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THE INVERSE SPECTRAL SOLUTION, MODULATION THEORY AND LINEARIZED STABILITY ANALYSIS OF N-PHASE, QUASI-PERIODIC SOLUTIONS OF THE NONLINEAR SCHRODINGER EQUATION

The Ohio State University

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AND LINEARIZED STABILITY ANALYSIS OF N-PHASE,
QUASI-PERIODIC SOLUTIONS OF THE NONLINEAR
SCHRODINGER EQUATION

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of the Ohio State University

By

Jong-Eao John Lee, B.S., M.S.

* * * * *

The Ohio State University
1986

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To My Parents
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PUBLICATIONS


FIELD OF STUDY

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ABSTRACT

We study the nonlinear Schrodinger (NLS) equation under periodic boundary conditions. First we focus on the exact theory of the periodic and quasiperiodic NLS solutions which can be expressed in terms of Riemann theta functions. Next, we study certain perturbations of these solutions; namely, the slow modulations in space and time, perhaps in the presence of external perturbations, and perturbation of initial conditions (linearized instabilities). We use Floquet theory and scattering theory to investigate the periodic NLS spectrum of the AKNS linear system and use the Daté technique to develop the inverse spectral transformation. We find exact representations, with a minimal set of parameters, for N-phase NLS wavetrains. We give precise formulas for the physical wavenumbers and frequencies of these solutions. We then use multiphase averaging, a method due to Whitham and Flaschka, Forest and McLaughlin, to derive the modulation equations for N-phase NLS wavetrains. The modulation equations are then diagonalized, with explicit formulas for Riemann invariants and their characteristic speeds. We follow the work of Ercolani, Forest and McLaughlin on the sine-Gordon equation to approach the linearized problem for all N-phase NLS wavetrains, from both geometric and analytic viewpoints. We characterize the stable and unstable modes. The growth rates are computed explicitly.
CHAPTER I

INTRODUCTION

The goal of this dissertation is a study of the exact and perturbative structure of quasiperiodic solutions of the nonlinear Schrödinger (NLS) equation. Before beginning, we place this study in the context of previous developments of this field.

NLS is one of several soliton equations; others include Korteweg-deVries (KdV), sine-Gordon (s-G), sinh-Gordon (sh-G) equations,

\[ q_t - 6qq_x + q_{xxx} = 0 \quad \text{(KdV)}, \quad (I.1) \]

\[ u_{tt} - u_{xx} + \sin u = 0 \quad \text{(s-G)}, \quad (I.2) \]

\[ iq_t + q_{xx} + 2|q|^2q = 0 \quad \text{(NLS)}. \quad (I.3) \]

These equations have been derived in many physical contexts, as the model equations for a specific balance of physical influences. KdV is a generic weakly nonlinear, dispersive wave equation, modelling long waves in shallow water [58, 67]. Sine-Gordon is a special field theory equation, consisting of the wave equation with a special choice...
of nonlinearity. It has been derived in the dislocation theory of crystals [67], in superconductivity [33] and the geometry of surfaces of constant negative curvature. The NLS equation is a model envelope equation, combining dispersion and weak nonlinearity. It has been derived in many physical contexts and is perhaps the most physically important of the soliton equations. Some examples of applications of NLS are in optical fiber transmissions, lasers [33, 76, 78, 79], and deep water theory [67]. In particular, NLS is easily derived in a slowly varying envelope approximation from KdV and s-G.

Since these equations (and NLS in particular) are generic model equations, their solution spaces and perturbations of these solutions are worthy of study. These equations are truly special, however, in that exact representations of classes of solutions have been found. These include the famous soliton solutions [25, 26, 36, 38, 67, 73], the rational solutions [77], and the theta function (N-phase) solutions [41, 42, 43, 46, 59, 2-6, 51, 53]. Our focus is the N-phase wavetrains of NLS.

Previous results on the exact theta function solutions have appeared in Its and Kotlyarov [51], Ablowitz and Ma [52], Previato [54], and Tracy [53]. We will only improve these results slightly to the extent that we give the sharpest representation, with a sharp count of free parameters in these solutions, and precise formulas for the physical wavenumbers and frequencies of these wavetrains.
Modulational instabilities are well-known for NLS on the $\infty$-line [58, 67]. Here, these long wavelength instabilities are known to saturate by the formation of NLS solitons. Tracy [53] has exhibited these long wavelength instabilities for the NLS plane wave, and done the analysis using the inverse spectral structure. We aim to carry out this linear analysis for arbitrary N-phase NLS solutions.

For stability considerations of N-phase waves we require further results on another equivalent representation of these exact N-phase solutions, the so-called $\tilde{\mu}$-representation. We follow Ercolani and Forest [4] and Forest and McLaughlin [3] to establish the topological structure of these dynamical coordinates for NLS isospectral sets. These results are crucial in the modulation theory and linearized stability analysis of NLS solutions, and are new.

Next, we proceed to study the perturbation theory of this class of NLS solutions. The modulation theory is developed to the same level of mathematical structure as the previous studies for KdV [1] and sine-Gordon and sinh-Gordon [3, 5]. The linearized analysis is then carried out using the $\tilde{\mu}$-representation and the technique of multiphase averaging. All linearized instabilities are identified, and their growth rates are explicitly derived.
CHAPTER II

THE PERIODIC NLS SPECTRAL THEORY

It is now well known that a soliton equation under vanishing boundary conditions can be integrated by the inverse scattering transformation (IST) [38]. The vanishing boundary conditions eventually determine the scattering data to recover the potential. For the periodic boundary conditions, we need the inverse spectral transformation. Unlike the periodic KdV equation, where the linear Schrodinger operator is selfadjoint and the spectrum is easily understood, the periodic s-G and NLS spectrum are much more complicated. Forest and McLaughlin ([2,61]) analyzed the periodic s-G spectrum in detail. We now follow [2,61] to develop the periodic NLS spectrum. This method is based on Floquet theory and scattering theory, which we now describe.

II.1 Floquet Theory

The NLS equation,

\[ i q_t + q_{xx} + 2 |q|^2 q = 0 \quad (II.1) \]

arises as the compatibility condition for the Z-S [36] or AKNS [38] system:
\[
\begin{align*}
\begin{cases}
\psi_1 = -iE\psi_1 + q\psi_2 , \\
\psi_2 = -r\psi_1 + iE\psi_2 ,
\end{cases}
\tag{II.2a}
\end{align*}
\]

and
\[
\begin{align*}
\begin{cases}
\psi_1 = i(q^r - 2E^2)\psi_1 + (iq_x + 2Eq)\psi_2 , \\
\psi_2 = (ir_x - 2Er)\psi_1 + i(2E^2 - qr)\psi_2 ,
\end{cases}
\tag{II.2b}
\end{align*}
\]

where
\[
r = q^* ,
\tag{II.2c}
\]

( ) * denotes complex conjugate.

A Straightforward Fact: if \( \phi(x,t;E) = (\phi_1(x,t;E),\phi_2(x,t;E))^T \) is a solution of (II.2a,c), so is \( \bar{\phi}(x,t;E) \) where
\[
\bar{\phi}(x,t;E) = (\phi_2^*(x,t;E)^*,-\phi_1^*(x,t;E^*))^T ;
\tag{II.3}
\]

moreover, \( \phi(x+L,t;E) \) is also a solution of (II.2a,c) provided that
\[
q(x+L,t) = q(x,t) , \quad -\infty < x < \infty .
\tag{II.4}
\]

We are interested in the spatial periodic NLS solution \( q(x,t) \), (II.4), which, of course, is an isospectral flow of the system (II.2a,c). To investigate the periodic NLS spectrum, it is enough to consider the initial system of (II.2a,c),
\[ \begin{aligned}
\frac{d}{dx} \psi_1 &= -iE \psi_1 + q_0 \psi_2, \\
\frac{d}{dx} \psi_2 &= -r_0 \psi_1 + iE \psi_2,
\end{aligned} \tag{II.5a} \]

where

\[ q_0(x) = q(x,0), \quad q_0(x + L) = q_0(x) \quad \text{and} \quad r_0(x) = q_0^*(x). \tag{II.5b} \]

We now fix \( x_0 \) in the \( x \)-axis and consider the fundamental basis \( \{ \phi, \phi \} \) for the system (II.5) where \( \phi = (\phi_1, \phi_2)^\top \) and

\[ \phi(x=x_0;E) = (1,0)^\top; \quad \bar{\phi}(x=x_0;E) = (0,-1)^\top. \tag{II.6} \]

From the above fact (II.3,4) and a straightforward calculation, we find

\[ (\phi(x+L,E), \bar{\phi}(x+L;E))^\top = \mathbf{T}(E)(\phi(x;E), \bar{\phi}(x;E))^\top \tag{II.7a} \]

where the transfer matrix \( \mathbf{T}(E) = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \) is given by

\[ T_{11}(E) = \phi_1(x_0 + L;E), \quad T_{12}(E) = -\phi_2(x_0 + L;E), \quad T_{21}(E) = -T_{12}(E^*), \quad T_{22}(E) = T_{11}(E^*). \tag{II.7b} \]

The eigenvalues of \( \mathbf{T}(E), \rho_{\pm}(E) \) satisfy
\[ \rho^2(E) - \Delta(E) \rho(E) + \det T(E) = 0 , \quad (\text{II.8a}) \]

where

\begin{align*}
(\text{i}) & \quad \Delta(E) = \text{trace } T(E) = \rho_+(E) + \rho_-(E) = T_{11}(E) + T_{11}^*(E) \\
& \quad = \phi_1(x_0+L;E) + \phi_1^*(x_0+L;E) , \quad (\text{II.8b}) \\
(\text{ii}) & \quad \rho_+(E) \cdot \rho_-(E) = \det T(E) = 1 , \quad (\text{II.8c}) \\
(\text{iii}) & \quad \rho_\pm(E) = \frac{\Delta(E) \pm \sqrt{\Delta^2(E) - 4}}{2} . \quad (\text{II.8d})
\end{align*}

The function \( \Delta(E) \) is called the Floquet discriminant and is fundamental to the entire periodic NLS spectral theory, as indicated by

\[ \sigma = \text{spectrum of (II.5)} = \{ E \in \mathbb{C} , \ -2 \leq \Delta(E) \leq 2 \} , \quad (\text{II.9}) \]

which is derived from (II.7,8) where \(-2 \leq \Delta(E) \leq 2 \) iff \( |\rho_\pm(E)| = 1 \) iff the corresponding eigenfunctions of \( T(E) \) (which are solutions of (II.5)) are bounded. Indeed, the NLS isospectrality is now captured by

\[ \frac{d}{dt} \Delta(E|q(x,t)) = 0 , \quad q(x,t=0) = q_0(x) , \quad (\text{II.10}) \]

which is verified by the scattering representation of \( \Delta(E) \) (refer to (II.22)).
We now investigate \( \Delta(E) \) to realize the periodic NLS spectrum.

**Lemma II.1** (Basic properties of the Floquet Discriminant \( \Delta(E) \))

(i) \( \Delta(E^*) = \Delta^*(E) \).

(ii) For real \( E \), \(-2 \leq \Delta(E) \leq 2\).

These are straightforward results of (II.8).

**II.2 Floquet Theory and Scattering Theory**

The Floquet discriminant \( \Delta(E) \) can be represented in terms of well known scattering data of (II.5a) for certain localized potential \( \hat{q}_0 \), from which we are able to analyze \( \Delta(E) \).

Consider a localized initial NLS potential \( \hat{q}_0 \),

\[
|\hat{q}_0(x)| \to 0 \quad \text{as} \quad |x| \to \infty; \quad \hat{r}_0(x) = \hat{q}_0(x).
\]  

The Z-S system, (II.5a,13), as \( |x| \to \infty \), is deduced to

\[
\begin{cases}
\frac{d}{dx} \psi_1 = -iE \psi_1, \\
\frac{d}{dx} \psi_2 = iE \psi_2,
\end{cases}
\]  

which admits the fundamental basis \( \{ \psi, \bar{\psi} \} \),

\[
\begin{cases}
\psi(x;E) = (1,0)^T e^{-iE(x-x_0)}, \\
\bar{\psi}(x;E) = (0,-1)^T e^{iE(x-x_0)}.
\end{cases}
\]
For the exact system (II.5a,13), there are two independent bases, 
{F,F} and {G,G}, known as the Jost functions, such that

\[ F \sim \psi, \quad F' \sim \psi \text{ as } x \text{ near } \infty, \quad (II.16) \]

\[ G \sim \psi, \quad G' \sim \psi \text{ as } x \text{ near } -\infty. \]

Suppose that \( q_0 \) has compact support, AKNS [38] showed \{F,F,G,G\} are entire functions of \( E \), and

\[ (F,F)^T = M(E)(G,G)^T \quad (II.17a) \]

where the transfer matrix \( M(E) = [M_{ij}]_{2 \times 2} \) is given by

\[ M_{11}(E) = \text{Wronskian}(F,G), \quad M_{12}(E) = \text{Wronskian}(G,F), \]

\[ M_{21}(E) = -M_{12}^*(E), \quad M_{22}(E) = M_{11}^*(E), \quad (II.17b) \]

and

\[ \det M = 1. \quad (II.17c) \]

Moreover, \( \{M_{ij}(E)\} \) are entire functions and

\[ M_{11}(E), M_{22}(E) \sim 1, \]

\[ M_{12}(E), M_{21}(E) \sim 0, \quad \text{as } |E| \text{ near } \infty. \quad (II.18) \]
We now take \( \hat{q}_0 \) to be the truncated potential from the L-periodic potential \( q_0 \), (II.5b),

\[
\hat{q}_0(x) = \begin{cases} 
q_0(x), & x_0 \leq x < x_0 + L, \\
0, & \text{otherwise.}
\end{cases}
\]

(II.19)

Then the asymptotic arguments, (II.14, 16, 18), are now applied to \( x \) near \( x_0 \) and \( x_0 + L \), corresponding to \(-\infty\) and \( \infty \). By straightforward calculation, (II.17) now becomes

\[
G(x=x_0+L;E) = M_{11}(E)e^{-iEL} \cdot G(x=x_0;E) + M_{12}(E)e^{iEL} \cdot G(x=x_0;E),
\]

(II.20)

\[
G(x=x_0+L;E) = -M_{12}^{*}(E) \cdot G(x=x_0;E) + M_{11}^{*}(E)e^{iEL} \cdot G(x=x_0;E).
\]

By observing the two systems, (II.5) and (II.5a, 19), they are identical in the period \( x_0 \leq x < x_0 + L \); moreover, from the asymptotic behaviors of \( \{G, \bar{G}\} \) at \( x_0 \), \( \{G, \bar{G}\} \) is precisely the fundamental basis, (II.6), in this specified period. Consequently, the two transfer matrices, (II.7) and (II.20), are identical, which immediately yield the following relations,

\[
T_{11}(E) = M_{11}(E)e^{-iEL}, \quad T_{12}(E) = M_{12}(E)e^{iEL},
\]

(II.21)

\[
T_{21}(E) = -T_{12}^{*}(E), \quad T_{22}(E) = T_{11}^{*}(E).
\]
Lemma II.2 (Scattering Representation of the Floquet Discriminant $\Delta(E)$)

(i) $\Delta(E)$ is an entire function which can be expressed in terms of the scattering data:

$$\Delta(E) = M_{11}(E)e^{-iEL} + M_{11}^*(E)e^{iEL} \quad \text{(II.22a)}$$

where

$$M_{11}(E), M_{11}^*(E) \sim 1 \text{ as } |E| \text{ near } \infty. \quad \text{(II.22b)}$$

(ii) For real $E$,

$$-2 \leq \Delta(E) = 2|M_{11}(E)|\cos(\text{ph}(M_{11}(E)) - EL) \leq 2, \quad \text{(II.23a)}$$

where

$$|M_{11}(E)| \leq 1 \text{ and } M_{11}(E) = |M_{11}(E)|\exp i(\text{ph}(M_{11}(E))) \cdot \text{(II.23b)}$$

(iii) If $E_k$ is real and $\Delta(E_k) = \pm 2$, then $\Delta'(E_k) = 0$ along the real $E$-line. \quad \text{(II.24)}$

Remarks on the Proof: (i) (II.22) follows (II.8b, 21). $\{M_{ij}(E)\}$ are entire functions implies $\Delta(E)$ is also an entire function. The scattering representation of $\Delta(E)$, (II.22), is most essential in understanding the periodic NLS spectrum. It determines the asymptotic behaviors of $\Delta(E)$ near $|E| = \infty$ and shows
whenever \( q \) evolves according to NLS, which is due to that \( M_{11}(E|q(x,t)) \) and \( M_{11}^*(E^*|q(x,t)) \) are invariant of \( t \) (a well known fact in inverse scattering theory where \( \frac{1}{M_{11}} \) is the transmission coefficient). Moreover, under the Hamiltonian structure, the periodic spectrum, \( \sum(q(x,t)) = \{E_k, \Delta(E_k) = \pm 2\} \), provides the action variables in action-angle coordinates (see \([70, 61, 6]\)). (ii) For real \( E \), \( \det M(E) = |M_{11}(E)|^2 + |M_{12}(E)|^2 = 1 \), i.e., (II.17c), which implies \( |M_{11}(E)| \leq 1 \). (iii) (II.24) is verified by straightforward calculation of \( \Delta(E) = T_{11}(E) + T_{11}^*(E^*) \) in (II.8b) (also is proved in \([52]\)).

**Lemma II.3 (Asymptotic Behaviors of the Floquet Discriminant \( \Delta(E) \) near \( |E| = \infty \))**

\[
\Delta(E) \sim e^{iEL} + e^{-iEL} = 2 \cos(EL) \tag{II.26a}
\]

\[
= 2[\cos(ReE\cdot L)cosh(ImE\cdot L) + i \sin(ReE\cdot L)sinh(ImE\cdot L)]
\]

where

\[
E = ReE + iImE. \tag{II.26b}
\]

In particular,

(i) \( \Delta(E) \sim 2 \cos(EL) \) when \( E \) is real and \( E \sim \infty \). \tag{II.27a}
(ii) $\Delta(E) \sim 2 \cosh(|E|L)$ when $E$ is purely imaginary and
$|E| \sim \infty$.  \hspace{1cm} (II.27b)

(iii) $|\Delta(E)| \rightarrow \infty$ when $|\text{Im}E| \rightarrow \infty$. \hspace{1cm} (II.27c)

(iv) $|\Delta(E)|$ is bounded and $\Delta(E)$ oscillates as (II.26) when
$|\text{Im}E| \leq \infty$ and $|E| \sim \infty$. \hspace{1cm} (II.27d)

(v) $|\Delta(E)| > 2$ when $\Delta(E)$ is real, $E$ is nonreal and
$|E| \sim \infty$. \hspace{1cm} (II.27e)

Remark: these results are straightforward from the scattering
representation of $\Delta(E)$, (II.22).

II.3 The Periodic NLS Spectrum

Theorem II.4 (The periodic NLS Spectrum)

Assume that $q(x,t)$ is a solution of NLS, (II.1), with spatial
period $L$. Then

$\sigma(q) = \text{spectrum (II.2a,c)} = \{E \in \mathbb{C}, \Delta(E|q) \in \mathbb{R}, -2 \leq \Delta(E) \leq 2\}$. \hspace{1cm} (II.28a)

In particular, if

$\Sigma(q) = \{E_k \in \sigma(q), \Delta(E_k) = \pm 2\}$, the periodic spectrum, \hspace{1cm} (II.28b)

$\Sigma^s(q) = \{E_k \in \Sigma(q), \Delta'(E_k) \neq 0\}$, the simple spectrum, \hspace{1cm} (II.28c)
and

\[ \Sigma^d(q) = \{ E_k \in \Sigma(q), \Delta'(E_k) = 0 \} \] the double spectrum, \((\text{II.28d})\)

then (i) \(\sigma(q) \subseteq \Sigma(q) = \Sigma^s(q) \cup \Sigma^d(q)\). Each of these sets, say \(Q\), satisfies the symmetry:

\[ E \in Q \iff E^* \in Q. \quad (\text{II.29}) \]

(ii) The real line \(R\) is a continuous spectrum and has no simple point, i.e.,

\[ R \subseteq \sigma(q) \quad \text{and} \quad \Sigma^s(q) \cap R = \emptyset. \quad (\text{II.30}) \]

(iii) \(\Sigma^d(q)\) has at most finite elements in \(\mathbb{C} \setminus R\).

(iv) \(\Sigma^s(q)\) has exactly \(N\) distinct complex conjugate pairs if \(q\) is an \(N\)-phase NLS wavetrain which is \(2\pi\)-periodic in each of the phases (refer to (11.39)).

(v) There is a one to one correspondence between degrees of freedom in \(q\) and critical points of \(\Delta(E|q)\). Each double point \(E_d \in \Sigma^d\) labels a closed (unexcited) degree of freedom, while each complex conjugate pair of \(\Sigma^s(q)\) corresponds to an open (excited) degree of freedom. For generic \(q\),

\[ \Sigma^d(q) = \emptyset. \quad (\text{II.31}) \]
Moreover, when a closed degree of freedom is excited, the corresponding double point $E_d$ splits into a pair of simple spectra.

(vi) The function theory of $q$ takes place on the Riemann surface $R$ of $\sqrt[4]{\Delta^2(E|q) - 4}$, with branch points $E_j$ precisely at the $2N$, $N \leq \infty$, elements of $\Sigma^S(q)$. If $N$ is finite, then the appropriate curve is

$$R^2(E) = \prod_{1}^{2N} (E - E_j), \quad E^*_{2j-1} = E_{2j}. \quad (II.32)$$

Remarks on the Proof: (II.29, 30) are straightforward results of Lemma II.1, II.2. Property (iii) is proved in [7]. This may also be argued from the asymptotic behaviors of $\Delta(E)$, (II.27e). This property (iii) places a bound on the number of linearized instabilities for given NLS solutions in Chapter V. Properties (iv,vi) are verified in Chapter III for the N-phase NLS wavetrains. Property (v) is essential in multiphase averaging technique and perturbation argument in Chapters IV, V, and is proved in Chapters III, V (also see [2, 6] for the complete description of s-G problem). For finite $N$, the NLS spectrum is illustrated in Figure 1.

Remark: in addition to the simple spectrum, $\Sigma^S(q)$, we will investigate, in Chapter III, the Dirichlet eigenvalues, $\{\mu_j\}$, of the Z-S eigenvalue problem (II.2a,c) to carry out the inverse spectral transformation for N-phase NLS wavetrains and stability analysis of these wavetrains. The $\mu$-variables are not isospectral and behave very complicated (refer to (III.23)), yet there are simple homologous curves for $\mu$-variables, as in Figure 2.
II.4 A Special Example

Consider the NLS plane wave solution \( q(x,t) \),

\[ q(x,t) = ae^{i\alpha t}, \quad (\text{II.33}) \]

where \( \alpha \) is real, and let \( L \) be the spatial period of \( q \). The Z-S eigenvalue problem (II.5) now is easy to integrate \( (q_0(x) = a, \text{ a constant}) \). In particular, the fundamental basis is given by

\[ \phi(x;E) = \begin{pmatrix} E-\lambda \\ -2\lambda \end{pmatrix}, \begin{pmatrix} ia \\ 2\lambda \end{pmatrix} e^{i\lambda x} + \begin{pmatrix} \frac{E+\lambda}{2\lambda} \\ -\frac{i\alpha}{2\lambda} \end{pmatrix}, e^{-i\lambda x}, \quad (\text{II.34a}) \]

\[ \phi(x;E) = \begin{pmatrix} \frac{-\lambda}{2\lambda} \\ 2\lambda \end{pmatrix}, \begin{pmatrix} E-\lambda \\ 2\lambda \end{pmatrix} e^{-i\lambda x} + \begin{pmatrix} \frac{\lambda}{2\lambda} \\ \frac{E+\lambda}{2\lambda} \end{pmatrix}, e^{i\lambda x}, \quad (\text{II.34b}) \]
where

$$\lambda = \sqrt{E^2 + a^2}. \quad \text{(II.34c)}$$

From (II.8), we find

$$\Lambda(E) = 2 \cos(\sqrt{E^2 + a^2} \cdot L), \quad \text{(II.35)}$$

and the periodic NLS spectrum, $\sigma(q = a \, e^{ia \cdot t})$, satisfies

(i) $E_k \in \Sigma$ if $E_k = \pm \sqrt{\frac{k^2 \pi^2}{L^2} - a^2}, \quad \text{(II.36)}$

in which case, $E_k$ is real or purely imaginary. In particular, $\Sigma$ has only finite purely imaginary elements.

(ii) $\Sigma^s = \{\pm ia\} = \{E_0, E_0^*\}. \quad \text{(II.37)}$

(iii) If $\frac{L^2}{a^2} < \frac{\pi^2}{a^2}$, $\Sigma^d$ has no purely imaginary element; if

$$\frac{n^2 \pi^2}{a^2} < \frac{L^2}{a^2} < \frac{(n+1)^2 \pi^2}{a^2}, \quad \text{then} \quad \Sigma^d \quad \text{has precisely} \quad N \quad \text{pairs}

of complex conjugates in purely imaginary line.

The periodic NLS spectrum is depicted in Figure 3. We will return to this example in the analysis of linearized instabilities in Chapter V.
The spectrum of NLS plane wave $q = a e^{ia^2 t}$ with spatial period $L$.

* denotes $E_j \in \Sigma^s = \{\pm ia\}$

○ denotes $E_d \in \Sigma^d = \left[ \sqrt{\frac{k \pi}{2} \frac{2}{L^2} - a^2}, k \neq 0 \right]$
CHAPTER III

THE N-PHASE NLS WAVETRAINS

For quasiperiodic solutions of soliton equations, there is a straightforward construction, due to Date [59]. Ercolani, Forest and McLaughlin [2-4] have carried out this analysis for s-G. We now follow [59], [2-4] to develop the NLS inverse spectral transformation. In particular, if the simple spectrum \( \Sigma^S \) has finite \((2N)\) elements, we find the inverse spectral solutions have precisely \(N\) degrees of freedom and are \(N\)-phase wavetrains, where \(N-1\) degrees of freedom are parameterized by \(N-1\) \(\mu\)-variables (Dirichlet eigenvalues of \(Z-S\) (II.2a)) and the remaining one corresponds to the plane wave factor attached to the theta function representation of those \(N\)-phase NLS wavetrains (see (III.39)).

III.1 2N Simple Spectra; Determination of a Riemann Surface

Given a prescribed simple spectrum \( \Sigma^S \), we now construct a Riemann surface where the function theory of the corresponding periodic NLS solutions take place. The eigenvalues (II.8) of \(Z-S\) eigenvalue problem (II.5) indicate that the proper Riemann surface \(R\) is given by the function \(\hat{R}(E)\),

\[
\hat{R}(E) = \left( \frac{1}{2} (E - 4) \right)^2 ,
\]

(III.1a)
which has branch points precisely at the simple spectrum $\Sigma^S$. In particular, if $\Sigma^S$ has only $2N$ (finite) elements, $\{E_k\}^{2N}_{k=1}$ with $E_{2k-1} = E_{2k}$, then $\hat{R}(E)$ can be written (Appendix L, also see [2,61] for s-G) as

$$\hat{R}(E) = S(E) \cdot \left( \prod_{k=1}^{2N} (E - E_k) \right)^2,$$  

where $S(E)$ is a single-valued function. We focus on the hyperelliptic curve, $(E, R(E))$, where

$$R^2(E) = \prod_{k=1}^{2N} (E - E_k).$$  

We now pose the finite degree of freedom inverse problem: given $N$ distinct pairs of complex conjugates $\{E_k\}^{2N}_{k=1}$, determine the most general periodic NLS solutions with simple spectrum $\{E_k\}^{2N}_{k=1}$. This can be achieved by the inverse spectral transformation we now develop.

### III.2 Quadratic Eigenfunctions

We now employ an equivalent system of quadratic eigenfunctions $\{f, g, h\}$ for the Z-S eigenvalue problem (II.2),

$$f = \frac{1}{2}(\phi_1 \psi_2 + \phi_2 \psi_1), \quad g = \phi_1 \psi_1, \quad h = \phi_2 \psi_2,$$  

where $\phi = (\phi_1, \phi_2)^T$ and $\psi = (\psi_1, \psi_2)^T$ are solutions of (II.2). By straightforward calculation, we find
\[
\begin{align*}
\begin{cases}
  f_x &= -rg + qh, \\
  g_x &= 2qf - 2iEg, \\
  h_x &= -2rf + 2iEh,
\end{cases} 
\tag{III.4a}
\]

and

\[
\begin{align*}
\begin{cases}
  f_t &= i(r - 2Er)g + i(q + 2Eq)h, \\
  g_t &= 2(iq + 2Eq)f + 2i(qr - 2E^2)g, \\
  h_t &= 2(ir - 2Er)f + 2i(qr - 2E^2)h,
\end{cases}
\tag{III.4b}
\]

where

\[ r = q^* \tag{III.4c} \]

It is easy to verify that

\[
(f, g, h)_{xt} = (f, g, h)_{tx} \text{ iff } iq_t + q_{xx} + 2|q|^2q = 0
\tag{III.5}
\]

and \( r = q^* \).

The quadratic eigenfunction system (III.4) is fundamental to the theory of the NLS equation, as indicated in

**Lemma III.1 (Basic Properties of Quadratic Eigenfunctions)**

Suppose that \( \{f, g, h\} \) is a solution of (III.4), then

(i) \( P(E) = f^2 - gh \) is a first integral, i.e., \( \frac{\partial P}{\partial x} = \frac{\partial P}{\partial t} = 0 \).
\( \tag{III.6} \)
(ii) \( g, h \) satisfy the linearized NLS equation about an arbitrary solution \( q \):

\[
\begin{align*}
ig_t + g_{xx} &= 2q^2 h - 4qrg, \\
qh_t - h_{xx} &= 4qr - 2r^2 g,
\end{align*}
\]

(III.7)

(iii) \( \{f, g, h\} \) satisfies the fundamental conservation law:

\[
f_t - [4Ef + i(qh + rg)]_x = 0.
\]

(III.8)

These results are consequences of straightforward calculations from (III.4, 5).

Some Remarks: (i) The first integral \( P(E) \) introduces the hyperelliptic curve (III.2) in the context of the periodic NLS solutions with \( N \) degrees of freedom in Sec 3. (ii) Clearly, the quadratic eigenfunctions \( \{f, g, h\} \) will play a fundamental role in local perturbation analysis of the NLS equation. We analyze linearized instabilities of the NLS solutions in Chapter V. (iii) The fundamental conservation law (III.8) actually provides the entire infinite family of NLS conservation laws. Moreover, this representation is central to the modulation theory of the \( N \)-phase NLS wavetrains in Chapter IV.

We now normalize the linear system (III.4) by the normalization condition:

\[
f^2 - gh = 1,
\]

(III.9)
which, from (III.4), implies that

\[
\begin{align*}
\mathbf{f} &\sim 1 + \sum_{j=2}^{\infty} \mathbf{f}_j(x,t)E^{-j}, \\
\mathbf{g} &\sim \sum_{j=1}^{\infty} \mathbf{g}_j(x,t)E^{-j}, \quad \text{as } E \text{ near } \infty. \quad (\text{III.10}) \\
\mathbf{h} &\sim \sum_{j=1}^{\infty} \mathbf{h}_j(x,t)E^{-j},
\end{align*}
\]

Inserting (III.10) into (III.4) yields the following equations for the coefficients \(\{\mathbf{f}_j, \mathbf{g}_j, \mathbf{h}_j\} :\)

\[
\begin{align*}
\mathbf{g}_1 &= -i\mathbf{q}, \quad \mathbf{h}_1 = -i\mathbf{r}, \\
(f)_j x &= -r\mathbf{g}_j + q\mathbf{h}_j, \\
(g)_j x &= 2q\mathbf{f}_j - 2i\mathbf{g}_{j+1}, \\
(h)_j x &= -2r\mathbf{f}_j + 2i\mathbf{h}_{j+1}, \quad j \geq 1,
\end{align*}
\]

and

\[
\begin{align*}
(f)_j t &= ir\mathbf{g}_j - 2r\mathbf{g}_{j+1} + i\mathbf{q}_x \mathbf{h}_j + 2q\mathbf{h}_{j+1}, \\
(g)_j t &= 2i\mathbf{q}_x \mathbf{f}_j + 4q\mathbf{f}_{j+1} + 2iqr\mathbf{g}_j - 4i\mathbf{g}_{j+2}, \quad (\text{III.11b}) \\
(h)_j t &= 2ir\mathbf{f}_j + 4r\mathbf{f}_{j+1} - 2iqr\mathbf{h}_j + 4i\mathbf{h}_{j+2}, \quad j \geq 1.
\end{align*}
\]
From the normalization condition (III.9), it is straightforward that one can generate recurrence formulae for \( \{ f_j, g_j, h_j \} \) in terms of the potential \( q, r = q^* \) and their spatial derivatives. For convenience, we list the first few:

\[
\begin{align*}
  f_0 &= 1, \quad g_1 = -iq, \quad h_1 = -ir, \quad f_1 = 0, \\
  f_2 &= -\frac{1}{2} qr, \quad g_2 = \frac{1}{2} q_x, \quad h_2 = -\frac{1}{2} r_x, \\
  f_3 &= \frac{i}{4} (rq_x - rq_x), \quad g_3 = \frac{i}{2} r^2 + \frac{i}{4} q_{xx}, \quad h_3 = \frac{i}{2} r^2 + \frac{i}{4} r_{xx}, \\
  f_4 &= \frac{3}{8} q^2 r^2 - \frac{3}{8} q_x r + \frac{1}{8} (rq_x + qr_x)_x, \\
  g_4 &= -\frac{3}{4} rq_x - \frac{1}{8} q_{xxx}, \quad h_4 = \frac{3}{4} rqr + \frac{1}{8} r_{xxx}.
\end{align*}
\]

(III.12)

We now insert (III.10) into the fundamental conservation law (III.8), which yields the infinite family of conservation laws:

\[
\sum_{j=2}^{\infty} \left\{ (f_j)_t - [4f_{j+1} + i(qh_j + rg_j)]_x \right\} E^{-j} = 0 \quad \text{as} \quad E \rightarrow \infty, \quad (III.13a)
\]

i.e.,

\[
(f_j)_t - [4f_{j+1} + i(qh_j + rg_j)]_x = 0, \quad j \geq 2. \quad (III.13b)
\]

From (III.12), the first three conservation integrals corresponding to (III.13) are:
which correspond to the number of particles, the momentum and energy respectively. In general,

\[
C_0 = \int_{x_0^-}^{x_0^+} \left( -\frac{1}{2} \right) qr \, dx ,
\]

\[
C_1 = \int_{x_0^-}^{x_0^+} \left( \frac{i}{4} \right) (qr_x - r q_x) \, dx ,
\]

\[
C_2 = \int_{x_0^-}^{x_0^+} \left( \frac{3}{8} \right) (q^2 r_x^2 - q r_x r_{xx}) \, dx ,
\]

where \( q \) is a potential and \( r = q^* \), \( q(x+L,t) = q(x,t) \), which correspond to the number of particles, the momentum and energy respectively. In general,

\[
C_{j-2} = \int_{x_0^-}^{x_0^+} f_j(x,t) \, dx , \quad j \geq 2 .
\]  

III.3 Construction of Special Solutions; \( \mu \)-Representation of N-Phase Wavetrains

We now restrict a periodic NLS spectrum with exactly \( N \) distinct complex conjugate pairs of simple spectrum \( \{E_k\}_{k=1}^{2N} \). Date [59] showed that N-phase NLS solutions have \( \Sigma = \{E_k\}_{k=1}^{2N} \) and these potentials \( q \) correspond to a polynomial ansatz on quadratic eigenfunctions up to a constant factor.
Date Ansatz: assume that

\[ f = \sum_{j=0}^{N-1} f_j(x,t)E^j, \]

\[ g = \sum_{j=0}^{N-1} g_j(x,t)E^j = g_{N-1} \prod (E - \mu_j), \quad (III.16) \]

\[ h = \sum_{j=0}^{N-1} h_j(x,t)E^j = h_{N-1} \prod (E - \bar{\mu}_j). \]

Lemma III.2 (Existence of Polynomial Quadratic Eigenfunctions)

Polynomial solutions (III.16) of the quadratic eigenfunction system (III.4) exist if and only if the NLS potential \( q \) has \( N \) degrees of freedom. Moreover, \( \langle x \, q \rangle_x \) can be represented in terms of \( \{E_k\}_{1}^{2N} \) and the \( N-1 \) zeros of \( g \).

To prove this Lemma, we investigate the quadratic eigenfunctions \( \{f,g,h\} \) in (III.16). Inserting (III.16) into (III.4) yields the following equations for coefficients \( \{f_j,g_j,h_j\} : \)

\[ g_{N-1} = -iq, \quad h_{N-1} = -ir, \quad f_N = 1, \]

\[ (f_j)_x = -rg_j + qh_j, \quad (III.17a) \]

\[ (g_j)_x = -2ig_j - 2qf_j, \]

\[ (h_j)_x = 2ih_j - 2rf_j. \]
and

\[
(f_j)_t = ir_{x_j}g_j - 2rg_{j-1} + iq_{x_j}h_j + 2qh_{j-1},
\]

\[
(g_j)_t = 2iq_{x_j}f_j + 4qf_{j-1} + 2iqr_{x_j}g_j - 4ig_{j-2}, \quad (III.17b)
\]

\[
(h_j)_t = 2ir_{x_j}f_j - 4rf_{j-1} - 2iqr_{x_j}h_j - 4ih_{j-2}.
\]

The connection between \( \{E_k\}_{1}^{2N} \) and the first integral \( f^2 - gh \) with the choice (III.16) is

\[
f^2 - gh = \sum_{1}^{2N} (E - E_k). \tag{III.18}
\]

We now normalize \( \{f,g,h\} \) by \( \sqrt{f^2 - gh} = \sqrt{\sum_{1}^{2N} (E - E_k)} \) to connect the normalization condition \( f^2 - gh \equiv 1 \), (III.9). This makes sure that the corresponding conservation laws are the exact physical ones and generates the recursion formulae for \( \{f_j,g_j,h_j\} \), where the first few coefficients for the ansatz (III.16) are:

\[
f_N = 1, \quad f_{N-1} = (-\frac{1}{2}) \sum_{1}^{2N} E_k, \quad g_{N-1} = -iq, \quad h_{N-1} = -ir,
\]

\[
f_{N-2} = \frac{1}{2} \left[ \sum_{i > j} E_i E_j - qr - \frac{1}{4} \left( \sum_{1}^{2N} E_k \right)^2 \right], \tag{III.19a}
\]

\[
g_{N-2} = \frac{1}{2} q_x + \frac{i}{2} q \left( \sum_{1}^{2N} E_k \right),
\]

\[
h_{N-2} = -\frac{1}{2} r_x + \frac{i}{2} r \left( \sum_{1}^{2N} E_k \right).
\]
From the fundamental conservation law (III.8), the generating functions $f$ and $4Ef + i(qh + rg)$ satisfy

(i) $f = \sum_{j=0}^{N} f_j E^j$ where $f_N = 1, f_{N-1} = (-\frac{1}{2}) \sum_{k=1}^{2N} E_k$. (III.19b)

(ii) $4Ef + i(qh + rg) = \sum_{j=0}^{N+1} d_j E^j$ where $d_{N+1} = 4, d_N = -2 \sum_{k=1}^{2N} E_k, d_{N-1} = 2 \sum_{i>j} E_i E_j - \frac{1}{2}(\sum_{k=1}^{2N} E_k)^2$.

We notice that $\{f_N, f_{N-1}, d_{N+1}, d_N, d_{N-1}\}$ are constants, independent of $x$ and $t$. This fact becomes essential when we investigate modulations of $N$-phase NLS solutions in Chapter IV.

From (III.19a), we find the preliminary inversion formula:

$$
\begin{align*}
q_X^2 & + i(\sum_{k=1}^{2N} E_k)q = g_{N-2}, \\
q_t + 2i[\frac{3}{4}(\sum_{k=1}^{N} E_k)^2 - \sum_{j>k} E_j E_k]q = 2(\sum_{k=1}^{2N} E_k)g_{N-2} + 4g_{N-3}.
\end{align*}
$$

(III.20)

From (III.16) and (III.19a), $g$ can be represented as

$$
g = (-iq)^{N-1} (E - \mu_j),
$$

(III.21)

where the zeros $\{\mu_j\}_{1}^{N-1}$ of $g$ turn out of the Dirichlet eigenvalues of the Z-S system (II.2a). We defer the proof in Appendix M, a method due to Forest and McLaughlin [61]. From (III.16, 18, 21), we summarize a series of facts and leave the proofs in Appendices A and B.
Lemma III.3 (Properties of Polynomial Quadratic Eigenfunctions)

Assume that \( \{f, g, h\} \) (III.16) is a solution of the quadratic eigenfunction system (III.4), and the normalization condition (III.18) holds. Then

(i) \( f_j \in \mathbb{R}, 0 \leq j \leq N; \ f(\mu_k) = \left( \prod \left( \mu_k - E_j \right) \right)^{\frac{1}{2}} \). \hspace{1cm} (III.22a)

(ii) \( h_j = -g_j^*, \ 0 \leq j \leq N-1 \). \hspace{1cm} (III.22b)

(iii) \( h(x,t;E) = -g(x,t;E^*) = (-ir) \prod (E - \mu_j^*(x,t)) \), \hspace{1cm} (III.22c)

\( r = q^* \).

(iv) (Generating Functions in terms of \( \mu \)-Variables)

\[
f = \left( E - \frac{1}{2} \sum \frac{E_k}{1} + \sum \frac{\mu_j}{1} \right) \left( \prod (E - \mu_j) \right),
\]

\[
rg = -i|q|^2 \prod (E - \mu_j), \quad qh = -i|q|^2 \prod (E - \mu_j^*),
\]

where

\[
|q|^2 = -\frac{i}{2} \frac{q_t}{q} - \frac{1}{2} \frac{q_{xx}}{q} \quad \text{with} \quad \frac{q_{xx}}{q} = \left( \frac{q}{q} \right)_x + \left( \frac{q}{q} \right)_x^2.
\]
Lemma III.4  (Dynamical System for the $\mu$-Variables)

Under the assumption of Lemma III.3, $\{\mu_k\}_{1}^{N-1}$ satisfy

\[
\frac{R(\mu_k)}{\prod_{j \neq k} (\mu_k - \mu_j)} , \quad 1 \leq k \leq N-1 .
\]

\[
(u_k)_{x} = (-2i) \frac{R(\mu_k)}{\prod_{j \neq k} (\mu_k - \mu_j)} , \quad 1 \leq k \leq N-1 .
\]  

(III.23a)

\[
(u_k)_{t} = (u_k)_{x} \cdot \left( \sum_{l=1}^{2N} E_l - 2 \sum_{j \neq k} \mu_j \right) .
\]  

(III.23b)

Lemma III.5  ($\alpha_n q$) has $N-1$ Degrees of Freedom)

Under the assumption of Lemma III.3, the NLS potential $q$ satisfies

\[
\frac{\partial q}{\partial t} = i(2 \sum_{1}^{N-1} \mu_j - \sum_{1}^{2N} E_k) ,
\]  

(III.24a)

\[
\frac{\partial q}{\partial t} = 2i[ \sum_{j > k} E_j E_k - \frac{3}{4} (\sum_{1}^{2N} E_k)^2 ]
\]  

\[- 4i\left[ (-\frac{1}{2} \sum_{1}^{2N} E_k \sum_{1}^{N-1} \mu_j) + \sum_{i>j} \mu_i \mu_j \right] .
\]

(III.24b)

Some Remarks: (III.22) is important in developing the modulation theory in Chapter IV where, in particular, the averaging technique
heavily depends on the $\tilde{\mu}$-symmetric structure of the generating functions (III.22d); moreover, the structure of $\{f_j, g_j, h_j\}$ (III.22a,b,c), eventually, assures the modulational instability of N-phase NLS solutions. (III.23) indicates that $\tilde{\mu}$-variables have complicated dynamical behaviors, just like s-G [2], yet, as shown in [4], $\{\mu_j\}_{1}^{N-1}$ are homologous to N-1 simple closed curves on the Riemann surface $R(E)$, (III.2), (refer to sec. 4) from which we carry out the stability argument of the perturbed NLS solutions. (III.24) is referred as the $\tilde{\mu}$-representation for the NLS solution $q$. This $\tilde{\mu}$-representation shows that $(\ln q)_x$ has N-1 degrees of freedom since the complete $(x,t)$ dependence is determined by $\{\mu_j\}_{1}^{N-1}$. We will show that $q$ has exactly N degrees of freedom where the remaining mode corresponds to the plane wave factor attached to the theta function representation of those N-phase wavetrains (see (III.39)).

III.4 Theta Function Representation of N-Phase NLS Wavetrains

We now perform the Abel-Jacobi transformation [68] to linearize the $\tilde{\mu}$ o.d.e's and develop the inverse spectral transformation to integrate the NLS potential $q$ with $2N$ simple spectra $\{E_k\}_{1}^{2N}$. To define the Abel-Jacobi transformation, we introduce "canonical a-b cycles" on the Riemann surface $R$ of $R(E)$,
\[ R^2(E) = \frac{2N}{1} \prod (E - E_k). \quad \text{(III.25)} \]

In the special case, \( N = 3 \), with the branch cuts given by the NLS spectrum as in Figure 1, a-cycles and b-cycles are depicted in Figure 4,

![Homology basis on \( \mathbb{R} \), \( N = 3 \)](image)

Fig. 4

We now introduce \( N-1 \) Abelian differentials of the first kind,

\[ dU_j = \sum_{v=1}^{N-1} c_{jv} E^{N-1-v} \frac{dE}{R(E)}, \quad 1 \leq j \leq N-1, \]

where the constant matrix \( C = [C_{ij}] \) is given by the normalization condition,

\[ \int_{a_i} dU_j = \delta_{ij}, \quad \delta_{ij} \text{ is the Kronecker symbol, } 1 \leq i,j \leq N-1 \quad \text{(III.27)} \]

Next, we define the period matrix \( B = [B_{ij}] \),
It is clear that the matrices $B, C$ are completely determined by 
\{E_j\}_{j=1}^{2N}$, independent of $q(x,t)$.

**Lemma III.6 (Properties of Constant Matrix $C$ and Period Matrix $B$)**

(i) $C_{ij} \in \mathbb{R}, 1 \leq i,j \leq N-1.$  

(ii) $B_{ij} \in \mathbb{R}, \text{Re}(B_{ij}) = -\frac{1}{2}$ for $i \neq j$,  

\[1 \leq i,j \leq N-1.\]

We defer the proof in Appendix C, which is based on the methods 
developed by Ercolani, Forest and McLaughlin [2-4], and Date [59].

We now define the Abel-Jacobi map from $\mu$ to $\xi$ where

\[\mu = (\mu_1, \ldots, \mu_{N-1})^T \in \mathbb{R}^{N-1},\]  

(iii.31a)

and

\[\xi = (\xi_1, \ldots, \xi_{N-1})^T \in \mathbb{C}^{N-1}/\{e_j, B_j, 1 \leq j \leq N-1\},\]  

(iii.31b)

where $e_j$ denotes the orthonormal vectors in $\mathbb{R}^{N-1}$ and $B_j$ is the

j-th column of the period matrix $B$:

\[\xi_j(\mu^+) = \sum_{k=1}^{N-1} \int_{\mu_k}^{\mu_k} dU_j, 1 \leq j \leq N-1.\]  

(iii.32)
Using the Lagrange Interpolation formulae (see [52]), we show, in Appendix D, that \( \{ \xi_j \} \) are linear in both \( x \) and \( t \), i.e.,

**Lemma III.7 \( \xi \)-Variables are linear in \( x \) and \( t \)**

(i) \( \xi_j^x = (-2i)C_{j1} \), \hspace{1cm} (III.33a)

(ii) \( \xi_j^t = (-2i)[(\sum_k E_k)C_{j1} + 2C_{j2}] \), \hspace{1cm} (III.33b)

\[ 1 \leq j \leq N-1. \]

(iii) \( \xi_j(x,t) = (-2i)[C_{j1}x + (\sum_k E_k)C_{j1} + 2C_{j2}] + \xi_j^o \), \hspace{1cm} (III.34a)

(iv) \( \xi_j(x,t) - \xi_j^o \in \mathbb{R} \), \hspace{1cm} (III.34b)

As indicated in (III.24), for the NLS potential \( q \), \( (\xi n q)_x \) is completely determined by \( \{ E_k \}_{1}^{2N} \) and \( \{ u_j \}_{1}^{N-1} \). To find an explicit formula for \( \vec{\mu}(x,t) \), where the \((x,t)\) dependence is given by the Abel-Jacobi map (III.32) restricted to the NLS flow (III.34), we need to solve the so-called "Abel-Jacobi inversion problem" [68], whose solution is in terms of Riemann-theta functions [69]. This NLS inverse spectral solution has been formally derived ([51, 53, 66]), which provides the theta function representation for the NLS potential \( q \), in the modified form (III.39), yet two fundamental questions were still remaining:

(1) What are the real, \( 2\pi \)-periodic phases of \( q(x,t) \)?
q(\theta_1, \ldots, \theta_j + 2\pi, \ldots, \theta_N) = q(\theta_1, \ldots, \theta_j, \ldots, \theta_N), \quad 1 \leq j \leq N

and \( \theta_j \in \mathbb{R} \) ? \hspace{1cm} (III.35)

(2) What is the N-phase isospectral class of \( \{E_j\}_{j=1}^{2N} \), the set of all NLS potentials satisfying

\[ r(\theta_1, \theta_2, \ldots, \theta_N) = q^*(\theta_1, \theta_2, \ldots, \theta_N) ? \hspace{1cm} (III.36) \]

Remark: Ercolani, Forest, and McLaughlin [6] have shown that the complete isospectral sets for non-selfadjoint theories (in particular, s-G) may include separatrices. The same is true for NLS. The theta function solutions do not capture these components, which are related to complex double periodic spectra. The analogous theory for NLS will appear in another place.

The above two fundamental questions (III.35, 36) will be answered (for the first time) through the remaining chapter. We now quote and modify the theta function representation of the NLS solution \( q \) given by Its and Kotlyarov [51], Matveev [66], and Tracy [53]. First, we define the Riemann theta function, \( \vartheta(z|B) \), for \( N-1 \) variables,

\[ \vartheta(z|B) = \sum_{\mathbf{m} \in \mathbb{Z}^{N-1}} \exp\{\pi i[2\mathbf{m}^\dagger \mathbf{m} + 2z^\dagger \mathbf{m}]\} . \hspace{1cm} (III.37) \]

The "periodicity" properties are [69]:

(i) \( \vartheta(z + \mathbf{e}_j|B) = \vartheta(z|B) \) , \hspace{1cm} (III.38a)

(ii) \( \vartheta(z + \mathbf{B}_j|B) = [\exp(-2\pi iz_j - \pi iB_{jj})] \cdot \vartheta(z|B) , \hspace{1cm} (III.38b) \)
(iii) \[ \theta(z) = \exp\{-i\sum_{n=0}^{N-1} m_n \theta_n \} \]  

\[ \times \theta(z) \),  

(III.38c)

where \( m_n \in \mathbb{Z} \), \( 1 \leq k \leq N-1 \).

Theorem III.7 (Theta Function Representation of N-phase NLS Wavetrain)

A quasi-periodic NLS solution \( q \) which yields the 2N simple spectrum \( \{E_k\}_{k=1}^{2N} \) is an N-phase wavetrain,

\[ q(\mathbf{\theta}_1, \mathbf{\theta}_2, \ldots, \mathbf{\theta}_N) = q(x=0,t=0) \frac{\phi}{\phi(x,t)} \frac{\phi(\frac{k}{2\pi} \mathbf{\theta}(x,t)+\mathbf{\theta})}{\phi(\frac{k}{2\pi} \mathbf{\theta}(x,t)+\mathbf{\theta})} e^{i\theta_N(x,t)} \]  

(III.39a)

where \( \mathbf{\theta}, \mathbf{\xi}, \mathbf{d} \) are \((N-1)\)-vectors and \( \theta_N \) is scalar such that

\[ \phi(x,t) = (\theta_1, \ldots, \theta_{N-1})^T \quad \mathbf{\theta}(x,t) = k_j x + \omega_j t \]  

\( 1 \leq j \leq N \)  

(III.39b)

where

\[ k_j = (-4\pi i) \mathbf{C}_{j1} \quad \omega_j = (-4\pi i) \left[ \sum_{k=1}^{2N} \mathbf{E}_k \mathbf{C}_{j1} + 2 \mathbf{C}_{j2} \right] \]  

\( 1 \leq j \leq N-1 \)  

(III.39c)

and \( \mathbf{k}_N \mathbf{\omega}_N \) are certain constants (refer to (IV.15c,d)).
\[ (iii) \quad \delta_j = \frac{1}{2} B_{jj} - \sum_{k=1}^{N-1} a_k \int_a^b \frac{du_j}{u_j} + \sum_{k=1}^{N-1} \delta_k, \quad 1 \leq j \leq N-1. \]  

\[ (III.39e) \]

\[ (iii) \quad d_j = \int_{-\infty}^{\infty} u_j, \quad 1 \leq j \leq N-1. \]  

\[ (III.39f) \]

Remarks: (1) Kotlyarov and Its [51] have shown that \( k_N, \omega_N \) are real constants, independent of \( x \) and \( t \). Previato [54] provided rather simple forms for \( k_N \) and \( \omega_N \) (see (IV.15c,d)).

(2) From Abel-Jacobi map (III.32, 33, 34),

\[ \ell_j(x,t) = \frac{1}{2\pi} \theta_j(x,t) + \delta_j, \quad 1 \leq j \leq N-1 \]  

\[ (III.40a) \]

and

\[ \theta_j, \quad k_j, \quad \omega_j \in \mathbb{R}, \quad 1 \leq j \leq N-1. \]  

\[ (III.40b) \]

(3) The theta function representation of \( q \), (III.39a), immediately shows that \( q \) is \( 2\pi \)-periodic in each real phase \( \theta_j \), \( 1 \leq j \leq N \),

\[ q(\theta_1, \ldots, \theta_j + 2\pi, \ldots, \theta_N) = q(\theta_1, \ldots, \theta_j, \ldots, \theta_N) \]  

\[ (III.41) \]

This completely answers the first fundamental question, (III.35).

We notice that the representation (III.39a) of \( q \) with \( 2N \) simple spectrum shows \( q \) has \( N \) degrees of freedom, which are represented by \( N \) real phases \( \{\theta_j\}_{j=1}^N \). On the other hand, from Abel-Jacobi inverse map: \( \psi^{-1}: \hat{\mu} \rightarrow \hat{\nu} \) with \( \hat{\nu} = \frac{1}{2\pi} \sum_{j=1}^{N-1} \theta_j \hat{e}_j + \hat{\lambda} \), we find
Consequently, $\psi^*$-variables are homologous to $\alpha$-cycles [4]. This property plays the fundamental role in our modulation and linearized instability analysis (refer to Chapters IV, V). The Abel-Jacobi inverse map also shows that $q$ is parameterized by $\{\psi_j\}_{1}^{N-1}$ and $\theta_N$.

III.5 NLS Constraints

We recall the two fundamental questions (III.35) and (III.36). The real phases are computed in Sec. 4. We now derive the inverse spectral representation of $r(x,t) = q^*(x,t)$. This theta function representation of $r(x,t)$ can be established through the inverse spectral transformation as for the theta function representation of $q$ except that $\{\psi_j\}$ are now replaced by $\{\psi^*_j\}$, and $\{\ell_j\}$ by $\{\ell^*_j\}$ where

$$\ell^*_j(\psi^*) = \sum_{k=1}^{N-1} \int_{\mu_k^*}^{\mu_k} dU_j, \quad 1 \leq j \leq N-1. \quad (III.43)$$

The above argument is based on (III.22c), which shows that $(\ln r)_x^t$ can be represented in terms of $\{\psi^*_j\}$ just as $(\ln q)_x^t$ in terms of $\{\psi_j\}$. Consequently, using the same argument as for $q$, we find $r$ also can be represented in terms of same phases $\{\theta_j\}_{1}^{N}$ in
Theorem III.8 (Theta Function Representation of $r(x,t) = q^*(x,t)$)

Under the assumption of Theorem III.7, $r(x,t) = q^*(x,t)$ has the inverse spectral representation,

$$r(x_1, \ldots, x_N) = r(x=0, t=0) \frac{\Theta(\beta - [\beta + \mu]) - i \Theta(x(t) + \beta - x)}{\Theta(\beta - x) - i \Theta(\beta(x(t) + \beta - [\beta + \mu]))}$$

(III.44a)

where $\hat{\beta}, \hat{x}, \hat{\mu}$ and $\Theta_N$ are given in Theorem III.7, and the $(N-1)$-vector $\hat{x}$ is

$$\hat{x} = \sum_{k=1}^{N-1} \int_{\mu_k}^{\mu_k^*} d\lambda_j, \quad 1 \leq j \leq N-1.$$

(III.44b)

Now, we evaluate $\hat{x}, \hat{\mu}$ and $\hat{\beta}$ in Appendix E with results in

Theorem III.9

Let $\hat{x}, \hat{\mu}$ and $\hat{\beta}$ be $(N-1)$-vectors as given in (III.39e, f) and (III.44b). Then
We use these quantities to evaluate the complex conjugate of the theta function representation (III.39) of $q$ and find which is exactly identical to the theta function representation of $r$, (III.44),

$$r(\theta_1, \ldots, \theta_N) = \hat{q}(\theta_1, \ldots, \theta_N),$$  

and, finally, the question (III.36) is answered.
We now consider an NLS wavetrain, \( q_n(\theta_1, \ldots, \theta_N; E_1, \ldots, E_{2N}) \), which appears locally as an exact N-phase solution (see (III.39)), but whose parameters, \( \{E_k\}_{1}^{2N} \), vary on large scales in both \( x \) and \( t \): \( E_j(x, t), X = \epsilon x, T = \epsilon t \), \( 0 < \epsilon << 1 \). We call this object a modulating N-phase NLS wavetrain. We do not prove the existence of such a solution to NLS, rather we assume it (see McLaughlin [62] for such questions). We now follow the methods of Ercolani, Flaschka, Forest and McLaughlin [1, 3, 5] for KdV, sinh-Gordon and s-G to approach the NLS modulations. Our results show that the structure of the modulation equations of \( q_n(\theta_1, \ldots, \theta_N; E_1, \ldots, E_{2N}) \) is completely understood and the modulations are unstable for all NLS wavetrains to which this theory applies. We remark that this formalism, originally due to Whitham [58], only applies to long wavelength solutions.

IV.1 Averaged Generating Functions

We consider a conservation law, \( \frac{\partial F}{\partial t} + \frac{\partial \chi}{\partial x} = 0 \), for an arbitrary NLS solution. We now evaluate \( \frac{\partial F}{\partial t} + \frac{\partial \chi}{\partial x} \) on a modulating N-phase NLS wave, \( q_n(\theta_1, \ldots, \theta_N; E_1, \ldots, E_{2N}) \),
\[
\frac{\partial}{\partial t} \mathcal{F}(q_N) + \frac{\partial}{\partial x} \mathcal{X}(q_N) = (\omega_j \frac{\partial}{\partial \theta_j} + \epsilon \frac{\partial}{\partial \phi_j}) \mathcal{F}(q_N) + (k_j \frac{\partial}{\partial \phi_j} + \epsilon \frac{\partial}{\partial x}) \mathcal{X}(q_N),
\]

(IV.1)

and demand that the phase average of this expression vanish. This yields, upon exploiting the $2\pi$-periodicity in each phase $\theta_j$, the averaged conservation law:

\[
\frac{\partial}{\partial t} <\mathcal{F}(q_N)> + \frac{\partial}{\partial x} <\mathcal{X}(q_N)> = 0 ,
\]

(IV.2a)

where the phase average $<\cdot>$ of any function $F$ of $q_N$ is defined by

\[
<F(q_N)> = \frac{1}{(2\pi)^N} \int_0^{2\pi} \cdots \int_0^{2\pi} d\theta_1 \cdots d\theta_N F(q_N(\theta_1,\ldots,\theta_N,E_1,\ldots,E_{2N})) ,
\]

(IV.2b)

and is computed for frozen values of $\{E_k(X,T)\}_{k=1}^{2N}$. We notice that the NLS densities $\mathcal{F}_j$ and fluxes $\mathcal{X}_j$ are independent of the plane wave phase $\theta_N$, as can be inspected from (III.12, 13, 22d). Thus (IV.2b) collapses to an $(N-1)$-fold integral in the calculations to follow. It is, therefore, that the averaged density $<\mathcal{F}>$ and the averaged flux $<\mathcal{X}>$ depend only on $\{E_k(X,T)\}_{k=1}^{2N}$. To obtain a closed system of modulation equations, we must average $2N$ independent conservation laws.

Recall the fundamental conservation laws (III.8),

\[
f_t - [4Ef + i(qh + rg)]_x = 0 ,
\]

(IV.3)
which generates all conservation laws (III.13) under the normalization condition (III.9) as \( E \) near \( \infty \). We want to average these terms, thereby averaging the entire family of conservation laws. We normalize the polynomial quadratic eigenfunctions \( \{f, g, h\} \) of \( q_N(\theta; E_1, \ldots, E_{2N}) \) by

\[
\sqrt{f^2 - gh} = R(E) = \sqrt{\frac{2N}{\pi} (E - E_k)}, \tag{IV.4}
\]

then density and flux terms are derived from (III.22d) in the \( \mu \)-coordinates, as given in

**Lemma IV.1 (Averaged Generating Functions in terms of \( \mu \)-Coordinates)**

(i) \( \langle f \rangle = \left\{ (E - \frac{1}{2} \sum_{k=1}^{N-1} E_k) \langle \prod_{j=1}^{N-1} (E - \mu_j) \rangle + \sum_{j=1}^{N-1} \mu_j \langle \prod_{j=1}^{N-1} (E - \mu_j) \rangle \right\} \frac{1}{R(E)}. \) \tag{IV.5a}

(ii) \( \langle q_h \rangle = \langle r_g \rangle , \)

\[
i \langle q_h + r_g \rangle = \left\{ \left[ \sum_{i,j=1}^{2N} E_i E_j - \frac{1}{2} \left( \sum_{k=1}^{2N} E_k \right)^2 \right] \langle \prod_{j=1}^{N-1} (E - \mu_j) \rangle - 2 \sum_{i=1}^{2N} E_i \langle \sum_{j=1}^{N-1} \mu_j \langle \prod_{j=1}^{N-1} (E - \mu_j) \rangle \rangle + 4 \left[ \sum_{i,j=1}^{N-1} \mu_i \mu_j + \sum_{i=1}^{N-1} (\mu_i)^2 \right] \right\} \frac{1}{R(E)}. \tag{IV.5b}
\]
(iii) \( \langle 4Ef + i(qh + rg) \rangle \)
\[
= \frac{1}{2N} \left[ (4E^2 - 2 \sum_{k=1}^{2N} E_k) \cdot E + 2 \sum_{i>j} E_i E_j \cdot \frac{1}{2} \left( \sum_{k=1}^{2N} E_k \right)^2 \right]
\]
\[
\times \left[ \langle \prod_{j=1}^{N-1} (E - \mu_j) \rangle + (4E - 2 \sum_{k} E_k) \cdot \langle \sum_{j=1}^{N-1} \mu_j \cdot \prod_{j=1}^{N-1} (E - \mu_j) \rangle \right]
\]
\[
+ 4 \left[ \sum_{i>j} \mu_i \mu_j + \sum_{j=1}^{N-1} \mu_j^2 \right] \cdot \langle \prod_{j=1}^{N-1} (E - \mu_j) \rangle \frac{1}{R(E)} .
\] (IV.5c)

Next, we compute (IV.5) by changing variables of integration from \( \hat{\theta} \) to \( \hat{\mu} \). From (III.32, 33, 34, 40), the Abel-Jacobi map induces the Jacobian,

\[
\frac{\partial \hat{\mu}}{\partial \hat{\theta}} = (2\pi)^{N-1} \frac{\det(\frac{\mu_k}{R(\mu_k)})}{\det(\int_{a_i}^1 E^{-1-j} dE/R(E))} .
\] (IV.6)

The product structure of (IV.5) and (IV.6), together with the \( \hat{\mu} \)-homology result (III.42), allows the \((N-1)\)-fold integrals to be factored into products of one-dimensional integrals [5]. We defer this rather long and nontrivial calculation to Appendix F with results in

Lemma IV.2 (Iterated Integral Representation of Averaged Generating Functions)

\[
(i) \langle f \rangle = \left\{ \sum_{j=0}^{N+1} D_j^{(-)} E^j \right\} \frac{1}{R(E)} ,
\] (IV.7a)

where
\[ D^{(-)}_{N+1} = 0, \quad D^{(-)}_N = 1, \quad D^{(-)}_{N-1} = \frac{1}{2} \left( \sum_{k=1}^{2N} E_k \right), \quad (IV.7b) \]

and

\[ D^{(-)}_j = \frac{\det M^{(N-j-1,N)}}{\det M} - \frac{1}{2} \left( \sum_{k=1}^{2N} E_k \right) \cdot \frac{\det M^{(N-j-1,N-1)}}{\det M}, \quad (IV.7c) \]

\[ 0 \leq j \leq N-2. \]

(ii) \[ \langle 4E_f + i(qh + rg) \rangle = \left\{ \sum_{j=0}^{N+1} D^{(+)}_j E_j \right\} \frac{1}{R(E)}, \quad (IV.8a) \]

where

\[ D^{(+)}_{N+1} = 4, \quad D^{(+)}_N = -2 \left( \sum_{k=1}^{E_j} E_k \right), \quad D^{(+)}_{N-1} = (2 \sum_{i>j} E_i E_j - \frac{1}{2} \sum_{k=1}^{E_k} E_k)^2, \quad (IV.8b) \]

and

\[ D^{(+)}_j = \{-4 \frac{\det M^{(N-1-j,N+1)}}{\det M} + 2 \left( \sum_{k=1}^{E_k} E_k \right) \cdot \frac{\det M^{(N-j-1,N)}}{\det M} \]

\[ + \left[ \frac{1}{2} \left( \sum_{k=1}^{E_k} E_k \right)^2 - 2 \sum_{i>j} E_i E_j \right] \cdot \frac{\det M^{(N-j-1,N-1)}}{\det M} \}, \quad (IV.8c) \]

\[ 0 \leq j \leq N-2. \]

(iii) \[ D^{(\pm)}_j \in \mathbb{R}, \quad 0 \leq j \leq N+1. \quad (IV.9) \]

Here, \[ M = C^{-1} \] where \[ C \] is the constant matrix given in (III.26, 27) with
Some Remarks: (1) \( D_{N+1}^{(\pm)} , D_N^{(\pm)} , D_{N-1}^{(\pm)} \) turn out to be the coefficients of the exact generating functions (III.19b), which is essential in understanding the structure of \( 2N \) modulation equations of \( \{ E_{k}(X,T)\}_{1}^{2N} \) as we describe later. (2) The real values of \( \{ D_j^{(\pm)} \} \), (IV.9), are crucial in determining the stabilities of these modulation equations.

Inserting (IV.7, 8) into (IV.2), we find the first order NLS N-phase modulation equations, a result of multiphase averaging technique,

\[
\frac{\partial}{\partial T} \left\{ \sum_{0}^{N+1} D_j^{(-)} \frac{E_j}{R(E)} \right\} + \frac{\partial}{\partial X} \left\{ \sum_{0}^{N+1} D_j^{(+)} \frac{E_j}{R(E)} \right\} = 0. \tag{IV.11}
\]

The averages of the usual conservation laws (III.13) for NLS are obtained by expanding this expression (IV.11) in the local coordinates \( \xi = E^{-1} \) near \( E = \infty \), denoted as \( < f_j >_T + < x_j >_X \), \( j \geq 2 \). There are \( 2N \) variables \( \{ E_{k}(X,T)\}_{1}^{2N} \), so we presumably choose any \( 2N \) of these, arriving at the system of \( 2N \), quasilinear, first-order P.d.e's. Next, we investigate the \( 2N \) modulation equations.
IV.2 An Invariant Representation of the Averaged Generating Functions

For KdV, sinh-Gordon and s-G, Ercolani, Flaschka, Forest and McLaughlin [1, 3, 5] have found a remarkable connection between averaged generating functions and certain Abelian differentials on these Riemann surfaces, from which they completely analyzed the modulation behaviors. This connection also exists for NLS, as we now describe.

We now define two unique meromorphic differentials on the Riemann surface \( R \) of \( R(E) \):

\[
\Omega^{(k)} = \sum_{j=0}^{N+1} D_j^{(k)} E_j \frac{dE}{R(E)}, \quad k = 1, 2, \quad (IV.12a)
\]

where

\[
\begin{align*}
(i) \quad & D^{(1)}_{N+1} = 0, \quad D^{(1)}_N = 1, \quad D^{(1)}_{N-1} = -\frac{1}{2} \sum_{k=1}^{2N} E_k; \\
& D^{(2)}_{N+1} = 4, \quad D^{(2)}_N = -2 \sum_{k=1}^{2N} E_k, \quad D^{(2)}_{N-1} = [-\frac{1}{2} \sum_{1}^{2N} E_k^2 + 2 \sum_{i>j} E_i E_j]. \\
(ii) \quad & \int_a^b \Omega^{(k)} = 0, \quad k = 1, 2, \quad 1 < j < N-1. \quad (IV.12c)
\end{align*}
\]

We note that \( D^{(k)}_{N+1}, D^{(k)}_N, D^{(k)}_{N-1}, \quad k = 1, 2, \) are taken to be \( D^{(\pm)}_{N+1}, D^{(\pm)}_N, \) and \( D^{(\pm)}_{N-1}, \quad (IV.7, 8), \) for the averaged generating functions.
It turns out that $\Omega^{(k)}$, $k = 1, 2$, are fundamentally connected to the averaged generating functions, the wave numbers $k$ and frequencies $\omega$ of $q_N$, the averaged conservation laws and modulations of $q_N$, and the linearized instability of $q_N$.

Lemma IV.3 (Invariant Representation of Averaged Generating Functions)

\[ \Omega^{(1)} = \langle f \rangle dE \]  
\[ \Omega^{(2)} = \langle 4Ef + i(qh + rg) \rangle dE \]

Some Remarks: (1) (IV.13) shows that the averaged generating functions actually define $\Omega^{(k)}$, and $\Omega^{(k)}$ are unique. We use linear algebra to show (IV.13) in Appendix G. (2) From (IV.7, 8), we find

(a) $\Omega^{(k)}$ has a pole of order $k+1$ at $\pm$, on $\Re$, and no other poles;

(b) The expansion of $\Omega^{(k)}$ in the local coordinates $E = \xi^{-1}$ near $\pm$ are:

\[ \Omega^{(1)} \sim \pm \left[ -\frac{1}{\xi^2} d\xi + \text{holomorphic part} \right] \quad \text{(IV.14a)} \]

with $E = \frac{1}{\xi}$ near $\pm$.

\[ \Omega^{(2)} \sim \pm \left[ -\frac{4}{\xi^3} d\xi + \text{holomorphic part} \right] \quad \text{(IV.14b)} \]

In Appendix H, we use Riemann bilinear identity for the Abelian differentials of first and second kind [68] to show
Lemma IV.4 (Physical Contacts of Averaged Generating Functions)

(i) \( \int_b^a \Omega^{(1)} = -k_j \) \hspace{1cm} (IV.15a)

(ii) \( \int_b^a \Omega^{(2)} = -\omega_j \) \hspace{1cm} (IV.15b)

Remark: Previato [54] showed

\[ \int_{E_1}^E \Omega^{(1)} \sim E - i \frac{K}{2} + O(E^{-1}) , \hspace{1cm} (IV.15c) \]

as \( E \to \infty \).

\[ \int_{E_1}^E \Omega^{(2)} \sim 2E^2 + \omega_N + O(E^{-1}) , \hspace{1cm} (IV.15d) \]

IV.3 An Invariant Representation of the Modulation Equations;

Consequences of the Invariant Representation

We now consider

\[ \Omega = \Omega^{(1)} - \Omega^{(2)} \] \hspace{1cm} (IV.16)

By straightforward calculation, we find

Lemma IV.5 (Basic Properties of \( \Omega \))

(i) \( \Omega = \left[ \sum_{j=0}^{N+1} (D^{(1)}_j - D^{(2)}_j) \frac{dE}{R(E)} + \frac{1}{2} \sum_{k=1}^{2N} \frac{\Omega^{(1)}_k - \Omega^{(2)}_k}{E - E_k} \right] \).

(IV.17a)
(ii) Near $E = \frac{1}{\xi} = \pm 1$, the local representation of $\Omega$ is:

$$\Omega \sim \frac{1}{\xi^n} \left[ \sum_{j=2}^{\infty} \left( \begin{array}{c} \zeta_j \langle \zeta_j \rangle - \langle \zeta_j \rangle \end{array} \right) \xi^{-2j} \right] d\xi \quad \text{as} \quad E = \frac{1}{\xi} \quad \text{near} \quad \pm 1.$$  \hspace{1cm} (IV.17b)

(iii) Near $E - E_k = \xi^2 = 0$, the local representation of $\Omega$ is:

$$\Omega \sim \frac{1}{\xi^2} \left[ \left( \Omega_k^{(1)} - \Omega_k^{(2)} \right) \frac{d\xi}{\xi^2} + \text{holomorphic part} \right] \quad (IV.17c)$$

as $E - E_k = \xi^2$ near 0,

where

$$\Omega^{(1)} = \sum_{j=0}^{N+1} D_j \left( \begin{array}{c} 2 \end{array} \right) \xi^j \quad \text{and} \quad \Omega^{(2)} = \sum_{j=0}^{N+1} D_j \left( \begin{array}{c} 2 \end{array} \right) \xi^j$$

$$= 2 \left[ \prod_{k \neq k} \frac{(E_k - E_{\ell})}{E_k - E_{\ell}} \right], \quad 1 \leq k \leq 2N. \quad (IV.17d)$$

We now use the Riemann-Roch theorem and Riemann bilinear identity to derive the main result with proof given in Appendix K, a method due to Flaschka, Forest and McLaughlin [1,3].

**Theorem IV.6 (An Invariant Representation of the Modulation Equations)**

Assume the first $2N$ averaged conservation laws are satisfied by

$$\{E_k(X,T)\}_{1}^{2N},$$
\( \langle \mathcal{J} \rangle_T + \langle \mathcal{J} \rangle_X = 0 \), \( j = 2, \ldots, 2N+1 \). (IV.18a)

Then all higher conservation laws are satisfied, and the modulation equations (IV.18a) take the equivalent form, via differentials on,

\[
\Delta \equiv \Delta^{(1)} - \Delta^{(2)} = 0 .
\] (IV.18b)

**Theorem IV.7 (Consequences of the Invariant Representation (IV.18))**

(i) **(Riemann Invariants):** The invariant form (IV.18) is equivalent to the Riemann invariants,

\[
E_k^T - S^{(k)} E_k^X = 0 , \ 1 \leq k \leq 2N ,
\] (IV.19a)

where

\[
S^{(k)} = \sum_{j=0}^{N+1} D_j^{(2)} E_j^k / \sum_{j=0}^{N+1} D_j^{(1)} E_k^j .
\] (IV.19b)

(ii) **(Modulational Instability):** The modulation equations are elliptic, and the modulational instability is predicted for all quasiperiodic NLS wavetrains.

(iii) **(Conservation of Waves):**

\[
(\omega_j - (\omega_j)_T) = 0 \), \( 1 \leq j \leq N-1 .
\] (IV.20)
Remarks on the Proof: (i) The equivalent Riemann invariance (IV.19) is an immediate result of (IV.17). (ii) Since $D_j^{(1)}$, $D_j^{(2)}$, $0 \leq j \leq N+1$, are real (IV.9), while $E_k \not\in \mathbb{R}$, the characteristic speeds $S^{(k)}$, $1 \leq k \leq 2N$, are nonreal; consequently, the modulations are unstable. (iii) The conservation of waves (IV.20) is due to (IV.15).

Remarks: for NLS wavetrains with periodic boundary conditions in $x$ of fixed period $L$, there are $N$ commensurability constraints on the parameters $E_1, \ldots, E_{2N}$:

$$\frac{L}{2\pi} = \frac{n_j}{k_j}, \quad j = 1, 2, \ldots, N, \quad n_j \in \mathbb{Z}, \quad (IV.21)$$

which reduce the system to $N$ degrees of freedom. Clearly, $k_j$ must all be identically constant or these commensurability conditions are broken. Consequently,

$$(k_j)_\chi = 0, \quad j = 1, \ldots, N. \quad (IV.21a)$$

From conservation of waves, (IV.20),

$$(\omega_j)_\chi = 0, \quad j = 1, \ldots, N, \quad (IV.21b)$$
i.e., **no spatial modulation in the presence of periodic boundary conditions for fixed period.** Moreover, in the absence of external modulations, there are no terms to balance temporal modulations of \( \{ E_k \} \); from (IV.19),

\[
(E_k)_T = - S^{(k)}(E_k)_x = 0 , \quad 1 \leq k \leq 2N , \quad (IV.22)
\]

consequently, **no modulations, temporal or spatial, are possible for x-periodic wavetrains of fixed period in the absence of external perturbations.**
Ercolani, Forest and McLaughlin [6] have completely analyzed the linearization about an N-phase s-G wavetrain by certain geometric methods; in particular, they showed exponential instability occurred quite often. Independently, Tracy [53] used the inverse spectral theory to study these modulational instabilities for the periodic NLS solutions in the neighborhood of the NLS plane wave. We now follow [6] to investigate the N-phase NLS wavetrains. We will (1) connect the classical argument of linearization with Floquet theory for the NLS plane waves; (2) show that the geometry of the NLS $\mu$-cycles characterizes the linearized instability; (3) present an analytic argument to verify the geometric predictions.

V.1 The Linearized Instabilities of the NLS Plane Waves

We now consider the NLS plane wave $q_0$,

$$q_0 = ae^{ia^2t}, \ a \in \mathbb{R}.$$  

(V.1a)

We seek a neighborhood solution,
\[ q = q_0 + q_1 \text{ where } q_1 = be^{ia^2t}, \ |b| \ll 1. \quad (V.1b) \]

The linearized NLS about an arbitrary solution \( q_0 \) is

\[ iq_1 + q_1 + 4|q_0|^2 q_1 + 2q_0 \cdot q_1^* = 0. \quad (V.2a) \]

For \( q_0, q_1 \) in (V.1), we find

\[ (i\partial_t + \partial_{xx} + 2a^2)b = -2a^2 b, \quad (V.2b) \]

\[ (i\partial_t + \partial_{xx} + 2a^2)b^* = -2a^2 b, \]

which decoupled into a simple equation of \( b \),

\[ (\partial_{tt} + \partial_{xxxx} + 4a^2)b = 0. \quad (V.2c) \]

Now, let \( b(x,t) = e^{\sigma t + ikx} \), and we find the linearized growth rate \( \sigma \):

\[ \sigma^2 = -k^2(k^2 - 4a^2) \quad (V.3) \]

\[ k^2_0 = 4a^2 : \text{instability cut off.} \]

\[ k^2_0 = 2a^2 : \text{most unstable.} \]

Fig. 5
(V.3) shows long waves \(0 < k^2 < 4a^2\) are unstable.

We now impose the boundary condition with fixed spatial period \(L\). Then only a discrete set of unstable modes \(k^2 = \frac{4n^2 \pi^2}{L^2}\) is allowed:

(i) \(q_0 = ae^{2ia^2t}\) is linear stable for \(4a^2 < k^2 = \frac{4\pi^2}{L^2}\). (V.4)

(ii) \(q_0 = ae^{2ia^2t}\) is unstable and there are \(2n\) Fourier exponentially unstable modes, \(k^2 = \frac{4n^2 \pi^2}{L^2}\), provided

\[
\frac{n \pi}{a^2} < L^2 \leq \frac{(n+1) \pi}{a^2}.
\] (V.5)

This classical argument shows the number of unstable modes is a function of the spatial period \(L\).

On the other hand, we have shown, in Sec. II.4, for the NLS plane wave \(q_0 = ae^{ia^2t}\) with spatial period \(L\), the double points \(\Sigma, \{E_k \in C, \Delta(E_k) = \pm 2, \Delta'(E_k) = 0\}\), are completely determined by \(L\). It turns out that the number of Fourier unstable modes and the number of nonreal double points are equal; moreover, each nonreal double point labels a unique unstable mode, as indicated in Figure 3 and (V.5). This is, essentially, the result of Tracy [16].
We generalize the above instability results and the connection with complex double points to arbitrary NLS periodic N-phase solutions next.

V.2 \( \mu \)-Coordinates; A geometric Realization of Linearized Instability

We recall from Chapter II that for an arbitrary NLS solution \( q : (1) \) for generic \( q \), there are no double points, \( \Sigma^d(q) \); \( (2) \) when \( q \) is degenerate, e.g., an N-phase wavetrain \( q_N \), the (open and closed) degrees of freedom in \( q_N \) are in 1:1 correspondence with the critical points of \( \Delta \); moreover, when a closed degree of freedom is excited, the corresponding double point \( E_d \) splits into a pair of simple spectra, which induces an open degree of freedom. We also show in Chapter III that all open degrees of freedom, \( \{E_k\}_{1}^{2N} \) in \( q_N \) are completely determined by \( \{\mu_j\}_{1}^{N-1} \). In [80], Ercolessi, Forest and McLaughlin have shown that complex double points correspond to non-torus components in the isospectral set. We will not focus on this point of view here. So we assume the remaining \( \mu \)-variables are tied to the double points \( \Sigma^d(q_N) \). Consequently, when we open certain closed degrees of freedom for \( q_N \) and investigate the stability problem, we may study the \( \mu \)-geometry instead. The homology between \( \alpha \)-cycles and \( \mu \)-variables, (III.42), allows us to use the simple closed curve \( \{a_j\} \).

We now open a simple double point \( E_d^0 \) of \( \Sigma^d(q_N) \) to order \( \epsilon \), where we have assumed the associated \( \mu \)-variable is tied at \( E_d^0 \). Two cases are considered: \( E_d^0 \in \mathbb{R} \) and \( E_d^0 \notin \mathbb{R} \).
Case 1: $E_d \in R$, with the corresponding $\mu = E_d^0$, as in Figure 6a. $E_d^0$ splits into

$$E_d^e = E_d^0 + O(\varepsilon), \quad E_d^e,$$

and the corresponding $\mu^e$-cycle is homologous to a $a$-cycle, as given in Figure 6b;

\[
\begin{align*}
E_d^0 & : \mu^e \\
E_d^e & = E_d^0 + O(\varepsilon).
\end{align*}
\]

This picture indicates

$$|\mu^e - \mu^0| = O(\varepsilon) \quad \text{for both } x \text{ and } t \text{ flows}, \quad \text{(V.7)}$$

which suggests linearized stabilities of both $x$ and $t$ flows.
Case 2: \( E^0_d \notin \mathbb{R} \). The symmetry of the NLS spectrum shows \( E^0_d \) is also a double point, and the corresponding \( \mu \)-variables are \( \mu^0_1 \equiv E^0_d \) and \( \mu^0_2 \equiv E^0_d \), as given in Figure 7a. \( E^0_d \) splits into

\[
E^e_1 = E^0_d + O(\epsilon), \quad E^e_2 = E^0_d + O(\epsilon).
\]

(V.8a)

\( E^0_d \) splits into \( E^e_1 \) and \( E^e_2 \). The corresponding \( \mu^e_1, \mu^e_2 \) have homologous \( a_1, a_2 \) cycles as given in Figure 7b.

This picture, Figure 7, indicates

\[
|\mu^e_k - \mu^0_k| = O(1), \quad 1 \leq \nu, k \leq 2,
\]

(V.8b)

which suggests linearized instability.
Actually, as we will show in Section 3, such a perturbation to the degenerate NLS potential $q_N$ yields modulational instability in the t-flow. The geometric evidence is achieved by taking the linear combination of $a_1, a_2$ cycles corresponding to the x-flow and t-flow, as given in Figure 8a,b.

![Fig. 8a: t-flow](image1)

![Fig. 8b: x-flow](image2)

**V.3 The Analytic Argument of Linearized Instability**

We now verify the geometric result that the degenerate NLS N-phase wavetrains are modulationally unstable. Consider a NLS N-phase wavetrain $q_N(x,t)$, $N < \infty$, with $2N$ simple spectrum $\{E_k\}_{k=1}^{2N}$, $N-1$ $u$-variables $\{\mu_k\}_{k=1}^{N-1}$ and the hyperelliptic curve $R^2_N(E) = \Pi_k (E-E_k^0)$. Our object is to compute the linearized growth rate corresponding to opening a closed degree of freedom (a double
point $E_d$ to order $\varepsilon$, and calculate the effect on the associated dynamical $\mu$ variable which, for $\varepsilon = 0$, is assumed locked at the double point.

Case 1: $E_d \in \mathbb{R}$: $\mu^0_N = E_d$ and $E_d$ breaks into $E_{2N+j}^*$, $j = 1, 2$,

$$E_{2N+j} = E_d + \varepsilon E_{2N+j}^{(1)}, \quad j = 1, 2; \quad E_{2N+2} = E_{2N+1}^*,$$  \hspace{1cm} (V.9a)

with

$$\mu^*_N = E_d + \varepsilon \mu^{(1)}_N.$$

Case 2: $E_d \notin \mathbb{R}$: $\mu^0_N = E_d$, $\mu^0_{N+1} = E_d^*$. They break into

$E_{2N+j}^*$, $1 \leq j \leq 4$,

$$E_{2N+j} = E_d + \varepsilon E_{2N+j}^{(1)}, \quad j = 1, 2,$$

and

$$E_{2N+j}^* = E_{2N+j}^*, \quad j = 1, 2,$$  \hspace{1cm} (V.9b)

with

$$\mu^*_N = E_d + \varepsilon \mu^{(1)}_N, \quad \mu^*_N = E_d + \varepsilon \mu^{(1)}_{N+1}.$$

Now, the perturbed $q_{N+\delta}$, $\delta = 1, 2$, has $2N + 2\delta$ simple spectrum $\{E_k\}$, $N-1+\delta$ open $\mu$-variables $\{\mu_k\}$ and the associated curve

$$R^2_{N+\delta} = \prod_{k} (E - E_k),$$  \hspace{1cm} where
\[ E_k = E_k^0 + \varepsilon E_k^{(1)} , \quad 1 \leq k \leq 2N , \quad (V.9c) \]

and

\[ \mu_k = \mu_k^0 + \varepsilon \cdot \mu_k^{(1)} , \quad 1 \leq k \leq N-1 . \]

We do not insert (V.9) into the \( \mu \)-dynamical system (III.23), expand in \( \varepsilon \), and retain terms to order \( \varepsilon \):

\[
\begin{align*}
\text{\( \varepsilon \)-term:} & \quad (-2i)R_{N+\delta}(\mu_\delta^p)\left[ \sum_{1}^{2N+2\delta} \sum_{j\neq p}^{N-1+\delta} \frac{E_k^0 - \sum_{j\neq p}^{N-1+\delta} \mu_j}{2N+2\delta} \right]_{\varepsilon=0} , \quad \delta = 1 \text{ or } 2 \\
= & \quad (-2i)R_{N+\delta}(\mu_\delta^p)\left[ \sum_{1}^{2N+2\delta} \sum_{j\neq p}^{N-1+\delta} \frac{E_k^0 - \sum_{j\neq p}^{N-1+\delta} \mu_j}{2N+2\delta} \right]_{\varepsilon=0} , \quad \delta = 1 \text{ or } 2 \\
= & \quad \begin{cases} 
0, \quad \text{for } \mu_N^0, \mu_{N+1}^0 \\
(-2i)R_{N+\delta}(\mu_\delta^p)\left[ \sum_{1}^{2N+2\delta} \sum_{j\neq p}^{N-1+\delta} \frac{E_k^0 - \sum_{j\neq p}^{N-1+\delta} \mu_j}{2N+2\delta} \right]_{\varepsilon=0} , \quad \text{i.e. (III.23), for } 1 \leq p \leq N-1 . 
\end{cases}
\end{align*}
\]
\[ \epsilon' \text{ term: we need evaluate } R'_{N+\delta}^{\nu_p | |_{\epsilon=0}} \] and

\[
\begin{align*}
2N+2\delta - N-1+\delta & \\
\left( \sum_{k=1}^{2N+2\delta} - 2 \sum_{j \neq p} \mu_j \right)
\end{align*}
\]

\[
\left[ \frac{1}{N-1+\delta} \sum_{j \neq p} (\mu_p - \mu_j) \right]'. \quad \text{The results are}
\]

\[ \epsilon = 0 \]

(i) \[ R'_{N+\delta}^{\nu_p | |_{\epsilon=0}} = \frac{1}{2} R_{N+\delta}^{\nu_p | |_{\epsilon=0}} \left[ \sum_{k=1}^{2N+2\delta} \frac{\mu_k' - E_k'}{p_k} \right], \quad (V.11a) \]

\[ \delta = 1 \text{ or } 2. \]

(ii) \[ \left[ \frac{1}{N-1+\delta} \sum_{j \neq p} (\mu_p - \mu_j) \right]' = \frac{1}{N-1+\delta} \left[ \sum_{j \neq p} (\mu_p - \mu_j) \right]'. \]

\[ \cdot \left[ \left( \sum_{k=1}^{2N+2\delta} \frac{\mu_k' - E_k'}{p_k} \right) - \left( \sum_{k=1}^{2N+2\delta} \frac{\mu_k - E_k}{p_k} \right) \right], \quad (V.11b) \]

\[ \cdot \sum_{j \neq p} \frac{(\mu_p - \mu_j)}{p_p - \mu_j} \]

We insert these quantities into \( \epsilon' \text{ term with } \epsilon = 0 \), we find:
Lemma V.1 (Linearized $\mu$-Dynamical System)

(i) $\left( \mu_k^{(1)} \right)^x = (-2i)^k \frac{R_N(\mu_k^0)}{t} \left[ \frac{2N}{N-1} \sum_{l \neq 0} \sum_{j \neq k} \left( \sum_{l \neq 0} 2 \sum_{j \neq k} \mu_j^{0} \right) \right]$

- \frac{1}{2} \sum_{l=1}^{2N} \frac{\mu_k}{E_l} - \sum_{j \neq k} \frac{\mu_j}{0} + \sum_{j \neq k} \frac{\mu_j^{0} - \mu_j^{0}}{0} \right)

+ \left( \sum_{l \neq 0} \sum_{j \neq k} \mu_j^{0} - 2 \sum_{j \neq k} \mu_j^{0} - 2 \sum_{j \neq k} \mu_j^{0} \right)_{\mu N-1+\delta}^{0}, 1 \leq k \leq N-1.

(ii) $\left( \mu_{N-1+\delta}^{(1)} \right)^x = \left[ (-2i)^k \frac{R_N(\mu_{N-1+\delta}^0)}{t} \right]$

Remark: $\mu_{N-1+\delta}$ are decoupled with other $\mu_j^{0}$ terms. Moreover, (III.22b,c) and (V.12b) show $\mu_N^{(1)}(x)$ and $\mu_{N+1}^{(1)}(x)$ move with complex conjugate pattern under complex conjugate initial conditions, i.e.,

$(\mu_N^{(1)}(x))^* = \mu_{N+1}^{(1)}(x)$, an analytic evidence of Figure 8.b. On the other hand, $\mu_N^{(1)}(t)$ and $\mu_{N+1}^{(1)}(t)$ do not move in such a pattern, as
evaluated from the $t$ Floquet exponent of (V.12b), which indicates the 
$\mu_N^{(1)}$, $\mu_{N+1}^{(1)}$ $t$-flows in Figure 8.a.

We now investigate the $\mu_{N-1+\delta}^{(1)}$ behavior(s) which is related to 
the Banjamine-Feir instability (see [6]). By applying classical 
Floquet theory, we evaluate $x,t$ Floquet means instead of Floquet 
exponents,

$$
\mu_{N-1+\delta}^{(1)} : (-2i)R_N(\mu_{N-1+\delta})\overset{1}{\underset{1}{\Pi}} \frac{1}{\nu_{N-1+\delta} - \nu_j} 
$$

(V.13a)

$$
\mu_{N-1+\delta}^{(1)}(t) : (-2i)R_N(\mu_{N-1+\delta})[\left( \frac{1}{\nu_{N-1+\delta} - \nu_j} \right)^2 + 2\mu_{N-1+\delta}^{(1)}]^{-\frac{1}{2}} 
$$

(V.13b)

where $<\cdot>$ is the multiphase averaging usued in Chapter IV. The 
linearized flow is stable iff the corresponding Floquet mean is purely 
imaginary; the linearized growth rate is given by the real part of the 
corresponding Floquet mean.

Ercolani, Forest and McLaughlin [6] have shown the Floquet means 
of $s$-G linearized $\Gamma$-variables can be represented as integrals of
certain unique Abelian differentials. Based on their method, we also develop a technique to evaluate the Floquet means with the same results:

Theorem V.2 (The Floquet Means as Integrals of \( \Omega \))

\[
\text{The x-flow Floquet mean, (V.13a) } = \int_0^{\mu_{N-1+\delta}} \left( \begin{array}{c} R_N \\ \Omega \end{array} \right)_{(1)} \\
\text{The t-flow Floquet mean, (V.13b) } = \int_0^{\mu_{N-1+\delta}} \left( \begin{array}{c} R_N \\ \Omega \end{array} \right)_{(2)}
\]

We leave the derivation of (V.14) in Appendix J.

The integrals (V.14) are nontrivial as evaluated in Appendix K, and the results are:

Theorem V.3 (Modulational Instability of N-Phase NLS Wavetrain)

(i) (Linearized x-Flows are Stable)

Floquet exponents of \( \mu_{N-1+\delta}^{(1)} \) are purely imaginary and are equal to \( ik_{N+\delta} \); \( \mu_{N-1+\delta}^{(1)} \) x-flows are stable.

(ii) Linearized t-Flows are Stable iff \( E_d \in R \).

When \( E_d \not\in R \), the linearized growth rate is real and non-zero, given by

\[
\text{Im}[\int (E_d^{-R_N}) \left( \begin{array}{c} \Omega \end{array} \right)_{(2)}] = \int_0 \left( \begin{array}{c} \Omega \end{array} \right)_{(2)}
\]

(V.15)
where $\gamma_d$ is a path from $E_d^*$ to $E_d$ on the first sheet of the Riemann surface $\mathbb{R}_N$ such that $\Omega^{(2)}$ does not vanish.

Remark: as mentioned in Chapter II, there are at most finitely many nonreal double points. These results can be summarized in

**Theorem V.4**

Given $q_N$, an $N$-phase periodic NLS solution, and its associated double periodic spectrum $\Sigma^d(q_N)$. Let $p$ be the number of complex pairs of double points, $\Sigma^d(q_N) \cap \{\mathbb{C} \setminus \mathbb{R}\} = \{E_d^*, E_d^*, \ldots, E_d^p, E_d^p\}$. Then $q_N$ has $2p$ linearly unstable modes, with growth rates given by

$$\text{Im}[\int_{(E_d^*, R_N)} \Omega^{(2)}] = \text{Im}[\int_{(E_d^*, -R_N)} \Omega^{(2)}], \quad E_d \in \{E_d^*, E_d^*, \ldots, E_d^p, E_d^p\},$$

(V.16)

where $\gamma_d$ is a path from $E_d^*$ to $E_d$ as in Theorem V.3. Moreover, if $\text{Re}(E_d^e) = \text{Re}(E_d^k)$, $1 \leq e, k \leq p$, then the $t$-growth rate of $E_d^e > t$-growth rate of $E_d^k$ if $\text{Im}(E_d^e) > \text{Im}(E_d^k)$.
CHAPTER VI

CONCLUDING SUMMARY; FUTURE RESEARCH

VI.1 Concluding Summary

The Riemann-theta function representation (III.39) of the inverse spectral solution $q_N$ which yields $2N$ simple spectra $\{E_k\}$ shows $q_N$ has $N$ degrees of freedom and is parameterized by $N$ real phases $\{\theta_j\}$ with $2\pi$-period in each phase $\theta_j$. The physical wave number $k_j$ and frequency $\omega_j$ of $\theta_j$ are the integrals of unique Abelian \( \frac{1}{2} \) differentials $\Omega_j$ over $b_j$-cycles, i.e., (IV.15).

The $\mu$-representation of $(\ln q_N)_t$, (III.24), along with the Abel-Jacobi inverse map shows $q_N$ is also parameterized by the $N-1$ Dirichlet eigenvalues $\{\mu_j\}$ and the plane wave phase $\theta_N$.

While those phases $\{\theta_j\}$ precisely describe the behavior of $q_N$, the $\{\mu_j\}$ variables are fundamental to the perturbations of $q_N$.

The modulation of the modulating $N$-phase NLS wavetrain $q_N(\hat{\theta};\hat{E})$ is prescribed by $2N$ modulation equations, the first $2N$ averaged conservation laws. The homology, $a_j \sim \mu_j$, simplifies the \( \frac{1}{2} \) averagings. Due to the connection, (IV.13), between $\Omega_j$ and the
averaged generating functions, the modulation equations are then
analyzed by \( \Omega^2 \) and represented in an invariant form (IV.18),
\[
(1) (2) \quad \Omega = \Omega_{\Omega} - \Omega_{\Omega} = 0 . \quad \text{Expanding } \Omega = 0 \text{ at } E - E_k = \xi^2 \sim 0 , \quad \text{the Riemann invariants (IV.19) are derived. Evaluating } \int_{\Omega} \Omega = 0 , \quad \text{the conservations of waves, (IV.20), are obtained. For } x\text{-periodic } q_N , \quad \text{there are no spatial or temporal modulations without external perturbations.}

To analyze the linearized instability of the exact N-phase wavetrain \( q_N \), double points \( E_d \) are opened up (to order \( \varepsilon \)) and the linearized \( \mu \)-dynamical equations are investigated. The Floquet means of the linearized \( \mu \)-equations are shown to be the integrals of \( \Omega^2 \), (V.14). From which, the unstable modes are identified and associated to the nonreal double points \( E_d \), and all linearized growth rates are explicitly calculated, i.e., (V.16).

**VI.2 Future Research**

Numerical studies by Overman, McLaughlin and Bishop [64] have conclusively indicated that the damped, driven sine-Gordon equation,
\[
q_{tt} - q_{xx} + \sin q = \varepsilon [-\alpha q_t + r \cos(\omega t)] , \quad (VI.1)
\]
with periodic boundary conditions, has chaotic temporal dynamics, while, at the same time, the spatial structure is coherent. This temporal chaos is determined by the usual time series analysis,
including Poincaré sections, Lyapunov exponents, and the Grassberger-Proccaccio dimension of the attractor. The spatial coherence is quantified via a precise measurement of the s-G periodic mode content in the spatial profile at each instant in time. That is, the full Pde (VI.1) is integrated numerically and, at each time step, the resulting $q(x,t)$ is placed into the Takhtajan-Faddeev eigenvalue problem, from which $\Delta(E;q(x,t))$ is computed numerically, and then the periodic spectrum is calculated. Even in parameter regimes of $\epsilon$, $\alpha$, $\Gamma$ where the t-flow is chaotic, there are only a low, finite number (2-8) of s-G modes excited.

The natural question is: how can an integrable equation, under small perturbations, become chaotic? According to modern dynamical systems theory, existence of homoclinic orbits in the unperturbed problem is a setup for chaos under small perturbations. Following the analysis of Ercolani, Forest and McLaughlin on sine-Gordon [80], we want to use Bäcklund transformations to explicitly find homoclinic orbits in the periodic NLS phase space. These orbits are realized as the global $\mu$-paths associated to complex double points. We will then investigate the perturbations of these orbits, under damping and driving, via the Melnikov technique [81]. This work will require extensive numerical calculations.
Lemma: Suppose $f_j \in \mathbb{R}$ and $g_j = -h_j^*$, then

(i) $g_{j-1} = h_{j-1}^*$, \hspace{1cm} (A.1)
(ii) $f_{j-1} \in \mathbb{R}$, \hspace{1cm} (A.2)

Recall (III.17a), $g_{jx} = -2ig_{j-1} + 2qf_j$, $h_{jx} = 2ih_{j-1} - 2rf_j$, $r = q^*$. This implies $g_{jx} + h_{jx} = -2i(g_{j-1} + h_{j-1}^*) + 2q(f_j - f_j^*)$, i.e., (A.1).

To prove (A.2), we recall (III.18),

$$f^2 - gh = \prod_{\ell=1}^{2N} (E - E^\ell_k) = \sum_{\ell=0}^{2N} K^\ell_k E^\ell_k . \hspace{1cm} (A.3)$$

Since $E^*_{2k-1} = E^*_{2k}$, consequently, $K^\ell_k \in \mathbb{R}$, $0 \leq \ell \leq N$.

Expanding $f^2 - gh$ with $f = \sum_{j=0}^{2N} f_j E^j$, $g = \sum_{j=0}^{2N} g_j E^j$, $h = \sum_{j=0}^{2N} h_j E^j$,

(III.16), we find

$$K^N_{2N-j} = 2f_j f_N - j + \sum_{\ell=0}^{2N-j} f_k f_{N-k} + \sum_{\ell=1}^{N-1} \sum_{k=0}^{N-j} (g_{j,k} + g_{j,k}^*) \in \mathbb{R} . \hspace{1cm} (A.4)$$

$N-j < \ell, k \leq N-1 \hspace{1cm} N-j < \ell, k \leq N-1 \hspace{1cm} k+\ell = 2N-j \hspace{1cm} \ell+k = 2N-j$
Suppose $f_N, \ldots, f_{N-j+1} \in \mathbb{R}$ and $g_{\ell} = -h_{\ell}^*$, $N-j+1 \leq \ell \leq N-1$, then, since $K_{2N-j} \in \mathbb{R}$ and $g_{\ell}h_{k} + g_{k}h_{\ell} = -2\text{Re}(g_{\ell}g_{k}^*)$, $N-j+1 \leq \ell, k \leq N-1$, we find, from (A.3,4), $f_{N-j} \in \mathbb{R}$, i.e., (A.2).

We now combine (III.19a) and (A.1,2), and use Mathematical Induction, so (III.22a,b) is completely verified.

We now prove (III.22c). Recall (III.16, 19a, 21),

$$g = (-i\epsilon)^{\frac{N-1}{2}} (E-\mu_j) = f_{N-1}(\sum_{0}^{\frac{N-1}{2}} \frac{g_{\ell}}{g_{N-1}} E^\ell)$$

Then, for real $E$,

$$-g(x,t;E) = -i\epsilon^\frac{N-1}{2} (E-\mu_j^*) = h_{N-1}(\sum_{0}^{\frac{N-1}{2}} \frac{h_{\ell}}{h_{N-1}} E^\ell).$$

Here, we use the fact $g_{\ell} = -h_{\ell}^*$. Consequently, for real $E$,

$$-g(x,t;E) = h(E) = \sum_{0}^{\frac{N-1}{2}} h_{\ell} E^\ell = -i\epsilon^\frac{N-1}{2} (E-\mu_j^*) .$$

Now, for complex $E$,

$$g(E^*) = (-i\epsilon)^{\frac{N-1}{2}} (E^*-\mu_j), \quad g(E^*) = i\epsilon^\frac{N-1}{2} (E-\mu_j^*),$$

i.e.,

$$-g(E^*) = -i\epsilon^\frac{N-1}{2} (E-\mu_j^*) = h(E).$$
Here we use the fact that \( h \) is continuous in \( E \). This completes the proof of (III.22).
APPENDIX B

PROOF OF (III.23,24)

Proof of (III.23a): recall (III.17a, 21),

\[ g_x = 2qf + (-2iE)g , \quad (B.1) \]

\[ g = -i \eta \prod_{k=1}^{N-1} (E - \mu_k) . \quad (B.2) \]

Inserting (B.2) into (B.1) yields

\[ (-i \eta_x) \prod_{k=1}^{N-1} (E - \mu_k) + (-i \eta) \sum_{j \neq k} \prod_{k=1}^{N-1} (E - \mu_j) \]

\[ = 2qf + (-2iE)g . \quad (B.3) \]

Let \( E = \mu_p \) in (B.3), (III.23a) is obtained.

Proof of (III.24a): from (III.16,21), we find

\[ g_{N-2} = i \eta \sum_{k=1}^{2N-1} \mu_k . \quad (B.4) \]

From (III.19a),

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\[ g_{N-2} = \frac{1}{2} q_x + \frac{i}{2} q \cdot \sum_{1}^{2N} E_k. \]  

(B.5) and (B.5) yield (III.24a).

Proof of (III.23b): recall (III.17b),

\[ g_t = 2(iq_x + 2E_q)f + 2i(qr - 2E^2)g. \]  

(B.6)

Inserting (B.2) into (B.6) yeilds

\[
\begin{align*}
(-iq_t) \prod (E-\nu_j) + (-iq) \sum_{1}^{N-1} (-\nu_k) \prod (E-\nu_j) \\
&= 2(1q_x + 2Eq)f + 2i(qr - 2E^2)g.
\end{align*}
\]  

(B.7)

Let \( E = \nu_p \) in (B.7), we find

\[
(\mu_p) = \frac{2}{\prod_{j \neq p} (\nu_p - \nu_j)} \cdot \frac{(iq_x + 2\nu_p q)}{\nu_p} \cdot f(\nu_p). \]  

(B.8)

Inserting (III.24a) into (B.8) yields (III.23b).

Proof of (III.24b): recall (III.17b, 19a, 21), \( g_{N-1} = -iq \),

\[ g_{N-3} = (-iq) \sum_{i > j} \nu_i \nu_j, \]

\[ q_t = (g_{N-1} g_t = i[2iq_x f_{N-1} + 4qf_{N-2} + 2iqrg_{N-1} - 4ig_{N-3}]. \]  

(B.9)
Since \( f_{N-1} = -\frac{1}{2} \cdot \sum_{k=1}^{2N} E_k \) and \( f_{N-2} = \frac{1}{2} \left[ \sum_{i>j} E_i E_j - qr - \frac{1}{4} \left( \sum_{k=1}^{2N} E_k \right)^2 \right], \)

we insert into (B.9) and yield (III.24b).
APPENDIX C

HOLOMORPHIC AND ANTIHOLOMORPHIC INVOLUTIONS: PROOF OF (III.29, 30)

We define the antiholomorphic involution \( \sigma \) on the Riemann surface \((E, R(E))\), \( R^2(E) = \pi (E - E_k) \),

\[ \sigma(E, R) = (E^*, R^*). \quad (C.1) \]

Clearly \( \sigma^2 \equiv \text{identity} \) and \( R^*(E) = \pm R(E) \). Either taking + or - signs gives the same answer, we now take + sign. Next, we define "pull back" of the differentials \( \{dU_j\} \). Recall (III.26),

\[ dU_j = \sum_{1}^{N-1} C_{jk} \frac{E^{N-k-1}}{R(E)} \, dE, \quad (C.2) \]

ten the pull back is

\[ \sigma^*(dU_j) = \sum_{1}^{N-1} C_{jk} \frac{(E^*)^{N-k-1}}{R(E^*)} \, dE^* . \quad (C.3) \]

Consequently,

\[ (\sigma^*(dU_j))^* = \sum_{1}^{N-1} C_{jk} \frac{E^{N-k-1}}{R(E)} \, dE . \quad (C.4) \]
Let \( \gamma \) be a one cycle on the Riemann surface. Then
\[
(f_{\gamma} dU_j)^* = \int_{\gamma} (\sigma (dU_j))^*,
\]
which can be seen from the Riemann sum of the integral. Since \( dU_j \) is holomorphic, so is \((\sigma (dU_j))^\star\), which then can be expanded on a holomorphic basis. We recall "a-b cycles" on this Riemann surface. From the cut structure of E-plane and a-b cycles construction, as in Figure 4, we find
\[
\sigma(a_j) = -a_j, \quad \sigma(b_k) = b_k + \sum_{j \neq k} a_j.
\]
Recall (III.26, 27), we find
\[
(f_{a_i} dU_j)^* = \int_{a_i} dU_j = \delta_{ij} = \int_{a_i} \sum_{k=1}^{N-1} \frac{C_{jk} E}{R(E)} dE,
\]
and
\[
(f_{a_i} dU_j)^* = \int_{\sigma(a_i)} (\sigma (dU_j))^* = \int_{-a_i} (\sigma (dU_j))^*.
\]
From (C.3, 7) we find
\[
C_{jk} = -C_{jk}^*, \quad \text{i.e.,} \quad C_{jk} \in i\mathbb{R},
\]
and
\[
dU_j = -(\sigma (dU_j))^*.
\]
This verifies (III.29), i.e., (C.8).

Recall (III.28), $B_{ij} = \int b_{ij} \, du_j$, then

\[
(f_{b_{ij}})^* = \int \sigma(b_{ij})(\sigma(du_j))^* = \int b_{ij} + \sum_{k \neq i} a_k
\]

\[
= \int b_{ij} \, du_j - \int_{N-1} \sum_{k \neq i} a_k \delta_{kj}.
\]

Consequently,

\[
(B_{ij})^* = -B_{ij} - \sum_{k \neq i} \delta_{kj},
\]

i.e., $B_{jj} \in i\mathbb{R}$ and $\text{Re}(B_{ij}) = -\frac{1}{2}$ for $i \neq j$, which is (III.30).
APPENDIX D

PROOF OF (III.33)

Proof of (III.33a): recall (III.26, 32, 23a). We now calculate

\[ (\xi_j) \]

\[ N-1 \sum_{k=1}^{N-1} \sigma_{\chi} (1 \frac{\mu_k}{\prod_{p \neq k} (\mu_k - \mu_p)}) \]  \hspace{1cm} \text{(D.1a)}

\[ = (-2i) \sum_{k=1}^{N-1} \sigma_{\chi} (1 \frac{\mu_k}{\prod_{p \neq k} (\mu_k - \mu_p)}) \]  \hspace{1cm} \text{(D.1b)}

The Lagrange Interpolation formula is

\[ \sum_{k=1}^{N-1} \frac{\mu_k}{\prod_{p \neq k} (\mu_k - \mu_p)} = \delta_{N-1-v,N-2}, \hspace{0.5cm} 1 \leq N-1-v \leq N-2, \hspace{1cm} \text{(D.2a)} \]

\[ = \begin{cases} 1 \text{ if } v = 1, \\ 0 \text{ otherwise}. \end{cases} \hspace{1cm} \text{(D.2b)} \]

From (D.1, 2), we have verified (III.33a), i.e.,

\[ (\xi_j) = -2i \sigma_{\chi} j_1 \]  \hspace{1cm} \text{(D.3)}
Proof of (III.33b): recall (III.23b). We now calculate \((\varepsilon_j)_t\).

\[
(\varepsilon_j)_t = \sum_{1}^{N-1} C_{jv} \left( \sum_{1}^{N-1-v} \frac{\mu_k}{R(\mu_k)} (\mu_k)_t \right) \quad (D.4a)
\]

\[
= (-2i) \sum_{1}^{N-1} C_{jv} \left[ \sum_{1}^{N-1-v} \frac{\mu_k}{(N-1)} \left( \sum_{1}^{N-1} \frac{E_{i} - 2 \sum_{\ell \neq k} \nu_{\ell}}{\prod (\mu_k - \mu_p)} \right) \right] \quad (D.4b)
\]

\[
= (-2i) \cdot \left( \sum_{1}^{N-1} E_{i} \right) \cdot \left[ \sum_{1}^{N-1} C_{jv} \cdot \left( \sum_{1}^{N-1-v} \frac{\mu_k}{\prod (\mu_k - \mu_p)} \right) \right] \quad (D.4c)
\]

\[
+ (4i) \cdot \sum_{1}^{N-1} C_{jv} \left[ \sum_{1}^{N-1-v} \frac{\mu_k}{\prod (\mu_k - \mu_p)} \cdot \sum_{1}^{N-1} \frac{\nu_j - \nu_k}{\prod (\mu_k - \mu_p)} \right] \quad (D.4d)
\]

\[
= -2i \left( \sum_{1}^{N} E_{i} \right) C_{jv} + 4i \cdot \sum_{1}^{N-1} C_{jv} \left[ \sum_{1}^{N-1-v} \frac{\mu_k}{\prod (\mu_k - \mu_p)} \cdot \sum_{1}^{N-1} \frac{\nu_j}{\prod (\mu_k - \mu_p)} \right] \quad (D.4d)
\]

\[
- 4i \sum_{1}^{N-1} C_{jv} \left[ \sum_{1}^{N-1-v} \frac{\mu_k}{\prod (\mu_k - \mu_p)} \right] \quad (D.4d)
\]
We notice that the last term in RHS of (D.4f) is for $v \geq 2$, so $N-v < N-2$, and the Lagrange Interpolation formula applies. We find

\[
\xi_{jt} = -2i(\sum_{1}^{2N} E_j j_1 c_j) + 4i(\sum_{1}^{N-1} \mu_j)c_j j_1
\]

\[
- 4i C_{j1} \left( \sum_{1}^{N-1} \frac{\mu_k}{\prod_{p \neq k} (\mu_k - \mu_p)} \right) - 4i \sum_{v=2}^{N-1} C_{jv} \delta_{N-v,N-2}
\]

\[
= -2i(\sum_{1}^{2N} E_j j_1 c_j) - 4i C_{j2} .
\]

This verifies (III.33b), i.e., (D.5b).
APPENDIX E

PROOF OF THEOREM III.9

First, we evaluate $\beta$. Recall (III.44b),

$$
\beta_j = \sum_{1}^{N-1} \int_{\mu_k}^{\mu_k^*} d\mu_j, \quad 1 \leq j \leq N-1.
$$

Let $\gamma$ be the path from $\mu_k^*$ to $\mu_k$ as given in Figure 9.1.

![Diagram of a path $\gamma$ from $\mu_k^*$ to $\mu_k$.](image)

$\gamma :$ from $\mu_k^*$ to $\mu_k$.

Fig. 9.1

We use the integral technique in Appendix C and find
\[
(f_\gamma dU_j)^* = \int_{\sigma(\gamma)} (\sigma^*(dU_j))^* = -\int_{-\gamma} dU_j = \int_{\gamma} dU_j,
\] (E.2)

i.e.,

\[
\int_{\gamma} dU_j \in \mathbb{R},
\] (E.3)

which implies, from (E.1), \( \beta_j \in \mathbb{R} \).

Now, we evaluate \( d \). Recall (III.39f),

\[
d_j = \int_{-\infty}^{\infty} dU_j, \quad 1 \leq j \leq N-1.
\] (E.4)

Let \( \gamma \) be the path from \( \infty^+ \) to \( \infty^- \) as given in Figure 9.2,

\[\gamma: \text{from } \infty^+ \text{ to } \infty^- .\]

\[\text{Fig. 9.2}\]

then
\[ \int_{-\infty}^{\infty} dU_j = \sum_{1}^{N-1} \left( -\frac{1}{2} \right) a_k dU_j + \int_{\gamma}^{N-1} dU_j - \sum_{1}^{N-1} \left( -\frac{1}{2} \right) a_k \]

where \( \gamma_1 = \gamma - \sum_{1}^{N-1} \left( -\frac{1}{2} \right) a_k \) is a real path.

\[ (\int_{\gamma_1} dU_j)^* = \int_{\sigma(\gamma_1)} dU_j - \int_{\gamma_1} dU_j, \]

i.e.,

\[ \int_{\gamma_1} dU_j \in i \mathbb{R}, \]

from which (E.5b) implies \( \text{Re}(\int_{-\infty}^{\infty} dU_j) = \frac{1}{2} \). We now evaluate \( \text{Re}(\xi_j - \frac{1}{2} \beta_j) \). Recall (III.39e, 44b),

\[ \xi_j = \frac{1}{2} \beta_j - \sum_{1}^{N-1} \left( \int_{-\infty}^{\infty} dU_j dU_k + \int_{1}^{N-1} \mu_k dU_j \right), 1 \leq j \leq N-1. \]

\[ \beta_j = \sum_{1}^{N-1} \left( \int_{-\infty}^{\infty} dU_j \mu_k \right), 1 \leq j \leq N-1. \]
Then

\[ \frac{1}{2} \beta_j = \frac{1}{2} \beta_{jj} - \sum_{1}^{N-1} a_k (\int_0^p dU_j) dU_k \]  

\((E.10)\)

\[ + \sum_{1}^{N-1} \Re \mu_k \int_0^p dU_j + \sum_{1}^{N-1} \Re \mu_k - \frac{1}{2} \sum_{1}^{N-1} \mu_k \]

First step: we evaluate \( \sum_{1}^{N-1} a_k (\int_0^p dU_j) dU_k \) in \((E.10)\). For each \( k \),

\[ [\int a_k (\int_0^p dU_j) dU_k]^* = \int \sigma(a_k) [\sigma^* [\int \gamma_j dU_j dU_k]]^* \]  

\((E.11)\)

Let \( \gamma_p \) be any fixed path from \( \omega^+ \) to \( p \), then

\[ (E.11) = \int \sigma(a_k) [\sigma^* [\int \gamma_p dU_j dU_k]]^* \]  

\((E.12a)\)

\[ = \int \sigma(a_k) [(\int \gamma_p dU_j) * \sum_{1}^{N-1} C_{kv} E \frac{dE^*}{R(E^*)}]^* \]  

\((E.12b)\)

\[ = \int_a a_k [\int_{\gamma_p} (\sigma^* (dU_j))^* (-dU_k)] \]  

\((E.12c)\)

\[ = \int_a a_k [\int_{\gamma_p} (-dU_j)](-dU_k) \]  

\((E.12d)\)
i.e.,

\[ \int_{a_k}^{\infty} (\int_{j}^{p} dU_j) dU_k \in \mathbb{R}. \quad (E.12e) \]

**Second step:** we now evaluate \[ N-1 \sum_{l=1}^{N-1} \int_{+}^{+} dU_j \] in (E.10). For each \( k \), let \( \gamma \) be the path from \( \infty^+ \) to \( \text{Re} \mu_k^o \) as given in Figure 9.3,

\[ \gamma : \text{from } \infty^+ \text{ to } \text{Re} \mu_k^o. \]

**Fig. 9.3**
The path $\gamma_1$ is a real path, as in (E.6),

$$\int_{\gamma_1} \, dU \in i\mathbb{R}.$$  \hspace{1cm} (E.14)

(E.13, 14) implies

$$\text{Re}(\sum_{k=1}^{N-1} \Re \circ \mu_k) = \frac{1}{2} \text{ or } 0. \hspace{1cm} (E.15)$$

Third step: we now evaluate

$$\sum_{k=1}^{N-1} \int_{\Re \circ \mu_k} \, dU_j - \frac{1}{2} \sum_{k=1}^{N-1} \int_{\Re \circ \mu_k} \, dU_j$$

in (E.10). Fixed $k$, consider the paths $\gamma_1$ and $\gamma_2$ from $\circ^{\ast} \mu_k$ to $\Re \circ \mu_k$ and $\Re \circ \mu_k$ to $\circ \mu_k$ respectively as given in Figure 9.4,
\( \gamma_1 \): from \( \mu_k^* \) to \( \Re \mu_k \); \( \gamma_2 \): from \( \Re \mu_k \) to \( \mu_k \).

Fig. 9.4

\[
\int_{\mu_k^*}^{\mu_k} dU_j \frac{1}{2} \int_{\mu_k^*}^{\mu_k} dU_j = \int_{\gamma_2}^{\gamma_2} dU_j - \frac{1}{2} \int_{\gamma_1+\gamma_2}^{\gamma_2} dU_j ,
\]

(E.16)

and

\[
(\int_{\gamma_2}^{\gamma_2} dU_j)^* = \int_{\sigma(\gamma_2)}^{\gamma_2} dU_j = \int_{-\gamma_1}^{-\gamma_1} dU_j = \int_{\gamma_1}^{\gamma_1} dU_j .
\]

(E.17)

(E.16, 17) implies

\[
\int_{\gamma_2}^{\gamma_2} dU_j - \frac{1}{2} \int_{\gamma_1+\gamma_2} dU_j = \frac{1}{2} (\int_{\gamma_2}^{\gamma_2} dU_j - \int_{\gamma_1}^{\gamma_1} dU_j )
\]

(E.18a)

\[
= \frac{1}{2} (\int_{\gamma_2}^{\gamma_2} dU_j - (\int_{\gamma_2}^{\gamma_2} dU_j)^*)
\]

(E.18b)

\[
= i \Im(\int_{\gamma_2}^{\gamma_2} dU_j)
\]

(E.18c)
i.e.,

\[
\int_{y_2} dU_j - \frac{1}{2} \int_{y_1+y_2} dU_j \in i\mathbb{R}, \quad (E.19)
\]

i.e.,

\[
\sum_{\mu_k}^{N-1} \left( \int_{\Re U_j} dU_j - \frac{1}{2} \int_{\Re U_j} dU_j \right) \in i\mathbb{R}, \quad 1 \leq j \leq N-1. \quad (E.20)
\]

We now use the fact \( B_{jj} \in i\mathbb{R} \) and (E.10, 12e, 15, 20) and find

\[
\Re(\ell_j - \frac{1}{2} \beta_j) = \frac{1}{2} \text{ or } 0. \quad (E.21)
\]
APPENDIX F

PROOF OF (IV.7, 8, 9)

By observing (IV.5), we need to evaluate \( \langle \prod (E - \mu_j) \rangle \),

\[
\begin{align*}
\langle \sum \mu_j \prod (E - \mu_j) \rangle, & \quad \langle \sum \mu_i \mu_j \prod (E - \mu_j) \rangle, \quad \langle (\sum \mu_j)^2 \cdot \prod (E - \mu_j) \rangle, \\
\langle \sum \mu_j \prod (E - \mu_j) \rangle, & \quad \langle \sum \mu_j \prod (E - \mu_j) \rangle. 
\end{align*}
\]

Lemma F.1 (A Useful Formula)

Let \( G \) be a \( g \times g \), \( g = N-1 \), matrix with \( G_{ij} = \mu_i^{g-j} \). Then

\[
\prod (\mu_i - \mu_j) = \det G = \sum_{i<j} \text{sign}(i_1, \ldots, i_g) \mu_i^{i_1} \mu_2^{i_2} \cdots \mu_g^{i_g}, \quad (F.1)
\]

where \( (i_1, \ldots, i_g) \) is a premutation of \((g-1, g-2, \ldots, 2, 1, 0)\), and sign is positive "+" if even, otherwise, negative "-".

Lemma F.2 (First Fundamental Formula for the Averaging)

\[
\langle \ast \rangle = \frac{1}{\det M} \int_a^g \left( \prod_{i<j} (\mu_i - \mu_j) \right) \frac{d\mu}{\prod R(\mu_j)}, \quad g = N-1. \quad (F.2)
\]
Proof of Lemma F.2: from (IV.2b, 6),

\[<\star> = \frac{1}{(2\pi)^{N-1}} \int_{\mathbb{R}^{N}} (\star) \prod_{\mu} \left| \frac{\partial \theta}{\partial \mu} \right| d\mu \]  

\[= \int_{\mathbb{R}^{N}} (\star) \frac{\det M}{\det \mathcal{P}} d\mu \]  

\[= \left[ \int_{\mathbb{R}^{N}} (\star) \det G \frac{d\mu^+}{g} \right] \frac{1}{\det M} \]  

\[= \left[ \int_{\mathbb{R}^{N}} (\star) \prod_{i < j} (\mu - \mu_j) \frac{d\mu^+}{g} \right] \frac{1}{\det M}. \]  

Lemma F.3 (Second Fundamental Formula for Averaging)

Assume that \( F \) is a function of \( \mu = \{\mu_k\}_{k=1}^{g} \), then

\[<F(\mu) \prod (E-\mu_j)> \frac{1}{R(E)} = \frac{g}{p(\xi)} \sum_{j=0}^{2N} F_j \xi^j, \]  

where \( \xi = \frac{1}{E} \) and \( p(\xi) = \sqrt{\prod_{k=1}^{2N} (1 - E_k \xi)} \), and \( \{F_j\} \) are given in the proof.
Proof of Lemma F.3:

\[ <F(y) \prod (E - u) > \frac{1}{R(E)} \]

\[ = \frac{\varepsilon}{p(\xi)} \frac{1}{\det M} \int_{\mathbb{A}} [F(\mu) \prod (1 - u \cdot \xi)] \prod_{i<j} (\mu_i - \mu_j) \frac{d\mu}{\prod R(\mu_j)} \]  

\[ = \frac{\varepsilon}{p(\xi)} \frac{1}{\det M} \int_{\mathbb{A}} [F(\mu) \prod (\mu_i - \mu_j)] \prod_{i<j} (1 - u \cdot \xi) \frac{d\mu}{\prod R(\mu_j)} \]  

\[ = \frac{\varepsilon}{p(\xi)} \frac{1}{\det M} \sum_{j=0}^g F_j \xi^j, \]  

where

\[ F_0 = \frac{1}{\det M} \int_{\mathbb{A}} F(\mu) \prod (\mu_i - \mu_j) \frac{d\mu}{\prod R(\mu_j)} \]  

\[ = \frac{1}{\det M} \int_{\mathbb{A}} F(\mu) \det G \cdot \frac{d\mu}{\prod R(\mu_j)} \]  

and

\[ F_k = (-1)^k \frac{1}{\det M} \int_{\mathbb{A}} F(\mu) \sum_{i_1 \leq \ldots \leq i_k} \mu_{i_1} \mu_{i_2} \ldots \mu_{i_k} \det G \frac{d\mu}{\prod R(\mu_j)} \]  

\[ 1 \leq k \leq g. \]
Definition F.4

(i) \( G^{(j,k)} \) is identical to \( G \) except that the \( j \)-th column is \((\mu_1, \mu_2, \ldots, \mu_g) \).

(ii) \( G^{(p_1, p_2, \ldots, p_g)} \) is a \( g \times g \) matrix where the \( j \)-th column is \((\mu_1, \mu_2, \ldots, \mu_g) \), \(1 \leq j \leq g\).

Lemma F.5

(i) Let \( \mu_k \cdot \text{sign}(i_1, i_2, \ldots, i_g) \cdot \mu_1 \ldots \mu_g \) be an element of \( N^{-1} \). Let \( \det G \). Then no other element in \( \sum_{1 \leq j \leq N-1} \mu_j \cdot \det G \) can cancel it iff \( i_1 = g-1 \).

(ii) Let \( \mu_k \mu_l \cdot \text{sign}(i_1, \ldots, i_g) \mu_1 \ldots \mu_g \) be an element of \( g \). Let \( \det G \). Then no other element in \( \sum_{i<j} \mu_i \mu_j \cdot \det G \) can cancel it iff \( \{i_1, i_l\} = \{g-1, g-2\} \).

(iii) Let \( \mu_k \cdot \text{sign}(i_1, \ldots, i_g) \mu_1 \ldots \mu_g \) be an element of \( g \). Let \( \det G^{(1,g)} \) where \((i_1, \ldots, i_g)\) is a permutation of \((g, g-2, \ldots, 1, 0)\). Then no other element of \( \sum_{1 \leq j \leq g} \mu_j \cdot \det G^{(1,g)} \) can cancel it iff \( i_k = g \) or \( i_k = g-2 \).
(iv) Let $\mu_1 \cdots \mu_g \sign(i_1, \cdots, i_g) \mu_1 \cdots \mu_g$ be an element of $\sum_{j_1}^{g} \cdots \mu_{j_k} \cdot \det G$. Then no other element in $\sum_{j_1}^{g} \cdots \mu_{j_k} \cdot \det G$ can cancel it iff $\{j_1, \cdots, j_k\} = \{g-1, \cdots, g-k\}$.

Remarks on the Proof of Lemma F.5: we now prove (i) in detail. Similar arguments apply to (ii), (iii), and (iv).

Let $\nu_k \sign(i_1, \cdots, i_g) \mu_1 \cdots \mu_g$ be an element of $\sum_{j_1}^{g} \nu_j \cdot \det G$. Suppose $i_k \neq g-1$, then $0 \leq i_k \leq g-2$. Let $i_0 = i_k + 1$. Then $1 \leq i_0 \leq g-1$. The translation between the two permutations,

\begin{align*}
(i, \cdots, i_k, \cdots, i_0, \cdots, i) \\
\text{(F.11a)}
\end{align*}

and

\begin{align*}
(i, \cdots, i_0, \cdots, i_k, \cdots, i) \\
\text{(F.11b)}
\end{align*}

need perform $2|j_0 - k + 1| - 3$ permutations. Consequently, the two elements,

\begin{align*}
\sign(i_1, \cdots, i_k, \cdots, i_0, \cdots, i_g) \nu_k \cdot \nu_1 \cdots \nu_{i_0} \cdots \nu_g \\
\text{(F.12a)}
\end{align*}

and

\begin{align*}
\sign(i_1, \cdots, i_0, \cdots, i_k, \cdots, i_g) \nu_{i_0} \cdot \nu_1 \cdots \nu_{i_k} \cdots \nu_g \\
\text{(F.12b)}
\end{align*}

vanishes each other, i.e., for $i_k \neq g-1$, all $(\pm)\nu_k(\nu_1 \cdots \nu_{i_k} \cdots \nu_g)$.
dissapear in \( \sum_{j=1}^{g} \mu_j \cdot \det G \).

On the other hand, for \( i = g-1 \),

\[
\text{sign}(i_1, \ldots, i_g) \mu_k \cdot \mu_1 \cdots \mu_g = \text{sign}(i_1, \ldots, i_g) \mu_1 \cdots \mu_k \cdots \mu_g.
\]  

(F.13)

There is no other element in \( \sum_{j=1}^{g} \mu_j \cdot \det G \) can cancel this element since

\[
0 \leq i < g-2 \quad \text{for} \quad p \neq k, \quad \text{and} \quad 1 \leq i+1 < g-1.
\]  

(F.14)

This completes the proof of Lemma F.5(i).

We now use Lemma F.5 to evaluate the basic elements stated in the beginning of this Appendix. It is straightforward and the results are in

Lemma F.6

(i) \[
\sum_{1 \leq i < j \leq g} \mu_i \cdot (\mu_i - \mu_j) = \sum_{j=1}^{g} \mu_j \cdot \det G = \det G^{(1,g)}.
\]  

(ii) \[
\sum_{i > j \leq 1 \leq g} \mu_i \cdot \mu_j \cdot \det G = - \det G^{(2,g)}.
\]  

(F.15)

(F.16)

(iii) \[
(\sum_{i < j} \mu_i) \cdot (\sum_{i < j} \mu_i - \mu_j) = (\sum_{i < j} \mu_i) \cdot (\sum_{i < j} \mu_i) \cdot \det G
\]

\[
= (\sum_{i < j} \mu_i) \cdot \det G^{(1,g)} = \det G^{(1,g+1)} - \det G^{(2,g)}.
\]  

(F.17)
(iv) \[ \sum_{1 \leq j_1 < j_2 < \ldots < j_k \leq g} \mu_{j_1} \mu_{j_2} \cdots \mu_{j_k} \prod_{i \leq j} (\mu_i - \mu_j) = (-1)^{k-1} \det G(k,g). \quad (F.18) \]

(v) \[ \sum_{i > j} \mu_i \mu_j \prod_{i \leq j} (\mu_i - \mu_j) \sum_{1 \leq i_1 < \ldots < i_k \leq g} \mu_{i_1} \cdots \mu_{i_k} = \det G(2,1,\ldots,1,0,\ldots,0) + \det G(2,\ldots,2,0,\ldots,0) \]

\[ + \det G(1,\ldots,1,0,\ldots,0) \quad (F.19a) \]

\[ = (-1)^{k-2} \det G(2,0\ldots0,k,0\ldots0) + (-1)^{k-1} \det G(k,g+1) \]

\[ + (-1)^{k+1} \det G(k+2,g). \quad (F.19b) \]

(vi) (Special Case of (v)) \[ \sum_{i > j} \mu_i \mu_j \prod_{i \leq j} (\mu_i - \mu_j) \sum_{i_1} \mu_{i_1} = -\det G(2,g+1) + \det G(3,g). \quad (F.20) \]

(vii) (Special Case of (v)) \[ \sum_{i > j} \mu_i \mu_j \prod_{i \leq j} (\mu_i - \mu_j) \sum_{i_1} \mu_{i_1} = (-1)^{g-2} \det G(2,0,\ldots,0,g). \quad (F.21) \]
(viii) $\left( \sum_{j} \mu_j \right)^2 \cdot \prod_{i,j} (\mu_i - \mu_j) \sum_{1 < i_1 < i_2 < \ldots < i_k < g} \mu_{i_1} \ldots \mu_{i_k} = (-1)^{k-1} \det G_{(k,g+2)} + (-1)^k \det G_{(k+1,g+1)} + (F.19b)$. \hspace{1cm} (F.22)

(ix) (Special Case of (viii))

$g \left( \sum_{j} \mu_j \right)^2 \cdot \prod_{i,j} (\mu_i - \mu_j) (\sum_{j} \mu_j) = \det G_{(1,g+2)} + (-1) \det G_{(2,g+2)} + (F.20)$. \hspace{1cm} (F.23)

(x) (Special Case of (viii))

$g \left( \sum_{j} \mu_j \right)^2 \cdot \prod_{i,j} (\mu_i - \mu_j) (\mu_1 \ldots \mu_g) = (-1)^{g-1} \det G_{(g,g+2)} + \det G_{(2,2,1,...,1)}$. \hspace{1cm} (F.24)

Lemma F.7 (Averagings of these basic elements)

(i) $g \left( \prod_{j} (E_{-\mu_j}) \right) \frac{1}{R(E)} = \left\{ E^g - \sum_{0} \frac{\det M_{(g-j,g)}}{\det M} E^j \right\} \frac{1}{R(E)}$. \hspace{1cm} (F.25)

(ii) $g \sum_{j} \mu_j \cdot \prod_{j} (E_{-\mu_j}) \frac{1}{R(E)} = \left( \sum_{1} \frac{\det M_{(N-j,g)}}{\det M} E^j \right) - \left( \sum_{0} \frac{\det M_{(g-j,N)}}{\det M} E^j \right) \frac{1}{R(E)}$. \hspace{1cm} (F.26)

(iii) $\left\{ \sum_{j} \mu_i \mu_j \cdot \prod_{j} (E_{-\mu_j}) \right\} = \left\{ \sum_{0} A^{-j} E^j \right\} \frac{1}{R(E)}$, \hspace{1cm} (F.27a)

where

$$A_0 = - \frac{\det M_{(2,g)}}{\det M}, \hspace{1cm} A_g = \frac{\det M_{(2,0,...,0,g)}}{\det M}.$$ \hspace{1cm} (F.27b)
and

\[ A_k = \frac{1}{\det M} \left\{ \det M_{(2,0,\ldots,0,k,0\ldots0)} - \det M^{(k,N)} - \det M^{(k+2,g)} \right\}, \]

\[ 1 \leq k \leq g-1. \]  

(F.27c)

(iv) \[ \langle \sum_j \mu_j \rangle^2 \prod_{i=1}^{g} \left( E - \mu_j \right) \frac{1}{R(E)} = \left\{ \sum_{i=0}^{g} B_{-i} E^i \right\} \frac{1}{R(E)}, \]  

(F.28a)

where

\[ B_0 = A_0 + \frac{\det M^{(1,N)}}{\det M}, \quad B_g = A_g - \frac{\det M^{(g,N+1)}}{\det M}, \]  

(F.28b)

and

\[ B_k = A_k - \frac{\det M^{(k,N+1)}}{\det M} + \frac{\det M^{(k+1,N)}}{\det M}, \quad 1 \leq k \leq g-1. \]  

(F.28c)

(v) \[ \langle \sum_j \mu_j \rangle^2 \prod_{i=1}^{g} \left( E - \mu_j \right) \frac{1}{R(E)} = \left\{ \sum_{i=0}^{g} C_{-i} E^i \right\} \frac{1}{R(E)}, \]  

(F.29a)

where

\[ C_j = B_j - 2A_j. \]  

(F.29b)

(vi) \[ \left[ \sum_{i>j} \mu_i \mu_j + \langle \mu_j \rangle^2 \right] \prod_{i=1}^{g} \left( E - \mu_j \right) \frac{1}{R(E)} = \left\{ \sum_{i=0}^{g} p_{-i} E^i \right\} \frac{1}{R(E)}, \]  

(F.30a)

where
\[ p_j = A_j + C_j = B_j - A_j = \frac{\det M^{(1,N)}}{\det M} , \quad j = 0 . \quad (F.30b) \]

\[ = - \frac{\det M^{(j,N+1)}}{\det M} + \frac{\det M^{(j+1,N)}}{\det M} , \quad 1 \leq j \leq g-1 . \quad (F.30c) \]

\[ = - \frac{\det M^{(g,N+1)}}{\det M} , \quad j = g . \quad (F.30d) \]

Remarks on the Proof: we prove (F.25) in detail. Similar arguments apply to the remaining equations. From (F.2, 7), we have

\[ \left< \prod_j (E - \mu_j) \right> \frac{1}{R(E)} = \frac{\varepsilon}{p(x)} \sum_0^{g} R_0 \xi^j , \quad (F.31a) \]

where

\[ R_0 = \left[ \int_{\tilde{a}} \prod_{i<j} (\mu_i - \mu_j) \frac{d\mu}{g} \prod_{i<j} R(\mu_j) \right] \frac{1}{\det M} \quad (F.31b) \]

\[ = \left[ \int_{\tilde{a}} \det G \frac{d\mu}{g} \prod_{i<j} R(\mu_j) \right] \frac{1}{\det M} \quad (F.31c) \]

\[ = \left[ \det \left( \int_{\tilde{a}} \frac{g^{-j} \frac{d\mu}{R(\mu)}}{\mu_i} \right) \right] \frac{1}{\det M} = 1 . \quad (F.31d) \]
Consequently,

\[
R_k = \left\{ \int_a^\infty \det G \sum_{i_1 < \ldots < i_k} \left. \frac{d^+}{g \prod R(\mu_j)} \right|_{i_1}^{i_k} \right\} (-1)^k \frac{\det M}{\det M} \quad (F.31e)
\]

\[
= \left\{ \int_a^\infty \det (k) \frac{1}{g \prod R(\mu_j)} \right\} (-1)^k \frac{\det M}{\det M} \quad (F.31f)
\]

\[
= (-1)^k \det \left( \int_a^\infty \frac{g-1}{\sum R(\mu)}(k) \frac{1}{\det M} \right) \quad (F.31g)
\]

\[
= (-1)^k \frac{\det M}{\det M}(k), \quad 1 \leq k \leq g. \quad (F.31h)
\]

Consequently,

\[
\left< \prod_{\mu} (E-\nu_j) \right> \frac{1}{R(E)} = \frac{\xi}{p(\xi)} \left[ 1 - \sum \frac{\det M}{\det M}(k) \cdot \xi \right] \quad (F.32a)
\]

\[
= \frac{1}{R(E)} \left[ E^g - \sum_0^{g-1} \frac{\det M}{\det M}(g-k, g) \cdot E^k \right]. \quad (F.32b)
\]

This completes the proof of (F.25). We now insert (F.25, 26, 27, 28, 29, 30) into (IV.5). The results are (IV.7, 8).

We now verify (IV.9). To show $D_j^{(\pm)}$ are real, it is enough to show that $M_{ij}$ and $M_{ij}^{(\ell, k)}$ have purely imaginary values. We use the integral technique in Appendix C. Each element (entry) of $M$ and $M^{(\ell, k)}$ has the form

\[
\int_a^\infty \frac{E^j}{R(E)} \quad (F.33)
\]
and

\[(\int_{a_i} E^j \frac{dE}{R(E)})* = \int_{\sigma(a_i)} [\sigma^* (E^j \frac{dE}{R(E)})]^* = \int_{-a_i} [E^j * \frac{dE^*}{R(E^*)}]* \]  

\[= \int_{-a_i} E^j \frac{dE}{R(E)}, \]  

i.e.,

\[\int_{a_i} E^j \frac{dE}{R(E)} \in i\mathbb{R}, \]  

i.e.,

\[\frac{\det M^{(2,k)}}{\det M} \in \mathbb{R}. \]  

This completes the proof of (IV.9).
APPENDIX G

PROOF OF (IV.13)

Proof of (IV.13a): recall (IV.12c), \( \int_{a_j}^{\Omega^{(1)}} = 0 \). Consequently,

\[
\sum_{0}^{N} D^{(1)} \int_{a_i}^{E^j} \frac{dE}{R(E)} = 0, \quad 1 \leq i \leq N-1, \quad (G.1)
\]
i.e.,

\[
\sum_{0}^{N-1} D^{(1)} \int_{a_i}^{E^j} \frac{dE}{R(E)} = -D^{(1)} \int_{a_i}^{E^N} \frac{dE}{R(E)} - D^{(1)} \int_{a_i}^{E^{N-1}} \frac{dE}{R(E)},
\]

\(1 \leq i \leq N-1\), \quad (G.2)
i.e.,

\[
\begin{align*}
D^{(1)}_{N-2} \int_{a_1}^{E^N} \frac{dE}{R(E)} &- \int_{a_1}^{E^g} \frac{dE}{R(E)} \\
M &+ \frac{1}{2} \sum_{k=1}^{2N} E_k \int_{a_1}^{E^g} \frac{dE}{R(E)} \\
D^{(1)}_0 \int_{a_g}^{E^N} \frac{dE}{R(E)} &- \int_{a_g}^{E^g} \frac{dE}{R(E)}
\end{align*}
\]

where \( g = N-1 \). Consequently,
\[ D_j^{(1)} = -\frac{\det M_{N-j+1,N}}{\det M} + \frac{1}{2} \sum_{k=1}^{2N} \frac{\det M_{N-j,g}}{\det M}, \quad 0 \leq j \leq N-2. \quad (G.4) \]

This completes the proof of (IV.13a).

Proof of (IV.13b): again, we recall (IV.12c), \( \int_{a_i} \Omega^{(2)} = 0 \). This implies

\[ M = -D_{N+1}^{(2)} V^{(N+1)} - D_N^{(2)} V^{(N)} - D_{N-1}^{(2)} V^{(N-1)}, \quad (G.5) \]

where

\[ V_i^+(\ell) = \int_{a_i} E \frac{dE}{R(E)}, \quad \ell = N+1, N, N-1. \quad (G.6) \]

From (G.5, 6), we find (IV.13b).
APPENDIX H

PROOF OF (IV.15)

The Riemann bilinear identity for the abelian differentials of the first and second kind implies

\[
\sum_{1}^{N-1} \left[ \int_{a_j}^{} dU_k \int_{b_j}^{} \Omega^2 - \int_{a_j}^{} \Omega^2 \int_{b_j}^{} dU_k \right] = \int_{\Gamma} (\int_{E}^U dU_k) \Omega^2, \tag{H.1}
\]

\[1 \leq k \leq N-1,\]

where the integration path \( \Gamma \) is the boundary of the normal polygon of the Riemann surface of \( R(E) \). Since \( \int_{a_j}^{} dU_k = \delta_{jk} \) and \( \Omega^2 \) has only poles at \( E = \infty^\pm \), we find

\[
\int_{b_j}^{} \Omega = 2\pi i \left[ \text{Res}(\int_{E}^U dU_j) \Omega^2; E = \infty^+ \right] + \text{Res}(\int_{E}^U dU_j) \Omega^2; E = \infty^- \right] \right], \tag{H.2}
\]

From (III.26), the local expansion of \( \int_{E}^U dU_j \) as \( E \) near \( \pm \) are

\[
\int_{E}^U dU_j \sim (\pm) \sum_{1}^{N-1} C_j \left( -\frac{1}{E} - \frac{1}{2(1+v)} E^{-(1+v)} + ... \right), \tag{H.3}
\]

as \( E \) near \( \pm \).
For $E = \frac{1}{\xi} = \pm$, then

$$\int_E^{dU_j} \sim (\pm) \sum_{j=1}^{N-1} C_{j\nu} \left( -\frac{1}{\nu} \xi^\nu + \frac{1}{-2(1+\nu)} \xi^{1+\nu} + \ldots \right),$$

as $E \to -\infty$. From (IV.14),

$$\left( \int_E^{dU_j} \right)^{(1)} \sim C_{j1} \frac{d\xi}{\xi} + \text{holomorphic part, as } E = \frac{1}{\xi} \text{ near } -\infty,$$

$$\left( \int_E^{dU_j} \right)^{(2)} \sim \left[ \sum_k E_k C_{j1} + 2C_{j2} \right] \frac{d\xi}{\xi} + \text{holomorphic part},$$

as $E = \frac{1}{\xi} \text{ near } -\infty$.

Consequently, from (H.2, 5, 6) and (III.39C), we find

$$\int_{b_j} \Omega^{(1)} = 4\pi i C_{j1} = -k_j,$$

$$\int_{b_j} \Omega^{(2)} = 4\pi i \left[ \sum_{k=1}^{2N} E_k C_{j1} + 2C_{j2} \right] = -\omega_j,$$

for $1 \leq j \leq N-1$.

This completes the proof of (IV.15).
APPENDIX I

PROOF OF (IV.18)

Assume that (IV.18a) is satisfied, i.e.,

\[ \langle \mathfrak{F}_j \rangle_T + \langle \mathfrak{X}_j \rangle_X = 0, \quad j = 2, \ldots, N+1. \]  \hspace{1cm} (I.1)

From (IV.17b), we have

\[ \Omega \sim \sum_{N+2}^{\infty} \left[ \langle \mathfrak{F}_j \rangle_T - \langle \mathfrak{X}_j \rangle_X \right] \xi^{j-2} \, d\xi \quad \text{as} \quad \xi = \frac{1}{E} \sim \pm \infty. \]  \hspace{1cm} (I.2)

We also recall the basic properties of \( \Omega \) from (IV.12c, 17),

(i) \( \Omega \) has the only singularities occurred at \( \{E_k\}^{2N}_1 \). \hspace{1cm} (I.3)

(ii) \( \Omega \) has at most double poles at \( \{E_k\}^{2N}_1 \). \hspace{1cm} (I.4)

(iii) \( \Omega \) vanishes at a-cycles: \( \int_{a_i} \Omega = 0, \quad 1 \leq i \leq N-1 \). \hspace{1cm} (I.5)

Lemma I.1

The space of ableian differentials satisfying (I.2, 4) has complex dimension \( N-1 \).

Proof of Lemma I.1: our proof is based on the Riemann-Roch theorem. Define a divisor \( \delta \),

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\[ \delta = \left( \prod_{k=1}^{2N} E_k^{-2} \right) \cdot (\omega^+)^{2N} \cdot (\omega^-)^{2N}. \quad (I.6) \]

It is clear \( d(\delta) = 0 \). Define

(i) \( L(\delta^{-1}) \) = space of meromorphic functions with at least double zeros at \( \{E_k\}_{k=1}^{2N} \) and at most \( 2N \)-fold poles at \( \infty^\pm \). \quad (I.7)

(ii) \( r(\delta^{-1}) = \dim(L(\delta^{-1})) \). \quad (I.8)

(iii) \( \mathcal{L}(\delta) \) = space of Abelian differentials with at most double poles at \( \{E_k\}_{k=1}^{2N} \) and at least \( 2N \)-fold zeros at \( \infty^\pm \). \quad (I.9)

(iv) \( i(\delta) = \dim(\mathcal{L}(\delta)) \). \quad (I.10)

We want to show \( i(\delta) = N-1 \). By the Riemann-Roch theorem,

\[ r(\delta^{-1}) = d(\delta) + i(\delta) - (N-1) + 1 = i(\delta) - N \quad (\text{since } d(\delta) = 0). \quad (I.11) \]

We now show \( r(\delta^{-1}) = 1 \), i.e., \( i(\delta) = N-1 \). Let \( f \in L(\delta^{-1}) \). Let \( g(E) = \frac{f(E)}{\prod_{k=1}^{2N} (E-E_k)^{-1}} \). Since \( f \) has double zeros at \( \{E_k\} \), \( g \) has no finite poles. Also \( f \) has at most \( 2N \)-fold poles at \( \infty^\pm \), say \( j \)-fold finite poles for \( f \), we consider \( E = \frac{1}{\xi} \sim \infty^\pm \),
\[ g(E) = \frac{f(E)}{2N} \sum_{k=1}^{2N} (\pm)(C_{-j}^E - j + C_{-j+1}^E - j+1 + \ldots) \]

\[ \times (\xi^{2N} + K\xi^{2N+1} + \ldots) \quad \text{as} \quad E = \frac{1}{\xi} \sim \pm , \]

i.e.,

\[ g(E) \sim (\pm)(C_{-j}^E \xi^{2N-j} + \ldots) \quad \text{as} \quad E = \frac{1}{\xi} \sim \pm , \quad (I.12b) \]

with

\[ 2N-j \geq 0 \quad (\text{since} \quad j \leq N) . \quad (I.12c) \]

Consequently, \( g \) has no pole at \( E = \pm \), from which, if \( j < 2N \), then \( g \equiv 0 \), and if \( j = 2N \), then \( g \equiv \text{constant} \) (By Liouville theorem). Thus \( L(\delta^{-1}) \) is spanned by \( \prod_{k=1}^{2N} (E-E_k) \). We have

\[ r(\delta^{-1}) = \dim(L(\delta^{-1})) = 1 , \quad \text{i.e.}, \quad i(\delta) = N-1 . \quad (I.13) \]

This completes the proof of Lemma I.1.

Next, we use Lemma I.1 and (1.5) to show \( \int_B \Omega = 0 \), from which we can prove \( \Omega \equiv 0 \).
Lemma 1.2 (Ω vanishes at b-cycles)

\[ \int_{b_j} \Omega = 0, \quad 1 \leq j \leq N-1. \]

Proof of Lemma 1.2: we define

\[ z_j = E_j^{-1} \frac{dE}{R^3(E)}, \quad j = 1,2, \ldots, N-1. \tag{I.14} \]

It is clear that \( \{z_j\}_{j=1}^{N-1} \) are linearly independent. Moreover, \( z_j \) satisfies

\[ z_j \in \mathcal{E}(\delta), \quad 1 \leq j \leq N-1. \tag{I.15} \]

We now verify (I.15). Let \( \xi^2 = E - E_k \) for fixed \( k \). Then local expansion at \( \xi^2 = E - E_k = 0 \) is

\[ z_j \sim \{E_j^{-1} \text{ constant} \cdot \frac{d\xi}{\xi^2} + \text{holomorphic part}\} \tag{I.16} \]

as \( \xi^2 = E - E_k \sim 0 \).

Moreover, for \( E = \frac{1}{\xi} \) near \( \mp \),

\[ z_j \sim (\pm)(\xi^{-3N-j-1} + d_1 E^{-j} + \ldots)(-d\xi) \quad \text{as} \quad E = \frac{1}{\xi} \sim \mp. \tag{I.17} \]
Since \( 1 \leq j \leq N-1 \), so \( 3N-j-1 \geq 3N-N = 2N \). Consequently, \( z_j \) has at least \( 2N \)-fold zero at \( E = \infty^\pm \), so \( z_j \in \mathcal{L}(\delta) \). We complete the proof of (I.15). Now, \( \{z_j\}_{j=1}^{N-1} \) form a basis of \( \mathcal{L}(\delta) \). Let

\[
\Omega = \sum_{j=1}^{N-1} p_j z_j ,
\]

for certain choice of constant \( \{p_j\}_{j=1}^{N-1} \). Let

\[
q_j = E^{j-1} R(E) dE .
\]

From the Riemann bilinear identity,

\[
\sum_{j=1}^{N-1} \left[ \int_{a_j}^{b_j} q_j \Omega - \int_{a_j}^{b_j} \Omega q_j \right] = \int_{\partial R} (f^\Omega q_j) \Omega ,
\]

where \( \partial R \) is the boundary of the normal form of the Riemann surface \( R \). From (I.5), we have

\[
\sum_{j=1}^{N-1} \int_{a_j}^{b_j} q_j \Omega = \int_{\partial R} (f^\Omega q_j) \Omega .
\]

We observe \( (f^\Omega q_j) \Omega \) at \( E = \infty^\pm \) and \( E = \frac{1}{\xi} \), we find \( q_j \) has pole of order \( (j-1+N) \) at \( \infty^\pm \), so \( f^\Omega q_j \) has pole of order \( (j+N) \) at \( \infty^\pm \), i.e., \( (f^\Omega q_j) \Omega \) has no pole at \( E = \infty^\pm \) since \( \Omega \) has at least \( 2N \)-fold zero at \( E = \infty^\pm \). Next, we consider \( E \) at \( E_k \).
\[ \int_{\mu} q_j \text{ is finite and } \Omega = \sum_{j=1}^{N-1} P_j z_j \text{ with } z_j \text{ having pole of order } 2 \]

at \( E_k \) without residue. By Cauchy theorem,

\[ \oint (\int_{\mu} q_j) \Omega = 0 , \quad (I.22) \]

consequently, from (I.21),

\[ \sum_{j=1}^{N-1} \oint_{a_i} \oint_{b_i} q_j \Omega = 0 . \quad (I.23) \]

Since

\[ \det(\oint_{a_i} q_j) \neq 0 , \quad (I.24) \]

we find

\[ \oint_{b_i} \Omega = 0 , \quad 1 \leq i \leq N-1 . \quad (I.25) \]

This completes the proof of Lemma 1.2.

We now know \( \Omega \) vanishes at \( a \) and \( b \)-cycles, so \( \Omega \) is exact, i.e., there is a meromorphic function \( f \) such that

\[ \Omega = df \text{ on } \Gamma . \quad (I.26) \]
Since $\Omega$ has at most double pole at $E_k$, $f$ has at most a simple pole at $E_k$. Since $\Omega$ has at least $2N$-fold zero at $E = \infty^\pm$, $f$ has at least $(2N+1)$-fold zero at $E = \infty^\pm$. Consider $G = R(E)f$. Then $G$ has no finite poles, and at $E = \infty^\pm$, $G$ has zeros of order at least $N+1$. By Liouville theorem, $G \equiv 0$, so $f \equiv 0$, i.e., $\Omega \equiv 0$. This completes the proof of (IV.18).
APPENDIX J

PROOF OF (V.14)

We do \( \mu_N \). Similar argument applies to \( \mu_{N+1} \). We use multiphase averaging technique and find

\[
(i) \quad \frac{1}{N-1} \prod \left( E_d - \mu_j \right) = \frac{\text{det } \mathcal{M}}{\text{det } M},
\]

\[
(ii) \quad \frac{1}{N-1} \prod \left( E_d - \mu_j \right) = \frac{\text{det } \mathcal{M}'}{\text{det } M},
\]

where \( M, \mathcal{M}, \mathcal{M}' \), are \((N-1) \times (N-1)\) matrices with

\[
M_{ij} = \int_{a_i}^{E_{N-1-j}} \frac{dE}{R_N(E)},
\]

\[
\mathcal{M}_{ij} = \int_{a_i}^{E_{N-1-j}} \frac{dE}{E - E_d R_N(E)},
\]

\[
\mathcal{M}'_{ij} = \mathcal{M}_{ij} \quad \text{for } j \neq 1 \quad \text{and} \quad \mathcal{M}'_{i1} = \int_{a_i}^{E_{N-1}} \frac{dE}{E_d R_N(E)}.\]

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We perform elementary row transformations to find a similar matrix of $\mathcal{M}$ with each $i$-th row given by

$$
\left( \begin{array}{c}
\frac{u}{E_d} \\
\frac{d\mu}{R(\mu)} \\
\frac{d\mu}{R(\mu)} \\
\ldots \\
\frac{d\mu}{R(\mu)}
\end{array} \right),
\int a_i^{N-3} \frac{d\mu}{R(\mu)}, \ldots, \int a_i \frac{d\mu}{R(\mu)},
$$

(J.6)

and still call $\mathcal{M}$. We also define a matrix $\mathcal{M}''$ such that each $i$-th row is given by

$$
\left( \begin{array}{c}
\frac{u}{E_d} \\
\frac{d\mu}{R(\mu)} \\
\frac{d\mu}{R(\mu)} \\
\ldots \\
\frac{d\mu}{R(\mu)}
\end{array} \right),
\int a_i^{N-2} \frac{d\mu}{R(\mu)}, \int a_i^{N-4} \frac{d\mu}{R(\mu)}, \ldots, \int a_i \frac{d\mu}{R(\mu)}.
$$

(J.7)

By straightforward calculation with $\mathcal{M}$ given by (J.6), we find

$$
\det \mathcal{M}' = E_d \cdot \det \mathcal{M} - \det \mathcal{M}'',
$$

(J.8)

from which, we find

$$
\frac{1}{N-1} \prod_{j=1}^{N-1} (E_d - \mu_j^0) \qquad = \frac{\det \mathcal{M}}{\det M},
$$

(J.9)

$$
\frac{1}{N-1} \prod_{j=1}^{N-1} (E_d - \mu_j^0) \sum_{j=1}^{N-1} \mu_j^0 \qquad = E_d \frac{\det \mathcal{M}}{\det M} - \det \mathcal{M}'' \frac{\det \mathcal{M}}{\det M}.
$$

(J.10)
We now consider the following algebraic system of \( \{x_i\}_{i=1}^{N-1} \),
\[
\left( \int_{a_i}^{E_{N-2}} \frac{dE}{R_N(E)} \right) x_1 + \left( \int_{a_i}^{E_{N-3}} \frac{dE}{R_N(E)} \right) x_2 + \cdots + \left( \int_{a_i}^{E} \frac{dE}{R_N(E)} \right) x_{N-1} = \int_{a_i}^{E} \frac{d\mu}{E_d - \mu}, \quad 1 \leq i \leq N-1.
\]
(J.11)

Let \( \mathbf{x} = (x_1, \ldots, x_{N-1})^T \) be the solution of (J.11), we find
\[
x_1 = \frac{\det \mathbf{M}}{\det M} \quad \text{and} \quad x_2 = \frac{\det \mathbf{M}''}{\det M}.
\]
(J.12)

Consequently, from (J.9, 10),
(i) Floquent exponent \( = (-2i)R_N(E_d)x_1 \).
(J.13)
(ii) Floquent exponent \( = (-2i)R_N(E_d) \left[ \sum_{k=1}^{2N} E^0_k x_1 + 2x_2 \right] \).
(J.14)

We now define an Abelian differential on the Riemann surface of \( R_N(E) \),
\[
\mathbf{w}_3 = \left[ \frac{\mu}{\mu - E_d} + x_1 \mu^{N-2} + x_2 \mu^{N-3} + \cdots + x_{N-1} \mu \right] \frac{d\mu}{R_N(\mu)}.
\]
(J.15)

By straightforward calculation, we find \( \mathbf{w}_3 \) is an Abelian differential of the third kind whose only singularities are simple poles at \( \mu = E_d^\pm \), and the local representation near \( E - E_d^\pm = \xi = 0 \) is
\[ w_3 \sim \pm \left( \frac{1}{R_N(E_d)} \frac{d\xi}{\xi} + \text{holomorphic part} \right), \quad E - E_d^\pm = \xi \text{ near } 0, \quad (J.16) \]

and

\[ \int_a^w w_3 = 0, \quad 1 \leq i \leq N-1, \quad (J.17) \]

and the local representation of \( w_3 \) near \( E = \frac{1}{\xi} = \pm \) is

\[ w_3 \sim (\pm) \left[ \sum_{j=0}^{\infty} C_j \xi^j d\xi \right], \quad E = \frac{1}{\xi} \text{ near } \pm, \quad (J.18a) \]

where

\[ C_0 = -x_1 \quad \text{and} \quad C_j = (-\frac{1}{2})^j \sum_{k=1}^{2N} E_k^0 x_1 - x_2. \quad (J.18b) \]

We now use Riemann bilinear identities of second and third kind

\[ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \]

for \( \Omega \) and \( w_3 \) and find

\[ 2\pi i \cdot \text{Res}(w_3; \mu = E_d) \int_{(E_d, -R_N)}^{(E_d, R_N)} \frac{1}{\Omega} \frac{1}{(E_d, -R_N)} = -\int_{(E_d, -R_N)}^{(E_d, R_N)} \frac{1}{\Omega} \frac{1}{(E_d, -R_N)} \quad (J.19a) \]

\[ = -2\pi i \left[ \text{Res}(\int w_3 \Omega; \xi = \pm) + \text{Res}(\int w_3 \Omega; \xi = \mp) \right]. \quad (J.19b) \]

Since
\[ \int_{E} w_3 = \pm \left\{ \left( -x_1 \xi \right) + \frac{1}{2} \left[ -\frac{1}{2} \left( \sum_{1}^{2N} E_k^0 \right) x_1 - x_2 \right] \xi^2 + \ldots \right\} \quad \text{as } E = \frac{1}{\xi} \sim \pm, \]

(J.20)

and

\[ \Omega^1 - \left( \pi \left( \frac{1}{\xi} \right) d\xi + \text{holomorphic part} \right) \quad \text{as } E = \frac{1}{\xi} \sim \pm, \quad \text{(J.21)} \]

and

\[ \Omega^2 - \left( \pi \left( \frac{4}{\xi^3} \right) d\xi + \text{holomorphic part} \right) \quad \text{as } E = \frac{1}{\xi} \sim \pm, \quad \text{(J.22)} \]

we find

\[ \text{Res} \left( (\int_{E} w_3)^{\Omega^1} ; E = \pm \right) = x_1, \quad \text{(J.23)} \]

and

\[ \text{Res} \left( (\int_{E} w_3)^{\Omega^2} ; E = \pm \right) = \left( \sum_{1}^{2N} E_k^0 \right) x_1 + 2x_2. \quad \text{(J.24)} \]

From (J.13, 14, 19, 23, 24), (V.14) is a straightforward result.
APPENDIX K

PROOF OF THEOREM V.3

First, we show that the $x$-flow of $\mu_N^{(1)}$ has purely imaginary Floquet mean, i.e., $\lim_{t \to \infty} \int_{(E_d,-R_N)}^{(E_d,R_N)} \omega \, \Omega^{(1)} = i k_{N+1}$, i.e., $\Omega^{(1)}$ is purely imaginary. But, from (IV.15),

$$\int_{(E_d,-R_N)}^{(E_d,R_N)} \omega \, \Omega^{(1)} = \lim_{E_d \to \infty} \int_{E_N+j + E_d}^{E_{N+1}} \omega \, \Omega^{(1)} = i k_{N+1}, \quad (K.1)$$

where $k_{N+1}$ is the induced fixed wave number (since $q_{N+1}$ is spatial periodic with same period as $q$ for all $q_{N+1}$). This proves all linearized $x$-flows, $\mu_N^{(1)}(x)$, are stable.

Second, we evaluate the $t$ Floquet exponent, i.e., $\int_{(E_d,-R_N)}^{(E_d,R_N)} \omega \, \Omega^{(2)}$. Let $\sigma$ be the antiholomorphic involution on $\mathbb{R}$ as in Appendix C,

$$[\sigma (\Omega^{(2)})(x)]^* = \frac{N+1}{\sum_j D_j^{(2)}} E^* \frac{dE^*}{R (E)^*} = \frac{N+1}{\sum_j D_j^{(2)}} E^* \frac{dE}{R (E)} = \Omega^{(2)}, \quad (K.2)$$

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where we use the fact that $D_j^{(2)} \in \mathbb{R}$ and $\Re (E) = \Re (E^*)$ as in Appendix C.

We now evaluate

$$\text{Im}(\int_{(E_d, R)}^{(E_d, -R)} \Omega^{(2)}),$$

which determines the real part of the Floquet means, i.e., the linearized growth rate. Consider a path $\gamma$ from $E_d^-$ to $E_d^+$, as given in Figure 10.1 where $\{E_0, E_0^*\}$ is the nearest branch pair of $E_d$.

Then $2i \text{Im}[(\int_{[y]}^{(2)} \Omega^{(2)})] = \int_{[y]}^{(2)} \Omega^{(2)} - (\int_{[y]}^{(2)})^* = \int_{[y]}^{(2)} \Omega^{(2)} - \int_{\sigma([y])}^{(2)} \Omega^{(2)}$

$$= \int_{[y-\sigma([y])]}^{(2)}.$$

Also, $\sigma(\Gamma_2) = \Gamma_2$, $\sigma(\Gamma_4) = \Gamma_4$ and $\sigma(\Gamma_3) = \Gamma_3^*$ and $\Gamma_3 - \sigma(\Gamma_3)$ is a $-\Gamma$-cycle. Consequently, (K.3) becomes
\[ 2i \text{Im}(\int_{\gamma_1}^{\Omega(2)}) = [\int_{r_1 - \sigma(r_1)} + \int_{r_5 - \sigma(r_5)}]^{\Omega(2)} = 2 \int_{E_d, R_N}^{(*)} \Omega(2), \quad (K.4) \]

which can be seen from Figure 10.2,

\begin{center}
\begin{tikzpicture}
    \draw[->] (-2,0) -- (2,0);
    \draw[->] (0,-2) -- (0,2);
    \draw[dashed] (0,0) -- (0,2);
    \draw[dashed] (0,0) -- (2,0);
    \node at (0,2) {$E_d$};
    \node at (2,0) {$+$};
    \node at (0,-2) {$-$};
    \node at (2,0) {$+$};
    \node at (0,2) {$E_d$};
    \node at (0,0) {$x$};
    \node at (-1,0) {$r_1$};
    \node at (1,0) {$r_5$};
    \node at (0,-1) {$-\sigma(r_1)$};
    \node at (0,1) {$-\sigma(r_5)$};
    \node at (0,0) {$x$};
    \node at (0,0) {$\gamma_1$};
    \node at (0,0) {$\gamma_5$};
\end{tikzpicture}
\end{center}

Fig. 10.2

path : \( \gamma_1 - \sigma(\gamma_1) + \gamma_5 - \sigma(\gamma_5) \).

From (K.4), it is clear that when

\[ E_d \in R, \text{Im} \left[ \int_{E_d, R_N} \Omega(2) \right] = 0, \text{ i.e., the t-flow of } \nu_N^{(1)} \text{ is stable}, \quad (K.5) \]

We now evaluate (K.4) for \( E_d \notin R \). Since \( \Omega^{(2)} \) has only finite zeros on the Riemann surface of \( R_N(E) \), we always can choose a homotopic path of \( \gamma_5 - \sigma(\gamma_5) \), say \( \gamma_1 \), such that \( \Omega^{(2)} \) does not
vanish on $\gamma_1$; consequently, the integral (K.4) is monotone on this path, and therefore,

$$\text{for } E_d \notin \mathbb{R}, \text{ Im}\left[\int (E_d^* R_N) \Omega^{(2)} \right] \neq 0,$$

i.e., the t-flow of (K.6) is unstable. 

Combining (K.1, 5, 6), this completely verifies Theorem V.3. Moreover, if $\{E_d^1, E_d^2\}$ are two pairs of nonreal double points,

$$\text{Re}(E_d^1) = \text{Re}(E_d^2),$$

then t-growth rate of $E_d^1 >$ t-growth rate of $E_d^2$ iff

$$\text{Im}(E_d^1) > \text{Im}(E_d^2).$$
APPENDIX L

INFINITE-PRODUCT REPRESENTATION OF $\Delta^2(E) - 4$

Recall the Floquet discriminant $\Delta(E)$, (II.8b). We may assume that $\Delta^2(E) - 4$ has at most double zeros and at most countable zeros (for such problems, see McKean [57], McKean and van Moerbeke [42], Forest and McLaughlin [61]). Also, for convenience, none of the zeros $\{E_k\}_{k=1}^\infty$ vanish and $\{E_k\}_{k=1}^{2N}$ are simple points, $\Sigma_s(q)$.

Lemma L.1

$$\Delta^2(E) - 4 = s(E;E_k) \prod_{k=1}^{2N} (E - E_k) \quad (L.1a)$$

where

$$s(E;E_k) = \sum_{n=2N}^{\infty} E^2 \cdot \left( \prod_{n>2N, k=1} E_k \right)^2 \quad (L.1b)$$

for some constant $s_0$.

Proof of Lemma L.1: Recall (II.26a),

$$\Delta(E) \sim 2 \cos(EL) \text{ as } |E| \to \infty. \quad (L.2a)$$

From (L.2a),

$$\Delta^2(E) - 4 \sim -4 \sin^2(EL) \text{ as } |E| \to \infty. \quad (L.2b)$$
This implies

\[ E_k \sim \pm \frac{n\pi}{L} \quad \text{as} \quad |n| \to \infty. \]  

(L.3)

We define

\[ P_\infty(E) = \prod_{1}^{2N} (1 - \frac{E}{E_k}) \prod_{n>2N}^{\infty} (1 - \frac{E}{E_k})^2. \]  

(L.4)

Since

\[ \sin(EL) \sim E \prod_{1}^{\infty} (1 - \frac{(EL)^2}{n^2\pi^2}) \quad \text{as} \quad |E| \to \infty, \]  

(L.5a)

so

\[ \sin^2(EL) \sim E^2 \prod_{1}^{\infty} (1 - \frac{(EL)^2}{n^2\pi^2})^2 \quad \text{as} \quad |E| \to \infty. \]  

(L.5b)

Consequently,

\[ P_\infty(E) \sim C \frac{\sin^2(EL)}{E^2} \quad \text{as} \quad |E| \to \infty. \]  

(L.6)

From (L.2b), we find

\[ \Delta^2(E) - 4 \sim C_1 E^2 P_\infty(E) \quad \text{as} \quad |E| \to \infty. \]  

(L.7)

By observing, \( \frac{\Delta^2 - 4}{E^2 P_\infty(E)} \) is pole free and bounded everywhere in \( |E| < \infty \). By Liouville theorem, this function is a constant, which
implies

\[ \Delta^2(E) - 4 = C_1 E^2 P_\infty(E) \]  \hspace{1cm} (L.8)

where \( P_\infty(E) \) is given by (L.4). This completes the proof of Lemma L.1, i.e., (III.1b).
APPENDIX M

CONNECTION BETWEEN DATA \( \nu_j \) VARIABLES AND THE "\( \nu \) SPECTRUM"

We recall the transfer matrix \( T(E) \), (II.7) of the Z-S eigenvalue problem, (II.2a,c). Here, we show that

\[
T_{21}(\nu_j) = 0 \text{ iff } g(\nu_j) = 0 \quad 1 < j < N-1 , \tag{M.1}
\]

where \( g \) is the quadratic eigenfunction given in (III.16).

We consider the Z-S eigenvalue problem, (II.2a,c),

\[
\begin{align*}
(\psi_1)_x &= -iE_1 \psi_1 + q \psi_2 \\
(\psi_2)_x &= -r \psi_1 + iE_2 \psi_2 .
\end{align*} \tag{M.2}
\]

Since the system is independent of the initial location \( x = x_0 \), it follows that if \( \psi(x,x_0;E) \) is a solution of (M.2), so is

\[
\frac{\partial}{\partial x_0} \psi(x,x_0;E) = \psi(x,x_0;E) . \tag{M.3}
\]

We also recall the fundamental basis \( \{ \phi(x,x_0;E), \tilde{\phi}(x,x_0;E) \} \), (II.6), of (M.2). From (II.7),

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\[
T(L;x_0;E) = \begin{pmatrix}
\phi_1(x_0 + L, x_0; E) & -\phi_2(x_0 + L, x_0; E) \\
-\phi_1(x_0 + L, x_0; E) & \phi_2(x_0 + L, x_0; E)
\end{pmatrix} \tag{M.4}
\]

and

\[
\Delta(E;x_0) = \phi_1(x_0 + L, x_0; E) - \phi_2(x_0 + L, x_0; E) . \tag{M.5}
\]

For \( \psi(x,x_0;E) \) being a solution of (M.2), we find

\[
\psi(x,x_0;E) = (\psi_1)_{x_0} x_0, \psi(x_0,x_0;E) = \phi(x,x_0;E)
\]

\[
-(\psi_2)_{x_0} x_0, \psi(x_0,x_0;E) = \phi(x,x_0;E) . \tag{M.6}
\]

If \( \psi(x,x_0;E) \) has prescribed data independent of \( x_0 \), i.e.,

\( \psi(x=x_0,x_0;E) = (a,b)^T \) with \( a, b \) independent of \( x_0 \), then

\[
\frac{d}{dx_0} \psi(x_0,x_0;E) = \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) \psi(x,y;E) \bigg|_{x = x_0, y = y_0} = 0 , \tag{M.7a}
\]

i.e.,

\[
\psi(x_0,x_0;E) = -\psi(x,x_0;E) \bigg|_{x = x_0} . \tag{M.7b}
\]

From (M.2),
Since the fundamental basis \( \{ \phi(x,x_0;E), \overline{\phi}(x,x_0;E) \} \) has initial data independent of \( x_0 \), we find, from (M.6, 8),

\[
\phi_{x_0}(x,x_0;E) = iE\overline{\phi}(x,x_0;E) - r(x_0)\overline{\phi}(x,x_0;E),
\]

(M.9)

\[
\overline{\phi}_{x_0}(x,x_0;E) = q(x_0)\phi(x,x_0;E) - iE\phi(x,x_0;E).
\]

We now define

\[
\phi(x_0;E) = \phi(x_0 + L, x_0;E),
\]

(M.10)

\[
\overline{\phi}(x_0;E) = \overline{\phi}(x_0 + L, x_0;E).
\]

From (M.2, 9) and the fact that \( q, r \) are spatial periodic with period \( L \),

\[
\frac{\partial}{\partial x_0} \phi(x_0;E) = \phi_y(y,x_0;E)|_{y=x_0 + L} + \phi_{x_0}(y,x_0;E)|_{y=x_0 + L} \tag{M.11a}
\]

\[
= (q(x_0)\phi_2(x_0;E) - r(x_0)\phi_1(x_0;E))
- [r(x_0)(\phi_1(x_0;E) + \phi_2(x_0;E)) + 2iE\phi_2(x_0;E)] \tag{M.11b}
\]
and
\[
\frac{\partial}{\partial x_0} \phi(x_0;E) = \left( -2iE \phi_1(x_0;E) + q(x_0)(\phi_1(x_0;E) + \phi_2(x_0;E)) \right).
\]
(M.12)

From (M.11, 12),
\[
\frac{\partial}{\partial x_0} \Delta(E;x_0) = \frac{\partial}{\partial x_0} (\phi_1(x_0;E) - \phi_2(x_0;E)) = 0.
\]
(M.13)

We define
\[
F(x_0;E) = \frac{1}{2} (\phi_1(x_0;E) + \phi_2(x_0;E)),
\]
(G(x_0;E) = \bar{\phi}_1(x_0;E),
H(x_0;E) = \phi_2(x_0;E).
(M.14)

By straightforward calculation, we find \{F,G,H\} satisfies the quadratic eigenfunction system, (III.4a). Moreover,
\[
F^2 - GH = \frac{1}{4} (\Delta^2(E) - 4).
\]
(M.15)

From (L.1) and the normalization condition (III.18),
\[
F^2 - GH = \frac{1}{4} \left( \int_{E}^{E_k} \int_{1}^{2N} s(E;E_k^*) \frac{2N}{(E - E_k^*)^2} \right.
\]
\[
= \frac{1}{4} \left. \int_{E}^{E_k} s(E;E_k^*) (f^2 - gh) \right.
\]
(M.16a)
(M.16b)
Thus, \{F,G,H\} are identical to the quadratic eigenfunctions \{f,g,h\} up to a constant multiplier. This completes the proof of (M.1).
LIST OF REFERENCES


     __________, Physica D, 9, 324-331 (1983), paper I and IV to follow.


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