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PACKING AND COVERING PROBLEMS

The Ohio State University  Ph.D.  1986

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PACKING AND COVERING PROBLEMS

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree of Philosophy in the Graduate
School of the Ohio State University

By

Andras Bezdek, B.S., M.S.

1986

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ACKNOWLEDGEMENTS

I express sincere appreciation to my adviser, Dr. Hans Zassenhaus for his guidance and suggestions throughout writing the thesis. Thanks go to the other member of my advisory committee, Dr. Alan C. Woods for his comments. Gratitude is expressed to Dr. Laszlo Fejes Toth for drawing my attention to very nice open questions and Drs. Karoly Bezdek and Karoly Boroczky for their support throughout the research. The help of Sue Staats at the proofreading is greatfully acknowledged.
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<th>Year</th>
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<tr>
<td>December 30, 1956</td>
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INTRODUCTION

Let $S$ be a system of closed sets in the $n$-dimensional Euclidean (or spherical or hyperbolic) space. If each point of the space belongs to the interior of at most one member of $S$ then $S$ is said to be a packing. If, on the other hand, each point of the space belongs to at least one member of $S$ then we say that $S$ is a covering.

Packing and covering theory is a special part of discrete geometry, which itself is a rather new branch of mathematics. One of the earliest results is due to A. Thue, who determined the densest packing of congruent circles in 1910 [TA1]. Since that time a great variety of problems were studied and solved mostly by mathematicians of the American, Hungarian, German and Russian school. For basic notations as well as for earlier literature the reader is referred to the books by L. Fejes Toth [FL2] and C.A. Rogers [RC1].

Here, we are interested in extreme value problems, which generally can be described in the following way:

First of all, we consider only a special subclass of packings and coverings (for example those consisting only of congruent circles,
balls, triangles, cubes... or packings which satisfy certain properties.) Then we ask which one among them is best with respect to certain quantified conditions (which is most economical, densest, thinnest, closest...).

This thesis contains the solutions of related problems. The origin of each problem together with the earlier results and other possible open questions will be discussed at the beginning of each chapter.

Using the advantage of discrete geometry that its problems usually can be formulated in an intuitive way, we give here a rough sketch of the questions which we are going to solve later.

- Consider a large table on which there is a coin packing such that each coin is touched by six others (this is the densest packing). Let us remove one coin from the table. Can we rearrange some of the rest such that the new packing will be essentially different from any packing derived from the original packing merely by removal of one coin? We will show the answer to be negative for the corresponding question in the hyperbolic plane.

- Suppose we have a very large table, but we are short of coins. How few coins do we need to construct a packing on the table such that there is no place for another coin without touching or overlapping two coins lying on the table?
- Suppose we have a collection of squares, but we don't know their number and their sizes. Knowing just the total area of the squares when can we state with certainty, that they can cover a unit square?

- Find the most economical coin packing such that any two coins are separable by a straight line that avoids all other coins.

- We will solve a problem about compact packings. Since the way it is formally posed does not have a markedly different intuitive interpretation, we would like to refer to the first paragraph of chapter 5.

- Suppose we have \( n \) circles of areas \( 1/n \), \( 2/n \), \( \ldots \), \( n/n \). What is the smallest square into which we can move those circles without overlapping?
CHAPTER I

SOLID PACKINGS OF CIRCLES

Introduction

Let us recall the fact [FL2] that the face-incircles of the regular trihedral tilings \( \{2,3\}, \{3,3\}, \{4,3\}, \{5,3\}, \) and \( \{6,3\} \) form a densest packing of 3, 4, 6, 12 and infinitely many equal circles, respectively. Can these results be extended to the hyperbolic tilings \( \{7,3\}, \{8,3\}, \ldots \)? This question started some investigations [FL3], [BK1] which led to the perception that in the hyperbolic plane it is impossible to define any "reasonable" notion of density. In order to avoid this difficulty L. Fejes Toth [FL7] introduced the notion of solidity of a packing: A packing of convex discs is said to be solid if no finite subset of the discs can be rearranged (by a series of rigid motions) so as to form, together with the rest of the discs, a packing not congruent with the original one. With this notion a nice result [FL9], [IM1] can be phrased as follows: For any integer \( p > 1 \) the face-incircles of \( \{p,3\} \) constitute a solid packing. The conjecture that also the face-incircles of any spherical, Euclidean or
hyperbolic trihedral Archimedean tiling form a solid packing, has been confirmed in many cases [FL7], [FG1].

In this chapter we want to deal with another conjecture of L. Fejes Toth according to which the face-incircles of a tiling \( \{p,3\} \) with \( p \geq 6 \) form a strongly solid packing in the sense that after removal of one of the circles the remaining circles continue to be solidly packed. This conjecture was suggested by the corresponding observation for the sphere. Removal of one of the face-incircles of \( \{2,3\} \), or \( \{3,3\} \) the remaining three circles will have "free" play. On the other hand, removing one of the six face-incircles of \( \{4,3\} \) the remaining five circles will have only rather restricted play: two of the circles must be centered, say, in the north- and south-pole, while the rest of the centres must lie on the equator. Finally, the packing of eleven circles arising by removing one of the twelve face-incircles of \( \{5,3\} \) is, apart from rotations of the sphere, uniquely determined. Thus the face-incircles of \( \{5,3\} \) form a strongly solid packing. Even more so that is expected for \( \{6,3\}, \{7,3\}, \ldots \). We will prove

**Theorem 1.1**

The face-incircles of a tiling \( \{p,3\} \) with \( p \geq 8 \) form a strongly solid packing.
The problem is still open for \( p = 6 \) and \( 7 \).

Proof of Theorem 1.1

First, we consider the case when \( p = 8 \).

Let \( S = \{ c_1, c_2, \ldots \} \) be the set of circles that arises from the face-incircles \( c_0, c_1, \ldots \) of \( \{8,3\} \) by removing \( c_0 \). We say that a subset \( s = \{ c_1, \ldots, c_n \} \) of \( S \) is good if there is no nontrivial rearrangement of the circles \( c_1, \ldots, c_n \) that results in a packing congruent to \( S \). If the property of being good is transmitted from \( s \) to the union of \( s \) and \( c_{n+1} \), we say that \( c_{n+1} \) can be joined to \( s \). Starting with a set \( s_0 \) consisting of a single circle of \( S \), which is obviously good, we shall successively join to \( s_0 \) new circles of \( S \) so as to obtain an infinite set of enlarged sets \( s_0, s_1, \ldots \) so that \( s_0 S_s_1 S_s_2 S \ldots = S \). We start with the following . . .

Remark 1.1

If the circle \( c_n \) cannot be joined to \( \{ c_1, \ldots, c_{n-1} \} \) then in a non-trivial rearrangement of \( c_1, \ldots, c_n \) at least two circles overlap the original place of \( c_n \). Overlap means the two circles are not the same but have interior points in common.

To see this we observe that in a non-trivial rearrangement at
least two circles are not inscribed in faces of \( (8,3) \). Thus, supposing that only one (or none) of the rearranged circles overlap the original place of \( c_n \), we put this circle (or any of the rearranged circles) to this place obtaining a non-trivial rearrangement of \( c_1, \ldots, c_{n-1} \). This contradicts the tacit assumption that \( \{c_1, \ldots, c_{n-1}\} \) is good.

**Lemma 1.1**

If from among the circles touching \( c_n \) at most two belong to \( \{c_0, c_1, \ldots, c_{n-1}\} \) then \( c_n \) can be joined to \( \{c_1, \ldots, c_{n-1}\} \).

**Proof of lemma 1.1**

Suppose that \( c_n \) cannot be joined to \( \{c_1, \ldots, c_{n-1}\} \) and consider the locus of the centres of those circles which overlap \( c_n \) without overlapping \( c_{n+1}, c_{n+2}, \ldots \). If from the circles \( c_0, \ldots, c_{n-1} \) only one or only two not adjacent circles touch \( c_n \) then this locus is empty. On the other hand, if among \( c_0, \ldots, c_{n-1} \) there are only two adjacent circles which touch \( c_n \) then the locus is an open point-set whose diameter is equal to \( 2r \), the diameter of a circle \( c_i \) (Figure 1). Thus, in contradiction to the above remark, we have at most one circle overlapping \( c_n \).
Lemma 1.2

Let the circles in the following triplets \((n > 5)\) mutually touch one another: 
\((c_1, c_2, c_n), (c_2, c_3, c_n), (c_1, c_2, c_4), (c_2, c_3, c_5)\). If in the set \(\{c_0, c_1, \ldots, c_{n-1}\}\) there is, apart from \(c_4, c_2, \text{ and } c_5\), no circle touching \(c_1\) or \(c_n\) or \(c_3\), then \(c_n\) can be joined to \(\{c_1, \ldots, c_{n-1}\}\).

Proof of Lemma 1.2

We suppose that \(c_n\) cannot be joined to \(\{c_1, \ldots, c_{n-1}\}\) and consider the circles \(c_1, \ldots, c_n\) in a nontrivial position \(X\). 

Figure 1  To the proof of Lemma 1.1
Since \( \{ c_1, \ldots, c_{n-1} \} \) is good, \( \{ c_2, c_3, c_4, \ldots, c_{n-1} \} \) is also good. By Lemma 1.1, \( c_n \) can be joined to \( \{ c_2, \ldots, c_{n-1} \} \), but because of our supposition, \( c_1 \) cannot be joined to \( \{ c_2, \ldots, c_n \} \).

Thus at least two circles of \( X \) overlap the original place of \( c_1 \).

![Figure 2](image)

Figure 2 To the proof of Lemma 1.2

Let \( O_i \) be the centre of \( c_i \). The locus of the centres of
circles which overlap \( c_1 \) without overlapping \( c_{n+1}, c_{n+2}, \ldots \) is the open "triangle" \( T = O_0 O_1 O_n \) bounded by arcs of circles of radius \( 2r \) concentric with \( c_1 \) and the circles other than \( c_2 \) touching on the one hand \( c_1 \) and \( c_4 \), on the other hand \( c_1 \) and \( c_n \) (Fig. 2). Denote the set \( C \cap T \) by \( T_1 \) and the set \( T \setminus C \) by \( T_2 \), where \( C \) denotes the circle of radius \( 2r \) concentric with \( c_4 \). Since the diameter of \( T_1 \) is equal to \( 2r \), only one of the centres of the circles overlapping \( c_1 \) lies in \( T_1 \). The centre of another circle of \( X \) lies in \( T_2 \).

We claim that this circle contains the point of contact \( P \) of \( c_2 \) and \( c_n \). Let \( O_k \) be the centre of the circle containing the side \( O_1 O_n \) of \( T \). Since \( \angle(O_1 O_4 O_k) = 3\pi / 4 < \pi \), the circle \( C \) intersects the side \( O_1 O_n \) in a point, say \( Q \). By some computation we obtain

\[
\sh PQ = (1+\sqrt{2})^{1/2}/3 < 2^{-1/2} = \sh r ,
\]

i.e., \( PQ < r \), showing that the circle of radius \( r \) centred at \( P \) contains \( T_2 \). This proves the assertion.

The same argument shows that in \( X \) there is a circle overlapping \( c_3 \) and containing \( P \). But the circles containing \( P \) and overlapping on the one hand \( c_1 \), and on the other hand \( c_3 \) must be
identical. No other circle of $X$ can overlap $c_n$, because then either the circular-arc-triangle $0_12_n$ or its image $0_32_n$ reflected in the line $Q_2$ would contain two points at distance $> 2r$. This is impossible since the diameter of $0_12_n$ is equal to $2r$. Thus, in view of our Remark, we got a contradiction to the assumption that $c_n$ cannot be joined to $\{c_1, \ldots, c_{n-1}\}$. This proves the Lemma 1.2.

We write $Z_0 = \{c_0\}$ and define the $m$th zone $Z_m$ of $c_0$ as the set of those circles of $S$ which touch at least one circle of $Z_{n-1}$ but do not belong to a zone $Z_k$ with $k < n$. We consider the union $U = Z_1 \cup Z_2 \cup \ldots \cup Z_n$. Since to any finite subset $s$ of $S$ there is an index $N$ such that $s$ is in $U_N$, the theorem will be proved by showing that for any integer $n > 0$ $U_n$ is good.

In order to obtain a better insight into the structure of a zone, it is convenient to consider, in addition to $\{8,3\}$, the dual tiling $\{3,8\}$. Let $V_1$ be the union of the triangles of $\{3,8\}$ meeting at $O_0$. Add to $V_1$ all triangles of $\{3,8\}$ which have a boundary-point in common with a triangle of $V_1$, obtaining $V_2$, and so on. The triangles of $V_1$ fill out a regular octagon without overlapping and without interstices. Since the triangles added to $V_1$ fill out a ring without overlapping and without interstices, the triangles of $V_2$ form a simple polygon. Since the angles of this polygon are equal
either to $2\pi/4$ or to $3\pi/4$, the polygon is convex. The same holds for the $n$-th polygon $P_n$ formed by the triangles of $V_n$.

Figure 3 shows a part of $\{3,8\}$. The horizontal lines represent the boundaries of $P_1', P_2', P_3'$ and $P_4'$. The vertical edges issuing downwards from the vertices of $P_1$ meet at $O_0$. The construction of the figure can easily be continued. Since the vertices of $P_n$ are the centres of the circles of $Z_n$, and the centres of adjacent circles are connected with edges of $\{3,8\}$, we see that each circle of $Z_n$ has either one or two neighbours in $Z_{n-1}$.

![Figure 3: The $\{3,8\}$ mosaic](image)
Calling the respective circles of type 1 and type 2, we observe that in any zone \( Z_n \) with \( n > 1 \) between two consecutive circles of type 2 there are either two or three circles of type 1.

Now, we start with a circle of \( Z_1 \). By Lemma 1.1, we can join to this circle the other circles of \( Z_1 \) successively in their cyclic order except the last one. The last circle can be joined to the previous ones by Lemma 1.2, showing that \( Z_1 \) is good. Referring to Lemma 1 we consecutively join to \( Z_1 \) all circles of \( Z_2 \) of type 2, then all circles of \( Z_2 \) of type 1 adjacent to those of type 2. The remaining circles of type 1 which lie between two circles of type 1 can be joined to the previous ones by Lemma 1.2. Thus \( Z_2 \) is also good.

In \( Z_n \) with \( n > 2 \) there are, besides consecutive circles of type 21112, also circles of type 2112. Now we proceed as follows. Using Lemma 1, again we first join to \( U_{n-1} \) all circles of \( Z_n \) of type 12. Then, in all quadruples of type 2112, we join one of the circles of type 1. The other circle of type 1 can be joined by Lemma 1.2. The rest of the circles of type 1 can be joined similarly as in \( Z_2 \).

This completes the proof of the theorem for \( p = 8 \).

The above proof can be applied also in the case when \( p > 8 \). Now, the inequality \( PQ < r \) can easily be proved without
computation. Let 0 be the centre of the triangle $O_1O_2O_n$. Since
\[ \text{angle}(O_4O_2O) = 3\pi/p \leq \pi/3 = \text{angle}(O_400_2), \]
we have $O_4O < O_2O_4 = 2r$, so that 0 lies within the circle with centre $O_4$ and radius $2r$.

It follows that $QP < OP$. But because of $\text{angle}(00_2P) = \pi/p \leq \pi/9 < \pi/3 = \text{angle}(O_2OP)$, we have $OP < PO_2 = r$, and thus $QP < r$.

Finally, let us observe that in a zone $Z$ with $n > 1$ generated by $(p,3)$ there are between two circles of type 2 either $p-5$ or $p-6$ circles of type 1. Therefore, there are for $p > 8$ no quadruples of type 2112. Which slightly simplifies the proof.
CHAPTER II

DOUBLE-SATURATED PACKINGS OF CIRCLES

Definition

We shall use the terminology of L. Fejes Toth and A. Heppes [FL1]. Let us consider a set of closed circles of radius \( r \) in the Euclidean plane. If no two circles have interior points in common the set is called a packing. If any closed circle of radius \( r \) intersects at least \( k \) circles of the set then the set is said to be \( k \)-saturated (in case of \( k=2 \) we speak about double-saturated set). How small can the density be of a \( k \)-saturated set (or packing) of unit circles?

Introduction

Let us recall the following elementary facts mentioned in the introduction of [FL1]. Replacing each circle of a \( k \)-saturated set \( S \) (of a \( k \)-fold covering \( S \)) of unit circles by a concentric circle of radius \( 2 \) (of radius \( 1/2 \)), we get a \( k \)-fold covering (a \( k \)-saturated packing). The density of the new system is 4-times (1/4-times) the density of \( S \). Thus, the problem of finding the thinnest \( k \)-saturated
set of unit circles is equivalent to the problem of finding the thinnest k-fold covering.

Figure 4  The thinnest 1-saturated circle packing

Since the latter problem is solved for $k=1$ (and only for this value) [1] we can state that the density of the thinnest 1-saturated set of unit circles is equal to $\pi/(6\sqrt{3})$. Since the extremal configuration is automatically a packing (Figure 4), we have found the thinnest 1-saturated packing too. It is easy to see that in the Euclidean plane a $k$-saturated packing of unit circles with $k > 3$ does not exist.

L. Fejes Toth and A. Heppes proved in [FL1] that the density of a
3-saturated packing of unit circles is at least $\pi/(\sqrt{3}+2)$. Equality holds for the circle system containing circles centred at the vertices of the Archimedean tiling $(3,3,4,3,4)$ with side length 2 (Fig. 5). They also proved that the density of a 2-saturated packing of unit circles is greater than 0.5898. The exact lower bound was conjectured to be $\pi/(3\sqrt{3}) = 0.60459\ldots$. It is attained by the incircles of the regular triangular tiling $(3,6)$ (Fig. 6).

Figure 5  The thinnest 3-saturated circle packing

In the present chapter we shall prove this conjecture. Further problems and results about saturated sets are contained in [FL5], [FL1], [EH1], [BR2], [BR3], [DV2] and [FL8]
Theorem 2.1

The incircles of the regular tiling \{3,6\} of side-length $3/\sqrt{3}$ form a thinnest 2-saturated packing of unit circles.

Consider a 2-saturated packing of circles of radii $1/2$. Let $\{0_i\}$ be the centre-system of the circles. Obviously, on each circle, the greatest "free" arc, which contains no point of contact with other circles of the packing, is smaller than a semi-circle. Thus, connecting the centres of circles touching each other by straight segments, we obtain a tiling containing convex cells of unit sides.

It follows from the definition that the cells are covered twice by the
unit circles. However, we can state a bit more. Each cell is covered by the unit circles centred at its vertices. Suppose that there exists a point \( P \) contained in a cell \( C \) and a circle centred at the point \( O_i \) such that \( O_i \) does not belong to the cell \( C \) and \( PO_i \leq 1 \). Denote by \( O_{jk} \) the side of \( C \) which has a common point with \( PO_i \). It is enough to prove that both segments \( PO_j \) and \( PO_k \) are \( \leq 1 \).

Suppose, on the contrary, that \( PO_j > 1 \). Since \( O_iO_j, O_iO_k > 1 \), we have

\[
\angle O_iO_jP < \angle PO_jO_i \leq \angle O_iO_kO_j < \angle PO_iO_k < \angle O_iO_jP,
\]

which is a contradiction. The following proposition states that each cell has at most 7 sides.

**Proposition I.**

A domain satisfying the conditions

(i) it is a convex,

(ii) it is bounded by at least 7 sides of unit length,

(iii) it is double covered by the unit circles centered at the vertices,

(iv) the vertices that are not neighbouring have distances greater than 1,

must be a 7-gon.
Proof of Proposition I

The mark \( \bar{\text{?}} \) after a statement will denote that the proof is trivial and we will omit it. We shall very often refer to the following elementary fact:

(1) If \( O,A,B,C,D \) are points such that \( A \) belongs to \( OB \), \( C \) belongs to \( OD \) and \( AB = CD \) then \( CA < BD \). \( \bar{\text{?}} \)

Consider a domain \( D \) satisfying the conditions of Proposition I.

Notations

We assign to each side \( e \) the triangle \( T_e \), which is enclosed by the line of \( e \) and the lines of the neighbouring sides of \( e \). The triangle \( T_e \) will be called the hat of the side \( e \) and the angle lying opposite to the side \( e \) is called the hat-angle of the hat \( T_e \). It follows from ii), iii) and (1) that any side divides its hat and the domain \( D \). On the figures we shall use some special signs.

If an angle is greater than \( 60^\circ \), then it will be denoted by an empty circular sector at its vertex on the figures. On the other hand, if the angle is less than or equal to \( 60^\circ \), then it will be denoted by a black circular sector. Notice that, two angles may be different even though both are denoted by the same type of circular sector.

Two points will be connected a) by a wavy b) by a straight
c) by a broken line according to whether they have a distance a) less than b) equal to c) greater than 1.

Very often we will reflect a certain vertex $A_i$ in the midpoint of the segment determined by the neighbouring vertices of $A_i$. The image will be denoted by $A_i'$. Proposition I will be proved via the claims (2)-(12).

(2) Suppose that two lines containing sides of D form an angle greater than $60^\circ$. Let $P$ be the intersection of these lines. Then to each side visible from $P$ belongs a hat-angle greater than $60^\circ$.

(3) Let $A$ be a vertex of D. The point $A'$ is contained in the interior of the domain D and, what's more even in the convex hull of the vertex $A$ and the first and second adjacent vertices of $A$.

(4) If the domain $D$ has more than 7 (perhaps infinitely many) sides, then it has 3 adjacent sides having hats with angles $> 60^\circ$.

Proof of (4)

Suppose, on the contrary, that $D$ doesn't satisfy (4). It is easy to see that, in this case, there exist two sides $e,f$ with hat-
angles $< 60^\circ$ such that they are separated by one or two sides. The union of these 3 or 4 sides will be called the "short arc". The two adjacent sides of this "short arc" form an angle $> 60^\circ$, because each of them form angles $\leq 60^\circ$ with the segment $AB$, where $A$ and $B$ denote the vertices of the hats belonging to the sides $e$ and $f$.

In view of (2) we conclude that all of the sides which are visible from the intersection have hat-angles $> 60^\circ$. The number of those sides is $= \{ \text{total number of sides minus 5} \}$ or $\{ \text{total number of sides minus 6} \}$. So, according to the indirect supposition $D$ cannot have more than 8 sides and if it has 8 sides, then the adjacent sides have hat-angles $\leq 60^\circ$ and $> 60^\circ$ in alternate pairs. Suppose that $A_1A_2A_3...A_8$ is an 8-gon of this type such that the sides with hat-angle $> 60^\circ$ are adjacent to the vertices $A_1$ or $A_2$ (Figure 7).

We may assume that $\angle(A_1A_3A_5) > \angle(A_4A_5A_6)$. Then

(5) $A_1'A_5 > 1$.

Proof of (5)

Otherwise, applying (1) for the quadrangles $A_2A_4A_5A_1'$ and $A_8A_1'A_5A_6$, we obtain that the intersections $B_1', B_2'$ of the lines $A_1'A_8, A_4A_5$ and $A_1'A_2, A_6A_5$ respectively are separated by the line $A_1'A_5$. 
Thus, the domain $A_1A_1'B_2A_5$ is a quadrangle and we have

$$\angle A_2A_1A_8 = \angle B_1A_1'B_2 < 2\pi - \angle B_1A_5B_2 = \angle A_4A_5A_1$$

which contradicts the supposition.

If the segments $A_1'A_4$ and $A_1'A_6$ were $> 1$ then there would exist a point on the line $A_1A_1'$ near $A_1'$ which is covered only by one circle. Without loss of generality, we may assume that $A_1'A_4 < 1$. Thus, in view of (1), the half-lines $A_1A_8$ and $A_3A_4$ either
intersect each other or are parallel. Draw a line $f$ parallel with $e = A_3A_4$ through $A_1$. According to the foregoing

(6) The lines $f$ and $A_7A_6$ form an angle $\alpha \leq 60^\circ$.

We shall show that (6) is a contradiction. Or rather, we shall consider $D$ as a framework (the sides of $D$ are inextendible, incompressible rods which are joined but rotate freely at the vertices) and give a continuous motion satisfying the following four conditions. During the motion

a) the vertices $A_1, A_2, A_3$ and the lines $e,f$ should remain invariant,

b) the domain $D$ remains convex and lies in the region between the lines $e$ and $f$,

c) the angle $\alpha$ decreases,

d) at the end of the motion each vertex, with the exception of $A_2$ and $A_6$ will lie on the lines $e$ and $f$.

Thus, we would obtain $\alpha > \alpha_{\text{end}} > 60^\circ$ which contradicts (6). Now we state two lemmas.

(7) Denote by $A_0, O$ two different points on the plane and $k$ a circle of radius $r$ centred at $O$. For any $0 < \alpha < \pi$ let $P_\alpha$ be choosen on the circle $k$ such that the angle $AOP_\alpha = \alpha$. The distance between the points $A,P_\alpha$ is an increasing function of $\alpha$. 

Let ABCD be a framework such that AB = CD, BC < DA and the vertices A, D are fixed. While the framework remains convex, the vector BC rotates in the opposite direction as the vectors AB and DC do.

Proof of (8)

Denote by $ABCD$ the position of the framework when it forms a symmetrical trapezium (Figure 8). The angle $\alpha$ between AB and $AB^\circ$ determines the whole system. Denote by $AB\alpha CD$ the position of the vertices belonging to $\alpha$.

Suppose now that there exist angles $\alpha, \beta$ such that $\angle(ADC^\alpha) < \angle(ADC^\beta)$. Neither the quadrangles $AB\alpha CD$ nor $AB\beta CD$ contains the other, since both the quadrangles have the same perimeter. Suppose that the point $C^\beta$ is not in $AB\alpha CD$. In view of (4) we have $B^\beta C^\beta < B^\alpha C^\alpha \leq B^\alpha C^\beta$ which contradicts the equality $B^\alpha C^\beta = B^\beta C^\beta$. Thus the vectors AB and DC rotate in the same direction.

To show that the vectors AB and BC rotate in the opposite direction we can restrict ourselves to the case of $\alpha < 0$. It is enough to prove that for any $0 \geq \alpha > \beta$ the inequality $m(B^\alpha C^\alpha) \geq 0$ implies the inequality $m(B^\beta C^\beta) > m(B^\alpha C^\alpha)$ where $m(PQ)$ denotes the slope of the vector PQ.
Let $B_1^\alpha$ (and $B_2^\alpha$ respectively) be the image obtained by reflecting $B^\alpha$ over the midpoint of the segment $AC^\alpha$ (and $B^\beta C^\alpha$ respectively). Applying (7) such that first the triplet $(D, B_1^\alpha, AB)$ and then the triplet $(D, B_2^\beta, BC)$ takes the role of $(A, O, r)$ we get that $DB_2^\alpha < DC$ and $m(B_2^\beta C') > m(B^\alpha C')$ respectively. Now we describe the motion by an algorithm, which is represented in table 1. Each row of the table contains sufficient information to carry out the corresponding steps of the algorithm. They consist of the following:

a) serial number of the operation,

b) only this part of the polygon will be moved,
c) this vector is rotated about its starting point in the given direction (subject to the restriction of b.) the motion of the whole framework is uniquely determined,

d) this property will be satisfied by the polygon after c) will be carried out as maximal as possible preserving the properties of convexity and of remaining between the lines e, f (see (9)),

e) serial number of the next operation.

Table 1 Algorithm for the proof of (6)

<table>
<thead>
<tr>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>A_7A_6A_5A_4</td>
<td>A_7A_6</td>
<td>A_5 is on the line e</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>A_1A_8A_7A_6A_5</td>
<td>A_6A_7</td>
<td>the vertices A_8', A_7, A_6 are on a line</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>or</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>A_8 is on the line f</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>A_1A_8A_6A_5</td>
<td>A_5A_6</td>
<td>k_4, A_8 is on the line f</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>A_8A_7A_6A_5</td>
<td>A_5A_6</td>
<td>k_5, A_7 is on the line f</td>
<td></td>
</tr>
</tbody>
</table>

(9). The choice of the property in d) is based on the following fact. Since 2 > A_1A_3 > A_8A_4 > A_7A_5, the vertices A_7, A_6, A_5 of the convex 8-gon cannot be on a line.

(10) If the domain D has more than 7 (perhaps infinitely many)
sides, then it has 5 adjacent sides such that each of the middle 3 sides should have hats with angles > 60°, and there exist vertices $A_i$ and $A_j$ different from $A_1, A_2, \ldots, A_6$ with the properties $A_3 A_1 \leq 1$ and $A_4 A_j \leq 1$ (Figure 9).

![Figure 9](image)

Figure 9 To the proof of (9)

**Proof of (10)**

In view of (4) there exist 5 adjacent sides satisfying the first part of the above statement. Denote their vertices by $A_1 \ldots A_6$.

(11) The proposition (10) follows from the inequalities $A_5 A_3' > 1$

and $A_1 A_4' > 1$. 
Proof of (11)

First of all, the following four inequalities are obviously true:

\[ A_1 A_3' > 1 \quad \quad A_5 A_3' > 1 \]
\[ A_6 A_4' > 1 \quad \quad A_2 A_4' > 1 \]

These inequalities together with the inequalities of (11) mean that a sufficiently small neighbourhood of the points \( A_3' \) and \( A_4' \) cannot be double covered by the circles centred at the vertices \( A_1 \ldots A_6 \).

(11) can be formulated the following way:

(12) It is enough to find 5 adjacent sides with vertices \( A_1, \ldots, A_6 \), such that each of the middle three sides has a hat-angle \( > 60^\circ \) and the half lines i) \( A_4 A_5, A_2 A_1 \) and ii) \( A_3 A_2, A_5 A_6 \) neither intersect nor are parallel to each other.

On the contrary, without loss of generality, we may suppose that the half-lines \( A_4 A_5, A_2 A_1 \) intersect each other or are parallel (Figure 10).

The half-lines \( A_5 A_6 \) and \( A_3 A_2 \) diverge, since otherwise comparing the quadrangles \( A_5 A_1 A_2 A \) and \( A_4 A_3 A_2 \) we get the inequality \( A_6 A_1 \leq A_3 A_4 = 1 \), which is a contradiction. By using (1) for the quadrangle \( A_6 A_5 A_4 A_3' \) we get \( A_6 A_3' > 1 \). Thus by (11), there exists a vertex \( A_k \) with \( k > 6 \) such that \( A_k A_3' \leq 1 \). We will prove
that the vertices $A_2A_4...A_7$ satisfy (12).

Otherwise, consider the supporting line $g$ parallel with the line $A_3A_4$. If $A_n$ belongs to $g$ then $n > k$. Otherwise, by using (1) for the quadrangle $A_nA_{n+1}A_2A_3$, we get $A_{n+1}A_2 < 1$.

After verifying that the angles between the lines $A_3A_4$ and $A_1A_2$ and between the lines $A_3A_4$ and $A_5A_4$ are $> 60^\circ$ it is easy to see that $A_2...A_7$ satisfy (12). Thus (10) is proved. Due to the convexity there must be a vertex $A_1$ and a half line $A_1A_{i+1}$ containing a side and intersecting the line $A_3A_4$.

Since $i+1 \neq 2$, we can show by using (1) that $A_{i+2}A_2 < 1$ which contradicts the supposition that the domain has more
than 7 sides.

**Proposition II**

The area of a 7-gon satisfying the conditions of Proposition I cannot be greater than the total area of two squares and three equilateral triangles of unit side length.

**Proof of Proposition II**

It is enough to prove that the 7-gon has a vertex $A$ such that the segments connecting the point $A'$ with the vertices lying not further from $A'$ than 1 divide the seven-gon into two quadrangles and 3 triangles. 

Denote the vertices by $A_1 \ldots A_7$ and the angles at the vertices $A_i$ ($i=1, \ldots, 7$) by $\alpha_i$. We shall need some simple statements:

(13) If $A_1 A_5' \leq 1$, then $\alpha_5 < \alpha_1$.

**Proof of (13)**

Applying (1) for the quadrangles $A_1 A_2 A_4 A_5'$ and $A_1 A_5' A_6 A_7$, we obtain that the intersections of the half-lines $A_2 A_1$, $A_5 A_6$ and of the half lines $A_1 A_7$, $A_4 A_5$ are separated by the line $A_1 A_5'$. Thus $\alpha_5 < \alpha_1$. 

(14) The inequalities \( A_1 A'_5 \leq 1 \), \( A_1'A_5 \leq 1 \) exclude each other.

(15) There exist adjacent sides with hat-angles greater than \( 60^\circ \).

Proof of (15)

By (2), each hat-angle cannot be \( \leq 60^\circ \). Suppose that the hat-angles of the sides \( A_{i-1}A_i \), \( A_iA_{i+1} \) are \( \leq 60^\circ \) and \( > 60^\circ \) resp. Depending on whether the hat-angle of the side \( A_{i+1}A_{i+2} \) is \( > 60^\circ \) or \( \leq 60^\circ \), the side-pair \( A_iA_{i+1}A_{i+2} \) or \( A_{i+3}A_{i+4}A_{i+5} \) satisfies (15).

Suppose that the angle \( \alpha_4 \) is the greatest of the angles of the 7-gon formed by the sides with hat-angles \( > 60^\circ \). We distinguish two types of 7-gons.

a) \( \alpha_4 < 120^\circ \);  
b) \( \alpha_4 \geq 120^\circ \)

Case of \( \alpha_4 < 120^\circ \)

\( \alpha_4 < 120^\circ \) means that the angle \( (A_4A_5A_4') > 60^\circ \). On the other hand, the condition that the side \( A_4A_5 \) has a hat-angle \( > 60^\circ \) means that angle \( (A_6A_5A_4) > 60^\circ \) (Figure 11). Thus angle \( (A_6A_5A_4) > 120^\circ > \alpha_4 \). From the definition of \( \alpha_4 \), it follows that the hat-angle of the side \( A_5A_6 \) is \( \leq 60^\circ \). So
In view of (7), we have

\[(16) \quad A_6 A_6' \leq 1.\]

Similarly as in the case of the side $A_5 A_6$, we can say the hat-angle of the side $A_2 A_3 \leq 60^\circ$. Consider the quadrangle enclosed by the lines $A_1 A_2, A_3 A_4, A_4 A_5, A_6 A_7$. Since a quadrangle cannot have
3 angles $\leq 60^\circ$.

(18) the side $A_7A_1$ has a hat-angle $> 60^\circ$.

Since $\alpha_4 < 120^\circ$, the section $A_4' A_4 > 1$. Both segments $A_1A_4'$ and $A_7A_4'$ must be $\leq 1$. Otherwise, would exist a point near $A_4'$ that is covered only by one circle. Using (13), both angles $\alpha_1$, $\alpha_7$ are $> \sigma_4$. Thus, by (18) and the definition of $\sigma_4$, the hat-angle of the side $A_6A_7$ is $\leq 60^\circ$. That is

(19) $A_1A_6' \leq 1$.

In view of (16), (17) and (19), $A_6$ is the vertex we wanted to find.

Case of $\sigma_4 \geq 120^\circ$

First we show that

(20) $A_1'A_6 > 1$.

Suppose, on the contrary, that the side $A_1A_7$ has a hat-angle $\leq 60^\circ$.

Consider the convex 7-gon $B_1B_2B_3B_4B_5B_6B_7$ of unit sides which can be defined by the property that the diagonal $A_1A_6$ divides the 7-gon into a regular triangle and a regular hexagon (Fig.12). Let us project
the vertices of this 7-gon and of $D$ onto the oriented line $A_4A_5$.

(Let the direction $A_4A_5$ be positive.) Because of the definition of $\alpha_4$ and $\alpha_4 \geq 120^\circ$ we get the following inequalities:

Length of the projection of $B_3A_4 \leq$ Length of the projection of $A_3A_4$

and

Length of the projection of $B_2B_3 \geq$ Length of the projection of $A_2A_3$.

So the projection of $A_3$ is in the positive direction compared with the projection of $A_2$.

Figure 12 To the proof of (15) in case b)
Similarly, we find that

\[ \text{Length of the projection of } A_5 B_6 < \text{Length of the projection of } A_5 A_6 \]

and

\[ \text{Length of the projection of } B_6 B_7 < \text{Length of the projection of } A_6 A_7. \]

So the projection of \( B_7 \) is in the negative direction compared with the projection of \( A_7 \). Thus, the length of the projection of \( A_2 A_7 > 2 \), which is a contradiction.

Using (1) for the quadrangles \( A_7 A_6 A_5 A_3 \) and \( A_1 A_2 A_3 A_4 \), we have

\[ A_1 A_5 > 1 \]

and

\[ A_1 A_4' < 1. \]

Thus, using (14), we obtain

\[ A_1 A_4' > 1. \]

Because of (20), (21) and (23), a small neighbourhood of the point \( A_1' \) can be double covered by the circles if and only if

\[ A_1 A_3' < 1. \]
(25) We will prove that \( A_7A_4' \leq 1 \) follows from the inequality (24) (Figure 13). First we verify that

\[ A_7A_3 \cap A_4A_4' = \emptyset. \]

**Proof of (26)**

Otherwise, the strip bounded by the lines \( A_4'A_3 \) and \( A_4A_5 ' \) would contain the broken line \( A_4'A_5A_6A_7 \). Let us consider this broken line as a framework. Rotate the triangle \( A_7A_5A_6 \) about \( A_5 \) in the negative direction as far as \( A_7 \) lies on the line \( A_3A_4' \). Because of (7), the length of \( A_4'A_7 \) decreases. Move the vertex \( A_7 \) towards
$A_4$ on the line $A_3A_4'$ as far as $A_6$ lies on the line $A_4A_5$. The length of $A_4'A_7$ decreases. Since the width of the strip $\leq \sqrt{3}/2$, the side $A_4'A_7$ of the obtained symmetrical-trapezoid $A_7A_6A_5A_4'$ is $\geq 2$. This is a contradiction, since at the beginning we had

$$A_4'A_7 < A_4'A_1 + A_1A_7 \leq 2.$$ 

Thus,

$$(26) \quad A_1A_3 \cap A_4A_4' = \emptyset.$$ 

Rotate the point $A_4'$ about $A_3$ the positive direction. According to (7), (25) and (26) the distances $A_1A_4'$, $A_7A_4'$ increase. If $A_1A_4' = 1$, the distance $A_7A_4'$ equals to $A_1'A_3$, which was unchanged during the motion. Thus (24) is proved.

In view of (22) and (24) $A_4$ is a vertex we wanted to find. Thus Proposition II is proved.

Returning to the original problem, consider a double-saturated packing of unit circles $\{C_i\}$. We shall denote a disc and its area with the same symbol.

Let $C(R,0)$ be a circle of radius $R$ centred at the point $O$. We have to prove
This inequality can be formulated in the following way.

There exists $R_0$ for any $h > 0$ such that if $R > R_0$ we can choose a subset $T$ of $C(R,0)$ satisfying the conditions:

$$T < R^2 h \pi \quad \text{and} \quad \lim \inf \frac{\bigcup \{ C_i \cap C(R,0) \}}{C(R,0)} \geq \frac{\pi}{3 \sqrt{3}}.$$

We will show how to choose $T$. First, we define an $\varepsilon$-figure.

Draw a concentric circle in the unit circle such that the ring will have an area $\leq \varepsilon \pi$. Let the number $k$ large enough so that the regular $k$-gon inscribed the unit circle contains the middle-circle of the ring. Let $R_0 > \frac{14(k-6)}{\varepsilon \pi}$. After enlarging the $\varepsilon$-figure $R_0$-times, the ring should have a width $\geq 6$.

If $R > R_0$, let us enlarge the $\varepsilon$-figure $R$-times. Then let us leave out all the cells (see the definition of the tiling at the beginning of this paper) that have no common point with the $k$-gon. The total area of those cells is obviously $< R^2 h \pi$.

Let us recall a consequence of Euler's theorem [10]:

If a convex $k$-gon is divided into $n$ convex polygons so that $n_i$ of them is $i$-gon then
\[ \sum_{i=1}^{n} m_i \leq 6n+k-6. \]

Next we will guarantee that the average side number of the intersection of the cells with the k-gon is at most 6, and none of the remaining cells have more than 7 sides. (In the bracket it will stay an obvious upper bound concerning the total area of the deleted part of \( C(R,O) \)).

Let us

- \( k-6 \) cells whose intersection with the regular k-gon is at least a 7-gon \( (R^2 h\pi) \).

- All the cells whose intersection with the regular k-gon is at least a 8-gon. Divide the intersections of the cells with the k-gons into groups by associating the triangles (quadrangles and pentagons) 3-3 (2-2 and 1-1 resp.) 7-gons. Of course, all the 7-gons can be used. The rest of the polygons form one-element-groups \( (R^2 h\pi) \).

- If a group contains a truncated cell, then let us leave out all the cells belonging to this group \( (12R^2 h\pi) \).

In summary, after deleting cells of total area \( 16R^2 h\pi = \varepsilon \pi \), the union of the cells remaining have average side number \( \leq 6 \). The inequality (27) can be verified by groups. For the purpose of the computation, we have to maximize the area of the cells. If the cell
is a 7-gon, we use Proposition II which states that the area is \( < \frac{(8+3\sqrt{3})}{4} \). Otherwise, the area of the i-gons \( i=3,4,5,6 \) will be estimated by the area of the regular i-gons. Thus, the density in a group of typ

- \((3,7,7,7)\) is at least \( \frac{8\pi}{(24+10\sqrt{3})} \) \( = 0.1932 \ldots \)

- \((4,7,7)\) is at least \( \frac{6\pi}{(20+6\sqrt{3})} \) \( = 0.1973 \ldots \)

- \((5,7)\) is at least \( \frac{4\pi}{(5\sin54^\circ+8+3\sqrt{3})} \) \( = 0.1992 \ldots \)

- \((6)\) is at least \( \frac{2\pi}{(6\sqrt{3})} \) \( = 0.1925 \ldots \)

- \((5)\) is at least \( \frac{3\pi}{(10\tan54^\circ)} \) \( = 0.1932 \ldots \)

- \((4)\) is at least \( \frac{\pi}{4} \) \( = 0.25 \ldots \)

- \((3)\) is at least \( \frac{\pi}{(2\sqrt{3})} \) \( = 0.2887 \ldots \)

It can be seen that each bound is \( > \frac{\pi}{(3\sqrt{3})} \), which completes the proof.
We will denote a closed convex set (solid) in the n-dimensional Euclidean space $E^n$ and its volume by the same symbol.

The main goal of this chapter is to give sufficient conditions for the existence of a covering of a given solid by solids of different shapes and sizes. We have to make precise the meaning of "different shape and size", because otherwise we cannot expect a reasonable answer. We will restrict ourselves to the case when the covering consists of homothetic solids.

Fix the solids $K$ and $K'$ of volume 1 in the n-dimensional Euclidean space. Denote by $f_n(K',K)$ the smallest number such that if a set $S$ consisting of $K$-homothetic solids has a total volume $\geq f_n(K',K)$, then $S$ always can cover $K'$. If $K'$ and $K$ are the same solids then we write simple $f_n(K)$. The problem of determining the function $f_n$ was raised by L. Fejes Toth. Denote the unit cube in $E^n$ by $C$. In this chapter, we will prove the following statements.
Theorem 3.1

\[ 2^n - 1 \leq f_n(C) \leq 2^n \]

Using a much longer argument we prove

Theorem 3.2

\[ f_n(C) = 2^n - 1 \]

In dimension 2, for an arbitrary convex disc \( K \) we will prove

Theorem 3.3

\[ 2 \leq f_2(K) \leq 12 \]

Remark 3.1

Since the problem is affine, in Theorem 3.1 and 3.2 the term "unit cube \( C \)" can be replaced by the term "parallelotop of volume one".

Remark 3.2

Using Theorem 3.2 and a theorem from Blaschke (see [HH1] pp.154-155) it can easily be proved that for any convex solid \( K \) in \( \mathbb{E}^n \), the number \( f_n(K) \) exists.

Theorem 3.4 (with Z. Füredi [BA3])

If \( T \) is a triangle of unit area in \( \mathbb{E}^2 \) then \( f_2(T) = 2 \)
This result led to the following

**Conjecture 3.1 (by L. Fejes Tóth)**

For any disc of unit area in $\mathbb{E}^2$ the inequalities $2 \leq f_2(K) \leq 3$ hold.

Let $T_0$ be the image of the triangle $T$ upon reflection in a point. Since the proof of the following theorem is similar to the proof of Theorem 3.2, it won't be included in this chapter.

**Theorem 3.5 (with K. Bezdek and E. Rozsahegyi)**

$$f_2(T_0, T) = 4$$

**Conjecture 3.2 (by K. Böröczky)**

Let $C_r$ denote the image of the unit square upon rotation through the angle $\pi/4$. Then $f_2(C_r, C) = 5/2$.

**Conjecture 3.3 (by M. Bognar)**

Let $C^*$ be a square in arbitrary position. Then $f_2(C^*, C) \geq 2$ and 2 is the greatest number with this property.

**Proof of Theorem 3.1**

Proof of the lower bound: Suppose, on the contrary, that $f_n(C) = h(2^n - 1)$, where $0 < h < 1$. By the definition of $f_n(C)$, $C$ should be
able to be covered by \(2^{n-1}\) cubes of side length \(\frac{n}{k}\) such that each of them is homothetic to \(C\). Since each cube can cover only one vertex of \(C\), this is impossible.

Proof of the upper bound: Let \(\{C_i \mid i=1,\ldots,k\}\) be a set of cubes such that \(C_1 + \ldots + C_k \geq 2^n\). We may suppose that \(1 > C_1 > C_2 > \ldots > C_k\). Let \(n_i\) be the number of cubes in this set of volume between \(2^{-in}\) and \(2^{-(i+1)n}\). Our algorithm goes as follows: Divide the cube \(C\) into \(2^n\) congruent cubes. Cover the first \(n_1\) cubes by the first \(n_1\) largest cubes of our system. Then divide each of the uncovered cubes into \(2^n\) congruent cubes and cover \(n_2\) of the small cubes by the next \(n_2\) cubes of our system. Continue this process until there are uncovered cubes in the subdivision. Since each cube is used for covering a cube of volume not smaller than the \(1/2^n\)-th part of his volume, we can not run out of cubes before the procedure stops.

Proof of Theorem 3.2

We will prove the following stronger theorem.

Theorem 3.4

If a set \(S\) of cubes \(\{C_i \mid i=1,\ldots,k\}\) satisfies the inequalities

\[C_1 \geq C_2 \geq \ldots \geq C_k\]

and

\[C_1 + \ldots + C_k \geq 2^{n-1} + C_1 + \ldots + C_{2^{n-1}-1},\]
then there is a rearrangement of the cubes of $S$ such that the cubes are in parallel position with $C$ and cover it.

We need the following Lemma.

**Lemma 3.1**

Let $P$ be a rectangular parallelepiped of edge lengths $a_1, \ldots, a_n$ and $C_1, \ldots, C_k$ be cubes of edge lengths $e_1 \geq e_2 \geq \ldots \geq e_k$ in $E^m$. If $\sum_{i=1}^{k} C_i \geq \prod_{j=1}^{n} (a_j + e_j)$, then there is a rearrangement of the cubes such that the cubes are in parallel position with $P$ and cover $P$.

It is sufficient to prove the following stronger lemma.

**Lemma 3.2**

Suppose that the conditions of Lemma 3.1 hold. Let $1 \leq N \leq k$ be the largest index such that

$$\sum_{i=1}^{N} C_i \leq \prod_{j=1}^{n} (a_j + e_j).$$

Then there is a rearrangement of the cubes $C_1, \ldots, C_N$ such that the cubes are in parallel position with $P$ and cover it.

**Proof of Lemma 3.2**

We will use induction on $n$. First let $n = 2$. Choose the
indices \( i_1, i_2, \ldots, i_m \) such that they satisfy the following inequalities:

\[
0 \leq \left[ \sum_{j=1}^{i_1} e_j \right] - a_1 < e_{i_1}
\]

\[
0 \leq \left[ \sum_{j=i_1+1}^{i_2} e_j \right] - a_1 < e_{i_2}
\]

\[
0 \leq \left[ \sum_{j=i_m+1}^{i_{m-1}} e_j \right] - a_1 < e_{i_m}
\]

\[
\left[ \sum_{j=i_{m-1}+1}^{N} e_j \right] - a_1 < 0
\]

From these inequalities we get

\[
\sum_{j=1}^{m} e_{i_j} > a_2 \quad \text{(3.1)}
\]

Suppose, on the contrary, that \( \sum_{j=1}^{m} e_{i_j} \leq a_2 \), then

\[
\sum_{i=1}^{k} e_{i_{j}} ^2 < e_{i_1} (a_{1} + e_{i_1}) + e_{i_1} (a_{1} + e_{i_2}) + \ldots + e_{i_1} (a_{1} + e_{i_{m-1}}) + e_{i_1} a_1
\]

\[
\leq e_{i_1} (a_{1} + e_{i_1}) + e_{i_1} (a_{1} + e_{i_2}) + \ldots + e_{i_1} (a_{1} + e_{i_{m-1}}) + e_{i_1} a_1
\]

\[
< (a_{1} + e_{i_1})(a_{2} + e_{i_1})
\]

which is a contradiction. Since for each \( j = 1, \ldots, m \), the rectangle
of edges $e_i$ and $a_1$ can be covered by the cubes $C_{i-1}^{i+1}$, $C_{i-1}^{i+1}$, $C_{i-1}^{i+1}$, $C_{i-1}^{i+1}$.

Lemma 3.2 immediately follows from (3.1).

Now suppose that Lemma 3.2 is proved for all dimensions less than $n$. Let $i_1$ be the largest index such that

$$e_{i_1}^{n-1} + \ldots + e_{i_1}^{n-1} < (a_2 + e_1)(a_3 + e_1)\ldots(a_n + e_1).$$

Suppose that we already have determined the indices $i_1, i_2, \ldots, i_{j-1}$.

Let $i_j$ be the largest index such that

$$e_{i_1}^{n-1} + \ldots + e_{i_j}^{n-1} < (a_2 + e_{i_j}^{n-1})(a_3 + e_{i_j}^{n-1})\ldots(a_n + e_{i_j}^{n-1}).$$

Suppose that we have determined in this way $m$ indices. Let us mention that the induction hypothesis means the following:

For each $j = 1, \ldots, m$, the $n-1$ dimensional rectangular parallelepiped of edges $a_1, a_2, \ldots, a_n$ can be covered by the $n-1$ dimensional cubes of edges $e_{i_1}^{n-1}, \ldots, e_{i_j}^{n-1}$.

From the above inequalities we get

$$\sum_{j=1}^{m} e_{i_j}^{n-1} > a_1$$

(3.2)

Since otherwise, we have

$$\sum_{i=1}^{k} e_i^{n} < e_1^{n} \prod_{j=2}^{n}(a_j + e_1) + e_{i_1+1}^{n} \prod_{j=2}^{n}(a_j + e_{i_1+1}) + \ldots + e_m^{n} \prod_{j=2}^{n}(a_j + e_m^{n})$$
\[ \leq (e_1 + e_{i_1} + \ldots + e_{i_{m+1}}) \prod_{j=2}^{n} (a_j + e_1) \]
\[ \leq (e_1 + e_{i_1} + \ldots + e_{i_m}) \prod_{j=2}^{n} (a_j + e_1) \]
\[ \leq \prod_{j=1}^{n} (a_j + e_1) \], which is a contradiction. Lemma 3.2 follows immediately from (3.2) and the induction hypothesis.

Proof of Theorem 3.4

We will use induction on the dimension again. First let \( n = 2 \).

Suppose we have \( k \) squares \( S_1, S_2, \ldots, S_k \) of sides \( e_1 \geq e_2 \geq \ldots \geq e_k \) in \( E^2 \) such that \( e_1^2 + \ldots + e_k^2 \geq 2 + e_1^2 \). We want to show that the unit square \( S \) can be covered by these squares. Without loss of generality, we may suppose that \( e_1 = e_2 \). If \( e_1 = e_2 \leq 1/2 \) then \( 2 + e_1^2 \geq (1 + e_1)^2 \). So we can apply Lemma 3.2 which says that a covering of the type we wanted exists.

Suppose \( 1 > e_1 = e_2 > 1/2 \). Denote the vertices of \( S \) by \( Q_1 Q_2 Q_3 Q_4 \). Cover the corners at \( Q_1 \) and \( Q_2 \) by the squares \( S_1 \) and \( S_2 \) as it is shown on the Figure 14. Let the rectangle \( Q_3 Q_4 F_1 F_2 \) be the uncovered part of \( S \). Let \( m \) be the largest index such that \( e_{m} \geq 1 - e_1 \) (it can happen that such an index does not exist).

If \( e_1 + \ldots + e_m \geq 1 \) then starting at the edge \( Q_3 F_2 \), arranging the squares \( S_1, \ldots, S_m \) side by side, we get a covering. Otherwise, a
small rectangle $Q_1 F_1 F_3 F_4$ remains uncovered, where
$Q_3 F_4 = e_1 + \ldots + e_m$. It is clear that no point of $S$ is covered three
times by the squares $S_2, \ldots, S_m$, and the total area of the uncovered
parts of the square $Q_1 Q_2 Q_3 Q_4$ is not smaller than $F_1 F_3 F_4 Q_4 + (1-e_1)^2$.

![Figure 14  To the proof of Theorem 3.4](image)

Let $S_0$ be the total area of the unused squares. $S_0$ is at least
$2 F_1 F_3 F_4 Q_4 + 2(1-e_1)^2$. We can write this sum in the following form:
\[ [(1-e_1) + (1-e_1)][Q_4 F_4 + (1-e_1)] \]. Since each square has side length $\leq 1-e_1$, we can apply Lemma 3.1, and we are done.

Let $n \geq 3$, and suppose that Theorem 3.4 is proved for all dimensions
less than $n$. Let $C$ be the unit cube of the $n$ dimensional Cartesian
coordinate system, i.e. each vertex of $C$ can be given by an $n$-tuple of 0's and 1's. Further, suppose that the cubes $C_1, C_2, \ldots, C_k$ satisfy the conditions of Theorem 3.4. We have to show that the cube $C$ can be covered by the cubes $C_1, C_2, \ldots, C_k$. Without loss of generality, we may suppose that

$$e_1 = e_2 = \ldots = e_{n-1} = \frac{1}{2}$$. If $e_1 \leq \frac{1}{2}$ then

$$(1 + e_1)^n \leq \left[ \frac{3}{2} \right]^n = \left[ \frac{3}{2} \right] \left[ \frac{3}{2} \right] \ldots \left[ \frac{3}{2} \right] < 2 \cdot 2 \cdot \ldots \cdot 2 = 2^{n-1} < \sum_{i=1}^{k} C_i.$$.

According to Lemma 3.1, the cube $C$ can be covered by the cubes $C_1, \ldots, C_k$. From now on we can suppose that

$$\frac{1}{2} < e_1 = e_2 = \ldots = e_{n-1} < 1.$$  \hspace{1cm} (3.3)

Cover the corners of $C$ at the vertices of coordinates $(0, h_2, \ldots, h_n)$ where $h_i = 0$ or 1 by the first $2^{n-1}$ cubes. An $n$ dimensional rectangular parallelepiped of edges 1, 1, ..., 1, 1-$e_1$ remains uncovered.

If $\sum_{j=2^{n-1}+1}^{k} C_j > 2^{n-1} - e_1^n \geq (1 + e_1)^{n-1}$, then according to Lemma 3.1, the above rectangular parallelepiped can be covered by the cubes $\{ C_j \}_{j=2^{n-1}+1}^{k}$. Therefore, we may suppose that

$$(1 + e_1)^{n-1} > 2^{n-1} - e_1^n.$$  \hspace{1cm} (3.4)
It is easy to check that $e_1 > 0.8$ follows from (3.4) and $n > 3$.

Let $i_0$ be the largest index such that $e_{i_0} > 1 - e_1$ (if $i_0$ exists then it is automatically $> 2^{n-1}$). Choose the indices $i_1, \ldots, i_m$ such that they satisfy the following inequalities

\[ 0 \leq \left[ \sum_{j=i_0+1}^{i_1} e_j^n \right] - 2(1 - e_1) < (1 - e_1)^n \]

\[ 0 \leq \left[ \sum_{j=i_1+1}^{i_2} e_j^n \right] - 2(1 - e_1) < (1 - e_1)^n \]

\[ \vdots \]

\[ 0 \leq \left[ \sum_{j=i_{m-1}+1}^{i_m} e_j^n \right] - 2(1 - e_1) < (1 - e_1)^n \]

\[ \sum_{j=i_m}^{i_{m+1}} e_j^n < 2(1 - e_1)^n \]

According to Lemma 3.1, the $n$ dimensional rectangular parallelepiped of edges $e_1, e_1, \ldots, e_1, 1-e_1$ can be covered by the cubes $\{ C_{i_{t+1}} \}_{i_1}^{i_{m+1}}$ where $0 \leq t < m$. On the other hand, considering (3.4), we get

\[ \sum_{j=i_{m+1}}^{i_m} e_j^n < 2(1 - e_1) + (1 - e_1)^n = (1 - e_1) (2 + (1 - e_1)^{n-1} ) \]

\[ < (1 - e_1)(2^{n-2} + 2^{n-3}(1 + e_1) + \ldots + (1 + e_1)^{n-2}) = 2^{n-1} - (1+e_1)^{n-1} \]

\[ < e_1^n \]  

(3.5)

Now, we have two possibilities
If the inequality (3.6) holds, then
\[ \sum_{j=2^{n-1}+1}^{i_0} e_j^n \leq 2 \quad (3.6) \]

or
\[ \sum_{j=2^{n-1}+1}^{i_0} e_j^n > 2 \quad (3.7) \]

If the inequality (3.6) holds, then
\[ \sum_{j=i_0+1}^{k} e_j^n > 2^{n-1} - 3 > (2 - e_1)^{n-1} 2(1 - e_1) \quad (3.8) \]

Since \( 2(1-e_1) < 1/2 \) it is enough to check that \( (2 - e_1)^{n-1} < 2^{n-6} \). By Lemma 3.1, the inequality (3.8) means that the n-dimensional rectangular parallelepiped of edges 1, 1, ..., 1, 1-e_1 can be covered by the cubes \( \{C_j\}_{i_0+1}^{k} \). So the proof is done in the case of (3.6). Suppose (3.7) holds. Using the inequalities (3.3) and (3.7), the following results are trivial:
\[ \sum_{j=2^{n-1}+1}^{i_0} e_j^n \leq e_1 \left[ \sum_{j=2^{n-1}+1}^{i_0} e_j^n \right] \quad (3.9) \]
\[ \sum_{j=i_1+1}^{k} e_j^n < \left[ \frac{1}{e_1} - 1 \right] \left[ \sum_{j=2^{n-1}+1}^{i_0} e_j^n \right] \quad (3.10) \]

In view of (3.5), (3.7), (3.9) and (3.10), we have
\[ 2^{n-1} - 1 < 2^{n-1} - e_1^n \leq \]
On account of the induction hypothesis, the n dimensional rectangular parallelepiped of edges 1, 1, ..., 1, 1-e_1 can be covered by the cubes \( \{ C_j \}_{j=2^{n-1}+1} \).

**Proof of Theorem 3.3**

Suppose that the parallelogram \( P \) has the greatest area among those which are contained by \( K \). It is well known that the vertices of \( P \), i.e. A, B, C and D lie on the boundary of \( K \). Let \( P^* \) be the smallest \( P \)-homothetic parallelogram containing \( K \). Suppose that the supporting lines parallel to AB touch \( K \) at the points \( P_1 \) and \( P_2 \) (Figure 15). Let \( m^* \) be the distance between the above two supporting lines. Let \( m \) be the distance between the lines AB and DC. We want to show that

\[ m^* \leq 2m \]  

(3.10)
Suppose, on the contrary, that \( m^* > 2m \). Let \( \Delta m = m^* - 2m \). Fix an integer \( n \) and define the parallelogram \( A'B'C'D' \) contained by \( K \) in the following way: \( A' \parallel AP_1 \), \( |AA'| = |AP_1|/n \), \( B' \parallel BP_1 \), \( |BB'| = |BP_1|/n \), \( C' \parallel CP_2 \), \( |CC'| = |CP_2|/n \), \( D' \parallel DP_2 \), \( |DD'| = |DP_2|/n \).

Let \( m' \) be the distance between the lines \( A'B' \) and \( C'D' \).

![Diagram](image)

**Figure 15** To the proof of Theorem 3.3

Obviously,

\[
m' = m + (m^* - m)/n = (n+1)m/n + \Delta m/n
\]

and

\[
|A'B'| = |AB|(n-1)/n
\]

So we have

\[
|A'B'|m' = |AB|m (n-1)/n^2 + |AB|\Delta m(n-1)/n^2
\]
If \( n \) is large enough then \((n-1)\Delta m > m\), so \( |A'B'|m' > |AB|m\) which contradicts our supposition. Repeating the above argument for the other parallel side pair of \( P \) we get an inequality similar to (3.10). So we have that

\[
P^* \leq 4P
\]

(3.11)

Now we prove that \( f_2(K) \leq 12 \). Suppose that \( \{ K_i \}_{i=1}^{k} \) is a set of \( K \)-homothetic regions with a total area at least 12. Let \( P_i \) be the \( P \)-homothetic parallelogram with the maximal area inscribed in \( K_i \). Using (3.11), we have that

\[
\sum_{i=1}^{k} P_i = P \sum_{i=1}^{k} K_i \geq 12P \geq 3P^*
\]

Using Theorem 3.2 we have that the parallelogram \( P \) can be covered by the parallelograms \( P_i \) and so we are done.

The inequality \( 2 \leq f_2(K) \) is trivial. It is not hard to prove that a convex domain cannot be covered by two homothetic but smaller copies of \( b \).
CHAPTER IV

LOCALLY SEPARABLE CIRCLE PACKINGS

Introduction

A set of open domains arranged in the Euclidean plane is said to be totally separable if any two of them can be separated by a straight line avoiding all of the domains. This notion was introduced in a joint paper by G. Fejes Tóth and L. Fejes Tóth [FG2]. Let us recall the following theorem [FG2]. The (upper) density $d$ of the densest totally separable packing of the plane with congruent convex discs $s$ satisfies the inequality $d \leq A(s)/A(q)$ where $A(s)$ denotes the area of the discs $s$ and $A(q)$ the area of a quadrangle of least area containing $s$. As an immediate consequence we phrase the following corollary: The density of a totally separable packing of equal circles is at most $\pi/4$. Equality holds for the face-incircles of the regular tiling (4,4).

We introduce the notion of local separability. A triplet of disjoint open domains is said to be locally separable if there is a straight line not intersecting any of them, but containing on both
sides one. We say a packing is locally separable if any triplet of the domains is locally separable. Obviously, any totally separable packing of domains is locally separable.

We shall prove that the above corollary remains true by replacing the term "totally separable" by "locally separable". On the other hand, we shall give a construction to show that the statement arising from the above theorem by replacing the term "totally separable" by "locally separable" is false. Our counter-example will contain domains being in non-parallel position. It may be conjectured that the theorem under consideration is true under the restriction that the packing consists only of translates of a convex domain.

Theorem 4.1

The density of a locally separable packing of congruent circles is at most \( \pi/4 \).

First we remark that equality holds for the face incircles of the regular tiling \{4,4\}. But this isn't the only extremal configuration. For example, we get other extremal configurations if we translate a row of circles of the above packing in the direction of the row through a certain distance, or if we delete a set of circles of density 0.
Proof of Theorem 4.1

Let us consider a locally separable packing of unit circles \( \{C_i\} \) with centres \( \{O_i\} \). Without loss of generality we may suppose that any circle \( C \) of radius 4 contains a center point \( O_i \). Otherwise, the unit circle concentric with \( C \) can be added to the packing preserving the property of local separability. Under this condition, we construct the tiling \( L^* \) in the following way. Associating with the point \( O_i \) the set \( D_i \) of all points lying nearer to \( O_i \) than to any other point \( O_j \), we obtain the Dirichlet cell \( D_i \). Considering all the Dirichlet cells \( \{D_i\} \) we get a partition of the plane in cells \( \{D_i\} \) forming a \( d \)-tiling (Dirichlet-tiling \[DG1\]). Connecting the centres \( O_i, O_j \) of neighbouring Dirichlet cells \( D_i, D_j \) we obtain the dual \( L \)-tiling (Delaunay-Voronoi-tiling \[DB1\]) of the plane. Dividing the faces of the \( L \)-tiling of more than three sides by non-intersecting diagonals we obtain the \( L' \)-tiling \[MJ1\]. If an edge \( O_iO_j \) is a side of a triangle \( T \) in \( L' \), such that \( T \) and the center point \( V \) of its circumcircle is separated by the line \( O_iO_j \), then replace the side \( O_iO_j \) by the broken line \( O_iVO_j \). The new tiling will be denoted by \( L^{*} \). Let the triangle \( \triangle \) be a face of \( L' \). Denote by \( \triangle^* \) the face of \( L^{*} \) derived from \( \triangle \). Let \( P \) be a polygon. We consider those vertices of \( P \) which belong to the set \( O_i \). Let \( s \) be the sum of angles of \( P \) at these vertices. We
define the density of the circles relative to $P$ by

$$d(P) = \pi s/A(P).$$

It is sufficient to prove, that $d(\Delta^*) < \pi/4$. We distinguish three types of faces $\Delta^*$

1) $\Delta$ is an acute or right angled triangle and the radius of its circumcircles is less than or equal to 2 (This means that $\Delta^* = \Delta$).

2) $\Delta$ is an acute or right angled triangle not of type 1.

3) $\Delta$ is an obtuse angled triangle.

Type 1

Since the packing is locally separable, $\Delta$ has an altitude greater than or equal to 2. Thus $d(\Delta^*) < \pi/4$.

We shall need

**Lemma 4.1**

Let $ABC$ and $A'B'C'$ be triangles so that $AB = BC > A'B' = B'C'$ and $AC > A'C'$ then

$$\frac{\angle(BAC)}{A(ABC)} \leq \frac{\angle(B'A'C')}{A(A'B'C')}$$

**Proof of Lemma 4.1**

Consider the triangle $A''B''C''$ such that $A''B'' = B''C'' = AB$ and $A''C'' = A'C'$. If $A''$, $B''$ are centres of unit circles then
\[
\frac{\text{angle}(BAC)}{A(ABC)} = \frac{\text{angle}(BAC)}{\frac{AC}{2} \sin(\text{angle}(ABC))} 
\leq \frac{\text{angle}(B''A''C'')}{\frac{A''C''}{2} \sin(\text{angle}(B''A''C''))} =
\]
\[
= \frac{\text{angle}(B''A''C'')}{\frac{A''C''}{2} \tan(\text{angle}(B''A''C''))} 
\leq \frac{\text{angle}(B'A'C')}{\frac{A'C'}{2} \tan(\text{angle}(B'A'C'))} =
\]
\[
= \frac{\text{angle}(B'A'C')}{A(A'B'C')}.
\]

Type 2

Let \( V \) be the centre of the circumcircle of \( \Delta^* = O_1O_2O_3 \).

Obviously,
\[
d(\Delta^*) \leq \max \{ d(\Delta^* \cap O_iV_{oi+1}) | i = 1, 2, 3 \}.
\]

It follows from Lemma 4.1 that \( d(\Delta^* \cap O_iV_{oi+1}) \leq d(O_iV_{oi+1}) \). Since \( O_iV \geq \sqrt{2} \) and \( O_iO_{i+1} \geq 2 \) we have by Lemma 4.1, \( d(O_iV_{oi+1}) \leq \frac{\pi}{4} \).

Type 3

Without loss of generality, we may suppose that the side \( O_1O_3 \)
separates \( V \) from \( \Delta \). Thus
\[
d(\Delta^*) \leq \max \{ d(\Delta^* \cap O_iV_{oi+1}) | i = 1, 2 \}.
\]

The rest of the proof is similar to that of type 2.

Now, we describe the counter-example mentioned in the introduction (Figure 18). The packing consists of centro-symmetric hexagons congruent to \( H = H_1H_2 \ldots H_6 \) inscribed in the rectangle \( R = R_1R_2R_3R_4 \)
of sides \( R_1R_2 = 1 \) and \( R_2R_3 = 10 - \sqrt{2} = k \), as depicted in Figure 16.

The vertex \( H_1 \) is the midpoint of \( R_1R_2 \) and the vertices \( H_2 \) and \( H_3 \) are on \( R_2R_3 \) so that \( R_2H_2 = R_3H_3 = 1/2 + \sqrt{2} \). It is easy to see
that \( R \) is the parallelogram of least area containing \( H \). Let

\( R' = R_1'^...R_4'^ \) ( \( R'' = R_1''...R_4'' \) respectively) be the parallelogram derived from \( R \) by rotating the straight lines \( R_1R_2 \) and \( R_3R_4 \) about \( H_1 \) and \( H_4 \) through \(+45^\circ\) (resp. \(-45^\circ\)).

![Figure 16 The hexagon H](image)

Obviously, \( R, R' \) and \( R'' \) have the same area. Since \( \sqrt{2} \) and 1 are incomparable, there exist integers \( n,m \) for \( h_0 > 0 \) (\( h_0 \) will be chosen afterwards) such that \( 0 < kn/\sqrt{2} - m = 2h < h_0 \). Let the rectangle \( A = A_1...A_4 \) be the union of 10 translates of \( R \) such that \( A_1A_2 = k \) and \( A_2A_3 = 10 \) (Figure 17). Further, let the parallelogram \( B = B_1...B_4 \) be the union of \( n \) translates of \( R'' \) such that \( B_1B_2 = nk \) and \( B_2B_3 = \sqrt{2} \). At last, let the parallelogram \( C = C_1...C_4 \) be the union of \( m \) translates of \( R' \) such that \( C_1C_2 = m\sqrt{2} \) and \( C_2C_3 = k \). Let \( A,B,C \) be placed in such a way that

1) the midpoint \( E \) of \( B_1B_4 \) should be on \( A_4A_3 \).

2) \( EA_3 = h \).
3) \( C_1C_4 \) is on \( A_2A_3 \).

4) \( C_3C_4 \) is on \( B_1B_2 \).

Figure 17  The structure of the counter example

Denote by \( D_h \) the union of \( A,B \) and \( C \). Let \( S \) be a point on \( A_4A_3 \) such that \( A_4S = h \) and let \( T \) be the midpoint of \( B_2B_3 \).
Translate $D_h$ together with the parallelograms by the lattice vectors of the lattice generated by the vectors $A_{14}$ and $ST$.

Inscribe in each parallelogram a hexagon. The set of hexagons has density greater than $A(H)/A(R)$, since the parallelograms cover the plane and some of them overlap (Fig. 18). Choose $h_0$ small enough such that the triplet of hexagons denoted by star in Figure 18 is separable. It is easy to verify that the set of hexagons form a packing and it is locally separable.

---

Figure 18  The counter example
In the Euclidean plane, we shall consider a set of closed convex domains. If no two domains have inner points in common then the domains are said to form a packing. Two domains are said to be neighbours if they touch one another. L. Fejes Toth introduced the notion of compact packing.

A packing of closed convex domains is said to be compact if the conditions i) - iii) are satisfied:

i) any domain has a positive and finite number of neighbours,

ii) the neighbours of any domain S can be assigned an order such that each neighbour touches its successor and the last touches the first,

iii) the union of the neighbours of any domain S contains the boundary of a polygon containing S.

Roughly speaking, a packing is compact if the neighbours of any domain S form a chain containing S. If iii) is not required, the packing shown in Figure 19 could be considered to be compact.
We shall prove the following theorem which was conjectured by L. Fejes Toth.

**Theorem 5.1**

If $d$ denotes the density of a compact packing of the Euclidean plane by homothetic convex discs such that the quotient of the areas of any two discs is greater than a positive number, then $d > 1/2$.

**Remark 5.1**

Equality may hold for any packings of homothetic triangles. Equality holds for the lattice packing of triangles where the triangles touch each other at the vertices (Figure 20). But this isn't the only extremal configuration. For example, we get other extremal configuration, if we replace the union of $n(n+1)/2$ triangles contained by an $n$-times greater triangle with their convex
hull (Figure 21). Obviously we may carry out this modification simultaneously on other parts of the packing. If the triangles affected by the modification form a packing of density $0$, then the new triangle system still has density $1/2$.

![Figure 20 Illustration to Remark 5.1 A](image)

![Figure 21 Illustration to Remark 5.1 B](image)
Remark 5.2

In the above theorem, the condition concerning the quotient of the areas of any two discs is greater than a positive number is not negligible. Consider the incircles of the triangles of the mosaic \((3,p)\) \(p > 6\), which is represented in the Poincaré model (Figure 22). Since all circles lie in a bounded region, the density is equal to 0.

Figure 22  Circle packing in the Poincare model

Remark 5.3

L. Fejes Toth proposed the following sketch to prove the theorem. Consider a compact packing of homothetic convex domains \(\{ S_1, S_2, \ldots \}\) with centroids \(\{ C_1, C_2, \ldots \}\). Connect the points \(C_i, C_j\) of the
neighbouring domains $S_i, S_j$ by the broken line $C_i V_{ij} C_j$ where $V_{ij}$ is one of the common points of $S_i$ and $S_j$. The broken lines divide the plane into hexagons. L. Fejes Toth conjectured that

**Lemma 5.1**

The density of the domains $\{S_i\}$ in each hexagon is at least $1/2$.

First K. Böröczky proved Lemma 5.2 which is a slightly stronger form of Lemma 5.1. Let $H$ be one of the hexagons and suppose that it is overlapped by the domains $S_1, S_2, S_3$.

**Lemma 5.2**

The area of the hexagon $H$ is greater than twice the area of the triangle $V_{12} V_{23} V_{31}$.

In this chapter, we give a short proof of the Theorem 5.1. From this proof it does not follow Lemma 5.2, but it has an advantage that it states that Lemma 5.1 remains true if in the above sketch we replace the term "centroids" by the term "arbitrarily chosen homothetic inner points".

**Proof of Theorem 5.1**

Consider a compact packing of homothetic convex domains
\( \{ S_1, S_2, \ldots \} \) with arbitrarily chosen homothetic inner points \( \{ O_1, O_2, \ldots \} \). Connect the points \( O_i, O_j \) of the neighbouring domains \( S_i, S_j \) by the broken line \( O_i V_{ij} O_j \) where \( V_{ij} \) is one of the points of \( S_i \) and \( S_j \). Since the quotient of the areas of any two discs is greater than a positive number, the broken lines divide the plane into not necessary convex hexagons.

Figure 23 To the proof of Theorem 5.1

We shall prove the Theorem 5.1 by showing that the density of the
domains \( \{S_i\} \) in each hexagon is at least \( 1/2 \).

Let one of the hexagons be \( H \) and suppose that it is overlapped by the domains \( S_1, S_2, S_3 \). Transform the configuration consisting of \( S_1, S_2, S_3, H \) by an area preserving affinity such that the triangle \( V_{12}V_{23}V_{31} \) is regular. Denote the images by the same symbols (Figure 23). Let the points \( W_1, W_2 \) of \( S_1 \) (and \( Z_1, Z_2 \) of \( S_1 \) resp.) be homothetic with the points \( V_{12}, V_{23} \) of \( S_2 \) (and \( V_{31}, V_{23} \) of \( S_3 \) resp.). We leave to the reader the proof of the obvious fact that the hexagon \( W_{12}V_{23}V_{31} \) is convex. The lines \( W_{12}, Z_{12}, V_{31} \) determine a regular triangle of side length \( b \).

Since the point \( O_1 \) is not necessarily contained by this triangle we can only state that

\[
\sum_{1}^{3} m_i \geq \sqrt{3} b/2 \tag{5.1}
\]

where \( m_1, m_2, m_3 \) denote the distances of \( O_1 \) from the lines \( V_{12}V_{31}, W_{12}, Z_{12} \). Denote \( A, B, C, D, E, F \) the areas of the triangles \( V_{12}V_{31}, V_{12}V_{O1}, V_{12}O_{23}, W_{O1}, V_{23}O_{23}, V_{23}O_{31}, Z_{12} \). Finally, \( a, d, f \) denote the lengths of the sections \( V_{12}V_{23}, W_{12} \), \( Z_{12} \). Using (5.1) we have

\[
B + C + E = B + \frac{a^2}{d^2} + \frac{f^2}{d^2} = \\
= a^2 \left( \frac{m_1}{a} + \frac{m_2}{d} + \frac{m_3}{f} \right)/2 \geq 
\]
\[ \geq a^2 \left( m_1 + m_2 + m_3 \right)b/2 \geq \]
\[ \geq \left( \sqrt{3}/4 \right)a^2 = T \]

Obviously, none or one (for example the triangle \( T_1 \)) of the triangles \( T_1, T_2, T_3 \) is contained by the triangle \( T \). In the first case, the just proven inequality \( T_1 + T_2 + T_3 \geq T \) (in the second case, the transformed form \( T_2 + T_3 \geq T - T_1 \)) implies that the density of the domains \( S_1, S_2, S_3 \) in \( H \) is at least 1/2. It is easy to see that equality may hold only for packings consisting of triangles.
Consider \( n \) circles of area \( 1/n, 2/n, \ldots, n/n \). What is the smallest square (or disc similar to a given one) such that we can arrange the circles inside that square without overlapping? And how can the most economical packing be described? We want to deal with the limit case, when \( n \) tends to infinity. We will construct economical packings for the cases that 1/ the radii, 2/ the areas of the circles are uniformly distributed in the interval \((0,1)\).

The homogenity of a set \( S \) of circles is defined by \( \inf |s|/\sup |s| \), where \(|s|\) denotes the area of the circle \( s \). Consider all possible circle packings of homogenity \( h \). Let \( d(h) \) be the supremum of the upper densities of these packings.

J. Molnar gave lower bounds for \( d(h) \) by different constructions in the interval \(((2/\sqrt{3})-1,1)\). Later they were improved by A. Heppes in several subintervals. The Figures 24/1-8 show some of their constructions. Denote those circle packings in natural order by \( MH_1, MH_2, \ldots, MH_8 \). Let the radii of the larger circles be always one, let
the radii of the smaller circles be \( r_1, r_2, \ldots, r_8 \), the ratio of the number of the small and the number of the large circles be \( v_1, v_2, \ldots, v_8 \), the homogeneity of the circles be \( h_1 = (r_1)^2, \ldots, h_8 = (r_8)^2 \), the densities be \( d_1, d_2, \ldots, d_8 \). The actual values of these parameters are contained in Table 2.

Table 2 Parameters of the MHi packings

<table>
<thead>
<tr>
<th>( i )</th>
<th>( r_i )</th>
<th>( r_i' )</th>
<th>( v_i )</th>
<th>( v_i' )</th>
<th>( h_i )</th>
<th>( d_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2/( \sqrt{3} - 1 )</td>
<td>2</td>
<td>.02393</td>
<td>.95031</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1/6</td>
<td>1/3</td>
<td>6</td>
<td>2</td>
<td>.02777</td>
<td>.94469</td>
</tr>
<tr>
<td>3</td>
<td>1+( 2-\sqrt{2} + \sqrt{8} )</td>
<td>4</td>
<td>.04702</td>
<td>.93312</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>(( \sqrt{17} - 3 ))/4</td>
<td>2</td>
<td>.07884</td>
<td>.93190</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>( \sin 15^\circ/(1-\sin 15^\circ) )</td>
<td>6</td>
<td>.12194</td>
<td>.92015</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>( \sqrt{2} )</td>
<td>1</td>
<td>.17157</td>
<td>.92015</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>that ( r_1 ) ot of ( 8x^3 + 3x - 2x - 1 = 0 ) ( \text{which is in (0,1)} )</td>
<td>.53330</td>
<td>2</td>
<td>.28440</td>
<td>.91418</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>that ( r_2 ) ot of ( x - 10x^2 - 8x + 9 = 0 ) ( \text{which is in (0,1)} )</td>
<td>.63756</td>
<td>1</td>
<td>.40648</td>
<td>.91068</td>
<td></td>
</tr>
</tbody>
</table>

If we decrease the homogeneity starting at \( h=1 \), the Molnar-Heppes's lower bound \( a(h) \) is equal to the constant \( d(1) = \pi/\sqrt{12} \) for a while.
This corresponds to the fact, that using circles of not too different size, we cannot construct a denser packing, than that using only congruent circles. The smallest value of $h$, for which $a(h) = d(l)$ is close to 0.41. While we continue to decrease $h$, $a(h)$ increases, but for $h > h_8$ it will be equal to the constant $d(h_8)$ for a while. Lastly we find that $a(h)$ is a never increasing function such that the strictly decreasing parts start at $h_1, \ldots, h_8$ (Figure 25).

![Figure 25 Density bounds](image-url)
The density of three mutually touching circles of radii $h, h, 1$ in the triangle determined by their centerpoints gives a good upper bound for $a(h)$ [FA1],[FL6]. The function $a(h)$ approximates $d(h)$ for $h = h_6, h_7, h_8$ with an error less than .0001. The following construction is based on the packings $M_{hi}$. For the sake of clarity of exposition and of shortness we will make only a rough sketch of the constructions.

Suppose that we have a very large number of circles the areas of which are uniformly distributed in the interval $(0,1)$. Consider the unit square $0 \leq x \leq 1, 0 \leq y \leq 1$ of the Cartesian $xy$ coordinate-system. Represent the circles in that square by a uniformly distributed point-system such that a circle of area $a$ is represented by a point, the $x$-coordinate of which is equal to $a$. Divide the unit square into very thin stripes by lines parallel to the $y$ axis such that the circles belonging to the same stripe are "practically" congruent. Suppose that the number of the circles is large enough to have a large number of circles in each stripe. We are going to group the circles before we start to build Molnar-Heppes packings. The appropriate grouping is exhibited in Figure 26. First leave the circles of area $\leq h_1$ out of consideration. It will turn out that they will fit into the gaps of the packing of the larger circles. Consider those circles with area between $h_1$ and $h_2$. For every 6 circles of area
A in this set, join 2 circles of area 4A and 1 circle of area 36A. Actually, this assignment can be realised by the circles which are represented by points belonging to the regions denoted by 2 (there are 3 such regions). Divide each of them into, let us say, 1000 congruent stripes by lines parallel to the y axis. Let $k$ be an integer
not greater then 1000. Consider the k-th stripe in each of the 3 above regions. As was said before, the circles represented by points belonging to the same stripe are practically congruent. Further, the number of the circles is large enough and the ratios of the sizes and the ratios of the numbers of the noncongruent circles are equal to those in the packing MH2, so the circles can be arranged with density $d_2$ in any rectangle provided that the shortest side is also large. The sides of the 1000 rectangles (k runs from 1 to 1000) can be choosen such that the union of them is still a rectangle. With other words, we can move all the circles represented in our 3 regions in a rectangle such that they form a packing with a density close to $d_2$. The density can be arbitrarily close to $d_2$, if "1000" is replaced by a large enough number and in each stripe there is still a large number of points representing circles.

In a similar way, the circles represented in the regions denoted by $i$ (i = 3, 4, 6, 7, 8) can be arranged in a rectangle with a density arbitrarily close to $d_i$. As it may be observed, we did not use the construction MH5. The reason for this will be given later. The circles represented in the region 9 will be packed in a rectangle with density close to $d_9 = \pi/\sqrt{12}$. The circles with area between $h_{1,8}$ and $h_1$ can be assigned in one-to-one fashion to the circles represented in the rectangle defined by $h_8 < x < 1$, $0 < y < h_1/2$. 
This rectangle is contained in the region 9 "where" we used the densest packing, which means that the "small" circles in question fit into the gaps of the packing of the "large" circles. The same can be done with the circles of area less than \( h_1 h_8 \), but this time we will use those circles of the region 9 which are represented in the rectangle defined by \( h_8 < x \leq 1, h_1/2 < y \leq h_1/(2(1-h_5)) \).

Let \( S_i \) be the total area of the circles represented in the regions denoted by \( i \). Let \( S \) be the total area of the circles. Let \( S_5 = 0 \). According to the above construction, all the circles can be packed in a rectangle with density arbitrarily close to \( d = \frac{S}{\sum_{i=2}^{9} S_i/d_i} \).

Since the total area of those circles represented in a rectangle \( 0 < x_1 \leq x \leq x_2 \leq 1, 0 < y_1 \leq y \leq y_2 \leq 1 \) is proportional to \( (x_2^2 - x_1^2)/(y_2^2 - y_1^2) \), we can determine the ratios \( S_i/S \):

\[
\begin{array}{cccccccc}
i & 2 & 3 & 4 & 6 & 7 & 8 & 9 \\
S_i/S & 0.00166 & 0.00909 & 0.02940 & 0.15676 & 0.14190 & 0.29182 & 0.35881 \\
\end{array}
\]

Using the above constants we get \( d > 0.9126 \). If we modify the above construction by using the packing MH5, then we need circles which were used in a MH4 packing. Since \( v_5 = 3v_4 \), in MH5 we must use only circles a third as large as large those of MH4. In this way
Figure 27 Illustration to the second construction

the number of large circles arranged only with density $d_9$ is increasing and this fact may destroy all what we gain by using a packing with density $d_5 > d_4$. The following calculation shows that that really will happen. Suppose that the total area of the small circles in the packing MH5 is $t$. This means that $t/(6h_5)$ is the
total area of the great circles which are used in the MH5 packing. 

\[ \frac{s_i}{d_i} \] will change by the following value.

\[
t[\left(1+\frac{1}{2h_4}\right)\frac{1}{d_4} + \left(1+\frac{1}{6h_5}\right)\frac{1}{d_5} + \left(\frac{1}{6h_5} - \frac{1}{2h_4}\right)\frac{1}{d_9}] \approx 0.267t
\]

Since this number is positive \( d \) will decrease.

The above construction for circles with uniform radius distribution gives a density \( d \geq 0.9223 \). The corresponding grouping is represented in Figure 27.

We could improve the construction taking the strictly increasing parts of \( a(h) \) under consideration, but that would not change the first 4 decimal digit in \( d \). Roughly speaking our construction is the best we can expect using the MHi packings. On the other hand, applying a general theorem of [FL4] the density of a circle packing with uniform area-, radius-distribution is less than 0.9202 and 0.9376 respectively.
LIST OF REFERENCES


