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INVariANTS OF ASSOCIATION SCHEMES

Dissertation

Presented in Partial Fulfillment of the Requirements for
the Degree of Doctor of Philosophy in the Graduate
School of The Ohio State University

By

Nachimuthu Manickam

****

The Ohio State University

1986

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Chapter 1

Introduction

The notion of distribution invariants was first introduced by Thomas Bier [4] while attempting to answer the following questions in the algebra of real numbers:

(1) For what triples \((a,b,c)\) does there exist a real-bilinear map \(f : \mathbb{R}^a \times \mathbb{R}^b \rightarrow \mathbb{R}^c\) with \(\|f(x,y)\| = \|x\| \cdot \|y\|\) (for \(z \in \mathbb{R}^d, \|z\| = z_1^2 + \ldots + z_d^2\))?

(2) For what triples \((a,b,c)\) does there exist a real-bilinear map \(f : \mathbb{R}^a \times \mathbb{R}^b \rightarrow \mathbb{R}^c\) with the property \(f(x,y) = 0 \implies x = 0\) or \(y = 0\) ?

Later in [5], Thomas Bier and Delsarte generalized the definition of distribution invariants to any symmetric association scheme and also derived certain lower and upper bounds in terms of \(T\)-designs of the association scheme. This dissertation is mainly concerned with calculating
the first distribution invariants of the Johnson scheme, the Hamming scheme and the q-analogues of them. In particular, we prove that the upper bound is attained in most of these cases. In this chapter we introduce the necessary basic materials.

In chapter 2 we prove that the first distribution invariant of the Johnson scheme \( J(n,d) \) is bounded between \( (\frac{n-r}{d-1}) \) and \( \binom{n}{d-1} \) if \( n = qd + r \), where \( 0 \leq r < d \). Also we establish that it is equal to \( \binom{n}{d-1} \) if \( n \geq d(d-1)d(d-2)+d(r+1)+d(d-1)^2(d-2) \). As a byproduct, we prove that if \( a_1, a_2, \ldots, a_n \in \mathbb{R} \), \( \sum_{1 \leq i \leq n} a_i = 0 \) and \( \sum_{i \in A} a_i \neq 0 \) for every \( d \)-subset \( A \subset \{1,2,\ldots,n\} \), where \( 2d \leq n \), then there exist at least \( \binom{n}{d-1} \) \( d \)-subsets of \( \{1,2,\ldots,n\} \) having positive sums, provided \( n \) is sufficiently large compared with \( d \). In the case of \( J(n,3) \), the first distribution invariant is proved to be equal to \( \binom{n-1}{2} \) if \( n \geq 11 \). The fact that \( J(n,2) \) has its first distribution invariant equal to \( n-1 \) if \( n \geq 6 \) is also established.

In chapter 3, by using the idea of \( d \)-spreads, we show that the first distribution invariant of the q-analogue Johnson scheme \( J_q(n,d) \) is \( \binom{n-r}{d-1}_q \) if \( d \) divides \( n \), and is bounded, in general, between \( \binom{n-r}{d-1}_q \) and \( \binom{n}{d-1}_q \), if \( n = qd + r \) with \( 0 \leq r < d \). In chapter 4 we show that the first distribution invariant of the Hamming scheme \( H(n,q) \) is \( q^{n-1} \) for all \( q \) and \( n \). It is shown in chapter 5, by using the existence of singleton-systems
in $H_q(n,d)$, that the q-analogue Hamming scheme $H_q(n,d)$ has $q^{n(d-1)}$ as its first distribution invariant for all $q,n$ and $d$. The definition of $W$-complement $d$-spreads and their existence is also discussed in this chapter.

1.1 Association schemes and Distance regular graphs

The definitions and notations used in this section generally follow those used by Bannai and Ito [1].

**Definition 1.1.1:** An association scheme $X = (X,\{R_i\}_{0 \leq i \leq d})$ consists of a finite set $X$ of $n$ points together with $d+1$ relations $R_0, R_1, \ldots, R_d$ defined on $X$ which satisfy

1. $R_0 = \{(x,x) | x \in X\}$
2. $R_0 \cup R_1 \cup \ldots \cup R_d = X \times X$, $R_i \cap R_j = \emptyset$ if $i \neq j$
3. $^tR_i = R_j$ for some $j \in \{0,1,\ldots,d\}$ where $^tR_i = \{(x,y) | (y,x) \in R_i\}$
4. If $(x,y) \in R_k$, then the number of $z$ such that $(x,z) \in R_i$ and $(z,y) \in R_j$ is a constant depending on $i$, $j$, $k$ but not on the choice of $x$ and $y$. This constant is denoted by $p^k_{ij}$. 
An association scheme $X$ is said to be symmetric or of Bose-Mesner type if $^t\mathcal{R}_i = \mathcal{R}_i$ for all $i$. It is called commutative if $p^k_{ij} = p^k_{ji}$ for all $i$, $j$, $k$.

Note: All the association schemes which will be considered in this dissertation are assumed to be symmetric. Henceforth, by an association scheme we will always mean a symmetric association scheme.

The $i^{th}$ adjacency matrix $A_i$ of $X$ is defined to be the $n \times n$ matrix whose rows and columns are indexed by the elements of $X$ and whose $(x,y)$ entries are

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } (x,y) \in \mathcal{R}_i \\ 0 & \text{otherwise} \end{cases}$$

Using the definition of $X$, it can be easily checked that

(i) $A_i$ is symmetric for every $i$

(ii) $A_0 + A_1 + \ldots + A_d = J$ (the all-ones matrix)

(iii) $A_0 = I$

(iv) $A_i A_j = \sum_{0 \leq k \leq d} p^k_{ij} A_k = A_j A_i$ for $i,j = 0,1,\ldots,d$.

Let $\mathcal{A}$ be the algebra generated by $A_0, A_1, \ldots, A_d$ over $\mathbb{R}$. The con-
ditions (i)-(iv) stated above imply that $A$ is a commutative algebra of degree $d+1$ over $\mathbb{R}$, and that every matrix in $A$ is symmetric. $A$ is called the Bose-Mesner algebra of $X$ over $\mathbb{R}$.

Let $X = (X, \{R_i\}_{0 \leq i \leq d})$ be an association scheme. Let us fix an ordering of $X$ and write it as $X = \{x_1, \ldots, x_n\}$. For $x = x_i \in X$, let $e_x = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{R}^n$ with 1 in the $i$th place and 0 elsewhere. Let $V$ be the vector space $\mathbb{R}^X$ generated by $\{e_x\}_{x \in X}$ over $\mathbb{R}$. Since the matrices in the Bose-Mesner algebra are pairwise commutative normal matrices, they can be simultaneously diagonalized by an invertible orthogonal matrix.

Let $A$ act on $V$. Then, using the standard inner product, we can write $V = V_0 \perp V_1 \perp \ldots \perp V_d$ where each $V_i$ is a common eigenspace of $A_0, A_1, \ldots, A_d$. Here we choose $V_i$ to be maximal in the sense that if $i \neq j$, then there exists $A_k$ such that the eigenvalue of $A_k$ on $V_i$ is different from that on $V_j$. Since the matrix $J$ belongs to $A$ and has $\langle (1,1,\ldots,1) \rangle$ as its eigenspace belonging to the eigenvalue $n$, $V_i = \langle (1,1,\ldots,1) \rangle$ for some $i$. Without loss of generality, let us assume that $V_0 = \langle (1,1,\ldots,1) \rangle$.

**Distance regular graphs**

A graph $\Gamma$ is an ordered pair $(V(\Gamma), E(\Gamma))$ consisting of a nonempty
finite set $V(\Gamma)$ of vertices and a subset $E(\Gamma)$, the edge set, of the set of unordered pairs of vertices in $V(\Gamma)$. Any pair of vertices $x$ and $y$ are defined to be adjacent if $\{x,y\} \in E(\Gamma)$. A path of length $n$ from $x$ to $y$, for $x,y \in X$, is a sequence of vertices $x_0 = x, x_1, \ldots, x_{n-1}, x_n = y$ such that $x_{i-1}, x_i$ are adjacent for all $i$ with $1 \leq i \leq n$. The distance $\delta(x,y)$ between $x$ and $y$ is the minimum length among all paths from $x$ to $y$. The maximum of $\{\delta(x,y) \mid x,y \in V(\Gamma)\}$ is called the diameter of the graph $\Gamma$. We denote it by $d$. Let $R_i = \{(x,y) \mid x,y \in V(\Gamma) \text{ and } \delta(x,y) = i\}$. $\Gamma$ is said to be a distance regular graph if $X = (X_i, R_i)_{0 \leq i \leq d}$ becomes an association scheme. $X$ is clearly a symmetric association scheme.

**Lemma 1.1.2** $\Gamma$ is a distance regular graph if and only if $A_i = P_i(A_1)$ for some polynomials $P_i(x)$ of degree $i$ in $\mathbb{R}[x]$.

**Proof** See [1] page 190.

**Definition 1.1.3:** An association scheme $X = (X_i, R_i)_{0 \leq i \leq d}$ is a $P$-polynomial scheme if there exist polynomials $P_i(x)$ of degree $i$ in $\mathbb{R}[X]$ such that $A_i = P_i(A_1)$ for $1 \leq i \leq d$. 
1.2 Distribution Invariants and T-designs

As in the previous section, let us assume that $X = (X, \{R_i\}_{0 \leq i \leq d})$ is a symmetric association scheme and $V$ its corresponding vector space over $\mathbb{R}$. For any vector $w$ in $V$, define

\[ X^+_w = \{ x \in X \mid \langle w, e_x \rangle > 0 \} \]
\[ X^-w = \{ x \in X \mid \langle w, e_x \rangle < 0 \} \]
\[ X^0_w = \{ x \in X \mid \langle w, e_x \rangle = 0 \} \]

Here $\langle ., . \rangle$ denotes the standard inner product in $V$. Thus $X^+_w$, $X^-w$, $X^0_w$ consist of the positive, the negative, and the zero coordinates of the vector $w$.

**Definition 1.2.1:** A vector $w \in V$ is said to be a general vector if and only if $X^0_w = \emptyset$.

**Definition 1.2.2:** If $I \subseteq \{1, 2, \ldots, d\}$, where $d$ denotes the number of classes in the association scheme, then we define the distribution invariant $\nu_t(X)$ to be equal to the minimal cardinality of the sets $X^+_w$, where $w$ ranges over all general vectors in $V_I = \bigcap_{i \in I} V_i$. i.e.

\[ \nu_t(X) = \min \{ |X^+_w| \mid w \in V_I \text{ and } w \text{ general} \}. \]

It is clear that if $I_1, I_2 \subseteq \{1, 2, \ldots, d\}$ and $I_1 \subseteq I_2$, then $\nu_{I_1}(X) \geq \nu_{I_2}(X)$. If $I = \{1, 2, \ldots, d\}$, then $\nu_t(X) = 1$. 
When \( I = \{i\} \) for \( 1 \leq i \leq d \), we denote \( v_{t_i}(X) \) by \( v_{t_i}(X) \) and call it the \( i \)th distribution invariant.

The above definitions and the following results in this section are extracted mainly from the work of T. Bier and Delsarte in [5].

For a subset \( Y \) contained in \( X \), let \( \phi(Y) \) denote the characteristic vector of \( Y \). i.e \( \phi(Y) = \sum_{y \in Y} e_y \).

**Definition 1.2.3:** Let \( T \subseteq \{1,2,...,d\} \). A subset \( Y \subseteq X \) is said to be a \( T \)-design if the characteristic vector \( \phi(Y) \) is orthogonal to the space \( V_T \).

If \( Y \subseteq X \) is a \( T \)-design and if \( w \in V_T \) is any general vector, then
\[
<w, \phi(Y)> = 0 \Rightarrow \sum_{y \in Y} <w, e_y> = 0 \Rightarrow \text{there exists at least one } y \in Y \text{ such that } <w, e_y> > 0.
\]
Hence for any \( T \)-design \( Y \) and a general vector \( w \) in \( V_T \), \( Y \cap X_+(w) \) is non-empty.

(1)

**Theorem 1.2.4** (Bier & Delsarte): If \( Y_I \) and \( Y_J \) represent \( I \) and \( J \) designs, where \( I \subseteq \{1,2,...,d\} \) and \( J = \{1,2,...,d\} \setminus I \), then

(i) \( v_{t_I} \leq |Y_J| \) and

(ii) If \( X \) admits a transitive automorphism group, then \( |X|/|Y_I| \) is less than or equal to \( v_{t_I}(X) \).
Proof: Since $Y_j$ is a J-design, the characteristic vector $\Phi(Y_j)$ belongs to the subspace $V_0 \perp V_1$.

$$\Rightarrow \Phi(Y_j) = \alpha \Phi(X) + w \text{ for some } w \in V_1$$

$$\Rightarrow <\Phi(Y_j), \Phi(X)> = \alpha |X|$$

$$\Rightarrow |Y_j| = \alpha |X| \Rightarrow \alpha = |X|/|Y_j|$$

For any $x$ in $X$, $<w, e_x> = 1-\alpha$ if $x$ is in $Y_j$ and is equal to $-\alpha$ otherwise. If $I \neq \emptyset$, then $\alpha < 1$. Therefore $X_+(w) = Y_j$ and hence $vt_1(X)$ is bounded above by $|Y_j|$.

To prove part(ii), let $w$ belonging to $V_1$ be any general vector. Let $G$ be an automorphism group which acts transitively on $\mathcal{X}$ (i.e $G$ preserves the relations and acts transitively on the vertices). Let $N$ denote the number of triples $(x, y, g)$ where $x \in X_+(w)$, $y \in Y_1$, $g \in G$ and $y^g = x$. On one hand, if we fix $x$ and $y$, then there exists $|G|/|X|$ elements in $G$ satisfying $y^g = x$.

$$\Rightarrow N = |X_+(w)| \cdot |Y_1| \cdot |G|/|X|$$

(2)

On the otherhand, for any $g \in G$, $g(Y_1)$ is also an I-design. Therefore, by (1) we have $g(Y_1) \cap X_+(w) \neq \emptyset$. Hence there exists $y$ in $Y_1$ such that
$y^g = x$ for some $x$ in $X_+(w)$. This implies that for every $g$ in $G$ we have at least one triple and hence we conclude that

$$N \geq |G|$$

(3)

(2) & (3) $\Rightarrow |X_+(w)| \geq |X|/|Y|$

Since this is true for any arbitrary general vector $w$, we get the desired lower bound in (ii).
Chapter 2

The First Distribution Invariant of J(n,d)

2.1 Introduction

Let S be a set of cardinality n and let $X = \{ T \subseteq S \mid |T| = d \}$ with $2d \leq n$. Let $d_j : X \times X \to \mathbb{N} \cup \{0\}$ be the distance function defined by $d_j(T_1, T_2) = d - |T_1 \cap T_2|$. We define the i-th distance relation $\mathcal{R}_i$ on $X$ by

$$\mathcal{R}_i = \{ (T_1, T_2) \mid d_j(T_1, T_2) = i \}.$$ 

Then $\chi = (X, \{\mathcal{R}_i\}_{0 \leq i \leq d})$ becomes a symmetric association scheme and is called the Johnson scheme or the triangular association scheme. Let $V = V_0 \perp V_1 \perp \ldots \perp V_d$ be the orthogonal decomposition of the vector space $V = \mathbb{R}^X$ with each $V_i$ being the maximal common eigenspace of the adjacency matrices of $J(n,d)$ belonging to the eigen-
value \( \lambda_i = d(n-d) - i(n+1) + i^2 \) of the first adjacency matrix \( A_1 \). We can give a certain nice algebraic expression to each \( V_i \) as explained in the next two pages.

Let \( M_{id} \) denote the \( \binom{i}{i} \times \binom{d}{d} \) matrix whose rows are indexed by the \( i \)-subsets of \( S \), whose columns are indexed by the \( d \)-subsets of \( S \), and where the entry in row \( Y \) and column \( T \) is 1 if \( Y \subseteq T \) and is 0 otherwise. Thus \( M_{0d} \) is the row vector of all ones of length \( \binom{d}{d} \) and \( M_{dd} \) is the identity matrix of order \( \binom{d}{d} \). Let \( U_i \) denote the row space of \( M_{id} \) for \( i = 0,1, \ldots, d \).

Lemma 2.1.1: \( U_{i-1} \subseteq U_i \) for \( 1 \leq i \leq d \).

Proof: Let \( X_i \) be the set of all \( i \)-subsets of \( S \), for \( 1 \leq i \leq d \), and for \( x \in X_d \) let \( x \in \mathbb{R}_d^{\binom{d}{d}} \) be the row \( x \) of \( M_{dd} \). For \( y \in X_i \), denote by \( y \) the row of \( y \) in \( M_{id} \). Then

\[
y = \sum_{y \subseteq x, x \in X_d} x.
\]

Now let \( y \in X_{i-1} \). We need to show \( y \in U_i \). For any given \( i-1 \) set in a \( d \)-set, there are \( (d-i+1) \) \( i \)-subsets of the \( d \)-set containing the \( (i-1) \) set. Therefore

\[
\sum_{y \subseteq w, w \in X_i} w = (d-i+1)y \Rightarrow y \in U_i, \text{ where the sum varies over } w.
\]

Lemma 2.1.2: \( U_i = V_i \perp U_{i-1} \) for \( 1 \leq i \leq d \).
Proof: Let $W_i$ be the orthogonal complement of $U_{i-1}$ in $U_i$. Any vector $v \in U_i$ can be written as $v = v_1 + v_2$ where $v_1 \in W_i$ and $v_2 \in U_{i-1}$.

Define $\Pi : U_i \to W_i$ by $\Pi(v) = v_1$.

To prove the lemma, we need to show $W_i = V_i$.

Claim $Aw = \lambda_i w$, $\forall w \in W_i$, where $\lambda_i$ is the eigenvalue of $A = A_1$ on $V_i$. We know that $\lambda_i = d(n-d) - i(n+1) + i^2$ see[1].

Let $u \in X_i$. We want to show $A(\Pi u) = \lambda_i(\Pi u)$. By the definition of $u$,

\[
Au = A \sum_{u \subseteq x} x \quad (x \in X_d)
\]

\[
\quad = u \sum_{x \subseteq u} Ax
\]

\[
\quad = \sum_{u \subseteq x} \sum_{y \cap x} y = d-1 \quad (y \in X_d) \quad \text{(by the definition of adjacency)}
\]

\[
\quad = \sum_{u \subseteq x} (|u|y | = i-1)(d-i+1) + \sum_{x \subseteq y} (d-i)(n-d)
\]

\[
\quad = \sum_{u \subseteq x} \sum_{v \subseteq X_i-1} \sum_{y \in X_d, y \cap u = v} y (d-i+1) + u(d-i)(n-d)
\]

\[
\quad = \sum_{v \subseteq u} \sum_{v \subseteq y} y (d-i-1) + u(d-i)(n-d)
\]
\[
\sum_{v \leq u} v - i(d-i+1)u + u(d-i)(n-d) = \sum_{v \leq u} v + u(d(n-d)-i(n+1)+i^2)
\]

Since \( \sum_{v \leq u} v \in U_{i-1} \) and \( \lambda_i = d(n-d) - i(n+1) + i^2 \), we obtain \( \Lambda(\Pi u) = \Pi(Au) = \Pi(u\lambda_i) = \lambda_i(\Pi u) \) and hence the claim.

Since \( J(n,d) \) is a P-polynomial association scheme, by lemma 1.1.2, \( W_i \) becomes a common eigenspace of the adjacency matrices of \( J(n,d) \) with \( \lambda_i \) as the eigenvalue of \( A_1 \). As \( V = U_d \), we get \( V = W_d \perp W_{d-1} \perp \ldots \perp W_0 \) where \( W_0 = V_0 \). From the uniqueness of the decomposition of \( V \) as \( V = V_0 \perp V_1 \perp \ldots \perp V_d \) (up to ordering) we conclude that \( W_i = V_i \) and hence \( U_i = V_i \perp U_{i-1} \).

Let \( r_1, \ldots, r_n \) denote the row vectors of \( M_{1d} \). Let \( T_1, \ldots, T_{\binom{n}{d}} \) be an enumeration of the \( d \)-subsets of \( S \) and assume that the \( i \)-th column of \( M_{1d} \) is indexed by \( T_i \) for \( 1 \leq i \leq \binom{n}{d} \). Then
\[
U_1 = \{ \sum_{1 \leq i \leq n} a_i r_i \mid a_i \in \mathbb{R} \} = \{ (\sum_{j=1}^{\binom{n}{d}} a_j)_{1 \leq j \leq \binom{n}{d}} \mid a_j \in \mathbb{R} \}
\]
Since \( U_1 = V_1 \perp U_0 = V_1 \perp V_0 \), we note that
\[
V_1 = \{ \sum_{1 \leq i \leq n} a_i r_i \mid \langle \sum_{1 \leq i \leq n} a_i r_i , (1,1, \ldots, 1) \rangle = 0 \}
\]

But
\[
\langle \sum_{1 \leq i \leq d} a_i r_i , (1,1, \ldots, 1) \rangle = 0
\]
\[ \langle \left( \sum_{j=1}^{d} a_i \right)_{1 \leq i \leq \binom{n}{d}}, (1,1,\ldots,1) \rangle = 0 \]

\[ \text{for } i < j < n \] 

\[ \sum_{i=1}^{n-1} a_{i} = 0 \]

Hence we see that a vector \( w \) belongs to \( V_1 \) if and only if the components of \( w \) are all the \( d \)-subsets of a fixed \( n \)-tuple \( (a_1, \ldots, a_n) \) belonging to \( \mathbb{R}^n \) with the property that \( \sum_{i=1}^{n} a_{i} = 0 \). Also the map which takes \( w \) to \( (a_1, \ldots, a_n) \) gives an isomorphism between \( V_1 \) and \( \delta_n^1 \), where \( \delta_n = (1,1,\ldots,1) \in \mathbb{R}^n \). Therefore by identifying \( V_1 \) with \( \delta_n^1 \), we see that a vector \( a \) belonging to \( \mathbb{R}^n \) is a general vector if and only if \( \sum_{i=1}^{n} a_{i} = 0 \) and for every \( d \)-subset \( T \) contained in \( \{1,2,\ldots,n\} \), \( \sum_{i \in T} a_{i} \neq 0 \).

**Remark 2.1.3:** Since the vector \( (n-1,-1,-1,\ldots,-1) \) belonging to \( V_1 \) is a general vector, the first distribution invariant of the Johnson scheme cannot exceed \( (n-1) \). (This fact can also be seen from the upper bound given in theorem 1.3.4). Therefore proving that the first distribution invariant of \( J(n,d) \) is equal to \( (n-1) \) is equivalent to proving the following statement:

If \( a_1,a_2,\ldots,a_n \) belong to \( \mathbb{R} \) and satisfy

1. \( \sum_{i=1}^{n} a_{i} = 0 \) and

2. \( \sum_{i \in T} a_{i} \neq 0 \) for every \( d \)-set \( T \subseteq \{1,2,\ldots,n\} \).
then there exist at least \( \binom{n-1}{d-1} \) d-subsets of \( \{1,2,\ldots,n\} \) giving positive sums. Proving this fact is going to be our main aim in the forthcoming sections.

\[ \Diamond \]

2.2 The first distribution invariant when d divides n

In this section we prove that the first distribution invariant of \( J(n,d) \) is \( \binom{n-1}{d-1} \) if \( d \) divides \( n \) and obtain a lower bound for the general case.

**Definition 2.2.1:** For any finite multiset \( A \) (i.e with repeated entries allowed) of real numbers, we define the weight of \( A \) to be equal to the sum of its elements and denote it by \( \text{wt}A \).

**Theorem 2.2.2:** Let \( d,n \) be positive integers such that \( d \) divides \( n \) and \( 2d \leq n \). Then \( V_{t1}(J(n,d)) = \binom{n-1}{d-1} \).

We give two different proofs for this theorem. The first proof is a direct consequence of Baranyai's theorem. The arguments to be given in the second proof are important as similar type of arguments can be carried over to the q-analogue Johnson scheme.

**First Proof:** Let \( (a_1,a_2,\ldots,a_n) \in V_1 \) be any general vector. Let \( N \)
Define a hypergraph on the elements of $\mathcal{N}$ by defining the edges to be all d-submultisets of $\mathcal{N}$. Then Baranyai's theorem [2] states that the minimum number of colors needed to color the edges in such a way that any two edges with the same colour are disjoint and that every $a_i$ belongs to an edge of each colour is precisely $\binom{n}{d-1}$. If we take all the edges of the same colour, then we get a partition of $\mathcal{N}$. Since $(a_1,a_2,\ldots,a_n)$ is a general vector $\text{wt}_N = 0$ and $\text{wt}_T \neq 0$ for any d-submultiset $T$ contained in $\{a_1,a_2,\ldots,a_n\}$. Therefore there exists at least one d-submultiset in every partition of $\mathcal{N}$ having a positive weight. Since all these d-submultisets are distinct, the number of d-submultisets having positive weights must be greater than or equal to $\binom{n-1}{d-1}$. Hence by remark 2.1.3, we conclude that $\nu_1(J(n,d)) = \binom{n-1}{d-1}$. ◊

**Second Proof:** Prior to getting into the proof of the theorem, we need the following lemma:

**Lemma 2.2.3:** Let $d$, $n$ be positive integers such that $d$ divides $n$. Let $\mathcal{N}$ be a multiset of order $n$ and let $\mathcal{F}$ be a collection of d-submultisets of $\mathcal{N}$ with $|\mathcal{F}| \geq \binom{n-1}{d-1}$. Then $\mathcal{F}$ contains a d-partition of $\mathcal{N}$, where d-partition stands for a partition of $\mathcal{N}$ into d-submultisets.

**Proof:** Suppose $\mathcal{F}$ does not contain any d-partition of $\mathcal{N}$. We count the number of pairs $(A,P)$ where $A$ belongs to $\mathcal{F}$ and $P$ is a d-partition of $\mathcal{N}$ containing $A$. For any fixed $A \in \mathcal{F}$, we have
\[(\binom{n-d}{d})(\binom{n-2d}{d}) \cdot \cdot \cdot (\binom{d}{d}) / [(n/d)-1]!\]

such pairs. Therefore, in total we have

\[|\mathcal{F}| \left\{ (\binom{n-d}{d})(\binom{n-2d}{d}) \cdot \cdot \cdot (\binom{d}{d}) \right\} / [(n/d)-1]! \text{ pairs}\]

(1)

On the other hand, by our assumption, in any d-partition of \(N\) we have at most \((n/d - 1)\) elements from \(\mathcal{F}\). Therefore, the number of pairs \((A, \mathcal{P})\) with \(A \in \mathcal{F}\) and \(\mathcal{P}\) a d-partition is at the most

\[\left\{ (\binom{n-d}{d}) \cdot \cdot \cdot (\binom{d}{d}) / (n/d)! \right\} \{(n/d)-1\}\]

(2)

Comparing (1) and (2) we see that

\[|\mathcal{F}| \leq \binom{n}{d}(n/d - 1) / (n/d)\]

\[= \left\{ n! / d! (n-d)! \right\} (n-d)/n = (n-1)! / \{ d! (n-d-1)! \} = \binom{n-1}{d-1}\]

which is a contradiction to the hypothesis that \(|\mathcal{F}| > \binom{n-1}{d-1}\). Therefore, there exists a d-partition of \(N\) in \(\mathcal{F}\).

Proof of Theorem 2.2.2: Let \((a_1, a_2, \ldots, a_n) \in V_1\) be any general vector. Let \(N = \{ a_1, a_2, \ldots, a_n \}\). Note that any d-submultiset of \(N\) has a non-zero weight as \(w\) is a general vector. Let
\[ \mathcal{F} = \{ A \subseteq N \mid |A| = d \text{ and } \text{wt}A < 0 \} \]

We claim that \(|\mathcal{F}| \leq \binom{n-1}{d} \). Suppose \(|\mathcal{F}| > \binom{n-1}{d} \). Then we know, by lemma 2.2.3, that \(\mathcal{F}\) contains a \(d\)-partition of \(N\). Since each element in this partition is a negatively weighted multiset, the weight of \(N\) should be negative, a contradiction to the fact that \(\text{wt}N = 0\). Hence \(|\mathcal{F}| \leq \binom{n-1}{d} \). This means that the number of positively weighted multisets is greater than or equal to \(\binom{n}{d} - \binom{n-1}{d} = \binom{n-1}{d-1}\)

and hence \(\text{Vt}_1(J(n,d)) = \binom{n-1}{d-1}\) by remark 2.1.1. \(\diamondsuit\)

Note that the above proofs work out all right if we replace the fact that \(\text{wt}N = 0\) by \(\text{wt}N \geq 0\) and hence we have

**Corollary 2.2.4**: If \(N = \{a_1, a_2, \ldots, a_n\} \subseteq \mathbb{R}\) satisfies

(i) \(\text{wt}N \geq 0\) and

(ii) \(\text{wt}T \neq 0\) for any \(d\)-submultiset \(T\) of \(N\),

then there exists at least \(\binom{n-1}{d-1}\) positively weighted \(d\)-submultisets of \(N\).

**Corollary 2.2.5**: If \(n = qd + r\) with \(0 \leq r < d\), then

\[ \text{Vt}_1(J(n,d)) \geq \binom{n-r-1}{d-1}. \]

**Proof**: Let \((a_1, a_2, \ldots, a_n) \in V_1\) be any general vector. Without
loss of generality we can assume that \( a_1 \geq a_2 \geq \ldots \geq a_m \geq 0 > a_{m+1} \geq \ldots \geq a_n \). Let \( \mathcal{N} = \{a_1, a_2, \ldots, a_n\} \setminus \{a_{m+1}, \ldots, a_{m+r}\} \). Then \( \mathcal{N} \) satisfies the hypothesis of Corollary 2.2.4 and hence the number of positively weighted \( d \)-submultisets of \( \mathcal{N} \) is at least \( \binom{|\mathcal{N}| - 1}{d} = \binom{n-r-1}{d-1} \) and therefore by remark 2.1.1 \( Vt_1(J(n,d)) \geq \binom{n-r-1}{d-1}. \)

**Corollary 2.2.6:** The first distribution invariant of \( J(2d+1,d) \) is equal to \( \binom{2d-1}{d-1} \).

**Proof:** By Cor 2.2.5, \( \binom{2d-1}{d-1} \leq Vt_1 \). Consider the vector \( (2,2, \ldots, 2, 1-2d, 1-2d) \in \mathbb{R}^{2d+1} \). The only \( d \)-submultisets of \( \{2, 2, \ldots, 1-2d, 1-2d\} \) having positive weights are the \( d \)-multisets not containing \( 1-2d \). They are precisely \( \binom{2d-1}{d-1} \) in number. But \( \binom{2d-1}{d-1} = \binom{2d-1}{d-1} \). Therefore \( Vt_1 \leq \binom{2d-1}{d-1} \) and hence \( Vt_1(J(n,d)) = \binom{2d-1}{d-1}. \)

**Corollary 2.2.7**

The first distribution invariant of \( (J(n,2)) \) is equal to \( \binom{n-1}{d-1} = n-1 \) for \( n \neq 5 \) and \( Vt_1(J(5,2)) = 3. \)

**Proof:** Let \( n = 2q + 1 \), since the case \( n = 2q \) has already been given. As in the proof of the Cor 2.2.5 let \( a = (a_1, a_2, \ldots, a_n) \in V_1 \) be a general vector and let us assume that \( a_1 \geq a_2 \geq \ldots \geq a_m \geq 0 > a_{m+1} \geq \ldots \geq a_n \).
case 1: \( m \leq q \). From Cor. 2.2.5 we see that there exist \((n-2)\) 2-submultisets among \( \{a_1, \ldots, a_n\} \) such that each multiset has a positive weight and none of these pairs involve \( a_{m+1} \). We claim that \( \omega t(a_1, a_{m+1}) > 0 \). If not then we would have \( 0 > \omega t(a_1, a_{m+1}) \geq \omega t(a_2, a_{m+2}) \geq \ldots \geq \omega t(a_m, a_{2m}) \) and by adding these \( m \) pairs and using \( m \leq q \) we get a contradiction to \( \omega t(a_1, \ldots, a_n) = 0, a_{2m+1} < 0, \ldots, a_n < 0 \). Thus \( \omega t(a_1, a_{m+1}) > 0 \), and \( Vt_1 = n-1 \) in this case.

case 2: \( m \geq q + 1 \). Since \( \binom{m}{2} \geq (q+1)q/2 \geq 2q \) for \( q \geq 3 \) we have at least \( 2q = n-1 \) pairs among the nonnegative \( a_i \) with positive weights. Thus \( Vt_1 = n-1 \) in this case, provided \( n \geq 6 \).

Case 3: \( m = 3 \) and \( n = 5 \). By Cor. 2.2.5, \( Vt_1(J(5,2)) \geq 3 \). Since \((2,2,2,-3,-3) \in V_1 \) is a general vector, we get that \( Vt_1(J(5,2)) = 3 \). \( \diamond \)

Remark \( Vt_1(J(n,d)) < \binom{n-1}{d-1} \) if \( n = 3d+1 \) and \( d > 3 \).

Proof Let \( a > 0 \) and \( b < 0 \) be real numbers satisfying \((3d-2)a + 3b = 0\). Then the vector \( (a, \ldots, a, b, b, b) \in \mathbb{R}^{3d+1} \) belongs to \( V_1 \) of \( J(3d+1,d) \). Since \((d-1)a+b = (d-1)a+(2-3d)/3 = -a < 0\), the only \( d \)-submultisets of \( \{a, \ldots, a, b, b, b\} \) having positive weights are the \( d \)-submultisets involving only \( a \). Therefore, we have exactly \( \binom{3d-2}{d} \) \( d \)-submultisets having positive weights. But \( \binom{3d-2}{d} < \binom{3d}{d-1} \). \( \diamond \)
2.3 Main Theorem

In this section we show that for sufficiently large \( n \) we always have \( V_t(J(n,d)) = \binom{n-1}{d-1} \). Since in corollary 2.2.7 we have treated the case \( d = 2 \), we assume in this section \( d \geq 3 \). Let \( n = qd + r, 0 \leq r < d \).

Lemma 2.3.1: Let \( w = (a_1, \ldots, a_n) \) be a general vector in \( V_1 \) with \( a_1 \geq a_2 \geq \ldots \geq a_m \geq 0 > a_{m+1} \geq \ldots \geq a_n \). If \( m \leq \text{minimum of } \{(n-r-1)/(d-1)(d-2), (n-r)/d\} \), then \( |X_+(a)| \geq \binom{n-1}{d-1} \).

Proof: Let \( A = \{a_1, \ldots, a_n\} \subseteq \mathbb{R}^n \) with \( \text{wt} A = 0 \) and let

\[ N = \{a_1, \ldots, a_m, a_{m+r+1}, \ldots, a_n\} \]

be the multiset already considered in the proof of Cor 2.2.5. There we have shown that there exist at least \( \binom{n-r-1}{d-1} \) \( d \)-submultisets of \( N \) having positive weights.

We now consider those \( d \)-submultisets which contain elements of \( \{a_{m+1}, \ldots, a_{m+r}\} \). For this let \( J \subseteq \{1,2, \ldots, r\} \) be a fixed submultiset of cardinality \( j, 1 \leq j \leq r \). Define the multisets \( A_{m+J} = \{a_{m+k}\}_{k \in J} \).

Consider the following \( d \)-submultisets of \( A \)

\[ S_1 = \{a_1\} \cup A_{m+J} \cup \{a_{n-(d-j-1)+1}, \ldots, a_n\} \]
We show that at least one of the multisets in (1) must have positive weight. By adding the weights of all the multisets in (1) and using the fact that $\text{wt} A = 0$, we get that

$$\sum_{1 \leq i \leq m} S_i = m(\text{wt} A_{m+J}) - \text{wt}\{a_{m+1}, a_{m+2}, \ldots, a_{n-m(d-j-1)}\}. \tag{2}$$

The condition $m \leq (n-r)/d$ ensures that $n - m(d-j-1) \geq m + r + 1 + j(m-1)$ so that the above sum (2) may be rewritten as

$$|\text{wt}(A_{m+J}) - \text{wt}\{a_{m+1}, \ldots, a_{m+r}\}| + [(m-1)\text{wt}(A_{m+J}) - \text{wt}\{a_{m+r+1}, \ldots, a_{m+r+1+j(m-1)}\}] + \text{a non-negative number.}$$

Since for $k \in J$, $a_{m+k} \geq a_{m+r+1} \geq \ldots \geq a_{m+r+1+j(m-1)}$, the second square bracket is nonnegative. Obviously the first square bracket
is also nonnegative. This shows that the sum of the weights of $S_i$ in (1) is nonnegative, and thus (a being a general vector) at least one of the multisets in (1) must have a positive weight.

Let $i = i(J)$ be the index of $a_i$ for which the multiset in (1) has positive weight. As we may replace the multiset $\{a_{n-i(J)}(d-j-1)+1, \ldots, a_{n-i(J)(d-j-1)}\}$ with any other $(d-j-1)$-submultiset of $\{a_i(J)+1, \ldots, a_m\}$ and as we further may replace $a_i(J)$ by any of $a_1, a_2, \ldots, a_i(J)$, still obtaining a multiset with positive weight in this way, we altogether get

$$i(J)^{(a-r-1-(i(J)-1))(d-j)}(d-j-1)}$$

$d$-submultisets having positive weights and containing the multiset $A_{m+j}$.

In the case $i(J) = 1$ for all $J$ we see that we altogether get

$$\sum_{1 \leq j \leq r} \binom{n-r-1}{a_j-1} = \binom{n-r-1}{d-j-1} \geq \binom{n-r-1}{d-j-1}$$

positively weighed $d$-submultisets. So we have proved the lemma if we show that

$$i(J)^{(a-r-1-(i(J)-1))(d-j)}(d-j-1)} \geq \binom{n-r-1}{d-j-1}$$

for all $i(J) \in \{2, \ldots, m\}$ and $j = 1, 2, \ldots, r$. For this we make an estimate of the following quotient
\[
\frac{(n-r-1-(i(J)-1)(d-j))}{d-j-1}
\]

\[
\frac{1}{d-j-1}
\]

\[
\prod_{i(J)=2}^{m} \frac{[(n-r-l-(i(J)-1)(d-j)+2)]}{[(n-r-l)(n-r-2) \ldots (n-r-1-(d-j-2))]} \]

\[
\prod_{i(J)=2}^{m} \frac{[(1-(i(J)-1)(d-j)/n-r-l) \ldots (1-i(J)(d-j)-2/n-r-1)]}{[1 \ldots (d-j-2/n-r-1)]} \]

which is \( \geq 1 \) for all \( i(J) \in \{2, 3, \ldots, m\} \) if and only if \( m < \frac{(n-r-1)}{(d-j)(d-j-1)} \). Since \( j = 1, 2, \ldots, r \) the lemma follows. \( \diamond \)

**Theorem 2.3.2**

\[ V_{t_1} \left( J(n,d) \right) = \binom{n-1}{d-1} \] if

(i) \( d = 3 \) and \( n \geq 93 \)

(ii) \( d > 3 \) and \( n \geq d(d-1)^d(d-2)^d + d(d-1)^2(d-2) + d(r+1) \).

**Proof:** In case (i) for \( d = 3 \) the condition of the lemma 2.3.1 be-
comes \( m \leq \frac{n-r}{3} \). Thus if \( a = (a_1, \ldots, a_n) \in \mathbb{R}^n \) is a vector as in lemma 2.3.1 and \( m \leq \frac{n-r}{3} \), then according to lemma 2.3.1
\[
|X_+(a)| \geq \binom{n-1}{d-1} = \binom{n-1}{2}.
\]

But if \( m > \frac{n-r}{3} \) then we have at least \( \binom{m}{3} \) positively weighted 3-multisets among the numbers \( \{a_1, a_2, \ldots, a_m\} \). We want this number to be greater than or equal to \( \binom{n-1}{2} \), so we estimate
\[
\frac{\binom{n}{3}}{\binom{n-1}{2}} = \frac{m(m-1)(m-2)}{3(n-1)(n-2)} > \frac{(n-r)(n-r-3)(n-r-6)}{81(n-1)(n-2)}
\]
which is > 1 for \( n \geq 93 \).

In case (ii) for \( d > 3 \) the condition of the lemma 2.3.1 becomes
\[ m \leq \frac{n-r-1}{(d-1)(d-2)} \]. Thus if \( a = (a_1, \ldots, a_n) \in \mathbb{R}^n \) is the vector in lemma 2.3.1 and \( m \leq \frac{n-r-1}{(d-1)(d-2)} \), then according to lemma 2.3.1
\[
|X_+(a)| \geq \binom{n-1}{d-1}.
\]

But if \( m > \frac{n-r-1}{(d-1)(d-2)} \) then we have at least \( \binom{m}{d} \) positively weighted \( d \)-multisets among the elements \( \{a_1, a_2, \ldots, a_m\} \). In order to compare this number to \( \binom{n-1}{d-1} \) we form the quotient and give an estimate:
\[
\frac{\binom{m}{d}}{\binom{n-1}{d-1}} = \frac{m(m-1) \cdots (m-d+1)}{(n-1)(n-2) \cdots (n-d+1)}
\]
\{(n-r-1)(n-r-1-(d-1)(d-2)). . .(n-r-1-(d-1)(d-1)(d-2))\}
> \frac{n\{(1-(r+1)/n)(1-(r+1+(d-1)(d-2))/n). . .(1-((r+1)+(d-1)(d-1)(d-2))/n)\}}{d(d-1)(d-2)^d(1-(1/n)). . .(1-(1-1)/n)\}}

and this is $\geq 1$ if and only if

$$n \geq d(d-1)^d(d-2)^d + d(r+1) + d(d-1)^2(d-2). \diamond$$

2.4 An alternative approach

Now we take a different approach towards the calculation of the first distribution invariants of $J(n,d)$. First we prove that if the first distribution invariant of $J(n,d)$ is $(\alpha, . . ., \alpha)$, then for any positive integer $k$, $J(n+d,d)$ and $J(kn,d)$ have $\left(\frac{n+d-1}{d-1}\right)$ and $\left(\frac{kn-1}{d-1}\right)$ as their first distribution invariants respectively. As a corollary we also prove that

$$J(n,3) = \left(\frac{n-1}{2}\right) \text{ if } n \geq 11.$$
Let \( A = \{a_1, \ldots, a_{n+d}\} \). \( N = \{ \text{negatively weighted } d\text{-submultisets of } A \} \).

**Claim:** \( |N| \leq (n^d + d-1) \).

Let \( f : \{a_1, \ldots, a_n\} \rightarrow \{1, \ldots, n\} \) be the function defined by \( f(a_i) = i \) for \( i = 1, \ldots, n \). We say that two submultisets \( A, B \) of \( \{a_1, \ldots, a_n\} \) are disjoint if \( f(A) \) and \( f(B) \) are disjoint. Now we count the number of pairs \((A, B)\) where \( A \) is any \( d\)-submultiset of \( A \) and \( B \) belongs to \( N \), and satisfying the property that \( A \) and \( B \) are disjoint. We count it in two ways.

On one hand, for any element \( B \) in \( N \), we have \( d \) such pairs. Therefore, the total number of pairs \((A, B)\) = \( |N|\cdot\binom{n}{d} \).

(1)

On the other hand, if we fix \( A \) and vary \( B \) over the elements of \( N \), we have the following two cases:

**Case 1:** when \( A \in N \).

Since \( \text{wt } A < 0 \), the multiset \( A \setminus A \) must be positively weighted. Therefore, by our assumption that \( Vt_1(J(n,d)) = \binom{n-1}{d-1} \) and by Cor 2.2.4, we get that the number of \( d\)-submultisets of \( A \setminus A \) in \( N \) cannot exceed \( \binom{n-1}{d-1} \).
Therefore for a fixed $A \in N$, there exists at most $\binom{n-1}{d}$ pairs $(A,B)$. Hence, the total number of pairs $(A,B)$ with both $A$, $B$ in $N$ is at most

$$|N|\binom{n-1}{d}. \quad (2)$$

**case 2:** When $A \notin N$. In this case, by straight counting, the total number of pairs $(A,B)$ with $A \notin N$ and $B \in N$ and $A \cap B = \emptyset$ is at most

$$\binom{n+d}{d} - |N|\binom{n}{d} \quad (3)$$

From (2) and (3) we see that the total number of pairs $(A,B)$

$$\leq \left[ \binom{n+d}{d} - |N|\binom{n}{d} \right] + |N|\binom{n-1}{d} \quad (4)$$

By comparing (1) and (4) we conclude that

$$|N|\binom{n}{d} \leq \left[ \binom{n+d}{d} - |N|\binom{n}{d} \right] + |N|\binom{n-1}{d}$$

$$\Rightarrow |N|\binom{n}{d} \leq |N|\binom{n+d-1}{d}$$

$$\Rightarrow |N| \leq \binom{n+d-1}{d}$$
and hence the claim is proved. Therefore, the number of positively weighted d-submultisets of \( A \) must be greater than or equal to

\[
\binom{n+d}{d} - \binom{n+d-1}{d-1} = \binom{n+d-1}{d-1}
\]

\[
\Rightarrow Vt_1(J(n+d,d)) = \binom{n+d-1}{d-1}
\]

by remark 2.1.1.

**Theorem 2.4.2:** If the first distribution invariant of \( J(n,d) \) is \( (\binom{n-1}{d-1}) \), then \( J(kn,d) \) has \( (\binom{kn-1}{d-1}) \) as its first distribution invariant for any integer \( k \geq 1 \).

**Proof:** As in theorem 1, let \( (a_1, \ldots, a_{kn}) \) be any general vector and let \( A = \{a_1, \ldots, a_{kn}\} \). \( N = \{ \text{negatively weighted submultisets of } A \} \).

**Claim:** \( |N| < \binom{kn-1}{d} \)

This time, we count the pairs \( (P,A) \) where \( P \) is an \( n \)-partition of \( A \) and \( A \) is an element of \( N \) satisfying the property that \( A \) is contained in one of the \( n \)-multisets in the partition \( P \). We count this pair \( (P,A) \) in two ways. First fix \( A \in N \). Then we can take \( P \) to be any partition satisfying the above property and hence, for any fixed \( A \), we have the total number of pairs \( (P,A) = \binom{kn-d}{n-d} \binom{kn-n}{n-1} \binom{n}{n} / (k-n)! \). Therefore by varying \( A \) over the elements of \( N \) we conclude that the number of pairs \( (P,A) = |N| \binom{kn-d}{n-d} \binom{kn-n}{n-1} \binom{n}{n} / (k-1)! \).
On the other hand, we notice that in any given n-partition $P$ of $A$ there exists at least one n-multiset $M$ having non-negative weights. By our assumption that $Vt_1(J(n,d)) = (\binom{n-1}{d-1})$ and by Cor 2.2.4, we see that $M$ can contain at most $(\binom{n-1}{d})$ elements of $N$. Therefore for a fixed n-partition $P$, we have at most $[(k-1)\binom{n}{d} + (\binom{n-1}{d})]$ pairs $(P,A)$. Hence the total number of pairs $(P,A)$

$$\leq \left[\binom{kn}{n} \cdot \overline{(n)} / k!\right] \left[(k-1)\binom{n}{d} + (\binom{n-1}{d})\right]$$

From (1) and (2) we get that

$$|N|\binom{kn-d}{n-d}\binom{kn-n}{n} \cdot \overline{(n)} / (k-1)! \leq \left[\binom{kn}{n} \cdot \overline{(n)} / k!\right] \left[(k-1)\binom{n}{d} + (\binom{n-1}{d})\right]$$

$$\Rightarrow |N| < (\binom{kn-1}{d})$$ as we claimed.

Hence the number of positively weighted submultisets of $A$ is

$$\geq (\binom{kn}{d}) - (\binom{kn-1}{d}) = (\binom{kn-1}{d-1})$$

$$\Rightarrow Vt_1(J(kn,d)) = (\binom{kn-1}{d-1})$$ by remark 2.1.3. ◊

Indeed, by using a similar technique as in theorem 2.4.1 and 2.4.2, we can prove theorem 2.4.4..

Consider the following statement:
**Statement 2.4.3:** Let $S$ be an $n$-multiset and $N$ a collection of $d$-submultisets of $S$. If $|N| > \binom{n-1}{d-1}$, then there exist non-negative numbers $\{\alpha_A\}_{A \in N}$ such that $\sum_{A \in N} \alpha_A X_A = (11 \ldots 1)$ where $X_A$ is the characteristic vector of $A$ (of length $n$).

**Theorem 2.4.4:** If above statement 2.4.3 is true for positive integers $n$ and $d$, then it is also true for

(i) $n+d$ and $d$

(ii) $kn$ and $d$ for any positive integer $k$.

The following is the Corollary of theorem 2.4.2.

**Corollary 2.4.5:** $V_1(J(n,3)) = \binom{n-1}{2}$ if $n \geq 11$.

To prove this corollary we need the lemmas 2.4.6 and 2.4.7.

**Lemma 2.4.6:** $V_1(J(11,3)) = \binom{10}{2}$

**Proof:** Let $(a_1, a_2, \ldots, a_{11}) \in V_1$ be any general vector. Let $A = \{a_1, \ldots, a_{11}\}$. Then $\text{wt}A = 0$ and no 3-submultiset has weight zero.

By remark 2.1.3, we need to show that $A$ contains at least $\binom{10}{2}$ 3-submultisets with positive weights. If $\text{wt}\{a_1, a_{10}, a_{11}\} > 0$, then obviously
we have \( \binom{10}{2} \) positive 3-submultisets of \( \Lambda \). Therefore, we can assume that 
\[ \text{wt}\{a_1,a_{10},a_{11}\} < 0. \]
Without loss of generality, let us assume that 
\[ a_1 \geq \ldots \geq a_m \geq 0 > a_{m+1} \geq \ldots \geq a_{11}. \]
If \( m \geq 9 \), then \( \binom{m}{3} > 45 = \binom{10}{2} \) and hence \( \Lambda \) possesses the needed number of positively weighted 3-submultisets in this case. If \( m \leq 3 \), then again we have enough positively weighted 3-submultisets of \( \Lambda \) by lemma 2.3.1. Therefore, we need to check for the cases \( m = 4,5,6 \) and 7.

**Case i:** When \( m = 4 \). Out of \( \{a_1,a_2,a_3,a_4\} \) we have four 3-submultisets with positive weights. We need to produce 41 more positive 3-submultisets of \( \Lambda \) involving at least one negative \( a_i \). Write

\[
\{a_1,a_2,a_3,a_4,a_7,a_8, \ldots ,a_{11}\} = P_1 \cup P_2 \cup P_3 \text{ where}
\]
\[
P_1 = \{a_3,a_4,a_{11}\}, \quad P_2 = \{a_2,a_9,a_{10}\} \quad \text{and} \quad P_3 = \{a_1,a_7,a_8\}.
\]

If \( \text{wt}P_1 > 0 \), then replacing \( \{a_3,a_4\} \) by any 2-submultiset of \( \{a_1,a_2,a_3,a_4\} \) and \( a_{11} \) by any element of \( \{a_5,a_6, \ldots ,a_{11}\} \) still ends up in a positively weighted 3-multiset and we get \( (6)(7) = 42 \) such multisets. Therefore we certainly have an adequate number of positively weighted 3-submultisets.

If \( \text{wt} P_2 > 0 \), then we get 30 new positive 3-multisets involving two elements from \( \{a_5,a_6, \ldots ,a_{10}\} \) and one element from \( \{a_1,a_2\} \). The rest of
the 11 needed positive 3-multisets can be produced by taking two non-negative \( a_i \)'s (with at least one of them \( a_1 \) or \( a_2 \)) and the remaining element to be any element from \( \{a_5, a_6, \ldots, a_{11}\} \).

If \( \text{wt} \, P_3 > 0 \), then we need to do more work than what we did in the above cases. By corollary 2.2.5, we know that we have \( \binom{6}{2} \) positive 3-submultisets among \( \{a_1, a_2, a_3, a_4, a_7, a_8, \ldots, a_{11}\} \). Therefore, we need to produce 17 more positive 3-submultisets of \( \Lambda \), perhaps involving \( a_5 \) or \( a_6 \).

Since \( \text{wt} \, P_3 > 0 \), we get that

(i) \( \text{wt}(a_1, a_6, a_i) > 0 \) for \( i = 2, 3, 4, 5, 7 \) and 8.

(ii) \( \text{wt}(a_1, a_5, a_j) > 0 \) for \( j = 2, 3, 4, 7 \) and 8.

We need 6 more positive 3-multisets.

Claim: \( \text{wt}(a_2, a_4, a_6) > 0 \). If not, write

\[ \Lambda = \{a_2, a_4, a_6\} \cup \{a_1, a_10, a_{11}\} \cup \{a_3, a_8, a_9\} \cup \{a_5, a_7\} \]

Since \( \text{wt}\{a_2, a_4, a_6\} < 0 \), we have \( \text{wt}\{a_3, a_8, a_9\} < 0 \) and \( \text{wt}\{a_5, a_7\} < 0 \). As \( \text{wt}\{a_1, a_10, a_{11}\} \) is assumed to be positive, we get that \( \text{wt}\Lambda < 0 \), a contradiction to the fact that \( \text{wt}\Lambda = 0 \) and hence

\[ \text{wt}(a_2, a_4, a_6) > 0 \]

(1)
If \( \text{wt}\{a_2,a_5,a_7\} > 0 \), then

(i) \( \text{wt}(a_2,a_3,a_i) > 0 \) for \( i = 3,4,6,7 \) and

(ii) \( \text{wt}(a_2,a_6,a_j) > 0 \) for \( j = 3,4 \) hence we have 6 new positive 3-multisets for a total of 17.

If \( \text{wt}\{a_2,a_5,a_7\} < 0 \), then consider

\[
\Lambda = \{a_2,a_5,a_7\} \cup \{a_1,a_9,a_{10}\} \cup \{a_3,a_6,a_8\} \cup \{a_8,a_{11}\}
\]

\( \text{wt}\{a_2,a_5,a_7\} < 0 \Rightarrow \text{wt}\{a_3,a_6,a_8\} < 0 \). Also \( \text{wt}\{a_3,a_4,a_{11}\} < 0 \Rightarrow \text{wt}\{a_4,a_{11}\} < 0 \). Therefore, we conclude that \( \text{wt}\{a_1,a_9,a_{10}\} > 0 \Rightarrow \text{wt}\{a_1,a_6,a_i\} > 0 \) and \( \text{wt}\{a_1,a_5,a_i\} > 0 \) for \( i = 0,10 \).

Now (1) gives \( \text{wt}\{a_2,a_4,a_i\} > 0 \) and \( \text{wt}\{a_2,a_3,a_i\} > 0 \) for \( i = 5,6 \).

Therefore, in total we have more than 6 new positive 3-submultisets and hence the result is true when \( m = 4 \).

Case (ii): When \( m=5 \). We need to produce 17 positive 3-multisets involving \( a_6 \) or \( a_7 \). Consider the partition \( \Lambda = Q_1 \cup Q_2 \cup Q_3 \cup Q_4 \) where

\( Q_1 = \{a_4,a_5,a_7\} \), \( Q_2 = \{a_1,a_9,a_{10}\} \), \( Q_3 = \{a_3,a_8\} \) and \( Q_4 = \{a_2,a_6,a_{11}\} \)

Sub case(i): If \( \text{wt} Q_1 > 0 \), then by replacing \( a_7 \) by \( a_6 \) or \( a_7 \) and \( a_4,a_5 \) by any 2-submultiset of \( \{a_1,a_2,..,a_5\} \), we get 20 positive 3-submultisets involving \( a_6 \) or \( a_7 \).
Sub case(ii): If $\text{wt} Q_2 > 0$, we make a direct count as follows.

Replacing $a_9$ and $a_{10}$ by any 2-submultiset of $\{a_2, \cdots, a_{10}\}$ results in a positive 3-submultiset. Therefore we get 36 of them this way and all of them involve $a_1$. But from $\{a_2, \cdots, a_5\}$ we get 4 positive 3-submultisets. Therefore, we need five more positive 3-submultisets involving at least one negative $a_i$ but not $a_1$.

We claim that $\text{wt}\{a_2, a_5, a_7\} > 0$. Suppose not. Consider the partition

\[ \Lambda = \{a_2, a_5, a_7\} \cup \{a_3, a_8\}, \{a_4, a_9\} \cup \{a_1, a_{10}, a_{11}\} \]

$\text{wt}\{a_2, a_5, a_7\} < 0 \Rightarrow \text{wt}\{a_3, a_8\} < 0 \text{ wt}\{a_1, a_{10}, a_{11}\} < 0 \text{ wt}\{a_4, a_9\} < 0. \Rightarrow \text{wt}\Lambda < 0$, a contradiction.

Therefore, $\text{wt}\{a_2, a_5, a_7\}$ must be positive. Now replacing $a_5$ by $a_3, a_4$ or $a_7$ by $a_6$ or $a_7$ we see that we have at least 6 more positive 3-submultisets as needed.

Sub case(iii): When $\text{wt}\{a_3, a_8\} > 0$.

Since $\text{wt}\{a_3, a_8\} > 0$, we get that

(i) $\text{wt}\{a_i, a_3, a_7\} > 0$ for $i = 1, 2, 4$ and 5
(ii) $\text{wt}\{a_i,a_3,a_6\} > 0$ for $i = 5, 4$ and 7

(iii) $\text{wt}\{a_i,a_2,a_6\} > 0$ for $i = 5, 4$ and 7.

(iv) $\text{wt}\{a_i,a_1,a_7\} > 0$ for $i = 5, 4$

(v) $\text{wt}\{a_i,a_1,a_6\} > 0$ for $i = 5, 4$

Therefore, we get 18 of them in total. We needed only 17 positive 3-submultisets anyway.

Sub case (iv): When $\text{wt}\{a_2,a_6,a_{11}\} > 0$.

$\text{wt}\{a_2,a_6,a_{11}\} > 0 \Rightarrow \text{wt}\{a_i,a_6,a_j\} > 0$ for $i = 1, 2$ and $j = 3, 4, 5, 7, 8, \ldots, 11$ and hence together with $\{a_1,a_2,a_6\}$ we do have 17 positive 3-submultisets in this case.

Case (iii) When $m = 6$: Consider the partition $\Lambda = P_1 \cup P_2 \cup P_3 \cup P_4$ where $P_1 = \{a_1,a_{11}\}$, $P_2 = \{a_2,a_{10}\}$, $P_3 = \{a_3,a_4,a_9\}$ and $P_4 = \{a_5,a_6,a_8\}$.

Subcase (i) When $\text{wt}P_1 \geq 0$. $\text{Wt}P_1 \geq 0 \Rightarrow \text{wt}\{a_1,a_i,a_j\} > 0$ for $i = 2, \ldots, 6$ and $j = 7, 8, \ldots, 11$. They are 25 of them in total. Out of $\{a_1, \ldots, a_6\}$ we get 20 positively weighted 3-submultisets and hence, in total, we have the necessary positively weighted 3-submultisets of $\Lambda$.

Subcase (ii) When $\text{wt}P_2 \geq 0$. $\text{Wt}P_2 \geq 0 \Rightarrow \text{wt}\{a_i,a_j,a_k\} > 0$ for
i=3,4,5,6, j=1,2 and k=7,8,9,10. We have 32 of them involving exactly one negative a_i. These positively weighted 3-submultisets together with the ones from \{a_1, \ldots, a_6\} account for more than 45 positively weighted 3-submultisets.

Subcase(iv) When \(\text{wt}P_4 > 0\). \(\text{wt}P_4 > 0 \Rightarrow \text{wt}\{a_i,a_j,a_k\} > 0\) for \(\{i,j\} \subseteq \{1,2,\ldots,6\}\) and \(k=7,8\). They are 24 in total. Since we needed only 17 positively weighted 3-submultisets of \(A\) involving \(a_7\) or \(a_8\), we are done in this case also.

Subcase(iii) When \(\text{wt}P_3 \geq 0\). \(\text{wt}P_3 > 0 \Rightarrow \text{wt}\{a_i,a_j,a_k\} > 0\) for \(\{i,j\} \subseteq \{1,2,3,4\}\) and \(k=7,8\). They total up to 12 and we need 5 more positively weighted 3-submultisets involving \(a_7\) or \(a_8\) or both. We can now assume that \(\text{wt}P_1 < 0\) for \(i=1,2,4\). Consider the partition

\[ \Lambda = \{a_1,a_7,a_8\} \cup \{a_5,a_6,a_9\} \cup \{a_3,a_4,a_{11}\} \cup \{a_2,a_{10}\}. \]

\(\text{wt}P_4 < 0 \Rightarrow \text{wt}\{a_5,a_6,a_9\} < 0\). Also \(\text{wt}P_2 < 0\) and hence we must have \(\text{wt}\{a_1,a_7,a_8\} + \text{wt}\{a_3,a_4,a_{11}\} > 0\). Therefore, at least one of them must have positive weight. If \(\text{wt}\{a_3,a_4,a_{11}\} > 0\) then \(\text{wt}\{a_i,a_j,a_k\} > 0\) for \(\{i,j\} \subseteq \{1,2,3,4\}\) and \(k=5,6,\ldots,11\). They are \((6)(7)=42\) in total. These together with \(\{a_i,a_5,a_6\},i=1,2,3,4\), give 46 positively weighted 3-submultisets. If \(\text{wt}\{a_1,a_7,a_8\} > 0\), then \(\text{wt}\{a_i,a_i,a_j\} > 0\) for \(i=5,6\) and \(j=7,8\). These give the five needed positively weighted 3-submultisets.
Case iv: When $m = 7$. We have 35 positively weighted 3-submultisets out of $\{a_1, a_2, \ldots, a_7\}$. We need to produce 10 more of them with at least one negative $a_i$. Consider the partition

$$
\Lambda = \{a_1, a_9\} \cup \{a_2, a_3, a_{11}\} \cup \{a_4, a_5, a_8\} \cup \{a_6, a_7, a_{10}\}.
$$

If $\text{wt}\{a_1, a_9\} \geq 0$, then $\text{wt}\{a_1, a_i, a_j\} > 0$ for $i = 2, 3, \ldots, 7$ and $j = 8, 9$.

If $\text{wt}\{a_2, a_3, a_{11}\} > 0$, then $\text{wt}\{a_i, a_j, a_k\} > 0$ for $\{i, j\} \subseteq \{1, 2, 3\}$ and $k = 8, 9, 10, 11$.

If $\text{wt}\{a_4, a_5, a_8\} > 0$, then $\text{wt}\{a_i, a_j, a_k\} > 0$ for $\{i, j\} \subseteq \{1, 2, 3, 4, 5\}$.

If $\text{wt}\{a_6, a_7, a_{10}\} > 0$, then $\text{wt}\{a_i, a_j, a_{10}\} > 0$ for $\{i, j\} \subseteq \{1, 2, \ldots, 7\}$.

Note that in each of the above cases we have at least 10 3-submultisets having positive weights. ◊

Lemma 2.4.7: $V_1(J(13, 3)) = \binom{12}{2}$.

Proof: As usual let $V_1$ denote the 1st eigenspace. Let $(a_1, \ldots, a_{13}) \in V_1$ be any general vector. Let $\Lambda = \{a_1, \ldots, a_{13}\}$. Then the weight of $\Lambda$ is zero and any 3-submultiset of $\Lambda$ has non-zero weight. By remark 2.1.1, we need to show that $\Lambda$ contains at least $\binom{12}{2}$ 3-submultisets with positive weights. If $\{a_1, a_{12}, a_{13}\}$ has positive weight then so have
\{a_1, a_i; a_j\} for \{i,j\} \subseteq \{2, \ldots, 13\}, hence \Lambda \text{ has } \binom{13}{2} \text{ positively weighted } 3\text{-submultisets in this case. Therefore, let us assume that } \text{wt}\{a_1, a_{12}, a_{13}\} \text{ is negative. Without loss of generality, we assume that } a_1 \geq \ldots \geq a_m > 0 \geq a_{m+1} \geq \ldots \geq a_{13}.

Any 3-submultiset of \{a_1, \ldots, a_m\} has positive weight. If } m \geq 9, \text{ then } \binom{m}{3} > 66 = \binom{12}{2}. \text{ Therefore, we need to show the existence of } \binom{12}{2} \text{ positively weighted } 3\text{-submultisets in the cases } m=5, 6, 7 \text{ and } 8. \text{ The cases } m \leq 4 \text{ are taken care of by Lemma 2.3.1.}

\textbf{Case (i)} : when } m = 5.

By Corollary 2.2.4, we notice that the multiset \(\Lambda \setminus \{a_6\}\) possesses at least \(\binom{11}{2}\) positively weighted 3-submultisets. Our aim now is to produce \(\binom{12}{2} - \binom{11}{2} = 11\) new positively weighted 3-submultisets of \(\Lambda\), perhaps involving \(a_6\).

Partition \(\Lambda \setminus \{a_6\}\) as

\[\Lambda \setminus \{a_6\} = P_1 \cup P_2 \cup P_3 \cup P_4\text{ where}\]

\[P_1 = \{a_1, a_{12}, a_{13}\}, P_2 = \{a_2, a_{10}, a_{11}\},\]

\[P_3 = \{a_3, a_8, a_9\}\text{ and } P_4 = \{a_4, a_5, a_7\}\]
Since wt\(\{\Lambda\{a_6}\}\) is positive, at least one of the \(P_i\) must have positive weight. We have already assumed that wt\(P_1 < 0\).

**Sub-case:** when wt\(P_2 > 0\). wt\(P_2 > 0 \Rightarrow wt\{a_2, a_6, a_i\} > 0\) and wt\(a_1, a_6, a_i\} > 0\) for \(i = 3, 4, 5, 7, 8, \ldots, 11\). Hence we get 16 new positively weighted 3-submultisets involving \(a_6\).

**Sub-case:** when wt\(P_3 > 0\). wt\(P_3 > 0 \Rightarrow wt\{a_3, a_6, a_i\} > 0\) and wt\(a_2, a_6, a_i\} > 0\) for \(i = 1, 2, 4, \ldots, 8, 9\)

So we certainly have more than 11 positively weighted 3-submultisets of \(\Lambda\) with \(a_6\) as one of the elements.

**Sub-Case:** when wt\(P_4 > 0\).

wt\(P_4 > 0 \Rightarrow wt\{a_1, a_j, a_6\} > 0\) for any 2-submultiset \(\{i, j\} \subseteq \{1, 2, \ldots, 5\}\).

Therefore, we have \(\binom{5}{2} = 10\) positively weighted 3-submultisets of \(\Lambda\) involving \(a_6\). We need to produce 1 more. If wt\(a_1, a_6, a_7\} \) > 0 then we are done. Suppose not. Then consider the partition \(\Lambda\{a_6\} = Q_1 \cup Q_2 \cup Q_3 \cup Q_4\) where \(Q_1 = \{a_1, a_7, a_8\}, Q_2 = \{a_2, a_9, a_{10}\}, Q_3 = \{a_3, a_{11}, a_{12}\}\) and \(Q_4 = \{a_4, a_{15}, a_{13}\}\).

wt\(a_1, a_6, a_7\} > 0 \Rightarrow wtQ_i < 0\) for \(i = 1, 2\) and 3. But wt\(\Lambda\{a_i\}\) > 0 and hence we must have wt\(Q_4 > 0\). wt\(Q_4 > 0 \Rightarrow wt\{a_1, a_j, a_k\} > 0\) for \(\{i, j\} \subseteq \{1, 2, \ldots, 5\}\) and \(j \in \{6, 7, \ldots, 13\}\) and hence \(\Lambda\) contains \(\binom{5}{2}/1 = \)
80 positively weighted 3-submultisets. Therefore, we have more than the needed \( \binom{12}{2} \) positively weighted 3-submultisets in this case.

**Case:** when \( m = 6 \). partition \( \Lambda \setminus \{a_7\} \) as

\[
\Lambda = P_1 \cup P_2 \cup P_3 \cup P_4 \text{ where} \\
P_1 = \{a_1, a_{12}, a_{13}\}, \; P_2 = \{a_2, a_{10}, a_{11}\}, \; P_3 = \{a_3, a_4, a_9\} \text{ and } P_4 = \{a_5, a_6, a_8\}
\]

Since \( \text{wt}(\Lambda \setminus \{a_7\}) > 0 \), at least one of the \( P_i \) must have positive weight.

As we have assumed that \( \text{wt} P_1 < 0 \), let us consider the other cases. When \( \text{wt} P_2 > 0 \) or \( \text{wt} P_4 > 0 \), similar proofs as in subcases when \( \text{wt} P_2 > 0 \) and \( \text{wt} P_4 > 0 \) of \( m = 5 \) case can be carried over. Therefore, we need to check only the case when \( \text{wt} P_3 > 0 \) and \( \text{wt} P_i < 0 \) for \( i = 1, 2, 4 \). \( \text{wt} P_3 > 0 \Rightarrow \text{wt}\{a_7, a_i, a_j\} > 0 \) for any 2-submultiset \( \{i, j\} \subseteq \{1, 2, 3, 4\} \) and hence we have 6 new positively weighted 3-submultisets.

Now we claim that \( \text{wt}\{a_3, a_6, a_7\} > 0 \). Suppose not. partition \( \Lambda \) as

\[
\Lambda = \{a_3, a_6, a_7\} \cup \{a_4, a_8\} \cup \{a_5, a_6\} \cup \{a_2, a_{10}, a_{11}\} \cup \{a_1, a_{12}, a_{13}\}
\]

\[
\text{wt}\{a_3, a_6, a_7\} < 0 \Rightarrow \text{wt}\{a_4, a_8\} < 0 \text{ wt}\{a_5, a_6\} < 0.
\]

Since we have assumed that \( \text{wt} P_2 < 0 \) we must have \( \text{wt}\{a_2, a_{10}, a_{11}\} < 0 \). Therefore we conclude that \( \text{wt} \Lambda < 0 \) which is a contradiction. Hence \( \text{wt}\{a_3, a_6, a_7\} > 0 \).

\[
\text{Wt}\{a_3, a_6, a_7\} > 0 \Rightarrow \text{the multisets } \{a_3, a_7\}, \{a_2, a_6, a_7\}, \{a_2, a_5, a_7\},
\]
\{a_1,a_3,a_7\} and \{a_1,a_5,a_7\} all have positive weights and hence we have produced 12 positively weighted 3-sub multisets of \(\Lambda\) involving \(a_7\) and we needed only 11. Therefore the \(m = 6\) case is done.

**Case:** when \(m = 7\)

By corollary 2.2.4 we know that we have \(\binom{11}{2}\) positively weighted 3-sub multisets out of \(\{a_1,a_2,\ldots,a_7,a_9,\ldots,a_{11}\}\). Therefore we need to produce 11 new positively weighted 3-sub multisets of \(\Lambda\) (perhaps involving \(a_8\)).

Partition \(\Lambda \setminus \{a_8\}\) as

\[
\Lambda \setminus \{a_8\} = \{a_2,a_{12}\} \cup \{a_3,a_{11}\} \cup \{a_4,a_{10}\} \cup \{a_5,a_6,a_9\} \cup \{a_1,a_7,a_{13}\}.
\]

Since \(\operatorname{wt}(\Lambda \setminus \{a_8\}) > 0\), at least one of the above multisets must have positive weight.

If \(\operatorname{wt}\{a_2,a_{12}\} > 0 \Rightarrow \operatorname{wt}\{a_2,a_8\} > 0 \Rightarrow \operatorname{wt}\{a_i,a_2,a_8\} > 0\) for \(i \in \{1,3,4,5,6,7\}\) Therefore, we get 6 new positively 3-sub multisets. Now replacing \(a_2\) by \(a_1\) and \(a_i\) by \(a_j\) for \(j \in \{3,4,5,6,7\}\), we get 5 more positively weighted 3-sub multisets and hence we have 11 of them in total. The above argument also settles the cases when \(\operatorname{wt}\{a_3,a_{11}\} > 0\) and \(\operatorname{wt}\{a_4,a_{10}\} > 0\).
If \( \text{wt}\{a_5, a_6, a_9\} > 0 \) \( \Rightarrow \) \( \text{wt}\{a_1, a_j, a_8\} > 0 \) for \( \{i, j\} \subseteq \{1, 2, \ldots, 6\} \). So we have \( \binom{6}{2} = 15 \) new positively weighted 3-submultisets. Hence we are done in this case also.

Now we are left with the case \( \text{wt}\{a_1, a_7, a_{13}\} > 0 \). We note that out of \( \{a_1, a_2, \ldots, a_7\} \) we have \( \binom{7}{3} = 35 \) positively weighted 3-submultisets. If we produce 31 more positively weighted 3-submultisets with each one of them involving at least one negative \( a_i \), then we would be done in this case also. Since \( \text{wt}\{a_1, a_7, a_{13}\} > 0 \), replacing \( a_7 \) by any \( a_i \) from \( \{a_2, \ldots, a_7\} \) and \( a_{13} \) by any \( a_i \) from \( \{a_8, a_9, \ldots, a_{13}\} \) still ends up in a positively weighted 3-submultiset, and we get 36 of them this way.

Case: when \( m = 8 \).

The multiset \( \{a_1, a_2, \ldots, a_8\} \) contains \( \binom{8}{3} = 56 \) positively weighted 3-submultisets. Therefore we have to produce 10 more 3-submultisets of \( A \) with at least one negative \( a_i \) and with positive weight. Partition \( A \backslash \{a_9\} \) as

\[
A \backslash \{a_9\} = Q_1 \cup Q_2 \cup Q_3 \cup Q_4 \quad \text{where} \quad Q_1 = \{a_4, a_5, a_{10}\}, \quad Q_2 = \{a_6, a_7, a_{11}\}, \quad Q_3 = \{a_1, a_8, a_{12}\} \quad \text{and} \quad Q_4 = \{a_2, a_3, a_{13}\}
\]

If \( \text{wt}(Q_1) > 0 \), then \( \text{wt}\{a_9, a_i, a_j\} > 0 \) for any 2-submultiset \( \{i, j\} \subseteq \{1, 2, 3, 4, 5\} \). Therefore, we have 10 positively weighted 3-submultisets involving \( a_9 \).
If \( \text{wt}(Q_1) < 0 \), then \( \text{wt}(Q_2) < 0 \). So we consider the cases \( \text{wt}(Q_3) > 0 \) and \( \text{wt}(Q_4) > 0 \).

\( \text{wt}(Q_3) > 0 \Rightarrow \text{wt}\{a_i, a_j, a_k\} > 0 \) for any \( i \in \{2,3,\ldots,7,9,10,11,12\} \) and we have 10 of them in total. If \( \text{wt}(Q_4) > 0 \) then \( \text{wt}\{a_i, a_j, a_k\} > 0 \) for \( \{i,j\} \subseteq \{1,2,3\} \) and \( k \in \{9,10,11,12,13\} \).

Therefore, we get 15 positively weighted 3-submultisets of \( A \) involving one negative \( a_i \) and hence the \( m = 8 \) case is also done.

Therefore we conclude that \( V_{t_1}(J(13,3)) = \binom{12}{2} \). ◇

**Proof of corollary 2.4.6:**

Since 3 divides 12, by theorem 2.2.2, the first distribution invariant of \( J(12,3) \) is \( \binom{12}{3} \). By Lemmas 2.4.6 and 2.4.7

\[ J(n,3) = \binom{n-1}{2} \] if \( n = 11 \) and 13.

Therefore, by theorems 2.4.1 and 2.4.2, we conclude that \( J(n,3) \) has \( \binom{n-1}{2} \) as its first distribution invariant if \( n \geq 11 \). ◇
Chapter 3
The First Distribution Invariant of $J_q(n,d)$

3.1 Introduction

Let $n$, $d$ be positive integers with $2d \leq n$. Let $V$ be a vector space of dimension $n$ over $GF(q)$, a finite field of $q$ elements, and let $X$ be the set of all $d$-dimensional subspaces of $V$. Let $\mathcal{R}_0, \mathcal{R}_1, \ldots, \mathcal{R}_d$ be subsets of $X \times X$ defined by

$$\mathcal{R}_i = \{ (A,B) \mid \dim (A \cap B) = d - i \} \text{ for } 1 \leq i \leq d.$$ 

Then $\mathcal{X} = (X, \{ \mathcal{R}_i \}_{0 \leq i \leq d})$ becomes a symmetric association scheme and is called the $q$-analogue of the Johnson scheme (also called the generalized Johnson scheme), and is written as $J_q(n,d)$. Let $V = V_0 \perp V_1 \perp \ldots \perp V_d$ be the orthogonal decomposition of the vector space $V = \mathbb{R}^X$ with each $V_i$ being the maximal common eigenspace of
the adjacency matrices of $J_q(n,d)$. As in the Johnson scheme, we can give an algebraic expression for $V_p$, as explained in the next paragraph.

In what follows, the notation $\left[ \begin{array}{c} n \\ m \end{array} \right]_q$ will stand for the Gaussian integer

$$\left[ \begin{array}{c} n \\ m \end{array} \right]_q = \frac{[(q^n-1)(q^n-q^2) \ldots (q^n-q^{m-1})]}{[(q^m-1)(q^m-q^1) \ldots (q^m-q^{m-1})]}$$

Let $M_{i,d}$ denote the $[i] \times [d]$ matrix whose rows and columns are indexed by $i$-dimensional and $d$-dimensional subspaces of $V$ respectively. Let $U_i$ denote the row space of $M_{i,d}$.

**Lemma 1:** $U_{i-1} \subseteq U_i$ for $1 \leq i \leq d$.

**Proof:** Let $X_i$ be the set of all $i$-dimensional subspaces of $S$, for $1 \leq i \leq d$. For $x \in X_d$, let $x$ be the column of $x$ in $M_{d,d}$. For $y \in X_i$, denote by $y$ the column of $y$ in $M_{i,d}$. Then $y = \sum_{y \in x} x$ (summation varies over $x$ and $x \in X_d$)

Now let $y \in X_{i-1}$. We need to show $y \in U_i$. For any given $(i-1)$ dimensional subspace in a $d$-dimensional space $W$, there are $\binom{d-i+1}{i}$ $i$-dimensional subspaces of $W$ containing the given $(i-1)$ dimensional subspace. Therefore

$$\sum_{y \leq w \in X_i} \binom{d-i+1}{i} y \Rightarrow y \in U_i.$$
**Lemma 2:** Let $V$ be a vector space of dimension $n$ over $GF(q)$ and let $U, W$ be $i$ and $d$ dimensional subspaces of $V$ respectively. If $U \subseteq W$, then there exists $\binom{d-i}{1}\binom{n-d}{1}q$ $d$-dimensional subspaces of $V$ containing $U$ and having $(d-1)$ as the dimension of the intersection space with $W$. If $\dim(U \cap W) = i-1$, then there exists $\binom{d-i+1}{1}$ $d$-dimensional subspaces of $V$ containing $U$ and having $(d-1)$ as the dimension of the intersection space with $W$.

**Proof:** Straight forward.

**Lemma 3:** $U_i = V_i \perp U_{i-1}$ for $1 \leq i \leq d$.

Let $W_i$ be the orthogonal complement of $U_{i-1}$ in $U_i$. Any vector $v \in U_i$ can be written as $v = v_1 + v_2$ where $v_1 \in W_i$ and $v_2 \in U_{i-1}$.

Define $\Pi : U_i \rightarrow W_i$ by $\Pi(v) = v_1$.

To prove the lemma, we need to show $W_i = V_i$.

**Claim** $\lambda w, \forall w \in W_i$, where $\lambda_i$ is the eigenvalue of $A = A_1$ on $V_i$. We know that $\lambda_i = \binom{d-i}{1}\binom{n-d}{1}q - \binom{i}{1}\binom{d-i+1}{1}$ (see [1]).

Let $u \in X_i$. We want to show $A(\Pi u) = \lambda_i(\Pi u)$. By the definition of $u$,
\[ Au = A \sum_{x \subseteq X} (x \in X_d) \]

\[ = \sum_{u \subseteq x} \sum_{y \subseteq x} (y \in X_d) \text{ (by the definition of adjacency)} \]

\[ = \sum_{u \subseteq x}(\sum_{y \subseteq X_{d-1}} (y \cap x \neq X_{d-1})) (y \in X_d) \]

\[ = \left( \sum_{y \subseteq X_{d-1}} \right)^{d-i+1} + \left( \sum_{u \subseteq y} \right)^{d-i} \mid |1|^{n-d} \mid q \]

(by lemma 3.1.2)

\[ \sum_{v \subseteq u, v \subseteq X_{d-1}} (y \subseteq X_d, y \cap u = v) \mid |1|^{d-i+1} + u \mid |1|^{d-i} \mid |1|^{n-d} \mid q \]

\[ = \sum_{v \subseteq u} (v \subseteq y - u \subseteq y) \mid |1|^{d-i+1} + u \mid |1|^{d-i} \mid |1|^{n-d} \mid q \]

\[ = \sum_{v \subseteq u} v - [i] \mid |1|^{d-i+1} + u \mid |1|^{d-i} \mid |1|^{n-d} \mid q \]

\[ = \sum_{v \subseteq u} v + u([i] \mid |1|^{n-d} \mid q - [i] \mid |1|^{d-i+1}) \]

Since \( \sum_{v \subseteq u} v \in U_{i-1} \) and \( \lambda_i = [i] \mid |1|^{n-d} \mid q - [i] \mid |1|^{d-i+1} \), we obtain \( A(\Pi u) = \Pi(u \lambda_i) = \lambda_i(\Pi u) \) and hence the claim.

Since \( J_q(n,d) \) is a \( P \)-polynomial association scheme, by lemma 1.1.2, \( W_i \) becomes a common eigenspace of the adjacency matrices of \( J_q(n,d) \) with \( \lambda_i \) as the eigenvalue of \( A_1 \). As \( V = U_d \), we get \( V = W_d \perp W_{d-1} \perp \ldots \perp W_0 \) where \( W_0 = V_0 \). From the uniqueness of the decomposition of \( V \) as \( V = V_0 \perp V_1 \perp \ldots \perp V_d \) (up to ordering) we conclude that \( W_i = V_i \) and hence \( U_i = V_i \perp U_{i-1} \).
Let \( I = \{v_1, \ldots, v_{[n]}\} \) be the set of all projective points of \( V \). Let \( \{W_1, \ldots, W_{[n]}\} \) be the collection of all \( d \)-dimensional subspaces of \( V \). Assume that the \((i, j)^{\text{th}}\) entry of \( M_{id} \) is indexed by \( v_i \) and \( W_j \) for \( 1 \leq i \leq [n] \) and \( 1 \leq j \leq [n] \).

From the definition of \( U_j \), we can express it as

\[
U_i = \sum_{1 \leq j \leq [n]} a_i m_i \quad | \quad a_i \in \mathbb{R}
\]

where \( m_1, m_2, \ldots, m_{[n]} \) in the first sum represent the row vectors of \( M_{id} \).

Since \( U = V_1 \perp U_0 = V_1 \perp V_0 \) and \( V_0 = \langle (1, 1, \ldots, 1) \rangle \subseteq \mathbb{R}^{[d]} \), we note that

\[
V_1 = \{ (v_i \sum_{j=1}^{[n]} a_i)_{1 \leq j \leq [d]} \mid \langle (v_i \sum_{j=1}^{[n]} a_i)_{1 \leq j \leq [d]}, (1, 1, \ldots, 1) \rangle = 0 \}
\]

Since each projective point is contained in exactly \([d-1] \) \( d \)-dimensional subspaces,

\[
\langle (v_i \sum_{j=1}^{[n]} a_i)_{1 \leq j \leq [d]}, (1, 1, \ldots, 1) \rangle = 0
\]

\[
\Leftrightarrow [d-1] (\sum_{1 \leq i \leq [n]} a_i) = 0
\]

\[
\Leftrightarrow \sum_{1 \leq i \leq [n]} a_i = 0,
\]

and conversely every \([n]-\)tuple \((a_1, \ldots, a_{[n]})\) of reals satisfying the property that \( \sum_{1 \leq i \leq [n]} a_i = 0 \) gives rise to a vector in \( V_1 \) whose \( j^{\text{th}} \) com-
ponent is $\sum_{i=1}^{J} a_i$ for $1 \leq j \leq \frac{n}{d}$. Hence $V_1$ is isomorphic to \\
$\{ \sum_{v \in I} a_v e_v = (a_v)_{v \in I} \mid \sum_{v \in I} a_v = 0 \}$, where $e_v$ denotes the vector of length $\frac{n}{d}$ with 1 in the $v^{th}$ place and 0 elsewhere.

**Remark 3.1.1:** By identifying $V_1$ with $\{ \sum_{v \in I} a_v e_v \mid \sum_{v \in I} a_v = 0 \}$, we conclude that a vector $w = (a_v)_{v \in I}$ is a general vector if and only if it satisfies

(i) $\sum_{v \in I} a_v = 0$ and

(ii) $\sum_{v \subseteq W} a_v \neq 0$ for every $d$-dimensional subspace $W$ of $V$.

For any given general vector, first we determine the minimum number of $d$-dimensional subspaces of $V$ whose corresponding sum as in (ii) above is positive. The first distribution invariant is the minimum of these minimum numbers for all general vectors.

**Definition 3.1.2:** Let $(a_v)_{v \in I}$ be any general vector in $V_1$. Then for any subspace $W$ of $V$, we define the weight of $W$ with respect to the vector $(a_v)_{v \in I}$ to be equal to $\sum_{v \subseteq W} a_v$, where $v$ varies over the projective points of $W$, and write it as $wtW$.

**Proposition 3.1.3** The first distribution invariant $V_{t_1}$ of $J_q(n,d)$ is less than or equal to $\frac{n}{d-1}$. 
Proof: Let \( \mu \) be a fixed projective point of \( V \) and let \( N = |I| = (q^n-1)/(q-1) \).

Let \( w = (N-1)e_\mu - \sum_{i \in I \setminus \mu} e_i \). Then \( w \in V_1 \) and \( w \) satisfies the conditions stated in the remark 3.1.1. Hence \( w \) is a general vector. Note that any \( d \)-dimensional subspace \( W \) has a positive weight with respect to \( w \) if and only if it contains the projective point \( \mu \).

Therefore, in total, we have \( \left\lfloor \frac{n-1}{d-1} \right\rfloor \) \( d \)-dimensional subspaces having positive weights with respect to \( w \). Now, from the definition of the first distribution invariant, it follows that \( V_{t_1} \leq \left\lfloor \frac{n-1}{d-1} \right\rfloor \).

Remark 3.1.4: From the above proposition, it is clear that to prove \( V_{t_1}(J_q(n,d)) = \left\lfloor \frac{n-1}{d-1} \right\rfloor \), we need to show that if \((a_v)_{v \in I}\) is any general vector in \( V_1 \), then there exist at least \( \left\lfloor \frac{n-1}{d-1} \right\rfloor \) \( d \)-dimensional subspaces of \( V \) having positive weights with respect to \((a_v)_{v \in I}\).

3.2 First distribution invariants and \( d \)-spreads

In this section we prove that the first distribution invariant of \( J_q(n,d) \) is equal to \( \left\lfloor \frac{n-1}{d-1} \right\rfloor \) if \( d \) divides \( n \).

Definition 3.2.1: A \( d \)-spread \( \xi \) of \( V \) is a set of \( d \)-dimensional sub-
spaces of $V$ such that each projective point lies in one and only one $d$-dimensional subspace contained in $\xi$ (see [9]).

**Lemma 3.2.2:** If $s$ denotes the total number of $d$-spreads and if $r$ is the total number of $d$-spreads containing a fixed $d$-dimensional subspace, then

$$s/r = \frac{\left[ \frac{n}{d} \right] / (q^{d-1}) / (q^n - 1)}{\left[ \frac{n}{d} \right] / (q^{d-1}) / (q^n - 1)}$$

**Proof:** Observe that

$$s/r = \frac{\text{total number of $d$-dimensional subspaces}}{\text{total number of $d$-dimensional subspaces in a spread.}}$$

$$= \left[ \frac{n}{d} \right] / (q^n - 1)$$

$$= \left[ \frac{n}{d} \right] / (q^{d-1})$$

**Theorem 3.2.3:** If $d|n$ and if $D$ is a collection of $d$-dimensional subspaces of an $n$-dimensional space $V$ over $GF(q)$ with $|D| > \left[ \frac{n}{d} \right] - \left[ \frac{n}{d} - 1 \right]$, then $D$ contains a $d$-spread of $V$.

**Proof:** Let us assume to the contrary that $D$ does not contain any $d$-spread of $V$. Now we count the number of pairs $(\xi, A)$ where $\xi$ is a $d$-spread and $A$ is an element of $D$ which is contained in $\xi$. Let $r$ and $s$ be as in Lemma 3.2.2.
For any given d-dimensional subspace \( A \) contained in \( D \), the number of pairs \((\xi, A)\) with the property as described above is equal to the number of d-spreads containing \( A \) and this number is independent of \( A \). Therefore, total number of pairs \((\xi, A)\)

\[
|D|(r)
\] (1)

On the other hand, since we assumed that there does not exist a d-spread consisting of d-subspaces only from \( D \), any given d-spread \( \xi \) contains at the most

the number of d-dimensional subspaces in a d-spread - 1, which is equal to \( \{(q^n-1)/(q^d-1)\} - 1 \) elements from \( D \).

Therefore the total number of pairs \((\xi, A)\) \( \leq s\{(q^n-1)/(q^d-1)\} - 1\}

(2)

Using (1) and (2) we see that

\[
|D| \leq (s/r)\{(q^n-1)/(q^d-1)\} - 1
\]

From Lemma 3.2.2, we know that

\[
s/r = \left[ \frac{\mathcal{R}}{\mathcal{Q}} \right] / \{(q^n-1)/(q^d-1)\}
\]

Therefore, \( |D| \leq \left[ \frac{\mathcal{R}}{\mathcal{Q}} \right] / (q^n-1/q^d-1) \left[ q^n-1/q^d-1 - 1 \right] \)
which is a contradiction to the fact that $|D| > |\begin{bmatrix} n \\ 3 \end{bmatrix} - |d - 1|$. Therefore, our assumption that D does not contain a d-spread is false.

**Corollary 3.2.4:** As in Theorem 3.2.3, we assume that $d,n$ are positive integers with $d|n$ and V is a vector space of dimension $n$ over GF(q). Let $I$ denote the collection of all projective points of V. If each projective point $v$ of V is assigned a weight $a_v$ satisfying

(i) $\sum_{v \in I} a_v \geq 0$ and

(ii) $wtW = \sum_{v \leq W} a_v \neq 0$ for every $d$-dimensional subspace $W$ of $V$.

Then there exists at least $|\begin{bmatrix} n \\ d \end{bmatrix} - 1|$ $d$-dimensional subspaces having positive weights.

**Proof:** Let $D=\{\text{negatively weighted } d\text{-dimensional subspaces of } V\}$. Then we must have $|D| \leq |\begin{bmatrix} n \\ 3 \end{bmatrix} - |d - 1|$. Otherwise, if $|D| > |\begin{bmatrix} n \\ 3 \end{bmatrix} - |d - 1|$, then by the above theorem, D contains a d-spread. Since each $d$-dimensional subspace in this spread is negatively weighted, the total weight of the d-spread $\sum_{v \in I} a_v$ must be negative, contradicting (i).

Hence the number of positively weighted $d$-dimensional subspaces of $V$ must be $\geq |\begin{bmatrix} n \\ 3 \end{bmatrix} - |D| = |\begin{bmatrix} n \\ d \end{bmatrix} - 1|$. 
**Theorem 3.2.5:** If \( d \) divides \( n \), then \( V_{t_1} = \left[ \frac{n-1}{d} \right] \).

**Proof:** Immediate from proposition 3.1.1 and Corollary 3.2.4.

**Lemma 3.2.6:** Let \( w = (a_v)_{v \in I} \) be a general vector in \( V_1 \). Then for any \( s \) with \( 1 \leq s \leq n \) there exists a subspace \( W \) of \( V \) of dimension \( s \) such that the weight of \( W \) (with respect to \( w \)) = \( \sum_{v \subseteq W} a_v > 0 \) (ie. summing over the projective pts of \( W \)).

**Proof** For any non-zero vector \( u \) in \( V \), there exist \( \left[ \frac{n-1}{s-1} \right] \) \( s \)-dimensional subspaces containing the projective point \( u \).

Therefore, \( \sum_W \sum_{v \subseteq W} a_v = \left[ \frac{n-1}{s-1} \right] \left( \sum_{v \in I} a_v \right) = 0 \), where the first sum on the left hand side varies over the set of all \( s \)-dimensional subspaces of \( V \). Since \( \sum_W \sum_{v \subseteq W} a_v = 0 \), there exists at least one \( s \)-dimensional subspace \( W \) with the property that \( wtW = \sum_{v \subseteq W} a_v > 0 \).
3.3 Certain transitivities on $V_{t_1}$

In this section we obtain a lower bound for the first distribution invariant of $J_q(n,d)$. We also prove that if $V_{t_1}(J_q(n,d)) = \lfloor \frac{n-1}{d-1} \rfloor$ then $V_{t_1}(J_q(kn,d)) = \lfloor \frac{kn-1}{d-1} \rfloor$ for any positive integer $k$.

**Theorem 3.3.1:** $V_{t_1} \geq \lfloor \frac{n-r-1}{d-1} \rfloor$ if $n = pd + r$ with $0 \leq r < d$.

**Proof:** Let $w = (a_v)_{v \in \mathbb{I}}$ be any general vector in $V_1$. Let $s = n - r$. By Lemma 3.2.6 there exists a subspace $W$ of $V$ of dimension $s$ satisfying $\sum_{v \in W} a_v \geq 0$. Since $d$ divides $s$, by corollary 3.2.4, there exist at least $\lfloor \frac{s-1}{d-1} \rfloor$ $d$-dimensional subspaces of $W$ having positive weights. These are also $d$-subspaces of $V$ having the same weight, hence $V_{t_1} \geq \lfloor \frac{s-1}{d-1} \rfloor$.

$s = n - r \Rightarrow V_{t_1} \geq \lfloor \frac{n-r-1}{d-1} \rfloor$.

**Theorem 3.3.2:** For any positive integer $k$, the first distribution invariant of $J_q(km,d)$ is $\lfloor \frac{km-1}{d-1} \rfloor$ if $J_q(m,d)$ has $\lfloor \frac{m-1}{d-1} \rfloor$ as its first distribution invariant.

**Proof:** Let $w$ be any general vector in the first eigenspace of $J_q(km,d)$. Let

$N = \{ W \mid W \text{ is a } d\text{-dimensional subspace of } V \text{ and } wtW < 0 \}$,

where $wtW$ represents the weight of $W$ with respect to $w$. 
Claim: \(|N| \leq \binom{k}{d} \cdot \binom{k-1}{d-1}\)

Count the number of pairs \((M,A)\) where \(M\) is an \(m\)-spread of \(V\), \(A\) is a \(d\)-dimensional subspace in \(N\), and \(A\) is contained in one of the \(m\)-dimensional subspaces in \(M\). For any fixed \(d\)-dimensional subspace \(A\), the number of pairs of \((M,A)\) is exactly equal to

\[
\binom{k-m-d}{m-d} \{\text{number of } m\text{-spreads containing a fixed } m\text{-dimensional subspace}\}.
\]

Therefore, by varying \(A\) over the elements of \(N\) we conclude that in total we have

\[
|N| \binom{k-m-d}{m-d} \{\text{the number of } m\text{-spreads containing a fixed m-dimensional subspace }\}
\]

such pairs.

On the other hand, suppose we fix an \(m\)-spread \(M\) and count the number of pairs \((M,A)\). \(M\) contains \((q^m-1) / (q^m-1) = r\) (say) \(m\)-dimensional subspaces of \(V\). Let \(W_1, \ldots, W_r\) be all of them and let

\[
N_i = N \cap \{\text{d-dimensional subspaces of } W_i\}
\]

As \(\sum_{1 \leq i \leq r} \text{wt}W_i(w.r.t \, w) = (\text{wt } w)\), here \(\text{wt } w\) stands for the sum of
all the components of $w$ (refer definition 3.1.2). Since $w$ is a vector in $V_1$, $wt\ w \geq 0$. This implies

$$\sum_{1 \leq i \leq r} wt\ W_i \ (w.r.t.\ w) \geq 0.$$  

This implies that there exists at least one $j$ such that $wtW_i$ with respect to $w$ is non-negative. As we have assumed that $Vt_1(J_{q}(m,d)) = \begin{bmatrix} m-1 \end{bmatrix}_{d-1}$, we have at least $\begin{bmatrix} m-1 \end{bmatrix}_{d-1}$ d-dimensional subspaces of $W_j$, with each of them having positive weight with respect to $w$. Hence the cardinality of $N_j$ is at most $\begin{bmatrix} m \end{bmatrix}_d - \begin{bmatrix} m-1 \end{bmatrix}_{d-1}$. Therefore, for any fixed m-spread $M$, we have at most $\{(q^{km-1}) / (q^{m-1}) - 1\} \begin{bmatrix} m \end{bmatrix}_d + \begin{bmatrix} m \end{bmatrix}_d - \begin{bmatrix} m-1 \end{bmatrix}_{d-1}$ pairs $(M,A)$, which is equal to $\{(q^{km-1})/(q^{m-1})\} \begin{bmatrix} m \end{bmatrix}_d - \begin{bmatrix} m-1 \end{bmatrix}_{d-1}$.

Now, by varying $M$ over the set of all m-spreads of $V$, we conclude that the total number of pairs $(M,A)$

$$\leq (\text{total number of m-spreads})\{(q^{km-1})/(q^{m-1})\} \begin{bmatrix} m \end{bmatrix}_d - \begin{bmatrix} m-1 \end{bmatrix}_{d-1}$$  

(2)

Let $s = \text{the total number of m-spreads and } t = \text{the total number of m-spreads containing a fixed m-dimensional subspace.}$

By comparing (1) and (2) we see that
\[ |N| = \left\{ \frac{(s/t)((q^{km-1})/(q^{m-1}))|^m \cdot [L_{-1}]}{\left[\frac{m-1}{d-1}\right]} \right\} / \left[\frac{km-d}{m-d}\right] \] (3)

By Lemma 3.2.2,

the total number of \( m \)-spreads

the total number of \( m \)-spreads containing a fixed \( m \)-dimensional subspace

the total number of \( m \)-dimensional subspace

\[ = \left[\frac{km}{m}\right] \frac{(q^m-1)/(q^{km-1})}{(q^{km-1})} \] (4)

Substituting (4) in (3) we get that

\[ |N| \leq \left[\frac{km}{m}\right] \frac{((q^m-1)/(q^{km-1}))}{((q^{km-1})/(q^{m-1}))} \left[\frac{[L_{-1}]}{[d-1]}\right] \left[\frac{[km-d]}{[m-d]}\right] \]

\[ = \left[\frac{km}{d}\right] - \left[\frac{km-1}{d-1}\right] \]

and hence the claim that \( |N| \leq \left[\frac{km}{d}\right] - \left[\frac{km-1}{d-1}\right] \) is settled.

Remark 3.3.3: By applying techniques similar to the above, we could also prove that if the following statement(*) is true for \( m \) then it is also true for \( km \), for any non-negative integer \( k \).

Statement (*): Let \( V \) be a vector space of dimension \( m \) over a finite field \( GF(q) \). Let \( S \) be a collection of \( d \)-dimensional subspaces of
V. If $|S| > \frac{m}{d} - \frac{m-1}{d-1}$, then there exist non-negative numbers $\{\alpha_A\}_{A \in S}$ such that $\sum_{A \in S} \alpha_A X_{|A|} = X_{|V|}$ where $|V|$, $|A|$ are the sets of all projective points of $V$ and $A$ respectively and $X_{|A|}$, $X_{|V|}$ are the characteristic vectors of length $\frac{m}{1}$.
Chapter 4

The First Distribution Invariant of $H(n,q)$

Let $S$ be a set of cardinality $q$ ($q \geq 2$) and $X = \{(x_1, \ldots, x_n) \mid x_i \in S, 1 \leq i \leq n\}$. Let $d_H : X \times X \to \mathbb{N} \cup \{0\}$ be the distance function defined by

$$d_H(x,y) = |\{ i \mid x_i \neq y_i, 1 \leq i \leq n\}|$$

for $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. We define the $i$th distance relation $R_i$ on $x$ by

$$R_i = \{(x,y) \in X \times X \mid d_H(x,y) = i\}$$

Then $X = (X \{R_i\}_{0 \leq i \leq n})$ becomes a symmetric association scheme. It is called the Hamming scheme or the hypercubic association scheme and written as $H(n,q)$. In fact, $H(n,q)$ is isomorphic to the direct sum of $n$ copies $J_1(q,1)$ of the Johnson scheme $J(q,1)$, where for any two elements $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in $\bigoplus_{1 \leq i \leq n} J_1(q,1)$ the distance between them is defined to be $\sum_{1 \leq i \leq n} d_J(\{x_i\}, \{y_i\})$ [6].
In this chapter we prove that the first distribution invariant of $H(n,q)$ is $q^{n-1}$. This fact has also been proved independently by Thomas Bier. But the method used in his proof is totally different from the one given here.

Let $V = V_0 \perp V_1 \perp \ldots \perp V_n$ be the orthogonal decomposition of $V = \mathbb{R}^X$ into the maximal common eigenspaces of the adjacency matrices of $H(n,q)$. Then $V_1 \cong V_{11} + V_{12} + \ldots + V_{1n}$, where $V_{1i}$ represents the first eigenspace of the $i$-th copy of $J(q,1)$ (when $H(n,q)$ is written as $\bigoplus_{1 \leq i \leq n} J(q,1)$).

From the description given for the first eigenspace of $J(n,d)$ in chapter II (page 14), we deduce that any element in the first eigenspace of $H(n,q)$ is of the form $(a_1, \ldots, a_n)$ with the property that

$$\sum_{1 \leq i \leq n} a_i = 0$$

for $1 \leq i \leq n$.

Let $Y = \{y_1, \ldots, y_n\}$ be the vertices of $J(q,1)$. For the $i$-th copy of $J(q,1)$ denote by $e_y^i$ the element $(0, \ldots, 0, 1, 0, \ldots, 0)$ belonging to $\mathbb{R}^q \times \ldots \times \mathbb{R}^q$ (n copies of $\mathbb{R}^q$), where 1 occurs in the $y$-th position (according to the ordering $Y = \{y_1, \ldots, y_n\}$ of the $i$-th copy of $\mathbb{R}^q$).

Let $V^{(i)}$ be the vector space of the $i$-th copy of $J(q,1)$ generated by $\{e_y^i\}_{y \in Y}$. Let $\Pi_1^i$ denote the projection of $V^{(i)}$ onto $V_{11}$. It can be easily
checked that \( \Pi_i (e_i^y) = e_i^y - (1/q)\delta_i \) where \( \delta_i = \sum_{y \in Y} e_i^y \). For \( x = (x_1, \ldots, x_n) \in X \), we define \( \Pi_1(e_x) = \prod_{1 \leq j \leq n} \Pi_i (e_i^x_j) \). Again, it is easy to check that \( \Pi_1 \) is the projection of \( V \) onto \( V_1 \).

Let \( w \in V_1 \) be any general vector. Since \( w \in V_1 \), we can write it as \( w = \sum_{1 \leq i \leq n} \sum_{y \in Y} a_i^y e_i^y \) with \( \sum_{1 \leq i \leq n} a_i^y = 0 \) for \( 1 \leq i \leq n \). Since \( w \) is a general vector, we have \( \langle w, \Pi_1(e_x) \rangle \neq 0 \) for every \( x \) in \( X \). Let \( x = (x_1, \ldots, x_n) \in X \) be any arbitrary element. Then

\[
0 \neq \langle w, \Pi_1(x) \rangle
= \langle \sum_{1 \leq i \leq n} \sum_{y \in Y} a_i^y e_i^y, \Pi_1 (e_i^x) \rangle
= \sum_{1 \leq i \leq n} \langle \sum_{y \in Y} a_i^y e_i^y, e_i^x \rangle
= \sum_{1 \leq i \leq n} a_i^x
\]

Hence we conclude that if \( 1 \leq i \leq n \sum_{y \in Y} a_i^y e_i^y \in V_1 \) is a general vector, then \( a_i^x \neq 0 \) for any element \( x = (x_1, \ldots, x_n) \in X \). Conversely, if \( \{a_j^1, \ldots, a_j^q, a_j^2, \ldots, a_j^n, \ldots, a_j^n\} \) is contained in \( R \) and satisfies \( \sum_{1 \leq j \leq q} a_j^i = 0 \) for \( 1 \leq i \leq n \), then \( \sum_{1 \leq i \leq n} \sum_{y \in X} a_i^y e_i^y \in V_1 \) and is a general vector. (\( a_i^y \) stands for \( a_i^y \) if \( y = y_j \))

**Proposition 4.1.1:** The first distribution invariant of \( H(n,q) \) is less than or equal to \( q^{n-1} \).
Proof: Consider the vector \( w = \sum_{1 \leq i \leq n} [ (q-1)e_i^1 - \sum_{y \neq y \in Y} e_y^1 ] \). \( w \) belongs to the first eigenspace of \( H(n,q) \) and is a general vector. The only elements \( x = (x_1, \ldots, x_n) \) in \( X \) with the property that \( \langle w, E_i(x) \rangle > 0 \) are the elements with at least one of the \( x_i \) equal to \( y_1 \). There are exactly \( q^{n-1} \) such elements in \( X \) and hence \( V_{t_1}(H(n,q)) \leq q^{n-1} \).

To prove \( V_{t_1}(H(n,q)) = q^{n-1} \), we need to prove the following lemma.

Lemma 4.1.2: Let \( a_1^1, \ldots, a_q^1, a_1^2, \ldots, a_q^2, \ldots, a_1^n, \ldots, a_q^n \) be real numbers such that \( \sum_{1 \leq i \leq q} a_i^j = 0 \) for \( i = 1, \ldots, n \) and \( \sum_{1 \leq i \leq n} a_i^j \neq 0 \) for every \( n \)-tuple \((j_1, \ldots, j_n)\) with \( 1 \leq j_i \leq q \) for \( 1 \leq i \leq n \). Then there exist at least \( q^{n-1} \) such \( n \)-tuples with \( \sum_{1 \leq i \leq n} a_i^j > 0 \).

Proof: Let \( \Lambda = \{a_1^1, \ldots, a_q^1, \ldots, a_1^n, \ldots, a_q^n\} \). Consider the partitions \( B = \{A_1, \ldots, A_q\} \) of \( \Lambda \) where each \( A_i \) is an \( n \)-tuple of the type \( \{a_1^j, \ldots, a_n^j\} \) with \( \sum_{1 \leq i \leq n} a_i^j \neq 0 \).

\[ (1) \]

We claim that there exists at least \( q^{n-1} \) such disjoint partitions of \( \Lambda \). We prove it by induction on \( n \). If \( n = 1 \), the claim is obviously true. Let us assume that it is true for \( n-1 \), i.e. if \( \Lambda^1 = \{a_1^1, \ldots, a_q^1, a_1^2, \ldots, a_q^2, \ldots, a_1^n, \ldots, a_q^n\} \).
\(a_1^2, \ldots, a_{q-1}^n, a_q^n\) then there exist \(q^{n-2}\) disjoint partitions of \(\Lambda^1\) as described in (1) (partition into \((n-1)\)-tuples).

Let \(B_1 = \{A_1^1, \ldots, A_q^1\}\) be any partition of \(\Lambda^1\). Then \(B = \{A_1^1 \cup \{a_1^n\}, \ldots, A_q^1 \cup \{a_q^n\}\}\) is a partition of \(\Lambda\). By cyclically moving the elements of \(\{a_1^n, \ldots, a_q^n\}\) in \(B\) we see that we get at least \(q\) new disjoint partitions of \(\Lambda\) from \(B_1\). Hence we get \(q^{n-2} \cdot q = q^{n-1}\) partitions of \(\Lambda\) satisfying the conditions as in (1). From the construction, it is clear that they are all disjoint and hence our claim is true by induction.

For any partition \(B\) as in (1), we must have at least one \(n\)-tuple \(A\) having positive sum, hence in total we have at least \(q^{n-1}\) \(n\)-tuples \(\{a_1^n, \ldots, a_q^n\}\) with positive sums.

**Theorem 4.1.3:** The first distribution invariant of \(H(n,q)\) is equal to \(q^{n-1}\) for any \(n\) and \(q\).

**Proof:** Immediate from proposition 4.1.1 and lemma 4.1.2.
Chapter 5
The First Distribution Invariant of $H_q(n,d)$

5.1 Introduction

Let $X$ be the set of all $d \times n$ matrices over $GF(q)$ ($d \leq n$). Define

$$\mathcal{R}_i = \{ (y,z) \in X \times X \mid \text{rank}(y-z) = i \}$$

Then $\mathcal{X} = (X, \{\mathcal{R}_i\}_{0 \leq i \leq d})$ becomes a symmetric association scheme. Indeed, it is a P-polynomial association scheme. It is usually called the generalized hamming scheme or the $q$-analogue of the Hamming scheme and is written as $H_q(n,d)$. It is pertinent here to note that $X$ can also be considered as the collection of all bilinear forms $f : W \times W' \to GF(q)$, where $W$ and $W'$ are vector spaces over $GF(q)$ of dimensions $d$ and $n$ respectively.
In this chapter, we introduce a new concept known as W-complement d-spreads of a vector space $H$ of dimension $n+d$ over $GF(q)$ and establish a one-to-one correspondence between the set of all W-complement d-spreads of $H$ and the set of all $(d,n,l,q)$-Singleton systems of order $q^n$. We also prove that we prove that the first distribution invariant of $H_q(n,d)$ is $q^{n(d-1)}$ for all $n,d$ and $q$. The proof of this statement depends on the facts that $H_q(n,d)$ is isomorphic to the adjacency graph of a certain attenuated space and that there always exist W-complement d-spreads of $H$.

5.2 W-Complement d-Spreads

Let $H$ be a vector space of dimension $n+d$ over the finite field $GF(q)$. Assume that $d \leq n$. Let $W$ be a fixed $n$-dimensional subspace of $H$.

**Definition 5.2.1.** A collection $\Lambda$ of $d$-dimensional subspaces of $H$ is said to be a W-complement d-spread of $H$ if

1. $A \in \Lambda \Rightarrow A \cap W = \{0\}$

2. If $0 \neq v \in H$ and $<v> \cap W = \{0\}$, where $<v>$ denotes the 1-dimensional subspace (projective point) generated by $v$, then $<v>$ is contained in one and only one $d$-dimensional subspace lying in $\Lambda$. 
Lemma 5.2.2. Let C be the complement of W in H so that $H = C \oplus W$. Let $\{c_1, ..., c_d\}$ be a base of C. Then for any d-dimensional subspace A of H which intersects W trivially, there exists a basis $\{c_1 + x_1, ..., c_d + x_d\}$ of A for some uniquely chosen $x_1, ..., x_d$ in W.

Proof: Straightforward. \( \diamond \)

Lemma 5.2.3. Let $\Lambda = \{U_1, ..., U_{qn}\}$ be a collection of d-dimensional subspaces of H such that $U_i \cap W = \{0\}$ for $1 \leq i \leq q^n$. Let $\{c_k + u_{ik}\}_{1 \leq k \leq d}$ be bases of $U_i$, for $1 \leq i \leq q^{n}$, as obtained in lemma 1. Then $\Lambda$ is a W-complement d-spread of H if $\{u_{ik} - u_{jk}\}_{1 \leq k \leq d}$ is linearly independent for every pair $i, j$ with $1 \leq i, j \leq d$ and $i \neq j$.

Proof: To prove $\Lambda$ is a W-complement d-spread of H, it is sufficient to show that $U_i \cap U_j = \{0\}$ for any two arbitrary elements $U_i$ and $U_j$ in $\Lambda$. Suppose $u \in U_i \cap U_j$.

$u \in U_i \Rightarrow u = \sum_{1 \leq k \leq d} r_k (c_k + u_{ik})$

$u \in U_j \Rightarrow u = \sum_{1 \leq k \leq d} s_k (c_k + u_{jk})$ for some $r_k, s_k \in GF(q)$.

$\Rightarrow \sum_{1 \leq k \leq d} (r_k - s_k)c_k = \sum_{1 \leq k \leq d} (s_k u_{jk} - r_k u_{ik})$

The sum in the left hand side gives an element in C whereas the
right hand side sum is an element in $W$. Since $C \cap W = \{0\}$, we must have 
\[ \sum_{1 \leq k \leq d} (r_k - s_k)c_k = 0. \]
But $\{c_1, \ldots, c_d\}$ are linearly independent implies $r_k = s_k$ for $1 \leq k \leq d$. Hence 
\[ \sum_{1 \leq k \leq d} (s_ku_{jk} - r_ku_{ik}) = \sum_{1 \leq k \leq d} r_k(u_{jk} - u_{ik}) = 0. \]
As we have assumed $\{u_{ik} - u_{jk}\}$ to be linearly independent, we get that $r_k = 0$ for $k = 1, 2, \ldots, d$. So $U_i \cap U_j = \{0\}$. Thus $\Lambda$ is a $W$-complement $d$-spread of $H$. \hfill $\Diamond$

Theorem 5.2.4. If $D$ is a collection of $d$-dimensional subspaces of $H$ having trivial intersection with $W$ and if $|D| > q^{nd} - q^{n(d-1)}$, then $D$ contains a $W$-complement $d$-spread of $H$.

Proof: Let us assume to the contrary that $H$ does not contain a $W$-complement $d$-spread of $H$. Count the number of pairs $(\Lambda, A)$ where $\Lambda$ is a $W$-complement $d$-spread of $H$ and $A$ is an element of $D$ which is contained in $\Lambda$. For any given $d$-dimensional subspace $A$ contained in $D$, the number of such pairs $(\Lambda, A)$ involving $A$ is equal to the number of $W$-complement $d$-spreads of $H$ containing $A$ and this number is independent of $A$. Let $t$ denote the total number of pairs $(\Lambda, A)$, and let $r$ denote the number of $W$-complement $d$-spreads of $H$ containing $A$. Then

\[ t = |D|r \] (1)

On the other hand, since we assumed that there does not exist a $W$-complement $d$-spread of $H$ consisting of $d$-subspaces only from $D$, any
given \( W \)-complement \( d \)-spread \( \Lambda \) of \( H \) contains at the most the number of elements in \( \Lambda - 1 = (q^n-1) \) elements from \( D \). Therefore

\[
t \leq (\text{number of } W \text{-complement } d \text{-spreads of } H \text{ in total})(q^n-1)
\]

(2)

Let \( s = \text{number of } W \text{-complement } d \text{-spreads of } H \text{ in total.} \) Using (1) and (2) we see that

\[
|D| \leq \frac{s}{r}(q^n-1)
\]

Since \( (s/r) = \frac{\text{(number of } d \text{-dimensional subspaces in the comple­ment of } W)}{\text{(number of } d \text{-dimensional subspaces in a } W \text{-complement } d\text{-spread)}} = \{q^{nd}/q^n\} \), we have

\[
|D| \leq \{q^{nd}/q^n\}(q^n-1) = q^{nd}q^n(d-1), \text{ which is a contradiction to the assumption that } |D| \leq q^{nd}q^n(d-1). \text{ Hence } D \text{ contains a } W \text{-complement } d \text{-spread of } H. \diamond
\]
5.3 Singleton systems and $W$-complement $d$-spreads

Let $U$, $W$ be vector spaces of dimensions $d$ and $n$ respectively over $GF(q)$. As mentioned in section 5.1, we may consider the vertex set of $H_q(n,q)$ as the set of all bilinear forms $f : U \times W \to GF(q)$.

**Definition 5.3.1.** A subset $Y \subseteq X$ is a $(d,n,t,q)$-Singleton system, for $t \in \{0,1,...,d\}$, if for each $t$-dimensional subspace $U_0$ of $U$ and for each bilinear map $g : U_0 \times W \to GF(q)$, there exists a unique bilinear form $f \in Y$ whose restriction to $U_0 \times W$ equals $g$. [7]

**Theorem 5.3.2.** Let $H$ be a vector space of dimension $n+d$ over $GF(q)$ such that $W \subseteq H$ and $H = W \oplus C$ for some fixed $d$-dimensional subspace $C$. Then there exists a one-to-one correspondence between the set of all $W$-complement $d$-spreads of $H$ and the set of all $(d,n,1,q)$-Singleton systems of size $q^n$.

**Proof:** Let $\Omega = \{f_1, \ldots, f_q\}$ be a $(d,n,1,q)$-Singleton system. Observe that if $f, g \in \Omega$, then $f-g$ must be of full rank. Let $A = (a_{ij})$ and $B = (b_{ij})$ be the matrices of $f$ and $g$ respectively. Let $\alpha_i = \sum_{1 \leq k \leq n} a_{ik}w_k$ and $\beta_i = \sum_{1 \leq k \leq n} b_{ik}w_k$ for $i=1,2,\ldots,d$, where $\{w_1, \ldots, w_n\}$ is a fixed basis of $W$. Since $f-g$ is of full rank, $\{\alpha_i-\beta_i\}_{1 \leq i \leq d}$ must be linearly independent. Let $U_f = \langle c_1+\alpha_1, \ldots, c_d+\alpha_d \rangle$ and $U_g = \langle c_1+\beta_1, \ldots, c_d+\beta_d \rangle$. Clearly $U_f \cap U_g = \{0\}$. 
Let $A = \{ U_f \mid f \in \Omega \text{ and } U_f \text{ is defined as above } \}$. Then $A$ becomes a $W$-complement $d$-spread of $H$.

Conversely, let $\Gamma = \{ U_1, \ldots, U_{q^n} \}$ be a $W$-complement $d$-spread. If $U_i \in \Gamma$, then from lemma 1.1 we know that $U_i$ has a basis $\{ c_k + u_{ik} \}_{i=1}^d$ for some uniquely chosen vectors $u_{i1}, \ldots, u_{id}$ in $W$. Let $\{ e_1, \ldots, e_d \}$ and $\{ w_1, \ldots, w_n \}$ be fixed bases of $U$ and $W$ respectively. Since $u_{i1}, \ldots, u_{id} \in W$, we can write each $u_{ij}$ as $u_{ij} = \sum a_{ij}^k w_k$ for $a_{ij}^k \in \text{GF}(q)$ and for $1 \leq k \leq n$.

Define $f_i : U \times W \rightarrow \text{GF}(q)$ by $f_i(e_j, w_k) = a_{ij}^k$ for $1 \leq k \leq n$, $1 \leq j \leq d$. Then $f_i$ becomes a bilinear form. We call $f_i$ to be the bilinear form belonging to $U_i$. Let $f_1, \ldots, f_{q^n}$ be the bilinear form belonging to $U_1, \ldots, U_{q^n}$ respectively. We claim that $\{ f_1, \ldots, f_{q^n} \}$ is a $(d,n,1,q)$-Singleton system. If possible, let there be a non-zero vector $u \in U$ and bilinear forms $f_i, f_j$, $i \neq j$, satisfying $f_i|_{<u>} \times W = f_j|_{<u>} \times W$. Extend $\{ u \}$ to a basis $\{ u = u_1, u_2, \ldots, u_d \}$ of $U$. With respect to this basis, the first row of the matrix of $f_i$ is equal to the first row of the matrix of $f_j$. Thus $U_i \cap U_j \neq \{ 0 \}$, a contradiction to our assumption that $\Gamma$ is a $W$-complement $d$-spread of $V$. Hence $\{ f_1, \ldots, f_{q^n} \}$ is a $(d,n,1,q)$-Singleton system of order $q^n$.

Now it can be easily seen that $\{ f_1, \ldots, f_{q^n} \} \leftrightarrow \{ U_1, \ldots, U_{q^n} \}$ is a one-to-one correspondence. \diamondsuit
Existence of \((d,n,t,q)\)-Singleton system for any \(t \in \{0,1,...,d\}\) is proved in [7]. We give a sketch of a proof of this fact in the following theorem for the sake of completeness.

**Theorem 5.3.3 (Delsarte):** For any \(t \in \{0,1,...,d\}\), there always exists a \((d,n,t,q)\)-Singleton system.

**Proof:** Let \(W' = GF(q^n)\) and let \(W\) be a fixed \(d\)-dimensional subspace of \(W'\). Denote by \(T\) the trace form \(T : W' \rightarrow GF(q)\) defined by \(T(y) = \sum_{0 \leq i \leq n-1} y_i^q\). For any \(d\)-tuple \(w = (w_0, w_1, \ldots, w_{d-1})\), \(w_i \in W'\), define \(f_w : W \times W' \rightarrow GF(q)\) by

\[
f_w(x, x') = \sum_{0 \leq j \leq d-1} T(w_i x_j x').
\]

It can be checked that \(f_w\) is a bilinear form. Define

\[
Y = \{ f_w \mid w=(w_0, w_1, \ldots, w_{d-1})\text{ and } w_j = 0 \text{ for all } j \geq t \}.
\]

Then \(Y\) is a \((d,n,t,q)\)-Singleton system. \(\diamondsuit\)

**Theorem 5.3.4:** For any vector space \(H\) of dimension \(n+d\) over \(GF(q)\) and \(W\) a fixed subspace of \(H\) of dimension \(n\), \((d \leq n)\), there always exists a \(W\)-complement \(d\)-spread of \(H\).

**Proof:** Straightforward application of the preceding two theorems. \(\diamondsuit\)
5.4 Distribution numbers and W-complement d-spreads

By applying the idea of W-complement d-spreads, we show here that the first distribution invariant of $H_q(n,d)$ is equal to $q^{n(d-1)}$. As in the previous sections, let us assume that $H$ is a vector space of dimension $n+d$ over $GF(q)$, $W$ be a fixed subspace of $H$ of dimension $n$, $C$ be the complement of $W$ so that $H = W \oplus C$. Let $\{c_1,\ldots,c_d\}, \{w_1,\ldots,w_n\}$ be bases of $C$ and $W$ respectively. Let

$$X_i = \{ A \mid A \subseteq H \text{ is an i-space and } A \cap W = \{0\} \}$$

Then $(X_d, X_{d-1}, \supseteq)$ becomes a semilinear incidence structure. Any incidence structure isomorphic to $\Pi = (X_d, X_{d-1}, \supseteq)$ is called an $(n,q,d)$-attenuated space.

Theorem 5.4.1: With respect to a fixed basis of $W$, $H_q(n,d)$ is isomorphic to the adjacency graph of $\Pi = (X_d, X_{d-1}, \supseteq)$.

Proof: Here we give only the sketch of the proof. For a detailed proof the reader is refered to page 6 of [13] or page 12 of [10].

It is known that each element $A$ in $X_d$ has a basis $\{c_1+v_1,\ldots,c_d+v_d\}$ for uniquely chosen vectors $v_1,\ldots,v_d$ in $W$. Let $v_i = \sum_{1 \leq i \leq n} a_{ij}w_j$. Let $\sigma(A)$ be the matrix
Then $H_q(n,d)$ is isomorphic to the adjacency graph of the attenuated space II via the correspondence $A \rightarrow \sigma[A]$ above. ◇

Remark 5.4.2: Let $X_1$ be the set of all 1-dimensional subspaces of $V$ that are not contained in $W$. Let $M_{1d}$ be the matrix whose rows are indexed by the elements of $X_1$ and columns are indexed by the elements of $X_d$, and the $(x,y)$th entry of $M_{1d}$ is 1 if $x \subseteq y$ and 0 otherwise. Let $U_1$ be the row space of $M_{1d}$. Following the same techniques as in lemmas 3.1.1 and 3.1.2, we can show that $U_1 = V_0 \perp V_1$ where $V_1$ is the first eigenspace of $H_q(n,d)$. Hence, from the definition of $M_{1d}$, we see that

$$V_1 \cong \{ (a_v)_{v \in X_1} \in \mathbb{R}^{|X_1|} : \sum_{v \in X_1} a_v = 0 \}$$

Therefore, a vector $w = (a_v)_{v \in X_1} \in V_1$ is a general vector if and only if $\sum_{v \subseteq A} a_v \neq 0 \ \forall \ A \in X_d$. Note that $X_+(w) = \{ A \in X_d : \sum_{v \subseteq A} a_v \text{ positive} \}$. Since $(|X_1|-1,-1,...,-1)$ is a general vector in $V_1$, $V_{t_1}(X) \leq q^{n(d-1)}$.

Theorem 5.4.3 The first distribution number of $H_q(n,d)$ is equal to $q^{n(d-1)}$. 
Proof: Let \( w = (a_v)_{v \in X_1} \in V_1 \) be a general vector. Let 
\[
D = \{ A \in X_d \mid \sum_{v \in A} a_v < 0 \}. 
\]
If \(|D| > q^{nd} - q^n(d-1)\), then by Theorem 5.2.4 D contains a W-complement d-spread \( \Psi \). So 
\[
0 = \sum_{v \in X_1} a_v = \sum_{\Lambda \in \Psi} (\sum_{v \in \Lambda} a_v) < 0, 
\]
a contradiction. Hence \(|D| \leq q^{nd} - q^n(d-1)\). Thus \(|X_+(w)| = |\{ A \in X_d \mid \sum_{v \in A} a_v > 0 \}| \geq q^n(d-1)\). Since \( w \) was any arbitrary general vector, we conclude that \( V_{t_1}(\mathcal{X}) \geq q^n(d-1)\). Therefore, from the remark preceding the theorem, we get that \( V_{t_1}(\mathcal{X}) = q^n(d-1) \). \( \diamond \)
References


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