\
    h_2(X) &= \sin \theta_1(d_2 \sin \theta_2 + d_3 \sin(\theta_2 + \theta_3)) \\
    h_3(X) &= d_1 + d_2 \cos \theta_2 + d_3 \cos(\theta_2 + \theta_3)
\end{align*}
\]

For, if the output channels exhibit a desired behaviour as discussed before and additionally the manipulator is stable, it is possible to come up with functions
\[ \ddot{b}_1(X) \text{ and } \ddot{a}_{11}(X) \text{ as below} \]

\[ \ddot{b}_1(X) = c_{11}L_F h_1(X) \quad (4.48a) \]
\[ \ddot{a}_{11}(X) = \text{constant} \neq 0 \quad (4.48b) \]

\[ \ddot{b}_2(X) = c_{20}h_2(X) + c_{21}L_F h_2(X) - c_{20}y_{20} \quad (4.49a) \]
\[ \ddot{a}_{22}(X) = 0 \quad (4.49b) \]

\[ \ddot{b}_3(X) = c_{30}h_3(X) + c_{31}L_F h_2(X) - c_{30}y_{30} \quad (4.50a) \]
\[ \ddot{a}_{33}(X) = 0 \quad (4.50b) \]

Eq. (4.48) allows the implementation of a feedback compensator as given in Eq. (4.46) to control end-effector motion along the \( Y_1 \) axis. The feedback functions in Eqs. (4.49) and (4.50), however, stabilize the end-effector at the off-set line \((y_{20}, y_{30})\). Furthermore, to prohibit any disturbances entering via input \( v_2 \) and \( v_3 \) the coefficients \( \ddot{a}_{22} \) and \( \ddot{a}_{33} \) are set to zero, i.e. the closed loop system has only one input. The remaining coefficients in Eqs. (4.46) and (4.48)-(4.50) are selected such that the desired closed loop pole loci are obtained.

The issue of stability of a manipulator, although important, is only a single entity in the field of robotic control. An equally important problem is that of designing a properly parameterized reference trajectory. Consider again the three link manipulator as introduced repeatedly in this chapter. Movements of the tip of this manipulator along the \( Y_1 \) coordinate are controlled by a particular reference trajectory \( y_1^{\text{ref}}(t) \). The reference trajectory \( y_1^{\text{ref}} \) shall be a parameterized function of time, twice differentiable. Moreover, the reference trajectory shall be capable of guiding the manipulator from an initial equilibrium to some final equili-
librium position. A suitable reference trajectory that satisfies the latter conditions is

\[ y_{1 \text{ ref}} = C_1 t + C_2 - \frac{A}{\alpha^2} \sin(\alpha t) \]  

(4.51)

where \( C_1, C_2, A, \alpha \in \mathbb{R} \) are yet to be determined. The first and the second time derivative respectively yields

\[ \dot{y}_{1 \text{ ref}} = C_1 - \frac{A}{\alpha} \cos(\alpha t) \]  

(4.52)

\[ \ddot{y}_{1 \text{ ref}} = A \sin(\alpha t) \]  

(4.53)

Implementing Eq. (4.41) allows a manipulator to begin and end in an equilibrium position. Define \( T_{ex} \) to be the total execution time for performing a movement from an initial point \( Y_I \) to a final point \( Y_F \) on \( S_T \), the scalars \( C_1, C_2, A \) and \( \alpha \) are as given below

\[ \alpha = \frac{2\pi}{T_{ex}} \]

\[ C_1 = \frac{Y_F - Y_I}{T_{ex}} \]

\[ C_2 = Y_I \]

\[ A = \frac{2\pi}{T_{ex}^2} (Y_F - Y_I) \]  

(4.54)

Substituting the expressions above into Eq. (4.51) yields

\[ y_{1 \text{ ref}} = \frac{Y_F - Y_I}{T_{ex}} t + Y_I - \frac{Y_F - Y_I}{2\pi} \sin \left( \frac{2\pi}{T_{ex}} t \right) \]  

(4.55)

Eq. (4.55) allows the manipulator tip to move from point \( Y_I = (y_{10}, y_{20}, y_{30}) \) to point \( Y_F = (y_{10}, y_{20}, y_{30}) \) along a cartesian path, which is collinear to the axis \( Y_1 \).

Next, consider the more general case as depicted in Fig. 20, which shows two arbitrary points \( Y_I \) and \( Y_F \), which are connected by a cartesian path \( P_C \). Let
Figure 20: Cartesian path motion along arbitrarily oriented line.
\( \xi \) be the frame of the coordinates \((\bar{Y}_1 \bar{Y}_2 \bar{Y}_3)\) and \(e\) be the inertial frame or work coordinates. Define a nonunique coordinate rotation \(T(\xi)\) which transforms \(e\) into \(\bar{e}\) such that the path \(P_C\) is parallel with the coordinate \(\bar{Y}_1\). The rotation matrix \(T(\xi)\) may be derived as shown in Section 3.2 with \(\xi = (\xi_1, \xi_2, \xi_3)\) denoting Bryant or Euler angles. Define \(\tilde{y} = (\tilde{y}_1 \tilde{y}_2 \tilde{y}_3) = (\bar{h}_1 \bar{h}_2 \bar{h}_3)\) a vector of output functions in the reference frame \(\bar{e}\) with \(\tilde{y}_2 = \tilde{y}_20 = \text{constant}\) and \(\tilde{y}_3 = \tilde{y}_30 = \text{constant}\). Thus, using \(y = (h_1 h_2 h_3)\) from Eq. (4.47) an output transformation may be established that relates \(Y\) to \(\bar{Y}\) as given by

\[
\bar{Y} = T(\xi)Y
\] (4.56)

Eq. (4.56) yields a new set of output functions \((\bar{h}_1(X) \bar{h}_2(X) \bar{h}_3(X))\) which shall replace \((h_1(X) h_2(X) h_3(X))\) in Eqs. (4.48)-(4.50) to obtain a new decoupling and stabilizing control. As in the previous example, \(\bar{y}_1\,\text{ref}\) will control movements along the path \(P_C\).

An extension to other task oriented surfaces is now obvious and it only depends on finding a set of output functions \((h_1(X) h_2(X) h_3(X))\) which specify a particular surface \(S_T\). The implementation will then be analogous to the method presented for cartesian path control.

This section is concluded by emphasizing an important problem that may show up when \(\text{span}\{dh_1(X)\, dh_2(X)\, dh_3(X)\} < \ell\), where \(\ell\) denotes the number of output channels. The latter condition arises if the matrix \(A(X)\) in Eq. (4.31) is not invertible for some \(X \in R^{2n}\). Claude [143] suggests a remedy to maintain input-output decoupling even in the case when \(\text{det}(A(X)) = 0\). However, at the singularities of \(A(X)\) the output feedback as defined by Eqs. (4.48)-(4.50) may fail to stabilize the dynamics of the manipulator. Clearly if \(A^{-1}(X)\) fails to exist the observability space \(\Omega |_{\mathcal{F}}\) of the closed loop system defined by \(\Omega |_{\mathcal{F}} = \{\)
\[ \omega = L^T_{\tilde{x}} \lambda, 0 < k < \infty, \lambda_i = dh_i, 1 \leq i \leq 6 \] may not satisfy the observability rank condition \( \rho(\Omega | \tilde{x}) = 2n \). For, if \( \rho(\Omega | \tilde{x}) < 2n \) and the unobservable states are unstable then output feedback, as defined in this section, cannot stabilize the manipulator. A simple solution to this problem, however, is possible if additional state feedback is added to the system whenever \( \text{det} (A(X)) = 0 \). The latter strategy requires a decision scheme on a higher level for choosing the appropriate feedback control law.

### 4.3.2 Concatenation of Task Oriented Surfaces

The objective of the upcoming discussion is to demonstrate how different task oriented surfaces can be pieced together in order to allow a manipulator to perform more sophisticated movements. In this context it is clear that higher level decision schemes must be implemented in the controller. The task of a higher level decision mechanism is to select proper nonlinear feedback gains according to some pre-specified task oriented surface and to match boundary conditions between task oriented surfaces, so one can guarantee smooth transitions between them as well.

To demonstrate this control strategy, a concrete example is stated as follows. Consider the three link robot arm from Chapter 3 and refer to Figure 29. Starting from some initial equilibrium position \( Y_I \) with respect to the inertial frame \( Y \), the manipulator shall move along the \( Y_1 \) axis (cartesian path) until it reaches point \( Y_{S1} \). At point \( Y_{S1} \), the task oriented surface will be changed to allow circular motion of the end-effector with constant radius \( \rho \) and height \( y_3 \). This second portion of the task oriented surface may be represented by a cylindrical coordinate system that is centered in the \( (Y_1, Y_2) \) plane and which is specified later. After the end-effector has moved through an angle \( \phi_c \) in cylindrical coordinates it reaches the point \( Y_{S2} \). Thereafter, cartesian straight line motion is performed in a new coordinate
system $\bar{Y}$, finally reaching the initial position $Y_f$. The points $Y_f$, $Y_{S1}$, and $Y_{S2}$ are specified as $Y_f = (0.2m, 0m, 1.2m)$, $Y_{S1} = (1.2m, 0m, 1.2m)$, $Y_{S2} = (1m + 0.2/\sqrt{2} m, -0.2/\sqrt{2} m, 1.2m)$, respectively. The points $\bar{Y}_{S2}$ and $Y_{S2}$ may be related by a positive coordinate rotation about the $Y_3$ axis by an angle of $\pi/4$. From the geometry of the two straight line segments and the necessity to connect boundary points together between pieces of different task oriented paths, the radius $\rho$ and the center of the cylindrical coordinate system become $\rho = \tan(\pi/8)m, (y_{1C}, y_{2C}) = (1.2m, \rho m)$.

For each segment of the task oriented coordinate path, a set of output functions can be identified which will be denoted $H_I(X), I = \{1, 2, 3\}$. For the first portion of the task oriented surface as discussed the set of output functions $H_1(X)$ is as given by Eq. (4.47). The output functions $H_2(X)$ and $H_3(X)$ for the second and third portion of the task oriented surface are

$$h_1(X) = \left\{ \left[ \cos \theta_1 (d_2 \sin \theta_2 + d_3 \sin(\theta_2 + \theta_3)) - y_{1C} \right]^2 + \left[ \sin \theta_1 (d_2 \sin \theta_2 + d_3 \sin(\theta_2 + \theta_3)) - y_{2C} \right]^2 \right\}^{\frac{1}{2}}$$

(4.57a)

$$h_2(X) = \arctan \left( \frac{\sin \theta_1 (d_2 \sin \theta_2 + d_3 \sin(\theta_2 + \theta_3)) - y_{2C}}{\cos \theta_1 (d_2 \sin \theta_2 + d_3 \sin(\theta_2 + \theta_3)) - y_{3C}} \right)$$

(4.57b)

$$h_3(X) = d_1 + d_2 \cos \theta_2 + d_3 \cos(\theta_2 + \theta_3)$$

(4.57c)

where $\rho = h_1(X) = constant$ and $h_3(X) = 1.2m$. Eq. (4.57) represents cylindrical coordinates which are centered at $(y_{1C}, y_{2C})$; and

$$h_1(X) = (d_2 \sin \theta_2 + d_3 \sin(\theta_2 + \theta_3))(\cos \theta_1 + \sin \theta_1)/\sqrt{2}$$

(4.58a)

$$h_2(X) = (d_2 \sin \theta_2 + d_3 \sin(\theta_2 + \theta_3))(\cos \theta_1 - \sin \theta_1)/\sqrt{2}$$

(4.58b)

$$h_3(X) = d_1 + d_2 \cos \theta_2 + d_3 \cos(\theta_2 + \theta_3)$$

(4.58c)

where $h_2(X) = -0.2/\sqrt{2} m = constant$, $h_3(X) = 1.2m$. Eq. (4.58) are the output equations for the rotated coordinate system $\bar{Y}$ as proposed in Eq. (4.56) with the
orthogonal matrix $T(\xi)$ as shown

$$T(\xi) = \begin{pmatrix}
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\
-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\
0 & 0 & 1
\end{pmatrix}$$

(4.59)

Having defined a set of output functions $H = \{H_I, I = 1, 2, 3\}$ which specify a complete path of end-effector movements, a suitable time-parameterization of the reference trajectory has yet to be defined. In recent years, a number of methodologies have been developed for establishing proper time-parameterization incorporating physical limitations and optimization methods [94], [100]-[103]. The method presented in the sequel does not take into account bounds on input torques or any optimization criterion. It, rather, will demonstrate the decision mechanism for selecting set of fictitious output functions and for matching boundary conditions. A block-diagram of the nonlinear feedback loop and the decision mechanism for choosing proper sets of functions $H_I(X)$ and $K_I(X)$ in order to implement task oriented surfaces is shown in Fig. 21. The task oriented path as previously described is shown in Figure 29, where the points $Y_{S_1}$ and $Y_{S_2}$ indicate the connection points of the three different task oriented coordinate systems. A possible time-parameterization algorithm for the path as shown in Figure 29 is given next. Suppose $\lambda(t)$ is the length of the traversed path at time $t$ along the surface $S_T$ and let $t_{ex}$ be the total time to move the end-effector from its initial to its final position. Then the total path length $\lambda(t_{ex}) = \lambda_o$ is

$$\lambda_o = ||y_{S_1} - y_I|| + 5\pi/4 \times \tan(\pi/8) + ||y_{S_2} - y_I||$$

(4.60)

or

$$\lambda_o = 2 ||y_{S_1} - y_I|| + 5\pi \tan(\pi/8)/4$$

(4.61)
Figure 21: Block structure of the feedback loops and higher level decision mechanisms.
Define a set of functions which are twice differentiable in terms of $\lambda(t)$ as below

\[
\dot{\lambda}(t) = \begin{cases} 
    A \sin \alpha t & ; 0 \leq t < t_a \\
    0 & ; t_a \leq t < t_f + t_a \\
    A \sin \alpha (t - t_f) & ; t_f + t_a \leq t \leq t_{ex}
\end{cases} 
\]  

\[
\lambda(t) = \begin{cases} 
    A/\alpha (1 - \cos \alpha t) & ; 0 \leq t < t_a \\
    2A/\alpha & ; t_a \leq t < t_f + t_a \\
    A/\alpha (1 - \cos \alpha (t - t_f)) & ; t_f + t_a \leq t \leq t_{ex}
\end{cases} 
\]

Let $t_f$ be the total intermediate time for which $\lambda(t) = \text{constant}$ and also satisfies $t_f + 2t_a = t_{ex}$. Having specified $t_{ex}$ and $t_f$, the coefficients $\alpha, A, ..., D$ yield

\[
\alpha = \frac{2\pi}{t_{ex} - t_f} \tag{4.65a}
\]

\[
A = \frac{\pi}{(t_{ex} - t_f)(t_f + t_a)} \tag{4.65b}
\]

\[
B = 0 \tag{4.65c}
\]

\[
C = -\frac{\lambda_0 t_a}{2(t_f + t_a)} \tag{4.65d}
\]

\[
D = \frac{\lambda_0 t_f}{2(t_f + t_a)} \tag{4.65e}
\]

From the task oriented contour as depicted in Fig. 29 and from Eqs. (4.51)-(4.54) the sequence of the exact reference trajectories $(y_1 \text{ ref}, \phi_{\text{ref}}, \dot{y}_1 \text{ ref})$ can be established. First, the switching instances $t_{S1}$ and $t_{S2}$ are computed to determine the range within which the reference trajectories are valid.

Using Eq. (4.64) and knowing that $\lambda(t_{S1}) = 1m$ yields
and because of symmetry $t_{S2}$ becomes

$$t_{S2} = t_{ex} - \frac{t_f + t_a}{\lambda_o} \left( 1 + \frac{\lambda_o t_a}{2(t_f + t_a)} \right)$$  \hspace{1cm} (4.67)$$

Eqs. (4.55) and (4.56) are employed to appropriately select the three different coordinate systems that describe the entire path of the end-effector. In the present scheme the sequence of the controlled output channels is $(Y_1(X), \phi(X), \dot{Y}_1(X))$ as indicated in Fig. 29. For the first reference trajectory $Y_1(X)$ one gets

$$y_1 \text{ ref} = \begin{cases} 
A_1/\alpha(t - \sin \alpha t) + B_1 & ; 0 \leq t < t_a \\
2A_1/\alpha t + C_1 & ; t_a \leq t < t_{S1}
\end{cases}$$  \hspace{1cm} (4.68a)$$

$$\dot{y}_1 \text{ ref} = \begin{cases} 
A_1(1 - \cos \alpha t) & ; 0 \leq t < t_a \\
2A_1/\alpha & ; t_a \leq t < t_{S1}
\end{cases}$$  \hspace{1cm} (4.68b)$$

where $A_1 = A, B_1 = 0.2m, C_1 = 0.2m + \frac{\lambda_o t_a}{2(t_f + t_a)} m$ and $Y_{20} = 0m, Y_{30} = 1.2m$.

$$\phi_1 \text{ ref} = 2A_2/\alpha - t + C_2 ; t_{S1} \leq t < t_{S2}$$  \hspace{1cm} (4.69a)$$

$$\dot{\phi}_1 \text{ ref} = 2A_2/\alpha ; t_{S1} \leq t < t_{S2}$$  \hspace{1cm} (4.69b)$$

where $A_2 = \frac{A}{\rho}, C_2 = \left( \frac{\pi}{2} + \frac{\lambda_o t_a t_{ex}}{\rho(t_f + t_a)} \right), \rho = \text{radius of the cylinder}$ as indicated in Fig. 29, $\rho_{20} = \tan(\pi/8)m, z_{30} = 1.2m$. Finally the reference trajectory in the $\hat{Y}$ coordinate system is

$$\ddot{y}_1 \text{ ref} = \begin{cases} 
2A_3/\alpha t + C_3 & ; t_{S2} \leq t < t_{ex} t_a \\
A_3/\alpha(t - \sin \alpha(t - t_f)) + D_3 & ; t_{ex} - t_a \leq t \leq t_{ex}
\end{cases}$$  \hspace{1cm} (4.70a)$$

$$\dot{\ddot{y}}_1 \text{ ref} = \begin{cases} 
2A_3/\alpha & ; t_{S2} \leq t < t_{ex} - t_a \\
A_3/\alpha(1 - \cos \alpha(t - t_f)) & ; t_{ex} - t_a \leq t \leq t_{ex}
\end{cases}$$  \hspace{1cm} (4.70b)$$
where

\[ A_3 = -A, \ C_3 = (1 + \frac{0.2}{\sqrt{2}})m + \frac{\lambda_{0f}e_2}{(t_a + t_f)}m \]

\[ D_3 = 0.2/\sqrt{2}m + \frac{\lambda_{0f}t_e}{2(t_a + t_f)}m \]

and

\[ \bar{y}_{20} = -0.2/\sqrt{2}m, \ \bar{y}_{30} = 1.2m \]

Eqs. (4.68) to (4.70) fully specify the reference path of Fig. 29.

In the following section two digital computer simulations are presented to demonstrate the nonlinear decoupling approach and the decision scheme presented in this chapter.

### 4.4 Digital Computer Simulations and Results

In this section the results of two digital computer simulations are provided and discussed. The simulation results confirm qualitatively the following assertions:

a) Nonlinear input-output decoupling yield more accurate gripper trajectories even for relatively fast moving end-effectors.

b) The method of task oriented surfaces allows more complex movements of the end-effector especially when the task oriented path planning mechanisms are supported by some decision strategy.

The first simulation in this section addresses assertion a., while the aspect of designing more sophisticated gripper movements is studied using a second simulation. The manipulator model from Section 3.2 will be employed for the upcoming simulation.
4.4.1 Manipulator Motion Along a Straight Line

The results of the forthcoming digital computer simulation shall verify if a nonlinear input-output decoupling scheme and the principle of task oriented surfaces renders accurate gripper trajectories. The selected task oriented surface is a cartesian path between the points \( Y_I = (1.4m, 0m, 2m) \) and \( Y_F = (0.4m, 0m, 1.0m) \) along which the end-tip of the manipulator shall move. The desired reference trajectories for the manipulator end-effector is shown in Fig. 22. For the purpose of consistency with our discussion in Section 4.3, a coordinate transformation \( A(\xi) \) ought to be specified that yields one coordinate axis, say \( \tilde{Y}_1 \), to be parallel to the cartesian path as determined by the points \( y_I \) and \( y_F \). The homogeneous coordinate rotation \( A(\xi) \) is not unique and a possible choice of \( A(\xi) \) is

\[
A(\xi) = \begin{pmatrix}
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\
-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

The coordinate rotation as suggested above allows to specify only one reference trajectory in the \((\tilde{Y}_1, \tilde{Y}_2, \tilde{Y}_3)\) coordinate system. However, the reference trajectory \( \tilde{y}_{1\text{ ref}} \) has not been plotted. The desired execution time of the cartesian path movement is \( t = 1s \) with an integration step size of 10ms. The integration of the nonlinear differential equation was accomplished with the 4th order Runge-Kutta integration algorithm [159]. The results of the cartesian path simulation are given in Fig. 23 to Fig 27. The end-effector trajectories versus time are shown in Fig. 23. More interestingly, however, is Fig. 24 (a) which shows the actual end-effector path in the reference frame \((Y_1, Y_3)\). The latter figure indicates almost complete absence of any deviation from an ideal cartesian path. Fig. 24 (b) shows a sequence of positions of a stick model of the manipulator in the plane \((Y_1, Y_3)\).
Figure 22: Desired reference trajectory in $(Y_1, Y_2, Y_3)$ coordinate system versus time.
Figure 23: Gripper trajectory in \((Y_1, Y_2, Y_3)\) coordinate system.
Figure 24: (a) Gripper trajectory $y_3$ versus $y_1$ in $(Y_1, Y_2, Y_3)$ coordinate system. (b) Stick model of the manipulator showing gripper trajectory $y_3$ versus $y_1$. 
Figure 25: Trajectories of the joint angles versus time.
Figure 26: Trajectories of the angular velocities versus time.
Figure 27: Trajectories of the joint input torques versus time.
Like Fig. 24 (a) the aforementioned figure provides a good qualitative picture of the gripper motion along the cartesian path.

The trajectories of the state vector $X$ are shown in Fig. 25 and 26. The three joint angles $\theta_1$, $\theta_2$, $\theta_3$ versus time are depicted in Fig. 25 and the joint angular velocities are plotted in Fig. 26. The latter figures show that the manipulator is stable and that its state variables reach their equilibrium point at about 1.1s. The three input torques $u_1$, $u_2$, and $u_3$ are drawn in Fig. 27. The largest input torque is $u_2$ with about $|u_2| \approx 300 Nm$, while the torque actuator at joint three only needs to provide a torque of about $|u_3| \approx 40 Nm$. In association with the cartesian path as shown in Figure 24a, there exists a path in terms of the state variables $\Theta = (\theta_1, \theta_2, \theta_3)$. Since $\theta_1 = 0$ for this particular simulation the submanifold $\theta_3$ versus $\theta_2$ is plotted in Fig. 28.

To summarize, the results clearly show the merit of the nonlinear decoupling feedback as well as the principle of a task oriented surface defined in end-effector coordinates. The simulation indicates no discernible deviation from the prespecified cartesian path.

4.4.2 Manipulator Motion Along a Closed Path

The digital computer simulation presented in the sequel are aimed at verifying the discussion on concatenation of task oriented surfaces as has been set forth in Section 4.3.2.

Reconsider Fig. 29 which shows the specified contour of the manipulator end-effector in the standard reference frame $Y = (Y_1, Y_2, Y_3)$. It was pointed out in Section 4.3.2 that a suitable parameterization can be established according to Eqs. (4.51) to (4.53) after $t_{ex}$ and $t_f$ has been specified. For this simulation let $t_{ex} = 4s$ and $t_f = 2.4s$. From Eqs. (4.50) and (4.51) one can completely
Figure 28: Submanifold of $\Theta$ induced by the surface $S_T$ that rendered cartesian path motion.
Figure 29: Desired path of the gripper of the three-link robot arm.

Figure 30: Path length $\lambda$ versus time as traversed along connected portions of the task oriented surface.
determine the coefficients in Eqs. (4.62)-(4.64) and (4.68)-(4.70). The trajectories of the latter three equations, i.e., path length $\lambda$ along desired contour Fig. 29 and its first and second time derivatives are shown in Fig. 30 to Fig. 32. Also indicated in Fig. 32 are the time intervals $t_0, t_f, t_{ee}$ to better clarify their meaning. The trajectory of the reference position in the three coordinate systems $(Y_1, Y_2, Y_3) (\varphi, \rho, z)$, and $(\tilde{Y}_1, \tilde{Y}_2, \tilde{Y}_3)$ is shown in Fig. 33. The following figure, Fig. 34, depicts the trajectory of the reference velocity which is specified in the aforementioned set of different coordinate systems. The results of the computer simulation are provided in Figs. 35-41. The end-effector position in the usual reference frame, versus time, is plotted in Fig. 35. These graphs, however, do not provide too much insight as to whether the end-effector accurately follows the desired path except for the height of the end-effector, which should be maintained at a constant level of 1.2 m; which in fact, it does. The latter plots also indicate that the manipulator tip reaches its final equilibrium almost exactly at $t = 4s$ as desired. Beyond this, no discernable overshoot is evident. The trajectories of the joint angles $\theta_1, \theta_2,$ and $\theta_3$ of the manipulator are shown in Fig. 36. The first time derivatives of the latter trajectories are plotted in Fig. 37. The time histories of the angular velocities, however, do show very lightly overshoot for $\dot{\theta}_1(t)$ and almost none for $\dot{\theta}_2(t)$ and $\dot{\theta}_3(t)$. The input torques $u_1, u_2,$ and $u_3$ of the simulated manipulator model are given in Fig. 38. All input torques, of which $u_1$, is most prominent, indicate a rapid change of magnitude when the reference coordinates are switched. This happens at the switching instances $t_{s1}$ and $t_{s2}$ as indicated in Fig. 34. In fact, this may suggest that the input torque vector $u$ is not a continuous function of time at the switching points $t_{s1}$ and $t_{s2}$. Nevertheless, $u$ clearly is piecewise continuous.

The $Y_2$ versus $Y_1$ plane is plotted in Fig. 39. This figure shows the actual
Figure 31: Desired velocity along reference path versus time.

Figure 32: Desired acceleration along reference path versus time.
Figure 33: Reference position of the gripper in the prespecified coordinate systems versus time.

Figure 34: Reference velocity of the gripper in the prespecified coordinate systems versus time.
Figure 35: Position of the end-effector in the standard reference frame versus time.
Figure 36: Trajectories of the joint angles versus time.
Figure 37: Trajectories of the angular joint velocities versus time.
Figure 38: Trajectories of the joint input torques versus time.
Figure 39: Actual end-effector path in standard reference frame.

Figure 40: Perspective view of a stick model of the manipulator moving along the specified end-effector path.
Figure 41: Submanifold of $\Theta$ induced by the task oriented surface $S_T$ (concatenated path).
motion of the end-effector along the desired path as shown in Fig. 29. A comparison of Fig. 39 with Fig. 29 does not show any noticeable deviation from the reference path. A perspective three dimensional drawing of a stick model of the robot model is shown in Fig. 40. The stick model is repeatedly drawn to indicate its attitude while moving along the desired task oriented surface. One interesting result is brought out in Fig. 41. The graph of the angular position $\theta_3$ versus $\theta_2$ surprisingly turns out to be a straight line. Or in other words, the submanifold which is inducted by the task oriented surface as given in Fig. 39 is a flat closed contour.

4.5 Summary

In this chapter recent nonlinear system-theoretical results were utilized to obtain input-output decoupling of a manipulator. The output channels, which represent the position and orientation of the manipulator’s end-effector were specified in terms of some desired but essentially arbitrary output coordinates. Together with the notion of task oriented surfaces, a particular path along which the end-effector shall move can be specified by a set of output channels. The freedom of designing any type of task oriented surface makes it extremely easy to establish a feedback law for cartesian path motion. Moreover, a decision strategy was outlined in this chapter that allows a more sophisticated creation of (task oriented surfaces or) end-effector paths.

Two digital computer simulations were conducted to verify qualitatively the merit of the nonlinear decoupling methodology in accordance with the principle of task oriented surfaces. In addition to the latter objective, the second simulation demonstrates the decision mechanism for specifying task oriented surfaces in a sequential order and to match their boundary conditions as well. The simulation
results exhibit a behaviour of the manipulator which matches extremely well the predicted theoretical outcome.
CHAPTER V

Compensation of Disturbances in Robotic Systems

5.1 Introduction

Robot manipulators, like many dynamic systems, are susceptible to external disturbances which may alter their behaviour so significantly that certain specifications cannot be maintained. Two types of disturbances are identified in this chapter.

The first one, which usually is not considered to be a disturbance input is the gravitational force that acts upon any manipulator. The gravitational force may be seen as a constant additive disturbance, which, at revolute joints, has position dependent effects on the manipulator. Traditionally, gravitational forces, which act on the manipulator structure are eliminated by computing feedforward biasing torques. This method fails, however, when the mass of a manipulator link or a load is not exactly known. In this chapter a simple but effective feedback strategy is proposed which asymptotically provides proper biasing torques despite possible parameter variation in the plant or an unknown load. It is only necessary that the overall closed loop system be asymptotically stable.

Additive signals which satisfy a linear differential equation with constant coefficients are the second type of disturbance which may enter a manipulator and corrupt its behaviour. All disturbance signals which satisfy a linear differential equation with constant coefficients are termed the class of linear disturbances.
From linear servocompensator theory [148], [149] one knows that linear disturbances can be asymptotically rejected if the linear plant satisfies certain criteria. These criteria are reviewed in the next section. However, for robot manipulator, which are extremely nonlinear the former criteria can not be applied. Yet, it is shown in this chapter that linear disturbances can be asymptotically rejected by employing a standard linear servocompensator. In linear system theory the proof that the error \( e \to 0 \) as \( t \to \infty \) is shown via Laplace transform. Unfortunately, the Laplace transform is not applicable to nonlinear systems, so that a different method should be sought. In the discussion ahead, it will be shown that the Volterra series is a suitable mathematical tool to verify error convergence. In fact, robot manipulators, which are linear in the input have a recursive Volterra series representation which simplifies the convergence proof significantly.

Two digital computer simulations qualitatively and quantitatively illustrate the closed loop behaviour of planar multi-link manipulators subject to gravitation and linear disturbance signals, respectively. The simulation results are illustrated and discussed.

5.2 A Review of the Servomechanism Problem

In this section the general servomechanism problem as it pertains to disturbance rejection is briefly summarized. This review will omit the tracking problem, since it is not relevant to the subject matter of this chapter.

Consider a linear time invariant system that can be described by the following set of linear equations

\[
\dot{X} = AX + BU + E\omega
\]

\[
Y = CX + DU + F\omega
\]  

(5.1)
where $X \in \mathbb{R}^n$ is the state vector of the system, $U \in \mathbb{R}^m$ is the control input, $Y \in \mathbb{R}^r$ is the output vector, and $\omega \in \mathbb{R}^d$ is the disturbance vector. The constant coefficient matrices are of appropriate sizes. Define an error $e \in \mathbb{R}^r$ as

$$e = Y - Y_{ref}$$

which denotes the error between the output of the system of Eq. (5.1) and some specified reference trajectory $Y_{ref}(t)$. In the theory of a general servomechanism tracking problem $Y_{ref}(t)$ is a function of time (including $Y_{ref} = constant$) that satisfies a linear differential equation. When disturbance rejection is considered only, $Y_{ref} = 0$. The disturbance $\omega$ is assumed to satisfy a linear time invariant homogeneous differential equation that is given by

$$\dot{Z} = HZ$$

$$\omega = GZ$$

where $Z \in \mathbb{R}^q$ is the state of the disturbance and $(G, H)$ is an observable pair. It is assumed that the initial disturbance $Z_0 = Z(0)$ may or may not be known.

The objective of the servocompensator design is to find a controller such that the error $e$ satisfies $e \to 0$ as $t \to \infty$, for all initial disturbances $Z(0) \in \mathbb{R}^q$. It is assumed that the matrices are of maximal possible rank, because a rank defect in any of the latter matrices suggests the elimination of the dependent vectors. Also it is assumed that all eigenvalues of the matrix $H$ are located in the closed right half plane. Otherwise the solution of the servocompensator problem is trivial, because the disturbance $\omega$ will eventually go to zero.

Let the set of eigenvalues of $H$ be $\sigma = \{\lambda_1, \lambda_2, \ldots, \lambda_p\}$ where $p$ is the number of distinct eigenvalues of $H$. Davison [150] showed a robust compensator can be designed if
i) the triple \((C, A, B)\) is observable and stabilizable

ii) \(m \geq r\), i.e., there are at least as many inputs as outputs.

iii) the intersection of the set of transmission zeros of \((C, A, B, D)\) and the set of eigenvalues of the disturbance \(\sigma\) is the empty set.

Under the assumption that the above conditions are satisfied, a servocompensator can be constructed. Thus the feedback law becomes

\[
U = KX + K_I \xi
\]  

(5.4)

where \(\xi \in \mathbb{R}^q\) is the state vector of the dynamic compensator. \(K\) and \(K_I\) are constant gain matrices of appropriate sizes.

The actual dynamics of the servocompensator is obtained by using the internal model principle. This means that the dynamics of the real external disturbance is replicated and then implemented as part of the feedback loop. This particular feedback scheme is robust in some sense and causes the external disturbances to be eliminated externally [120], [147], [148]. Let the characteristic polynomial of the disturbance matrix \(H\) be given by

\[
\lambda^q + a_{q-1} \lambda^{q-1} + \ldots + a_1 \lambda + a_0 = \prod_{i=1}^{p} (\lambda - \lambda_i)^{k_i}
\]  

(5.5)

where \(a_j\) are real constants and the \(k_i\)'s denote the multiplicity of root \(i, i = 1, \ldots, p, q = \sum_{i=1}^{p} k_i\). Construct a matrix \(\hat{A}\) which possess an identical characteristic polynomial as the disturbance matrix \(H\), i.e., \(\hat{A}\) must satisfy Eq. (5.5). If more than one disturbance is present, the characteristic polynomial of the matrix \(\hat{A}\) must be the least common characteristic polynomial of all the disturbances together. Writing the matrix \(\hat{A}\) in companion form yields
In fact, any matrix $\hat{A}$ may be constructed as long as its eigenvalue spectrum satisfies the characteristic polynomial in Eq. (5.5). The final step of the servocompensator design procedure comprises of constructing a matrix $A_c$ of size $rq \times rq$ and a matrix $B_c$ of size $rq \times r$ such that the pair $(A_c, B_c)$ is controllable and has structures as given by

$$A_c = \begin{pmatrix}
\hat{A} \\
\hat{A} \\
\vdots \\
\hat{A}
\end{pmatrix} \quad (5.7a)$$

$$B_c = \begin{pmatrix}
\hat{B} \\
\hat{B} \\
\vdots \\
\hat{B}
\end{pmatrix} \quad (5.7b)$$

where $\hat{B} = (00\ldots01)^T$, $\hat{B} \in \mathbb{R}^{q \times 1}$. In order to find the constant gain matrices $K$ and $K_f$ the dynamics of the plant and the compensator are augmented to become

$$\begin{pmatrix}
\dot{X} \\
\dot{\xi}
\end{pmatrix} = \begin{pmatrix}
A & 0 \\
B_cC & A_c
\end{pmatrix} \begin{pmatrix}
X \\
\xi
\end{pmatrix} + \begin{pmatrix}
B \\
B_cD
\end{pmatrix} U + \begin{pmatrix}
E \\
BF
\end{pmatrix} \omega \quad (5.8)$$

Choosing a set of desired closed loop poles $\sigma_c$ for the augmented system above, the feedback matrices $K$ and $K_f$ can be found by any of the well known pole placement
The block diagram of the overall closed loop system Eq. (5.8) with state feedback as given by Eq. (5.4) is given in Fig. 1

5.3 Dynamic Feedback Compensation of Gravitational Disturbances

In this section an especially simple type of servocompensator is applied to planar interconnected rigid bodies for asymptotically eliminating the effects of the constant earth gravitational field. In the upcoming two subsections, the method is introduced using a simple pendulum and then applied to a chain of planar interconnected systems. Digital computer simulations are also provided in this section.

5.3.1 The Planar Simple Pendulum

Consider Fig. 2 which shows a single rigid body with one degree-of-freedom. The body is rigidly hinged to the ground. It has mass $m$, inertia $I$ about its center of gravity, and the distance from the hinge point to its center of mass is $k$. The tilt angle $\theta$ of the body is measured with respect to the vertical as indicated. The dynamical equation of the pendulum is

$$(I + mk^2)\ddot{\theta} - mgk \sin(\theta) = u$$

where $g \approx 9.81 \text{m/s}^2$ is the gravitational constant. Let Eq. (5.9) be rewritten in terms of its Taylor series expansion about some arbitrary point $\theta_0$; this yields

$$\ddot{\theta} - \alpha_1 \sin(\theta_0) - \alpha_1 \sum_{n=1}^{\infty} \frac{1}{n!} \left. \frac{d^n}{d\theta^n} \sin(\theta) \right|_{\theta=\theta_0} (\theta - \theta_0)^n = \alpha_2 u$$

where

$$\alpha_1 = \frac{mgk}{I + mk^2}, \quad \alpha_2 = \frac{1}{I + mk^2}$$
Figure 42: Block diagram of a closed loop system with servocompensator.
Figure 43: Single planar pendulum.

Figure 44: Illustration of Aizerman's conjecture and Hurwitz' region.
when \( \theta_0 \) shall assume the equilibrium point of the differential equation (5.10), then the term \( \alpha_1 \sin(\theta_0) \) appears as a constant disturbance for \( \theta_0 \neq n\pi, n \in \{\ldots,-2,-1,0,1,2,\ldots\} \). Let \( \omega \) be defined by \( \omega = \alpha_1 \sin(\theta_0) \) which is bounded by \( -\alpha_1 \leq \omega \leq \alpha_1 \) and satisfies a linear differential equation

\[
\dot{\omega} = 0
\]  

(5.11)

with initial condition \( \omega(0) = \alpha_1 \sin(\theta_0) \). Using the solution of Eq. (5.11) and substituting it into Eq. (5.10) gives

\[
\ddot{\theta} - \alpha_1 \sum_{n=1}^{\infty} \frac{1}{n!} \sin(n\theta) \frac{d^n}{d\theta^n} (\theta - \theta_0)^n = \alpha_2 u + \omega
\]

(5.12)

Let \( \ddot{\theta} = \theta - \theta_0 = 0 \) in Eq. (5.12) yields

\[
0 = \alpha_2 u + \omega
\]

(5.13)

Eq. (5.12) possesses an equilibrium point at \( \theta_0 \) only if the term of the right hand side of Eq. (5.13) is identical zero for all time. This means the input \( u \) must assume the value

\[
u = \frac{-1}{\alpha_2} \omega
\]

(5.14)

Since \( \omega \) depends on the equilibrium position \( \theta_0 \), the torque \( u \) in Eq. (5.14) provides the proper feedforward biasing torque. As is shown in the sequel, the biasing torque may be provided using a servo-compensator. Let us motivate the problem from a somewhat different perspective. A frequent control problem for the dynamics in Eq. (5.12) is to steer the pendulum to any arbitrary position by linear state feedback only. Define a linear state feedback law

\[
u = -k_1 \dot{\theta} - k_2 (\theta - \theta_0)
\]

(5.15)

where \( k_1, k_2 \in \mathbb{R} \). After substituting Eq. (5.15) into (5.9) gives
\[ \ddot{\theta} - \alpha_1 \sin(\theta) + \alpha_2 k_1 \dot{\theta} + \alpha_2 k_2 (\theta - \theta_0) = 0 \]  
(5.16)

Solving Eq. (5.16) for the equilibrium position yields

\[ -\alpha_1 \sin(\theta) + \alpha_2 k_2 (\theta - \theta_0) = 0 \]  
(5.17)

Define an error \( e \) as \( e = \theta - \theta_0 \) the former expression assumes the form

\[ \alpha_2 k_2 e - \alpha_1 \sin(e + \theta_0) = 0 \]  
(5.18)

For \( e = 0 \), Eq. (5.18) has a solution if \( \theta_0 = n\pi, n \in \mathbb{Z} \). For any other value of \( \theta_0 \) a finite steady state error \( e \neq 0 \) remains. Eq. (5.10) shows that locally the error \( e \) is due to a constant disturbance. This type of disturbance can be compensated for by a simply integral servocontroller [117]. Define a first order integral controller as

\[ \dot{\xi} = \theta - \theta_0 \]  
(5.19)

and let the new feedback law become

\[ u = -k_1 \dot{\theta} - k_2 (\theta - \theta_0) - k_3 \xi \]  
(5.20)

Substituting Eq. (5.20) into (5.16) renders

\[ \ddot{\theta} - \alpha_1 \sin(\theta) + \alpha_2 k_1 \dot{\theta} + \alpha_2 k_2 (\theta - \theta_0) + \alpha_2 k_3 \xi = 0 \]  
(5.21)

Define new state variables as \( \xi = \xi, \dot{\xi} = \theta - \theta_0, \ddot{\xi} = \dot{\theta}, \) and \( \dddot{\xi} = \ddot{\theta} \). Perform a change of state variables upon Eq. (5.21) renders

\[ \dddot{\xi} + \alpha_2 k_1 \dddot{\xi} + \alpha_2 k_2 \dddot{\xi} + \alpha_2 k_3 \xi - \alpha_1 \sin(\xi + \theta_0) = 0 \]  
(5.22)

The above differential equation can be made stable by choosing proper feedback gains \( k_1, k_2, \) and \( k_3 \). The equilibrium of this system gives
\[ \xi_0 = \frac{\alpha_1}{\alpha_2 k_3} \sin(\theta_0) \]  

(5.23)

where, of course, \( k_3 \neq 0 \epsilon R \).

In order to show that the dynamic system of the single inverted pendulum can be stabilized with linear state feedback, consider again the dynamics in Eq. (5.22) which is linear with the exception of a simple nonlinearity in the state variable \( \xi \). This fact makes the dynamics of the pendulum suitable for applying Aizerman's conjecture [124] in order to establish proper stability conditions on the gain constants \( k_1, k_2, \) and \( k_3 \).

Let the nonlinear term in Eq. (5.22) be replaced by two linear gains such that they are a bound of the nonlinearity for all values of \( \xi \). The nonlinearity and the linear gains are shown in Fig. 3. Replacing the constant gains by the nonlinearity \( \sin(\xi + \theta_0) \) renders Eq. (5.22) to assume the form

\[ \ddot{\xi} + \alpha_2^1 \dot{\xi} + \alpha_2^2 \xi + \alpha_2 k_1 \xi - \alpha_1 S \xi = 0 \]  

(5.24)

where

\[ S = \{1, -\frac{2}{3\pi}\} \]

Inspection of Eq. (5.24) and using the two possible values in the set \( S \) shows that stability only is adversely affected if \( S > 0 \). Consequently, it is possible to reject the negative value in \( S \) for a stability test. Applying Routh-Hurvitz stability test yields the following inequalities for an asymptotically stable system [117].

\[ k_1 > 0 \]

\[ k_2 > (k_3/k_1 + \alpha_1)/\alpha_2 \]

\[ k_3 > 0 \]  

(5.25)
According to Aizerman the coefficients in Eq. (5.25) are sufficient conditions for asymptotic stability. Because Aizerman's criterion is a conjecture the sufficient conditions may not hold for all types of systems. This means the closed loop system of Eq. (5.22) with gains satisfying Eq. (5.25) should be tested to be indeed stable. However, relying on Aizerman's conjecture a stability criterion for a stable closed loop inverted pendulum with PID-controller was established such that the error $e$ in Eq. (5.18) asymptotically approaches zero. This is independent of the desired position $\theta_0$. Moreover, it may easily be deduced that the system in Eq. (5.22) admits parameter variations either in the plant or the controller as long as the inequality constraints in Eq. (5.25) remain satisfied. Thus, the controller is robust. Fig. 4 shows a block diagram of this closed loop system.

5.3.2 The Planar Multi-Link Manipulator

In the previous subsection a feedforward integral type compensator was specified to eliminate the effect of gravitation by generating proper biasing torque. In this subsection the controller structure used for the single inverted pendulum is applied to a multi-link, planar rigid-body structure. Consider an arbitrary planar system of interconnected rigid bodies. As examples may serve a special purpose industrial manipulator or biomechanical models [14], [41]. The former system may be represented by the following system of ordinary differential equations

$$H_1(\Theta)\ddot{\Theta} + H_2(\Theta, \dot{\Theta}) + H_3(\Theta) = H_4U$$

(5.26)

where $\Theta, \dot{\Theta}, \ddot{\Theta}$, and $U \in R^n$. The matrix $H_1(\Theta)$ is of size $R^{n \times n}$, is positive definite, and has entries which are analytic functions of $\Theta$. The vectors $H_2(\Theta, \dot{\Theta})$ and $H_3(\Theta)$ are of size $R^n$ with analytic entries in $\Theta$ and $\dot{\Theta}$. Finally, $H_4$ is a constant invertible $n \times n$ matrix. Define a feedback law for the system given by Eq. (5.26) in an analogous fashion as shown in the previous subsection. Thence
Figure 45: Block diagram of the inverted pendulum with PID-controller.
\[ U = K_1 \Theta + K_2(\Theta - \Theta_0) + K_3 \xi \]  

(5.27)

where \( \xi \in \mathbb{R}^n \) is the state of the compensator dynamics

\[ \dot{\xi} = \Theta - \Theta_0 \]  

(5.28)

with \( \Theta_0 \in \mathbb{R}^n \) = constant.

Define a set of new state variables \( \xi \rightarrow \xi, \Theta - \Theta_0 \rightarrow \dot{\xi}, \dot{\Theta} \rightarrow \ddot{\xi}, \) and \( \Theta \rightarrow \dot{\Theta} \).

With these variables the closed loop system of Eq. (5.26) with feedback law (5.27) becomes

\[ H_1(\dot{\xi} + \Theta_0) \dot{\xi} + H_2(\dot{\xi} + \Theta_0, \dot{\xi}) + H_3(\dot{\xi} + \Theta_0) + H_4K_1 \dot{\xi} + H_4K_2 \dot{\xi} + H_4K_3 \xi = 0 \]  

(5.29)

From the structure of the dynamics above the equilibrium points in the state space are depending on \( \Theta_0 \). The equilibrium points of Eq. (5.29) are related to \( \Theta_0 \) by:

\[ \xi_0 = K_3^{-1}H_4^{-1}H_3(\Theta_0) \]  

(5.30)

where it is assumed that \( K_3 \) is an invertible matrix. Fig. 5 shows the overall block diagram of the closed loop planar, multi-link system with PID-controller.

Moreover, from the definition of the position error \( e = \Theta - \Theta_0 \) and \( \dot{\xi} = \Theta - \Theta_0 \), one obviously sees that the error vanishes at the equilibrium \( \xi_0 \). Up to this point it is shown that the error vanishes at the equilibrium point \( \xi_0 \). The stability of the system in Eq. (5.29) is not yet established. The objective in the sequel is, thus, to define conditions on the feedback gain matrices \( K_1, K_2, \) and \( K_3 \) such that the nonlinear system Eq. (5.29) is stable and satisfies \( e \rightarrow 0 \) as \( \xi \rightarrow \xi_0 \) and all time derivatives of \( \xi \) in Eq. (5.29) approach zero as well. Finding conditions on \( K_1, K_2, \) and \( K_3 \) which guarantees global stability of a planar dynamics and verifying it by using a Lyapunov function appeared to be impossible. This is mainly due to a lack of construction rules for Lyapunov functions, often making trial and error methods necessary. Since trial and error methods are frequently not successful
Figure 46: Block diagram of the planar multi-link system with PID-controller.
an alternative idea is pursued here. Instead of looking at global stability a local
stability criterion for feedback gains $K_1, K_2,$ and $K_3$ may be derived either about
an arbitrary equilibrium point or, more general, for any arbitrary point in state
state. Pursuing the latter idea and linearizing Eq. (5.29) about $\xi_0, \dot{\xi}_0, \ddot{\xi}_0,$ and
$\dddot{\xi}_0 = 0$ gives
\[
H_1(\xi_0 + \Theta_0) \dddot{\xi} + \frac{\partial H_2}{\partial \Theta} \frac{\partial \Theta}{\partial \xi} \dot{\xi} + \frac{\partial H_3}{\partial \Theta} \frac{\partial \Theta}{\partial \dot{\xi}} \ddot{\xi} + H_4(K_1 \dddot{\xi} + K_2 \dot{\xi} + K_3 \xi) = 0
\]  
(5.31)

where we have abused notation by letting $d\xi \rightarrow \xi$, $d\dot{\xi} \rightarrow \dot{\xi}$, $d\ddot{\xi} \rightarrow \ddot{\xi}$, and $d\dddot{\xi} \rightarrow \dddot{\xi}$.

Simplifying Eq. (5.31) renders
\[
H_1(\Theta_0) \dddot{\xi} + \left( \frac{\partial H_2}{\partial \Theta} \bigg|_{\Theta_0} + H_4K_1 \right) \dddot{\xi} + \left( \frac{\partial H_2}{\partial \Theta} + \frac{\partial H_3}{\partial \Theta} \bigg|_{\Theta_0} + H_4K_2 \right) \dot{\xi} + K_3 \xi = 0
\]  
(5.32)

Premultiplying Eq. (5.32) by $H_1^{-1}(\Theta_0)$ yields
\[
\dddot{\xi} + H_1^{-1}(\Theta) \left( \frac{\partial H_2}{\partial \Theta} \bigg|_{\Theta_0} + H_4K_1 \right) \dddot{\xi} + H_1^{-1}(\Theta) \left( \frac{\partial H_2}{\partial \Theta} + \frac{\partial H_3}{\partial \Theta} \bigg|_{\Theta_0} + H_4K_2 \right) \dot{\xi} + H_1^{-1}(\Theta_0)K_3 \xi = 0
\]  
(5.33)

The linearized system above can be stabilized by choosing proper feedback gain
matrices $K_1, K_2,$ and $K_3$. Hemami et al. [47] showed that for systems of the
type above pole placement and decoupling can be done simultaneously. Define
three diagonal $n \times n$ matrices $\hat{K}_1, \hat{K}_2, \text{and } \hat{K}_3$ and express the desired closed loop dynamics of Eq. (5.33) as

$$\ddot{\xi} + \hat{K}_1 \dot{\xi} + \hat{K}_2 \xi + \hat{K}_3 \xi = 0$$  \hspace{1cm} (5.34)

where the diagonal matrices are

$$K_1 = \begin{pmatrix} -\lambda_{11} - \lambda_{12} - \lambda_{13} & 0 & 0 \\ 0 & -\lambda_{21} - \lambda_{22} - \lambda_{23} & 0 \\ 0 & 0 & -\lambda_{31} - \lambda_{32} - \lambda_{33} \end{pmatrix}$$

$$K_2 = \begin{pmatrix} \lambda_{11} \lambda_{12} + \lambda_{11} \lambda_{13} + \lambda_{12} \lambda_{13} & 0 & 0 \\ 0 & \lambda_{21} \lambda_{22} + \lambda_{21} \lambda_{23} + \lambda_{22} \lambda_{23} & 0 \\ 0 & 0 & \lambda_{31} \lambda_{32} + \lambda_{31} \lambda_{33} + \lambda_{32} \lambda_{33} \end{pmatrix}$$

$$K_3 = \begin{pmatrix} -\lambda_{11} \lambda_{12} \lambda_{13} & 0 & 0 \\ 0 & -\lambda_{21} \lambda_{22} \lambda_{23} & 0 \\ 0 & 0 & \lambda_{31} \lambda_{32} \lambda_{33} \end{pmatrix}$$

where $\lambda_{ij}$ is the $j$-th eigenvalue of the $i$-th linearized link.

By comparing the coefficient matrices of Eq. (5.33) and Eq. (5.34), one obtains the gain matrices $K_1, K_2,$ and $K_3$ as shown

$$K_1 = H_4^{-1} \left( H_3(\Theta_0) \tilde{K}_1 - \frac{\partial H_3}{\partial \Theta} |_{\Theta_0} \right)$$  \hspace{1cm} (5.35a)

$$K_2 = H_4^{-1} \left( H_3(\Theta_0) \tilde{K}_2 - \frac{\partial H_3}{\partial \Theta} |_{\Theta_0} \right)$$  \hspace{1cm} (5.35b)

$$K_3 = H_4^{-1} H_1(\Theta_0) \tilde{K}_3$$  \hspace{1cm} (5.35c)

The above choice of feedback gain matrices $K_1, K_2,$ and $K_3$ locally stabilizes the dynamics in Eq. (5.29) everywhere in the state space $(\xi, \dot{\xi}, \ddot{\xi})$. When stability about an equilibrium point is considered only, the matrices $K_1, K_2,$ and $K_3$ are evaluated at $\dot{\xi}_0 \in \mathbb{R}^n$, and $\ddot{\xi}_0 = 0$. 

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In the next subsection the latter discussion is demonstrated by conducting two
digital computer simulations. A five link planar manipulator serves as a model for
the discussed compensator scheme.

5.3.3 Digital Computer Simulations

Two digital computer simulations are presented here. The intention of these
simulations is to show qualitatively the behaviour of a planar rigid body system
with PID controller. The planar model consists of five links. Each link has param­
eters as shown in Fig. 6. The numerical values of the body parameters are given
in Table 1 The entries in the matrices $H_1(\Theta)$, $H_2(\Theta, \dot{\Theta})$, $H_3(\Theta)$, and $H_4$ may be
found in [41]. It is assumed that the first link is hinged to the ground.

In the first computer simulation the desired position is kept constant in order
to be able to test the asymptotic performance of the servocompensator. The
desired position vector $\Theta_0$ is kept constant over the entire period of simulation.
The five link angles are chosen to be $\Theta_0 = (\pi/2, \pi/2, \pi/2, \pi/2, \pi/2)$. For obtaining
a meaningful simulation of the PID controller, the initial states of the compensator
are selected such that they do not coincide with the final state. The numerical
values of the initial compensator state is given by $\xi = (0, 0, 0, 0, 0)$. The sampling
period is chosen to be $T = 5 \text{ ms}$ (or units). The results of the first simulation are
given in Figures 7 - 9. Fig. 7 and 8 clearly show the initial perturbation of the
position $\Theta$ and the velocity $\dot{\Theta}$ of the planar system. The perturbations occur, of
course, due to insufficient initial biasing torque at the input of the open loop plant.
After a relatively short period of time the servocontroller provides sufficient input
torque so that the disturbances in the position $\Theta$ and the velocity $\dot{\Theta}$ diminish.
Fig. 9 shows the trajectories of the five states of the servocompensator, which all
depart from the origin and asymptotically approach their final values. The final
Figure 47: Free body diagram of the \( r^{th} \) planar link.
Table 3: Kinematic and Kinetic Parameters of the Five Link Planar Model.

<table>
<thead>
<tr>
<th>Link $i$</th>
<th>$d_i$</th>
<th>$k_i$</th>
<th>$m_i$</th>
<th>$I_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Units</td>
<td>mm</td>
<td>mm</td>
<td>g</td>
<td>gm$^2$</td>
</tr>
<tr>
<td>1</td>
<td>250</td>
<td>125</td>
<td>1.125</td>
<td>5.86</td>
</tr>
<tr>
<td>2</td>
<td>75</td>
<td>37</td>
<td>336</td>
<td>0.158</td>
</tr>
<tr>
<td>3</td>
<td>46</td>
<td>23</td>
<td>125</td>
<td>22e-3</td>
</tr>
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<td>28</td>
<td>14</td>
<td>60</td>
<td>3.92e-3</td>
</tr>
<tr>
<td>5</td>
<td>20</td>
<td>10</td>
<td>40</td>
<td>1.33e-3</td>
</tr>
</tbody>
</table>
Figure 48: Position trajectories of the multi-body system with gravity compensation versus time (static case).

Figure 49: Velocity trajectories of the multi-body system with gravity compensation versus time (static case).
Figure 50: Compensator state trajectories of the multi-body system with gravity compensation versus time (static case).
values of the vector $\xi$ can be predicted from the inverse plant for the given position $\Theta_0$.

The purpose of the second simulation is to test the behaviour of the same five-link manipulator as it performs point-to-point motion.

As in the first simulation the initial state of the manipulator and the servocompensator are chosen to be $\Theta_I = (\pi/2, \pi/2, \pi/2, \pi/2, \pi/2)$ and $\dot{\Theta}_I = (0, 0, 0, 0, 0)$. The final, desired angular position of the manipulator is $\Theta_F = (\pi/2, \pi/2, \pi/2, 3.49, 4.76)$. Both, at the initial and the final position the manipulator is in equilibrium. The manipulator movement is simulated in a time frame of $T_{ex} = 1s$. The results of this simulation are depicted in Fig. 10-13. The trajectories of the angular position vector $\Theta$ are given in Fig. 12. After slight initial perturbations $\Theta_1 - \Theta_3$ reach their steady state value after about 0.25. This coincides with the results of the first simulation. The trajectories of the angular position $\Theta_4$ and $\Theta_5$ reach their desired steady state value at about 0.95-1s of this simulation. Both trajectories exhibit a very slight overshoot. In order to better discern the initial perturbations of $\Theta_1 - \Theta_3$ from their desired position, their trajectories are plotted in Fig. 11 separately. In Fig. 12 the velocity trajectories of the five link system is shown. The velocity profiles clearly show the initial perturbations of link 1-3 and also show that link 4 and 5 slightly overshoot the desired, final position. However, one of the more interesting findings of this simulation is that the decoupling feedback gains which are obtained from the linearized system are well suited to also decouple the nonlinear system. This can be deduced from Fig. 10 - 12, because links 1-3 are not influenced by the motion of link 4 and 5 although they are coupled via the matrix $H_1(\Theta)$ and the vectors $H_2(\Theta, \dot{\Theta})$ and $H_3(\Theta)$. The five states of the servocompensator are shown in Fig. 13. The states $\xi_1, \xi_2$, and $\xi_3$ of the compensator, which correspond to link 1-3 start at $\xi = 0$ and reach their steady
Figure 51: Position trajectories of the multi-body system with gravity compensation versus time (dynamic case).

Figure 52: Position trajectories of the multi-body system with gravity compensation versus time (dynamic case, only $\theta_1 - \theta_3$).
Figure 53: Velocity trajectories of the multi-body system with gravity compensation versus time (dynamic case).

Figure 54: Compensator state trajectories of the multi-body system with gravity compensation versus time (dynamic case).
state value after about 0.2s. This asymptotic behaviour might be expected from the behaviour of the states $\xi_1, \xi_2,$ and $\xi_3$ of the first simulation. Note that the initial and the desired position of link 1 through 3 are identical. In contrast, link 4 and 5 are moved through a range of about 110 deg. and 180 deg, respectively. The trajectory profile of the servocompensator states $\xi_4$ and $\xi_5$ are shown in Fig. 13 as well. The states $\xi_4$ and $\xi_5$ reach their steady state value together with the angular position of link 4 and 5 as expected.

In this section it was demonstrated that a linear asymptotic servocompensator can be satisfactorily employed to a planar mechanical manipulator in order to eliminate the need of feedforward biasing torques. A servocompensator was used to substitute the feedforward compensator, i.e., inverse plant, and to provide the necessary biasing torques for eliminating the gravitational disturbance. Two digital computer simulation examples have shown that the proposed compensator scheme performs well. In addition and contrary to the inverse plant technique, the feedback compensator permits parameter variations in the plant, i.e., it is robust as long as the closed loop system remains stable.

5.4 Compensation for Input Disturbances

The purpose of this section is to show that linear servocompensator theory is applicable to robotic systems in order to obtain disturbance rejection. We will assume that the disturbance $\omega$ that enters the manipulator satisfies a linear homogeneous differential equation of the form

$$\omega^{(p)} + \alpha_p \omega^{(p-1)} + \alpha_{p-1} \omega^{(p-2)} + \ldots + \alpha_2 \dot{\omega} + \alpha_1 \omega = 0$$  \hspace{1cm} (5.36)

where the eigenvalues of Eq. (5.36) are all contained in the closed right half complex plane. Applying standard servocompensator theory as discussed by Davison [148].
[149], it is possible to achieve asymptotic disturbance rejection of the disturbance \( \omega \). The mathematical approach in dealing with this problem is based on the Volterra series expansion of the dynamics of a manipulator. To eliminate the need of proving convergence of the Volterra series, we will refer to Brockett [153]. In his former work it was shown that systems that are analytic in the state \( X \) and linear in the input \( U \) have a Volterra series representation provided there is no finite escape time. This means the systems may not satisfy \( \| X \| \to \infty \) as \( t \to t_1 \), with \( t_1 \in (t_0, \infty) \). The dynamical formulae of interconnected rigid body systems satisfy Brockett's criterion. Obviously, robot manipulators are a special class of general interconnected rigid body systems. A mathematical expression of the Volterra series for a general robot manipulator is prohibitively difficult to derive. Therefore, in the next subsection the Volterra series of a simple inverted pendulum is derived. In a following step a feedback servocompensator is implemented and it is shown that this type of servocompensator can asymptotically reject disturbances of the type discussed before.

In the sequel the Volterra series for a planar multi-link manipulator arm is presented generalizing the compensator control strategy to multi-input/multi-output systems. A digital computer simulation is conducted to acquire more insight into the resulting control strategy.

5.4.1 The Volterra Series of a Planar Pendulum and Disturbance Rejection

Consider the differential equation of a single pendulum with static state feedback as given by Eq. (5.16). For convenience this equation is repeated here

\[
\ddot{\theta} + \alpha_2 K_1 \dot{\theta} + \alpha_2 K_2 (\theta - \theta_0) - \alpha_1 \sin(\theta) = u \tag{5.37}
\]

where
without loss of generality let $\theta_0 = 0$ and let Eq. (5.37) be expressed in a Maclaurin series, which gives

$$\ddot{\theta} + \alpha_2 k_1 \dot{\theta} + (\alpha_2 k_2 - \alpha_1)\theta - \alpha_1 \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n + 1)!} \theta^{2n+1} = u$$

(5.38)

Eq. (5.38) can be expressed in terms of an infinite series of operators of the form

$$u(t) = T[\theta(t)] = T_1[\theta(t)] + T_r[\theta(t)]$$

(5.39)

where $T_1[\theta(t)]$ is a 1st order operator and $T_r$ represents all higher order operators, i.e.,

$$T_r[\theta(t)] = \alpha_1 \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n + 1)!} \theta(t)^{2n+1}$$

(5.40)

Schetzen [152] showed that if $T_1^{-1}[]$ is a stable operator and $|u| < M_u$ is bounded then $T_1^{-1}[]$ can be expressed by a Volterra operator. Define $H_1(s)$ be the Laplace operation of $T_1^{-1}[]$, i.e.,

$$H_1(s) = \mathcal{L}[T_1^{-1}[]]$$

(5.41)

then $H_1(s)$ becomes

$$H_1(s) = \frac{1}{s^2 + \alpha_2 k_1 s + (\alpha_2 k_2 - \alpha_1)}$$

(5.42)

$H_1(s)$ should have its poles in the open left half complex plane. Next, let the solution of the differential equation (5.38) be written in terms of a Volterra operator

$$\theta(t) = H[u(t)] = \sum_{n=1}^{\infty} H_n[u(t)]$$

(5.43)

Define $u(t) \rightarrow cu(t)$ with $c$ an arbitrary constant. Then Eq. (5.43) can be written as
\[ \theta(t) = \sum_{n=1}^{\infty} \mathcal{H}[c\theta(t)] = \sum_{n=1}^{\infty} c^n \mathcal{H}[u(t)] \]  

(5.44)

The coefficient \( c \) will be helpful in solving for \( \theta(t) \) in terms of \( u(t) \) by comparing equal powers of \( c \) and relating the \( n \)-th power of \( c \) to the \( n \)-th Volterra operator of Eq. (5.44).

Define

\[ \theta_n(t) = \mathcal{H}_n[u(t)] \]  

(5.45)

That is, \( \theta_n(t) \) is the solution of the \( n \)th order Volterra operator. Substitute Eq. (5.45) into Eq. (5.44) and then into Eq. (5.38) gives

\[ c\theta(t) = \sum_{n=1}^{\infty} c^n \left[ \bar{\theta}_n + \alpha_2 k_1 \bar{\theta}_n + (\alpha_2 k_2 - \alpha_1) \theta_n \right] \]

\[ -\alpha_1 \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k+1)!} \left[ \sum_{n=1}^{\infty} c^n \theta_n \right]^{2k+1} \]

(5.46)

After expanding the double summation into a form that is more familiar in terms of a Volterra series, one gets

\[ c\theta(t) = \sum_{n=1}^{\infty} c^n \left[ \bar{\theta}_n + \alpha_2 k_1 \bar{\theta}_n + (\alpha_2 k_2 - \alpha_1) \theta_n \right] \]

\[ -\frac{\alpha_1}{3!} \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \sum_{n_3=1}^{\infty} c^{n_1+n_2+n_3} \theta_{n_1}(t) \theta_{n_2}(t) \theta_{n_3}(t) \]

\[ +\frac{\alpha_1}{5!} \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \sum_{n_3=1}^{\infty} \sum_{n_4=1}^{\infty} c^{n_1+n_2+n_3+n_4} \theta_{n_1}(t) \theta_{n_2}(t) \theta_{n_3}(t) \theta_{n_4}(t) \]

\[ -\frac{\alpha_1}{7!} \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \sum_{n_3=1}^{\infty} \sum_{n_4=1}^{\infty} \sum_{n_5=1}^{\infty} c^{n_1+n_2+n_3+n_4+n_5} \theta_{n_1}(t) \theta_{n_2}(t) \theta_{n_3}(t) \theta_{n_4}(t) \theta_{n_5}(t) \]  

(5.47)

Let

\[ u(t) = \sum_{n=1}^{\infty} T[\theta_n(t)] \]  

(5.48)
and equate the coefficients of equal power of \( c \) in Eq. (5.47) with the \( n^{th} \) term in Eq. (5.48); Thus

\[
T[\theta_1(t)] = u(t) = \ddot{\theta}_1 + \alpha_2 k_1 \dot{\theta}_1 + (\alpha_2 k_2 - \alpha_1)\theta_1 
\]

(5.49)

\[
T[\theta_2(t)] = 0 = \ddot{\theta}_2 + \alpha_2 k_1 \dot{\theta}_2 + (\alpha_2 k_2 - \alpha_1)\theta_2 
\]

(5.50)

\[
T[\theta_3(t)] = 0 = \ddot{\theta}_3 + \alpha_2 k_1 \dot{\theta}_3 + (\alpha_2 k_2 - \alpha_1)\theta_3 
\]

\[
- \frac{\alpha_1}{3!} \theta_1^3 
\]

(5.51)

\[
T[\theta_4(t)] = 0 = \ddot{\theta}_4 + \alpha_2 k_1 \dot{\theta}_4 + (\alpha_2 k_2 - \alpha_1)\theta_4 
\]

\[
- \frac{\alpha_1}{2!} \theta_2^2 \theta_2 
\]

(5.52)

\[
T[\theta_5(t)] = 0 = \ddot{\theta}_5 + \alpha_2 k_1 \dot{\theta}_5 + (\alpha_2 k_2 - \alpha_1)\theta_5 
\]

\[
- \frac{\alpha_1}{2!} (\theta_2^2 \theta_3 + \theta_1 \theta_2^2) + \frac{\alpha_1}{5!} \theta_5^5 
\]

(5.53)

\[
T[\theta_6(t)] = 0 = \ddot{\theta}_6 + \alpha_2 k_1 \dot{\theta}_6 + (\alpha_2 k_2 - \alpha_1)\theta_6 
\]

\[
- \alpha_1 \theta_2 \theta_3 = \frac{\alpha_1}{3!} \theta_2^2 + \frac{\alpha_1}{4!} \theta_3^4 + \theta_2 \theta_3 
\]

(5.54)

\[
T[\theta_7(t)] = \ldots 
\]

etc.

Operating on each term \( T[\theta_n(t)] \) with \( H_1[u(t)] = T_1^{-1}[u(t)] \) gives

\[
\theta_1 = H_1[u(t)] 
\]

(5.55)

where the operator \( H_1 \) is assumed to be stable. Because \( H_1 \) is the first order operator, it has a Laplace transform as given in Eq. (5.42).

Operating on both sides of Eq. (5.50) with \( T_1^{-1} \) gives

\[
\theta_2(t) = T_1^{-1}[u(t)] = 0 
\]

(5.56)

Since Eq. (5.56) must be zero for all possible \( u(t) \) and all time \( t \), the kernel \( h_2(\tau_1, \tau_2) \) of the integral

\[
\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} h_2(\tau_1, \tau_2) u(\tau_1) u(\tau_2) d\tau_1 d\tau_2 
\]

(5.57)

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must satisfy

$$h_2(T_1, T_2) = 0$$  \hspace{1cm} (5.58)

Operating with $T_1^{-1}$ on Eq. (5.51) yields

$$\theta_3(t) = H_1[\frac{\alpha_1}{3!} \theta_1^3(t)]$$  \hspace{1cm} (5.59)

Continuing this procedure the next three higher terms are

$$\theta_4(t) = 0$$  \hspace{1cm} (5.60)

$$\theta_5(t) = H_1[\frac{\alpha_1}{2!} \theta_1^2 \theta_1^2 - \frac{\alpha_1}{5!} \theta_1^5]$$  \hspace{1cm} (5.61)

$$\theta_6(t) = 0$$  \hspace{1cm} (5.62)

\[ \vdots \hspace{1cm} \text{etc.} \]

Previously, it was stated that the terms $\theta_1(t), \theta_3(t), \theta_5(t)$ in Eqs. (5.55), (5.59), and (5.61) represent the first three terms of the Volterra series. To show that $\theta_1(t)$ is the first order term is trivial, because it is simply the convolution of the input $u(t) < M_u$ with the impulse response of the linearized pendulum dynamics.

Thus,

$$\theta_1(t) = \int_0^\infty Ce^{AT}Bu(t - T)dT$$  \hspace{1cm} (5.63)

where the triple $(C, A, B)$ is a realization of the differential equation (5.49). To obtain the next higher order term in the Volterra series, consider Eq. (5.59) which may be written as

$$\theta_3(t) = \frac{\alpha_1}{3!} \int_0^\infty Ce^{AT}B\theta_1^3(t - T)dT$$  \hspace{1cm} (5.64)

Because the input of Eq. (5.64) depends on the solution of Eq. (5.63), substituting the latter equation into Eq. (5.64) yields
\[ \theta_3(t) = \frac{a_1}{3!} \int_0^\infty C e^{AT} B \left[ \int_0^\infty C e^{AT} B u(t-T) dT \right]^3 dT \]  

\hspace{1cm} (5.65)

Introducing dummy variables \( T_1, T_2 \) and \( T_3 \) for \( T \) and changing the order of integration gives

\[ \theta_3(t) = \frac{a_1}{3!} \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty C e^{AT} B C e^{AT_1} B C e^{AT_2} B C e^{AT_3} B \]
\[ \times u(t-T-T_1) u(t-T-T_2) u(t-T-T_3) dT_1 dT_2 dT_3 dT \]  

\hspace{1cm} (5.66)

Let \( T_i = T + T_i, i = 1,2,3 \) and after a change of variables as well as the order of integration Eq. (5.65) may be written as

\[ \theta_3(t) = \int_0^\infty \int_0^\infty \int_0^\infty \frac{a_1}{3!} \int_0^\infty C e^{AT} B C e^{A(T_1-T)} B C e^{A(T_2-T)} B C e^{A(T_3-T)} B dT \]
\[ \times u(t-T_1) u(t-T_2) u(t-T_3) dT_1 dT_2 dT_3 \]  

\hspace{1cm} (5.67)

Define the kernel of the third order Volterra term as

\[ h_3(T_1, T_2, T_3) = \frac{a_1}{3!} \int_0^\infty C e^{AT} B C e^{A(T_1-T)} B C e^{A(T_2-T)} B C e^{A(T_3-T)} B dT \]  

\hspace{1cm} (5.68)

then Eq. (5.67) appears in familiar form as

\[ \theta_3(t) = \int_0^\infty \int_0^\infty \int_0^\infty h_3(T_1, T_2, T_3) u(t-T_1) u(t-T_2) u(t-T_3) dT_1 dT_2 dT_3 \]  

\hspace{1cm} (5.69)

In a similar fashion it is, of course, possible to compute all higher order terms.

With this development, Eqs. (5.55)-(5.62) show that the input \( u(t) \) only affects the linear term \( \theta_1(t) \) because the system is linear in the input. Moreover, one may notice that all higher order (nonlinear) terms of \( \theta_i \) depend on nonlinear forcing functions of lower order. Fig. 14 shows the block diagram of the first three non-zero terms of the dynamics of the pendulum. Clearly the response of \( \theta(t) \) can be considered to be the sum of the linear response \( \theta_1(t) \) plus a perturbation about \( \theta_1(t) \). The structure of the block diagram in Fig. 14 suggests that disturbances that enter the input of Eq. (5.38) and are of the class given in Eq. (5.36) may possibly be compensated for by employing a linear servocompensator. The pendulum dynamics with a linear servocompensator is shown in Fig. 15. This figure
Figure 55: Block diagram of the first three Volterra operators of the pendulum.
Figure 56: Block diagram of the pendulum with servocompensator.
shows the nonlinear part which is lumped together into one block, the linear part of the plant, and the servocompensator in the feedback loop. The purpose of the servocompensator is to compensate the modes of the disturbance signal \( u \). If this is feasible, then all higher order modes, which are introduced by the nonlinearity of the system will be eliminated as well.

Let the dynamics of the disturbance be

\[
\dot{X}_d = A_d X_d \\
\omega = C_d X_d
\]  \hspace{1cm} (5.70)

where

\[ X_d \in \mathbb{R}^d \]

We design a dynamic feedback compensator

\[
\eta = A_c \eta + B_c X \\
u = -K_f \eta + \omega
\]  \hspace{1cm} (5.71)

where

\[ A_c \in \mathbb{R}^{d \times d}, \, B_c \in \mathbb{R}^d, \, K_f \in \mathbb{R}^{1 \times d} \quad \text{and} \quad \sigma(A_c) = \sigma(A_d) \]

Here \( \sigma(\cdot) \) denotes the spectrum of a matrix. Combining the dynamics of the pendulum and the servocompensator yields

\[
\begin{pmatrix}
\dot{\eta} \\
\dot{\theta}
\end{pmatrix}
= 
\begin{pmatrix}
A_c & B_c & 0 \\
\vdots & \ddots & \vdots \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\eta \\
\theta
\end{pmatrix}
+ 
\begin{pmatrix}
0 \\
0 \\
\alpha_1 \sin(\theta)
\end{pmatrix}
\omega
\]  \hspace{1cm} (5.72)
The augmented dynamic system can be stabilized by choosing appropriate values for the feedback gains $K_I, K_1,$ and $K_2$, despite the nonlinearity in the dynamics. To show this let Eq. (5.72) be rewritten as

$$\dot{X} = A_a X + A_N + B_\omega$$

(5.73)

$$\theta = C_a X$$

Define a Lyapunov function

$$V = X^T Q X$$

(5.74)

Differentiating Eq. (5.74) with respect to time and substituting Eq. (5.73) into this result gives

$$\dot{V} = X^T (QA_a + A_a^T Q)X + 2X^T Q A_N$$

(5.75)

where $u = 0$ for this stability study. If the spectrum of $A, \sigma(A),$ contains only eigenvalues with negative real parts then $Q$ can be solved uniquely from the equation

$$QA_a + A_a^T Q = -2I$$

(5.76)

where $I$ is the unity matrix.

Lemma 5.1 The system Eq. (5.73) is asymptotically stable if all eigenvalues of $A_a$ have negative real parts and $\|q_f\| \cdot \alpha_1 \leq 1$. The vector $q_f$ is the last column of the matrix $Q$. 

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Proof The dynamics of Eq. (5.73) is asymptotically stable if
\[ \dot{V} < 0 \quad \text{for } x \neq 0 \]

From Eq. (5.72) and (5.76) \( \dot{V} \) assumes the form
\[
\dot{V} = -2X^T + 2X^T q \alpha_1 \sin(\theta)
\]  
(5.77)

By inspection of the term
\[ \|X\| - \|q\| \alpha_1 \sin(\|X\| \cos \zeta) \]
one notices that if \( \|q\| \cdot \alpha_1 \leq 1 \) the only zeros of the function \( \dot{V} \) occur at \( \|X\| = 0 \).
Hence, \( \dot{V} \) is negative definite.

Q.E.D.

Using Eq. (5.76) and (5.72), \( \|q\| \) can be solved in terms of the feedback gains \( K_I, k_1, \) and \( k_2 \). However, it should be pointed out that the norm condition on \( q \) is a sufficient condition only. Therefore it may be possible that a set a feedback gains \( k_1, k_2, \) and \( K_I \) stabilizes the closed loop dynamics but does not satisfy the norm condition on \( q \).

In the remaining part of this section a mathematical argument is provided to show that for any measurable or unmeasurable disturbance \( \omega \) described by Eq. (5.70) there exists a feedback compensator for the dynamics in Eq. (5.38) such that \( \theta \rightarrow 0 \) as \( t \rightarrow \infty \).
Reconsider the dynamics of the pendulum with dynamic compensator as given in Eq. (5.71) and Eq. (5.73). For the linear part of these equations it is straightforward to show that the position $\theta \to 0$ as $t \to \infty$. The linear part of Eq. (5.73) is

$$\dot{X} = \begin{pmatrix} A_{c} & B_{c} & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 1 \\ -K_{I} & -(\alpha_{2}k_{2} - \alpha_{1}) & \alpha_{2}k_{1} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} v + \omega \end{pmatrix} \quad (5.78)$$

where $X = (\eta, \theta, \dot{\theta})^T$ and $\eta \in \mathbb{R}^d$ is the state vector of the compensator. From linear servocompensator theory as discussed by Davison [149] for any $\omega$ satisfying Eq. (5.70). Whenever the system in Eq. (5.78) is stable, there exists a servocompensator described by Eq. (5.71) such that $\theta \to 0$ as $t \to \infty$. To show existence of a servocompensator for the dynamics of the plant given by Eq. (5.73) such that $\theta \to 0$ as $t \to \infty$, the following lemma is necessary.

**Lemma 5.2** A sufficient condition that there exists a robust linear-time invariant dynamic feedback controller for the system Eq. (5.38) such that $\theta \to 0$ as $t \to \infty$ for all disturbances $\omega$ described by Eq. (5.78), is that the following conditions hold.

(i) There exists a solution of the servocompensator problem of the linearized pendulum dynamics.

(ii) There exists a recursive Volterra series such that if $\theta(t) = \sum_{n=1}^{\infty} \theta_{n}(t)$ and $\theta_{n}$ is the $n$-th term in the Volterra series, then

$$\theta_{1} = \mathcal{H}_{1}[\omega(t)]$$
\[ \theta_n = H_1[F_n(\theta_1, \theta_2, \ldots, \theta_{n-1})] \]

where \( F_n(\ldots) \) is a homogeneous polynomial of degree \( n \) of the variables \( \theta_1, \ldots, \theta_{n-1} \).

**Proof** The proof of this lemma is simple. Condition (i) guarantees that the linear part of the nonlinear dynamics and the linear servocompensator satisfy \( \theta_1 \to 0 \) as \( t \to \infty \). In Eq. (5.78) it has been shown that this condition can be satisfied. Condition (ii) guarantees that if \( \theta_{i-1} \to 0 \) then \( \theta_i \to 0 \), because \( F_n(\theta_1, \ldots, \theta_{n-1}) = 0 \) when \( (\theta_1, \ldots, \theta_{n-1}) = 0 \). Consequently if (i) and (ii) are satisfied and each \( \theta_i \to 0 \) as \( t \to \infty \), then \( \theta \to 0 \) as \( t \to \infty \).

Q.E.D.

Using the previous lemma, it remains to show that the differential equation Eq. (5.73) can be expanded into a recursive Volterra series.

In order to accomplish this, let \( \omega(t) \) be replaced by \( c\omega(t) \) in Eq. (5.73). Then the solution of Eq. (5.73) may be written as

\[ X(t) = \sum_{n=1}^{\infty} c^n X_n(t) \quad (5.79) \]

where each \( X_n = H_n[\omega(t)] \) is the solution of the \( n \)-th order Volterra operator.

Substituting Eq. (5.79) into Eq. (5.73) yields

\[ \sum_{n=1}^{\infty} c^n \dot{X}_n(t) = A_0 \sum_{n=1}^{\infty} c^n X_n(t) + Bc\omega(t) \]
\[ + A_n \left( \sum_{n=1}^{\infty} X_n(t) \right) \quad (5.80) \]
Expanding the third term on the right hand side of Eq. (5.80) into a power series and comparing powers of the coefficient $c$ as in Eq. (5.47) yields

\[ \dot{X}_1(t) = A_a X_1(t) + Bw(t) \] (5.81)

\[ \dot{X}_3(t) = A_a X_3(t) + A N_3 \] (5.82)

\[ \dot{X}_5(t) = A_a X_5(t) + A N_5 \] (5.83)

\[ \dot{X}_7(t) = A_a X_7(t) + A N_7 \] (5.84)

\[ \vdots \]

where

\[ A_{N_n} = (0 \ldots 0, F_n(\theta_1, \ldots, \theta_{n-1}))^T = B \cdot F_n \]

and $F_n$ is as given in the former lemma. Solving Eq. (5.81) to Eq. (5.84) and so forth gives

\[ X_1(t) = \int_0^\infty e^{A_a(t-T)} B w(T) dT \] (5.85)

\[ X_3(t) = \int_0^\infty e^{A_a(t-T)} B F_3(X_1) dT \] (5.86)

\[ X_5(t) = \int_0^\infty e^{A_a(t-T)} B F_5(X_1, X_3) dT \] (5.87)

\[ \vdots \]

Clearly, Eq. (5.85) to Eq. (5.87) are the first three terms of a recursive Volterra series, which has non-zero terms only when $n = \text{odd}$. From Eq. (5.78) the angle $\theta$ is

\[ \theta_n(t) = C_a X_n(t) \] (5.88)

Using the latter relation one gets

\[ \theta_n(t) = \begin{cases} H_1[\omega(t)] & n = 1 \\ H_1[F_n(\theta_1, \ldots, \theta_{n-1})] & n > 1 \end{cases} \] (5.89)
Finally, since $H_1[\cdot] \to 0$ as $t \to \infty$ all $H_n[\cdot] \to 0$ as $t \to \infty$. The servocompensator presented in this discussion also is robust in the sense of Davison and Goldenberg [148].

In this treatise it was shown that a linear feedback servocompensator can be successfully employed to asymptotically reject measurable and unmeasurable disturbances of certain type which may perturb a simple nonlinear system. It was assumed that the disturbance satisfies a linear differential equation whose eigenvalues are presumed to be known as well. Because of the simplicity of the nonlinear system, one may doubt the ability to generalize the former discussion to multi-link system. Therefore a generalization of the planar pendulum to multi-link, planar rigid body systems is provided next.

### 5.4.2 The Servocompensator Applied to Planar Multi-body Mechanical Systems

In this section it is shown that the results of Section 5.4.1 may be generalized to multi-link, planar rigid body mechanical structures. In this discussion, however, the mathematical expressions are shortened substantially. The mathematical principles are analogous to the ones in Section 5.4.1.

Consider the dynamics of a general planar multi-link rigid body system with revolute joints, whose dynamic behaviour may be expressed in mathematical form as

$$H_1(\Theta)\ddot{\Theta} + H_2(\Theta, \dot{\Theta}) + H_3(\Theta) = H_4U$$

where $\Theta, \dot{\Theta}, \ddot{\Theta}, U \in \mathbb{R}^n$, $H_1$ is the inertia matrix, $H_2$ the centrifugal force vector, $H_3$ the gravitational force vector, and $H_4$ is an $n \times n$ constant matrix. The vectors $\Theta, \dot{\Theta},$ and $\ddot{\Theta}$ obviously are the joint angles as well as their first and second time derivative, respectively. The vector $U$ is the input torque vector. The entries of
the matrices $H_1$ and $H_4$, and vectors $H_2$ and $H_3$ are analytic functions of $\Theta$ and $\dot{\Theta}$. For a chain connected rigid body system, the elements of the former matrices and vectors are given in [14] as stated before.

Let $\omega \in \mathbb{R}^n$ be a measurable or unmeasurable disturbance that enters the dynamics of Eq. (5.90) and satisfies a linear differential equation with unknown initial conditions. Thus, Eq. (5.90) renders

$$H_1(\Theta)\ddot{\Theta} + H_2(\Theta, \dot{\Theta}) + H_3(\Theta) = H_4U + H_4\omega$$

Define a linear control law with compensator as

$$U = -K_1\dot{\Theta} - K_2\Theta - K_I\eta$$

where $\eta \in \mathbb{R}^{n \times d}$ is the state of the servocompensator and the matrices $K_1, K_2$ and $K_I$ are of appropriate sizes. Let the compensator be defined as

$$\dot{\eta} = A_c\eta + B_c\Theta$$

where the matrices $A_c$ and $G_c$ are constructed as discussed in [162, p. 88)]. Combining Eq. (5.91) to Eq. (5.93) yields

$$\begin{pmatrix} H_1(\Theta)\ddot{\Theta} \\ \eta \end{pmatrix} = \begin{pmatrix} -H_2(\Theta, \dot{\Theta}) - H_3(\Theta) - H_4(K_1\dot{\Theta} + K_2\Theta + K_I\eta) + H_4\omega \\ A_c\eta + B_c\Theta \end{pmatrix}$$

In the next step each analytic term in Eq. (5.94) is expanded into a power series about the origin and the linear part is separated from the higher order terms. Define

$$H_{11} = H_1(0)$$

$$H_{31} = \frac{\partial H_3}{\partial \Theta} \bigg|_{\Theta = 0}$$

$$H_{1R} = H_1(\Theta) - H_{11}$$

$$H_{2R}(\Theta, \dot{\Theta}) = H_2(\Theta, \dot{\Theta})$$
\[ H_{3R}(\Theta) = H_3(\Theta) - \frac{\partial H_3}{\partial \Theta} \bigg|_{\Theta=0} \]

The matrices or vectors \( H_{1R}, H_{2R}, \) and \( H_{3R} \) contain terms of higher order. Rewriting Eq. (5.94) yields

\[
\begin{pmatrix}
\dot{\Theta} \\
\dot{\eta}
\end{pmatrix} =
\begin{pmatrix}
0 & I & 0 \\
-H_{11}^{-1}(H_{31} + H_{4} K_2) & -H_{11}^{-1} H_{4} K_1 & -H_{11}^{-1} H_{4} K_I \\
B_\zeta & 0 & A_c
\end{pmatrix}
\begin{pmatrix}
\Theta \\
\eta
\end{pmatrix} +
\begin{pmatrix}
0 \\
H_{11}^{-1}(H_{1R}\dot{\Theta} + H_{2R} + H_{3R}) \\
0
\end{pmatrix} +
\begin{pmatrix}
0 \\
H_4 \omega(t)
\end{pmatrix}
\tag{5.95}
\]

Define the state of the composite dynamic system to be \( X = (\Theta \ \dot{\Theta} \ \eta)^T \), where \( X \in \mathbb{R}^{2n + nx d} \). Let the solution of Eq. (5.94) in terms of the disturbance \( \omega \) be

\[
X(t) = \sum_{n=1}^{\infty} H_n[\omega(t)]
\tag{5.96}
\]

where \( H_n \) is the \( n^{th} \) Volterra operator, e.g.

\[
H_1[\omega](t) = \int_0^\infty e^{A(t-\tau)} B_\omega(\tau) d\tau
\tag{5.97}
\]

Define \( X_n(t) = H_n(t) \) and let \( \omega(t) \) be replaced by \( c \omega(t) \), then the solution of Eq. (5.95) with \( X(t=0) = 0 \) assumes the form

\[
X(t) = \sum_{n=1}^{\infty} e^n X_n(t)
\tag{5.98}
\]

Substituting Eq. (5.97) into Eq. (5.94) and comparing powers of \( c \) yields

\[
\dot{X}_1(t) = A_\alpha X_1(t) + B_\alpha \omega
\tag{5.99}
\]
where $A_a$ is the coefficient matrix on the right hand side of Eq. (5.94) and $B_a = (0 \ I_{n\times n} \ 0)^T$.

\[
\begin{align*}
\dot{X}_2(t) &= A_a X_2(t) & (5.100) \\
\dot{X}_3(t) &= A_a X_3(t) + A_{N_3}(X_1) & (5.101)
\end{align*}
\]

where

\[
A_{N_3} = \begin{pmatrix}
0 \\
H_{11}^{-1}(H_{12}(\Theta_1)\bar{\Theta}_1 + H_{21}(\Theta_1)\dot{\Theta}_1^2 + H_{33}(\Theta_1))
\end{pmatrix}
\]

and $H_{1n}, H_{2n},$ and $H_{3n}$ are the $n$-th order polynomial terms of $H_1, H_2,$ and $H_3,$ respectively.

The terms of order higher than three may be obtained in a manner similar to the one shown in Section 5.4.1. More important is the fact, however, that the Volterra series of the solution of Eq. (5.94) can be expressed in terms of a recursive series. In fact, this allows us to apply Lemma 5.2 to guarantee asymptotic disturbance rejection of some disturbance $\omega$, provided the closed loop system is stable. A block diagram of the dynamics of Eq. (5.95) is shown in Fig. 16. This figure illustrates the decomposition of the planar multi-link system with servocompensator into a linear and a nonlinear part. It also shows how the nonlinear partition is excited by the linear one. As in the linear servocompensator case, closed loop stability of the overall system is necessary for achieving asymptotic disturbance rejection. With constant linear feedback gains as given in Eq. (5.92), global stability of the composite system Eq. (5.94) may be impossible to prove. In fact, many attempts to verify global asymptotic stability of a planar multi-link assemblage with servocompensator have failed. In contrast, local stability about the equilibrium point at the origin can be shown. This approach has already been discussed in Section 5.3.2.
Figure 57: Block diagram of the decomposed multi-body dynamics with servocompensator.
In order to illustrate qualitatively and quantitatively the merit of linear servocompensator to asymptotically reject disturbances, a digital computer simulation is presented in the sequel.

5.4.3 Computer Simulation and Results

The discussion in this section has shown that linear servocompensator theory is applicable to rigid body dynamic systems in order to obtain asymptotic disturbance rejection. As in linear system theory the disturbance may be measurable or unmeasurable, but must satisfy a linear differential equation with constant coefficients. Furthermore, constructing a dynamic compensator requires a priori knowledge of all eigenvalues of the disturbance signals. A digital computer simulation is conducted to assess the behaviour of the closed loop system of a multi-link structure composed with a linear servocompensator. As model of the multi-link structure serves a double inverted pendulum. The kinematic and kinetic parameters of this model are given in Table 2. The model is depicted in Fig. 17. The dynamic equations of this model are derived according to [41]. Two different disturbance signals are assumed to enter the pendulum dynamics at both joints. They are

\[ \omega = (10\sin(2\pi t + \pi/4)Nm, 0.5e^{jN}m)^T \]

The eigenvalue spectrum of \( \omega \) yields \( \sigma_\omega = \{+j, -j, +1\} \). Knowing the eigenvalues in the set \( \omega_\omega \) a possible compensator realization is
Figure 58: Double inverted pendulum.
The constant feedback gain matrices are computed such that the linearized dy-
namic system — linearized at the origin — is decoupled and possesses eigenvalue
locations for each link at \( \sigma = \{-4, -5, -6, -8, -10\} \). The resulting feedback gains
are

\[
K_1 = \begin{pmatrix}
816 & 275.4 \\
136 & 139.4
\end{pmatrix}
\]

\[
K_2 = \begin{pmatrix}
9848.3 & 3350.8 \\
1674.1 & 1676.7
\end{pmatrix}
\]

\[
K_I = \begin{pmatrix}
626,941 & -170,407 & 42,426 & 211,593 & -57,512 & 14,319 \\
104,490 & -28,401 & 7,070.9 & 107,102 & -29,111 & 7,247.7
\end{pmatrix}
\]

The initial state of the double inverted pendulum is slightly perturbed and is chosen
to be \( \Theta_I = (0.1 \text{rad} \ 0.2 \text{rad})^T \) and \( \dot{\Theta}_I = (0 \ 0)^T \). The state of the servocompensator
is initially set to zero. The simulation is conducted for a period of \( T_{ex} = 4s \) and
the integration step size is selected to be \( T = 10ms \). The results of this simulation
are plotted in Fig. 18-27. The angular position and velocity of the double inverted
pendulum are shown in Fig. 18 and 19. The plots show that the pendulum
returns from its initially perturbed position to its vertical rest position within
approximately \( 2s - 2.5s \).
Figure 59: Position trajectories of the double pendulum with disturbances versus time.

Figure 60: Velocity trajectories of the double pendulum with disturbances versus time.
This happens despite the disturbance $\omega$ at the input of the double pendulum, which indicates that the servocompensator is capable of compensating the disturbance signals at the input of the plant. The states of the compensator are depicted in Fig. 20-23.

Fig. 20 shows the states of the compensator which are excited by the position of the first link; i.e., $\Theta_1$. Fig. 21 shows the same three states on a different scale to better visualize their behaviour in steady state. The states of the compensator which are excited by $\Theta_2$ are shown in Fig. 22 and Fig. 23. The latter figure also is plotted on a different scale to obtain a better picture of the steady state behaviour of the states $\eta_4 - \eta_6$.

Fig. 24 depicts the time history of the total input torques applied to the double inverted pendulum. The total input torque is defined to be the sum of the feedback torques plus the torque due to the disturbance $\omega$. As expected the total input torques approach zero asymptotically.

The two disturbance signals that enter the pendulum dynamics are shown in Fig. 25. The disturbance $\omega_1(t)$ and $\omega_2(t)$ enter the plant at joint 1 and joint 2, respectively.

Fig. 26 and Fig. 27 depict the disturbance signals $\omega_3(t)$ and the feedback torque $-U_i(t)$. These two figures clearly manifest that the servocompensator generates two feedback signals that asymptotically cancel the input disturbances $\omega$.

5.5 Summary

In this chapter compensator schemes are discussed for eliminating the effects of two types of disturbances which significantly change the dynamic performance of rigid body systems. The two types of disturbances which were considered in this
Figure 61: Servocompensator states $\eta_1 - \eta_3$ that are excited by $\theta_1$ versus time.

Figure 62: Servocompensator states $\eta_1 - \eta_3$ that are excited by $\theta_1$ versus time (enlarged scale).
Figure 63: Servocompensator states $\eta_4 - \eta_6$ that are excited by $\theta_2$ versus time.

Figure 64: Servocompensator states $\eta_4 - \eta_6$ that are excited by $\theta_2$ versus time (enlarged scale).

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Figure 65: Input torques to the double pendulum versus time.

Figure 66: Time history of the disturbances $\omega_1$ and $\omega_2$ during simulation.
Figure 67: Disturbance signal $\omega_1$ and total feedback signal ($-u_1$) versus time.

Figure 68: Disturbance signal $\omega_2$ and total feedback signal ($-u_2$) versus time.
chapter are gravitation and input disturbance signals which satisfy a linear differential equation with constant coefficients. The gravitational effect (disturbance) is asymptotically eliminated by implementing an integral type controller in the feedback loop. Error convergence was shown via stability theory. Two computer simulations are provided to qualitatively study the behaviour of a compensated multi-body system. The simulation results verify that a simple servocompensator is capable of providing proper biasing torques.

The second type of disturbance which was considered in this chapter belongs to the class of linear disturbances, which may be measurable or unmeasurable. It was shown in this chapter that this type of disturbances can be eliminated by employing a standard linear servocompensator. A recursive Volterra series for planar manipulators was derived in order to verify that the error $e \to 0$ as $t \to \infty$. A computer simulation of a planar twolink manipulator with servocompensator was performed which asymptotically rejects two different input disturbance signals.
Table 4: Kinematic and Kinetic Parameters of the Two Link Planar Model.

<table>
<thead>
<tr>
<th>Link $i$</th>
<th>$d_i$</th>
<th>$k_i$</th>
<th>$m_i$</th>
<th>$I_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Units</td>
<td>m</td>
<td>m</td>
<td>kg</td>
<td>kgm$^2$</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0.5</td>
<td>20</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>0.8</td>
<td>0.4</td>
<td>10</td>
<td>2.5</td>
</tr>
</tbody>
</table>
CHAPTER VI

Nonlinear Observability of Planar Manipulators

6.1 Introduction

In controlling man made or natural manipulators it is in general necessary to obtain a measure of the state of the system. In natural systems the state usually is measured or estimated by a vast amount of sensory information, which are processed in the nervous system and then fed back to stimulate the muscular actuators and to achieve desired motion.

On the other hand man-made robots only utilize a relatively small number of sensors in order to determine the manipulator's state. Using a minimum number of sensors can be of serious consequences if one or several sensors fail to function. Of course one might argue that it is always possible to equip manipulators with physically redundant sensors in order to measure position, velocity, and maybe acceleration, which will reduce the risk of a single sensor failure.

Taking a different approach, it is reasonable to ask whether it is possible to estimate the information of a faulty sensor or several faulty sensors using the knowledge of the remaining ones. This problem is twofold. First, it is necessary to study whether the information of the faulty sensors is contained in the signals of the non-faulty ones. This is an observability problem [136, 138, 139]. Second, if the indirectly measured state information is contained in the remaining sensory signals then this information should be recovered. This is an estimation problem.
A somewhat different motivation for studying observability maybe presented when stability and trajectory control of redundant or non-redundant manipulators is being discussed. The most common problem with redundant manipulators is the indeterminacy problem that relates the joint states (i.e., angular position and angular velocities) to the end-effector trajectory. This nondeterminism may be circumvented if the knowledge of gripper position and velocity for stabilizing redundant manipulators and for controlling their end-effector movements is sufficient. This is because it is not necessary to know the actual trajectory of each link which a redundant manipulator performs as long as the end-effector follows the desired gripper path within a specified error. The suggested output stabilization scheme, however, may only be possible if the system is observable. In fact, the manipulator control methodology introduced in Chapter 4 relies on output stabilization which, in turn, requires that the states of the system are observable.

6.2 Basic Definitions and Theorems

In this section some basic definitions pertaining to nonlinear observability are given. Moreover, it will be shown that the results are consistent with the usual observability for linear systems.

Let a nonlinear system be described by

\[ \dot{x} = f(x, u) \]

\[ \sum: \]

\[ y = h(x) \]  \hspace{1cm} (6.1) \]

where \( x \in M, \ M \) is a \( C^\infty \) connected manifold of dim \( n, u \in \Omega \subset R^n, y \in R^l, \) and \( f(\cdot, \cdot) \) and \( h(\cdot) \) are \( C^\infty \) functions. The vectors \( x, U, \) and \( y \) respectively denote the state,
the inputs, and the output of the system.

**Definition 6.1** If for every bounded measurable input \( u \) and every initial state \( x(t_0) = x^0 \epsilon M \) there exists a solution of the differential equation (6.1) satisfying \( x(t) \epsilon M \ \forall \ t \epsilon R \), then the system is said to be complete [136]. Further, define \( \Phi(t,x^0,u,[t_0,t^1]) \) be the flow of \( \Sigma \) with initial state \( x^0 \) input \( u \), and \( t \epsilon [t_0,t^1] \). Hereafter, we will assume that the system \( \Sigma \) is complete.

**Definition 6.2** A pair of states \( x^0 \) and \( \tilde{x}^0 \) is said to be indistinguishable if the input-output map \( h_u(x^0,[t_0,t^1],u) = h_u(\tilde{x}^0,[t_0,t^1],u) \) for the same input \( u \epsilon \Omega \) and \( t \epsilon [t_0,t^1] \).

**Definition 6.3** The system \( \Sigma \) equation (6.1) is observable at \( x^0 \) if the set of indistinguishable states at \( x^0 \) is just \( x^0 \) itself, denoted by \( I(x^0) = \{x^0\} \). \( \Sigma \) is globally observable if \( I(x) = \{x\} \ \forall \ x \epsilon M \).

Note that the more general concept of (nonlinear) observability may depend on the input \( u \) as well, which seems to be in contrast to linear system theory. The global observability concept also implies that the time or distance travelled to distinguish \( x^0 \) from some other state \( \tilde{x}^0 \) may be considerably large.

**Definition 6.4** Let \( U \) be an open subset of \( M \) and let \( x^0, \tilde{x}^0 \epsilon U \), then \( x^0 \) is \( U \)-indistinguishable from \( \tilde{x}^0 \), denoted by \( (x^0 I_u \tilde{x}^0) \) if for every control \( u(t) \epsilon \Omega \) and \( t \epsilon [t_0,t^1] \) the output \( h(x^0,[t_0,t^1]) = h(\tilde{x}^0,[t_0,t^1]) \) and the trajectories of \( \Sigma \) remain in \( U \), i.e., \( \Phi(t,x^0,[t_0,t^1]), \Phi(t,\tilde{x}^0,[t_0,t^1]) \epsilon U \); see Fig. 69.

**Definition 6.5** The system Eq. (6.1) is locally observable at \( x^0 \) if for every open neighbourhood \( U \) containing \( x^0 \), we have \( I_U(x^0) = \{x^0\} \). If the latter holds for all

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Figure 69: Illustration of Definition 6.4.

\[ I(x^*) = \{ x^*, \tilde{x}^*, \hat{x}^* \}; \quad I(x^*) \cap U = \{ x^* \}. \]

Figure 70: Illustration of Definition 6.6 and 6.7.
$x \in M$ then $\Sigma$ is locally observable.

**Definition 6.6** The system $\Sigma$ is weakly observable at $x^0$ if there exist a neighbourhood $U$ such that $I(x^0) \cap U = \{x^0\}$ and it is weakly observable if it is so for all $x^0 \in M$; see Fig. 70.

Note that it is possible that a system $\Sigma$ is observable in the sense of Definition 6.6 but not necessarily according to Definition 6.3. In order to demonstrate this, let $x^0 \in U$ and $\bar{x}^0 \in U$, then it is possible that $I(x^0) = \{x^0, \bar{x}^0\}$ and $\Sigma$ is not observable at $x^0$. However, it will be weakly observable, since $I(x^0) \cap U = \{x^0\}$. Hence, the term weak observability is justified, since it restricts the observation to a region of the manifold $M$.

**Definition 6.7** The system $\Sigma$ is locally weakly observable at $x^0$ if there exists an open neighbourhood $U$ of $x^0$ such that for every open neighbourhood $V$ of $x^0, V \subset U, I_V(x^0) = \{x^0\}$ and it is locally weakly observable if $I_V(x) = \{x\}$ for every $x \in M$. The latter definition essentially means that one can distinguish $x^0$ from all its immediate neighbours. Refer also to Fig. 70 and Fig. 71.

In a later portion of this exposition, we will see that the concept of local weak observability will result in a conceptually relatively easy mathematical test. However, before reaching this point, let us introduce some more definitions.

Let $C^\infty(M)$ be the infinite dimensional real vector space of all $C^\infty$ real valued functions on $M$, and let $X(M)$ be the space of real vector fields. Then elements in $X(M)$ act on elements of $C^\infty(M)$ by Lie differentiation. Let $f(x) \in X(M)$ and $g \in C^\infty(M)$, then

$$L_f(g)(x) = \frac{\partial g}{\partial x}(x) \cdot f(x) \quad (6.2)$$

where
Locally weakly observable at \( x^0 \)

Figure 71: Illustration of Definition 6.7.

\[ \text{Lie derivative } L_{f_1}(x) = dh(x) f_1(x) \] -- dot product of vector fields.

Figure 72: Geometric illustration of the Lie derivative.
A geometrical interpretation of the Lie derivative is given in Fig. 72.

Let $H^0$ denote the subset of $C(M)^\infty$ functions containing the output functions $h_1(x), \ldots, h_\ell(x)$ and $F^0(x)$ the subset of $X(M)$ generated by the vector fields $f(x, u^i)$ with piecewise constant control $u^i \in \Omega$. Define the set $H$ to be the smallest linear subspace of $C^\infty(M)$ which is closed under Lie differentiation by elements of $T^0$. Two typical elements of $H$ are $L_{f_1}(h_i)(x); L_{f_1}(\ldots (L_{f_n}(h_j))\ldots)(x)$, etc. Elements in $H$ are closed not only under Lie differentiation by elements of $F^0$, but also under elements of the Lie algebra of $F^0$. The Lie algebra $F$ of $F^0$ is the smallest subset of $X(M)$ generated by $F^0$ which is closed under repeated use of the Lie bracket. In order to show that this is correct, let $f_1, f_2 \in F^0$ and $g \in C(M)^\infty$ then

$$L_{f_1}(L_{f_2}(g)) - L_{f_2}(L_{f_1}(g)) = L_{[f_1, f_2]}(g)$$ (6.3)

where the Jacobi bracket is defined as shown in Eq. (2.14). Since $[f_1, f_2](x) \in F$, one easily sees that $F^0 \subset F \subset X(M)$. The Lie algebra $F$ is of importance in studying local nonlinear controllability of $\Sigma$. If $\dim F = \dim T_\Sigma(M)$ for all $x \in M$, then $\Sigma$ is said to be locally reachable [131]. Here, $T_\Sigma(M)$ denotes the tangent space of $M$ at $x$.

Let the set of one-forms (gradients) of $H$ be defined by $dH = \{dH = \frac{\partial h}{\partial x}, h \in H\}$, where $dH$ is a $1 \times n$ row vector valued functions of $x$. The set $dH$ is a subspace of the covector field (or cotangent space) $T^*_\Sigma(M)$ on $M$ [137]. In a number of literature the covector field of a manifold is also called a codistribution [159]. Each element $dh(x)$ in $dH$ can be viewed as a direction at $x \in M$ along which the output function $h_i$ changes as the state trajectory moves on the manifold $M$. If $\omega \in T^*_\Sigma(M) = \{dg, g \in C^\infty(M)\}$ then Lie differentiation acts on one-forms as follows:
where \((^T)\) denotes transpose. The operation \(L_f \) and \(d\) commute if \(\omega = dg\), i.e.,

\[
L_f(dg)(x) = dL_f(g)(x) \quad (6.5)
\]

From Eq. (6.5), it follows that the set \(dH\) is invariant (closed) under Lie differentiation by elements of \(\mathcal{F}\) as well. In fact, the space \(dH\) is invariant under \(\mathcal{F}\) if the following holds. Let \(\omega \in dH\) and \(f(x) \in \mathcal{F}\), then \(L_f(\omega)(x) \in dH\). This means, there exist smooth functions \(\gamma_{ijk}(x)\) such that

\[
L_f(\omega_j)(x) = \sum_{k=1}^{\mu} \gamma_{ijk}(x)\omega_k(x) \quad (6.6)
\]

where \(\mu\) is the number of elements in \(dH\).

**Definition 6.8** The system \(\Sigma\) satisfies the observability rank condition at \(x^o\) if the dimension of the span of the set \(dH\), \(\dim(dH) = n\); it satisfies the observability rank condition if \(\dim(dH) = n\) for every \(x \in M\).

**Lemma 6.1** Let \(x^o\) and \(\tilde{x}^o\) be two elements of an open set \(\mathcal{V}\) and assume that \(x^o\) is locally indistinguishable from \(\tilde{x}^o\), i.e., \(x^oI_\mathcal{V}\tilde{x}^o\), then \(\gamma(x^o) = \gamma(\tilde{x}^o)\) for every element \(\gamma\) in \(H\).

**Proof** Hermann and Krener [136].

If \(x^o\) is indistinguishable from \(\tilde{x}^o\) then for any piecewise constant control \(u^1, \ldots, u^k \in \Omega\), \(k \geq 0\) and small \(\tau_1, \ldots, \tau_k \geq 0\) each output function \(h_i\), \(i = 1, \ldots, \ell\) will satisfy

\[
h_i \left( \Phi_{i_1}^2 \cdots \Phi_{i_k}^2 \cdot \Phi_{i_1}^1(x^o) \right) = h_i \left( \Phi_{i_1}^2 \cdots \Phi_{i_k}^2 \cdot \Phi_{i_1}^1(\tilde{x}^o) \right) \quad (6.7)
\]
where $\Phi^i_{\dot{\tau}_i}$ is the state trajectory of $\Sigma$ with control $u^i$. Differentiating Eq. (6.7) with respect to $\tau_i$ yields

$$L_{f_i} (\ldots (L_{f_k}(h_i)) \ldots)(x^0) = L_{f_i} (\ldots (L_{f_k}(h_i)) \ldots)(\dot{x}^0)$$

(6.8)

with $f_i = f(x, u^i)$. Since the above expression is a linear combination of elements in $H$, it follows that $\gamma(x^0) = \gamma(\dot{x}^0)$ for every $\gamma \in H$.

**Theorem 6.1** The system $\Sigma$ is locally weakly observable at $x^0$ if $\Sigma$ satisfies the observability rank condition at $x^0$.

**Proof** [136] If the dimension of $dH = n$ at $x^0$, then locally there exist $n$ functions $\gamma_1, \ldots, \gamma_n \in H$ such that $d\gamma_1(x^0), \ldots, d\gamma_n(x^0)$ are linear combinations of elements in $dH$ and $\dim(d\gamma_1(x^0), \ldots, d\gamma_n(x^0)) = n$. Define a local coordinate transformation $\Psi$ at $x^0$ by forming the Jacobian matrix $\Psi$ of the functions $\gamma_1(x), \ldots, \gamma_n(x)$ at $x^0$, i.e., $\Psi(x^0)$ is spanned by the elements of $dH$. Since the rank of $\Psi$, $\rho(\Psi) = n$ at $x^0$, there exists an open set $U$ which contours $x^0$ such that $\Psi$ is an injection within the neighbourhood $U$. Let $V$ be an open subset as in Definition 7, i.e., $V \subset U$, then from Lemma 6.1 $\gamma(x^0) = \gamma(\dot{x}^0)$ implies that $x^0 = \dot{x}^0$, and hence $I_V(x^0) = \{x^0\}$. So the dynamic system $\Sigma$ is locally weakly observable at $x^0$.

Having provided the basic definitions and a main theorem on observability, in what follows it will be shown that the results are consistent with the usual rank condition for linear systems.

Let

$$\dot{x} = Ax + Bu \quad \text{(6.9)}$$

$$y = Cx$$
where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $y \in \mathbb{R}^\ell$. The matrices $A$, $B$, and $C$ are constant matrices and are of appropriate sizes and of full rank. From our previous discussion, the set $dH^0$ is spanned by the rows of $C$, i.e., $C_j, j = 1, \ldots, \ell$. The set $\mathcal{F}^0$ is spanned by $Ax$ and the columns of $B_i$ say $b_i$, $i = 1, \ldots, m$. Observe that one must assume $u$ is piecewise constant. Next, construct the set $dH$ as follows: Take Lie derivatives of all possible combinations of vectors in $dH^0$ with those in $\mathcal{F}^0$. Then take Lie derivatives of all possible combinations of the resultant vectors with those in $\mathcal{F}^0$. Continue this until all newly generated vectors become linear combinations of the set of generating vectors.

For the linear system in Eq. (6.9) one gets the following

$$L_{b_i}(dh_j)(x) = L_{b_i}(c_j) = 0$$

$$L_{Ax}(dh_j)(x) = c_jA$$

$$L_{Ax}(L_{Ax}(c_j))(x) = c_jA^2$$

$$\vdots$$

$$L_{Ax}(\ldots L_{Ax}(c_j)\ldots)(x) = c_jA^{n-1}$$

By the Cayley-Hamilton Theorem one can show that any further Lie derivative of the covectors $c_j$ in the direction of $Ax$ will provide vectors that the linearly dependent on the previous vectors. Hence the space $dH$ is closed under Lie differentiation and generated by the following set of vectors.

$$dH = \{c_j, c_jA, c_jA^2, \ldots, c_jA^{n-1}, j = 1, \ldots, \ell\}$$

This set can be written in the usual observability form as
In the next section we will study whether a planar multi-body system is observable, assuming that the location of end-effector is considered to be the output of the dynamical system.

6.3 Observability of Planar Manipulators

In this section local weak observability of a planar manipulator is investigated. The position of the end-effector of the multi-body system is treated as the output channels. A typical link of a planar rigid body system is shown in Fig. 6. The physical quantities of each of these links are the distance between two consecutive joints $d_i$, the distance of the joint $i$ and the center of mass $k_i$, the mass $m_i$, and the inertia $I_i$ about the center of mass. The orientation of each link is measured from the horizontal position in clockwise direction. The dynamics of such a system has been stated in Chapter 5, Eq. (5.26). Arranging Eq. (5.26) into standard form gives

$$\dot{x} = f(x) + g(x)u$$

where $x = (\theta \ \dot{\theta})^T$, $x \in \mathbb{R}^{2n}$ and

$$f(x) = \begin{pmatrix} \dot{\theta} \\ -H_1^{-1}(H_2 + H_3) \end{pmatrix} \quad g(x) = \begin{pmatrix} 0 \\ H_1^{-1}H_4 \end{pmatrix}$$

As mentioned before, the output of the manipulator is defined to be the end-effector position and velocity. Consequently, one obtains the following four output
equations

\begin{align*}
y_1 &= \sum_{i=1}^{n} d_i \sin \theta_i \\
y_2 &= \sum_{i=1}^{n} d_i \cos \theta_i
\end{align*} \tag{6.11a} \tag{6.11b}

\begin{align*}
\dot{y}_1 &= \sum_{i=1}^{n} d_i \cos \theta_i \cdot \dot{\theta}_i \\
\dot{y}_2 &= -\sum_{i=1}^{n} d_i \sin \theta_i \cdot \dot{\theta}_i
\end{align*} \tag{6.12a} \tag{6.12b}

where the \(d_i\)'s are the length of each link. In order to employ consistent notation, let the output functions be rewritten as \(y_1 = h_1, \ y_2 = h_2, \ \dot{y}_1 = h_3, \) and \(\dot{y}_2 = h_4.\) Hence, the set \(H^o\) becomes \(H^o = \{h_1, h_2, h_3, h_4\}\) and the set \(\mathcal{F}^o\) assumes \(\mathcal{F}^o = \{f(x) \cup \text{columns of } g(x)\}.\) The latter two sets are the starting point for our analysis. In fact, since \(H_4\) is a constant one-to-one and onto mapping and it is assumed that the control input is piecewise constant, the generating vector field \(\mathcal{F}^o\) may assume \(\mathcal{F}^o = \{f(x) \cup \text{columns of } G'(x) = (0 : N_{n \times n})_{2n \times n}\};\) where \(N(\theta)\) is the inverse of \(H_1(\theta),\) i.e., \(N(\theta) = H_1^{-1}(\theta).\) In order to shorten notation for the derivation ahead, let \(c_i = \cos \theta_i, \ s_i = \sin \theta_i,\) and \(N^{ij}_{\theta_k} = \frac{\partial N^{ij}}{\partial \theta_k},\) with \(N^{ij}\) being the \(i, j\)-th element of \(N(\theta).\) Consider again Theorem 6.1 in Section 6.2 which states that in order to show that a dynamic system is locally weakly observable, one has to find \(2n\) linearly independent vectors in the set \(dH\) which span the covector field \(T^{\ast}_x(M),\) i.e.,

\[dH = T^{\ast}_x(M)\] \tag{6.13}

Because \(H^o\) is the initial set in \(H,\) let us begin with stating the one-form of \(dh_i \wedge dH^o, \ i = 1, \ldots, 4.\)

\[dh_i = (d_1 c_1 \ d_2 c_2 \ldots d_n c_n \ 0 \ldots 0_{1 \times n})\]
In the following steps new one-forms must be found by taking Lie derivatives of elements in \( dH^0 \) and included into the set \( dH \).

An easy check shows that

\[
L_{g_i}(dh_j)(x) = 0 \quad i = 1, \ldots, n; \quad j = 1, 2
\]  

(6.15)

The Lie derivatives in Eq. (6.15) evaluate to zero, because the vector fields \( g_i, i = 1, \ldots, n \) only act on the states \( \theta_i, i = 1, \ldots, n \) and not on \( \theta_i, i = 1, \ldots, n, \) but \( h_1(x) \) and \( h_2(x) \) are solely functions of the position vector \( \theta \).

In a similar way one obtains

\[
L_f(dh_1)(x) = dh_3(x)
\]  

(6.16a)

\[
L_f(dh_2)(x) = dh_4(x)
\]  

(6.16b)

To this point one will notice that Eq. (6.15) and Eq. (6.16) do not generate new linearly independent covector fields using the covectors \( dh_1 \) and \( dh_2 \).

However, taking Lie derivatives of \( h_3(x) \) and \( h_4(x) \) with respect to \( g_i \) yields the following functions

\[
L_{g_i}(h_3)(x) = \sum_{j=1}^{n} d_j c_j N^{ji} \quad i = 1, \ldots, n
\]

\[
L_{g_i}(h_4)(x) = -\sum_{j=1}^{n} d_j c_j N^{ji} \quad i = 1, \ldots, n
\]  

(6.17)

Computing the one-form of Eqs. (6.17) gives

\[
dL_{g_i}(h_3)(x) = \left( \sum_{j=1}^{n} d_j c_j N^{ji}_{\theta_i} - d_i s_i N^{ji}, \sum_{j=1}^{n} d_j c_j N^{ji}_{\theta_{ij}} - d_j s_j N^{2ji}, \ldots \right)
\]
The above expressions may be written in matrix form as

\[ dLg(h_3)(x) = \left[ N_\theta D \dot{\mathcal{C}} : N_\theta D \mathcal{C} : \ldots : N_\theta D \mathcal{C} \right] - NDC : 0_{1nx} \] 

(6.19a)

and

\[ dLg(h_4)(x) = \left[ N_\theta D\dot{\mathcal{S}} : N_\theta D\mathcal{S} : \ldots : N_\theta D\mathcal{S} \right] - NDC : 0_{1nx} \] 

(6.19b)

where the matrices in Eqs. (6.19) are defined as below

\[
N_\theta = \frac{\partial N}{\partial \theta_i}, \quad D = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ d_2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & d_n \end{pmatrix}, \quad \mathcal{S} = \begin{pmatrix} S_1 \\ S_2 \\ \vdots \\ S_n \end{pmatrix}
\]

\[
S = \begin{pmatrix} S_1 & 0 & \cdots & 0 \\ S_2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & S_n \end{pmatrix}, \quad C = \begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{pmatrix}, \quad \mathcal{C} = \begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{pmatrix}
\]

Before attempting to test the rank of Eqs. (6.19), one should observe that the velocity dependent partition is not spanned at all—all entries are zero. Consequently, let us take the Lie derivative of Eqs. (6.17) with respect to the vector field \( f(X) \). This operation yields the following two sets of equations

\[ L_fL_\theta(h_3)(x) = \left[ N_\theta D \dot{\mathcal{C}} : N_\theta D \mathcal{C} : \ldots : N_\theta D \mathcal{C} \right] - NDC \cdot \dot{\theta} \] 

(6.20a)

\[ L_fL_\theta(h_4)(x) = \left[ N_\theta D\dot{\mathcal{S}} : N_\theta D\mathcal{S} : \ldots : N_\theta D\mathcal{S} \right] - NDC \cdot \dot{\theta} \] 

(6.20b)
Computing the one-form (gradient) of Eq. (6.20) gives

\[
\begin{align*}
\frac{d}{d\theta} L_p(h_\theta)(x) &= \left( \frac{\partial}{\partial \theta} \left( N_{\alpha_i} D\hat{C} : N_{\alpha_i} D\hat{C} : \ldots : N_{\alpha_i} D\hat{C} \right) - N DS \right) \dot{\theta} : \\
&\quad \left[ N_{\alpha_i} D\hat{C} : N_{\alpha_i} D\hat{C} \right] - N DS \\
&= \left( \frac{\partial}{\partial \theta} \left( - N_{\alpha_i} D\hat{S} : N_{\alpha_i} D\hat{S} : \ldots : N_{\alpha_i} D\hat{S} \right) - N DC \right) \dot{\theta} : \\
&\quad - \left[ N_{\alpha_i} D\hat{S} : N_{\alpha_i} D\hat{S} : \ldots : N_{\alpha_i} D\hat{S} \right] - N DC \\
&= \left[ N_{\alpha_i} D\hat{C} : \ldots : N_{\alpha_i} D\hat{C} \right] - N DS \\
&\quad - \left[ N_{\alpha_i} D\hat{S} : \ldots : N_{\alpha_i} D\hat{S} \right] - N DC \\
\end{align*}
\]

(6.21a)

(6.21b)

At this point, let us consider the collection of covector fields in Eqs. (6.14), (6.19), and (6.21) and examine whether the present collection of row vectors (cotangent vectors) span the entire cotangent space \( T^*_x(M) \) at any point \( x \) on \( M \). This means that linear independence of cotangent vectors (as well as tangent vectors) is a pointwise concept. It therefore does not suffice to show that there does not exist a set of constant coefficients which are not all equal to zero. Instead, one has to show that there does not exist a set of smooth functions \( \gamma_i(x) \neq 0 \) for some \( x \in M \). 

Combining Eqs. (6.14), (6.19), and (6.21) and evaluating the equations at \( \dot{\theta} = 0 \) gives

\[
[dH'] = \left( \begin{array}{c|c|c|c|c|c|c|c|c|c|c|c|c}
C^\gamma D & : & 0 \\
-5^\gamma D & : & 0 \\
0 & : & C^\gamma D \\
0 & : & -5^\gamma D \\
\left[ N_{\alpha_i} D\hat{C} : \ldots : N_{\alpha_i} D\hat{C} \right] - N DS & : & 0 \\
-\left[ N_{\alpha_i} D\hat{S} : \ldots : N_{\alpha_i} D\hat{S} \right] - N DC & : & 0 \\
0 & : & \left[ N_{\alpha_i} D\hat{C} : \ldots : N_{\alpha_i} D\hat{C} \right] - N DS \\
0 & : & -\left[ N_{\alpha_i} D\hat{S} : \ldots : N_{\alpha_i} D\hat{S} \right] - N DC \\
\end{array} \right) \\
\]

(6.22)

If \( dH' \) is of rank \( 2n \), i.e., \( \rho(dH') = 2n \), then \( dH' \) is also of rank \( 2n \) for all \( x \in M \). This is true, because the rows in the set \( dH' \) span the cotangent space independently of the value of the vector \( \theta \in R^n \). Consequently, if \( \rho(dH') = 2n \) for
every $\theta \in \mathbb{R}^n$, then the system is locally weakly observable. If $dH'$ is not of rank $2n$ for some $\theta \in \mathbb{R}^n$ or all $\theta \in \mathbb{R}^n$, then new cotangent vectors have to be generated unless the present collection of vectors in $dH'$ is involutive. Suppose $dH'$ is involutive and $\rho(dH') < 2n$ for some $\theta \in \mathbb{R}^n$ then the system is not observable at these points. If $\rho(dH') < n$ for all $\theta \in \mathbb{R}^n$ then the manipulator is not locally weakly observable.

As one may easily notice, the difficulty in determining the rank of the matrix $[dH']$ depends significantly on the knowledge of the rank of

$$P = \begin{pmatrix} NDS - [N_{\theta_1}D\tilde{C} : \ldots : N_{\theta_n}D\tilde{C}] \\ NDC + [N_{\theta_1}D\tilde{S} : \ldots : N_{\theta_n}D\tilde{S}] \end{pmatrix} \quad (6.23)$$

In order to simplify the expression in Eq. (6.23) the matrix $N_{\theta_i}$ will be replaced by the identity below

$$\frac{\partial N}{\partial \theta_i}(\theta) = N(\theta) \frac{\partial H_1(\theta)}{\partial \theta} N(\theta) \quad (6.24)$$

Moreover, Eq. (6.23) will be premultiplied by a nonsingular matrix of rank $2n$, which is defined as

$$\begin{pmatrix} H_1(\theta) & 0 \\ \ldots & \ldots & \ldots \\ 0 & H_1(\theta) \end{pmatrix}$$

The new matrix becomes

$$P' = \begin{pmatrix} DS - [H_{\theta_1}NDC : \ldots : H_{\theta_n}NDC] \\ DC + [H_{\theta_1}NDS : \ldots : H_{\theta_n}NDS] \end{pmatrix} \quad (6.25)$$

To obtain an expression which reflects the rotational symmetry present in the system, premultiply the matrix $P'$ by

$$T = \begin{pmatrix} S & C \\ -C & S \end{pmatrix} \quad (6.26)$$
Because $T$ is an orthogonal matrix with $\det(T) = 1$ the rank of $TP'$ will remain invariant under this transformation. Carrying out the operation above yields

$$P'' = \begin{pmatrix} W \\ A \end{pmatrix}$$  \hspace{1cm} (6.27)

where $W$ and $Z$ are written in matrix form as given underneath

$$W = D - S \left[ H_{1\theta}, ND\bar{C} : H_{1\theta}, ND\bar{C} : \ldots : H_{1\theta}, ND\bar{C} \right]$$

$$+ C \left[ H_{1\theta}, ND\bar{S} : H_{1\theta}, ND\bar{S} : \ldots : H_{1\theta}, ND\bar{S} \right]$$  \hspace{1cm} (6.28)

$$Z = C \left[ H_{1\theta}, ND\bar{C} : H_{1\theta}, ND\bar{C} : \ldots : H_{1\theta}, ND\bar{C} \right]$$

$$+ S \left[ H_{1\theta}, ND\bar{S} : \ldots : H_{1\theta}, ND\bar{S} \right]$$  \hspace{1cm} (6.29)

Consider now the $i$-th partial derivative of the inertia matrix below

$$H_{1\theta} = \begin{pmatrix}
0 & -a_{i1}s_{i1} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
-a_{(i-1)j} s_{(i-1)j} & \vdots & \ddots & \vdots \\
an_{i1}s_{i1} & 0 & \cdots & 0 \\
0 & -a_{n1}s_{n1} & \cdots & 0
\end{pmatrix}$$

where $a_{ij} = a_{ji}$ and $s_{ij} = \sin(\theta_i - \theta_j)$.

The elements of the matrices $W$ and $Z$ may hence be given as

$$W_{kk} = d_k + \sum_{\substack{l=1 \\ l \neq k}}^n W_{kl} ; \hspace{0.5cm} k = l$$

$$W_{kl} = -d_{kl}s_{kl} \sum_{j=1}^n N^{lj}d_jS_{kj} ; \hspace{0.5cm} k \neq l$$  \hspace{1cm} (6.30)

and

$$Z_{kk} = d_k + \sum_{\substack{l=1 \\ l \neq k}}^n Z_{kl} ; \hspace{0.5cm} k = l$$
\[ Z_{kl} = -d_{kl} s_{kl} \sum_{j=1}^{n} N_{ij} d_j c_{kj}; \quad k \neq l \]  \hspace{1cm} (6.31)

where \( s_{ij} = \sin(\theta_i - \theta_j) \), \( c_{ij} = \cos(\theta_i - \theta_j) \), and \( N_{ij} \) is the \( i,j \)-th element of the matrix \( N \).

It may become clear at this point that due to the complexity of the algebraic expressions in the matrices \( W \) and \( Z \) not much can be said about the rank of these matrices except that

i) \( W = D \) and \( Z = Q \) if \( \theta_1 = \theta_2 = \ldots = \theta_n \)

ii) \( \rho(Z) \leq n - 1 \) for all \( \theta \) because the sum of the columns of \( Z \) is identical to zero for all \( \theta \).

For a manipulator with \( n = 2 \), however, it is easy to show that the manipulator is state observable. Constructing the observability matrix for \( n = 2 \) yields

\[
[dH_2] = \begin{pmatrix}
  d_1 c_1 d_2 c_2 & : & 0 \\
  -d_1 s_1 - d_2 s_2 & : & d_1 c_1 d_2 c_2 \\
  0 & : & -d_1 s_1 - d_2 s_2 \\
  W & : & 0 \\
  Z & : & W \\
  0 & : & Z
\end{pmatrix}
\]  \hspace{1cm} (6.32)

for \( \theta_1 \neq \theta_2 \) the rank of \([dH_2]\), \( \rho(dH_2) = 4 \), because the first four rows of Eq. (6.32) span \( dH_2 \). If \( \theta_1 = \theta_2 \) then the first four rows only span \( \text{dim} = 2 \), but since \( W = D \) and all \( d_i > 0 \; i = 1, 2 \), \( W \) will span \( dH_2 \). Hence the two link manipulator is locally weakly observable for all \( \theta \). At this point it is appropriate to emphasize that local observability clearly does not mean that the particular system here is also observable as it would be the case for linear systems. In fact, if \( \theta_i = \theta_i + 2n \pi \) and

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where \( n \) is any arbitrary integer, then these points cannot be distinguished from each other whatever input signals are applied to the systems. Consequently, the planar two-link system is not observable for \( \theta \in \mathbb{R}^2 \), but it will be so if the domain is restricted to the two dimensional unit circle \( S_1 \times S_1 \).

Planar multi-link manipulators may not be locally observable if \( \rho([dH']) < 2n \). This may occur if the rank of the matrix \( P'' \) becomes less than \( n \). If the latter rank defect occurs, then the system may only be locally unobservable at discrete points of the manifold \( M = \mathbb{R}^{2n} \). If one defines the domain to be \( s^n_1 = S_1 \times S_1 \times \ldots \times S_1 \ n \)-times, then the number of possible points at which the system may possibly not be observable is finite. In order to show this consider the determinant \( \det(P'') \) of the matrix \( P'' \). The determinant \( \det(P'') \) is a polynomial of the trigonometric function \( \sin(\theta_i - \theta_j) \) and \( \cos(\theta_i - \theta_j) \), \( i, j = 1, \ldots, n \) and because \( \det(P'')(\theta = 0) = \prod_{i=1}^{n} d_i > 0 \) there will only be at most a finite number of zeros of the \( \det(P'')(\theta) \) on the domain \( S^n_1 \).

Formally, if \( \rho([dH']) < 2n \) for some \( \theta \in \mathbb{R}^n \), additional covectors should be generated using the vectors already in the set \( dH' \). However, this should only be pursued if a definite statement can be made about the rank of the covectors in \( dH' \), Eq. (6.22), or \( P'' \), Eq. (6.27). Otherwise a number of superfluous covectors may be generated that do not increase the dimension of the covectors already in the set \( dH' \).

The differential geometric approach to analyze nonlinear analytic system appears to be theoretically very elegant, but in practice it may happen that for certain systems, as in the case for planar multi-link manipulators, the analysis becomes too cumbersome or impossible to be carried out.
6.4 Summary

In this chapter nonlinear observability concepts were introduced and applied to planar multi-link manipulators when the end-effector position and its time derivatives only are assumed to be known. Based on differential geometric concepts it was shown that a planar two-link pendulum is locally weakly observable. For redundant manipulators, i.e., $n > 2$, a definite statement whether the system is locally weakly observable could not be made due to difficulties in determining the dimension of the span of the generated covector fields in the set $dH'$. However, it was found that any redundant planar multi-link manipulator is locally weakly observable possibly excluding at a finite number of points on $M = S^n_1$. 
CHAPTER VII
Discussion and Conclusion

7.1 Summary

Recent issues in the control of manipulators are investigated in this dissertation. In Chapter 2, a survey of relevant literature from various disciplines in the field of robot modelling and control is compiled. The literature survey is divided into three sections, which comprise the area of general multi-body systems, topics on kinematics, dynamics, and control of robot manipulators, and a selection of pertinent literature on linear as well as nonlinear system theory.

Chapter 3 contains an exposition on modelling of a three link, three dimensional manipulator. This model serves as a testing tool in the subsequent chapter for newly developed control strategies. This chapter also contains a comparative discussion on manipulator dynamics that are based on Euler parameters or Bryant (Euler) angles. It is shown that the inherent stability problem during numerical integration of the equations of motion of the Euler parameter based manipulator can be solved by applying artificial, non-physically meaningful feedback for stabilizing the constraint condition that the Euler parameters must satisfy. The proposed feedback strategy guarantees that the Euler parameters constraint can be maintained within an \( \varepsilon \)-neighbourhood of that constraint if an upper bound of the angular velocities of the manipulator links are assumed to be known. Two digital computer simulations contrast the two different philosophies of utilizing either
Bryant angles or Euler parameters. It was found that for the particular simulated movement of the robot arm no discernible difference of either method could be established.

In Chapter 4, a new methodology for control and trajectory planning is proposed. This method integrates control, stability, and trajectory planning into a unified concept. The trajectory along which a manipulator end-effector shall move is specified as a task oriented surface $S_T$ or portions of different task oriented surfaces that are connected together. It is assumed that any task oriented surface may be specified in terms of invariant analytic functions of the states of the manipulator. Furthermore, it is assumed that the end-effector position on $S_T$ can be specified by analytic functions as well. A mutual nonlinear input-output decoupling is performed between the surface invariant analytic functions and those which specify points on $S_T$. Having performed this type of decoupling, end-effector motion on $S_T$ can be directly controlled via a reduced number of external inputs. This control strategy allows to control robot motion along a straight line or a circle with constant radius with just one control input. Obviously, this method simplifies the design of reference trajectories significantly. For illustration purposes, Chapter 4 includes two examples of manipulator movements on a task oriented surface. The first example demonstrates cartesian path motion between two arbitrary points on a specified straight line within the workspace of the robot. The second example shows how portions of task oriented surfaces can be connected together for obtaining complex end-effector movements of a robot. The two examples are simulated on a digital computer and the results are graphed. The finding of the latter two simulations are extremely encouraging in the sense that the proposed control scheme works well and simplifies the burden of designing proper reference trajectories. This methodology also permits assignment of arbitrary closed loop
Compensation of disturbances in robotic systems is studied in Chapter 5. Two classes of disturbances are considered in this chapter. The first type of disturbance is the gravitational field that acts upon robot manipulators. Locally, gravitational disturbances may be considered to be of constant type. Therefore a linear PID controller seems to be suitable for compensating the gravitational disturbance and providing the proper biasing torques to the input of manipulators. It is also shown that if the overall system is stable a PID controller is capable of providing the correct biasing torques at any arbitrary equilibrium point. This means proper biasing can be obtained globally. The second type of disturbances, that may perturb the behaviour of manipulators are assumed to satisfy a linear, time-invariant differential equation. It is verified in this chapter that the latter class of disturbances can be asymptotically rejected when a standard, linear servocompensator is implemented in the feedback loop of a manipulator. The proof of error convergence is based on the Volterra series representation of a manipulator. For both types of disturbances several illustrative examples are included. Two manipulator models are simulated to assess qualitatively their behaviour for the different types of disturbances.

In Chapter 6, observability of planar multi-link manipulators are studied under the assumption that only the end-effector position and its time derivatives as well as the input are known. It is found that planar two link pendulums are locally weakly observable. Redundant manipulators, this means manipulators with more than two links, are found to be locally weakly observable possibly except at a number of discrete points. An exact statement cannot be made, however, because of the inability to define precise conditions on the span of the covector distribution. This latter problem is essentially due to the complexity of the entries in the covectors.
7.2 Recommendations for Future Research

Studies in the field of manipulator control comprises multidisciplinary areas from various engineering and scientific fields. In this dissertation topics in the fields of modelling of robotic manipulator, numerical analysis, and control are addressed.

In Chapter 3, a numerical instability problem is solved by a feedback method and an error estimate using the Euler integration method is derived. More accurate error estimates could be obtained, however, if different integration algorithms are used to derive a particular error estimate. Also different feedback schemes should be tested to possibly obtain even better results.

Several future investigations can be suggested from Chapter 4. Sets of task oriented surfaces should be defined which aid in accomplishing some particular task. These surfaces should not be manipulator dependent for compatibility purposes. In a next step, manipulator dependent mappings that relate the end-effector coordinates to those task oriented surfaces should be defined. The latter mappings as well as the aforementioned set of task oriented surfaces shall serve as a data base for higher level control and decision mechanisms like CAD and/or CIM systems.

A second problem which may need further investigation is related to the nonlinear feedback gain computation. When resorting back to Chapter 4, one may recall that the computation of the decoupling feedback depends on the invertibility of some matrix $A(x)$. Although decoupling algorithms are known that do not rely on the invertibility of $A(x)$, they do not guarantee stability or do not allow arbitrary pole placement locally. These shortcomings may be solved in future studies as well.

In Chapter 5, the studies on disturbance rejection via servocompensator theory were carried out assuming that local stability about an equilibrium may suffice.
This assumption was deemed necessary, because no Lyapunov function could be found in order to prove global stability. Further investigations of global stability should be pursued possibly by obtaining a suitable Lyapunov function.

The results of nonlinear decoupling in Chapter 4 and servocompensator theory from Chapter 5 should be utilized to study not only disturbance rejection but also tracking.
APPENDIX A

Lyapunov Computation

In Section 3.4, the Lyapunov function in Eq. (3.41) was differentiated with respect to time yielding Eq. (3.42) which is given here for convenience as

\[ \dot{V} = \dot{R}^T \dot{R} + \theta^T (H_1 \dot{\theta} + H_2^*) - \dot{\theta} H_3 \quad (A.1) \]

It has been claimed in the same section that substitution of Eq. (3.40) into Eq. (A.1) yields

\[ \dot{V} = \dot{R}^T \dot{R} (T(\dot{\phi}) Q - K_n R^{-1} \dot{Q}) + \dot{\theta}^T (H_3 - K \dot{\theta}) - \dot{\theta} H_3 \quad (A.2) \]

If the former is indeed true one must be able to show that

\[ \dot{\theta}^T (H_1 \dot{\theta} + H_2^*) = \dot{\theta}^T (H_1 \dot{\theta} + H_2) \quad (A.3) \]

It will be shown further that \( H_2^* \neq H_2 \) for all possible choices of \( \theta \) and \( \dot{\theta} \), but

\[ \dot{\theta}^T H_2^* = \dot{\theta}^T H_2. \]

The scalars, vectors, and matrices are as defined in Section 3.2. Furthermore, the convention regarding the reference frames of any indexed vector or matrix is as discussed in the aforementioned section. The latter definitions are valid throughout this Appendix.

Let us begin with the kinetic energy function of a body \( i \) with respect to its center of gravity.

\[ T_i = \frac{1}{2} m_i \dot{X}_i^T \dot{X}_i + \frac{1}{2} w_i J_i w_i \quad (A.4) \]
Since the total energy $T$ is the sum of the kinetic energy of the individual bodies, one may write Eq. (A.4) in matrix form as

$$T = \frac{1}{2} W^T N_{11} W + \frac{1}{2} \dot{X}^T N_{21} \dot{X} \quad \text{(A.5)}$$

From Eq. (3.14) the velocity vector $X$ can be expressed as $X = M_1 W$ such that Eq. (A.5) becomes

$$T = \frac{1}{2} W^T (N_{11} + M_1^T N_{21} M_1) W \quad \text{(A.6)}$$

Furthermore, using Eq. (3.15), Eq. (A.6) can be written as

$$T = \frac{1}{2} \dot{\theta}^T [E_1^T (N_{11} + M_1^T N_{21} M_1) E_1 \dot{\theta}] \quad \text{(A.7)}$$

where the term in the brackets is the inertia matrix as in Eq. (3.21). Eq. (A.7) shows that the total kinetic energy comprised of translational and rotational energy of a constraint rigid body is

$$T = \frac{1}{2} \dot{\theta}^T H_1 \dot{\theta}.$$  

After taking the time derivative of Eq. (A.7) one gets

$$\dot{T} = \dot{\theta}^T E_1^T (N_{11} + M_1^T N_{21} M_1) E_1 \dot{\theta} + \dot{\theta}^T E_1^T (N_{11} + M_1^T N_{21} M_1) E_2$$

$$+ \dot{\theta}^T E_1 M_1^T N_{21} M_2 (E_1(\dot{\theta}))$$

$$\quad \text{(A.8)}$$

Consider next the differential equation (3.35) which can be written as

$$E_1^T (N_{11} + M_1^T N_{21} M_1) E_1 \ddot{\theta} + E_1^T (N_{11} + M_1^T N_{21} M_1) E_2$$

$$+ E_1^T M_1^T N_{21} M_2 (E_1(\dot{\theta})) + E_1^T N_{12} (E_1(\dot{\theta})) = H_3(\theta) + H_4 U \quad \text{(A.9)}$$

When Eq. (A.9) is substituted into Eq. (A.8) the following expression results

$$\dot{T} = -\dot{\theta}^T E_1^T N_{12} (E_1(\dot{\theta})) + \dot{\theta}^T H_3 + \dot{\theta}^T H_4 U \quad \text{(A.10)}$$

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Note that \( w = E_1 \dot{\theta} \) and \( N_{12}(f_1(w_1) \ f_2(w_2) \ f_3(w_3)^T) \). Define

\[
f_i(w_i) = w_i \times J_i w_i \tag{A.11}
\]

then it is trivial to show that when Eq. (A.11) is premultiplied by \( w_i^T \) one gets

\[
w_i^T f_i(w_i) = 0 \tag{A.12}
\]

Hence we have shown that \( H_2^* \neq H_2 \) because \( H_2 \) includes the term \( E_1^T N_{12}(E_1 \dot{\theta}) \) while \( H_2^* \) does not contain this term. However, \( \dot{\theta}^T H_2^* = \dot{\theta}^T H_2 \), since \( E_1^T N_{12}(E_1 \dot{\theta}) \) is annihilated by premultiplying the latter term by \( \dot{\theta}^T \).
APPENDIX B

Introductory Definitions and Theorems

In this Appendix some basic mathematical concepts of differential geometry are introduced such that the ideas developed in Chapter 4 essentially are self contained. Frequently, theorems and lemmas are stated without proofs, but references are provided. This makes it possible to maintain this Appendix compact. If a particular proof is short, it is stated.

Definition B.1 A tangent vector \( v \) at a point \( p \in M \) is a map \( v \) which maps \( C^\infty \) functions into a real value \( \mathcal{R} \) at \( p \); i.e., \( v : C^\infty(p) \to \mathcal{R} \). A tangent vector possesses linearity property and satisfies Leibniz's rule. A tangent vector \( v \) acts on \( C^\infty \) functions as follows. Let \( a \in \mathcal{R}, g(x) \in C^\infty, x \in \mathbb{R}^n, \) and \( v(x) \in \mathbb{R}^n \), then

\[
\begin{align*}
    a &= \langle v, \nabla g \rangle(p) \\
    &= \sum_{i=1}^{n} v_{i} \frac{\partial g}{\partial x_{i}}(x) \bigg|_{x=p} 
\end{align*}
\]  

(B.1)

Definition B.2 Suppose \( M \) is a smooth manifold. The tangent space to \( M \) at \( p \) expressed \( T_p(M) \) is the set of all tangent vectors at \( p \). The set \( T_p(M) \) is a vector space under the field \( \mathcal{R} \) which satisfy the linearity property. Let \( c_1, c_2 \in \mathcal{R}, \gamma \in C^\infty, \) and \( v_1, v_2 \in T_p(M) \) then

\[
(c_1v_1 + c_2v_2)(\gamma) = c_1v_1(\gamma) + c_2v_2(\gamma) 
\]  

(B.2)
Definition B.3 Let $M$ be a manifold of dimension $n$ and let us assume that for each point $p \in M$ there exists a mapping into a $m$-dimensional subspace $\Delta_p$ of the tangent space $T_p(M), m \leq n$. Moreover, suppose that in a neighbourhood $U$ of each $p \in M$ there exist $m$ linearly independent $C^\infty$-vector fields $\tau_1, \ldots, \tau_m$ which generate a basis of $\Delta_q$ for every $q \in U$. If the above is true, $\Delta$ will be denoted a smooth distribution on a manifold $M$.

Definition B.4 A distribution $\Delta$ is said to be nonsingular if there exists an integer $m$ such that $\dim \Delta_p = m$ for all $p \in M$. If the latter condition is satisfied for all $p \in U$, $U$ an open subset of $M$, then $\Delta$ is said to be nonsingular on $U$.

Definition B.5 A distribution $\Delta$ is said to be involutive if for any pair of vector fields $\tau_1, \tau_2, \epsilon \Delta$ the Lie bracket $[\tau_1, \tau_2]$ also belongs to $\Delta$.

This definition may be restated in the following form. A distribution $\Delta$ of dimension $m$ is involutive if and only if for each point $p \in U$ a local basis $\tau_1, \ldots, \tau_m$ defined on $U$ satisfies

$$[\tau_i, \tau_j] = \sum_{k=1}^{m} c_{ij}^k \tau_k$$  \hspace{1cm} (B.3)

where $c_{ij}^k$ are real valued smooth functions.

Definition B.6 Let $f$ be a vector field and $\Delta$ be a smooth distribution of $\text{dim} m \leq n$ on $M$, then one says that $\Delta$ is invariant under the action of $f$ if and only if

$$[f, \Delta] \subseteq \Delta$$  \hspace{1cm} (B.4)

that is $f$ belongs to $\Delta$. 

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Definition B.7 A codistribution $\Omega$ on a manifold $M$ is a mapping that assigns to every point $p \in M$ a subspace $\Omega_p$ of the cotangent space $T^*_p(M)$. Similarly, one can define a smooth codistribution as a dual of a smooth distribution given by Definition B.3.

Definition B.8 The annihilator of a distribution $\Delta$ or a codistribution $\Omega$ respectively is denoted $\Delta^\perp$ and $\Omega^\perp$, and defined as

$$\Delta^\perp = p \mapsto \{u \in T^*_p(M) : <u, \tau> = 0 \text{ for all } \tau \in \Delta_p\}$$

$$\Omega^\perp = p \mapsto \{\tau \in T_p(M) : <u, \tau> = 0 \text{ for all } u \in \Omega_p\}$$

A distribution and codistribution have a number of interesting properties, of which a few are given here for later use

i) $\dim \Delta + \dim \Delta^\perp = \dim M$.

ii) $\Delta_1 \subseteq \Delta_2 \iff \Delta_1^\perp \subseteq \Delta_2^\perp$.

ii) $(\Delta_1 \cap \Delta_2)^\perp = \Delta_1^\perp \oplus \Delta_2^\perp$.

Definition B.9 Let $h$ be a real-valued smooth function defined on $M$. For each such $h$, one can associate a distribution $\ker(h_x) = (\text{span}(dh))^\perp$ or

$$\ker(h_x) : p \mapsto \{\tau \in T_p(M) : h_x\tau = 0\}$$

where $dh$ is the one-form of $h$.

Theorem B.1 (Frobenius) A nonsingular distribution $\Delta$ on a manifold is completely integrable if and only if the distribution is involutive. For a proof see [132, 160].
Lemma B.1  Let $M$ be an $n$-dimensional manifold and let $\Delta$ be a nonsingular distribution of dimension $m \leq n$. Then the distribution $\Delta$ is integrable if and only if its annihilator is locally spanned by $n - m$ exact one-forms, $d\lambda_i$. That is, there exist functions $\lambda_i \in \mathcal{C}^\infty$, $i = 1, \ldots, n - m$ such that

$$d\lambda_i(q) \in \Lambda_q \quad i = 1, \ldots, n - m$$

and for all $v \in \Delta$ we have $< d\lambda_i, v > = 0$. Proof: See [160, pp. 17-21].

Definition B.10  Let $\Omega$ be a codistribution on $M$, then $\Omega$ is said to be invariant under a vector field $f$ if the Lie derivative of any covector field $\omega \in \Omega$ along $f$ is again a covector field which belongs to $\Omega$, or in short

$$L_f \Omega \subset \Omega \quad (B.5)$$

Lemma B.2  A codistribution $\Omega$ is invariant under $f$ if a smooth distribution $\Delta = \Omega^\perp$ is invariant under $f$ and vice versa.

Proof  Consider the following identity

$$< L_f \omega, \tau > = L_f < \omega, \tau > - < \omega, [f, \tau] > \quad (B.6)$$

Since $\Omega = \Delta^\perp$ and $\omega \in \Omega$, $\tau \in \Delta$, one gets

$$L_f < \omega, \tau > = 0 \quad (B.7)$$

Furthermore, if $\Delta$ is invariant under $f$ then for all $\tau \in \Delta \rightarrow [f, \tau] \subset \Delta$, thus

$$< \omega, [f, \tau] > = 0 \quad (B.8)$$

$$\Rightarrow$$

$$< L_f \omega, \tau > = 0 \quad (B.9)$$
Therefore, we can conclude that $L_f \omega$ belongs to the covector field $\Omega$.

The following two lemmas pertain to local decomposition of control systems and are relevant to developments in Chapter 4. These lemmas are useful in studying output invariance and input decoupling. Consider the nonlinear control system below

\[
\begin{align*}
\dot{x} &= f(x) + \sum_{i=1}^{m} g_i(x)u_i \\
y_i &= h_i(x) & i = 1, \ldots, \ell
\end{align*}
\]  

(B.10)

where $x \in M$ and the input $u_i$, and the output $y_i$ are real valued functions of time. The vector fields $f(x), g_1(x), \ldots, g_m(x)$ are smooth vector fields on $M$ and $h_i(x)$ are smooth functions on $M$.

**Lemma B.3** Suppose $\Delta$ is a nonsingular distribution of dimension $k$ and is invariant under the dynamics of Eq. (B.10). Furthermore, assume that the input distribution $\text{span}\{g_1, \ldots, g_m\} \subset \Delta$. Then, for every point $x_0 \in M$ there exists an open subset $U$ of $x_0$ and a local coordinate transformation $\xi = \xi(x)$ on $U$ such that the dynamics Eq. (B.10) can be transformed into a new coordinate representation as

\[
\begin{align*}
\dot{\xi}_1 &= f_1(\xi_1, \xi_2) + \sum_{i=1}^{m} g_{i1}(\xi_1, \xi_2)u_i \\
\dot{\xi}_2 &= f_2(\xi_2)
\end{align*}
\]

(B.11)

where $\xi$ is partitioned into $(\xi_1, \xi_2)$ and $\dim(\xi_1) = k$. Proof: [160, p. 29].

**Lemma B.4** Suppose $\Delta$ is a nonsingular distribution of dimension $k$ and is invariant under the dynamics of Eq. (B.10). Furthermore, assume that the output codistribution $\Omega = \text{span}\{dh_1, \ldots, dh_\ell\}$ satisfies $\Omega \subset \Delta^\perp$. Then, for every $x_0 \in M$
there exists an open set $\mathcal{U}$ containing $x_0$ and a local coordinate transformation $\xi = \xi(x)$ on $\mathcal{U}$ such that the dynamics of Eq. (B.10) can be transformed into a new coordinate representation as

$$
\begin{align*}
\xi_1 &= f_1(\xi_1, \xi_2) + \sum_{i=1}^{m} g_{1i}(\xi_1, \xi_2)u_i \\
\xi_2 &= f_2(\xi_2) + \sum_{i=1}^{m} g_{2i}(\xi_2)u_i \\
y_i &= h_i(\xi_2)
\end{align*}
$$

where $\xi$ is partitioned into $(\xi_1, \xi_2)$ and $\dim(\xi_1) = k$. Proof: See [160, p. 29].

From the previous two lemmas one sees that, under certain conditions, it is possible to locally achieve output-state and input-state decoupling of a nonlinear control system. In Chapter 4, the aforementioned tools will be employed as a basis to implement task oriented path planning schemes and input decoupling. The inputs will be decoupled such that a robot end-effector can be independently controlled to perform translational motion along a task oriented path and to change its orientation.
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