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The Ohio State University Ph.D. 1986

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VARIABLE STRUCTURE CONTROL SYSTEM MANEUVERING OF SPACECRAFT

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree Doctor Of Philosophy in the Graduate School of The Ohio State University

By

Osama A. Mostafa

The Ohio State University

1986

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To My Parents
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PUBLICATIONS


CONTENTS

DEDICATION ................................................................. ii
ACKNOWLEDGEMENT ......................................................... iii
VITA ........................................................................ iv
LIST OF FIGURES .......................................................... vii
LIST OF TABLES .......................................................... xii

CHAPTER PAGE
I  INTRODUCTION ...................................................... 1
  1.1 Introduction ......................................................... 1
  1.2 Literature Review .................................................. 1
  1.3 Features Of VSS ..................................................... 8

II  BASICS OF VARIABLE STRUCTURE CONTROL .............. 11
  2.1 Introduction ......................................................... 11
  2.2 Basics Of VSS ....................................................... 12
  2.3 Design Of Reaching Controls ................................. 19
    2.3.1 Hierarchical Control Approach ....................... 19
    2.3.2 Diagonalization Method ................................... 22
    2.3.3 Simultaneous Control ...................................... 24
  2.4 Robustness Properties .......................................... 25

III  APPLICATIONS I: BASIC VSS .................................... 28
  3.1 Introduction ......................................................... 28
  3.2 Maneuver of a Rigid Spacecraft .............................. 28
  3.3 Maneuvering of an Elastic Spacecraft ....................... 32

IV  CHATTER CONTROL .................................................. 66
  4.1 Introduction ......................................................... 66
  4.2 Boundary Layer Approach ...................................... 67
  4.3 Input Prefiltering .................................................. 70
  4.4 Asymptotic Reaching ............................................. 72
LIST OF FIGURES

2.1 Unstable Focus ................................................. 14
2.2 Unstable Saddle ................................................. 14
2.3 Variable Structure Control Trajectories ............... 14

3.1 NASA Standard Wheel Configuration ..................... 30
3.2 Rotor Angular Velocity $\Omega$ vs Time ..................... 33
3.3 Spacecraft Angular Velocity $\omega$ vs Time ............... 33
3.4 Euler's Angles $\alpha$ vs Time ................................. 34
3.5 Rotor Torques $U$ vs Time .................................... 34
3.6 Phase Plane Trajectory $\alpha_1$ vs $\omega_1$ .................... 35
3.7 Phase Plane Trajectory $\alpha_2$ vs $\omega_2$ .................... 35
3.8 Spacecraft Configuration ........................................ 36
3.9 Case 1 Attitude Error $\theta_e$ vs Time ....................... 42
3.10 Case 1 Angular Velocity $\dot{\theta}$ vs Time ................... 42
3.11 Case 1 1st Mode Amplitude $q_1$ vs Time ................. 43
3.12 Case 1 2nd Mode Amplitude $q_2$ vs Time ................ 43
3.13 Case 1 Phase Plane Trajectory $\theta$, vs $\dot{\theta}$ .................................................. 44
3.14 Case 1 Phase Plane Trajectory $q_1$, vs $\dot{q}_1$ .................................................. 44
3.15 Case 1 Phase Plane Trajectory $q_2$, vs $\dot{q}_2$ .................................................. 45
3.16 Case 1 Control Torque $T$ vs Time ............................................................. 45
3.17 Case 1 Control Force $F_1$ vs Time ............................................................. 46
3.18 Case 1 Control Force $F_2$ vs Time ............................................................. 46
3.19 Case 2 Attitude Error $\theta_e$ vs Time ............................................................. 47
3.20 Case 2 Angular Velocity $\dot{\theta}$ vs Time ............................................................. 47
3.21 Case 2 1st Mode Amplitude $q_1$ vs Time ............................................................. 48
3.22 Case 2 Phase Plane Trajectory $\theta$, vs $\dot{\theta}$ .................................................. 48
3.23 Case 2 Control Torque $T$ vs Time ............................................................. 49
3.24 Case 2 Control Force $F_1$ vs Time ............................................................. 49
3.25 Case 3 Attitude Error $\theta_e$ vs Time ............................................................. 50
3.26 Case 3 Angular Velocity $\dot{\theta}$ vs Time ............................................................. 50
3.27 Case 3 1st Mode Amplitude $q_1$ vs Time ............................................................. 51
3.28 Case 3 Phase Plane Trajectory $\theta$, vs $\dot{\theta}$ .................................................. 51
3.29 Case 3 Phase Plane Trajectory $q_1$, vs $\dot{q}_2$ .................................................. 52
3.30 Case 3 Control Torque $T$ vs Time ............................................................. 52
3.31 Case 3 Control Force $F_1$ vs Time ............................................................. 53
3.32 Case 4 Attitude Error $\theta_e$ vs Time ............................................................. 53
3.33 Case 4 Angular Velocity $\dot{\theta}$ vs Time ............................................................. 54
3.34 Case 4 Control Torque T vs Time .................................................. 54
3.35 Case 5 Attitude Error $\theta_e$ vs time ........................................ 55
3.36 Case 5 Angular Velocity $\dot{\theta}$ vs Time .................................. 55
3.37 Case 5 1st Mode Amplitude $q_1$ vs Time .................................... 56
3.38 Case 5 Phase Plane Trajectory $\theta_e$ vs $\dot{\theta}$ ......................... 56
3.39 Case 5 Control Torque T vs Time .................................................. 57
3.40 Case 5 Control Force $F_1$ vs Time ............................................. 57
3.41 Case 6 Attitude Error $\theta_e$ vs Time ........................................ 58
3.42 Case 6 Angular Velocity $\dot{\theta}$ vs Time .................................. 58
3.43 Case 6 2nd Mode Amplitude $q_2$ vs Time ................................... 59
3.44 Case 6 Phase Plane Trajectory $\theta_e$ vs $\dot{\theta}$ ......................... 59
3.45 Case 6 Control Torque T vs Time .................................................. 60
3.46 Case 6 Control Force $F_1$ vs Time ............................................. 60
3.47 Case 7 Attitude Error $\theta_e$ vs Time ........................................ 61
3.48 Case 7 Angular Velocity $\dot{\theta}$ vs Time .................................. 61
3.49 Case 7 Phase Plane Trajectory $\theta_e$ vs $\dot{\theta}$ ......................... 62
3.50 Case 7 Control Torque T vs Time .................................................. 62
3.51 Case 8 Angular Position $\theta$ vs Time ....................................... 63
3.52 Case 8 Angular Velocity $\dot{\theta}$ vs Time .................................. 63
3.53 Case 8 1st Mode Amplitude $q_1$ vs Time ................................... 64
3.54 Case 8 Control Torque T vs Time .................................................. 64
3.55 Control Force $F_1$ vs Time .................................................. 65

4.1 Maximum Allowable Parameter Variations For Stability vs P ........ 76

5.1 Case 1 Attitude Error $\theta_e$ vs Time ....................................... 82
5.2 Case 1 1st Mode Amplitude $q_1$ vs Time ................................. 82
5.3 Case 1 Phase Plane Trajectory $\theta_e$ vs $\dot{\theta}$ ......................... 83
5.4 Case 1 Control Torque $T$ vs Time ........................................ 83
5.5 Case 1 Control Force $F_1$ vs Time ........................................ 84
5.6 Case 2 Attitude Error $\theta_e$ vs Time ....................................... 84
5.7 Case 2 1st Mode Amplitude $q_1$ vs Time ................................. 85
5.8 Case 2 Phase Plane Trajectory $\theta_e$ vs $\dot{\theta}$ ......................... 85
5.9 Case 2 Control Torque $T$ vs Time ........................................ 86
5.10 Case 2 Control Force $F_1$ vs Time ........................................ 86
5.11 Case 3 Attitude Error $\theta_e$ vs Time ....................................... 87
5.12 Case 3 1st Mode Amplitude $q_1$ vs Time ................................. 87
5.13 Case 3 Phase Plane Trajectory $\theta_e$ vs $\dot{\theta}$ ......................... 88
5.14 Case 3 Control Torque $T$ vs Time ........................................ 88
5.15 Case 3 Control Force $F_1$ vs Time ........................................ 89
5.16 Case 4 Attitude Error $\theta_e$ vs Time ....................................... 90
5.17 Case 4 Phase Plane Trajectory $\theta_e$ vs $\dot{\theta}$ ......................... 90
5.18 Case 4 Control Torque $T$ vs Time ........................................ 91
5.19 Case 4 Control Force $F_1$ vs Time ............................................................... 91
5.20 Case 5 Attitude Error $\theta_e$ vs Time ............................................................ 92
5.21 Case 5 Phase Plane Trajectory $\theta_e$ vs $\dot{\theta}$ ........................................ 92
5.22 Case 5 Attitude Error $\theta_e$ vs Time ............................................................ 93
5.23 Case 5 Phase Plane Trajectory $\theta_e$ vs $\dot{\theta}$ ........................................ 93
5.24 Case 5 Attitude Error $\theta_e$ vs Time ............................................................ 94
5.25 Case 5 Phase Plane Trajectory $\theta_e$ vs $\dot{\theta}$ ........................................ 94

6.1 Case 1 Attitude Error $\theta_e$ vs Time ............................................................... 106
6.2 Case 1 1st Mode Amplitude $q_1$ vs Time ....................................................... 106
6.3 Case 1 2nd Mode Amplitude $q_2$ vs Time ....................................................... 107
6.4 Case 1 3rd Mode Amplitude $q_3$ vs Time ....................................................... 107
6.5 Case 1 Control Torque $T$ vs Time ................................................................. 108
6.6 Case 1 Control Force $F_3$ vs Time ................................................................. 108
6.7 Case 2 Attitude Error $\theta_e$ vs Time ............................................................... 109
6.8 Case 2 1st Mode Amplitude $q_1$ vs Time ....................................................... 109
6.9 Case 2 2nd Mode Amplitude $q_2$ vs Time ....................................................... 110
6.10 Case 2 3rd Mode Amplitude $q_3$ vs Time ...................................................... 110
6.11 Case 2 Control Torque $T$ vs Time ............................................................... 111
6.12 Case 2 Phase Plane Trajectory $\theta_e$ vs $\dot{\theta}$ ........................................... 111

A.1 Rigid Spacecraft Axes ................................................................................... 116

xi
B.1 Flexible Spacecraft Axes ........................... 133
LIST OF TABLES

3.1 Boundary Conditions ........................................... 32
CHAPTER I
INTRODUCTION

1.1 Introduction

Variable structure system (VSS) is a method of control by which different control laws are assigned according to the state of a dynamic system in a phase hyper-plane. The VSS was developed mainly in the USSR during the last 20 years. In the last decade western scientists began to realize the advantages of this method and began applying it to control of robotic systems, model following controls, and very recently to maneuvering of spacecraft.

This study is aimed at applying VSS to maneuvering of flexible spacecraft. Problems appeared in the applications to this new discipline, to which some solutions are proposed.

1.2 Literature Review

Our aim here is to review the papers that appeared on the VSS, and some of the papers on maneuvering spacecraft. The only text books available on this subject
are written by Russian authors.

The first book is written by V. I. Utkin [1]. In part one of this book he reviews the basic theory and its development. In part two the theory is applied to linear systems with scalar control. The applications to linear systems with vector control is given in part three.

The second and last book available is written by Itkis [2]. This book is a collection of lectures he has given at The University of Illinois as a visiting professor. The book deals mainly with scalar control of linear systems. The effects of noise, nonidealities in switching and time delays are also studied. Itkis also wrote two extensive survey papers [3] and [4]. In the first paper, the fundamentals of VSS, its advantages and current research topics are presented. In the second paper Itkis updates the first paper and gives an extensive list of publications primarily by the Russians.

In [5] Drazenovic proves the conditions for invariance of motion in sliding regimes in the case of time varying plants or a plant subjected to disturbances. It is shown that if the disturbance vector function belongs to the span of the columns of the control input influence matrix then the motion in the sliding regime is invariant. The ideas of this paper are included in [1] and are generalized to include the system parameter variations.

In [6] Utkin discusses various methods of finding the sliding motion equations. He discusses the case of vector controls and formalizes the method of equivalent
controls. The mathematical formulation is presented and the physical meaning of the equivalent control is discussed. Finally a comparison of the ideas discussed in this paper is made by Filippov's method. These ideas are included and are further developed in [1].

Bezvodinskaya and Sabaev derived in [7] and [8] and proved the necessary and sufficient conditions for the system states to reach the sliding regimes from any initial conditions. In [9] they formulate and prove the necessary and sufficient conditions for the stability of motion of linear systems using VSS. These results are also included in [1].

In [10] Utkin discusses some methods for choosing the constant coefficients matrix of the sliding regimes. Three methods are discussed in [10], namely: Eigenvalue placement; optimal sliding modes method, where a procedure for finding the optimal coefficients matrix by minimizing a quadratic performance index is given and the third method combines the second method with equivalent control optimal sliding modes. In this last method a performance index based on the states of the second method and the equivalent control is given. The conditions for minimizing this performance index are found to be Riccati like equations and have to be solved numerically.

K. K. Young published the first paper [11] applying the VSS to a robotic manipulator. The designed controls were used in a simulation which clearly showed the chatter phenomenon in VSS, and showed clearly that the system was
stable in spite of the chatter.

Young then used VSS to design a VSS model following control system \textsuperscript{12}. He proved that the variable structure controls have an inherent adaptation mechanism. He also showed that the error transient is invariant with respect to a class of parameter variations in the plant input matrix. A simulation was given for an aircraft control problem.

The same problem of model following control systems is studied in \textsuperscript{13},\textsuperscript{14} and \textsuperscript{19}. In \textsuperscript{15} the authors use the projector theory to design a closed loop system with arbitrary eigenvalues, and hence find the sliding matrix coefficients that satisfy these eigenvalues.

In \textsuperscript{16} Utkin uses VSS to control a distributed parameter system. The distributed parameter systems are modeled by an infinite dimensional partial differential equation. However, only a finite dimensional system is considered for control purposes, thus creating problems with the uncontrolled modes. In this paper Utkin discusses a method in which he uses Fourier's method of separation of variables, by which the distributed parameter system is approximated by a finite dimensional lumped system, describing the behavior of the first $N$ modes. The error due to this approximation was found, which gave a bound on $N$. Finally the ideas were applied to the problem of controlling heat distribution of a thin rod using a finite number of burners.

J. J. Slotine in \textsuperscript{17} gives an extended mathematical formulation of the VSS
basics, also pointing to the drawbacks of VSS, namely the chatter problem. He then develops the concept of a time varying sliding surface in the state space to solve the problem of robustness. He then proposes a solution to the problem of chatter, namely replacing the discontinuous control laws with continuous ones near the sliding surfaces using the saturation function as an approximation for the sign function. In [18] he ends the discussion of the methodology presented in his first paper. He first quantifies the trade off between robustness and tracking precision. To minimize the trade off, he discusses a method by which the perturbations are lowpass filtered and gives bounds on the filter break frequency. Finally Slotine extends the methodology to achieve optimal time varying tracking performance by using time varying boundary layer widths to account for time dependent parameter variations.

In [23] Calise and Kramer apply the VSS approach to the control of VTOL aircraft. First they give a short tutorial about variable structure control design. The summary of a VSS design that results in a velocity command system for the Harrier dynamics in hovering flight is given in [23]. A comparison between linear and nonlinear responses show that they are basically the same thanks to VSS.

In [24] Morgan and Özgüner discuss the application of VSS in conjunction with decentralized control to control a robotic manipulator. They use a modified form for the reaching conditions to reduce chattering while in the sliding mode, and they show that it is equivalent to using a filter. Applications are given for
two link and three link manipulator control.

In [25] Al Abbas and Özgüler extend the VSS theory to include the decentralized model reference adaptive control. They develop and prove the necessary conditions for decentralized controls. The chatter problem is addressed and a solution is proposed. Their method depends on using time variant sliding surfaces. An explicit form for the coefficients matrix is given. Simulations are presented for interconnected systems, using fixed and time variant sliding surfaces. Comparing both simulations, the chatter problem is considerably reduced.

The first application of VSS to a flexible spacecraft control problem was given by Öz and Özgüler in [20]. Öz and Özgüler used VSS in conjunction with the independent modal space control (IMSC). The advantages gained here, as discussed, were that the system modes are decoupled hence each mode is controlled independently which gives easier design methods. They showed that there arise switching lines and conjugate switching lines orthogonal to them, and a transformation is used to these switching lines. Implementation of VSS was discussed and finally these techniques were applied to a dual spin flexible gyroscopic spacecraft configuration.

Recently, Öz and Osama in [21] and [22] applied the VSS control to maneuvering of flexible spacecraft. References [21] and [22] are the results of research undertaken in this dissertation.

Another recent study of application of VSS to large angle maneuvers of rigid
spacecraft is presented by Vadali in [26]. Vadali finds the sliding surfaces using optimal control, where he introduces a performance index based on Euler parameters and angular velocities. The robustness of the control system to unmodeled dynamics, disturbance torques and parameter variations is shown.

All of the above cited work discuss the VSS theory and its applications, in what follows a brief review of papers on maneuvering spacecraft through optimal control is given. These papers are chosen due to the fact that they use the same spacecraft models as the ones chosen in this dissertation for application of VSS control to spacecraft maneuvers. This gives a basis for comparing the simulation results.

Turner in [28] and [29] formulates the problem for optimal large angle single axis maneuvers of a flexible spacecraft. The necessary conditions are found using Pontryagin's principle. A single stage relaxation process is proposed for solution of the two point boundary value problem (TPBVP). The initial approximation was found based on the solution of the linearized system costate variables. A relaxation process is started to increase the participation of the nonlinearity until full nonlinearity is considered and the solution of the nonlinear problem is found.

Vadali and Junkins in [30] considered the problem of a rigid spacecraft attitude maneuver. The open loop optimal control is considered first. A performance index based on the controls was defined and the necessary conditions for minimizing this index were found using Pontryagin conditions. These conditions led
to a TPBVP which when solved gave the optimal control history. The computations of feedback controls for the spacecraft was discussed next. The problem was formulated as a stability problem and Liapunov stability theory was used to find the feedback control structure. The methodology discussed was used in a numerical example.

In [31] Juang, Turner and Chun consider the problem of closed form solution of optimal control for maneuvering a linear spacecraft. A performance index depending on the output states and control was proposed. The necessary conditions for minimizing this index were found. Since this performance index does not depend on the final state, a new approach was discussed. The conditions of optimal solution were given by coupled Riccati like matrix differential equations. A closed form solution was found to these equations, which needed finding an inverse for two matrices at each time step. In [32] the authors extend the methodology to accomodate external disturbances and minimize interactions of residual modes.

### 1.3 Features Of VSS

In this dissertation the VSS control theory was chosen to maneuver spacecraft due to its advantages [21] over other control techniques for maneuvering of spacecraft. Currently, maneuvering of rigid and flexible spacecraft is addressed via optimal control theory [26-32]. Based on optimal control theory, formulations requiring
solutions of nonlinear Riccati or Riccati-like matrix differential equations are obtained. Generally, large angle single-axis maneuvers have been considered. A solution method based on "homotopy" or "relaxation method" [28] has been presented. Recently, closed-form solutions for optimal maneuvers with terminal constraints have been given [31]. Similar developments have been reported for disturbance accommodating tracking maneuvers [32].

A feature of all of the cited work is that considerable computational effort is required to obtain the optimal control gains. Furthermore, the control designs are characteristically off-line approaches. The controls are known only in "formulation" and their specific gain parameters have to be computed through extensive computational effort. Given the added complexities such as uncertainties, nonlinear effects and disturbances, a new cycle of computational effort is required. On the other hand, the variable structure control system (VSCS) maneuvering philosophy demonstrated in this dissertation requires no such endeavor. The structures of the control laws can be assigned in advance in consistence with reaching conditions. The qualitative behavior of the system is guaranteed to within the control parameters chosen by the control designer rather than obtained from some algorithmic procedures. Thus by careful choice of the control parameters, the response profile can be tailored to obtain an optimal behavior. The "Control Strategy" then becomes one of deciding which form of the control structure to activate as the trajectory evolves. The only programming effort is
to load the "switching logics" for the rules of structural changes.

In addition to the easiness of computations, VSS has other advantages that will be shown later as: insensitivity to variation in system parameters, modelling errors and neglected higher order dynamics. VSS is also insensitive to external disturbances.

The basic disadvantage of VSS is known as the "chatter phenomenon", which causes the states to chatter around the sliding surface. The chatter phenomenon is studied in this dissertation and three methods are proposed for solving this problem. The spillover problem associated with the control of flexible spacecraft is checked here and it was found that VSS has small effects on the uncontrolled modes. Then the basic disadvantages of applying VSS to maneuver spacecraft can be alleviated.
CHAPTER II

BASICS OF VARIABLE STRUCTURE CONTROL

2.1 Introduction

Variable Structure System (VSS) control technique is characterized by surfaces which guarantee asymptotic stability if the state trajectories move along them. These surfaces are called the sliding surfaces or sliding regimes. The motion along them is referred to as a "sliding mode". The aim of VSS is to design the control inputs in a way that the state of the system steers towards a sliding surface from any initial state, this is called the reaching phase. Once the trajectory point lies on the sliding surface, the control inputs are designed in a way that guarantees staying on the sliding surface thus guaranteeing asymptotic stability, this is called the sliding phase.

The VSS has attractive advantages of robustness to parameter uncertainties, neglected model dynamics and external disturbances. The main disadvantage of VSS arises from the fact that the controls are discontinuous which makes them vulnerable to nonidealities in switching and time delays. This disadvantage
causes what is known as the chatter phenomenon. The chatter phenomenon causes the controls to switch directions with a very high frequency. Therefore actuators with very high bandwidths are needed to implement the control inputs. Doubts have been expressed about the feasibility of realizing these control inputs at all.

2.2 Basics Of VSS

The basics of VSS theory can be best shown by using an example. Consider a time invariant system with scalar control

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -a_1 x_2 - U
\end{align*}
\]  

(2.1)

where \(x_1\) and \(x_2\) are state variables, \(a_1\) is a system parameter and \(U\) is a scalar control input. An overdot denotes time derivative. Let \(U\) be discontinuous on the straight line

\[s_0 = c x_1 + x_2 = 0, \quad c > 0\]  

(2.2)

i.e., let

\[
U = \begin{cases} 
U^+ = \alpha x_1 & \text{if } x_1 s > 0 \\
U^- = -\alpha x_1 & \text{if } x_1 s < 0
\end{cases}
\]  

(2.3)

In this case, \(s_0\) describes a one dimensional sliding surface (line) and \(c\) is a sliding surface coefficient which is picked by the control designer to guarantee a desirable
sliding motion. Let the trajectories of the system while applying \( U = \alpha x_1 \) be an unstable focus (Fig 2.1), and let the trajectories of the system with \( U = -\alpha x_1 \) be an unstable saddle point (Fig 2.2). According to the initial position, an appropriate \( U \) is applied which will move the trajectory point towards \( s_0 = 0 \) until it reaches it, hence the name the reaching phase. Once the trajectory crosses the line \( s_0 = 0 \) the control switches its direction thus forcing the trajectory towards \( s_0 = 0 \) again (Fig 2.3).

From Fig 2.3 it can be seen that the \( x_2 \) axis and the straight line \( s_0 = c_1 x_1 + x_2 \) divide the phase plane to four regions.

- region I: \( x_1 > 0, s > 0 \)
- region II: \( x_1 > 0, s < 0 \)
- region III: \( x_1 < 0, s < 0 \)
- region IV: \( x_1 < 0, s > 0 \)

Let the initial state lie in region I (\( x_1 s > 0 \)), so according to (2.3) \( U^- \) will be applied and the state will move an arc of a diverging spiral (unstable focus) towards the line \( s_0 \). At some time \( t_1 > t_0 \) the state will reach the line \( s_0 \) and enters region II. In region II (\( x_1 s < 0 \)) and according to (2.3) \( U^- \) will be applied and the state will move along an arc of a hyperbola (saddle point trajectory) towards the line \( s_0 \). After some time the state will leave region II and enter region I where the control is switched to \( U^+ \) again, thus forcing the state to
Figure 2.1: Unstable Focus

Figure 2.2: Unstable Saddle

Figure 2.3: Variable Structure Control Trajectories
region II again. It is seen that the state trajectory neither belongs to region I nor to region II and it has to move on the common border between the regions, in this case the straight line \( s_0 = c_1 x_1 + x_2 = 0 \). It is seen that the trajectory cannot lie outside the line \( s_0 = 0 \) and the only possible motion for the system trajectory is to stay on the surface \( s_0 = 0 \) and move on it. This motion is called the sliding motion and the line \( s_0 = 0 \) is called the sliding surface or the switching surface.

To generalize the previous ideas, consider the equations of motion of the flexible spacecraft (see Appendix B, Eq. B.43).

\[
\dot{X}_e = f(X_e) + BU + Dd
\]  

(2.4)

where \( X_e \) is a \( 2(n+1) \)-dimensional error vector, \( U \) is \( m \)-dimensional control inputs vector and \( d \) is a disturbance vector. Consider the equation of the sliding manifold to be

\[
S_0 = C X_e = 0
\]  

(2.5)

where \( C = m \times 2(n+1) \) dimensional constant (sliding) coefficients matrix. For any arbitrary spacecraft state, \( S = C X_e \) gives the position vector \( S \) relative to the intersection of \( m \) hyperplanes \( S_0 \) in the error state space. Note also that the origin of the error state space \( X_e \) lies in the sliding manifold \( S_0 = 0 \).

To study the stability of the system define a Lyapunov function

\[
\frac{1}{2} S^T S
\]  

(2.6)
It can be seen that $L$ will be a positive definite function for all $S$ except at the origin $S_0 = 0$. Evaluate the time derivative of $L$ along the trajectory $X_e$.

$$
\dot{L} = S^T \dot{S}
$$

(2.7)

The well known Lyapunov stability theory states that [35] if a positive definite function $V(x)$ can be found such that $\dot{V}(x)$ is negative definite, then the origin is asymptotically stable. It can be seen that in order to make the origin asymptotically stable we must have

$$
\dot{L} = S^T \dot{S} < 0
$$

(2.8)

The above inequality is known as the reaching condition [1]. Imposing this inequality means that the trajectory $X_e$ will converge to the origin $S_0 = 0$, i.e. reaching the sliding surface. The control inputs $U$ are found by using the inequality (2.8). (2.8) is referred to as the reaching condition(s) it will have the form, after substituting $\dot{S} = C \dot{X}_e$ and using (2.4):

$$
S^T C [f(X_e) + BU + Dd] < 0
$$

(2.9)

It is seen from the above inequality that depending on the sign of $S$, $U$ will have to change its value. Call these functions $U^+$ and $U^-$

$$
U = \begin{cases} 
U^+ & S > 0 \\
U^- & S < 0 
\end{cases}
$$

(2.10)

also notice that $U^+ \neq U^-$. These are called the reaching controls. Since $U$ is a
vector, finding explicit values for control components of $u_i^-$ and $u_i^-$ needs special approaches, these approaches are discussed in the next section.

Next, assume that at time $t_s$ the sliding manifold $S_0$ have been reached. The controls required to keep the system trajectories on the sliding manifold and move them towards the origin are called the equivalent controls $U_{eq}$. The equivalent controls can be found, at least theoretically, by imposing the constraint

$$\dot{S}_0 = C \dot{X}_e = C[f(X_e) + B U_{eq} + D \dot{d}] = 0 \quad (2.11)$$

Assuming that $[CB]^{-1}$ exist, then

$$U_{eq} = -[CB]^{-1} C[f(X_e) + D \dot{d}] \quad (2.12)$$

From now on it is assumed that the number of inputs is chosen equal to the number of controlled modes in the configuration space, that is $m = n + 1$. Further more choose the constant coefficient matrix $C$ as

$$C = \begin{bmatrix} C_1 & I \end{bmatrix} \quad (2.13)$$

$$C_1 = \text{diag}[c_1, \ldots, c_{n-1}] \quad (2.14)$$

where $c_i > 0$, and $I =$ identity matrix of dim$(n + 1)$. Once the sliding motion begins the trajectories of the system satisfy the equation of the sliding manifold

$$S_0 = C X_e = C_1 e + I \dot{e} = 0 \quad (2.15)$$

$$\dot{e} = -C_1 e \quad (2.16)$$
where $e$ is the configuration error vector. Equation (2.16) is the governing equation of motion of the error dynamics on the sliding manifold. This equation has the solution

$$e = e(t_s) \cdot \exp(-C_1(t - t_s))$$  \hspace{1cm} (2.17)

since $C_1$ is a design parameter, then by proper choice of these parameters, that is a positive definite $C_1$, the configuration error vector will vanish asymptotically

$$\lim_{t \to \infty} e = 0$$  \hspace{1cm} (2.18)

If $C_1$ is chosen according to (2.14) then equation (2.16) represents a set of $(n - 1)$ uncoupled first order linear systems, i.e. the system (2.4) is represented by a reduced order model in the sliding mode. The elements $-c_i$ are the eigenvalues of (2.16), so choosing $c_i$ as positive constants guarantees asymptotic stability of the errors, while large values for $c_i$ means faster convergence in the sliding regime.

From (2.16) the motion on the sliding manifold have the following properties:

1. The system is linearized, so it is insensitive to nonlinearities.

2. The motion is insensitive to external disturbances.

3. The motion depends only on the sliding regime coefficients matrix $C$ which is chosen by the designer.
2.3 Design Of Reaching Controls

In the previous section it was shown that the reaching condition from any initial states is

\[ S^T \dot{S} < 0 \]  (2.19)

Finding control inputs using the above inequality is not an easy task, since usually the controls are coupled. Three approaches are discussed here to solve this problem. The basic idea behind these methods is to decouple the inequalities and design one control input to satisfy one inequality.

2.3.1 Hierarchical Control Approach

The first step in designing the reaching controls [1] is to chose the sliding manifold equation in the form

\[ S = CX_e \]  (2.20)

where it is assumed that the \( i^{th} \) control \( u_i \) is discontinuous on the surface \( s_i = 0 \). where \( s_i \) is the \( i^{th} \) component of \( S \). In order to apply this method we assume a hierarchy of reaching the sliding surfaces for example \( s_1 - s_2 \cdots - s_m \), that is, \( s_1 \) is reached then \( s_2 \), then \( s_3 \), finally \( s_m \).

Now let \( k = m \) and assume that sliding motion has occured on surfaces \( s_i = 0 \), \( i = 1, \ldots, m-1 \). Since it was assumed that sliding motion has occured on these
surfaces the equivalent controls can be found from imposing the condition

\[ \dot{s}_i = 0 \quad i = 1, \ldots, m - 1 \tag{2.21} \]

so \( u_{eq} \), can be found as a function of the states, the sliding surface parameters \( c_i \) and the control \( u_k \). Now using the reaching condition, a sliding mode will be formed on the surface \( s_k = 0 \) iff

\[ s_k \dot{s}_k < 0 \tag{2.22} \]

from which \( u_k^+, u_k^- \) can be found. Next let \( k = m - 1 \), with \( u_m \) known and assuming that sliding have occurred on \( s_i = 0 \) and \( i = 1, \ldots, m - 2 \), find \( u_{m-1} \).

And so on until \( k = 1 \).

For an intermediate step equation (2.22) has the form

\[ b_{kk} u_k \leq -C_i^T f(X_e) - C_i^T Dd - \sum_{i=1}^{k-1} b_{ki} u_{eqi} - \sum_{i=k+1}^{m} b_{kl} u_l \cdots s_k \geq 0 \tag{2.23} \]

where \( u_{eq} \) is the equivalent control and is a function of \( u_k \). \( u \) is the control vector found from the previous steps, \( C_i^T \) is the \( i^{th} \) row of matrix \( C \), and \( b_k \) is the \( k^{th} \) row vector of \( B \). For example, these inequalities can be satisfied if

\[ u_k = b_{kk}^{-1} [1. - g_k \ \text{sign}(s_k) \ \text{sign}(F_k)] F_k \tag{2.24} \]

\[ F_k = -C_i^T f(X_e) - C_i^T Dd - \sum_{i=1}^{k-1} b_{ki} u_{eqi} - \sum_{i=k+1}^{m} b_{kl} u_l \tag{2.25} \]

where \( g_k > 0 \) are control parameters chosen by the designer.

This method through introducing hierarchy of reaching sliding surfaces will assign the control \( u_m \) which guarantees motion in sliding mode along the surface
$s_m = 0$ for any values $u_1, \ldots, u_{m-1}$. While the control $u_{m-1}$ will guarantee motion along the intersection of surfaces $s_m = 0$ and $s_{m-1} = 0$ for whatever values of $u_1, \ldots, u_{m-2}$. In the end the control inputs will guarantee reaching the intersection of all the sliding surfaces, which is the desired sliding manifold.

There are other types of hierarchical control approach. For example [21] for the maneuvering dynamics of spacecraft of appendix B, take $n=2$ and consider

$$s_3 = C_3 q_2 + \dot{q}_2, \quad s_2 = C_2 q_1 + \dot{q}_1, \quad s_1 = C_1 \theta + \dot{\theta}.$$  

Assign $F_2$ for $s_3$, $F_1$ for $s_2$ and $T$ for $s_1$. Next, assume the hierarchy

$$s_0_2 \rightarrow s_0_3 \rightarrow s_0_1.$$

The control design would proceed as follows.

Step 1: $k = 3$. Use $F_2$ to satisfy $s_3 \dot{s}_3 < 0$ and assume $\dot{s}_2 = \dot{s}_1 = 0$. One way of satisfying $s_3 \dot{s}_3 < 0$ is to use the simultaneous control approach of sec.2.3.3 to obtain $F_2$ via (2.45) which will require the values of $F_1$ and $T$. Since $\dot{s}_2 = \dot{s}_1 = 0$ is assumed, the needed values are the equivalent controls $F_1$ and $T$ which can be obtained from equations similar to (2.11).

Denoting the row vectors of $B$ corresponding to $\theta, q_1$ and $q_2$ dynamics by $b_1, b_2$ and $b_3$, respectively, from (2.45)

$$F_2 = b_3^{-1} \left[ -D_3 \text{sign}(s_3) - (c_3 q_2 + b_3 T_0 + b_{32} F_1) \right]$$  (2.26)

where $f_i$ is the $i + n + 1$ element of the vector of $f(X_e)$. 


$F_1$, and $T_0$ will have to be obtained from equivalent control equations similar to (2.11). The solutions for $T_0$ and $F_1$, will thus be obtained in terms of $F_2$. Therefore when $T_0$ and $F_1$, are introduced into (2.26), it will have to be rearranged for $F_2$.

Step 2: $k = 2$. Use $F_1$ to satisfy $s_2 \dot{s}_2 < 0$ and assume $\dot{s}_1 = 0$. Similar to (2.26)

$$F_1 = b_{22}^{-1} \left[ -P_2 \text{sign}(s_2) - (c_2 \dot{q}_1 + f_2 + D_2^T d + b_{21} T_0 + b_{23} F_2) \right] \quad (2.27)$$

where $T_0$ is obtained as in (2.11) in terms of $F_2$ and $F_1$. $F_2$ is available from step 1.

Step 3: $k = 1$. Use $T$ to satisfy $s_1 \dot{s}_1 < 0$:

$$T = b_{11}^{-1} \left[ -P_1 \text{sign}(s_1) - (c_1 \dot{q}_e + f_1 + D_1^T d + b_{13} F_2 + b_{12} F_1) \right] \quad (2.28)$$

where $F_1$ and $F_2$ are available from steps 1 and 2

2.3.2 Diagonalization Method

In this method [1] a linear transformation is used to decouple the controls in such a way that each sliding surface will depend only on one control input. Then using the reaching condition on each sliding surface we can find the control input that guarantees motion in sliding mode along the surface $s_i = 0$. Finally using the reverse linear transformation we can find the original control inputs.
To show the above procedure let us have the system

\[ \dot{X}_s = f(X_s) + BU + Dd \]  

and let the equation of the sliding surfaces be

\[ S_0 = CX_s = 0 \]  
\[ \dot{S} = C\dot{X}_s = Cf(X_s) + CBU + CDd \]

Let

\[ QU' = CBU \]

where \( Q = \text{diag}(q_1, \ldots, q_m) \) and \( q_i > 0 \). Writing each reaching condition separately

\[ s_i \dot{s}_i < 0 \]  
\[ s_i (C_i^T f(X_s) + q_i u_i + C_i^T Dd) < 0 \]  
\[ u_i \begin{cases} < & -q_i^{-1}(C_i^T f(X_s) + C_i^T Dd) \quad s_i > 0 \\ > & -q_i^{-1}(C_i^T f(X_s) + C_i^T Dd) \quad s_i < 0 \end{cases} \]

where \( C_i^T \) is the \( i^{th} \) row of matrix \( C \). Equation (2.35) can be clearly satisfied if we use the nonlinear controls

\[ u_i = 1 - g_i \text{sign}(F_i) \text{sign}(s_i) |F_i| \]  
\[ F_i = -q_i^{-1}[C_i^T f(X_s) + C_i^T Dd] \]

where \( g_i > 0 \) are control parameters chosen by the designer. Larger values for \( g_i \) mean stronger satisfaction of (2.35). For a linear system

\[ \dot{X}_s = AX_s + BU \]
\( u_i \) can be found as a linear state feedback of the form

\[
 u_i = \sum_{j=1}^{2(n-1)} G_{ij} X_e \tag{2.39}
\]

where \( G_{ij} \) are the control gains. Using the reaching condition (2.33), we get

\[
s_i [C_i^T A X_e + q_i \sum_{j=1}^{2(n-1)} G_{ij} X_e] < 0 \tag{2.40}
\]

Then satisfying this inequality element by element, we get

\[
 G_{ij} < -q_i^{-1} [CA]_{ij} \quad \cdots \quad s_i X_{e_i} > 0
\]

\[
 G_{ij} > -q_i^{-1} [CA]_{ij} \quad \cdots \quad s_i X_{e_i} < 0 \tag{2.41}
\]

Finally

\[
 U = [CB]^{-1} QU \tag{2.42}
\]

assuming of course that \( CB \)^{-1} exists.

### 2.3.3 Simultaneous Control

One method of satisfying the reaching condition \( S^T \dot{S} < 0 \), which has the explicit form

\[
 \sum_{i=1}^{m} s_i \dot{s}_i < 0 \tag{2.43}
\]

is by satisfying each term of the inequality separately. The reaching condition is written in an alternate form [24]:

\[
 \dot{s}_i = -P_i \text{sign}(s_i) \tag{2.44}
\]
where $P_i > 0$. Consider the system

$$\dot{X}_s = f(X_s) + BU + Dd$$

and the sliding surface equations

$$S_0 = CX_s = 0$$

substituting from the above equations, we get

$$\dot{S} = Cf(X_s) + CBU + CDd = -P\text{sign}(S) \quad (2.45)$$

The above equation has $m$ unknown control inputs, and $U$ can be found as

$$U = [CB]^{-1} [-P\text{sign}(s) - Cf(X_s) - CDd] \quad (2.46)$$

The above approach guarantees reaching each sliding surface independently, and once all the sliding surfaces are reached the motion will lie on the sliding manifold. It should be noticed that satisfying the reaching condition in general, does not require that each component of the summation to satisfy the inequality separately. Hence this is a conservative approach.

### 2.4 Robustness Properties

One of the basic requirements on any control scheme is its robustness to parameter uncertainties, external disturbances, unmodeled dynamics, etc. With the condition that the dimension of the control input is the same as the dimension of
that is $m = n - 1$, it was shown earlier that once the system begins moving on the sliding manifold it satisfies its equation, and the sliding motion is insensitive to nonlinearities, external disturbances and parameter uncertainties. It remains to show that robustness is guaranteed in the reaching phase of the maneuver.

Consider a system

$$\dot{X}_e = A(t)X_e + BU + \Delta f$$

where $\Delta f$ contains all the nonlinear terms, external disturbances and parameter uncertainties. Here it is assumed that these terms are available for measurement and their bounds are known. First design the controls based on the linear, deterministic, noise-free model of the spacecraft. As before, assume the sliding surface equations to be

$$S_0 = CX_e = 0$$

and use the reaching conditions to find the control vector $U$

$$C_i^TBU \geq -C_i^TAX_e \quad s_i \leq 0 \quad (2.47)$$

where $(\quad)_i$ is the $i_{th}$ row of the respective matrix. In order to overcome the $\Delta f$ term, subtract a constant term $\delta$ from the right hand side to get

$$C_i^TBU \geq -(C_i^TAX_e + \delta_i) \quad s_i \leq 0 \quad (2.48)$$

But the reaching condition for the actual system is

$$\dot{s}_i = C_iAX_e + C_iBU + C_i\Delta f < 0 \quad s_i > 0 \quad (2.49)$$
using the limiting value for $C_i^T B U$. and substituting from (2.48) in (2.49). the inequality simplifies to

$$\delta_i > C_i^T \Delta f \quad \cdots \quad s_i > 0$$  \hspace{1cm} (2.50)

but since it was assumed that the bounds for $\Delta f$ are known then

$$\delta_i > \pm \max C_i^T |\Delta f| \quad \cdots \quad s_i > 0$$  \hspace{1cm} (2.51)

These inequalities will insure satisfying the reaching conditions for the actual system and the reaching motion will be robust under nonlinearities, parameter uncertainties and external disturbances.

The $\delta_i$ terms result in stronger satisfaction of the inequalities, hence overcoming the undesirable effects on the system dynamics. We also note that the robustness will be affected by the degree of choice of sliding regime parameters.
CHAPTER III
APPLICATIONS I: BASIC VSS

3.1 Introduction

The basic concepts and ideas behind VSS were discussed in chapter II. The applications of these concepts are shown here as applied to a three dimensional maneuver of a rigid spacecraft. A more complicated problem is considered next, which is a single axis maneuver of a flexible spacecraft. Examples are given for regulation problems, tracking problems, and disturbance accomodations.

3.2 Maneuver of a Rigid Spacecraft

Appendix A shows that the system equations of motion are

\[
\dot{\mathbf{w}} = -[\mathbf{I}]^{-1} [\mathbf{w}] [I^* \mathbf{w} + \mathbf{A} \mathbf{\Omega}] + [B]^T \mathbf{U} \tag{3.1}
\]

\[
\dot{\alpha} = [\alpha]^{-1} \mathbf{w} \tag{3.2}
\]

\[
\dot{\mathbf{\Omega}} = [B][\mathbf{I}]^{-1} [\mathbf{w}] [I^* \mathbf{w} + \mathbf{A} \mathbf{\Omega}]
+ [\mathbf{J}_o]^{-1} \mathbf{U} \tag{3.3}
\]

28
where each symbol is defined therein.

In the absence of external forces the angular momentum is conserved, therefore we cannot control both rotor angular velocity ($\Omega$) and body angular velocity ($\omega$). But since we want to control the orientation angles ($\alpha_V$) we have to control the body angular velocity since they are related through equation (3.2). In consequence we have to accept whatever the final rotor angular velocities might be in order to satisfy the constraint of constant angular momentum in the new position.

From equation (3.2) we can find that $\alpha_{V_i}$ does not appear explicitly, i.e. it is an ignorable coordinate, so we have to feed it back. To do that let the first rotor torque be of the form

$$u_1 = u_{11} + u_{12}$$

$$u_{12} = k\alpha_{V_i}$$

(3.4)

(3.5)

Where $k$ is a constant coefficient to be chosen by designer. From now on we will drop the subscript $V$ for simplicity. Now transferring to the state space form let

$$e_1 = \Omega_1 \quad e_2 = \Omega_2 \quad e_3 = \Omega_3 \quad e_4 = \Omega_4$$

$$e_5 = \omega_1 - \omega_1 \quad e_6 = \omega_2 - \omega_2 \quad e_7 = \omega_3 - \omega_3$$

$$e_8 = \alpha_1 - \alpha_1 \quad e_9 = \alpha_2 - \alpha_2 \quad e_{10} = \alpha_3 - \alpha_3$$

(3.6)

where $(-)_{-}$ stands for the desired final states. Now we want to find $U$ that gives us the desired final states. The controls were designed using the variable structure theory. This problem is a three dimensional rotation, i.e. we have three rotational degrees of freedom for the spacecraft. Then we can only have
Figure 3.1: NASA Standard Wheel Configuration

three sliding surfaces, let their equations be

\[ s_{\alpha i} = c_{\alpha} e_{\alpha} + e_{\alpha} \]

\[ s_{02} = c_{2} e_{0} - e_{2} \]

\[ s_{03} = c_{3} e_{7} + e_{10} \]  \hspace{1cm} (3.7)

The NASA standard reaction wheel configuration \cite{30} was used (Fig 3.1). It has three rotors aligned with the three spacecraft reference axis, and a fourth rotor is situated such that its axis is equally inclined from the three reference axis. For our simulations, the spacecraft parameters used in \cite{30} were used here. The spacecraft parameters are; the mass of the spacecraft alone, \( M \), is 500 Kg; the mass of each wheel, \( m \), is 5 Kg; the unit length from the center of cluster.
is 0.2 m; spacecraft alone, principal moment of inertia about the x axis, \( I_1 \), is \( 86.215 \text{ Kg m}^2 \); about the y axis, \( I_2 \), is \( 85.07 \text{ Kg m}^2 \); and about the z axis, \( I_3 \), is \( 113.565 \text{ Kg m}^2 \); wheel axis moment of inertia, \( J_a \), is 0.5 Kg m\(^2\).  

\[
\begin{bmatrix}
    \hat{I} \\
    B
\end{bmatrix} = \begin{bmatrix}
    87.19238 & 0 & 0 \\
    0 & 85.94873 & 0 \\
    0 & 0 & 114.4996
\end{bmatrix} \quad (3.8)
\]

\[
\begin{bmatrix}
    \epsilon \\
    \eta
\end{bmatrix} = \begin{bmatrix}
    0.97999 & 0.19887 & -0.00871 \\
    -0.19892 & 0.97998 & -0.00836 \\
    0.00688 & 0.00993 & 0.99993 \\
    0.45492 & 0.68634 & 0.56745
\end{bmatrix} \quad (3.9)
\]

The diagonalization method (sec.2.3.2) was used to maneuver the spacecraft via VSS control. The matrix \( Q \) was chosen to be the identity matrix of dimension 3. Since the control vector \( U \) has dimension four while the vector \( U^* \) has dimension three, a minimum norm solution is sought [36]. To avoid the jump in controls initially a filter is used to shape the control inputs. The filter was given by

\[
m(t) = (1 - e^{-3\tau})
\]

where \( \tau = \frac{\dot{T}}{T_r} \) and \( T_r \) is the rise time chosen to be 10 sec. The initial and final states were chosen as shown in Table 3.1.

The desired maneuver time was 60 sec. We used a Runge Kutta integration scheme with time step 0.1 sec. The feedback error gain was \( k = 0.5 \) and the sliding surface parameters were \( c_1 = c_2 = c_3 = 1 \).

The results are shown in Figures 3.2-3.7. The phase plane trajectories Fig 3.6 and Fig 3.7 clearly show the sliding surfaces as straight lines. The results obtained
<table>
<thead>
<tr>
<th>State</th>
<th>Initial Conditions</th>
<th>Final Conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_1$</td>
<td>1.34</td>
<td>0</td>
</tr>
<tr>
<td>$\alpha_2$</td>
<td>1.34</td>
<td>0</td>
</tr>
<tr>
<td>$\alpha_3$</td>
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<td>0</td>
</tr>
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</tr>
<tr>
<td>$\Omega_2$</td>
<td>0</td>
<td>free</td>
</tr>
<tr>
<td>$\Omega_3$</td>
<td>0</td>
<td>free</td>
</tr>
<tr>
<td>$\Omega_4$</td>
<td>0</td>
<td>free</td>
</tr>
<tr>
<td>$\omega_1$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\omega_2$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\omega_3$</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 3.1: Boundary Conditions

Here were comparable to [30], in which the authors used optimal control and had to solve Riccati equations. Comparing both results, it can be seen (fig. 9 [30]) that the maximum torque value reached was 1.2 Nt-m in both cases. The torque profiles are almost the same. The rotor angular velocities (fig.9 [30]) also reached their maximum value of 120 rad/sec, while in our case our simulations gave a maximum of 95 rad/sec.

### 3.3 Maneuvering of an Elastic Spacecraft

This section considers the maneuvering of a flexible spacecraft. The spacecraft configuration is shown in Fig 3.8. The model consists of a central hub with four flexible appendages cantilevered from it. The same model was used in [31] and the same numerical data given there are used here.

The parameters of the spacecraft were: the moment of inertia $I=6764$ Kg $m^2$; the length of each appendage $L=35$ m; the radius of the rigid hub $R=1.0$ m;
Figure 3.2: Rotor Angular Velocity $\Omega$ vs Time

Figure 3.3: Spacecraft Angular Velocity $\omega$ vs Time
Figure 3.4: Euler's Angles $\alpha$ vs Time

Figure 3.5: Rotor Torques $U$ vs Time
Figure 3.6: Phase Plane Trajectory $\alpha_1$ vs $\omega_1$

Figure 3.7: Phase Plane Trajectory $\alpha_2$ vs $\omega_2$
the flexural rigidity of each cantilevered appendage $EI_A = 1500 \text{ Nt m}^2$ and the mass per length of the four identical elastic appendages $\rho = 0.09096 \text{ Kg m}$. The control inputs were as follows. a rigid body torquer $T$ was located on the central hub and two control forces $F_1$ and $F_2$ were located on each appendage at $0.29L$ and $0.84L$ respectively.

The comparison functions used to discretize the elastic displacements of the appendages were assumed to have the form

$$\phi_r(x) = 1 - \cos \left( \frac{r\pi x}{L} \right) + 0.5(-1)^{r-1} \left( \frac{r\pi x}{L} \right)^2$$

(3.11)

where $r = 1, \ldots, n$, and they satisfy the boundary conditions of the cantilevered
appendages

\[ \phi_r|_{z=0} = \phi'_r|_{z=0} = \phi''_r|_{z=L} = \phi'''_r|_{z=L} = 0 \]  

(3.12)

In our model, two elastic coordinates were considered, \( n = 2 \). A single axis maneuver was considered where the axis of rotation was normal to the plane of appendages. This resulted in a six order state space formulation. The three sliding surfaces were assumed to have the form

\[ s_0_1 = c_1 \theta_e + \dot{\theta}_e \]
\[ s_0_2 = c_2 q_1 + \dot{q}_1 \]
\[ s_0_3 = c_3 q_2 + \dot{q}_2 \]  

(3.13)

where \( \theta_e, q_1 \) and \( q_2 \) are the attitude error angle, and the first and second elastic appendage coordinates respectively.

Several case studies were done, each case is discussed below.

Case I: This case was a rest to rest maneuver done using hierarchical control method (sec. 2.3.1). We required a slewing maneuver of 0.4 rad. during a time of 14 sec. Thus \( \dot{\theta} = \ddot{\theta} = 0 \) and the initial error state vector is

\[ X^T(0) = [ -0.4 \ 0 \ 0 \ 0 \ 0 \ 0 ] \]  

(3.14)

The attitude angle error feedback gain was \( k = 500 \). The switching logic

\[ \dot{s}_i = -P_i \text{sign}(s_i) \]  

(3.15)
was used here with $P_1 = 0.5 \| \theta_e \|, P_2 = \| q_2 \|, P_3 = \| \dot{q}_2 \|$. This choice will smooth the control inputs as the origin is approached. The sliding regime parameters were chosen as $c_1 = 1.0, c_2 = c_3 = 1.0$. The order of reaching hierarchy was assumed to be $s_{0_2} \to s_{0_2} \to s_{0_1}$, which is the natural hierarchy since initially the second and third error states lie on their sliding surfaces. Figures 3.9-3.18 show the simulation results. From the phase plane trajectories, Figs 3.13-3.14, we can clearly see the reaching phase and the sliding phase corresponding to the straight line trajectory. In Fig 3.16 it is seen that the control inputs switch between $u^+ \theta$ and $u^- \theta$ to keep the system trajectory on the sliding surface.

Case 2: This was a rest to rest maneuver done using hierarchical control. The same initial state error of case 1 was assumed. The control parameters of equation (2.24) were chosen as $g_1 = g_2 = g_3 = 0.5$. The attitude error gain was $k = 3000$. The sliding surface parameters were $c_1 = 1.0, c_2 = c_3 = 1.0$. Figures 3.19-3.24 show the simulation results.

The required maneuver time was 14 sec. But the maneuver took 18.62 sec. The reason for this delay can be seen in Fig 3.22 where it was noticed that the slope of the sliding line was $c_1 = 0.3$ instead of the required $c_1 = 1.0$. In this case the system hit a natural asymptote and followed it to the origin, rather than the prescribed sliding surface which had a higher slope. The system is guaranteed to be stable in this case, since any deviation from the asymptote the reaching conditions would guarantee hitting the sliding surface.
Case 3: To show that this was really the reason for the delay in case 2, the same maneuver was repeated with \( c_1 = c_2 = c_3 = 0.1 \) so that the sliding surfaces will be reached first rather than the asymptotes. Figures 3.25-3.31 show the simulation results. From the phase plane trajectories Fig 3.28 and Fig 3.29 it can be seen that the slope of the sliding lines are \( c_1 = c_2 = 1.0 \). The chatter phenomena appears clearly in these figures and also in the control input time histories Fig 3.30 and Fig 3.31. Realizing these inputs would require high bandwidth actuators.

Case 4: In the previous cases, it is noticed that there is a jump in the controls at the beginning of the maneuver. Initial jumps in control inputs are undesirable because they cause large excitations in the unmodeled elastic modes. In order to eliminate this jump the control inputs were passed through a filter given by

\[
F(t) = (1 - e^{-2t})
\]  

(3.16)

The same maneuver done in case 2 was repeated here using the filter. Figures 3.32-3.36 show the simulation results. It can be seen from Fig 3.34 that the control torque is smoother. However the filter introduced a time lag which increased the maneuver time to 19.84 sec.

Case 5: In order to show the disturbance accommodation of VSS, we introduced a disturbance torque input

\[
T_d = 50(\sin 2\pi t + \sin 4\pi t)
\]

(3.17)

on the central hub. The chosen parameters were like case 2. Figures 3.37-3.40
show the simulation results. Comparing Fig 3.19 and Fig 3.35 it can be seen that
the disturbance has no effect on the attitude angle. But comparing the phase
plots Fig 3.22 and Fig 3.38 it can be seen that the system tries to keep moving
on the asymptote until it is finally disturbed away and reaches the sliding surface
where the trajectory is smooth, i.e. the motion is insensitive to the disturbance
on the sliding manifold.

Case 6: This is the same as case 5. In order to prevent the initial jump in the
control inputs the filter (3.16) was used. Figures 3.41-3.46 show the simulation
results. It is seen that the sliding surface was reached finally in Fig 3.44.

Case 7: In order to show the robustness of VSS to neglected nonlinearities
the control inputs were designed using a linearized version of the equations of
motion, and were applied to the nonlinear system. The diagonalization method
(sec.2.3.3) was used here and since the system is linear, a feedback gain matrix
was chosen as

\[ G_{ij} = \begin{cases} \alpha_{ij} & s_i X_{s_j} > 0 \\ -\alpha_{ij} & s_i X_{s_j} < 0 \end{cases} \]  \hfill (3.18)

where \( i = 1, 2, 3 \) and \( j = 1, \ldots, 6 \) and

\[ \alpha_{ij} = \begin{bmatrix} -0.22 & -0.028 & -0.037 & -1.1 & 0 & 0 \\ 0 & -0.375 & 27.703 & 0 & -1.1 & 0 \\ 0 & -0.372 & 11.907 & 0 & 0 & -1.1 \end{bmatrix} \]  \hfill (3.19)

and \( Q \) was chosen as the identity matrix. The controls satisfied the inequalities
strong enough that the nonlinear terms were accommodated. Figures 3.47-3.50
show the simulation results. From the phase plane trajectory Fig 3.49 it can be
seen that the linear and nonlinear systems behave differently during the reaching phase. The nonlinear system manages to reach the sliding surface inspite of the effects of the nonlinearities. Once on the sliding surface both systems had the same behaviour, which proves that the system is linearized around the sliding surface.

Case 8: The last case studied was a rest to spin up maneuver where the desired attitude spin rate was given by

$$\dot{\theta}^* = 0.04 \left(1 - e^{-t/3}\right)$$

Hierarchical control was used with the sliding parameters $c_1 = 1.0, c_2 = c_3 = 1.0$, the attitude error gain was $k = 3000$, and the control parameters were chosen as $g_1 = g_2 = g_3 = 0.99$. The results are shown in Figures 3.51-3.55.
Figure 3.9: Case 1 Attitude Error $\theta_e$ vs Time

Figure 3.10: Case 1 Angular Velocity $\dot{\theta}$ vs Time
Figure 3.11: Case 1 1st Mode Amplitude $q_1$ vs Time

Figure 3.12: Case 1 2nd Mode Amplitude $q_2$ vs Time
Figure 3.13: Case 1 Phase Plane Trajectory $\theta$ vs $\dot{\theta}$

Figure 3.14: Case 1 Phase Plane Trajectory $q_1$ vs $\dot{q}_1$
Figure 3.15: Case 1 Phase Plane Trajectory $q_2$ vs $\dot{q}_2$

Figure 3.16: Case 1 Control Torque $T$ vs Time
Figure 3.17: Case 1 Control Force $F_1$ vs Time

Figure 3.18: Case 1 Control Force $F_2$ vs Time
Figure 3.19: Case 2 Attitude Error $\theta_e$ vs Time

Figure 3.20: Case 2 Angular Velocity $\dot{\theta}$ vs Time
Figure 3.21: Case 2 1\textsuperscript{st} Mode Amplitude $q_1$ vs Time

Figure 3.22: Case 2 Phase Plane Trajectory $\theta$ vs $\dot{\theta}$
Figure 3.23: Case 2 Control Torque $T$ vs Time

Figure 3.24: Case 2 Control Force $F_1$ vs Time
Figure 3.25: Case 3 Attitude Error $\theta_e$ vs Time

Figure 3.26: Case 3 Angular Velocity $\dot{\theta}$ vs Time
Figure 3.27: Case 3 1st Mode Amplitude $q_1$ vs Time

Figure 3.28: Case 3 Phase Plane Trajectory $\theta$ vs $\dot{\theta}$
Figure 3.29: Case 3 Phase Plane Trajectory $q_1$ vs $q_2$

Figure 3.30: Case 3 Control Torque $T$ vs Time
Figure 3.31: Case 3 Control Force $F_1$ vs Time

Figure 3.32: Case 4 Attitude Error $\theta_z$ vs Time
Figure 3.33: Case 4 Angular Velocity $\dot{\theta}$ vs Time

Figure 3.34: Case 4 Control Torque $T$ vs Time
Figure 3.35: Case 5 Attitude Error $\theta_e$ vs time

Figure 3.36: Case 5 Angular Velocity $\dot{\theta}$ vs Time
Figure 3.37: Case 5 1st Mode Amplitude $q_1$ vs Time

Figure 3.38: Case 5 Phase Plane Trajectory $\theta$ vs $\dot{\theta}$
Figure 3.39: Case 5 Control Torque $T$ vs Time

Figure 3.40: Case 5 Control Force $F_1$ vs Time
Figure 3.41: Case 6 Attitude Error $\theta_e$ vs Time

Figure 3.42: Case 6 Angular Velocity $\dot{\theta}$ vs Time
Figure 3.43: Case 6 2nd Mode Amplitude $q_2$ vs Time

Figure 3.44: Case 6 Phase Plane Trajectory $\theta_e$ vs $\dot{\theta}$
Figure 3.45: Case 6 Control Torque $T$ vs Time

Figure 3.46: Case 6 Control Force $F_1$ vs Time
Figure 3.47: Case 7 Attitude Error $\theta_e$ vs Time

Figure 3.48: Case 7 Angular Velocity $\dot{\phi}$ vs Time
Figure 3.49: Case 7 Phase Plane Trajectory $\theta$ vs $\dot{\theta}$

Figure 3.50: Case 7 Control Torque $T$ vs Time
Figure 3.51: Case 8 Angular Position $\theta$ vs Time

Figure 3.52: Case 8 Angular Velocity $\dot{\theta}$ vs Time
Figure 3.53: Case 8 1st Mode Amplitude $q_1$ vs Time

Figure 3.54: Case 8 Control Torque $T$ vs Time
Figure 3.55: Control Force $F_1$ vs Time
CHAPTER IV

CHATTER CONTROL

4.1 Introduction

It was shown in the previous chapters that once the system states begin the sliding motion the system is linearized and is insensitive to the parameter uncertainties and external disturbances. The controls used in the sliding phase were called equivalent controls. The physical meaning of the equivalent controls is that they are the average between the discontinuous controls $u_i^+$ and $u_i^-$. This average is formed by switching between $u_i^+$ and $u_i^-$ with infinite frequency. In reality, there are time delays in the switching logic, the switching frequency is finite, this causes the system states to overshoot the sliding surfaces and after the delay time it senses the correct position with respect to the sliding surface and switches the control input. This kind of motion about the sliding manifold is known as a nonideal sliding motion in the phase plane. To realize these inputs would require high bandwidth actuators. The chatter phenomenon is undesirable in general, and especially in the area of controlling flexible spacecrafts, since it
might cause intolerable excitation of the unmodeled flexible modes. Hence we see the importance of studying this problem and trying to find a solution.

Three approaches are proposed here to reduce or possibly eliminate altogether the chatter problem. Two of them are based on insuring convergence of the states to the sliding manifold without overshoots, while the other approach tries to reduce the frequency of chatter.

4.2 Boundary Layer Approach

In this approach the domain around the sliding surface $S_0$ satisfying $\| S \| \leq \delta$ is called the boundary layer, where $\| S \|$ is the Euclidian norm. It was shown earlier that regardless of the system parameters or nonlinearities, once the system states lie on the sliding manifold the system behaves as a linear one. Then control inputs are designed according to the reaching condition $S^T \dot{S} < 0$ to force the system to enter the boundary layer. Inside the boundary layer the system is linearized around the manifold $S_0 = 0$, which is a particular desired solution for the trajectory. A control input $U_p$ is designed inside the boundary layer which will guarantee asymptotic convergence to the sliding manifold $S_0$.

To show the above approach consider the nonlinear state space dynamics of a system

$$\dot{X}_e = f(X_e) + BU$$  \hspace{1cm} (4.1)
and the equation of the sliding manifold

\[ S_0 = C X_s = 0 \] (4.2)

Assume that after applying the discontinuous controls the states of the system \( X_e(t) \) are within a distance \( \delta \) from the sliding manifold \( S_0 \). The control inputs \( U \) and the system states can be written as

\[
\begin{align*}
X_e &= X_0 + X_p \\
U &= U_0 + U_p
\end{align*}
\] (4.3)

where \( X_0 \) are the states of the system on the sliding manifold, \( U_0 \) are the equivalent controls required to keep the system on the sliding manifold, while \( X_p \) and \( U_p \) are perturbation vectors from the ideal sliding motion.

After substituting the above equations in the state equation and keeping the first order term in a Taylor expansion we get

\[
\dot{X}_p = A_p(X_0)X_p + BU_p
\] (4.5)

where

\[
A_p(X_0) = \frac{\partial f(X_e)}{\partial X_e} \bigg|_{X_e=X_0}
\] (4.6)

The above equation was reached by using

\[
\dot{S} = C \dot{X}_0 = C[f(X_0) + BU_0] = 0
\] (4.7)

\[
U_0 = [CB]^{-1}Cf(X_0)
\] (4.8)
For any trajectory point \( X_e(t) \), \( X_0 \) is the point of intersection of the norm from \( X_e(t) \) to the sliding manifold \( S_0 \). Then \( X_p \) plays the role of the normal distance from \( X_e(t) \) to the sliding manifold \( S_0 \).

It is noticed here that \( X_0 \) is a function of time, therefore (4.5) represents a nonautonomous linear system. It remains to design the boundary layer control \( U_p \), which is chosen to have the form

\[
U_p = - \left[ 0 \ B^{-1} \ |A(X_0) + \lambda|X_p \right] (4.9)
\]

where

\[
\lambda = \begin{bmatrix} 0 & I \\ \lambda^1 & \lambda^2 \end{bmatrix} (4.10)
\]

\[
\lambda^1 = \text{diag} \left[ \lambda_1^1 \ldots \lambda_{n-1}^1 \right]
\]

\[
\lambda^2 = \text{diag} \left[ \lambda_1^2 \ldots \lambda_{n-1}^2 \right]
\]

and \( \lambda_j > 0 \) for \( i = 1, 2, j = 1, \ldots, n - 1 \).

After substitution of the perturbation inputs in the equation of perturbation (4.5), it simplifies to

\[
\dot{X}_p = -\lambda X_p (4.11)
\]

which guarantees that

\[
\lim_{t \to \infty} X_p = 0 (4.12)
\]

hence the sliding manifold will be reached asymptotically without overshoots.

To summarize the above, proceed as follows: use any method to design the controls during the reaching phase, once the system trajectories are within \( \delta \) from
the sliding manifold (in another words the trajectory has entered the boundary layer) apply the control

\[ U = U_0 + U_p = [C \dot B]^{-1} C f(X_0) - \begin{bmatrix} 0 & \dot B \end{bmatrix}^{-1} \|A(X_0) + \lambda X_p \]  \hspace{1cm} (4.13)

which will guarantee reaching the sliding manifold without overshoots, then no chatter should occur.

Since outside the boundary layer the reaching controls are applied and inside \( U = U_0 - U_p \) are applied, which have different gains, a jump in controls might happen. To overcome this problem the gains are adjusted to have smooth profiles. One method is to match the system states and their derivatives just on the edge of the boundary layer, which must be done through simulation since the system is nonlinear.

### 4.3 Input Prefiltering

Chatter is due to high frequency switching between \( U^+ \) and \( U^- \) to satisfy the reaching condition, depending on whether the system states are on the \( S^- \) or \( S^+ \) side with respect to the sliding manifold \( S_0 \). Therefore this technique depends on passing the computed control law \( U \) through a lowpass filter to eliminate the high frequency components from the inputs. Denoting the output from the filters
as \( \mathcal{U} \), then it can be written in Laplace domain as

\[
\bar{u}_i = G_{F_i}(s)u_i(s)
\]  

(4.14)

where \( G_{F_i}(s) \) is the lowpass filter transfer function for input \( i \). A continuous-time first order lowpass filter can be assumed

\[
G_{F_i}(s) = \frac{a_i}{s + a_i}
\]  

(4.15)

But because the variable structure control laws are computed digitally, it is proposed to use a digital lowpass filter corresponding to \( G_{F_i} \). By using a backward differencing scheme [38], the Z-transform version of \( G_{F_i}(s) \) can be shown to be

\[
z = \frac{1}{1 - Ts}
\]

(4.16)

\[
H_i(z) = \frac{(1 - a_i)z}{z - a_i}
\]

(4.17)

\[
a_i^2 + (2 \cos \omega_c T - 4)a_i + 1 = 0
\]

(4.18)

\[
a_i = \frac{1 - \alpha_i}{\alpha_i T}
\]

(4.19)

where \( \omega_c \) is the desired bandwidth (BW) of the digital filter and \( T \) is the sampling time.

To design the filter, first choose the bandwidth frequency \( \omega_c \) based on the bandwidth of the available actuators, then proceed to find the filter parameters and desired sampling time. Since the lowpass filter introduces a phase lag into the system, some degradation of the sliding motion should be expected. This degradation can reach a point where choosing some bandwidth \( \omega_c \) can lead to
instability of the system. Since the control logic of the input $U$ is a nonlinear behaviour, the problem at hand would be to study the stability of a nonlinear system with filters. This problem is left for further research.

### 4.4 Asymptotic Reaching

It was shown before that chatter occurs due to overshooting the sliding manifold. Then if the sliding manifold can be reached without overshooting (asymptotically) no chatter would occur. To this end we propose to write the reaching condition in an equivalent form

$$\dot{S} = -PS$$  \hfill (4.20)

where

$$S = CX_e$$

$$P = \text{diag}(P_1 \cdots P_m)$$

and $P_i > 0$. The above condition satisfies both the reaching condition

$$S^T \dot{S} = -S^T PS < 0$$  \hfill (4.21)

and the sliding condition

$$\dot{S} = -PS = 0$$  \hfill (4.22)

therefore one control law can be used throughout the motion. Use (4.20) to find the controls for the system

$$\dot{X}_e = f(X_e) + BU$$  \hfill (4.23)
\[
\begin{align*}
\dot{S} &= C \dot{X}_e \\
&= C(f - BU) = -PCX_e \\
U &= -[CB]^{-1}[PCX_e + Cf] \\
\end{align*}
\]  

(4.24)  

(4.25)

It is noticed from the above equation that no switching of control inputs are needed and the controls are continuous functions.

It was shown earlier that once the system begins the sliding motion it is robust to parameter uncertainties and external disturbances. It remains to study the robustness of the system using controls with designs based on the asymptotic reaching approach. To this end assume that \( f(X_e) \) can be written as

\[
f(X_e) = AX_e + \Delta f
\]  

(4.26)

where \( \Delta f \) represents all the parameter uncertainties, nonlinear terms and the external disturbances. Design the controls based on the linear dynamic behaviour

\[
U = -[CB]^{-1}PC + CA X_e \\
\]  

(4.27)

Assuming that the bounds of \( \Delta f \) are known as

\[
\Delta f \leq \pm \max | \Delta f |
\]  

(4.28)

substitute from (4.27), (4.23) into (4.20) to get

\[
\begin{align*}
\dot{S} &= -PCX_e + C \Delta f \\
\dot{S} &= -PS + C \Delta f
\end{align*}
\]  

(4.29)
which has the steady state solution

\[ S_{ss} = \int_0^t e^{-P(t-\tau)}C \Delta f(\tau) d\tau \]

\[ S_{ss} = \pm P^{-1}C \max_{i} |\Delta f_i| \quad (4.30) \]

\( S_{ss} \) is the steady state error, i.e. the system will converge to within \( \| S_{ss} \| \) from the sliding manifold. However since \( P \) and \( C \) are diagonal matrices then equation (4.30) decouples and each error equation can be treated separately

\[ S_{ss} = \frac{C_i}{P_i} \max_{i} |\Delta f_i| \quad (4.31) \]

then by the proper choice of \( C_i, P_i \) the steady state error \( S_{ss} \) can be made negligible. It should be noticed that higher values of \( P \) means faster convergence and smaller reaching time and smaller steady state errors from the sliding manifold.

For linear time invariant systems values of \( P \) and \( C \) can be found using results of [39]. If we assume to have a system

\[ \dot{X}_e = AX_e - BU + EX_e \quad (4.32) \]

where \( E \) is an \( 2(n+1) \times 2(n+1) \) parameter variation matrix. Then using the asymptotic reaching method we can find the controls as

\[ U = -[CB]^{-1}[CA - PC]X_e \quad (4.33) \]

substituting from equation (4.33) into (4.32) we get

\[ \dot{X}_e = A_{e}X_e + EX_e \quad (4.34) \]
where
\[ A_{cl} = \begin{bmatrix} 0 & 1 \\ -PC_1 & -P - C_1 \end{bmatrix} \]  
(4.35)

is the closed loop matrix.

The system (4.34) will be stable if
\[ \| A_{cl} \| \leq \lambda_{max} \]

where \( U_n \) is an \( 2(n - 1) \times 2(n + 1) \) matrix whose entries are unity, \( |G| \) is the modulus of each element in the matrix, \( (G) \) is the symmetric part of the matrix, \( \sigma_{max} \) is the maximum singular values of the matrix and \( L \) is the solution of the Lyapunov equation
\[ A_{cl}^T L + LA_{cl} + 2I_n = 0 \]  
(4.37)

It should be noticed that the closed loop system depend on the values of \( P \) and \( C \) only, which are chosen by the designer. The previous procedure was used to find the change of \( |E_{ij}| \) with \( P \), the results are shown in Figure 4.1. Thus it was proven that with large \( P \), robustness of asymptotic reaching is guaranteed.
Figure 4.1: Maximum Allowable Parameter Variations For Stability vs $P$
CHAPTER V
APPLICATIONS II

5.1 Introduction

The methods proposed in chapter IV were used here to illustrate the VSS maneuvering of flexible spacecraft. The model in appendix B and the numerical data given in chapter III were used here. All examples presented below were rest to rest maneuvers through a 0.4 rad rotation, thus \( \theta^* = 0.4 \) and \( \dot{\theta} = \ddot{\theta} = 0 \).

This is a set point regulation problem with the initial error state vector

\[
X_e^T(0) = \begin{bmatrix} -0.4 & 0 & 0 & 0 & 0 \end{bmatrix}
\]  

(5.1)

The desired maneuver time was 14 sec. The attitude angle error feedback gain was \( k = 2000 \). There were three sliding regimes with the sliding regime parameters taken as \( c_1 = 1.0, c_2 = c_3 = 10.0 \).

All the cases studied used the diagonalization method (sec.2.3.2) for control design, the matrix \( Q \) was chosen as the identity matrix of dimension three and the control parameters were chosen to be \( g_1 = g_2 = g_3 = 0.1 \). Since the system is
nonlinear the response was found using a Runge Kutta integration scheme with time step 0.01 sec.

5.2 Examples

Case 1: This case is included for comparison reasons. Figures 5.1-5.6 show the simulation results. The chatter phenomena is clear in the phase plane Fig 5.3. From Fig 5.5 it can be seen that the torque input is smooth until it reaches the sliding surface where it begins to chatter. Since initially the first mode amplitude error lies on the sliding surface, the control force \( F_1 \) begins to chatter at the beginning of simulations.

Case 2: The boundary layer approach (sec.4.2) was used here. Since initially the elastic coordinates errors are zeros, i.e. they lie on the their respective sliding surfaces and they would begin to chatter, therefore the boundary layer thickness \( \delta = \| S \| \) was chosen large enough such that the initial states would lie inside the boundary layer. In this case

\[
\| S(t_0) \| = \| c_1 \theta_e(0) \| = 0.4
\]

and the boundary layer thickness was chosen to be \( \delta = 0.5 \). However the linearization process was not valid, and according to the previous discussions the values of \( \lambda \) had to be chosen large enough to overcome the effect of the high order
neglected nonlinear terms. The values of $\lambda_1$ and $\lambda_2$ were chosen as

$$
\lambda_1 = \text{diag} \begin{bmatrix} -2 & -2 & -2 \end{bmatrix}
$$

$$
\lambda_2 = \text{diag} \begin{bmatrix} -7 & -7 & -7 \end{bmatrix}
$$

The simulation results are given in Figures 5.6-5.10, from which we can see that the chatter had been completely eliminated and the robustness to the neglected nonlinearities was assured. Comparing Fig 5.2 and Fig 5.7 it can be seen that the first mode amplitude has been smoothened. The only disadvantage here was the higher control inputs needed because of the higher gains.

Case 3: In this simulation asymptotic reaching conditions were used to eliminate chatter. The constant coefficient vector $P$ was chosen as

$$
P = \text{diag} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}
$$

Figures 5.11-5.15 show the simulation results. From Fig 5.13 it is seen that the chatter has been eliminated. The control inputs had almost the same values like the original case. The maneuver time was considerably shorter 8.48 sec and the first mode amplitude deflection was much smoother. The phase plane plots show that the system trajectories move asymptotically towards the sliding surface and finally reaches it at the origin.

Case 4: In order to reduce the chatter frequency input prefiltering was used. For each input a continuous time lowpass filter was used with coefficient $a_i = 10$ and $i = 1, 2, 3$. A sampling time of 0.01 sec was used, which made the bandwidth
of each digital lowpass filter be $BW = 9.88$. Figures 5.16-5.19 show the simulation results.

From Fig 5.17 a considerable reduction in chatter frequency is noticed. Using zero initial conditions for the filters, initial jumps in controls, typical in VSS controls, have been eliminated also. The drawback of this method is seen in the higher mode amplitudes.

Case 5: In order to show the robustness of the approaches discussed in chapter 4 the model used for designing the control inputs had a moment of inertia $I=5000$ Kgm$^2$, while applied to the actual spacecraft with true moment of inertia $I=6764$ Kgm$^2$ which gave an error of about 25 percent.

Figures 5.20-5.25 show the simulation results. Fig 5.21 shows the system using controls designed according to sec.2.4. The control succeeded in accommodating the parameter uncertainty, but chatter clearly appears. In Fig 5.23 the control inputs were passed through the same filters of case 2 which resulted in decreasing the chatter frequency, but increased its amplitude. While the average of the motion is the theoretical sliding surface. In Fig 5.25 the asymptotic reaching method was applied. In order to minimize the steady state error, higher values of $P$ had to be chosen. The values

$$P = \text{diag}[2 \ 2 \ 2]$$  \hspace{1cm} (5.4)

gave acceptable results. It is seen that in all the cases the sliding surfaces were
reached, and the desired maneuver was accomplished in spite of the parameter variation.
Figure 5.1: Case 1 Attitude Error $\theta_e$ vs Time

Figure 5.2: Case 1 1st Mode Amplitude $q_1$ vs Time
Figure 5.3: Case 1 Phase Plane Trajectory $\dot{\theta}_e$ vs $\dot{\theta}$

Figure 5.4: Case 1 Control Torque $T$ vs Time
Figure 5.5: Case 1 Control Force $F_1$ vs Time

Figure 5.6: Case 2 Attitude Error $\theta_2$ vs Time
Figure 5.7: Case 2 1st Mode Amplitude $q_1$ vs Time

Figure 5.8: Case 2 Phase Plane Trajectory $\theta$, vs $\dot{\theta}$
Figure 5.9: Case 2 Control Torque $T$ vs Time

Figure 5.10: Case 2 Control Force $F_1$ vs Time
Figure 5.11: Case 3 Attitude Error $\theta_z$ vs Time

Figure 5.12: Case 3 1st Mode Amplitude $q_1$ vs Time
Figure 5.13: Case 3 Phase Plane Trajectory $\theta_e$ vs $\dot{\theta}$

Figure 5.14: Case 3 Control Torque T vs Time
Figure 5.15: Case 3 Control Force $F_1$ vs Time
Figure 5.16: Case 4 Attitude Error $\theta_e$ vs Time

Figure 5.17: Case 4 Phase Plane Trajectory $\theta_e$ vs $\dot{\theta}$
Figure 5.18: Case 4 Control Torque $T$ vs Time

Figure 5.19: Case 4 Control Force $F_1$ vs Time
Figure 5.20: Case 5 Attitude Error $\theta_e$ vs Time

Figure 5.21: Case 5 Phase Plane Trajectory $\theta_e$ vs $\dot{\theta}$
Figure 5.22: Case 5 Attitude Error $\theta_e$ vs Time

Figure 5.23: Case 5 Phase Plane Trajectory $\theta_e$ vs $\dot{\theta}$
Figure 5.24: Case 5 Attitude Error $\theta_e$ vs Time

Figure 5.25: Case 5 Phase Plane Trajectory $\theta_e$ vs $\dot{\theta}$
CHAPTER VI

MODAL APPROACH

6.1 Introduction

In this chapter the application of VSS in the modal domain is studied. In order to transform the states of the system from the configuration domain to the modal domain, we need to solve an eigenvalue problem, which can be done only for linear systems, hence the linearized equations of motion will be used. Here the expectation is that the nonlinear terms will be accommodated through robustness property of the VSS design.

6.2 Modal Approach

A common problem to all distributed parameter systems is the infinite dimensionality of the system such that one has to retain a large number of modes to have a good model. Due to the complexity of control design, a limited number of modes are chosen to be controlled while trying not to disturb the uncontrolled modes. This can be done in the modal space since the modes are uncoupled.
In what follows, a study of the VSS control in the modal space is considered. Only the linear time invariant equations of motion are considered. Let the equation of motion be (see appendix B)

$$M\ddot{\zeta} + \tilde{K}\zeta = \dot{B}U$$ \hspace{1cm} (6.1)

where

$$\zeta = \left[ \begin{array}{c} \theta - \theta^* \\ q^T \end{array} \right]^T$$

$$M = \left[ \begin{array}{cc} I & P^T \\ P & J \end{array} \right]$$

$$\tilde{K} = \left[ \begin{array}{cc} 0 & 0 \\ 0 & K_\delta \end{array} \right]$$

$$\dot{B} = \left[ \begin{array}{ccc} 1 & N(R + x_1) & \cdots & N(R + x_m) \\ 0 & N\phi(x_1) & \cdots & N\phi(x_m) \end{array} \right]$$

$$U = \left[ \begin{array}{c} T \\ F_1 \\ \cdots \\ F_m \end{array} \right]^T$$

The matrix $M$ is positive definite symmetric matrix and $\tilde{K}$ is symmetric positive semidefinite. Solve the eigenvalue problem

$$\lambda_r M y_r + \tilde{K} y_r = 0$$ \hspace{1cm} (6.2)

to get the eigenvalues $\lambda_r$ and the eigenvectors $y_r$. The eigenvalues and the eigenvectors are all real. Form the modal matrix

$$\tilde{M} = [y_1 \cdots y_{n+1}]$$ \hspace{1cm} (6.3)

subject to the normalizations

$$\tilde{M}^T M \tilde{M} = I$$ \hspace{1cm} (6.4)

$$\tilde{M}^T K \tilde{M} = \text{diag}[\lambda_1 \cdots \lambda_{n+1}]$$ \hspace{1cm} (6.5)
Use the linear transformation
\[
\zeta = \mathcal{M}Y
\]  
(6.6)
in (6.1) and premultiply by $\mathcal{M}^T$, also substitute from (6.4) and (6.5) to get
\[
\ddot{Y} + [\lambda]Y = U^*
\]  
(6.7)
\[
U^* = \mathcal{M}^TBU
\]  
(6.8)
Choosing to control $m$ modes, where $m < n + 1$, the modal vector can be divided to controlled modes and undisturbed modes
\[
Y = \begin{bmatrix} Y_c^T & Y_s^T \end{bmatrix}^T
\]  
(6.9)
where $Y_c$ is $m$-dimension vector of controlled modes and $Y_s$ is $(n+1-m)$-dimension vector of undisturbed modes, and equation (6.7) decouples to
\[
\ddot{Y}_c + \lambda_c Y_c = U_c
\]  
(6.10)
\[
\ddot{Y}_s + \lambda_s Y_s = U_s
\]  
(6.11)
\[
\lambda_c = \text{diag}\{\lambda_{c_1}, \ldots, \lambda_{c_m}\}
\]  
(6.12)
\[
\lambda_s = \text{diag}\{\lambda_{s_1}, \ldots, \lambda_{s_{n+1-m}}\}
\]  
(6.13)
transform to the state space
\[
e_1 = Y_c \quad e_2 = \dot{Y}_c \quad e_3 = Y_s \quad e_4 = \dot{Y}_s
\]  
(6.14)
\[
e_c = \begin{bmatrix} e_1 & e_2 \end{bmatrix} \quad e_s = \begin{bmatrix} e_3 & e_4 \end{bmatrix}
\]  
(6.15)
Then equation (6.7) could be written in matrix form as
\[ \dot{e} = Ae + \bar{U} \] (6.16)
where
\[ A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\lambda & 0 \end{bmatrix} \]
\[ \bar{U} = \begin{bmatrix} 0 \\ U^*_c \\ 0 \\ U^*_s \end{bmatrix} \]

The \( m \) controlled modes are usually chosen to be the lowest modes since they are easily excited. But if any higher order mode is found unstable, it should be controlled instead of a lower stable mode. It is assumed here that the undisturbed modes are stable ones. Since it is not desired to excite the undisturbed modes choose
\[ U^*_s = 0 \] (6.17)

It remains to design \( U^*_c \) using VSS. Let the sliding surfaces equation have the form
\[ S = Ce_c = C_1 e_1 + e_2 \] (6.18)
where
\[ C_1 = \text{diag}[c_1 \cdots c_m] \]

For example, use the asymptotic reaching condition
\[ \dot{S} = -PS \] (6.19)
where

\[ P = \text{diag}[P_1 \cdots P_m] \]

to find the control inputs. Doing the time derivatives and substituting from (6.18) and (6.14) in (6.19) we get

\[ U_c^* = [-\lambda_c - PC_1]e_1 + [-P - C_1]e_2 \] (6.20)

Substitute for \( U_c^* \) and \( U_j^* \) in (6.16) to get

\[ A_{cl} = \begin{bmatrix} A_c & 0 \\ 0 & A_r \end{bmatrix} \] (6.21)

where

\[ A_c = \begin{bmatrix} 0 & 1 \\ -PC_1 & -P - C_1 \end{bmatrix} \]

\[ A_r = \begin{bmatrix} 0 & 1 \\ -\lambda_r & 0 \end{bmatrix} \]

The closed loop eigenvalues are those of \( A_c \) and \( A_r \), which are stable by design. It should be noticed that the eigenvalues of the undisturbed modes are imaginary, i.e. there is no damping, therefore these modes will be conditionally stable; their amplitudes will depend on the initial conditions, but they will never destabilize the system.

Finally to find the real control inputs use (6.8)

\[ U = \left[ \tilde{M}^T \tilde{B} \right]^{-1} \begin{bmatrix} U_c^* \\ 0 \end{bmatrix} \] (6.22)

the above equation assumes that \( [\tilde{M}^T \tilde{B}] \) has dimension \((n + 1)\), i.e. the number of inputs is equal to the total number of modes considered in the model.
6.3 Spillover Problem

Since we retained a small number of modes in our model, inputs used to maneuver the spacecraft and damp the modeled modes work as excitation forces for the unmodelled and uncontrolled modes, this is known as control spillover. Usually the maneuver dynamics of the spacecraft is nonlinear, which means that the controlled and uncontrolled modes are coupled in such a way that the uncontrolled modes act like excitation forces for the controlled modes and vice versa. From here we can see the importance of considering the spillover problem.

The nonlinear equations of motion for a flexible spacecraft can be written as

\[ M\ddot{\zeta} - \bar{K}\zeta + \delta(\zeta) = \bar{B}U \quad (6.23) \]

where

\[ \zeta = [ \theta - \dot{\theta} \quad q^T ]^T \]
\[ M = \begin{bmatrix} I & P^T \\ P & J \end{bmatrix} \]
\[ \bar{K} = \begin{bmatrix} k & 0 \\ 0 & K_E \end{bmatrix} \]
\[ \bar{B} = \begin{bmatrix} 1 & N(R + x_1) & \cdots & N(R + x_m) \\ 0 & N\phi(x_1) & \cdots & N\phi(x_m) \end{bmatrix} \]
\[ U = \begin{bmatrix} T & F_1 & \cdots & F_m \end{bmatrix}^T \]
\[ \delta(\zeta) = \begin{bmatrix} 2(\dot{\theta} - \dot{\theta}^\cdot)\dot{q}^T Kq \\ (\dot{\theta} - \dot{\theta}^\cdot)^2 Kq \end{bmatrix} \quad (6.24) \]
then $\delta(\zeta)$ includes all the nonlinearities. The matrix $M$ is positive definite symmetric matrix and $K$ is symmetric positive semidefinite. Solve the eigenvalue problem

$$\lambda_r M y_r + K y_r = 0 \quad (6.25)$$

to get the eigenvalues $\lambda_r$ and the eigenvectors $y_r$. The eigenvalues and the eigenvectors are all real. Form the modal matrix

$$\tilde{M} = [y_1 \cdots y_{n+1}] \quad (6.26)$$

subject to the normalizations

$$\tilde{M}^T \tilde{M} \tilde{M} = I \quad (6.27)$$

$$\tilde{M}^T K \tilde{M} = \text{diag}[\lambda_1 \cdots \lambda_{n-1}] \quad (6.28)$$

Use the linear transformation

$$\zeta = \tilde{M} Y \quad (6.29)$$

and premultiply (6.23) by $\tilde{M}^T$ to get

$$\dot{Y} + \lambda Y + \tilde{M}^T \delta(\tilde{M} Y) = BU \quad (6.30)$$

Choosing to control $m$ modes, where $m < n - 1$, the modal vector can be divided to controlled modes and uncontrolled modes

$$Y = [Y_c^T Y_s^T]^T \quad (6.31)$$
where $Y_c$ is $m$-dimension vector of controlled modes and $Y_s$ is $(n - 1 - m)$-dimension vector of uncontrolled modes.

\[
\dot{Y}_c + \lambda_c Y_c + \Delta_1 = \bar{B}_c U
\]

\[
\dot{Y}_s + \lambda_s Y_s + \Delta_2 = \bar{B}_s U
\]

\[
\lambda_c = \text{diag} \{ \lambda_{c_1}, \ldots, \lambda_{c_m} \}
\]

\[
\lambda_s = \text{diag} \{ \lambda_{s_{n-1-m}}, \ldots, \lambda_{s_{n-1-m}} \}
\]

\[
\begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix} = \bar{M}^T \delta(\zeta) = \bar{M}^T \delta(\bar{M} Y)
\]

where $\Delta_1$ and $\Delta_2$ are the $m$ and $(n - 1 - m)$-dimensional partitions of the nonlinearity in terms of the controlled and uncontrolled modal coordinates. From the forms of $\Delta_1$ and $\Delta_2$ it is seen that the uncontrolled and controlled modes are mutually coupled because the nonlinearity. This is in contrast to linear systems where the controlled modal dynamics excites the uncontrolled modes but is itself unaffected by the uncontrolled modes (see (6.24)).

$\bar{B}$ is partitioned to $\bar{B}_c$ and $\bar{B}_s$ with dimension $m \times m$ and $(n - 1 - m) \times m$ respectively.

To design $U$ we use the hierarchy control method (sec.2.3.1) and take our plant to be

\[
\dot{Y}_c + \lambda_c Y_c = \bar{B}_c U
\]

The nonlinear term $\Delta_1$ can be written as

\[
\Delta_1 = \Delta_{1c} + \Delta_{1s}
\]
where $\Delta_{1c}$ depends on the controlled modes only and $\Delta_{1s}$ has the rest of the nonlinearities. The control $U$ is made robust with respect $\Delta_{1c}$ term by subtracting a term $\delta$ (sec.2.4) and we neglect $\Delta_{1s}$, since it is not available for measurement. We can only hope that the robustness properties of VSS can accommodate the $\Delta_{1s}$ term as well. Choose the sliding surface equation to be

$$S = C_Y \dot{Y} + \dot{Y}_c$$

Then using (2.23) we can find

$$b_{ci} u_i \leq -c_i \dot{Y}_c + \lambda c_i Y_c - \sum_{j=1}^{i-1} b_{cij} u_{eqj} - \sum_{j=i+1}^{m} u_j + \delta_i \cdots s_i \geq 0$$

(6.39)

where

$$\delta_i \geq c_i \Delta_{1c} \cdots s_i \geq 0$$

(6.40)

and $i = 1, \ldots, m$

Since it was assumed that $\Delta_{1c}$ is a function of $(Y_c, \dot{Y}_c)$, then it is available for measurement and a bound can be found for it.

### 6.4 Applications

For these simulations the same spacecraft model in appendix B and the numerical data given in chapter III were used here. All examples presented below were rest to rest maneuvers through a 0.4 rad rotation, thus $\theta^* = 0.4$ and $\dot{\theta}^* = \ddot{\theta}^* = 0$. This is a set point regulation problem with the initial error state vector

$$X^T_c(0) = [-0.4 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0]$$

(6.41)
Case 1: The modal approach discussed in sec.6.2 was applied here to show its effectiveness. The attitude error feedback gain was set to zero $k = 0$, since in using the asymptotic reaching method the attitude error appears explicitly in the formulations. The number of modes in the model were three $n = 3$. Figures 6.1-6.6 show the simulation results. We assumed to have a rigid body torquer $T$ located on the central hub and three control forces $F_1, F_2$ and $F_3$ on each appendage located at $0.29L, 0.84L$ and $0.8L$ respectively. The matrix $P$ was chosen to be $P = \text{diag}[1 \ 1 \ 1]$.

In this simulation the controls were calculated on the linearized model while they were applied on the nonlinear model. Robustness was guaranteed by choosing the matrix $P$ with high values. It was noticed from Fig 6.1 that the mode amplitudes were smaller in comparison to the previous cases.

Case 2: This simulation checks the effect of spillover to uncontrolled modes on maneuvering of the flexible spacecraft. The attitude error feedback gain was $k = 2000$. The number of modes in the model were three $n = 3$, while we choose to control the first two modes only, there is one uncontrolled mode. We assumed to have a rigid body torquer $T$ located on the central hub and two control forces $F_1$ and $F_2$ on each appendage located at $0.29L$ and $0.84L$ respectively. Hierarchical control method was used with control gains chosen as $g_1 = g_2 = g_3 = .5$. The sliding surfaces coefficients are $c_1 = c_2 = c_3 = 1.0$.

The controls were calculated using the controlled modes only, hence the third
mode is uncontrolled and we hoped that the VSS controls would overcome the effect of the uncontrolled nonlinearities. The simulation results are given in Figures 6.7-6.12. From Fig. 6.12 we can see that the attitude sliding surface is reached in spite of the neglected nonlinearities and the unmodeled third mode. From Figures 6.8-6.10 the effect of the unmodeled third mode is evident on the elastic coordinates. Once the attitude maneuver is reached, the controls are turned off and the residual excitation on the third mode persists and its steady oscillations disturbs the controlled elastic coordinates about their desired final (zero) equilibrium point. Thus both the spillover and spillback effects of unmodeled modes due to nonlinearity is evident. In reality, any structural damping present will damp out the residual excitation so that the small excitation shown in Figures 6.8-6.10 would be tolerable.
Figure 6.1: Case 1 Attitude Error $\theta_e$ vs Time

Figure 6.2: Case 1 1st Mode Amplitude $q_1$ vs Time
Figure 6.3: Case 1 2\textsuperscript{nd} Mode Amplitude $q_2$ vs Time

Figure 6.4: Case 1 3\textsuperscript{rd} Mode Amplitude $q_3$ vs Time
Figure 6.5: Case 1 Control Torque $T$ vs Time

Figure 6.6: Case 1 Control Force $F_3$ vs Time
Figure 6.7: Case 2 Attitude Error $\theta_e$ vs Time

Figure 6.8: Case 2 1st Mode Amplitude $q_1$ vs Time
Figure 6.9: Case 2 2\textsuperscript{nd} Mode Amplitude $q_2$ vs Time

Figure 6.10: Case 2 3\textsuperscript{rd} Mode Amplitude $q_3$ vs Time
Figure 6.11: Case 2 Control Torque T vs Time

Figure 6.12: Case 2 Phase Plane Trajectory $\theta_c$ vs $\dot{\theta}_c$
CHAPTER VII

CONCLUSIONS

This study was primarily concerned with the applications of Variable Structure Control to maneuvering of spacecraft. Many examples were given in chapter III on applications of the basic theory. The problem of chatter was studied and methods for its solution were proposed in chapter IV. Illustrative examples of chatter control were presented in chapter V. Chapter VI introduced the modal formulation and addressed the spillover problem for the maneuver dynamics. Again with illustrative examples.

The basic conclusions obtained in this study are:

- Variable Structure System Control is a powerful method for maneuvering of spacecraft.

- The control system is robust with respect to parameter uncertainties, unmodeled dynamics and external disturbances. No extra work is done to insure robustness, all that is needed is to have stronger satisfaction of the reaching inequalities.
• The VSS is easy to implement since it is a "strategic approach". No iterations or complex computations are required, hence it can be programmed using spacecraft onboard computers. This is a distinct advantage over the present day maneuver approaches.

• The chatter phenomenon can be eliminated completely by using the proposed methods of chapter IV. The resulting controls are continuous, which can be implemented using the current state of the art actuators with smaller bandwidths.

This study has opened many questions which we leave as topics for further research as

• The stability of nonlinear systems using the filtering method for chatter control needs to be studied.

• Control spillover for nonlinear systems is still an open question. In this dissertation we only studied the effect of VSS controls on the uncontrolled modes, no attempt was made to reduce the effect of controls on the uncontrolled modes.

This dissertation demonstrated that Variable Structure Control is a powerful, efficient alternative methodology for the control of spacecraft. Many other tools of control theory can be used in conjunction with the VSS to yield better designs.
For example, optimization theory can be used to find optimum sliding mode parameters. Variable Structure Adaptive Control techniques can also be studied with spacecraft applications in mind.
APPENDIX A
EQUATIONS OF MOTION FOR
A RIGID SPACECRAFT

To find the equations of motion for the spacecraft, let us assume to have an inertial frame of reference \((XYZ)\), a fixed body axis \((xyz)\) and \(i^{th}\) rotor axis \((x_R, y_R, z_R)\). It is assumed that \((x_R y_R z_R)\) is fixed relative to the spacecraft axis and does not rotate. Define the following notations:

- \(\Omega_i\): Angular velocity of rotor \(i\) relative to the spacecraft
- \(\omega_R\): Rotor angular velocity relative to its axis
- \(\omega\): Absolute angular velocity of spacecraft
- \(R_C\): Spacecraft center of mass inertial position vector
- \(r_{C_i}\): Position vector of rotor \(i\) center of mass \(C_R_i\) relative to the overall center of mass \(C\).
- \(r'\): Position vector of rotor element relative to its center of mass \(C_R_i\).
- \(r_j\): Position vector of vehicle element \(j\) relative to overall center of mass \(C\).

The angular momentum about \(C\) can be defined as

\[
H_C = \int_V r_j \times (v_C + \omega \times r_j) \, dm_j \\
+ \int_R (r_{C_i} + r) \times (v_{C_R_i} + \omega_R \times r') \, dm 
\]

(A.1)

where the integral with subscript \(V\) is over vehicle domain and the integral with subscript \(R\) is over rotor domain. But \(v_{C_R_i}\) is the translational velocity of \(C_{R_i}\).
Figure A.1: Rigid Spacecraft Axes
and can be expressed as

\[ v_{C,n} = v_C - \omega_v \times r_c, \]  

(A.2)

Perform the cross operation and regroup the terms to get

\[
H_C = \int_V r_j dm_j \times v_C + \int_V r_j \times (\omega_v \times r_j) dm_j
+ \int_R r_c dm \times v_{C,n} + \int_R r_c \times (\omega_R \times r') dm
+ \int_R r' dm \times v_{C,n} + \int_R r' \times (\omega_R \times r') dm
\]  

(A.3)

Introduce the total mass of the rotors \( M_R = \int_R dm \), and regroup the terms to get

\[
H_C = \left( \int_V r_j dm_j + r_c M_R \right) \times v_C + \int_V r_j \times (\omega_v \times r_j) dm_j
- r_{C,i} \times (\omega_v \times r_{C,i}) M_R + r_{C,i} \times (\omega_R \times \int_R r' dm)
+ \int_R r' dm \times v_{C,n} + \int_R r' \times (\omega_R \times r') dm
\]  

(A.4)

But

\[
\int r' dm = 0
\]  

(A.5)

because this is the location of the rotor center of mass relative to the rotor axis \((x_R y_R z_R)\) which is centroidal. Also

\[
\int_V r_j dm_j + r_c M_R = 0
\]  

(A.6)

because this is the location of the complete spacecraft center of mass (including the rotors) relative to point C which is the total center of mass. Then

\[
H_C = \int_V r_j \times (\omega_v \times r_j) dm_j + \int_R r' \times (\omega_R \times r')
\]
\[ -r_{C_i} \times (\omega_V \times r_{C_i})M_R \]  

(A.7)

Introduce the moment of inertia of the spacecraft alone excluding the momentum wheels \( I_V \) and the moment of inertia of rotor \( R_i \) about its own axis \( I_{C_{R_i}} \). It should be noticed that we have \( N \) rotors. Recombining certain terms noting that \( \omega_R = \omega_V + \Omega \) and rearranging we get

\[ H_C = \left[ I_V + \sum_{i=1}^{N} I_{R_i} \right] \omega_V + \sum_{i=1}^{N} I_{C_{R_i}} \Omega_i \]  

(A.8)

where \( I_{R_i} \) is the moment of inertia matrix of rotor \( R_i \) about the spacecraft axis, it is a combination of \( I_{C_{R_i}} \) and a transfer term from \( C_{R_i} \) to \( C \). But

\[ I' = I_V + \sum_{i=1}^{N} I_{R_i} \]  

(A.9)

is the complete spacecraft moment of inertia matrix as if the rotors were locked.

Then

\[ H_C = I' \omega_V + \sum_{i=1}^{N} I'_{C_{R_i}} \Omega_i \]  

(A.10)

Let us define the complete spacecraft angular momentum vector when all the wheels are locked as the term

\[ H'_{C} = I' \omega_V \]  

(A.11)

and the angular momentum of rotor \( i \) about \( C_R \) relative to the spacecraft as

\[ H'_{R_{C_i}} = I'_{C_{R_i}} \Omega_i \]  

(A.12)
The angular momentum $H_C$ is expressed along the fixed body axis $xyz$, while the angular momentum $H'_R \_{i,R}$ is expressed along the rotor axis $x_R y_R z_R$. In order to have uniformity, transfer the rotor momentum to components along the fixed body axis. Using the direction cosine matrix $[33]$ between the two sets of axes

$$X = [b]^T X_R$$

(A.13)

where $X$ is the inertial frame of reference, $X_R$ is the rotating frame of reference and $[b]^T$ is the matrix of direction cosines, then

$$H'_{R,i} = [b]_i^T H'_{R,i}$$

(A.14)

the total angular momentum becomes

$$H_C = I \cdot \omega_V + \sum_{i=1}^{N} [b]_i^T I'_{C,R_i} \Omega_i$$

(A.15)

denote $[G] = [b]^T I'_{C,R}$. Then we can write

$$\frac{d}{dt} H_C = \dot{H}_C - \omega_V \times H_C = L_C$$

(A.16)

where $\frac{d}{dt}$ is inertial time derivative, $\dot{}$ is the derivative with respect to the rotating body axis and $L_C$ is the external torque about the center of mass $C$. Applying the inertial time derivative operator, that is

$$\frac{d}{dt} (-) = (\dot{}) + \omega_V \times (-)$$

we get

$$I \cdot \dot{\omega}_V + [G] \dot{\Omega} + \omega_V \times [I \cdot \omega_V + [G] \Omega] = L_C$$

(A.17)
using the skew symmetric matrix representation of cross products we can write

\[ I \ddot{\omega}_V + [G_i \Omega + [\omega_V I \dot{\omega}_V + [\omega_V I G_i \Omega = L_c \] (A.18)

\[ [\omega_V] = \begin{bmatrix}
  0 & -\omega_V_x & \omega_V_y \\
  \omega_V_x & 0 & -\omega_V_z \\
  -\omega_V_y & \omega_V_z & 0
\end{bmatrix} \] (A.19)

Hence the equations of motion are highly nonlinear in \( \omega_V \) and \( \Omega \).

The above equations are for the spacecraft and the rotor system, therefore the reaction forces between the spacecraft and the rotor supports are internal with respect to such a system and they do not appear in the system equation of motion. In order to include the motor torques (which will be the ultimate control inputs) in the equations of motion we must write the equations of motion for the rotors alone. The equation of motion of rotor \( i \) is given by

\[ \frac{d}{dt} h_{C_{R_i}} = L_{C_{R_i}} \] (A.20)

where \( h_{C_{R_i}} \) is the absolute angular momentum of rotor \( i \) relative to its own center of mass \( C_{R_i} \) and \( L_{C_{R_i}} \) is the external torque about the rotor center \( C_{R_i} \). The rotor angular momentum can be expressed as

\[ h_{C_{R_i}} = [I'_{C_{R_i}}] \omega'_{R_i} \] (A.21)

where \([I'_{C_{R_i}}]\) is the moment of inertia matrix of rotor \( i \) about its rotor axis. The dash denotes that it is in terms of components along the rotor axis. \( \omega'_{R_i} \) is the rotor absolute angular velocity in terms of components along the rotor axis.
Applying the inertial time derivative operator, that is

\[
\frac{d}{dt} (-) = (\dot{\cdot}) - \omega_V \times (-)
\]

we obtain

\[
[I'_{C_{R_i}}] \dot{\omega}'_{R_i} + [\omega'_{V}] [I'_{C_{R_i}}] \omega'_{R_i} = L_{C_{R_i}} \tag{A.22}
\]

where \(\omega_V\) is the angular velocity of the rotor axis, since it was assumed to be fixed with respect to the body axis. Since the rotors are usually axisymmetric bodies of revolution, then we can assume that

\[
[I'_{C_{R_i}}] = \begin{bmatrix} J_a & 0 & 0 \\ 0 & J & 0 \\ 0 & 0 & J \end{bmatrix} \tag{A.23}
\]

where \(J_a\) is the rotor moment of inertia about \(X_{R_i}\), \(J\) is the rotor moment of inertia about \(Y_{R_i}\). But we have

\[
\omega'_{R_i} = \omega'_{V} + \Omega'_{R_i} \tag{A.24}
\]

Substitute into (A.22) and perform the cross product to get

\[
[\omega'_{V}] [I'_{C_{R_i}}] \omega'_{R_i} = \left[ ((J_a - J)\omega'_{V_{z_i}} - J\Omega'_{s_i})\omega'_{V_{s_i}} \right] \hat{j}_{R_i} + \left[ ((J - J_a)\omega'_{V_{z_i}} + J\Omega'_{s_i})\omega'_{V_{s_i}} \right] \hat{k}_{R_i} \tag{A.25}
\]

We can notice that there is no \(\hat{i}_{R_i}\) component and in equation (A.22) \(\hat{i}_{R_i}\) component simplifies to

\[
J_a \omega'_{R_{z_i}} = u_{z_i} \tag{A.26}
\]
where $u_{z_i}$ is the $z$ component of the torque $L_{C_{n_i}}$ about the rotor shaft. This torque is applied by an electric motor. From now on we will use $u$ as the torque around the rotor shaft. The torque applied by the motor on the rotor creates a reacting torque on the spacecraft since the motor is fixed to it. Since

$$\dot{\omega}_{R_{n_i}} = \dot{\omega}_{V_{n_i}} + \dot{\Omega}_i$$

$$= b_i^T \omega_V + \dot{\Omega}_i$$  \hspace{1cm} (A.27)

where $b_i^T$ is the first row vector of $[b_i]^T$, then (A.26) becomes

$$J_{z_i}(\dot{\Omega}_i + b_i^T \omega_V) = u_i$$  \hspace{1cm} (A.28)

Since there are $N$ rotors, combining all the equations in matrix form we get

$$[J_a] \begin{bmatrix} \dot{\Omega} + [B] \omega_V \end{bmatrix} = U$$  \hspace{1cm} (A.29)

where

$$[J_a] = \text{diag}[J_{a_1}, \ldots, J_{a_N}]$$

$$[B]^T = [b_{z_1}, \ldots, b_{z_N}]$$

$$U^T = [U_1 \ldots U_N]$$

Returning to (A.15) we perform the matrix product in the last term

$$[b_i]^T I'_{C_{n_i}} \Omega_i = J_{a_i} b_{z_i} \Omega_i$$  \hspace{1cm} (A.30)

$$H_C = I' \omega_V + \sum_{i=1}^{N} J_{a_i} b_{z_i} \Omega_i$$  \hspace{1cm} (A.31)
define

\[ [A] = [B]^T [J_A] \]  \hspace{1cm} (A.32)

\[ H_C = I^* \omega_V - [A] \Omega \]  \hspace{1cm} (A.33)

Solve for \( \dot{\Omega} \) from (A.29) we get

\[ \dot{\Omega} = [J_A]^{-1} U - [J]^{-1} - a[B]\dot{\omega}_V \]  \hspace{1cm} (A.34)

Finally combining equations (A.18), (A.29) and (A.32) assuming that \( L_C = 0 \) we get

\[ I^* - [A][B] \dot{\omega}_V + [\omega_V] [I^* \omega_V + [A] \Omega] = -[A][J_A]^{-1} U \]  \hspace{1cm} (A.35)

From (A.32) we can see that

\[ [A][B] = [B]^T [J_A][B] \]  \hspace{1cm} (A.36)

\[ [A][J_A]^{-1} = [B]^T \]  \hspace{1cm} (A.37)

substituting from (A.37) and (A.36) into (A.35) we get

\[ [I^* - [B]^T [J_A][B]] \dot{\omega}_V = -[\omega_V] [I^* \omega_V + [A] \Omega] - [B]^T U \]  \hspace{1cm} (A.38)

let

\[ \dot{I} = I^* - [B]^T [J_A][B] \]  \hspace{1cm} (A.39)

\[ \dot{\omega}_V = -[\dot{I}]^{-1} \left( [B]^T U + [\omega_V] [I^* \omega_V + [A] \Omega] \right) \]  \hspace{1cm} (A.40)
Substitute (A.40) into (A.29) to obtain

\[
\dot{\Omega} = [B^{-1}]^T \omega_V + I \omega_V - A \Omega
\]

\[
+ [B^T + (J_s)^{-1}] U
\]

(A.41)

We note that in equations (A.40) and (A.41) there is no explicit mention of the spacecraft rotation with respect to the inertial reference frame. But for maneuvering purposes we must know the spacecraft angular rotations. To do that we use any set of Euler's angles, from which the kinematic relation between the spacecraft angular velocity and its angular rotations are [33]

\[
\omega_V = [\alpha_V] \dot{\alpha}_V
\]

(A.42)

where \(\alpha_V\) is the spacecraft angular rotations, and \([\alpha_V]\) is a 3 x 3 matrix

\[
[\alpha_V] = \begin{bmatrix}
\cos \alpha_2 \cos \alpha_3 & \sin \alpha_3 & 0 \\
-\cos \alpha_2 \sin \alpha_3 & \cos \alpha_3 & 0 \\
\sin \alpha_2 & 0 & 1
\end{bmatrix}
\]

(A.43)

where the order of the rotations is \(\alpha_1, \alpha_2, \alpha_3\). Thus the angular rotation rates can be found as

\[
\dot{\alpha}_V = [\alpha_V]^{-1} \omega_V
\]

(A.44)

The three dimensional spacecraft attitude motion can now be completely described by equations (A.40), (A.41) and (A.44)
Assume a spacecraft (see Figures B.1 and 3.8) with a rigid central hub and N cantilevered appendages put in symmetry around the hub. Each appendage has length \( L \), and we consider only the antisymmetric inplane deflections due to the rotation of the spacecraft about the \( Z \) axis. Let us have an inertial frame of reference \( (X, Y, Z) \), a vehicle reference frame \( (X_v, Y_v, Z_v) \) and \( (X_E, Y_E, Z_E) \) axes at the root of appendage \( i \). Denote:

- \( I_{Hub} \): Moment of inertia of the rigid hub
- \( \theta \): Spacecraft attitude angle with respect to the inertial reference frame
- \( r_P \): Position vector of point \( P \) on appendage
- \( u_P \): Elastic displacement of point \( P \) relative to the \( (X_E, Y_E, Z_E) \) system
- \( \Omega_E \): Absolute angular velocity of point \( E \) (center of elastic axes)
- \( v_E \): Absolute velocity of point \( E \)
- \( R \): Radius of rigid hub

The kinetic energy of the spacecraft can be written as

\[
T = \frac{1}{2} I_{Hub} \dot{\theta}^2 + \frac{1}{2} \int v^2(x, t) dm_E \tag{B.1}
\]

where \( v(x, t) \) is the velocity of an appendage point \( P \), and can be expressed as

\[
v(x, t) = v_E + \Omega_E \times (r_P + u_P) + \dot{u}_P \tag{B.2}
\]
and \( v(x,t) \) has the explicit form

\[
v(x,t) = \begin{bmatrix} 0 \\ \dot{\theta} R \\ 0 \end{bmatrix} - [r - u]\Omega_E - \begin{bmatrix} 0 \\ \dot{u} \\ 0 \end{bmatrix}
\]  \hspace{1cm} (B.3)

where

\[
[r + u] = \begin{bmatrix} 0 & 0 & u \\ 0 & 0 & -x \\ -u & x & 0 \end{bmatrix}
\]  \hspace{1cm} (B.4)

But

\[
[r + u]\Omega_E = \begin{bmatrix} u\dot{\theta} \\ -x\dot{\theta} \\ 0 \end{bmatrix}
\]  \hspace{1cm} (B.5)

\[
v(x,t) = \begin{bmatrix} -u\dot{\theta} \\ \dot{\theta} R + x\dot{\theta} + \dot{u} \\ 0 \end{bmatrix}
\]  \hspace{1cm} (B.6)

For \( N \) appendages the total kinetic energy can be expressed as

\[
T = \left[ \frac{1}{2} I_{HB}^V + \frac{N}{2} \int_0^L \rho \left[ u^2 - R^2 - x^2 + 2Rx \right] dx \right] \dot{\theta}^2 - \frac{N}{2} \int_0^L \rho u^2 dx
\]  \hspace{1cm} (B.7)

where \( \rho \) is the mass per unit length of the appendages.

The total potential energy is assumed to be due to elastic deflection of appendage and centrifugal stiffening of the appendages due to rotation about the \( Z \) axis

\[
V = \frac{1}{2} [u, u] + \frac{1}{2} \text{[centrifugal stiffening]} \]  \hspace{1cm} (B.8)

where \([u, u]\) is the potential energy due to the elastic deflection of appendages

\[
[u, u] = \int_0^L EI_A u''^2 dx
\]  \hspace{1cm} (B.9)
where $EI_A$ is flexural rigidity of the cantilevered appendage.

Centrifugal stiffening is equal to the work done by the centripetal force $P(x)$ at any appendage point $x$ due to attitude angular velocity $\theta$. If a small element of the appendage is taken, it can be shown that the work done is [34]:

$$ W = -\int_0^L \frac{1}{2} P(x)u'^2 dx $$

(B.10)

where $u' = \frac{du}{dx}$. For $N$ appendages the total kinetic energy becomes

$$ V = \frac{N}{2} [u, u] - \frac{N}{4} \int_0^L P(x)u'^2 dx $$

(B.11)

$$ P(x) = \int_x^L \rho \dot{\theta}^2 (R + \xi) d\xi $$

(B.12)

After performing the integration, the total potential energy becomes

$$ V = -\frac{N}{2} EI_A \int_0^L u''^2 dx - \frac{N}{8} \int_0^L \rho \dot{\theta}^2 u'^2 \left[ (R + L)^2 - (R - x)^2 \right] dx $$

(B.13)

The Lagrangian of the spacecraft is

$$ L = T - V $$

(B.14)

The rigid body rotation $\theta(t)$ depends on time alone. The elastic displacement $u(x, t)$ of a given flexible member of the spacecraft relative to the spacecraft axes represents a coordinate dependent on both space and time. The elastic displacement is discretized in space by using the assumed modes method [34]. Using this method we can express

$$ u(x, t) = \sum_{i=1}^n \phi_i(x)q_i(t) = \phi^T(x)q(t) $$

(B.15)
where \( x \) is spatial position, \( \phi(t) \) is an admissible function vector, \( q(t) \) is the generalized elastic coordinate vector, and \( n \) is the number of elastic coordinates.

Substituting from equations \((B.15),(B.13)\) and \((B.7)\) in equation \((B.14)\) the total Lagrangian is

\[
L = \frac{1}{2} \dot{\theta}^2 + \frac{N}{2} q^T f \int_0^L \rho \phi \phi^T d\gamma \dot{\theta}^2 + \frac{N}{2} \int_0^L \rho (R + x)^2 d\gamma \dot{\theta}^2 \\
+ \frac{N}{2} \dot{\gamma} \int_0^L \rho (2\phi^T R + 2\phi^T x) d\gamma q + \frac{N}{2} \frac{\dot{\theta}^T}{\dot{\theta}} \int_0^L \rho \phi \phi^T d\gamma q \\
- \frac{N}{2} q^T E\gamma \int_0^L \phi'' \phi''^T d\gamma q \\
- \frac{N}{8} q^T \int_0^L \rho \dot{\theta}^2 \left[(R + L)^2 - (R + x)^2\right] \phi' \phi'^T d\gamma q
\]  
\[(B.16)\]

Let

\[
I = I_{HUB} + N \int_0^L \rho (R + x)^2 d\gamma \\
P = N \int_0^L \rho (R + x) \phi^T d\gamma \\
J = N \int_0^L \rho \phi \phi^T d\gamma \\
K_e = \frac{N}{4} \int_0^L \rho \left[(R + L)^2 - (R + x)^2\right] \phi' \phi'^T d\gamma \\
K_E = N \int_0^L E\gamma \phi'' \phi''^T d\gamma
\]

Then

\[
L = \frac{1}{2} \dot{\theta}^2 + \frac{1}{2} q^T f J q \dot{\theta}^2 + \dot{\theta} P q \\
+ \frac{1}{2} \frac{\dot{\theta}^T}{\dot{\theta}} J q \dot{\theta} - \frac{1}{2} q^T k_E q - \frac{1}{2} q^T K_c q \theta^2
\]  
\[(B.22)\]
The Lagrange's equations of motion can be written in the general form

\[
\begin{align*}
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} &= Q_{\theta} \quad (B.23) \\
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} &= Q_{q} \quad (B.24)
\end{align*}
\]

Where \(Q_{\theta}, Q_{q}\) are generalized forces. Doing the above differentiations and grouping the terms, we get

\[
\begin{align*}
I \ddot{\theta} + P^T \ddot{q} + 2 \dot{\theta} \dot{q}^T (J - K_C) q + q^T (J - K) \ddot{q} &= Q_{\theta} \quad (B.25) \\
\ddot{\theta} P + J \ddot{q} + \left( K_E - \theta^2 (J - K_C) \right) q &= Q_{q} \quad (B.26)
\end{align*}
\]

It is assumed here that the deflections are small, hence the quadratic terms in \(q\) can be neglected. Let

\[
K = J - K_C \quad (B.27)
\]

Finally the equations of motion are found as

\[
\begin{align*}
I \ddot{\theta} + P^T \ddot{q} + 2 \dot{\theta} \dot{q}^T K q &= Q_{\theta} \quad (B.28) \\
\ddot{\theta} P + J \ddot{q} - \left( K_E - \theta^2 K \right) q &= Q_{q} \quad (B.29)
\end{align*}
\]

Assuming a generalized disturbance force \(d\) it can be added as \(d_{\theta}\) and \(d_{q}\) to the right hand side of the equations of motion to get

\[
\begin{align*}
I \ddot{\theta} + P^T \ddot{q} - 2 \dot{\theta} \dot{q}^T K q &= Q_{\theta} + d_{\theta} \quad (B.30) \\
\ddot{\theta} P + J \ddot{q} + \left( K_E - \theta^2 K \right) q &= Q_{q} + d_{q} \quad (B.31)
\end{align*}
\]
The generalized forces can be found using the virtual work expression as

\[ \delta W = T' \delta \theta + \int_0^L f(x,t)(\delta R_E + \delta \theta \times (r + u) + \delta u)dx \]  
(B.32)

where \( f(x,t) \) is a distributed external load density on the appendage and \( T' \) is a control torque around the rigid hub. Then

\[ Q_\theta = T' + N \int_0^L (R + x)f(x,t)dx \]  
(B.33)

\[ Q_q = N \int_0^L \phi f(x,t)dx \]  
(B.34)

Discretizing the distributed load to be \( m \) concentrated appendage control forces \( F_1, \ldots, F_m \) at locations \( x_i \), then

\[ f(x,t) = \sum_{k=1}^m F_k(t) \delta(x - x_k) \]  
(B.35)

where \( \delta \) is the spatial Dirac Delta function, substitute from (B.35) into equations (B.33) and (B.34) to get

\[ Q_\theta = T' + N \sum_{k=1}^m (R + x_k)F_k \]  
(B.36)

\[ Q_q = N \sum_{k=1}^m \phi_i(x_k)F_k \]  
(B.37)

Now let the configuration vector be

\[ \zeta = \begin{bmatrix} \theta & q^T \end{bmatrix}^T \]  
(B.38)

and the state vector

\[ X = \begin{bmatrix} \zeta^T & \dot{\zeta}^T \end{bmatrix}^T \]  
(B.39)
The equations of motion can be put in the form

\[
\begin{bmatrix}
\dot{\varsigma}
\dot{\xi}
\end{bmatrix} = \begin{bmatrix} f_1(\varsigma, \dot{\varsigma}) \\ f_2(\varsigma, \dot{\xi}) \end{bmatrix} + \begin{bmatrix} 0 \\ B \end{bmatrix} U + \begin{bmatrix} 0 \\ D \end{bmatrix} d(t)
\]

where

\[
f_1 = \begin{bmatrix} 0 & [1] \end{bmatrix} X
\]

\[
f_2 = M^{-1} \begin{bmatrix}
-2\dot{\theta}q^T K q \\
-(K_E - \dot{\theta}^2 K)q
\end{bmatrix}
\]

\[
M = \begin{bmatrix} I & P^T \\ P & J \end{bmatrix}
\]

\[
\bar{B} = M^{-1} \begin{bmatrix}
1 & N(R + x_1) & \cdots & N(R - x_m) \\
0 & N\phi(z_i) & \cdots & N\phi(x_m)
\end{bmatrix}
\]

\[
U = \begin{bmatrix} T' \\ F_1 \\ F_i \end{bmatrix}^T
\]

and \([1]\) is the identity matrix of dimension \((n+1)\). It is noticed that the equations of motion are nonlinear. Let the desired spacecraft configuration vector be

\[
\varsigma^* = [\theta^*(t) \quad q_1^*(t) \ldots q_n^*(t)]^T
\]

where \(\theta^*(t)\) is the desired attitude behavior, and \(q_i^*(t)\) is the required elastic behaviour. Since in a maneuvering problem no elastic displacement is required, therefore \(q_i^* = 0\). Introduce a state error vector

\[
e = \varsigma - \varsigma^* = [\theta_e \quad q]^T
\]

where \(\theta_e = \theta - \theta^*\) is the attitude error. It was noticed that the attitude error \(\theta_e\) is an ignorable coordinate, but it is needed as feedback for control purposes. To
solve this problem let the input torque be

\[ T = T' + k\theta_e \]

where \( k \) is a constant gain to be specified later. Finally the equations of motion take the form

\[
\begin{bmatrix}
\dot{e} \\
\dot{\theta}_e
\end{bmatrix} = \begin{bmatrix} f_{1e} \\ f_{2e} \end{bmatrix} + \begin{bmatrix} 0 \\ \bar{B} \end{bmatrix} U + \begin{bmatrix} 0 \\ \bar{D} \end{bmatrix} d
\]

where

\[
f_{1e} = \begin{bmatrix} 0 & 1 \end{bmatrix} X_e
\]

\[
f_{2e} = M^{-1} \begin{bmatrix} 2(\dot{\theta}_e + \dot{\theta}^*)^T K q + k\theta_e \\ -(K_E - (\dot{\theta}_e + \dot{\theta}^*)^2 K) q \end{bmatrix} + \begin{bmatrix} \dot{\theta}^* \\ 0 \end{bmatrix}
\]

Let \( X_e = [e^T \dot{e}^T]^T \), be the total spacecraft error state vector then the equations of motion have the compact form

\[
\dot{X}_e = f_e(X_e) + BU - Dd \quad (B.43)
\]
Figure B.1: Flexible Spacecraft Axes
BIBLIOGRAPHY


[38] Mayhan, R. J., Discrete-Time and Continuous Time Linear Systems, Addison-Wesley, Ma., 1984.