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Blenman, Lloyd Patrick

IMPLIED FORWARD AND FUTURES RELATIONS IN THE T-BILL MARKET

The Ohio State University

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Implicit Forward And Futures Relations

In The T-Bill Market

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By
Lloyd P. Blenman, B.Soc.Sc., M.A.

* * * * *

The Ohio State University
1986

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DEDICATION

To My Beloved Parents and Sisters

- ii -
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# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>DEDICATION</td>
<td>ii</td>
</tr>
<tr>
<td>ACKNOWLEDGEMENTS</td>
<td>iii</td>
</tr>
<tr>
<td>VITA</td>
<td>iv</td>
</tr>
<tr>
<td>LIST OF TABLES</td>
<td>vii</td>
</tr>
<tr>
<td>CHAPTER</td>
<td></td>
</tr>
<tr>
<td>I.</td>
<td>1</td>
</tr>
<tr>
<td>II. THEORIES OF FORWARD AND FUTURES PRICING</td>
<td>12</td>
</tr>
<tr>
<td>Time-Series And Martingale Models</td>
<td>16</td>
</tr>
<tr>
<td>Arbitrage Models</td>
<td>20</td>
</tr>
<tr>
<td>General Equilibrium Models</td>
<td>23</td>
</tr>
<tr>
<td>Contract Valuation. Contango and Normal</td>
<td>47</td>
</tr>
<tr>
<td>Backwardation</td>
<td></td>
</tr>
<tr>
<td>III. FORWARD-FUTURES DIFFERENTIALS IN THE TREASURY BILL MARKET</td>
<td>77</td>
</tr>
<tr>
<td>The Data</td>
<td>82</td>
</tr>
<tr>
<td>Model Selection. Interest Rate</td>
<td></td>
</tr>
<tr>
<td>Differentials And Their Explanatory Power</td>
<td>89</td>
</tr>
<tr>
<td>Specification Error Tests</td>
<td>112</td>
</tr>
<tr>
<td>Forward and Futures Prices as Maturity-Price Estimators</td>
<td>126</td>
</tr>
<tr>
<td>Tax Effects In The General Equilibrium Model</td>
<td>139</td>
</tr>
</tbody>
</table>
IV. PRICE VOLATILITY IN THE T-BILL MARKET . . . . . 148

Modelling Price Volatility . . . . . . . . . . 151
On The Distribution Of Price Changes . . . 156

V. SUMMARY . . . . . . . . . . . . . . . . . . . . . . . . . 158

BIBLIOGRAPHY . . . . . . . . . . . . . . . . . . . . . . . . 172
## LIST OF TABLES

<table>
<thead>
<tr>
<th>TABLE</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Forward-Futures Price Differentials At 30 Days To Maturity</td>
<td>87</td>
</tr>
<tr>
<td>2. Forward-Futures Price Differentials At 31 Days To Maturity</td>
<td>88</td>
</tr>
<tr>
<td>3. Regression of Interest-Rate Differentials On Log Forward/Futures Differentials</td>
<td>104</td>
</tr>
<tr>
<td>4. Regression of Interest-Rate Differentials On Log Forward/Futures Differentials</td>
<td>105</td>
</tr>
<tr>
<td>5. Regression of Interest-Rate Differentials On Log Forward/Futures Differentials</td>
<td>105</td>
</tr>
<tr>
<td>6. Regression Of Log Forward/Log Futures Differentials On Interest-Rate Differentials</td>
<td>107</td>
</tr>
<tr>
<td>7. Regression Of Log Forward/Log Futures Differentials On Interest-Rate Differentials</td>
<td>108</td>
</tr>
<tr>
<td>8. Regression Of Log Forward/Log Futures Differentials On Interest-Rate Differentials</td>
<td>109</td>
</tr>
<tr>
<td>9. Regression of Normalized Price Differentials On Financing-Cost Differentials</td>
<td>110</td>
</tr>
<tr>
<td>10. Regression of Normalized Price Differentials On Financing Costs Differentials And Weighted Functions Of Past Prices</td>
<td>111</td>
</tr>
<tr>
<td>11. Log Price Differentials On Term Interest Rate And Cumulative Spot Interest Rates</td>
<td>119</td>
</tr>
<tr>
<td>12. Log Price Differentials On Term Interest Rate And Cumulative Spot Interest Rates</td>
<td>120</td>
</tr>
<tr>
<td>13. Regression of Log Price Differentials On Term Interest Rate And Cumulative Spot Interest Rates</td>
<td>121</td>
</tr>
</tbody>
</table>
Futures and forward contracts are contracts for the deferred delivery and receipt of commodities or assets. Futures contracts are traded on organized exchanges and the contract specifications are standardized. If there exist any quality, quantity or timing options then those options are clearly detailed in the contract specifications. Forward contracts are generally not standardized and the terms are usually agreed upon by the contracting parties.

In futures markets, a Clearing House acts as an intermediary between buyers and sellers and ensures performance on all contracts entered. Hence the risk of non-delivery that is present in forward markets is not a major one in futures markets even though some fears have been expressed that the absence of alternative deliverable assets could facilitate a squeeze on a deliverable treasury bill (U.S. Treasury, Treasury/Federal Reserve Study of Treasury Futures Markets, Vol.1, May 1979). The reason for this fear lies in the fact that the more standardized the contract specifi-
fications are, the less the degree of flexibility in terms of the deliverable instrument and the more likely the possibility that the short sellers may be unable to acquire enough of the deliverable asset.

However, Clearing Houses possess the authority to redefine the deliverable instruments and to impose position limits or alter existing ones to counteract a squeeze on the market. In the case of the T-Bill futures markets, if it is determined that a shortage of deliverable 13 week U.S. Treasury bills exists, the Board of Governors of the Exchange can either:

(a) determine a cash settlement based on the current cash value of a three-month U.S. Treasury Bill as determined by using the current cash market yield curve of U.S. Treasury Securities on the last day of trading, or

(b) designate as deliverable any specified U.S. Treasury bill of a maturity other than, or in addition to, the maturity specified above and otherwise meeting the specifications and requirements stated in the rule of "Commodity Specifications".

As a funding source for the U.S. Government, the Treasury Bill market is of paramount importance. This role will likely increase in view of the recurring budget deficits.
The Treasury/Federal Reserve study addressed the effects of the existence of futures markets on the cash bill market. Such market interactions obviously affect the interest rates paid on these securities, in a way that is not clearly understood. Since T-Bills are traded in spot and futures markets it is possible to create a synthetic forward position in T-Bills and to compare the cost of the forward strategy with that of a futures market strategy.

Conventional wisdom holds that the key to understanding the relationship between the two markets may lie in being able to explain the relation between forward and futures pricing of T-Bills. Consequently, as a first step, we elaborate the distinctions between forward and futures contracts and discuss the implications for the pricing of contracts. The key propositions of various theoretical models are summarized, and we also explore the rationale for using the model we develop. We also believe that the issue of default risk which is addressed is not negligible. This is underscored by the recent collapse of the forward market in tin on the London Metal Exchange.

Futures contracts are not equivalent to forward contracts, even though they share some common features. Such differences as exist between forward and futures contracts are engendered by institutional characteristics such as the
posting of initial margin requirements, the imposition of daily price limits and the possibility of differential tax treatment of gains or losses deriving from positions in futures contracts or forward contracts.

Several writers have provided models of forward and futures pricing which attempt to establish some of the essential differences between forward and futures contracts. Capozza and Cornell (1979) argue, based on arbitrage considerations that in a perfect capital market forward and futures rates should be equivalent in order to eliminate riskless arbitrage opportunities. Their argument is confined to a world in which interest rates are deterministic, and there are no market imperfections. Poole (1978) theorizes, on the basis of a transactions cost argument, that for the contract that is closest to maturity, futures rates should lie below forward rates. Rendleman and Carabini (1979) develop a no-arbitrage condition for futures prices, but in so doing they, as did Poole (1978), ignore the daily settlement feature of futures contracts as well as the impact of margin requirements.

In an extension of a study by Black (1976), Morgan (1981) establishes by arbitrage arguments that forward and futures prices must differ when daily settlement on futures contracts occurs in undiscounted terms. In his argument,
initial margin requirements are irrelevant to the no-arbitrage condition between forward and futures prices, since he explicitly assumes that a T-bill is deposited as margin.

Kane (1980) establishes the general non-equivalence of forward and futures rates, in the absence of any settlement arguments, by considering various covered and uncovered trading strategies. Observable differentials between forward and futures rates are attributed to costly performance guarantees in futures markets, which include opportunity costs generated by settlement and margin requirements.

Recent papers by Cox, Ingersoll and Ross (1981) and Richard and Sundaresan (1981) demonstrate under assumptions of zero transactions costs, rational-expectations and zero margin requirements in continuous-time general-equilibrium models that forward and futures prices are not necessarily equal when interest rates are stochastic. French (1982) demonstrated equivalent results in a discrete time model. The work of all of these authors helps to establish conclusively that a futures contract is not simply a forward contract with a daily settlement feature. Forward contracts realize a net cash flow at delivery date depending on the relationship of the exercise price and the spot price of the underlying asset at the delivery date.
All value that exists in a forward contract thus accumulates until the delivery date. On the other hand, futures contracts generate a daily flow of undiscounted payments equivalent to the daily change in futures prices. Were the exchange authorities to institute a policy of daily payouts on forward contracts, such payments should be the discounted value of the daily change in forward prices.

There is thus a fundamental distinction between forward and futures contracts since a forward contract may be used to effectively fix the price today at which the underlying asset will be delivered at some time in the future. Holders of futures contracts are subject to cash flows, whenever the contracts are marked-to-market, which alters the effective price at which the asset or commodity is delivered.

This does not imply that the currently quoted price on a forward contract is the effective price at which the underlying goods or assets would be delivered, even if the contracting parties agreed on the current price. The currently quoted price and all other prices at which forward contracts are entered may only constitute effective prices if in the absence of perfect performance guarantees, the fortuitous event occurs that all delivery specifications are met.
Perfect performance guarantees either of an implicit or explicit nature are costly and may impose trading costs on both of the contracting parties. To the extent that it is asserted that current prices of forward contracts can be used to effectively fix the price at which the underlying asset will be delivered, we must assume the absence of any implicit or explicit costs. In the case of a futures contract, the issue is clear. We have no illusions about being able to determine beforehand the effective price at which the underlying good or asset is delivered. The reason being that future cash flows, the result of mark-to-market settlements, can earn interest at future market rates of interest, and the contract holder is free theoretically to pursue different investment strategies with these cash flows.

In the model which we extend in Chapter 2, and in those developed by several cited authors, a usual assumption is one of zero margin requirements. This implies that no explicit costs are imposed on either party. This assumption however does not rule out the set of implicit costs. It is obvious that performance guarantees that are provided costlessly cannot be perfect since there is no mechanism to ensure performance. In the absence of perfect performance guarantees, an element of uncertainty exists, concerning the quantity of goods deliverable on both types of contracts.
It is the performance guarantee role of margin requirements which effectively breaks the equivalence between forward and futures contracts even though parties to forward contracts may face significant implicit guarantee costs. Forward and futures contracts are contracts for the deferred delivery of goods and assets and the default risk associated with each type of contract in general varies precisely because their guarantee features differ. It is therefore possible that even in a world in which interest rates are deterministic that organized forward and futures markets in identical goods may exist simultaneously, because the traded contracts are subject to different degrees of default risk.

We cannot categorically assert that forward and futures prices are equivalent, in a world with deterministic interest rates, unless inter alia we make some explicit assumption about default risk. Since performance guarantees are required to ensure that the parties to a contract fulfill their obligations, a zero margin requirement per se provides no guarantee. Certainty of performance can however be improved with restrictions on trading strategies, credit checks and rules on position limits as well as price limits.
Implicit forward prices are mere transforms of the forward rates embedded in the structure of cash prices and it is generally known that these prices are influenced by fundamental factors which need not be market-specific. Equivalent prices in the T-Bill market are influenced by interest-rate considerations and all those factors which affect the determination of the rate of interest are obviously transmitted through the futures and cash markets for T-Bills. If we disregard the possibility of trader manipulation in these markets, the currently observed prices embody all market participants' expectations of the future course of behavior of both prices and interest rates.

Therefore from a general economic viewpoint it is of interest to know whether the information provided by implicit forward and futures prices is qualitatively equivalent. If the information contained in both estimators is qualitatively equivalent then they both should on average be either biased or unbiased estimators of maturity prices of contracts for a given data set. Because of the imposition of daily limits on price changes in futures markets, observed prices may not truly reflect a complete adjustment of prices to the received information. Under conditions in which the futures price moves the limit, we have another reason to expect to find significant differences in the sum of squared errors associated with the two estimators.
Apart from trying to ascertain the relative informational efficiency of forward and futures prices, the object of our research effort is to derive alternative models of forward and futures pricing, which incorporate some of the risk characteristics of these contracts. We feel that such models may be useful, since the major models that have been proposed and tested, ascribe the differences between forward and futures prices to interest-rate risk.

In these models the fact that default risk exists is not addressed, with the result that forward and futures prices are deemed equal if interest rates are deterministic. Moreover, since the models repose in an identical-consumers framework or are derived by arbitrage arguments, there is no explicit interaction between contract holding and wealth. In addition the results that have previously been derived in an equilibrium context, that we have referred to, are premised on the notion that the contract price is to be derived from the assumption of a zero-valued position in each contract by the representative consumer.

In Chapter 2 we develop a model of forward and futures pricing in which contracts are held by market participants. Past information on asset prices, available and known to contract holders, is embedded in current equilibrium prices. The consideration of an expanded information set is
shown to lead to alternative pricing models. The relations between the pricing models we derive and the received models are established. We also demonstrate within the context of the model that forward and futures prices on similar goods or assets may differ even in a world with deterministic interest rates.

This result facilitates our effort to provide a statistical explanation of forward-futures differentials in the T-Bill market that does not merely hinge on mean difference t-tests of interest rates. That is we provide potentially measurable variables that capture an element of default risk. Chapter 3 is devoted to the issue of model selection and the related criteria for discriminating between models. The chapter covers general specification error tests, and the role of forward and future prices as maturity price estimators. We also discuss the likely impact of taxation in the general model formulated in Chapter 2.

Chapter 4 discusses the theoretical implications of the model developed in chapter 2, for the volatility of forward and futures prices. A regression model of price volatility is formulated. This model is tested with T-Bill and repurchase rate data. Finally we evaluate the distribution of price changes in the data utilized. Chapter 5 is a summary of our theoretical and empirical findings.
Recently, models have been developed which seek to explain forward and futures pricing both in continuous and discrete time contexts. This new class of models has been tested by French (1982) and Sundaresan (1980), both in the context of commodity futures, under a wide range of assumptions, and both authors reject the model specification. Rendleman and Carabini (1979) also test Cox, Ross and Ingersoll's (1985, b) model with data on the Treasury Bill market, and they find that the differences implied by the model specification do not fully explain empirical differences observed between forward and futures prices in the T-Bill market.

Some authors hold that the reason why these unexplained divergences between forward and futures rates and prices exist rests in the singular nature of the Treasury Bill market. However, the fact that this result has also been found in commodity futures markets to some extent invalidates this argument. Our objective is to try to ascertain
if the explanatory power of these models can be increased by alternative model specifications.

We wish to determine how the pricing formulas are modified when we allow individual consumers to diversify their asset holdings across goods and contingent claims, in such a way that the optimizing problem is not constrained to corner solutions. We also wish to determine what are the implications for the intraday valuation of contracts, that are generated by the intertemporal asset pricing model. In this regard we explicitly extend the asset pricing model to incorporate cases where contingent-claim contracts are not necessarily zero-valued.

A contingent-claim contract is zero-valued whenever the issue price equals the current trading price. However, an individual who holds a position in the contract, either short or long, at a price other than the purchase price, has a non-zero-valued position relative to institutional payoff procedures in force. This extends the logical completeness of this type of model and provides a rationale for the use of the derived pricing formulas in the analysis of market data. If we only consider zero-valued cases, there is no reason for market participants to consider holding the associated contracts, and it is not likely that the derived pricing formulas can provide the basis for adequate analysis of market data.
We theorize that this model extension is potentially significant for the following reasons:

(1) In this model, the prices of goods, and the values of contingent claim contracts are linked to the stochastic fluctuations of the economy.

(2) In empirical studies, the current settlement price of a futures contract is not uniquely representative of market participants' expectations of the future course of prices and interest rates, since the majority of outstanding contracts were not traded at the settlement price.

As a first step, we embed the models of Richard and Sundaresan (1981) and French (1982) in a more general framework. We then derive general pricing formulas for both forward and futures prices, and demonstrate the equivalence under certain assumptions of our results and the received theoretical relations. These general formulas are then examined to ascertain their implications for the hypotheses of normal backwardation and normal contango. We also discuss the possible relation between the hedging properties of forward and futures contracts and the performance of their respective prices as estimators of future spot prices. These formulas are tested in Chapter 3, in order to see whether they provide a satisfactory explanation of forward-futures differentials observed in the Treasury Bill market.
This chapter is organized as follows. In section 1, we summarize the various time-series and martingale models and isolate a testable hypothesis regarding the behavior of futures prices. In section 2, we discuss a wide range of arbitrage models, with particular emphasis on those involving repurchase strategies. Section 3 is an extension of the intertemporal asset-pricing model to incorporate non-zero valued contracts and positions in contingent claims. Section 4 examines the issues of valuation, contango and backwardation, in this type of model.
An extensive body of research models dealing with time-series specifications of futures pricing exists. In this regard, it would be impossible for us to exhaustively survey this body of literature, and we therefore choose works that have been considered as representative of the existing body of knowledge in this area. A representative of these models is Samuelson's (1965) theory that futures prices constituted a martingale process. The properties of martingale processes are discussed by Malliaris (1981), Malliaris and Brock (1983), Ash (1972) and Doob (1971) among others.

The martingale property of forecast price sequences, which serves as a defining characteristic of market efficiency, does not by itself convey adequate information about the degree of error involved in utilizing its predictions of future spot prices as our estimates of the actual prices that will occur in the future. In addition it does not convey any policy conclusions about the usefulness of price constancy or variability (Alchian 1974).

This entire search for the presence or absence of martingale behavior in futures markets is inextricably linked to the issue of market efficiency. Several authors including Lucas (1978) have shown that failure to display
martingale-type properties does not necessarily imply that a market is inefficient. Rausser and Carter (1983) provide a useful and current review of the evidence in the context of futures markets, and reach a similar conclusion. The major theoretical work in this area is that of Samuelson (1965). This work has generated "the maturity effect hypothesis" in futures pricing. We discuss the underlying model in some detail. Samuelson (1965) proposed that the current futures price for a contract maturing $T$ periods in the future. $Y(T, t)$ obeyed the following axiom:

$$Y(T, t) = E_t[X_{t+T} | X_t, X_{t-1}, ...]$$

where $T = 1, 2, ..., \text{and } X_t, X_{t-1}, ...$ represent the sequence of spot prices. In conjunction with the AR (1) process $X_t = \alpha X_{t-1} + u_t$, where $u_t$ is a white noise process, this axiom creates a model in which there is increasing futures price variability as the contract approaches its maturity date, providing $|\alpha| < 1$. Rutledge (1976) posits a special AR (2) process for spot prices namely.

$$S_t - S_{t-1} = \beta(S_{t-1} - S_{t-2}) + u_t.$$

This process generates a random walk when $\beta = 0$. and in conjunction with Samuelson's axiom implies constant vari-
ability of futures prices. When $\beta < 0$, this model implies that futures price variability decreases as contracts approach their maturity dates. Tests by Rutledge on data from the wheat, soybean, silver and cocoa markets provided mixed results in relation to his attempt to reject the null hypothesis that the variability of futures prices is independent of the time to maturity of the contract. In a response to the work of Rutledge (1976), Samuelson (1976) indicates that even though transiently the variability of futures prices may rise, what is important is that ultimately futures prices should display monotonicity.

It is apparent that Samuelson's (1965) theory holds only in special cases where spot prices follow a stationary first-order autoregressive process and futures prices are martingales. However, a substantial body of evidence has accumulated demonstrating that the assumption of a strict martingale-type behavior for futures prices is not supported by empirical data.

Following the pioneering work of Samuelson (1965), several authors undertook studies to verify whether futures prices followed a random walk or other more complex processes. Additional investigations on this issue include the works of Cargill and Rausser (1975), and Stevenson and Bear (1970) and Rausser and Carter (1983). This research
indicates that it is apparently impossible to discriminate among competing autoregressive processes solely on the basis of a priori considerations. The work of these and other authors suggests that each individual futures market may have its own distinct time-series characteristics.

As a consequence the time-to-maturity effect associated with Samuelson's (1965) theory becomes a matter to be investigated for its empirical relevance to individual futures markets. The available evidence suggests that futures markets may display either increasing or decreasing price variability as contracts mature.
In this section, we focus on those models which are derived from purely arbitrage considerations. In some respects the trichotomy and the nomenclature we have employed are somewhat artificial. It can be shown that the results developed by the martingale and arbitrage models, emanate from particular restrictions imposed upon the general-equilibrium models. The arbitrage models are represented by those of Lang and Rasche (1978), Koch and Kwaller (1984), Kolb and Gray (1985), Elton, Gruber and Rentzler (1984) and Kane (1980) among others. In general these arbitrage models were derived by:

(1) combining positions in the cash market to achieve a forward position and comparing the yield on the forward position with that of a futures market position. Generally these strategies involved the shorting of T-Bills, Cornell and Capozza (1979), the use of repurchase agreements Koch and Kwaller (1984), Kane (1980) and "bond swap "strategies. 6 Elton, Gruber and Rentzler (1984).

(2) comparing positions in the futures market only, Kolb and Gray (1985). These authors compare the yields from holding four-91 day T-bill futures contracts in a strip with the yield from four 1-year T-Bill futures contracts.
We discuss the repurchase-agreement financing strategy employed by dealers in the market. Evidence obtained by Koch and Kwaller (1984) and Kamara and Lawrence (1985) indicates that this arbitrage strategy prices futures contracts at the margin. Generally the traders employ a cash-and-carry strategy. If the holding-period yield from carrying a deliverable T-Bill to the maturity of a futures contract is not in line with financing costs then traders would earn sure arbitrage profits.

Repurchase Strategy

The dealer:

1. borrows the purchase price of a deliverable bill under a repurchase agreement, and simultaneously purchases the bill as well as establishes a short futures position.

2. holds the deliverable bill until maturity, when he delivers it against the maturing futures contract.

3. clears his repurchase agreement.

The holding-period yield is calculated on the purchase price of the deliverable bill and the settlement price of the short futures position. The financing-period costs are
calculated on the basis of the proceeds of the repurchase agreement (i.e., the purchase price of the deliverable bill) and the price at which the security is repurchased. This strategy is employed whenever the holding-period yield exceeds the financing costs. If the relation is reversed the trader simply reverses his strategy and in effect finances a borrower. What these strategies all have in common is that they postulate that a particular set of rates or a particular set of prices should be equal so as to prevent riskless arbitrage profits. As such, they are seen to be particular results of the general-equilibrium models.
GENERAL EQUILIBRIUM MODELS

The general-equilibrium models which strive to model futures and forward prices, are related to the single-good models of Lucas (1978), Woodward (1983) and McCulloch (1980). These models demonstrate the role of marginal utility, dated for time preference, in the pricing of assets in the case of Lucas (1978) and in establishing the relationship between forward and future spot rates of interest (Woodward 1983, McCulloch 1980).

In a continuous-time framework, Richard and Sundaresan (1981), Cox, Ingersoll and Ross (1981, 1985 a, 1985 b) develop models for the pricing of contingent claims, and more specifically models for the pricing of futures and forward contracts. Current prices of these contracts are shown to be functions of the expected future spot price at the expiration date of the contract as well as the covariance of marginal utility at the expiration date with the maturity price of the contract.

In addition Cox, Ingersoll and Ross (1981) utilize a general valuation equation, in the form of a partial differential equation, which they demonstrate that all contingent claims must satisfy. (See Cox, Ingersoll and Ross 1985, b). This valuation equation is then used to develop pricing relations for forward and futures contracts. Their
results are special cases of those of Richard and Sundaresan (1981). A different approach, employing continuous processes, is that of Garbade and Silber (1983), who develop pricing utilizing a Black-Scholes methodology.

An analog of these models in term-structure theory is the work of Woodward (1983). In an extension of the work of McCulloch (1980), Woodward develops a series of propositions linking the existence of liquidity and solidity premia to the covariation of future interest rates and the marginal utility of consumption. The models of Cox, Ingersoll and Ross (1981, 1985 a, 1985 b) and Richard and Sundaresan (1981) are all couched in an identical-consumers economy, in which consumers hold goods and contingent claims. Since all consumers are identical the optimal number of contingent claims contracts held is zero and there is no riskless lending. All wealth is held in physical goods.

These models embody the martingale features of Samuelson's (1965) theory as well as the marginal utility of consumption features of the models of Lucas (1978), Woodward (1983), and McCulloch (1980) among others. We extend the model to the case of an economy with different groups of consumers. However each individual group is composed of homogeneous members. This assumption ensures that in equilibrium the optimal number of contingent claims held is not
constrained to zero, and that individual wealth is not necessarily held only in physical goods.

We proceed with a version of a general-equilibrium continuous-time, rational-expectations model of a multigood economy, formulated by Richard and Sundaresan (1981), in which individuals act to maximize expected utility. We assume that individuals hold contingent-claim contracts in a frictionless environment. The existence of explicitly costless guarantees in the markets where these contracts are exchanged, is also assumed. Moreover contract prices are not subject to institutionally determined limit moves.

The model is shown to be flexible in terms of the specification of consumer endowments and preferences, and the valuation formulas are not restricted to the identical-consumers case. This provides a justification for the testing of hypotheses with data based on contracts held by market participants.

The existence of traded contracts can be assured by postulating the presence of two groups of consumers differentiated via their endowments and risk preferences. Equilibrium pricing formulas are not affected by this modification, and individuals in different groups have different marginal utility of wealth in equilibrium, only if the
change of aggregate wealth with respect to individual wealth differs across individual groupings. (See Appendix A). Hence the identical-preferences, identical-endowments assumption is not a binding restriction. This allows us to simplify our analysis by deriving the valuation formulas in the context of an identical-consumers model. Moreover, since each consumer faces the same maximization problem as a result of being endowed with similar stocks of goods and sharing similar preferences, we may derive pricing relations as the solution to a single individual's maximization problem. The equilibrium prices that are derived are consistent with Pareto-optimality. We assume that each individual tries to maximize expected utility in the form,

\[ E \int_0^T e^{-\rho t} u(c(t)) dt. \] (1)

There are \( n \) goods in this economy and the rate of net production of goods is governed by a stochastic differential equation,

\[ dq = (h - c) \, dt + G \, dz. \]

d\( q \) is an \( n \times 1 \) vector of the rate of net output of goods. 

\( G \) is an \( n \times (k + n) \) diffusion matrix. 

d\( z \) is a \( (k + n) \times 1 \) vector of independent Weiner processes.
h is the vector of expected instantaneous rates of output.
c is the vector of instantaneous rate of consumption.

Each consumer has a proportional share of the initial stock of goods Q(0) and is able to invest in all productive processes. All goods are used as inputs in the production of good i, i=1, 2,...,n. The total amount of good j invested in production at time t is,

\[ q_j = \sum_{i=1}^{N} k_{ij}, \quad j = 1, 2, \ldots, n. \] (2)

\( k_{ij} \) represents the amount of the j-th good used in the production of good i; i=1, 2,...,n. Additionally \( G = G(k, Y) \) and \( h = h(k, Y) \) where \( k \) is the vector of goods used in the production of good i;

\( Y \) is the vector of technological state variables which influence productive activity. These technological state variables are governed by a stochastic differential equation.

\[ dY = \mu(Y, t) dt + S(Y) dz, \]

where \( Y \) is a \((k \times 1)\) vector and \( S \) is a \(k \times (n+k)\) matrix.
Stocks of output $Q(t)$ are subject to change via a diffusion process $dQ = B \, dt + D \, dz$.

$Q$ is an $(n \times 1)$ vector.

$D$ is an $n \times (n + k)$ diffusion matrix.

$B$ is an $(n \times 1)$ vector, and we assume that the summation of all endowments at time $t$ equals the stock $Q(t)$.

Finally we specify the stochastic differential equations governing the two types of contracts, and the prices of the $n$ goods associated with the two types of contracts. Even though we specify in an ex-ante fashion the processes governing the evolution of the prices of goods and contractual claims on those goods, the actual forms of these processes are determined endogenously, since all prices and valuation formulas are functions of the underlying state variables $Y$ and $Q$. In particular $V_k = V_k(Q, Y, t), k = 1, 2$. $V_k$ is an $n$-element vector of the values of type $k$ contracts on all $n$ goods. By Itô's Lemma we obtain the following relation for the $j$-th element of $V_k$.

\[
\begin{align*}
\ud V_j &= \left[ \sum_{i=1}^{N} V_{Q_i} B_i^j + \sum_{i=1}^{k} V_{Y_i} Y_i^j \mu_i + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} V_{Q_i} Q_j (DD')_{ij} \\
&\quad + \frac{1}{2} \sum_{i=1}^{k} \sum_{j=1}^{N} V_{Q_i} Y_j S_i D_j' + \frac{1}{2} \sum_{i=1}^{k} \sum_{j=1}^{k} V_{Y_i} Y_j (SS')_{ij} \right] \ud t \\
&\quad + \left[ \sum_{i=1}^{N} V_{Q_i} D_i^j + \sum_{i=1}^{k} V_{Y_i} S_i^j \right] \ud z.
\end{align*}
\]

(3)
This implies that \( \text{d}V_k = (\alpha_k - \phi_k) \, \text{d}t + H_k \, \text{d}z \) by restricting \((\alpha_k - \phi_k)\) and \(H_k\) to be the vector analogue of the terms in brackets above. We also note at this stage for future reference, that the differential operator \(L\) defined in equation (9) is such that \(L \, V_k = \alpha_k - \phi_k\). For purposes of simplicity we therefore specify that:

\[
\text{d}V_k = (\alpha_k - \phi_k) \, \text{d}t + H_k \, \text{d}z, \quad V_k \text{ is the vector of values of the type } k \text{ contracts, and } k=1 \text{ and } 2 \text{ respectively for forward and futures contracts.}
\]

Prices of goods are described by the vector of stochastic differential equations \( \text{d}p = \alpha_p \, \text{d}t + \sigma_p \, \text{d}z \). \( \alpha_k - \phi_k \) and \( \alpha_p \) are the respective instantaneous rates of change of the contracts of type \( k \), and prices of goods of type \( k \) contracts.

\( H_k \) and \( \sigma_p \) are their respective diffusion matrices. Each consumer is also assumed to be able to borrow and lend at the riskless rate of interest \( r(t) \) and to hold goods and contingent claims. Each individual therefore has an equation of wealth such that,

\[
p'q + \sum_{k=1}^{2} M_k V_k + \lambda = w. \quad (4)
\]
\( \lambda \) is the amount invested in riskless lending or borrowing. \( \mathbf{N}_k \) is a vector of the number of type \( k \) contracts held on each good. \( \mathbf{V}_k \) is an \( n \)-vector of values of each type \( k \) contract held on each good.

By application of Itô's Lemma to the wealth equation we have

\[
dw = p'q + q'dp = dp'dq + \sum_{k=1}^{2} M_k \mathbf{V}_k + \lambda rt.
\]

This implies that,

\[
dw = \left[ p'(h - c) + q'\sigma + tr \sigma \mathbf{G}' + \sum_{k=1}^{2} M_k (\alpha_k - \phi_k - \mathbf{V}_k) \right] dt + \left[ p'\mathbf{G} + q'\sigma \mathbf{P} + \sum_{k=1}^{2} M_k \mathbf{H}_k \right] dz.
\]

Let \( dw = \beta_w dt + \sigma_w dz \). \( \beta_w \) and \( \sigma_w \) being the terms in square brackets in (5). The consumer's problem is to maximize expected utility for appropriate choice of the control variables \( M_k, \phi_k \), and \( k_{ij} \) subject to the wealth equation and the state variables \( Q \) and \( Y \). Here we follow the analysis of Merton (1969, 1971), who showed that the consumer's problem can be reduced to,
subject to the budget constraint.

\[
dW = \beta_w \, dt + \sigma_W \, dz
\]

and \( c(t) \geq 0; \; W(t) > 0; \; W(0) = W_0 > 0. \)

\( I(Y(1), Q(1), W(1), T) \) in this context is simply a measure of the value function at the expected terminal time \( T. \) We next use a transformation \( J(Y, Q, W) = e^{pt} I(Y, Q, W, t) \) to eliminate the explicit dependence on the time variable. \( J \) is assumed to be concave in the wealth variable and the utility function is a von Neumann-Morgenstern utility function. The stochastic Bellman equation is then derived. Fleming and Rishel (1975) prove that if the control variables satisfy the Bellman equation then those control variables constitute optimal consumer decisions and \( J \) is the optimum value function. We implicitly assume that this condition is met. The requisite Bellman equation is:

\[
\begin{align*}
\max_{\mathbf{m}_k, \mathbf{c}_i, k, ij} & \quad \mathbf{u}(c(t)) + \beta_w J_w + \sum_{i=1}^{k} \mu_i J Y_i + \sum_{i=1}^{N} B_i J Q_i \\
& - \rho J + \frac{1}{2} J w \sigma w \sigma w' = \sum_{i=1}^{k} J w Y_i \sigma w S_i + \sum_{i=1}^{N} J w Q_i \sigma w D_i' \\
& + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} J Q_i Q_j (D D')_i j + \frac{1}{2} \sum_{i=1}^{K} \sum_{j=1}^{K} J Y_i Y_j (S S')_i j
\end{align*}
\]
or \( 0 = \max_{c_i, m_k, k_{ij}} \left[ LJ + u(c(t)) - \rho J \right] \tag{8} \)

where \( L \) is a differential operator such that,

\[
LJ = B_wJ + \sum_{i=1}^{K} \mu_iJy_i + \sum_{i=1}^{M} B_iJQ_i + Jt
\]

\[
+ \sum_{i=1}^{K} \mu_iJw_{ij} + \sum_{i=1}^{K} \mu_iJy_i (SS')_{ij} + \sum_{i=1}^{K} \mu_iJw_{ij} (DD')_{ij} + \sum_{i=1}^{K} \mu_iJQ_j (DS')_{ij}.
\tag{9}
\]

Let \( \psi = \max_{K_{ij}, c_i, m_k} LJ + u(c(t)) - \rho J = 0. \tag{10} \)

From the first order conditions derived from maximizing \( \psi \) subject to the control variables we obtain.

\[
\psi_c_i = u_i(c(t)) - \rho_iJ = 0. \quad i = 1, \ldots, N. \tag{11}
\]

\( \psi_c_i \) is the marginal utility of consumption of the \( i \)-th good.

\[
\psi_{m_k} = J_w \left[ (\alpha_k - \phi_k) - \gamma_{kr} \right] + J_{ww}H_k\sigma_w
\]

\[
+ \sum_{i=1}^{K} J_{w}y_i H_k S_i + \sum_{i=1}^{N} J_{w}Q_i H_k D_i = 0; \quad k = 1, 2. \tag{12}
\]

\[
\psi_{k_{ij}} = \left[ P_i + (\partial h_i/\partial k_{ij}) + \alpha P_j - rP_j + \sigma P_i (\partial G_i/\partial k_{ij}) \right]
\]

\[
+ \left[ P_j (\partial G_i/\partial k_{ij}) + \sigma P_j \right] (J_{ww}\sigma_w + \sum_{i=1}^{K} J_{w}y_i S_i + \sum_{i=1}^{N} J_{w}Q_i D_i) \tag{13}
\]
But we note that:

\[ L\mathcal{W} = \beta \mathcal{W} \mathcal{W} + \sum_{i=1}^{K} u_i \mathcal{W} \mathcal{Y}_i + \sum_{i=1}^{K} B_i \mathcal{W} \mathcal{Q}_i \]

\[ + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \mathcal{W} \mathcal{Q}_i \mathcal{Q}_j (\mathcal{D}^D)^{ij} + \frac{1}{2} \sum_{i=1}^{K} \sum_{j=1}^{K} \mathcal{W} \mathcal{Y}_i \mathcal{Y}_j (\mathcal{S}^S)^{ij} \]

\[ + \sum_{i=1}^{K} \sum_{j=1}^{N} \mathcal{W} \mathcal{Q}_i \mathcal{Y}_j (\mathcal{S}_i \mathcal{D}_j^D) + \sum_{i=1}^{N} \mathcal{W} \mathcal{W} \mathcal{Y}_i \sigma_w \mathcal{S}_i \]

\[ + \frac{1}{2} \mathcal{W} \mathcal{W} \mathcal{Y}_i \sigma_w \mathcal{D}_i^D. \] (14)

Taking the total derivative of the stochastic Bellman equation with respect to \( \mathcal{W} \), we obtain:

\[ 0 = \beta \mathcal{W} \mathcal{W} + r \mathcal{W} + \sum_{i=1}^{K} u_i \mathcal{W} \mathcal{Y}_i + \sum_{i=1}^{N} \mathcal{W} \mathcal{W} \mathcal{Q}_i \]

\[ - \rho \mathcal{W} + \frac{1}{2} \mathcal{W} \mathcal{W} \sigma_w \sigma_w^T + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \mathcal{W} \mathcal{Q}_i (\mathcal{D}^D)^{ij} \]

\[ + \frac{1}{2} \sum_{i=1}^{K} \sum_{j=1}^{K} \mathcal{W} \mathcal{Y}_i \mathcal{Y}_j (\mathcal{S}^S)^{ij} + \sum_{i=1}^{K} \sum_{j=1}^{K} \mathcal{W} \mathcal{W} \mathcal{Q}_i \sigma_w \mathcal{S}_i \]

\[ + \sum_{i=1}^{K} \mathcal{W} \mathcal{W} \mathcal{Q}_i \sigma_w \mathcal{D}_i^D + \sum_{i=1}^{N} \sum_{j=1}^{N} \mathcal{W} \mathcal{Y}_i \mathcal{Q}_j (\mathcal{S}_i \mathcal{D}_j^D). \] (15)

This implies that \( 0 = (r - \rho ) \mathcal{W} + L \mathcal{W}. \) (16)
Additionally, repeated differentiation of (12) with respect to \( M \) shows that it can be expressed as, in the case of \( k = 1, 2 \):

\[
J_w \left[ (\alpha_k - \phi_k) - \nu_k r \right] = 0. \tag{17}
\]

First we discuss the different treatment of forward and futures contracts and then proceed to solve equation (17) for the pricing formulas for futures and forward prices of good \( j \). Futures contracts are continuous payoff contracts, that have zero value after each payoff since the price of the contract is marked to market value. Hence the change in the value of a futures contract on good \( j \) between payoffs is

\[
d V = df_j. \tag{2j}
\]

Here \( df_j = f_j(t, T) - f_j(t_0, T) \), \( 0 \leq t_0 < t < T \).

Forward contracts are maturity-payoff contracts that are generally not zero-valued except at the time of issuance. The change in the value of a forward contract on the \( j \)-th good is

\[
d V = \rho(t, T) d F_j. \tag{jj}
\]

Here \( d F_j = F_j(t, T) - F_j(t_0, T) \) and \( 0 \leq t_0 < t \). \( \rho(t, T) \) is the present value of a dollar to be received with certainty at time \( T \), and \( 0 \leq t_0 \leq t \leq T \). Good \( 1 \) is the numeraire good and its price is set at 1 identically for all times \( t \in [0, 1] \). Hence \( \rho(t, T) \exp \int_t^T r(t, s) ds = 1 \) defines the structure of spot and forward rates of interest. \( r(t, T) = r(t) \) is the current spot rate of interest. We now proceed to derive pricing formulas for
forward and futures contracts on the j-th good, after some definitional issues.

\[ F(t_0, T) \] \( (f(t_0, T)) \) is the price at time \( t_0 \), of a forward (futures) contract entered at \( t_0 \), with maturity date \( T \).

\[ F(t, T) \] \( (f(t, T)) \) is the time-\( t \) price of a forward (futures) contract that has been held from time \( t_0 \), with maturity date \( T \).

\[ \nu_{ij}(t) = \nu_{ij}(t_0, t, T) \] is the value of the position held in a type \( i \) contingent claim contract on the j-th good from time \( t_0 \) to time \( t \).

\[ \nu_{ij}(T) = \nu(t_0, T, T) \] is the value of the position held in a type \( i \) contingent claim on the j-th good from time \( t_0 \) to time \( T \).

\[ \gamma(t) = \gamma(t_0, t, T) \] is the percentage change in the price of a forward contract from time \( t_0 \) to \( t \), with current price as base.

\[ \gamma(T) = \gamma(t_0, T, T) \] is the percentage change in the price of a forward contract from \( t_0 \) to \( T \), with current price as base.

Forward Contracts
We assume that these contracts have been held from some arbitrary time $t_0$ such that $0 \leq t_0 < t$. The value of a single forward contract on good $j$ at time $t$ is $V_{1j}(t) = \beta_1(t, T) (F_j(t, T) - F_j(t_0, T))$. $V_{1j}(t)$ therefore represents the cumulative change in the value of the contract, to its holder, since the contract was written.

Since $V_{1j}(t) = \beta_1(t, T) (F_j(t, T) - F_j(t_0, T))$, this implies that

$$V_{1j}(t) = \beta_1(t, T) \left[ 1 - \frac{F_j(t_0, T)}{F_j(t, T)} \right]$$

$$= \gamma(t) R_j(t, T)$$

(18)

We define $R_j(t, T)$ to be equal to $\beta_1(t, T) F_j(t, T)$ and $\gamma(t)$ is as previously defined. From (17) we note that the $j$-th element of $J_w \left[ (\alpha_j - \phi_1) - V_{1j} r \right] = 0$.

$$J_w \left( \alpha_{1j} - \phi_1 - V_{1jr} = 0 \right) \text{ can be expressed as}$$

$$L(J_w V_{1j}) - V_{1j} L J_w - V_{1jr} J_w = 0.$$  

(19)

By substituting for $V_{1j}(t)$ in (19), we obtain:

$$L(J_w \gamma(t) R_j(t, T)) - \gamma(t) R_j(t, T) L J_w$$

$$- r(t) \gamma(t) R_j(t, T) J_w = 0.$$  

(20)
Next we utilize (16) and substitute $LJ_w$ in (20) with an expression in $(J_wY(t)R_j(t,T))$. This implies that:

$$L(J_wY(t)R_j(t,T)) + (r - \rho)Y(t)R_j(t,T)J_w$$

$$-r(t)Y(t)R_j(t,T)J_w = 0. \quad (21)$$

By simplifying this equation we obtain.

$$L(J_wY(t)R_j(t,T)) - \rho Y(t)R_j(t,T)J_w = 0. \quad (22)$$

On the assumption that all the continuity conditions are satisfied, by Theorem 5.2 of Friedman (1975), the solution of (22) is:

$$J_wY(t)R_j(t,T) = \mathbb{E}(J_w(T)Y(T)B_1(T,T) \theta(W(T),Y(T),Q(T),T) \exp \int_0^T (r(t,s) - \rho) \, ds). \quad (23)$$

$\theta(W(T),Y(T),Q(T),T) = F(T,T)$ is the boundary condition at time $T$ on the $j$-th contract whose time $t$ price is $F(t,T)$. Since $B_1(T,T) = 1$ and $F(T,T) = P(T)$, the general solution of (22) can be simplified as follows:

$$F_j(t,T) = \mathbb{E}_t \left( \frac{J_w(T)}{J_w(t)} P_j(T) \frac{Y(T)}{Y(t)} \exp \int_t^T (r(t,s) - \rho) \, ds \right). \quad (24)$$
Finally we can utilize the relation in equation (11) relative to the numeraire good i.e. \( u_1(c(t)) - J_w = 0 \), since \( P_1 = 1 \) at all times, to express (24) in terms of the marginal utility of consumption of the numeraire good. This implies that:

\[
F_j(t,T) = E_t \left( \frac{u_1(c(T))\gamma(T)}{u_1(c(t))\gamma(t)} P_j(T) \exp \int_t^T (r(t,s) - \rho) ds \right).
\]

(25)

This relation holds for \( n-1 \) goods in our economy. Richard and Sundaresan (1981) in their derivation demonstrate that the guaranteed forward price of the \( j \)-th good \( F_j(t,T) \) is the present value of a known number \( \exp \int_t^T r(t,s) ds \) of the good to be delivered at time \( T \). In their formulation \( \frac{\gamma(T)}{\gamma(t)} = 1 \). Based on their analysis, they assert, as does French (1982) that holding a guaranteed forward contract is in essence only a gamble on the expected future spot price.

Our formulation of the general pricing relation shows that even in the absence of any other consideration, such as transactions costs for example, the forward price on a contract can be shown to be the present value of a random quantity of goods to be delivered at time \( T \). The element of randomness which is introduced is not merely that the expected price behavior of forward contracts over the current and future holding periods may be different.
In addition $\gamma(T)/\gamma(t)$ contains an implicit assumption about the behavior of prices from $t$ to maturity of the contract, relative to its past price behavior. It should be noted that there is no inherent contradiction between the results we have derived and those obtained by Richard and Sundaresan (1981). Here we explicitly consider the prices of forward contracts when such contracts are held by market participants from some arbitrary time $t_0$ until at least some time, possibly infinitesimally greater than time $t$.

As a result of our explicit assumptions that the values of the contracts were influenced by Weiner processes, a logical implication is that forward prices are also functions of the particular Weiner processes $Z_{\gamma}$ that we have specified implicitly. Our formulation admits that of Richard and Sundaresan (1981) under the explicit assumption that $F_{\gamma}(t_0, T) = 0$ but allows for the possibility that the stochastic sequence of forward prices did not originate at time $t$.

Our interpretation of the general pricing relation for forward prices shows that the mere uncertainty introduced by having to consider the expected movement of prices relative to their past history may possibly make a forward contract a de facto contract for a random quantity of goods.
However this does not indicate the full extent of the uncertainty and risks faced by forward contract holders. A complete treatment of those risks would then include the possibility of non-performance. (See Kane 1980). In the course of this analysis we explicitly assume the existence of costless guarantees. However our analysis clearly shows that residual risk has not been eliminated. Intuitively this result obtains because a guarantee that is provided without cost to the beneficiary and to the issuer cannot be perfect. \( \frac{\gamma(T)}{\gamma(t)} \) captures the element of risk involved.

**Futures Contracts**

From (17) we note that \( \sum_{j} a_{2j} - \phi_{2j} \) = 0 is the equation that must be solved to derive the futures pricing relationship for the j-th good, since the value of a futures contract equals zero after settlement. We note that

\[ a_{2j} - \phi_{2j} = L V_{2j} \]

This implies that \( L V_{2j} = L f_{j} f_{j} = f_{j}(t, T) \). Hence

\[ L(J_{w} V_{2j}) - V_{2j} L J_{w} = 0 \]

is equivalent to \( L(J_{w} f_{j}) - f_{j} L J_{w} = 0 \). only if we explicitly normalize futures prices also. Under this assumption, we utilize (16) and substitute for \( L J_{w} \) in our equation.
This implies that \( L(J_w f_j) + (r - \rho) J_w f_j = 0 \). (26)

An additional application of Friedman's (1975) theorem generates the following solution:

\[
f_j(t, T) = E_t \left( \frac{J_w(T)P_j(T)}{J_w(t)} \exp \int_t^T (r - \rho) ds \right).
\] (27)

Equivalently we can express (27) in terms of the marginal utility of the numeraire good by utilizing the relationship \( u_1(c(t)) = J_w(t) \). Hence we can state that,

\[
f_j(t, T) = E_t \left( \frac{u_1(c(T)) P_j(T)}{u_1(c(t))} \exp \int_t^T r(s) - \rho ds \right).
\] (28)

Versions of equations (25) and (27) provide the basis for tests on forward-futures price differences in chapter 3.

An intuitive interpretation of the pricing formula for futures prices is provided by Cox, Ingersoll and Ross (1981) utilizing a portfolio strategy termed a "forward plan". This strategy can be shown to generate the basic Richard and Sundaresan (1981) result under the assumption that marginal utility of consumption is identical in the current and terminal states. Since our formulation of the general pricing relation for forward prices indicates that such prices are the discounted present values of contracts for the delivery of random quantities of the underlying goods, we present a portfolio strategy that is consistent
with our assertions, under the assumption that the marginal utility of consumption is constant. At first we assume that the time-\(t\) price of a forward contract for good \(j\) is \(F_j(t, T)\) and that no implicit or explicit investment is required. Investors take a position in \(\frac{1}{\gamma(t)} \exp \int_t^T r(t,s)ds\) forward contracts and invest \(F_j(t,T)\) in riskless long term bonds maturing at time \(T\).

The current value of this portfolio at time \(t\) is \(F_j(t,T)\).

At time \(T\) the value of the bond holdings is \(F_j(t,T) \exp \int_t^T r(t,s)ds\).

The time-\(T\) payoff on each contract is \(P_j(T) - F_j(t,T)\).

The payoff at time \(T\) on all guaranteed forward contracts on good \(j\) is \(\frac{1}{\gamma(t)} \exp \int_t^T r(t,s)ds (P_j(T) - F_j(t,T))\).

Hence the value of the combined holdings of bonds and guaranteed forward contracts is

\[
F_j(t,T) \exp \int_t^T r(t,s)ds + \frac{1}{\gamma(t)} \exp \int_t^T r(t,s)ds [P_j(T) - F_j(t,T)]
\]

\[
= F_j(t,T) \exp \int_t^T r(t,s)ds \left(1 + \frac{P_j(T) - F(t,T)}{F_j(t,T) - F_j(t_0,T)}\right).
\]

Note that \(\frac{1}{\gamma(t)} = \frac{F_j(t,T)}{F_j(t,T) - F_j(t_0,T)}\).
The value of the combined holdings at time $T$ simplifies to:

$$ F_j(t, T) \exp \int_t^T r(t, s) ds \left( \frac{P_j(T) - F_j(t, T)}{F_j(t, T) - F_j(t_0, T)} \right) $$

(30)

However, $\gamma(T) = \left( \frac{P_j(T) - F_j(t, T)}{F_j(t, T) - F_j(t_0, T)} \right) \left( \frac{F_j(t, T)}{P_j(T)} \right)$ since $P(T) = F_j(T, T)$ by definition. This result shows that (30) is equivalent to:

$$ \gamma(T) \exp \int_t^T r(t, s) ds P_j(T). $$

(31)

**Implications**

Because an investment of $F_j(t, T)$ at time $t$ leads to the payoff shown in (31), $F_j(t, T)$ is its present value. $\gamma(T)$ is a random variable with respect to time-$t$ information. Hence a forward contract's price $F_j(t, T)$ can be interpreted as the present value of a random quantity $\gamma(T) \exp \int_t^T r(t, s) ds$ of the goods. We also note that our assumption of rational expectations ensures that at time-$t$, we expect that $\frac{\gamma(T)}{\gamma(t)}$ is non-negative. Otherwise economic rationality would be violated. At this junc-
ture, it is appropriate to emphasize that the particular relation developed for forward and futures prices implies the choice of a particular arbitrage strategy. This is implicit in the work of Cox, Ingersoll and Ross (1981) who develop a series of propositions on forward and futures prices in a continuous-time framework.

In the empirical literature, Koch and Kwaller (1984) and Kamara and Lawrence (1985) provide evidence that the financing strategies of professional dealers determine futures pricing at the margin. These authors' arbitrage strategies hinge on implicit assumptions that position limits, if any, are not violated. The larger the position in any forward or futures contract, the greater is the possibility of non-performance, in the event of adverse price movements, by the disadvantaged party, if the party who stands to benefit does not hold adequate guarantees. Hence the normalization condition $F_j(t_0, T) = 0$ is seen not only as an assumption about price movements but also as an assumption about the choice of arbitrage strategies, and of position limits.

Next we address the issue of the equivalence of forward and futures prices. Black (1976), Cox, Ingersoll and Ross (1981), Jarrow and Oldfield (1981) to name a few authors all develop propositions on the equivalence of forward and
futures prices in the context of nonstochastic interest rates. However the results we have shown indicate that the pricing relations in forward and futures markets depend on the particular arbitrage strategies deemed applicable to those markets. Hence the assumption of nonstochastic interest rates may be a necessary but not sufficient condition for price equivalency. Even when we disregard all the effects of market imperfections, which apply to futures and forward markets in general. In (34) and (45), see Section 2.4 of this chapter, we show that the following general relations hold for forward and futures prices respectively.

\[ F_j(t, T) = F_j(t_0, T) + E_t \left( \left( \frac{J_w(T)}{J_w(t)} \right) (P(T) - F(t_0, T)) \exp(\int_t^T r(s) - \rho)ds \right) \]

and,

\[ f_j(t, T) = f_j(t_0, T) - E_t \left( \left( \frac{J_w(T)}{J_w(t)} \right) (P(T) - f(t_0, T)) \exp(\int_t^T r(s) - \rho)ds \right) \]

Under an assumption of nonstochastic interest rates future spot rates equal forward rates implicit in the term structure, but \( F_j(t, T) \neq f_j(t, T) \) unless, in our model, \( F_j(t_0, T) = f_j(t_0, T) \). This result extends the work of the cited
It implies that consideration of default risk in these markets necessarily drives a wedge between futures and forward prices of similar goods. Unless the risk of default is the same in both types of markets.

Merton (1973), Fisher (1975), Cox, Ingersoll and Ross (1985, b) all show that individual optimal asset demands are functionally related to a measure of risk aversion. Merton also shows that there is a demand for the asset as a vehicle against unfavorable shifts in the opportunity set. Breeden (1984) agrees with Merton (1973) even though he disputes Merton's definition of hedging. The optimal demands for contingent claims on each good can be obtained in our model by solving the optimality equations (12) and (13). However, we do not explore this issue any further since it is not our primary point of focus. It is mentioned only as a logical prologue to the next section that establishes some precedents for our interest in the hedging properties of these derivative assets.
In the previous section of this chapter we utilized a version of an intertemporal asset-pricing model to derive valuation equations. A fundamental difference in our approach is that we explicitly consider the value of positions held in forward and futures contracts, in formulating our equation of wealth. This generalization of the wealth equation allows us to develop general valuation equations for both forward and futures contracts that are not restricted to the case of zero-valued contracts and do not entail any unduly restrictive assumptions on the sequence of past prices. As a result we show that the valuation equation derived by Richard and Sundaresan (1981), in the case of futures contracts, is only strictly applicable to a particular class of futures prices, namely settlement prices, and prices of newly issued contracts. It applies to intraday futures prices only as a limiting case where it is assumed that the rate of interest over each instant of time is identically zero. This is shown in Proposition 3.

One may wish to posit that in the context of a continuous-time model, the intraday trading period is infinitely divisible, and hence by considering infinitesimally short intervals of time, the approximate behavior of intra-
day prices is adequately addressed. However there are at least four objections to this argument:

(1) The shortest length of time over which interest is payable is one day and when we use discrete versions of the model for testing we are implicitly considering day-to-day price movements.

(2) Contracts are not marked-to-market continuously but rather at the end of each day of trading, and are subject to limits on price movements.

(3) Random discrete shocks to the price equilibrating process may occur during the intraday period, which are not directly correlated with the underlying state of the economy.

(4) Futures contracts are, in the main, not traded at settlement prices. Moreover observed settlement prices may not in fact be equilibrium prices, because of exchange-imposed limits.

Proposition 3 gives a general valuation formula for futures contracts between consecutive settlement dates. It however does not provide any clear indications on the causal factors that generate those price changes. Black (1976) correctly indicated that the intraday value of a futures contract is not necessarily zero, and emphasized
the distinction between a contract's value and its price. A contract may have an intrinsic value of zero but its price needs not be zero.

The settlement process in futures markets forces the intrinsic value of a futures contract to a prospective buyer to zero, as soon as settlement has been effected on existing contracts. However the value of a position in the same contract, that has been held prior to the last settlement date, may differ substantially from zero because of past cash flows. Two of the most recent studies on the intraday movements of futures prices are those of Trevino and Martell (1984) and Elton, Gruber and Rentzler (1984). Trevino and Martell (1984) summarize the works of several authors who addressed this issue using a variety of techniques, including runs analysis, spectral analysis and serial correlation, and we discuss their findings in some detail.

Specifically Trevino and Martell find evidence that (a) the intraday behavior of futures prices is strongly correlated with the volume of market activity, (b) the intraday behavior of prices is not homogenous over the life of a contract, (c) intraday serial correlation changes from small but significantly positive to not significantly different from zero, to strongly negative during the life of
the contract. These findings are corroborative evidence of the results of Kaen, Rosenman and Helms (1984) who also report evidence of persistent dependence in futures prices. These authors find nonperiodic cycles in both daily and intraday futures prices. This body of evidence indicates that one cannot disregard the possible links between past, current and future forward and futures prices.

The work of Elton, Gruber and Rentzler (1984) is also important since not only did they utilize intraday data in testing forward and futures relations, but they also used prices matched according to trading times. These authors examined a wide range of arbitrage strategies and found several instances where pure arbitrage profits, as opposed to quasi-arbitrage profits, were available. These findings together with those cited by Martell and Trevino's (1984) study cannot be attributed to the effects of market-imposed price limits, since only intraday data are utilized. Martell and Trevino (1984) make a strong argument that "transaction-to-transaction" price movements are dominated by temporary disequilibria and the way in which floor traders realign supply and demand. In summary there is compelling evidence that the volume and pattern of trading activity must be analyzed, if we wish to obtain a fuller understanding of intraday pricing.
In this section we explicitly recognize that the intraday value of a futures contract is generally not zero. This allows us to derive valuation equations for futures prices that cover,

(a) the time period between two settlement dates.

(b) futures prices across successive settlement dates.

We also derive general relations linking current forward and futures prices to expected future spot prices at the maturity of the respective contracts, to past prices. Sufficient conditions for the equivalence of our results with those obtained by other authors are also stated. These generalized relations provide the basis for alternative specifications of the link between contemporaneous forward and futures prices.

Keynes (1930) describes backwardation as the case which occurs when current spot prices exceed current forward prices. In his "Theory of The Forward Market", he explicitly considers a case where production takes time. The period for the productive process equals the time to maturity of the current forward contract. The price difference between current spot price and the forward price is the insurance premium paid by producers or hedgers, who are desirous of avoiding the risk of price fluctuations during
the production process. Hicks (1939) gives an analogous definition. On the other hand a contango is said to exist when current spot prices are less than current forward prices. This contango equals the cost of carrying stocks to the maturity date of the contract. A contango exists when there are excess current supplies of goods, in the Keynesian economy.

The modern versions, normal backwardation and normal contango, deal with the relation of current forward prices to expected future spot prices. In their current usage normal backwardation occurs when forward prices are less than expected spot prices i.e so that an increasing price trend is expected and a normal contango occurs when forward prices exceed expected future spot prices. The term-structure analogue of these arguments hinges on the existence of liquidity premia in interest rates. We cannot do justice to all of the authors who have attacked this problem, and so with this brief apologia, we summarize the main hypotheses: (1) the pure expectations hypothesis; (2) the liquidity preference hypothesis (3) the market segmentation hypothesis; (4) the preferred habitat theory.

Under the pure expectations hypothesis, expected future spot rates equal forward rates and there are no liquidity premia. In a pricing context, this corresponds to a mar-
tingale behavior of prices. The other hypotheses cited all assume the existence of some term premia for risk assumption. However McCulloch (1980), Woodward (1983) et al. demonstrate that either a liquidity or a solidity premium is compatible with interest rate uncertainty. These authors also show that both positive and negative liquidity premia, or even zero liquidity premia can be associated with risk-averse behavior by market participants.

This body of theoretical work is a clear indication that one should expect to find evidence of either normal backwardation or contango in futures and forward markets. There is no compelling evidence that the markets are pre-disposed to displaying a single characteristic. In the next subsection, we analyze the generalized version of the intertemporal model discussed in section 2.3 in order to ascertain what are the conditions, implied by the model, that would lead to the occurrence of contango or normal backwardation in forward and futures markets.

**Forward Contracts**

Previously we utilized the valuation equation:

$$J_w(\alpha_k - \phi_k) - V_{kr} J_w = 0$$

to derive a generalized version of a pricing relation for forward contracts on the $j$-th good, by using a simple transformation. Here we revert to
the valuation relation to show another extension of the results of Jarrow and Oldfield (1981) and Richard and Sundaresan (1981). We note the following results:

\[ J_w [a_{ij} - \phi_{1j} - V_{1jr}] - 0. \]

is equivalent to \( J_w L V_{1j} - V_{1jr} J_w = 0. \)

But \( J_w L V_{1j} = L (J_w V_{1j}) - V_{1j} L J_w \), and utilizing (16), we can easily show that \( J_w L V_{1j} - V_{1jr} J_w = 0 \) also equals.

\[ L (J_w V_{1j}) = \rho (J_w V_{1j}) = 0. \tag{32} \]

**Proposition 1**

The time-\( t \) value of a forward contract on good \( j \), written at time \( t \), with maturity date time \( T \), is

\[ V_{1j}(t) = E_t \left( \frac{J_w(T)}{J_w(t)} V_{1j}(T) \exp \int_t^T r(t,s) ds \right) \tag{33} \]

where \( V_{1j}(T) = P(T) - F(t_0,T) \), since we explicitly assume the convergence of forward and cash market prices.

and \( V_{1j}(t) = (F_j(t,T) - F_j(t_0,T)) \exp \int_t^T r(t,s) ds \)

are the respective boundary conditions.

\( 0 \leq t_0 < t < T \).

**Proof.** Apply Friedman's Theorem to equation (32).
The valuation relation (33) simply states that the current value to a long holder of a forward contract equals the discounted expected terminal value of the contract, adjusted for the rate of intertemporal substitution. It is assumed that the holders of contracts intend to hold their contracts to maturity.

From Proposition 1, we derive a relation linking current forward prices, past prices and the expected future spot price for the j-th good. From (33), we note that,

\[(F_j(t, T) - F_j(t_0, T)) \exp \int_t^T r(t, s) ds\]

\[= E_t \left( \frac{J_w(T)}{J_w(t)} \right)^{V_{lj}} \exp \int_t^T - \rho ds \]  \hspace{1cm} (34)

By definition:

\[\left( \exp \int_t^T r(s) - \rho \right) ds = \delta(t) \]  \hspace{1cm} (35)

\[1 - \left( \exp \int_t^T r(s) - \rho \right) ds \right) E_t \left( \frac{(J_w(T)/J_w(t))} = k(t) \right) \]  \hspace{1cm} (36)

\[\exp \int_t^T (r(t, s) - \rho) ds E_t \left( \frac{(J_w(T)/J_w(t))} = \delta^*_i (t) \right) \]  \hspace{1cm} (37)
Applying the covariance operator to the term in square brackets, we note that (34) implies that.

\[
F(t,T) - F(t_0,T) = \delta(t) E_t(P_j(T)) + \\
\delta(t) \text{cov}_t(J_w(T),P(T)) - F(t_0,T) \delta^*(t). \tag{38}
\]

We can further simplify (38) as:

\[
F_j(t,T) = F_j(t_0,T) \delta(t) + \delta(t) \text{cov}_t(J_w(T),P(T)) \\
+ \delta(t) E_t(P_j(T)) \tag{39}
\]

Equation (39) shows that in general the relation between forward and expected future spot price is linked to the past history of prices, the expected covariation of marginal utility and maturity price, the current structure of interest rates and the intertemporal rate of substitution, all adjusted for time preference. This extends the results of Richard and Sundaresan (1981) and French (1982), who demonstrate the influence of a hedging term, \( \text{cov}_t(J_w(T), P(T)) \) on the relation between forward and expected future spot price. These authors have stated results of the general form:

\[
F(t,T) = \frac{\text{cov}_t(J_w(T), P(T)) - F_j(t,T)}{E_t(J_w(T))} + E_t(P_j(T)). \tag{40}
\]
For further clarity we point out that (40) is equivalent to,

\[ F_j(t, T) = \text{cov}_t \left( \frac{J_w(T), P_j(T)}{E_t(J_w(T))} \right) + E_t(P_j(T)). \]  

(41)

Since \( \text{cov}_t(J_w(T), P(T)) = \text{cov}_t(J_w(T), P(T) - F_j(t, T)) \).

From (38), we note that if \( F_j(t, T) = F_j(t_0, T) \), the resulting equation after simplification, is equivalent to equation (40). \( F_j(t_0, T) = 0 \) is not a sufficient condition for the equivalence of our result and that specified in (40). The implication of (39), is that it clearly indicates that (40) is a special result which only holds for all currently zero-valued contracts, whether they were newly written or not. In the context of this model a sufficient condition for a forward contract to be currently zero-valued is that the initial trading price equal the current trading price. The price relation in (40) has been proposed as a measure of the existence of backwardation or contango in forward markets depending on whether forward contracts are good hedges against consumption and price risks.
Equivalently, we can restate this claim in terms of the sign of the covariance term. Normal backwardation occurs when the covariance term is negative and contango occurs when the covariance term is positive. If one is prepared to accord some influence to past prices on current prices, and to acknowledge that all forward contracts are not zero-valued, (39) indicates that knowledge of the hedging properties of forward contracts is not sufficient to determine whether normal backwardation or contango will occur in a particular forward market.

Futures Contracts

For a futures contract on good j, the appropriate valuation formula, when there are instantaneous stochastic payoffs is,

\[ J_w L V_{2j} = 0 \]  (42),

where \( L \) is the previously defined operator. See Equation (9). As we shortly show, if there is a lag between the change in the futures price and the payoff, the valuation formula is radically altered. The basic valuation formula when there are no lags between price changes and payoffs, (42) can be expressed as:

\[ L(J_w V_{2j}) + (r - \rho) V_{2j} J_w = 0 \]  (43)

This leads to the following proposition.
Proposition 2

The time-\( t \) realized value to its holder of a futures contract entered at time \( t \), with maturity date time \( T \) is:

\[
V_{2j}(t) = E\left[ \frac{J_w(T)}{J_w(t)} V_{2j}(T) \exp \int_t^T r(s) - \rho ds \right].
\] (44)

where \( V_{2j}(t) = f_j(t, T) - f_j(t_0, T) \).

and \( V_{2j}(T) = p_j(T) - f_j(t_0, T) \), are the respective boundary conditions and \( f_j(t, T) \) is a settlement price.

\( 0 \leq t_0 < t < T \).

Proof. Apply Friedman's Theorem to equation (43).

This version of the valuation equation strictly applies to futures contracts held over successive settlement dates. What is even more interesting is that Proposition 2 indicates that another version of the pricing equation is consistent with the basic model. This version is.

\[
f_j(t, T) = f_j(t_0, T) + E_t((J_w(T)/J_w(t))(P(T) - f(t_0, T)) \exp \int_t^T r(s) - \rho ds).
\] (45)
The above result can easily be verified by using a modified version of the forward plan with an initial investment of \( f_j(t,T) - f_j(t_0,T) \). Next we discuss the form of the valuation equation when we explicitly assume that there is a lag between price movements and settlements. This valuation equation is directly applicable to the intraday values of futures contracts. Under these assumptions, the valuation equation is

\[
J_W L v_{2j} = v_2 j r J_W = 0. \quad (46)
\]

Using the same substitution implied by (16), (46) can be expressed as

\[
L(J_W v_{2j}) - \rho (J_W v_{2j}) = 0. \quad (47)
\]

From (47), we derive Proposition 3.

**Proposition 3**

The intraday value of a futures contract at time \( t \), with successive settlement times \( t^* \) and \( t_2 \) is,

\[
v_{2j}(t) = E_t \int_{t_1}^{t_2} \frac{J_W(t_2)}{J_W(t_1)} v_{2j}(t) \exp^{t_2 - \rho ds} - t_1 
\]

\[
v_{2j}(t_1) = f_j(t_1, t_2) - f_j(t^*, t_2).
\]
and \( v_{2j}(t) = f_j(t_2, t_2) - f_j(t^*, t_2) \).

are the respective boundary conditions.

\[ t^* < t_1 < t_2. \]

Proof. Once again, Proposition 3, is just another application of Friedman's theorem.

The valuation relation in Propositions 2 and 3 differs fundamentally. The basic rationale for this difference is that the intraday value of a futures contract is an unrealized value. Realization of its value is only achieved by liquidation of the contract. Conversely, the valuation relation in Proposition 2, relates to realized gains or losses accruing from holding a futures position over successive settlement periods. We also note that interest rates do not explicitly enter the valuation equation when we are dealing with futures contracts held solely between settlement periods. Nevertheless, the rate of interest may enter implicitly into the analysis, in the way individuals form their expectations about the value of the contract at the next settlement date. Existing exchange-imposed price limits will also temper those expectations.

By definition,

\[
\exp \int_{t_1}^{t_2} - \rho \, ds = \delta. \tag{49}
\]
By using the covariance operator we can show that the relation implied by Proposition 3 is:

\[ f_j(t_1, t_2) = f_j(t^*, t_2) \kappa^* + \]

\[ \delta \text{ "} \text{Cov} \left( \frac{J_w(t_2)}{J_w(t_1)}, f_j(t_2, t_2) - f_j(t_1^*, t_2) \right) \delta \text{ "} \text{Et}_1(f_j(t_1, t_2)) \quad (52) \]

\( f_j(t_2, t_2) \) is the expected settlement price of the contract at the end of the day. \( f_j(t_2, t_2) - f_j(t^*, t_2) \) is the expected trading day movement of futures prices.

The current intraday futures price depends on the expected end-of-day price, the covariation of marginal utility with expected end-of-day profits, and the previous settlement price all weighted by a discount factor. By inspection, we note that (52) is merely a variant of (39). Proposition 3 and (52) jointly indicate that the intraday relation of futures prices closely conforms to the pricing relation we
derive for forward prices. We note that the hedging term against interest rate risk, no longer is applicable when we are dealing with an intraday position.

The reason for this is that individuals who are intraday traders are engaged in pure speculation. These traders do not hold any overnight positions in the market. For an individual who is a day trader, a futures contract, in isolation, can only provide a hedge against consumption risks and we cannot infer from the above relation whether or not these speculators will on average, make profits, since the direction of the bias of \( f_j(t_1, t_2) \) is also indeterminate. We complete our analysis in this subsection by establishing a general form of the relation between current, past and expected future spot prices for futures prices. We utilize (45) which is a general version of (28).

By definition,

\[
\exp \int_t^T (r - \rho) ds = \delta'(t) \tag{53}
\]

\[
1 - E_t(\exp \int_t^T (r - \rho) ds) E_t(\frac{J_W(T)}{J_W(t)}) = \kappa'(t) \tag{54}
\]

By definition.
After applying the covariance operator to (45) and simplifying, we obtain the following relation:

\[
\begin{align*}
E_t(\exp \int_t^T (r - \rho) ds) \frac{J_w(T)}{J_w(t)} &= \delta^{**}(t). \\
\end{align*}
\]  

(55)

Equation (56) is a result that holds for non-zero valued positions in futures contracts, and for zero-valued positions as a special case.

The results quoted by other authors derive from the assumption of a zero-valued position in futures contracts or equivalently that the contract is newly issued. In order to verify the equivalence of our result with that quoted in the literature, it is necessary to impose the following restriction. \( f_j(t, T) = f_j(t_0, T) \). Under this assumption (56) degenerates to,

\[
\begin{align*}
E_t(P_j(T)) + \text{cov}_t \left( \exp \int_t^T r(s) ds, \frac{J_w(T)}{J_w(t)} \right) + \delta^{**}(t) E_t(P_j(T)). \\
\end{align*}
\]  

(56)

\[
\begin{align*}
E_t(\exp \int_t^T r(s) ds) \frac{J_w(T)}{J_w(t)} &= \delta^{**}(t). \\
\end{align*}
\]  

(57)
Even in the extreme case where futures contracts are currently zero-valued, equation (57) shows that the prices of those contracts are generally not unbiased estimators of future spot prices. An obvious reason for this lies in the uncertainty regarding the future course of prices and interest rates even if we disregard the default risk issue. Nevertheless, this case is instructive.

\[ \text{cov}_t \left( \exp \int_t^T r(s) ds, (P_j(T) - f_j(t,T)) \right) \text{ is a measure of the covariation of interest rates with the expected profit on holding a contract to maturity. For interest rate contracts it is the case that this term will generally be negative. This is true for the Treasury Bill contract. However we cannot make any definitive statements about its sign when the relevant contracts are commodity futures contracts. Pricing of commodity futures contracts is generally dominated by factors other than the rate of interest. These other factors include inventory levels, production forecasts, buffer stocks as well as current market demand factors.} \]

The second covariance term, \( \text{cov} \left( (J_w(T), \exp \int_t^T r(s) ds (P_j(T) - f_j(t,T)))) \right) \) measures the covariation of marginal utility of wealth or of consumption, with the interest-rate adjusted profit from holding a futures contract. This term is the consumption-risk factor of Richard and Sundareshan (1981).
This term may be either positive or negative depending on whether futures contracts are good consumption hedges.

The issue of whether contango or backwardation will be observed cannot be definitively resolved, except with the imposition of severe restrictions. The numerical value and sign of \( \text{cov} \left( \left( J_{W}(T), \exp \int_{t}^{T} r(s) ds \right) (P_{j}(T) - f_{j}(t, T)) \right) \) may vary across contracts and will be related to economy-wide conditions. However, when we consider the more general case, the possibilities are that the covariance-weighted terms in (56) may offset, strengthen or nullify each other's influence. It therefore becomes clear that even if we know beforehand that futures contracts are or are not good consumption hedges, we cannot tell whether contango or backwardation will be observed. Hence the result stated by Richard and Sundaresan (1981), for zero-valued contracts, also holds for the case where these contracts are non-zero valued.

Normal Backwardation and Normal Contango

Equation (39) shows that the following general relation holds for forward prices:

\[
F(t, T) = F(t_0, T) + \delta(t) \text{cov}_w \left( J_{W}(T), P(T) \right)
\]
It is evident from the above relation that in general forward prices are not unbiased estimators of future spot prices. This result is already well established in the literature but our formulation gives an added reason for these biases, namely the influence of past prices. In order to negate the influence of past prices on current prices it is necessary to assume that either or both of $F(t_0, T)$ or $K(t)$ be identically zero. In terms of any empirical work $F(t_0, T) = 0$ is an untenable assumption, and we have no valid a priori reason to believe that $K(t) = 0$. In general $K(t)$ is either negative or positive, its numerical magnitude and sign depends on the level of interest rates, the inter-temporal rate of substitution, and the discount factor $p$. However, for forward prices to be unbiased estimators of future spot prices, we need not only to negate the influence of past prices but also to assume that $\text{cov}_t(J_w(T), P(T)) = 0$.

$\text{cov}_t(J_w(T), P(T))$ is a measure of the forward contract's role as a hedging instrument against consumption or price risk. In our framework, the conditions for the unbiasedness of forward prices become very stringent. Past prices must have no impact on currently observed prices, and the
forward contract must have no role as a hedging instrument, or at least its hedging value must be negated by the time-preference-adjusted interest rate factor $\delta(t)$. In addition $\delta^*(t)$ must equal 1.

The only logical conclusion that one can therefore arrive at is that either normal backwardation or contango is likely to be observed in forward markets depending on the relative signs and magnitudes of $\delta(t)$, $\delta^*(t)$ and $\text{cov}_t(J_w(T), P(T))$. As regards futures markets, we are also forced to conclude that models based on martingale properties of futures prices, may be criticized because they assume very stringent conditions. In equation (56), we showed that the following general relation held for futures prices, in this model,

$$f_j(t, T) = f_j(t_0, T)\kappa'(t) + E_t\left(\frac{J_w(T)}{J_{w}(t)}\text{cov}_t(\delta'(t), P_j(T))\right)$$

$$+ \text{cov}_t\left(\frac{J_w(T)}{J_{w}(t)}, \delta'(t) (P_j(T) - f_j(t_0, T)) + \delta^{**}(t) E_t(P_j(T)).\right)$$

Futures prices reflect the influence of past prices, as do forward prices, and similar arguments obviously apply. We have no a priori reason to expect that $\kappa = 0$. What is interesting is the fact that this relation shows that a futures contract provides a hedge not only against consump-
tion and price risk but also against interest rate risk. In addition, current futures prices are shown to be functionally related to past futures prices. At this juncture we introduce the following caveats: (a) the solutions we derive are not unique solutions and (b) no satisfactory way has been found to deal with intraday prices in the context of this model. What we have achieved is the specification of alternative representations of forward and futures prices. This allows us to test the basic model for the influence of what we believe to be misspecification. In addition, we demonstrate that knowledge of the hedging properties of these contracts may not be sufficient to determine whether contango or normal backwardation occurs in any market. We also derive and present portfolio strategies which lead to the general pricing results we state.

Finally, we show that the arbitrage strategies cited by other authors are only applicable to the pricing of forward and futures contracts when past prices are disregarded or contracts are newly issued. The results we derive do not violate any a priori expectations that one may logically hold. Moreover they emanate from the fundamental notion, that in the absence of perfect performance guarantees and position limits, default risk exists, and the delivery of goods on both types of contracts becomes a risky proposition.
Appendix A

Differentiated Time Preferences and Endowments in the General Equilibrium Model

We assume in complete generality, that there exists an arbitrary number $N$ of individuals in each group $f$, $f=1, 2$. Each individual in a given group has similar endowments and tastes to that of any other member of his group. Let $q_1$ be the endowment held by the representative individual of group 1. Similarly let $q_2$ be the endowment held by the representative individual of group 2. Both $q_1$ and $q_2$ are vectors. The representative individual of each group is assumed to hold his wealth at any point in time in contracts, goods and a riskless asset.

The productive process is such that each commodity is used in the production of all $n$ commodities. Debreu and Scarf (1962) demonstrate that if a competitive solution exists for an economy, whose members can be divided into different groups, each group being differentiated according to taste and endowments, with the same number of individuals in each group, then all individuals in a particular group will have the same competitive allocation.
We draw on this result to reduce the dimensionality of the maximizing problem by considering one individual from each group. Let \( k_{ij} \) be the amount of good \( j \) used in the production of the \( i \)-th good by an individual from group \( f \), \( f = 1, 2 \). This implies that \( q_j = \sum_{i=1}^{N} (k_{1ij} + k_{2ij}) \) is the amount of good \( j \) invested in production at any arbitrary time \( t \). By definition \( \sum q_j = w^* \), where \( w_1 \) and \( w_2 \) are the respective wealth levels of the 2 representative individuals. For the group 1 individual,

\[
p'q_{i1} + \sum_{k=1}^{2} M_k V_k + \lambda_1 = w_1. \tag{A1}
\]

Similarly for the group 2 individual we note that,

\[
p'q_{i2} + \sum_{k=1}^{2} L_k V_k + \lambda_2 = w_2. \tag{A2}
\]

\( M_k \) and \( L_k \) denote the vectors of the numbers of contracts of type \( k \), held by the representative individuals of each group. We now state that the vector of the quantities of each good invested in the production of each good \( i \) is \( k_i = k_{1i} + k_{2i} \). For each individual the amount of each good allotted to the productive process has an upper bound his particular endowment level. We define \( q_{1i} + q_{2i} \) to be equal to \( q^* \). Market equilibrium conditions require that

\[
\lambda_1 + \lambda_2 = 0. \tag{A3} \text{(equilibrium in the market for loanable funds)}
\]
This implies that \( p'q = W^* \). However all wealth of a representative individual is not necessarily in physical goods. A representative consumer may hold some of his wealth in contracts that are non-zero valued and may engage in riskless lending or borrowing. Furthermore each individual tries to maximize expected utility in the form,

\[
E_t \left( \exp \int_t^T - \rho_i(t) u(c_i^*(t)) \, dt \right)
\]

where \( i = 1, 2 \).

\( \rho_i \) is the subjective rate of time discount of the group \( i \) representative. Each representative individual, at any arbitrary time, has past information on the realized values of the stochastic process in the economy and on the basis of this information set forecasts the possible future values of the processes. Even though representative individuals' information sets are complete with respect to past and current information, they will likely estimate future values of the processes with error.
$c_1^*$ is the vector of instantaneous rate of consumption of the representative consumer $i$. By definition $c^* = c_1 + c_2$.

The stochastic differential vector equation covering output of goods is now defined by:

$$dq = (h - c^*) \, dt + G \, dz.$$  

See equations 3-9, as well as footnote 12 for the development of the stochastic Bellman equation and the differential operator $L$. Here we note that the revised equation of wealth is:

$$dW^* = \left[ p'(h - c^*) + q'q + tr \sigma_p G' + \sum_{k=1}^{2} (M_k + L_k)(\alpha_k - \nu_k r) + (W^* - p'q)r \right] \, dt$$

$$+ \left[ p'G + q' \sigma_p + \sum_{k=1}^{2} (M_k + L_k)H_k \right] \, dz. \quad (A7)$$

We define the equation of wealth as:

$$dW^* = \beta_{W^*} \, dt + \sigma_{W^*} \, dz. \quad (A8) \quad \beta_{W^*} \text{ and } \sigma_{W^*} \text{ being the terms in square brackets in (A7). Furthermore we define an indirect value function } J(W, Q, Y) = e^{\rho t} I(W, Q, Y, t), \text{ where } \rho \text{ is a "community" rate of time discount. The requisite Bellman equation is then derived. Since } N, \text{ the number of consumers in each group is arbitrary, we can without loss of generality set it equal to 1. The Bellman equation for the resulting maximizing problem then reduces to.}$$
where $L$ is the operator defined in (9).

We define $\exp(\rho - \rho_2)t$ as $\alpha^*_2(t)$ and $\exp(\rho - \rho_1)t$ as $\alpha^*_1(t)$ and then express the Bellman equation as,

$$0 = \max \alpha^*_1(t)u_1(c^*_1(t)) + \alpha^*_2(t)u_1(c^*_2(t)) + L\rho J - \rho J$$

(A9)

We denote the expression in (A9) to be $\psi^*$. The first order conditions are,

$$\psi^*_{c^*_i} = \alpha^*_i(t)u_1(c^*_i(t)) - p_iJw^* = 0; \ i = 1, 2, \ldots, N. \tag{A10}$$

$$\psi^*_{c^*_2} = \alpha^*_2(t)u_1(c^*_2(t)) - p_iJw^* = 0; \ i = 1, 2, \ldots, N. \tag{A11}$$

$$\psi^*_{M_k} = \psi_{L_k} = Jw^* \left( a_k - f_k - Vk^* \right) + Jw^*HkS^*w^* + \sum_{j=1}^{k} Jw^*Y_jHkS^j_j + \sum_{i=1}^{N} Jw^*Q_iHkD^i_j = 0. \tag{A12}$$
\[ \psi_{k_{i+j}}^1 = \psi_{k_{i+j}}^2 = 0 \] are the other first order conditions. Equation (A12) is the relevant valuation relation that generates our pricing formulas. This implies that, in equilibrium, the valuation equation for each type of individual is identical. Moreover \( J_{W_2} = J_{W_1} \) for the two representative individuals, who hold contracts and hence can be termed long and short holders. This can be made clear by the following result, since \( W^* = W_1 + W_2 \) by assumption. \[ J_{W_1} = J_{W_2} \frac{\partial W_2}{\partial W_1} \]

But \[ \frac{\partial W}{\partial W_1} = \frac{\partial W}{\partial W_2} = 1. \] This implies that \( J_{W_1} = J_{W_2} \)

Intuitively we note that a Pareto optimum may not be achieved if \( J_{W_1} \neq J_{W_2} \), since it may be possible, by means of a wealth redistribution to increase the welfare of the economic agents taken jointly. The representative individuals who hold contracts are termed long and short holders respectively. What this model extension permits is simply to allow for the possibility that equilibrium prices exist at prices other than the issue or initial trading price.

Under conditions in which all consumers are identical it is not possible that contracts exist in any meaningful way, since all economic agents will pursue identical portfolio
strategies, and they all cannot simultaneously lose or gain on their holdings of contingent-claim contracts. Since the results we derive do not preclude the basic pricing relations in the case of one class of economic agents, our model extension is not restrictive. Moreover, the analysis we conduct shows that a sufficient condition for individuals, in an equilibrium context, to hold long and short positions in contracts, is the existence of differentiated rate of time preferences. A more general approach would of course be to explicitly assume that individuals have different expectations or different information, about the nature of the underlying stochastic processes, and consequently at every temporary price equilibrium, short and long contract holders possess different beliefs about the future direction of prices.
CHAPTER III
FORWARD-FUTURES DIFFERENTIALS IN THE TREASURY BILL MARKET.

Several authors including Cornell and Capozza (1979), Arak and McCurdy (1980), Arak (1983), Lang and Kasche (1978), Rendleman and Carabini (1979), Kane (1980) and more recently Koch and Kwaller (1984), have detailed the existence of persistent divergences between forward and futures rates or equivalently between forward and futures prices in the T-Bill market. These interest rate differentials are generally an increasing function of the time to maturity of the relevant contracts and do not appear to have narrowed over time. Most of the older studies showed that futures rates always exceeded the forward rates implicit in cash-market prices, with the exception of the spot contract.

However, forward-futures price differences and interest rate differences vary not only with respect to the maturities of underlying contracts but also across sample periods because market conditions change. The relations between implicit forward and futures T-Bill prices, or their trans-
forms the forward and futures interest rates, are important for several reasons. In the first instance, these market prices contain information about the expected course of interest rates and their structure and is of especial interest to those concerned with the direction of monetary and fiscal policy matters as well as professional dealers, financial institutions and market speculators alike.

The second reason revolves around the price-discovery function of forward and futures markets in general. (See Garbade and Silber (1983). 1979). This is not logically separate from the first reason even though they are sometimes treated separately in the literature. The price-discovery role of futures markets as defined in Garbade and Silber (1983) hinges on whether new information is incorporated in futures prices before cash market prices or whether the order of information flows is reversed. Regardless of the order of information flows, it is known that futures, forward and cash markets are integrated to some degree. The nature of their integration depends on transactions costs and market liquidity in both cash and futures markets. However the very nature of this interaction can obscure the reliability of the price-discovery function. Arbitrage strategies in the markets may lead to price movements that are not related to market fundamentals.
The third reason is linked to the risk-transfer function of markets for deferred delivery of goods in general. Whenever forward and futures markets coexist, individuals will utilize that market in which they can transfer perceived risk at the lower cost. Hence, the relation between forward and futures prices is of paramount importance if we are to understand the actions of individuals engaged in risk-transferring arbitrage and speculative activities.

In studies which deal with the relationship between prices in different markets, there is substantial opportunity for timing effects to obscure the true relation. This concern is implicit in the work of Lang and Rasche (1978), in which the issue of quotation-date bias is addressed by a randomization procedure. In addition, the divergence between futures and cash market prices, due to carrying costs, tends to disappear as the settlement date approaches. The same is perhaps true for forward markets in general even though in this instance the cost-of-carry may not be readily estimable. If these postulated effects hold, it implies that the limiting value of implied forward and futures price differentials should also be zero. This suggests that it may be impossible to adequately explain forward-futures differentials if the time to maturity is too short.
In this study, interesting types of variation are those that are evident in intersample comparisons. The variation that is evident in intersample comparisons is mainly attributable to altered delivery rules, position limits, margin conditions as well as tax treatment. Studies in the former category have attempted to test for tax effects in futures pricing (Cornell (1981)), the relationship between volume and price variability (Cornell (1981) and Grammatis-Kos and Saunders (1986)), the narrowing of forward and futures price differences over time as open interest in the market increased (Lamp and Rasche (1978)), shifts in forward-futures relation (Arak (1983)), and the primacy of cash-and-carry arbitrage in futures pricing (Koch and Kwaller (1984)).

However, several unanswered empirical questions remain. What is the relation between futures price variability in the T-Bill market and the costs of funding positions in these markets? What is the explanatory role, if any, of interest rate differentials in equilibrium models of forward and futures pricing? This chapter is organized as follows. In Section 1, we describe our data and data sources. In Section 2 we present alternative model specifications of forward and futures prices and we derive testable hypotheses on parameters emanating from these different specifications. Section 3 is devoted to tests for specifi-
cation error in both models, since it is possible that neither model is the true model. Section 4 analyzes the performance of implicit forward and futures prices as estimators of maturity prices in the Treasury Bill market, and the forecast errors of these estimators are used to determine the stable properties of these prices. Section 5 addresses the issue of tax effects. The general-equilibrium model in Chapter 2 is also extended to incorporate taxation and we assess the model implications with and without wealth effects, but with tax exogeneity.
THE DATA

We utilize Treasury Bill futures data obtained from the Wall Street Journal to calculate the settlement prices of the respective contracts. T-Bill futures contracts are quoted at a discount from 100, and this differs from the actual settlement price. The value of the deliverable unit on a Treasury Bill is calculated according to the settlement pricing formula shown below.

Settlement price = 1,000.0000 \times (1 - \text{D1}(\text{TY})/360)

D1 is the number of days from delivery date to maturity date of the deliverable instrument.

TY is the treasury bill yield, which is the difference between the I.M.M. index and 100, multiplied by .01. The futures pricing data utilized are the end-of-trading quotations. The repurchase rates used are the rates of interest specified in repurchase agreements actually concluded.

Data series on repurchase rates were obtained from Bank of America Capital Markets Group.
These data series are 1-day, 30-day and 60-day repurchase rates. The repurchase rate series are only available on a daily basis from June 1984. Prior to this, the quoted repurchase rates are based on weekly quotations. No quotations are available in the intra-week period, nor on the bid-ask range from the Bank of America (B.O.A) data. Weekly B.O.A. quotations are available on a continuous basis from 13 February 1980 up to 27 June 1985.

Overnight (1-day) repurchase rates obtained from the New York Federal Reserve Bank, via Telerate, are available almost continuously on a daily basis from 1 January 1976 to 12 June 1985. Both bid and ask rates are quoted. Since neither the B.O.A. nor the New York Federal Reserve Bank (N.Y.F.R.B.) 1-day repurchase rate data series is complete, the data series are joined, with the bulk of the observations being from the N.Y.F.R.B. data series. We also use 1-day Federal Funds rates obtained from N.Y.F.R.B. This data series is used as an alternative to the 1-day repurchase rate, in the event that the dominant market participants' cost of funds may be more closely correlated with the Federal Funds rate. The Federal Funds rates used are closing rates.

Due to the limitations of the relevant data series, our analysis is restricted to the period from 13 February 1980
to 27 June 1985. To assess the behavior of the model, we conceive of the data set as being replications of futures and implicit forward pricing behavior at a specific time-to-maturity interval. This time interval is set at 30 days since it is the only time for which reasonably complete term rates series were obtained.

Data for the corresponding period on implicit forward prices on Treasury Bills are generated from end-of-day cash-market quotations collected from the Wall Street Journal. The implicit forward prices used are calculated from the means of bid-ask quotations. This is a second best procedure, because individuals cannot actually trade at the bid-ask mean prices. However, we explicitly set out to evaluate the performance of these models in a framework where theoretical implied forward prices are compared with futures prices.

The theoretical implied forward price is calculated according to

\[
(P(t, T+91)/P(t, T)) (1000000)
\]

where \( P(t, T) \) is the cash price of a T-Bill at time \( t \) with \( T \) days to maturity.
(b) \( P(t, T+91) \) is the cash price of a T-Bill at time \( t \) with \( T+91 \) days to maturity.

Cash prices in the T-Bill market are generated from the formula:

\[
d = \frac{(100 - P)}{\left(\frac{360}{n}\right)}
\]

where \( d \) is the quoted discount rate applicable.

\( n \) is the number of days to the maturity of the bill.

\( P \) is the price per $100 face value of the bill.

In periods where there were no bills of the requisite maturity needed to calculate the theoretical implied forward price, the mean bid-ask rates were estimated by linear interpolation.

We examine 22 different contracts, at 30 days to maturity, and find that with the exception of the September 1983 contract, the implicit forward price is generally lower than the futures price. The regularity of this relation substantiates the results of previous studies involving spot contracts. Those studies show that, for the near contract, forward rates exceeded the futures rates.

We also examine the behavior of the differential at 31 days to maturity and find that, as a generalization,
The same relation holds true. However, the frequency of occurrence of positive differentials is greater than for the previous case. The absolute variation of the differentials, as a percentage of their associated contract settlement price ranges from a high of 0.33% to a low of 0.00%. These results are shown in Tables 1-2.
Table 1

Forward-Futures Price Differentials At 30 Days To Maturity

<table>
<thead>
<tr>
<th>Contract</th>
<th>Differential</th>
<th>% of Settlement Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mar. 1980</td>
<td>-2405.50</td>
<td>-.25</td>
</tr>
<tr>
<td>Jun. 1980</td>
<td>-1080.68</td>
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</tr>
<tr>
<td>Sept. 1980</td>
<td>-56.12</td>
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<td>Dec. 1980</td>
<td>-1817.73</td>
<td>-.19</td>
</tr>
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<td>Mar. 1981</td>
<td>-3224.97</td>
<td>-.33</td>
</tr>
<tr>
<td>Jun. 1981</td>
<td>-882.93</td>
<td>-.09</td>
</tr>
<tr>
<td>Sept. 1981</td>
<td>-1716.66</td>
<td>-.18</td>
</tr>
<tr>
<td>Dec. 1981</td>
<td>-45.77</td>
<td>-.00</td>
</tr>
<tr>
<td>Mar. 1982</td>
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<tr>
<td>Jun. 1982</td>
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<td>-.09</td>
</tr>
<tr>
<td>Sept. 1982</td>
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</tr>
<tr>
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</tr>
<tr>
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</tr>
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<td>-.03</td>
</tr>
<tr>
<td>Sept. 1983</td>
<td>33.00</td>
<td>-.00</td>
</tr>
<tr>
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<tr>
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<tr>
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<tr>
<td>Dec. 1984</td>
<td>-1213.99</td>
<td>-.12</td>
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<td>Mar. 1985</td>
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<td>-.09</td>
</tr>
<tr>
<td>Jun. 1985</td>
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<td>-.05</td>
</tr>
<tr>
<td>Contract</td>
<td>Differential</td>
<td>% of Settlement Price</td>
</tr>
<tr>
<td>-----------</td>
<td>--------------</td>
<td>-----------------------</td>
</tr>
<tr>
<td>Mar. 1980</td>
<td>-1052.01</td>
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</tr>
<tr>
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<td>1003.43</td>
<td>.10</td>
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<td>Dec. 1982</td>
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<td>Jun. 1983</td>
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<td>.01</td>
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</tr>
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<tr>
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<tr>
<td>Mar. 1985</td>
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</tr>
<tr>
<td>Jun. 1985</td>
<td>-596.78</td>
<td>-.06</td>
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</table>
MODEL SELECTION, INTEREST RATE DIFFERENTIALS AND EXPLANATORY POWER

A theoretical premise, implicit in the work of Richard and Sundaresan (1981) and French (1982), on forward and futures pricing suggests that the differential between forward and futures prices should be related to the difference between short-term and long-term rates of interest. In this chapter we are concerned with the empirical issue of whether it can be demonstrated that the forward-futures price differential is explainable in terms of interest-rate differentials. French (1982) asserted, on the basis of his analysis of price data on the copper and silver markets, that interest-rate differences were not useful in explaining price differentials in those markets. However it is not obvious that those findings are not market-specific, or that they can be generalized to financial contracts.

Interest rates, even if they have a causal role in the pricing of copper and silver contracts, may not be the most dominant explanatory factor in the pricing of those contracts. We believe that the T-Bill market provides a better testing ground, since interest rates directly determine the prices of futures contracts on T-bills. We propose to
test the hypothesis using implied forward prices and futures prices, and repurchase rates which may closely parallel market participants' costs of funding their positions in these markets. The issue of which model specification is most appropriate for explaining forward-futures differentials is an unresolved one. Several of the models we discuss derive from the premise that particular riskless strategies can be utilized to price forward and futures contracts at the margin. Other studies which have pursued time-series and general-equilibrium explanations, have to a large extent also overlooked the issue of default risk.

Our work and that of Kane (1980), clearly shows that the foundations on which such models are built do not preclude the existence of default risk. Market completion costs, which must be considered, play a causal role in driving a wedge between forward and futures prices. Empirical work in this area has been difficult because of the absence of a reliable data series on repurchase rates. The latter problem has been alleviated, but tremendous difficulties remain in identifying cost functions for performance guarantees and transacting costs. Moreover, each class of models presents its own peculiar set of challenges. We do not claim to have a general econometric framework in which to resolve these issues.
What we attempt to do is more modest. We restrict the set of models to be tested to those of Richard and Sundaresan and our model, which is exposited in Chapter 2. We derive testable hypotheses for both models and then ascertain whether the implied parameter restrictions are supported by the empirical data. Hereafter we refer to the model of Richard and Sundaresan as model A and to our model as model B. The expectational assumptions underlying models A and B are that market participants form their expectations rationally and are aware of the underlying stochastic processes influencing prices in the economy. They however do not possess perfect foresight. The states of nature which they envisage based on current information may not occur. The markets are incomplete and individuals will generally err on the true prices that will prevail in the economy.

These errors occur in forecasting the prices of goods and a fortiori, the prices of contingent claims on those goods. Forecast errors based on expectations at time t, with respect to future values of the processes are likely to be randomly distributed rather than follow a systematic pattern. We do not expect contract holders to continuously err about the direction of movement of prices and still retain their positions in these contracts. It is sufficient for our model, that for the particular investment horizons
considered, long and short holders respectively believe that their particular expectations will be realized. Models A and B incorporate parameters whose values are not currently observable and parameters whose values are not readily extracted even from ex-post data.

The usual recourses in dealing with models involving future values of current variables are to forecast such values or to utilize the values actually realized at the future date. It is possible to merely utilize estimated future spot interest rates and prices from time-series data, but even this is a second-best procedure. This procedure would ignore the actual correlations among the explanatory variables. However, we have no means of quantifying, from market data, the interactions of these variables. In particular we do not have any means of acquiring estimates of the marginal utility of the numeraire good or of wealth.30

We also cannot judge beforehand the likely effect of any specification error, caused by our use of forecasted values of interest rates and prices as proxies, on the robustness of the estimators. Neither can we tell whether the magnitude of such specification errors is likely to be greater or less than in the case where we simply utilize realized values of observable variables as proxies. The case for
the latter alternative is implicitly supported by Cox, Ingersoll and Ross (1985 a), who state "if rationality requires that ex-post realizations not differ systematically from ex-ante views, then statistical tests can be made of ex-ante propositions by using ex-post data".

Use of the realized values of the explanatory variables as proxies simplifies the estimation problem. In particular we no longer require any explicit consideration of the wealth or consumption variable, which is eliminated from the regression equation. This procedure is utilized by French (1982) in his test of Model A with copper and silver data. For ease of comparison, this approach is used here also. We recall that the equations for forward and futures pricing in this case are:

\[ F_j(t,T) = E_t \left( \frac{J_w(T)}{J_w(t)} (P_j(T)) \exp \int_t^T (r(t,s) - \rho) ds \right) \]  
(25)

and

\[ f_j(t,T) = E_t (\left( \frac{J_w(T)}{J_w(t)} \right) (P(T)) \exp \int_t^T r(s) - \rho) ds \]  
(27)

Model A therefore implies that:

\[ F(t,T) - f(t,T) = \text{Cov}_t (J_w(T), P(T)) / E_t (J_w(T)) - \]
(\text{cov}_t (\exp_\mathcal{T} r(s)ds, (P(T) - f(t,T))) / \mathcal{E}_t (\exp_\mathcal{T} r(s)ds) - \\
\text{cov}_t (J_w(T), (\exp_\mathcal{T} r(s)dsP(T) - f(t,T))/ \\
\mathcal{E}_t \exp_\mathcal{T} r(s)ds \mathcal{E}_t (J_w(T))^\) 

Equation (58) specifies an exact relation involving forward and futures prices. However, this relation is neither analytically tractable nor easily testable. To obviate this difficulty, French (1982) assumed that the product of the marginal utility of the numeraire commodity and the maturity price of the contract is independent of the nominal rate of interest, over the holding period.

In the general-equilibrium model which we address, we note that \( P(T) \) reflects the time-\( t \) expectations of the settlement price of a T-Bill. \( P(T) \) is a random variable with respect to time-\( t \) information. The interest rate that determines \( P(T) \) is the 90-day T-Bill rate since the delivered instrument actually matures 90 days after the maturity of the contract.

If we assume risk-neutrality, the marginal utility of wealth is constant over time. In this case, utilizing the realized values of repurchase interest rates and the settlement price is an implicit assumption of expected cash
correlation between the 90-day T-Bill rate and the cumulated 1-day repurchase interest rates over the 30-day holding period prior to the contract's maturity.

Under this assumption, the ratio of forward to futures prices for the j-th pool simplifies to,

\[ \frac{F(t,T)}{f(t,T)} = \frac{\exp \int_t^T r(t,s)ds}{\mathbb{E}_t \left( \exp \int_t^T (r(s)ds) \right)} \]  

(59)

By definition \( \beta_1(t,T) \exp \int_t^T r(t,s)ds \)

\[ = \beta_1(t,T) \exp \left( (T-t)r_{T-t} \right) = 1. \]

where \( r_{T-t} \) is the per diem equivalent of the \((T-t)\)-day loan rate.

Since the future spot rates of interest are not observable at time \( t \), the use of realized future spot rates as proxies means that a discrete operational version of (59) is.

\[ \frac{F(t,T)}{f(t,T)} = \exp(T-t)r_{T-t})/\exp \left( \sum_{j=t}^{T-1} r(j) + c_t \right) \]  

(60)

is the error incurred in using the realized cumulative 1-day rates as forecasts of the cumulative future spot rates. By taking the natural logarithm of (60) and rearranging terms we obtain.

\[ (T-t)r_{T-t} - \sum_{j=t}^{T-1} r(j) = \ln F(t,T) - \ln f(t,T) + c_t. \]  

(61)

which forms the basis for regression A.
We define.

\[ y = (T-t)r_{T-t} - \sum_{j=t}^{T-1} r(j) \]

as the difference between funding at the long-term rate and on a daily basis over the holding period.

\[ x_{1t} = \ln F(t, T) - \ln f(t, T) \]

is the logarithmic premium of the forward price over the futures price.

Three versions of \( y \) are used in our regressions. These are:

- \( y_a \) which is based on B.O.A. 30-day repurchase rates and 1-day repurchase rates from N.Y.F.R.B..
- \( y_b \) which is based on B.O.A. 30-day repurchase rates and 1-day Federal Funds rates from N.Y.F.R.B..
- \( y_c \) which is based on B.O.A. 30-day repurchase rates and 1-day B.O.A. repurchase rates, but based on weekly quotations.

The hypothesized relation between theoretical implicit forward and futures prices and interest rates can be tested by analyzing the estimated coefficients of regression model A:

\[ y_{1t} = \beta_0 + \beta_1 x_{1t} + \epsilon_t \]
Under the null hypothesis $B_0 = 0$ and $B_1 = 1$. The error term is assumed to be normally and independently distributed. The alternative hypothesis is $B_0 = 0$ and $B_1 \neq 1$. Since we have no cogent arguments about the influence of $x_1$ on $y$.

The generalized versions of forward and futures pricing that we have developed in Chapter 2, present even more formidable difficulties in arriving at a testable model specification than does the Richard and Sundaresan model. If the alternative model specifications are to provide any improvement over the existing specification tested by French, it would seem desirable that the generalized model specifications converge to that of the basic model as $F(t,T) = F(t_0,T)$ and $f(t,T) = f(t_0,T)$. However we have not been able to establish conditions for the convergence of the two model specifications except in the trivial case where $F(t_0,T) = f(t_0,T) = 0$.

We recall that the generalized pricing formulas for forward and futures contracts are:

$$F(t,T) = F(t_0,T) +$$

$$E_t((J_w(T)/J_w(t))(F(T) - F(t_0,T)) \exp\int t_r(s) - \rho ds)$$

and
\[ f(t, T) = f(t_0, T) + \]

\[ E_t\left( \left( \frac{J_w(T)}{J_w(t)} \right)(P(T) - f(t_0, T)) \exp \int_t^T r(s) - \rho ds \right) \]

(45)

Under these assumptions (34) and (45) reduce to:

\[ F(t, T) - F(t_0, T) = \]

\[ E_t\left( \exp \int_t^T (r(t, s) - \rho) ds \right) E_t\left( (P(T) - F(t_0, T)) \right) \]

(63)

and,

\[ f(t, T) - f(t_0, T) = \]

\[ E_t\left( \exp \int_t^T r(s) - \rho ds \right) E_t\left( (P(T) - f(t_0, T)) \right) \]

(64)

If \( F(t_0, T) = f(t_0, T) = 0 \), which is an unacceptable assumption, we can generate the basic specification by taking the logarithm of the ratio of (63) and (64). But since \( F(t, T) - F(t_0, T) \) and \( f(t, T) - f(t_0, T) \) may assume both positive and negative values independently of each other the simple expedient of utilizing the realized values of the variables, in a discrete version and taking the natural logarithm of the ratio of (34) and (45) is not helpful.
The way that we proceed to test Model B is suggested by an alternative representation of Model A, which we call Model A1. We assume risk neutrality, \( \rho \) equals zero and utilize the realized values of the variables in (25) and (27) as proxies. Under this assumption, (25) and (27) reduce to:

\[ F(t, T) = P(T) \exp \int_t^T r(t, s) ds \]  

and

\[ F(t, T) = P(T) \exp \int_t^T r(s) ds \]  

We can motivate (61) at this stage by simply taking the logarithms of (65) and (66) respectively and taking their difference. The alternative approach we use normalizes (65) and (66) by \( P(T) \). We then take their differences and this leads to the alternative regression specification in discrete form.

\[ \frac{F(t, T) - f(t, T)}{P(T)} = \exp(T-t)r_{T-t} - \exp \sum_{j=t}^{T-1} r(j) + V_t. \]  

(67)

\( V_t \) is the sum of the error introduced into the model by using the realized values of future spot rates and price as proxies. We assume that the error introduced in the dependent variable by using a proxy for \( P(T) \) is not correlated with the regressors. The regressors are also measured with
error because of our use of realizations as proxies for expected interest rates. Further we assume that the sum of these errors $v_t$ is not correlated with the regressors. This assumption is made for all regressions involving model A1 and B.

The alternative form of the model, Model A1 is,

$$y_t = B_0 + B_1 z_t + v_t$$  \hspace{1cm} (68)

where, $y_t = (F(t,T) - f(t,T))/P(T)$

$$z_t = \exp((T-t)r_{T-t}) - \exp \sum_{j=t}^{T-1} r(j).$$

The implied hypothesis of model A1 is that the normalized difference of forward and futures prices is explained by the difference of the cumulative costs of term financing against financing on a 1-day basis. The null hypothesis is $B_0 = 0$, $B_1 = 1$, and the alternative hypothesis is $B_0 \neq 0$, $B_1 \neq 1$.

We proceed in a similar fashion in our analysis of Model B. We assume risk neutrality, $\rho$ equals zero and use proxies for expectations. Under these assumptions (34) and (45) reduce to (63) and (64). By collecting terms and rearranging we obtain,

$$F(t,T)/P(T) = (F(t_0,T)/P(T))(1 - \exp \int_r(t,s)ds) + \exp \int_r(t,s)ds.$$  \hspace{1cm} (69)

and

$$f(t,T)/P(T) = (f(t_0,T)/P(T))(1 - \exp \int_r(t,s)ds) + \exp \int_r(t,s)ds.$$
\[ \exp \int_t^T r(s) ds. \]  \hspace{1cm} (70)

We define.

\[ Z_2 = \left( \frac{F(t_0, T)}{P(T)} \right) (1 - \exp((T-t)r_{T-t})) \]  \hspace{1cm} (71)

\[ Z_3 = -\left( \frac{f(t_0, T)}{P(T)} \right) (1 - \exp \sum_{j=t}^{T-1} r(j)) \]  \hspace{1cm} (72)

We then take the difference of (69) and (70). This leads to a testable specification of model B. The hypothesized regression version of model B is

\[ y_{1t} = \beta_0 + \beta_1 Z_{1t} + \beta_2 Z_{2t} + \beta_3 Z_{3t} + v_t. \]  \hspace{1cm} (73)

The null hypothesis is \( \beta_0 = 0, \beta_1 = 1, \beta_2 = 1, \) and \( \beta_3 = 1. \) The alternative hypothesis is \( \beta_0 \neq 0, \beta_1 \neq 1, \beta_2 \neq 1, \) and \( \beta_3 \neq 1. \) The regression version of model B suggests that normalized forward-futures price differentials can be explained by the difference of the cumulative costs of term versus daily financing and weighting factors which are functions of past prices and interest rates.

We present our test results in Tables 3-10. In all 3 versions of regression model A in Tables 3-5, the constant term is not significantly different from zero, but the coefficient of the second regressor is significantly different from 1 at the 5% level. We cannot reject the assumption of no autocorrelation. However, the \( R^2 \) estimates are all in the vicinity of 1% and the calculated F-values from tests of the joint hypotheses are 73.3, 55.9, and 61.5.
respectively. Our test statistic is distributed $F(2, 20)$. We can therefore reject the model specification. Since this version of Model A does not perform well, we run regressions of log forward-log futures differentials on a single interest-rate differential as regressor. This version is called Model A2. We present this version for comparative purposes.

We run 3 versions of Model A2, with $y_a$, $y_b$, and $y_c$ as our regressors. These results are presented in Tables 6-8. The individual parameters are all significantly different from their hypothesized values. F-values of the joint hypotheses that $\beta_0 = 0$, and $\beta_1 = 1$ are 66.7, 56.0, and 62.0. This leads us again to reject the model specification.

We then test Model A1. We use 1 measure of interest-rate differential since there is apparently little difference in their explanatory power. The results of this regression are presented in Table 9. A standard F-test of the joint hypothesis generates an F-value of 120.

This leads us to reject this model specification also. The next regression model we test is model B. The test results are presented in Table 10. We cannot reject the null hypothesis of no autocorrelation. $R^2$ is .443. The individual parameter estimates are not individually significantly different from their hypothesized values. However
an F-test of the joint hypothesis generates a value of 4632. This provides strong evidence against the jointly hypothesized parameter values.

The magnitude of the F-statistic suggests, and this is confirmed in our regression, that the unexplained variation increases significantly when the coefficients are restricted.
Table 3
Regressions of Interest-Rate Differentials on Log Forward/Futures Differentials

Regressions Model A

\[ y_a = \beta_0 + \beta_1 x + \epsilon \]

<table>
<thead>
<tr>
<th>Estimates</th>
<th>S. Errors</th>
<th>T-statistics</th>
<th>H _o: \beta_1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \hat{\beta}_0 ) = .0003</td>
<td>.0005</td>
<td>.60</td>
<td>0</td>
</tr>
<tr>
<td>( \hat{\beta}_1 ) = -.2048</td>
<td>.4424</td>
<td>-2.72*</td>
<td>1</td>
</tr>
</tbody>
</table>

\[ R^2 = .0106 \]
\[ DW = 2.481 \]

* significant at 5% level.
Table 4

Regressions of Interest-Rate Differentials on Log Forward/Futures Differentials

Regressions Model A

\[ y_b = \beta_0 + \beta_1 x_1 + \epsilon \]

<table>
<thead>
<tr>
<th>Estimates</th>
<th>S. Errors</th>
<th>T-statistics</th>
<th>( H_0: \beta_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \hat{\beta}_0 ) = -0.0000</td>
<td>0.0005</td>
<td>0.00</td>
<td>0</td>
</tr>
<tr>
<td>( \hat{\beta}_1 ) = -1.004</td>
<td>0.4463</td>
<td>-2.47*</td>
<td>1</td>
</tr>
</tbody>
</table>

\( R^2 = 0.003 \)

\( DW = 2.642 \)

* significant at 5% level.
Table 5
Regression of Interest-Rate Differentials On Log Forward/Futures Differentials

Regression Model A

\[ y^c = \beta_0 + \beta_1 x + \epsilon \]

<table>
<thead>
<tr>
<th>Estimates</th>
<th>S. Errors</th>
<th>T-statistics</th>
<th>H_0: \beta_1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \hat{\beta}_0 ) = .0003</td>
<td>.0005</td>
<td>.60</td>
<td>0</td>
</tr>
<tr>
<td>( \hat{\beta}_1 ) = -.0899</td>
<td>.4129</td>
<td>-2.64*</td>
<td>1</td>
</tr>
</tbody>
</table>

\[ R^2 = .002 \]
\[ DW = 2.70 \]

* significant at 5% level.
Table 6
Repression Of Log Forward/Log Futures Differentials On Interest-Rate Differences

Repression Model A2
\[ x_1 = \beta_0 + \beta_1 y_a + \epsilon \]

<table>
<thead>
<tr>
<th>Estimates</th>
<th>S. Errors</th>
<th>T-Statistics</th>
<th>( H_0: \beta_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \hat{\beta}_0 = -.0009 )</td>
<td>.0002</td>
<td>-4.50**</td>
<td>0</td>
</tr>
<tr>
<td>( \hat{\beta}_1 = -.0517 )</td>
<td>.1119</td>
<td>-9.39**</td>
<td>1</td>
</tr>
</tbody>
</table>

\( R^2 = .016 \)

\( DW = 1.44 \)

** significant at both 1% and 5% levels.
Table 7

Repression Of Log Forward/Log Futures Differentials On Interest-Rate Differentials

Repression Model A2

\[ x_t = \beta_0 + \beta_1 y_b + \epsilon \]

<table>
<thead>
<tr>
<th>Estimates</th>
<th>S. Errors</th>
<th>T-statistics</th>
<th>H_0: \beta_1</th>
</tr>
</thead>
<tbody>
<tr>
<td>\hat{\beta}_0 = -.0009</td>
<td>.0002</td>
<td>-4.50 **</td>
<td>0</td>
</tr>
<tr>
<td>\hat{\beta}_1 = -.2516</td>
<td>.1118</td>
<td>-11.19 **</td>
<td>1</td>
</tr>
</tbody>
</table>

\[ R^2 = .003 \]

\[ DW = 1.43 \]

** significant at both 1% and 5% levels.
Table 8

Repression Of Log Forward/Log Futures Differentials On Interest-Rate Differentials

Regression Model A2

\[ x_1 = \beta_0 + \beta_1 y_c + \epsilon \]

<table>
<thead>
<tr>
<th>Estimates</th>
<th>S. Errors</th>
<th>T-Statistics</th>
<th>( H_0: \beta_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \hat{\beta}_0 )</td>
<td>-0.0008</td>
<td>0.0002</td>
<td>-4.00**</td>
</tr>
<tr>
<td>( \hat{\beta}_1 )</td>
<td>-0.0263</td>
<td>0.108</td>
<td>-8.49**</td>
</tr>
</tbody>
</table>

\( R^2 = 0.002 \)

\( DW = 1.46 \)

** significant at 1% and 5% levels.
Table 9
Repression of Normalized Price Differentials On Financing-Cost Differentials

Regression Model A1

\[ y_1 = \beta_0 + \beta_1 z_1 + \epsilon \]

<table>
<thead>
<tr>
<th>Estimate</th>
<th>S. Errors</th>
<th>T-Statistics</th>
<th>H_0: \beta_1</th>
</tr>
</thead>
<tbody>
<tr>
<td>\hat{\beta}_0</td>
<td>-0.0009</td>
<td>0.0002</td>
<td>-4.50**</td>
</tr>
<tr>
<td>\hat{\beta}_1</td>
<td>-0.0510</td>
<td>0.1104</td>
<td>-9.53**</td>
</tr>
</tbody>
</table>

R^2 = 0.011

DW = 1.44

** significant at 1% and 5% levels.
Table 10

Repression of Normalized Price Differentials On Financing Costs Differentials And Weighted Functions Of Past Prices

Repression Model B

\[ y_1 = \beta_0 + \beta_1 z_1 + \beta_2 z_2 + \beta_3 z_3 + \nu \]

<table>
<thead>
<tr>
<th>Estimates</th>
<th>S. Errors</th>
<th>T-Statistics</th>
<th>H_0: \beta_1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \hat{\beta}_0 )</td>
<td>-0.0011</td>
<td>0.006</td>
<td>1.83</td>
</tr>
<tr>
<td>( \hat{\beta}_1 )</td>
<td>-0.4833</td>
<td>1.4092</td>
<td>-1.05</td>
</tr>
<tr>
<td>( \hat{\beta}_2 )</td>
<td>-0.4029</td>
<td>1.4088</td>
<td>-0.99</td>
</tr>
<tr>
<td>( \hat{\beta}_3 )</td>
<td>0.6140</td>
<td>1.4080</td>
<td>-0.27</td>
</tr>
</tbody>
</table>

R^2 = 0.443

DW = 2.12
SPECIFICATION ERROR TESTS

The issue to be confronted here is that of model misspecification. We confront the possibility that neither Model A nor Model B is the true model, in the sense that the classical assumptions are satisfied. Of course there is no inherent reason why a correctly specified model should obey all the assumptions of the classical model, but we conduct this test in lieu of more powerful tests. Failure to reject the null hypothesis in either or both models is not conclusive evidence of the validity of the model specification. We note these facts in order to place our empirical results in an appropriate perspective.

Among the general procedures advanced for testing individual models for misspecification are Theil's (1971) test for nonlinearity and the Reset procedure suggested by Thursby and Schmidt (1977). These are similar procedures and we use the Reset method. Reset examines the possibility of the exclusion of explanatory variables by testing the coefficients associated with the additional regressors the analyst inserts in the original regression matrix. The regression matrix is usually augmented by the inclusion of squares and cubes of the explanatory variables. Here we simply include the squares of the explanatory variables as a diagnostic measure.
For all the versions of regression Model A we examine in section 3.2, the inclusion of the squares of the explanatory variables as additional regressors does not improve the performance of these models appreciably. If valid explanatory variables are omitted, Reset usually detects this by rejecting the hypothesis that the coefficients of the extra regressors are zero. In the cases we examine these coefficients are not significantly different from zero. This is not a failure of the Reset method. It is attributable, in part, to the fact that the explanatory variables omitted from the regression matrices are not highly correlated with the chosen nonlinear transformations of the original regressors, or that the original regressors are incorrectly chosen. Of course, our results may differ if we choose other nonlinear transformations. It however seems clear that the basic regression models are either misspecified by reason of the constraints imposed in deriving them or the presence of some condition that violates the maintained hypotheses.

Since our tests do not indicate a serious problem of autocorrelation, we first address the issue of the implicit coefficient restrictions imposed on the model specifications. Here, we choose a representative from the group of regressions in Model A and Model Al. We define

$$y^*_1 = (T-t)T-t$$ (74)
An unrestricted representation of the regression model in Table 6 is
\[ x_{1t} = \beta_0 + \beta_1 y_{1t} + \beta_2 y_{2t} + \epsilon_{1t} \]  \quad (77)
We term this model specification Model A3.

\[ x_{1t} = \beta_0 + \beta_1 y_{1t} + \epsilon_{1t}, \]  \quad (76)
\[ y_{1t} = \frac{1}{T-1} \sum_{j=t}^{T-1} r(j) \]  \quad (75)
\[ y_a = y_{1t} + y_{2t} \]

We test this hypothesis. The calculated F-statistic is 16.63 and we reject the implicit parameter restrictions. The coefficient estimates of regression equation (77) are reported in Table 11. In this case, we also find that all the regressors, except the intercept term, have coefficient estimates that are individually significantly different from their hypothesized values. The explanatory power of the model improves with the removal of the invalid restriction, but the model specification still cannot be validated. We also reestimate the equations in Tables 7 and 8 using the same interest-rate decomposition. The $R^2$ in both model versions improves to .461 and .451 respectively. These results are shown in Tables 12 and 13.

We cannot reject the hypothesis of no serial correlation in the error terms, but the regressors, with the exception
of the constant term are significantly different from their hypothesized values. Removal of the restriction on Model A and B also improves the performance of both models. We term these model specifications All and B1 respectively. The results are shown in Tables 14 and 15. We define,

\[ Z_{11} = \exp((T-t)\tau_{T-t}) \]  

and \[ Z_{12} = -\exp \sum_{j=t}^{T-1} r(j), \]  

The hypothesized regression specifications for Models All and B1 are,

\[ y_t = \beta_0 + \beta_1 Z_{11} t + \beta_2 Z_{12} t + \mu_t \]  

and \[ y_t = \beta_0 + \beta_1 Z_{11} t + \beta_2 Z_{12} t + \beta_3 Z_{3} t + \beta_4 Z_{4} t + \epsilon_t \]  

Their \( R^2 \) are .442 and .464 respectively. The calculated F-statistics for the joint hypotheses of the respective models are 147.5 and 3143. This is also strong evidence against both model specifications. This suggests to us that even though separating the interest-rate variables improves the explanatory power of all the models, it has an adverse effect on the standard errors of the coefficient estimates. This is shown by the increase in the standard errors of the
coefficients of the models. Regression results for Models All and B1 are shown in Tables 14 and 15.

However, we also note that when we test the specific implicit restriction that $\beta_1 = \beta_2$ in Model B1, our F-statistic is .45, which does not provide evidence against this restriction. This is not the case in Model All. In Model All it introduces near dependency among the regressors. This suggests a collinearity problem. We use the collinearity diagnostic procedures suggested by Belsey, Kuh and Welsch (1980) to check for its severity. If collinearity is severe, the regression matrix $X'$ is such that $X'X$ is near singular. The elements of its inverse are very large and this is reflected in the large standard errors of the coefficient estimates. The eigenvalues of $X'X$ are used to diagnose this condition.

Specifically, Belsey, Kuh and Welsch (1980) examine the square roots of the ratios of the largest eigenvalue to that of every other eigenvalue to determine condition indexes. These authors suggest that the data are ill-conditioned, reflects the adverse effects of collinearity when the condition index for any regressor exceeds 30. However, they also report weak dependencies with condition indexes as low as 1. We examine Models A1 and B since they are our best trial specifications. Both models show evi-
dence of collinearity. The usual methods for dealing with collinearity are, to name a few, (1) dropping one of the collinear variables (2) introducing additional regressors and (3) ridge regression. Ridge regression is premised on an implicit prior, namely that the coefficient vector is a null vector. (See Belsey, Kuh and Welsch 1980, p.212). Hence it is not appropriate in this case. We apply various data deletion procedures to Model B and find that the conditioning of the data is not improved. Collinearity diagnostics for Models A3 and B are reported in Tables 16 and 17. These tables provide evidence of the degradation of the parameter estimates by the presence of near dependencies among the regressors.

We also test for the effects of heteroskedasticity using Glejser's (1969) test procedure. The absolute values of the OLS residuals are regressed on various monotonic transformations of the regressors. Evidence of heteroskedasticity is usually detected if the coefficients of this regression are significantly different from zero. In both models we could not reject the hypothesis of homoskedasticity.

For example, in both models A3 and B, we regress the absolute values of the residuals on the explanatory variables of A3 and B. We find that none of the coefficients of the regressors in any model specification is significantly
different from zero. The calculated F-values resulting from tests of the joint hypothesis are 2.18 and 1.23 respectively. Our test statistics are distributed $F(3, 19)$ and $F(4, 18)$ respectively. We can reject the hypothesis that the error terms are heteroskedastic.
Table II

Long Price Differentials On Term Interest Rate And Cumulative Spot Interest Rates

Regression Model A3

\[ x_t = B_0 + B_1 y_{1t} + B_2 y_{2t} + \epsilon_t \]

<table>
<thead>
<tr>
<th>Estimates</th>
<th>S.Errors</th>
<th>T-Statistics</th>
<th>Ho: ( B_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \hat{B}_0 ) = .0012</td>
<td>.0005</td>
<td>2.40( \approx )</td>
<td>0</td>
</tr>
<tr>
<td>( \hat{B}_1 ) = -.0867</td>
<td>.0821</td>
<td>-13.24( ** )</td>
<td>1</td>
</tr>
<tr>
<td>( \hat{B}_2 ) = .1399</td>
<td>.0938</td>
<td>-9.16( ** )</td>
<td>1</td>
</tr>
</tbody>
</table>

DW = 2.24

\( R^2 = .496 \)

\( ** \) significant at 1 and 5% levels.

\( * \) significant at 5% level.
Table 12

Log Price Differentials On Term Interest Rate And Cumulative Spot Interest Rates

Regression Model A3

\[ x_{1t} = \beta_0 + \beta_1 y_{1t} + \beta_2 y_{2t} + \varepsilon_t \]

<table>
<thead>
<tr>
<th>Estimates</th>
<th>S.Errors</th>
<th>T-Statistics</th>
<th>( H_0: \beta_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \hat{\beta}_0 )</td>
<td>-0.0012</td>
<td>0.0005</td>
<td>2.00</td>
</tr>
<tr>
<td>( \hat{\beta}_1 )</td>
<td>-0.1428</td>
<td>0.0634</td>
<td>-16.71**</td>
</tr>
<tr>
<td>( \hat{\beta}_2 )</td>
<td>0.0701</td>
<td>0.0760</td>
<td>-12.24**</td>
</tr>
</tbody>
</table>

DW=2.21

\( R^2=0.451 \)

**Significant at 1% and 5% levels.
### Table 13

**Regression of Log Price Differentials On Term Interest Rate And Cumulative Spot Interest Rates**

**Repression Model A3**

\[ x_t = \beta_0 + \beta_1 y_t + \beta_2 y_3 + \varepsilon_t \]

<table>
<thead>
<tr>
<th>Estimates</th>
<th>S. Errors</th>
<th>T-Statistics</th>
<th>H₀: ( \beta_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \hat{\beta}_0 ) = .0011</td>
<td>.0006</td>
<td>1.83</td>
<td>0</td>
</tr>
<tr>
<td>( \hat{\beta}_1 ) = -.1544</td>
<td>.0687</td>
<td>-16.80**</td>
<td>1</td>
</tr>
<tr>
<td>( \hat{\beta}_2 ) = .0494</td>
<td>.0719</td>
<td>-13.22**</td>
<td>1</td>
</tr>
</tbody>
</table>

DW = 2.22

\( R^2 = .51 \)

**Notes:**
- **\( y_3 \)** is significant at 1% and 5% levels.
- **\( y_3 \)** is based on daily Federal Funds Rate.
Table 14
Regression Of Normalized Price Differentials On
Term Financing And Spot Financing Costs

Regression Model All

\[ y_t = \beta_0 + \beta_1 z_{1t} + \beta_2 z_{12} + \epsilon_t \]

<table>
<thead>
<tr>
<th>Estimates</th>
<th>S. Errors</th>
<th>T-Statistics</th>
<th>H₀: ( \beta_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \hat{\beta}_0 ) = .2126</td>
<td>.0557</td>
<td>3.82**</td>
<td>0</td>
</tr>
<tr>
<td>( \hat{\beta}_1 ) = -.0763</td>
<td>.0853</td>
<td>-12.62**</td>
<td>1</td>
</tr>
<tr>
<td>( \hat{\beta}_2 ) = .1531</td>
<td>.0976</td>
<td>-11.63**</td>
<td>1</td>
</tr>
</tbody>
</table>

\( R^2 = .142 \)
\( F = 1.11 \)

** significant at 1% and 5% levels.
Table 15

Regressions of Normalized Price Differentials on Financing Cost Differentials and Weighted Functions of Past Prices

Regressions Model B1

\[ y_{1t} = \beta_0 + \beta_1 z_{11t} + \beta_2 z_{12t} + \beta_3 z_{2t} + \beta_4 z_{3t} + \epsilon_t \]

<table>
<thead>
<tr>
<th>Estimates</th>
<th>S. Errors</th>
<th>T-statistics</th>
<th>H0: ( \beta_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \hat{\beta}_0 ) = 4.1196</td>
<td>4.6550</td>
<td>.88</td>
<td>0</td>
</tr>
<tr>
<td>( \hat{\beta}_1 ) = -4.3243</td>
<td>5.6986</td>
<td>-.93</td>
<td>1</td>
</tr>
<tr>
<td>( \hat{\beta}_2 ) = -2.057</td>
<td>2.9976</td>
<td>-.40</td>
<td>1</td>
</tr>
<tr>
<td>( \hat{\beta}_3 ) = -4.3090</td>
<td>5.7616</td>
<td>-.92</td>
<td>1</td>
</tr>
<tr>
<td>( \hat{\beta}_4 ) = 3.0066</td>
<td>3.0066</td>
<td>-.19</td>
<td>1</td>
</tr>
</tbody>
</table>

\( R^2 = .464 \)

D' = 1.95
Table 16  
**Collinearity Diagnostics For Model B**

<table>
<thead>
<tr>
<th></th>
<th>Eigen-Condition</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>value</td>
<td>index</td>
<td>0</td>
<td>Z1</td>
<td>Z2</td>
<td>Z3</td>
</tr>
<tr>
<td>1</td>
<td>3.084</td>
<td>.0068</td>
<td>.0001</td>
<td>.0000</td>
<td>.0000</td>
</tr>
<tr>
<td>2</td>
<td>.8702</td>
<td>.0025</td>
<td>.0037</td>
<td>.0000</td>
<td>.0000</td>
</tr>
<tr>
<td>3</td>
<td>.0460</td>
<td>.9897</td>
<td>.0001</td>
<td>.0003</td>
<td>.0004</td>
</tr>
<tr>
<td>4</td>
<td>.0001</td>
<td>.9961</td>
<td>.9961</td>
<td>.9995</td>
<td></td>
</tr>
</tbody>
</table>

Elements in columns of the variance proportions matrix do not sum to 1 because of rounding error.
### Table 17

**Collinearity Diagnostics For Model A3**

<table>
<thead>
<tr>
<th>Model A3</th>
<th>Variance Proportions</th>
</tr>
</thead>
<tbody>
<tr>
<td>E_index</td>
<td>B₀ * y_1 * y_2</td>
</tr>
<tr>
<td>2.939</td>
<td>1.00</td>
</tr>
<tr>
<td>0.0476</td>
<td>7.852</td>
</tr>
<tr>
<td>0.0131</td>
<td>14.97</td>
</tr>
</tbody>
</table>

Elements in columns of the variance proportions matrix do not sum to 1 because of rounding error.
A substantial body of theoretical work has accumulated demonstrating that forward and futures prices are not unbiased estimators of future spot prices. Evidence of this is provided by several authors including French (1982) and Richard and Sundaresan (1981). We test the relative performance of these estimators in the T-Bill market by comparing their forecast error relative to n-day forecasts of the maturity spot price, within a given sample period. Since we do not have any cogent a priori rationalization that permits us to hypothesize about the distribution of these residuals, this analysis is mainly an empirical exercise.

In addition to the effects of differential taxation and transactions costs, futures prices are influenced by limit moves, initial and maintenance margins, and also by the influence of trader activity. In some markets substantial non-stationarities in the form of contract month effects and other seasonality factors may also exist. To realistically evaluate the role of futures prices as maturity price estimators, these issues should not be ignored. However, that endeavor must await the advent of models with greater verisimilitude.
Nevertheless it may be reasonable for us to assume that such effects are incorporated in futures prices, and evidence of their impact may be gleaned by comparing the forecast errors associated with the futures price estimator with those from an implicit forward price estimator. Under this scenario, if the forecast errors associated with the futures price estimator deviate significantly from those of the forward price estimator, this may be construed to be prima facie evidence of such effects or of differential information in the two markets. Since the maturity price in forward and cash markets may not converge completely because of transactions and other costs, the two estimators may provide estimates of two different random variables. In this case the difference between the respective forecast errors may not be equal to the current forward-futures spread.

Another troublesome issue revolves around the notion of significant deviations between the forecast errors associated with the two estimators. How are admissible bounds for any deviations established if in fact they do exist? Does knowledge of the bounds of current forward/futures differentials translate into any meaningful hypotheses about the nature of forecast error differentials?
In answer to these questions, we note that it is possible that significant current price differentials may be consistent with established theories of forward and futures pricing. In the polar case of zero-valued contract positions, if the associated prices are utilized as estimators, (41) and (57) indicate that the bounds on the difference between the two contract prices may be defined by the difference between the covariance terms of the two prices. That is,

\[ F(t, T) - f(t, T) = \text{cov}_t \left( \frac{\langle J_w(T), P_j(T) - F_j(t, T) \rangle}{E_t(J_w(T))} \right) \]

\[ = \text{cov}_t \left[ \frac{\exp \int_r^t (r(t, s) ds, (P(T) - f(t, T)))}{E_t(\exp \int_r^t r(t, s) ds) ds} \right] \]

\[ - \text{cov}_t \left( J_w(T), P(T) - f_j(t, T) \right) \exp \int_r^t r(t, s) ds \]

\[ \frac{E_t(\exp \int_r^t r(s) ds) E_t(J_w(T))}{E_t(\exp \int_r^t r(s) ds) E_t(J_w(T))} \]

(82) indicates that the current price differential is dependent on the assessment of the relative expected hedging efficacy of the contracts based on current information. Surprisingly the rate of time discount factor \( p \) has no causative role in the determination of the price differential.
in the polar case of zero-valued contract positions. In discussing the more general case, where contract positions and contracts are not zero-valued we obtain slightly different results. Utilizing the results in (39) and (56), and a covariance decomposition we note that,

\[ F(t,T) = F(t_0,T) \]

\[ + \text{cov}_t \left[ \frac{J_{W}(T)}{J_{W}(t)} (P(T) - F(t_0,T)) \exp \int_{t}^{T} (r(t,s) - \rho) ds \right] \]

\[ + \mathbb{E}_t \left( \frac{J_{W}(T)}{J_{W}(t)} \right) \mathbb{E}_t ((P(T) - F(t_0,T)) \exp \int_{t}^{T} (r(t,s) - \rho) ds. \] (83)

and

\[ f(t,T) = f(t_0,T) \]

\[ + \text{cov}_t \left[ \frac{J_{W}(T)}{J_{W}(t)} (P(T) - f(t_0,T)) \exp \int_{t}^{T} (r(s) - \rho) ds \right] \]

\[ + \mathbb{E}_t \left( \frac{J_{W}(T)}{J_{W}(t)} \right) \mathbb{E}_t ((P(T) - f(t_0,T)) \exp \int_{t}^{T} (r(s) - \rho) ds. \] (84)

This implies, that
Interpreting our result, the general model attributes current price differentials to three main factors. These are past price differentials, differences in the hedging effectiveness of the contracts against profit risk as well as differences in the expected profits on the contracts adjusted by an intertemporal substitution factor. We note that, the rate of time discount factor $\rho$ enters economic agents' calculations since they actually expect to realize profits, even though their expectations may not be realized.

In accordance with earlier results we noted in chapter 2, both the sign and magnitude of these price differentials in (83) and (84) are indeterminate. They do depend on
empirical covariances, but these are not readily estimable, even if we resort to the expedient of specifying a particular choice of utility function. The answers to the questions posed are suggestive not conclusive. The analysis that has been conducted only generates local knowledge of the forward/futures price differentials, based on expectations of future price movements. It is not apparent that this type of local knowledge can be meaningfully extrapolated to future time periods, and therefore it is not possible to hypothesize on the nature of forecast error differentials.
The Distribution of The Forecast Errors

If we examine the distribution of the errors associated with the "naive" forward and futures price estimators, we may obtain additional information. McCulloch (1986) presents simple order statistics and tables for determining where in the class of stable distributions those residuals fit. A stable distribution is characterized by four parameters, namely a location parameter, a scale parameter, a characteristic exponent and a symmetry parameter. Let \( \alpha \) be the characteristic exponent, \( \beta \) be the symmetry parameter. The characteristic exponent and the symmetry parameter completely determine the shape of the distribution. When \( \alpha = 2 \), the distribution is normal and when \( \alpha = 1 \) we obtain a Cauchy distribution. The characteristic exponent lies in the range \((0, 2]\) and when \( \alpha < 2 \), the variance is infinite. We adopt the definition used by McCulloch (1986) since we utilize his estimation techniques for estimating \( \alpha \) and \( \beta \).

If \( \beta \) is positive, the distribution is skewed to the right. If it is negative it is skewed to the left. When \( \beta = 0 \), the distribution is symmetric. Let \( \hat{x}_p \) be the \( p \)-th population quantile and \( \hat{X}_p \) the corresponding sample quantile. McCulloch (1986) provides test statistics which are monotonic functions of \( \alpha \) and \( \beta \), and provide consistent estimators of \( \nu_\alpha \) and \( \nu_\beta \), the population parameters. The
inverse functions of $V_a$ and $V_B$ are derived to obtain estimates of $a$ and $b$.

$$
\hat{V}_a = \frac{x_{.95} - x_{.05}}{x_{.75} - x_{.25}}
$$

is a decreasing function of $a$.

$$
\hat{V}_b = \frac{x_{.95} - x_{.05} - 2\hat{x}_{.5}}{x_{.95} - x_{.05}}
$$

is an increasing function of $b$. We order the forecast errors in quantiles and use (86) and (87) to obtain $\hat{V}_a$ and $\hat{V}_b$ for forward and futures prices respectively. Then we use McCulloch's tables to obtain estimates of $a$ and $b$.

These results are shown in Tables 18 through 22. We use both the I.M.M. index of futures prices as forecasts of the future futures price index, and the current settlement price as "naive" forecasts of the future settlement price. Both generate almost identical information about the distribution of the errors from using these "naive" estimators. By linear interpolation on McCulloch's (1986) tables we infer that the forecast errors associated with both implicit forward and actual futures prices are both normally distributed. These results are sensitive to sampling error since the sample size is small.
Table 18
Forecast Errors at 30 days Using Futures Price Indices as Maturity Price Index Estimators: Their Distribution

March 1980-June 1985
Sample Quantiles

<table>
<thead>
<tr>
<th></th>
<th>(1)</th>
<th>(2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{x}_{0.05}$</td>
<td>-1.86</td>
<td>-1.80</td>
</tr>
<tr>
<td>$\hat{x}_{0.25}$</td>
<td>-0.92</td>
<td>-0.88</td>
</tr>
<tr>
<td>$\hat{x}_{0.51}$</td>
<td>-0.12</td>
<td>-0.11</td>
</tr>
<tr>
<td>$\hat{x}_{0.75}$</td>
<td>-0.16</td>
<td>-0.91</td>
</tr>
<tr>
<td>$\hat{x}_{0.95}$</td>
<td>2.61</td>
<td>2.30</td>
</tr>
<tr>
<td>$\hat{V}_a$</td>
<td>2.38</td>
<td>2.29</td>
</tr>
<tr>
<td>$\hat{V}_b$</td>
<td>0.23</td>
<td>0.18</td>
</tr>
</tbody>
</table>

(2) Adjusted for continuity by McCulloch's Method.
Table 19
Forecast Errors at 31 Days Using Futures Price Indices As Maturity Price Index Estimators: Their Distribution

March 1980 - June 1985

Sample Quantiles

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>(2)</td>
<td></td>
</tr>
<tr>
<td>( x_{.05} )</td>
<td>(-2.20)</td>
<td>(-2.10)</td>
</tr>
<tr>
<td>( x_{.25} )</td>
<td>(-1.07)</td>
<td>(-.95)</td>
</tr>
<tr>
<td>( x_{.5} )</td>
<td>(-.30)</td>
<td>(-.30)</td>
</tr>
<tr>
<td>( x_{.75} )</td>
<td>(-1.165)</td>
<td>(1.07)</td>
</tr>
<tr>
<td>( x_{.95} )</td>
<td>(3.06)</td>
<td>(2.40)</td>
</tr>
<tr>
<td>( \hat{V}_{\alpha} )</td>
<td>(2.36)</td>
<td>(2.23)</td>
</tr>
<tr>
<td>( \hat{V}_{B} )</td>
<td>(.28)</td>
<td>(.20)</td>
</tr>
</tbody>
</table>

(2) Adjusted for continuity by McCulloch's Method.
Table 20
Forecast Errors At 30 Days Using Actual Futures
Prices As Maturity Price Forecasts

March 1980 - June 1985

Sample Quantiles

<table>
<thead>
<tr>
<th></th>
<th>(1)</th>
<th>(2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_{.05}$</td>
<td>-4646.5</td>
<td>-4454.6</td>
</tr>
<tr>
<td>$x_{.25}$</td>
<td>-2300.0</td>
<td>-2200.0</td>
</tr>
<tr>
<td>$x_{.5}$</td>
<td>-300.0</td>
<td>-300.0</td>
</tr>
<tr>
<td>$x_{.75}$</td>
<td>-2393.7</td>
<td>-2275.0</td>
</tr>
<tr>
<td>$x_{.95}$</td>
<td>-5516.3</td>
<td>-6087.0</td>
</tr>
</tbody>
</table>

$\hat{V}_a = 2.38$  
$\hat{V}_b = 0.22$

(2) Adjusted for continuity by McCulloch's Method.
Table 21
Forecast Errors At 30 days Using Implied Forward Prices As Maturity Price Estimators

March 1980 - June 1985
Sample Quantiles

<table>
<thead>
<tr>
<th></th>
<th>(1)</th>
<th>(2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>X.05</td>
<td>-4997.2</td>
<td>-4975.8</td>
</tr>
<tr>
<td>X.25</td>
<td>-1771.7</td>
<td>-1744.5</td>
</tr>
<tr>
<td>X.50</td>
<td>631.0</td>
<td>632.0</td>
</tr>
<tr>
<td>X.75</td>
<td>3333.4</td>
<td>3063.9</td>
</tr>
<tr>
<td>X.95</td>
<td>6901.2</td>
<td>6309.8</td>
</tr>
<tr>
<td>V_a</td>
<td>2.33</td>
<td>2.35</td>
</tr>
<tr>
<td>V_b</td>
<td>.27</td>
<td>.23</td>
</tr>
</tbody>
</table>

(2) Adjusted for continuity by McCulloch's Method.
### Table 22

**Forecast Errors At 31 days Using Implied Forward Prices As Maturity Price Estimators**

<table>
<thead>
<tr>
<th>Sample Quantiles</th>
<th>(1)</th>
<th>(2)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \hat{x}_{0.05} ) = -5983.0</td>
<td>(-4817.8)</td>
</tr>
<tr>
<td></td>
<td>( \hat{x}_{0.25} ) = -2522.0</td>
<td>(-2435.1)</td>
</tr>
<tr>
<td></td>
<td>( \hat{x}_{0.50} ) = 634.3</td>
<td>634.1</td>
</tr>
<tr>
<td></td>
<td>( \hat{x}_{0.75} ) = 3152.3</td>
<td>3070.5</td>
</tr>
<tr>
<td></td>
<td>( \hat{x}_{0.95} ) = 5975.4</td>
<td>5704.1</td>
</tr>
<tr>
<td></td>
<td>( \hat{v}_{a} ) = 2.11</td>
<td>1.91</td>
</tr>
<tr>
<td></td>
<td>( \hat{v}_{b} ) = .11</td>
<td>.20</td>
</tr>
</tbody>
</table>

(2) Adjusted for continuity by McCulloch's Method.
It is generally accepted that the nature of the current tax laws are such that differential tax treatment creates clientele effects. Arak (1983) has shown that the existence of differential rates of taxation, based on differentiated incomes, generates varied portfolio strategies among economic agents. It is further hypothesized that the tax laws may have altered investor preferences and caused a reversal in the observed forward-futures relationship. Cornell (1981) suggests an explicit test for tax effects on market data. His test procedure is predicated on the assumption that either individual investors or dealers are marginal investors. The ex-gain day behavior of futures prices was tested for evidence of abnormality since investors lose the tax portion relative to the treatment of gains and losses on that day.

Cornell (1981) concluded that differential taxation did not explain the discrepancy he observed between forward and futures prices, and Arak (1983) that the future impact of the revised tax treatment of futures on forward-futures rates is indeterminate. In view of the conclusions arrived at by the above authors and the ongoing discussion about
appropriate modelling of pricing relations in these markets. we think that it is critically important to model the likely effects of taxation in a true equilibrium framework. what we try to accomplish in this section is to incorporate taxes in the general equilibrium model developed in chapter 2, and to determine whether we can gain any insights about the possible behavior of forward-futures spreads when profits and losses are taxed and any impact on the magnitude of the risk premiums inherent in those prices. we assume that gains and losses are taxed asymmetrically. moreover it is assumed that taxes are collected continuously and costlessly by an external economic agency.

all ensuing tax revenues are disposed of in a neutral manner and do not flow back to contract holders in the form of goods, numeraire or otherwise, during the holding period.

\( t \) is the known rate of taxation on gains and losses of type i contracts for the individual from the j-th group. this tax structure allows for the symmetric tax treatment of gains and losses in an investor group, but also permits differential rates of taxation across groups. the rate of taxation is exogenously determined and we assume for simplicity that there is no interaction of the tax structure and the asset valuation and pricing processes. economic
agents are assumed to maximize consumption subject to their
tax-adjusted wealth. The model is the same as that exposiT-
ed in Appendix A with a few modifications.

\[ W^* = W_1^* \quad \text{where} \quad W^* = \text{the tax-adjusted aggregate}
\text{wealth and} \quad W_i^* \text{is the tax-adjusted wealth of the group} \ i
\text{individual. For the group} \ 1 \ \text{individual}
\]

\[
p'q_1 + \sum_{k=1}^2 M_kV_k(1-t_k) + \lambda_1 = w_1^*.
\]

(83)

Similarly for the group 2 individual we note that

\[
p'q_2 + \sum_{k=1}^2 L_kV_k(1-t_k^2) + \lambda_2 = w_2^*.
\]

(39)

The equilibrium conditions specified in (A3), (A4) and (A5)
continue to hold, but \( n'q = w'' \).

The new stochastic differential equation for aggregate
wealth is

\[
dw^* = \left[ p'(h-c) + q'\alpha_p + tr\sigma_pG' + \sum_{k=1}^2 M_k(a_k - \phi_k - V_kr)(1-t_k^1) \right. \\
+ \sum_{k=1}^2 L_k(a_k - \phi_k - V_kr)(1-t_k^2) + (w^* - p'q)r \right] dt \\
+ \left[ p'G + q'\sigma_p + \sum_{k=1}^2 (M_k)H_k(1-t_k^1) + \sum_{k=1}^2 L_kH_k(1-t_k^2) \right] dz.
\]

(90)
\[ d\omega^* = B_{\omega^*} \sigma_{\omega^*} dz \quad (91) \]

where and are terms in square brackets in (90).

The functional \( J \) which is the objective function of the maximizing problem is now defined as \( J(W, Q, Y) = I(W, Q, Y, t) \) and the differential operator \( L \) in (9) is now defined with respect to \( W^* \) instead of \( W \). The first order conditions for the resulting optimizing problem are

\[ \psi_{c*1_i} = \alpha_1(t)u(c_1(t)) - p_iJ_{W^*1} = 0; \quad i = 1, 2, \ldots, n. \]

\[ (92) \]

\[ \psi_{c*2_i} = \alpha_2(t)u(c_2(t)) - p_iJ_{W^*2} = 0; \quad i = 1, 2, \ldots, n. \]

\[ (93) \]

\[ \psi_{H_k} = J_{W^*1} \left[ (\alpha_k - \phi_k) - V_k r \right] (1 - t^1_k) + \]

\[ (1 - t^1_k)J_{W^*1} W^*1 H_k \sigma_{W^*W^*} + \]

\[ \sum_{j=1}^{N} J_{W^*1} Y_j H_k S_j (1 - t^1_k) + \sum_{j=1}^{N} J_{W^*1} Q_j H_k D/j (1 - t^1_k) = 0. \]

\[ (94) \]

\[ \psi_{L_k} = J_{W^*} \left( \alpha_k - \phi_k - V_k r \right) (1 - t^2_k) + (1 - t^2_k)J_{W^*} H_k \sigma_{W^*W^*} + \]

\[ (95) \]
The other first order conditions remain unchanged. The pricing formulas are once again derived for the specific case of a non-stochastic investment opportunity set, at time $t$, that is $H_k = 0$. Since the post-tax factor $(1 - t_k)$ enters the valuation equations multiplicatively, we can obtain a few qualitative conclusions. However, first we recall that the equilibrium futures and forward pricing relations that are generated are (45) and (34) respectively.

**Case "A". Exogenously Specified Tax Structures and Zero-Valued Positions in Contingent-Claim Contracts.**

This case generates the same results as the no-taxation model. Under our assumptions, tax revenues would be zero over the interval of time $dt = t_0 - t$ and hence the wealth of economic agents would be unaffected, i.e., $w_1^* = w_1$ and $w_2^* = w_2$. In terms of the alternate equilibrium characterizations that have been proposed, we note the following. (1) If all economic agents have identical endowments and preferences, an exogenously determined tax structure logically should not alter the equilibrium paths of the prices of
these derivative assets. A fortiori, it would not matter whether profits and losses were taxed symmetrically or asymmetrically. (2) If economic agents do have diverse endowments and preferences, it still would not affect the current equilibrium path of these prices since the assumption of a zero-valued position in a contract means that the price of the contract is unvarying over time or that the contract has been issued at the current instant. We therefore conclude that the assumption of an exogenous tax structure together with zero-valued positions in forward and futures contracts adds nothing to our previous knowledge of the properties of the prices of these contingent claims. Equilibrium wealth is not perturbed in any state, tax revenues are zero, and the optimal number of contracts held is not changed.

Case B Exogenously Specified Tax Structures and Non-Zero Valued Positions in Contingent-claim Contracts

In this example, the effect of taxing profit and losses at differential rates is captured in the marginal utility of wealth term $J_W$, depending on whether we are dealing with long or short contract holders. In any event the ultimate effect of taxation is to alter the risk premiums associated with both forward and futures prices. The equilibrium paths and prices of claims on goods are not altered by the
the imposition of these taxes. In addition the optimal number of contracts held in equilibrium remains indeterminate. To clearly see this latter point we revert to equations (A12). These equations imply

\[ J_{w \times 1} (\alpha_k - \phi_k) - V_{kr} \] + \[ J_{w \times 1} w_{i} H_{k} \left( \frac{2}{\sum_{j=1}^{N} H_{k} M_{j}^{'}} + \sigma_{p} G + G_{p} \right) + \]

\[ \sum_{j=1}^{N} J_{w \times 1} y_{j} H_{k} S_{j}^{'}, \sum_{j=1}^{N} J_{w \times 1} y_{j} H_{k} D_{j}^{'}, i = 1, 2; k = 1, 2. \]

(96)

By definition \( \Delta_k = (\alpha_k - \phi_k) - V_{kr} \). This implies that

\[ J_{w \times 1} \Delta k + J_{w \times 1} w_{i} H_{k} H_{l} M_{1} + J_{w \times 1} w_{i} H_{k} H_{2} M_{2} \]

\[ + J_{w \times 1} w_{i} H_{k} (\sigma_{p} G - G_{p}) + \sum_{j=1}^{k} J_{w \times 1} y_{j} H_{k} S_{j}^{'}, \sum_{j=1}^{N} J_{w \times 1} Q_{j} H_{k} D_{j}^{'}, k = 1, 2. \]

(97)

\[ M_{1} = \frac{-J_{w \times 1} \Delta 2 - H_{1} H_{2} M_{2}}{J_{w \times 1} w_{1}} - \frac{H_{1} (\sigma_{p} G + G_{p})}{H_{1} H_{1}} \]

\[ - \left( \frac{\sum_{j=1}^{k} J_{w \times 1} y_{j} H_{1} S_{j}^{'}}{J_{w \times 1} w_{1} H_{1} H_{1}^{'}} \right) \]

(98)

and

\[ M_{2} = \frac{-J_{w \times 1} \Delta 2 - H_{2} H_{1} M_{1}}{J_{w \times 1} w_{1} H_{2} H_{2}} - \frac{H_{2} (\sigma_{p} G + G_{p})}{H_{2} H_{2}^{'}} \]

\[ \sum_{j=1}^{k} J_{w \times 1} y_{j} H_{2} S_{j}^{'}, \sum_{j=1}^{N} J_{w \times 1} Q_{j} H_{2} D_{j}^{'}, \]

(99)
$M_1^*$ and $M_2^*$ are the $n \times l$ vectors of the optimal number of long forward and futures contracts held by economic agents. Contract demand functions are interdependent and depend separately on the reciprocal of a measure of absolute risk aversion. However, the limiting values of $M_1^*$ and $M_2^*$ are both indeterminate as $H_1$ and $H_2$ converge to null vectors.

These demand functions for contingent claims shed light on their interdependent nature. The optimal number of futures contracts held by an economic agent on good $j$ is inversely related to the number of forward contracts that would be held by that agent on good $j$.

However, the model is not capable of providing information of more crucial relevance. Are futures and forward prices forced along higher or lower equilibrium paths as a result of taxation? Are the associated risk premiums increased or narrowed? In our view, the answers to these questions await the development of model extensions that endogenize the tax process. In conclusion, we note the following. The issue of taxation, exogenously or endogenously determined, is only relevant when economic agents have a desire to hold contracts. On the assumption that the wealth of agents is altered by taxation, the risk premia in the contracts may be altered also. But we cannot tell what is the potential impact on the optimal number of contracts.
that agents will hold. This is not an entirely negative finding, since it at least assures us that with wealth effects possible, the optimal number of contracts that ought to be held is not constrained to zero.
CHAPTER IV
PRICE VOLATILITY IN THE T-BILL MARKET

The behavior of futures prices has been the subject of considerable theoretical investigation. Some researchers have taken the route of assuming that futures prices constitute a martingale and then assuming a particular autoregressive process for spot prices. Such assumptions permit the categorization of possible futures price behavior as contracts approach their maturity dates. However, the current consensus viewpoint is that such a priori assumptions are unduly restrictive and constrain the theoretical behavior of futures prices in these models.

Not surprisingly, the empirical results from analyses of these models have been mixed, depending on whether the test data fitted particular autoregressive specifications. Current research efforts have been focused on model specifications that permit a wide range of volatility behavior and subsume the model associated with Samuelson (1965) as a special case. Sundaresan (1980) develops closed-form
solutions for futures and forward pricing, under various utility-function specifications, and establishes conditions in his model such that contract prices display increasing volatility as the contracts approach maturity. These conditions are shown by him to be valid only when the term structure of futures prices is downward sloping. However, he appropriately noted that the stated results might be model specific and not robust across model specifications.

Anderson (1982) in a state variable model implies that futures price volatility will be relatively high when the resolution of price uncertainty is significant. Price uncertainty may stem from either demand or supply uncertainty in the market.

From a completely general standpoint, the version of the intertemporal general equilibrium model of asset pricing formulated in Chapter 2, encompasses the state variable and maturity-effect hypotheses as special cases. Current futures price variability is influenced by the expected change in futures prices over the holding period, a subjective rate of time discount factor and the intertemporal marginal rate of substitution of contract holders.

Futures price variability may be relatively high because contract holders expect future price movements to be large. This may be, as Anderson suggested, because a significant
amount of price uncertainty has been resolved, thus leading to corrective price movements. However the same results are possible if contract holders are faced with increasing uncertainty. As a result, in the context of this model, no a priori grounds exist for assuming either increasing or decreasing price volatility in contracts as they approach their maturity dates. This is consistent with the conclusions of recent work by Grammatikos and Saunders (1986), who found no strong relationship between price variability and contract maturity, in their study of foreign currency futures.

This chapter is organized as follows. In Section 1 we discuss the volatility of T-bill futures when contracts are arbitrageable and use the data generated in Section 2, to evaluate the applicability of the regression specification implied by our theoretical model. Section 2 is an empirical examination of the distribution of futures price changes, involving tests for the normality of the sample data.
MODELLING PRICE VOLATILITY

In this section we attempt to address an issue that has not been previously evaluated in an empirical context. That is, what is the link if any, in the observed day-to-day variations in closing futures settlement prices, the cost of funding positions in the contracts and realized profits. The existence of a marked differential between implicit forward prices and futures prices has been well documented in previous studies, and is also confirmed in our sample, which has held maturity effects constant. This being so, we hypothesize the existence of a well defined relation between the daily variation in futures prices, repurchase rates and the realized returns from a position in the contract, which should be observed in repeated sampling of observations with maturity effects held constant.

What we do is to isolate the most important variables that influence price volatility in the interday range. We have previously shown that extant continuous time models analyzed cannot be used to explain the intraday variation in futures prices. Price volatility is precisely defined here as the square of the change in daily closing prices. In this respect, we follow Grammatikos and Saunders (1986).
Our approach differs from previously cited studies, where various measures of price volatility are regressed against volume of contracts traded, open interest, maturity variables or forward/futures price differentials.

The basis for our model specification is (45). We postulate that a discrete operational version should, in the type of sample analyzed, explain the price change when the time to maturity changes from 31 days to 30 days. We make a further assumption which may be plausible, that for the time period under consideration, the marginal utility of wealth is invariant over time, and that the repurchase rates used are not correlated with the profits realized on contract positions.

By definition,

\[ X_1 = \log \left( f(t, T) - f(t, 1) \right)^2 \] (100)

\[ X_2 = \log \left( P(T) - f(t, T) \right)^2 \] (101)

\[ Y = 2 \sum_{j=1}^{30} r_j \] (102)

\( X_1 \) defines the natural logarithm of the price variation over two closing periods.
X2 is the logarithm of the realized profits which is used as a proxy for expected profits.

Y is the cumulative interest rates over the assumed holding period.

T-t = 30 days   T-t = 31 days

Hence the hypothesized regression model is:

\[ Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \epsilon_t \] (103)

H : \beta_0 = 0, \beta_1 = 1, \beta_2 = 1

H : \beta_0 \neq 0, \beta_1 \neq 1, \beta_2 \neq 1

The null hypothesis implies that the model is subject to 3 restrictions. SSE(\hat{\beta}) is the error sum of squares of the restricted estimator. SSE(\hat{\beta}) is the error sum of squares of the unrestricted estimator. SSE(\hat{\beta}) = 83.31 and SSE(\hat{\beta}) = 416.19. The regression results are presented Table 23. We note that the coefficient estimates are generally significantly different from their hypothesized values. That is, all with the exception of the constant term.

This model specification also fails a joint test of its implied hypotheses. The calculated F-statistic of 23.5 leads us to reject the null hypothesis at both the 1% and 5% levels of significance. We also note that this F-statistic is the lowest reported in all our tests of the
implied hypotheses of various model specifications. It may be as our general pricing formula would indicate that these models would best explain price variation rather than price levels.

Nevertheless, our results indicate that this is not an entirely appropriate specification. The coefficient estimates here are also affected by collinearity in spite of the fact there is only one interest-rate regressor. The highest condition index is 12.9. Even though this condition index does not indicate evidence of serious collinearity, it exceeds the threshold level at which it is suggested that coefficient estimates may be affected.

We could not reject the hypothesis of no autocorrelation and Glejser's test for heteroskedasticity does not provide evidence against our maintained hypothesis. We also run this regression with interest rate as our dependent to determine the effect of any error-in-variables problem. In this case the $R^2$ are .301. This model specification also fails a joint test of its implied hypothesis.
Table 23
Regressions of Log of Price Variability on the Log
of Expected Profits and Financing Costs

\[ X_t = \beta_0 + \beta_1 Y + \beta_2 X_2 + \epsilon_t \]

<table>
<thead>
<tr>
<th>Estimates</th>
<th>S.Error</th>
<th>T-statistics</th>
<th>$H_0$: $\beta_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{\beta}_0$=4.751</td>
<td>2.513</td>
<td>1.89</td>
<td>0</td>
</tr>
<tr>
<td>$\hat{\beta}_1$=.2451</td>
<td>.1734</td>
<td>4.35**</td>
<td>1</td>
</tr>
<tr>
<td>$\hat{\beta}_2$=171.5</td>
<td>73.66</td>
<td>2.31*</td>
<td>1</td>
</tr>
</tbody>
</table>

$R^2$=.363

$D_w=1.87$

* significant at the 5% level.

** significant at the 1% level.
In this section we address the issue of the distribution of the futures price changes that comprise our sample. In section 4 of Chapter 3, we investigated the nature of the forecast errors associated with using futures prices as maturity price estimators. For the restricted data set, on which term and overnight repurchase rates were available, the results which should be interpreted with care because of the size of the sample. Our results indicate that these errors are normal, however an expanded data set provides evidence of non-normality. This data set based on 36 contracts shows that the forecast errors may be stably distributed but not normal. This highlights the sensitivity of the data set. Here we try to place in better focus the results of our regression specification in section 1 of this chapter.

David, Hartley and Pearson (1954) suggest that the w/s statistic, the ratio of the range to the standard deviation may be useful in detecting departure from normality, in small samples, if such a condition is indeed present. We reiterate that the results generated in this empirical exercise are not comparable with those in studies which
examine a sequence of futures prices, perhaps for several contracts, and generally find that such price changes are best described by stable processes. This exercise is directed at evaluating the validity of the normality assumption for our data set, when it is assumed that maturity effects are held constant.

If in fact our observations are drawn from a leptokurtic set, the maintained hypothesis of normality will generate coefficient estimates that are further contaminated by error. The usual t-statistics and F-statistics will be invalid. For the normalized price variable $y_1$, we find that $w$ is 0.0036 and $s$ is 0.00080. The value of the $w/s$ test statistic is 4.17. For $x_1$, the value of the test statistic is 4.18. By linear interpolation on David, Hartley and Pearson's (1954) tables, we find that the critical values of the test statistic are 5.01 and 4.72 at the 1% and 5% levels respectively. We cannot reject the null hypothesis of normality on the basis of this test. An analysis of the residuals of the regressions also support this conclusion. Hence it does not appear that our results are biased by a violation of the normality assumption.
CHAPTER V
SUMMARY

This dissertation examines various theoretical model specifications of forward and futures pricing and their implied hypotheses. The models cover a spectrum from discrete arbitrage models to continuous-time models. We derive and present explicit replicable strategies that indicate forward and futures contracts should not generally be equivalently priced even if interest rates are deterministic. In the limiting case, our general results are equivalent to those reported in the literature.

The consideration of an expanded information set and heterogeneous contract holders leads to an economy in which contract prices are free to deviate from their initial trading prices. We show that for any equilibrium, at a price other than the initial trading price, current forward and futures prices reflect not only the influence of past prices, but also of expectations of future movements in prices and interest rates. It is also shown that in the absence of explicit restrictions on position limits, price
movements and on the types of arbitrage strategies economic agents can pursue. Both forward and futures contracts are de facto contracts for the delivery of a random quantity of goods.

For the more general class of model we examine, we find that we could not analyze the hypotheses of backwardation or contango, without considering the past sequence of prices. This theoretical result seems to fit the empirical evidence that current futures and forward prices are affected by their past history as well as expectations of interest rates and price movements. This provides additional evidence from the contingent-claims literature that forward and futures premia over future spot prices may be either negative, zero or positive.

We test several model specifications to determine whether this class of models has any explanatory power relative to observed price differentials in the T-Bill market. We do find that that generalized versions of the models previously tested have some explanatory power. These models explain approximately 50% of the implicit forward-futures price differentials in the T-Bill market, in the analyzed sample.

We find that the regression model specification tested by French (1982) invalidly restricts the coefficients of
the interest-rate regressors to be equal. Removal of this restriction improves the explanatory of the basic model from 1% to 49%. However, this model, in all its tested versions, fails simple t-tests of its individual parameter estimates, as well as F-tests of its joint hypothesis.

Model B, which includes past forward and futures weighted by interest-rate factors as regressors, passes simple t-tests of its individual parameters. Even though this regression model also fails a joint test of its hypothesis, it appears to be more robust than Model A. This suggests that past forward and futures prices, appropriately weighted, are important in explaining currently observed price differentials. Their exclusion from the regression increases the potential for model failure.

However, all the model specifications that have been proposed and tested appear to be misspecified. The standard errors of the coefficient estimates in the various regressions is compromised by the effects of collinearity. Established corrective measures fail to reduce the severity of the problem. The only solution may be to find additional regressors that are uncorrelated with interest rates, and are not eliminated from the pricing relations when we take their differences. It is not known whether this type of problem accounts for the model failures reported in
other studies, but we find that even though the explanatory power of the models improves with removal of some of the implicit restrictions, they still could not be validated.

What our regression results suggest is that this class can in fact be useful for explaining intrasample variations in price differentials. The regression model specifications existing in the empirical literature have been improved, but more work needs to be done.

We did not address the issue of simultaneity. It may be that some of the regressors are not truly exogenously determined, as is implied by our regression specification. Hence our regression estimates may be subject to simultaneity bias.

The status of research work on this model, and its regression versions, does not validly permit general specification error tests of the types described by Pesaran (1979) or Davidson and McKinnon (1981). Since none of the models pass tests of their joint hypotheses, any attempt to choose between the non-nested regression specifications we discuss is without merit.

In future work, we will test these model specifications with commodity data, for which there are longer time series. We also propose to use other rates as proxies for
repurchase rates. We will also focus on the issue of simultaneity and whether other exogenous variables can lead to improved model performance. The models that have been tested to date presume that economy influences from variables such as output are imbedded in the observed variables in the model specification. We propose to relax this assumption. In commodity markets where production is not instantaneous, and harvest patterns, stockpiling and demand conditions lead to distinct price patterns, such an assumption may be particularly untenable.

As regards the empirical relation between forward and futures prices, we find that as we examine different maturity horizons the properties of the differentials change. However the pricing data, over a sample of 22 contracts, support the findings of earlier studies that forward rates are generally greater than futures rates for the spot contract. They also provide evidence that the variability of prices changes as the time-to-maturity changes.

We also examine the nature of the errors associated with using forward and futures prices as estimators of future spot prices. These errors are normally distributed but their mean is not zero. The evidence generated from our analysis supports the theoretical and empirical findings of several researchers. We also examine a model of price variability, and find weak evidence that the observed cur-
Rent price variability can be explained by expected future price variation as well as financing costs.
Footnotes


2. McCulloch also points out that the settlements in futures markets should be the discounted value of the current price movements. This should hold if the sole concern were one of settling future losses and profits between contract holders. The absence of such a discount rule suggests to us that other factors, such as market liquidity considerations, may be dominant.

3. By effective price, we mean the price the contract holder actually pays. This price includes both implicit and explicit costs.

4. See Telser (1981) for an explicit description of these costs, which may be incurred even if a performance guarantee is satisfied by the tendering of interest-bearing assets.

5. Brennan (1986) adduces evidence of the use of price limits as a partial substitute for margin requirements. Price limits on all interest rate contracts on the I.M.M. have recently been removed.
6. See Kane (1980) for a listing of other possible strategies.

7. In practice the dealer may provide collateral for the repurchase agreement in the form of any U.S. Government or Federal Agency Security. Here we discuss the repurchase agreement as if the collateral is the deliverable bill. The mechanics of the repurchase agreement are not fundamentally altered if the collateral and the deliverable bill are not identical securities.

8. For a detailed discussion of the repurchase agreement market, see Bowsher (1979), Lucas et al. (1977) and Smith (1978).

9. This ignores the trading and negotiation costs involved in negotiating the repurchase agreement and in establishing a short futures position. Such costs may be sufficient to preclude the execution of what appears to be a profitable strategy.

10. The pricing formulas for long and short holders of contracts differ with respect to their subjective rates of time discount. Obviously, since we assume that both groups intend to hold contracts to maturity, their current expectations about the future course of prices and interest rates must differ also. The discussion that follows, pro-
ceeds from the viewpoint of long holders of contingent-claim contracts.

11. Cox, Ingersoll and Ross (1985, b) also discuss the possible incorporation of dissimilar firms owned by individuals into the model. This extension is also shown by them to have no effect on equilibrium prices.

12. L is a partial differential operator on the function $J(Y, Q, W)$. A heuristic method suggested by Merton (1971) to obtain this operator is: (1) Take the differential of $J(Y, Q, W)$, applying Ito's Lemma.

   (2) Take the conditional expectation of the differential of $J(W, Q, W)$ and "divide" by $dt$. The resulting expression is the differential operator. This differential operator is also termed a weak infinitesimal operator.

13. For a rigorous derivation of the stochastic Bellman equation, see Dreyfus (1965). Merton (1969) shows that the resulting Bellman equation is valid $\forall t \in [0,T]$.

14. In the discussion that follows it should be noted that the general pricing formulas we derive are applicable to contingent-claim contracts on goods other than the numeraire good. Since we assume that $P_1 = 1$ identically at all times, the value of contingent claims on the numeraire good is zero at all times. In the case of the numeraire good, a
forward and a futures contract are priced equivalently. In this specific case, these contracts are redundant and the valuation formulas reduce to an identity.

15. $J_w$, $J_{Q_i}$, $J_Q$, $J_y$, etc are all partial derivatives of the value function with respect to its arguments. Note that $J_w$ and $J_t$ both equal zero, since $J$ does not explicitly depend on $t$.

16. To obtain the general result it is necessary that the utility weighted value of the investment at time $t$, equal the expectation of the discounted utility-weighted value of the terminal payoff.

17. Merton (1973) also states that the portfolio demands and the resulting equilibrium relationships will be a function of the specific trading intervals chosen.

18. It is trivially correct that a contract for the delivery of zero units of the underlying asset is zero-valued and is also zero-priced. However this serves to highlight the inadmissibility of assuming that contract prices are initially zero, since it would imply that economic agents contract for the delivery and receipt of zero units of the underlying goods for each contract.

19. See Cox, Ingersoll and Ross (1985, a) for a listing of works which derive analogous relations in an interest rate context.
20. For a more recent approach to the theory of the term structure, see Cox, Ingersoll and Ross (1985, a).

21. Richard and Sundaresan (1981) define a forward contract as a good hedge against consumption risk, in terms of the profit from holding a long position and the marginal utility of consumption. This is compatible with a definition that utilizes the expected maturity price and the marginal utility of wealth. If expected increases in prices are positively correlated with increases in wealth, thus tending to preserve welfare levels, the covariation between marginal utility of wealth and prices would be negative. On the other hand, if forward contracts are not welfare preserving the covariation between marginal utility of wealth and price would be positive. We have previously assured that $J(W, Q, Y)$ is concave in wealth.

22. See Richard and Sundaresan (1981) and French (1982) for details of this version of the forward plan.

23. See Inside T-Bill Futures, Rule 3208, for the daily price limits for T-Bill futures contracts.

25. These models should not rule out the possibility of sub-martingale or supermartingale type behavior.

26. For the strict martingale model to apply, sufficient parameter restrictions on equation (56) are.

(1) Either of $x(t)$ or $f_j(t_0, T)$ should be zero.

(2) $\text{cov}_t(\delta'(t), P_j(T)) = 0$.

(3) $\text{cov}_t(J_w(T), \delta'(t)(P_j(T) - f_j(t_0, T))) = 0$.

and.

(4) $\delta_s(t) = 1$.

Slightly weaker necessary conditions may hold. However we have no a priori reason to believe that $\delta^{**}(t) = 1$ and if $\delta(t) \neq 1$, the strict martingale result will not hold.

27. The term spot contract means the contract that is currently being traded for delivery at the nearest settlement date.

28. McCulloch has indicated linear interpolation may actually lead to biased estimates of forward rates and that other approximating functions such as cubic splines, McCulloch (19750), and grafted quadratic functions, McCulloch (1971), and Fuller (1969), may give better fits in general.
29. An alternative forward market has evolved in which T-Bills are traded on a when issued basis. This forward market is relatively limited, since the market is made by the purchase and sale of T-Bills during the period between the auction and the settlement for the instrument.

30. Another method would of course be to utilize survey data. Market participants could be asked to give information on their expectations of interest rates and future prices.

31. A substantial body of work generated by several authors including McCulloch (1984), Fama and Roll (1968, 1971), and Cornew, Town and Crowson(1984) indicates that futures prices and other financial price series are not well described by Gaussian processes. These authors have found that other stable processes provide a better fit.

32. Ex-gain day is defined as the day 6 months before the maturity of the contract when marginal investors lose the possibility of treating gains as capital gains. This option was a valuable one because of the different rates of taxation of capital gains and ordinary and short term gains.

33. These assumptions are necessary for the internal consistency of the model, and to ensure that tax-adjusted wealth is altered in all states for all contract holders. We rule
out any cash transfers from the external agency to the agents in the economy. The strategem of not endogenizing the actions of the third economic entity means that we eliminate the need for extensive discussions specifying this entity's utility function and how optimal taxation rates are determined. Hence solutions to the stochastic dynamic optimization problem of aggregate consumer utility subject to the wealth and asset valuation processes still characterize proper equilibrium relations.

This conclusion is invalid whenever taxes and tax rates are endogenously determined. In that case there must be interaction between the tax structure and asset valuation processes, and optimal contract holdings may altered in all states.
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