INFORMATION TO USERS

This reproduction was made from a copy of a manuscript sent to us for publication and microfilming. While the most advanced technology has been used to photograph and reproduce this manuscript, the quality of the reproduction is heavily dependent upon the quality of the material submitted. Pages in any manuscript may have indistinct print. In all cases the best available copy has been filmed.

The following explanation of techniques is provided to help clarify notations which may appear on this reproduction.

1. Manuscripts may not always be complete. When it is not possible to obtain missing pages, a note appears to indicate this.

2. When copyrighted materials are removed from the manuscript, a note appears to indicate this.

3. Oversize materials (maps, drawings, and charts) are photographed by sectioning the original, beginning at the upper left hand corner and continuing from left to right in equal sections with small overlaps. Each oversize page is also filmed as one exposure and is available, for an additional charge, as a standard 35mm slide or in black and white paper format. *

4. Most photographs reproduce acceptably on positive microfilm or microfiche but lack clarity on xerographic copies made from the microfilm. For an additional charge, all photographs are available in black and white standard 35mm slide format. *

*For more information about black and white slides or enlarged paper reproductions, please contact the Dissertations Customer Services Department.
Kuo, Hai-Perng

STABILITY AND FINITE AMPLITUDE NATURAL CONVECTION IN A SHALLOW CAVITY WITH HORIZONTAL HEATING

The Ohio State University

Ph.D. 1986

University Microfilms International 300 N. Zeeb Road, Ann Arbor, MI 48106

Copyright 1986 by Kuo, Hai-Perng All Rights Reserved
PLEASE NOTE:

In all cases this material has been filmed in the best possible way from the available copy. Problems encountered with this document have been identified here with a check mark •✓.

1. Glossy photographs or pages •
2. Colored illustrations, paper or print •
3. Photographs with dark background •
4. Illustrations are poor copy •
5. Pages with black marks, not original copy •
6. Print shows through as there is text on both sides of page •
7. Indistinct, broken or small print on several pages ✓
8. Print exceeds margin requirements •
9. Tightly bound copy with print lost in spine •
10. Computer printout pages with indistinct print •
11. Page(s) _______ lacking when material received, and not available from school or author.
12. Page(s) _______ seem to be missing in numbering only as text follows.
13. Two pages numbered ______. Text follows.
14. Curling and wrinkled pages •
15. Dissertation contains pages with print at a slant, filmed as received •
16. Other ______________________________________________________

___________________________________________________________________

University Microfilms International
STABILITY AND FINITE AMPLITUDE NATURAL CONVECTION
IN A SHALLOW CAVITY WITH HORIZONTAL HEATING

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By

Hai-Perng Kuo, B.S., M.S.

* * * * *

The Ohio State University
1986

Dissertation Committee:

Dr. Seppo A. Korpela
Dr. Donald R. Houser
Dr. Donald E. Richards
Dr. Kambiz Vafai

Approved By

Advisor
Department of Mechanical Engineering
ACKNOWLEDGEMENTS

The author wishes to express sincere appreciation to Professor Seppo A. Korpela for his continuous encouragement and thoughtful guidance through this research. The author is also extremely grateful to Professors Donald R. Houser, Donald E. Richards and Kambiz Vafai, for their service on the advisory committee.

A special thanks is extended to Professor Philip S. Marcus of Harvard University for making available to us the theoretical developments and a detailed structure of the implementation of pseudo-spectral methods. The author wishes also to thank Arnon Chait for making discussion during the course of this study.

Financial support was received from the National Science Foundation and the Ohio State University Department of Mechanical Engineering, and computer time from the Instruction and Research Computer Center at OSU on the IBM 3081 computer, and Cray-1 computer at the NASA Lewis Center. To these organizations a heartfelt thanks.

Finally, the author wishes to express his deepest love to his wife, Wei-Ying Hu, for her companionship and support through a period of seven years of study abroad and to his parents, Sheng-Fong and Nan-Shine Kuo, for their stable and endless love.
VITA

May 23, 1953 ............ Born - Kaohsiung, Taiwan, The Republic of China (ROC)

1976 .................. B.S., Chung-Hsing University Taichung, Taiwan, ROC

1978-1980 ............... Research Assistant, Department of Mechanical Eng., The Utah State University, Logan, Utah

1980 .................. M.S., The Utah State University Logan, Utah

1980-1986 ............... Research and Teaching Assistant, Department of Mechanical Eng., The Ohio State University, Columbus, Ohio

PUBLICATIONS


FIELDS OF STUDY

Major Field: Mechanical Engineering

Studies in Fluid Mechanics. Professor Seppo A. Korpela
Studies in Heat Transfer. Professor Seppo A. Korpela
Studies in Combustion. Professor Robert H. Essenhigh
# TABLE OF CONTENTS

ACKNOWLEDGEMENTS ..................................... ii
VITA .................................................................. iii
TABLE OF CONTENTS ...................................... iv
LIST OF TABLES ........................................... v
LIST OF FIGURES .......................................... vi
NOMENCLATURE .............................................. x

<table>
<thead>
<tr>
<th>CHAPTER</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. INTRODUCTION AND OVERVIEW</td>
<td>1</td>
</tr>
<tr>
<td>1.1 INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>1.2 LITERATURE REVIEW</td>
<td>2</td>
</tr>
<tr>
<td>1.3 RESEARCH OBJECTIVES</td>
<td>5</td>
</tr>
<tr>
<td>2. BASE FLOW INSTABILITY</td>
<td>7</td>
</tr>
<tr>
<td>2.1 FORMULATION</td>
<td>7</td>
</tr>
<tr>
<td>2.1.1 GOVERNING EQUATIONS</td>
<td>7</td>
</tr>
<tr>
<td>2.1.2 LINEAR STABILITY ANALYSIS</td>
<td>10</td>
</tr>
<tr>
<td>2.2 METHOD OF SOLUTION</td>
<td>14</td>
</tr>
<tr>
<td>2.2.1 GENERAL DESCRIPTION</td>
<td>14</td>
</tr>
<tr>
<td>2.2.2 CALCULATION OF EIGENVALUES</td>
<td>16</td>
</tr>
<tr>
<td>2.2.3 VERIFICATION STUDY</td>
<td>25</td>
</tr>
<tr>
<td>2.3 RESULTS AND DISCUSSION</td>
<td>28</td>
</tr>
<tr>
<td>2.3.1 CONDUCTING BOUNDARIES</td>
<td>28</td>
</tr>
<tr>
<td>2.3.2 INSULATED BOUNDARIES</td>
<td>46</td>
</tr>
<tr>
<td>2.3.3 THREE DIMENSIONAL DISTURBANCES</td>
<td>60</td>
</tr>
<tr>
<td>3. SECONDARY FLOW SIMULATION</td>
<td>65</td>
</tr>
<tr>
<td>3.1 FORMULATION</td>
<td>65</td>
</tr>
<tr>
<td>3.2 TIME-SPLITTING METHOD</td>
<td>67</td>
</tr>
<tr>
<td>3.3 VALIDITY OF THE SIMULATIONS</td>
<td>74</td>
</tr>
<tr>
<td>3.4 RESULTS AND DISCUSSION</td>
<td>78</td>
</tr>
<tr>
<td>4. CONCLUSIONS</td>
<td>96</td>
</tr>
<tr>
<td>4.1 SUMMARY</td>
<td>96</td>
</tr>
<tr>
<td>4.2 FURTHER WORK</td>
<td>98</td>
</tr>
</tbody>
</table>

APPENDIX

A. DATA FOR THE CRITICAL STATES AND CRITICAL WAVE NUMBERS .......... 100

LIST OF REFERENCES ....................................... 103
# LIST OF TABLES

<table>
<thead>
<tr>
<th>TABLE</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Non-dimensional amplification factor, $\sigma_f$, for $Pr=0, \alpha=2.65$</td>
<td>25</td>
</tr>
<tr>
<td>2. The critical Grashof no. as a function of the order of approximation and the Prandtl number for $\alpha=2.65$</td>
<td>26</td>
</tr>
<tr>
<td>3. Convergence study of the stability solutions, (transverse mode, conducting boundaries.)</td>
<td>27</td>
</tr>
<tr>
<td>4. Wave speeds for the travelling transverse mode in a cavity with conducting boundaries</td>
<td>42</td>
</tr>
<tr>
<td>5. Energy balance for transverse modes in a cavity with conducting walls</td>
<td>47</td>
</tr>
<tr>
<td>6. Energy balance for transverse modes in a cavity with insulated walls</td>
<td>54</td>
</tr>
<tr>
<td>7. Critical states and critical wave numbers in a cavity with conducting boundaries</td>
<td>101</td>
</tr>
<tr>
<td>8. Critical states and critical wave numbers in a cavity with insulated boundaries</td>
<td>102</td>
</tr>
</tbody>
</table>
# LIST OF FIGURES

<table>
<thead>
<tr>
<th>FIGURE</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. A sketch of shallow cavity</td>
<td>8</td>
</tr>
<tr>
<td>2. Non-dimensional amplification factor for $\alpha=2.11$, $\beta=0$, $Pr=0.25$ in a cavity with conducting boundaries</td>
<td>23</td>
</tr>
<tr>
<td>3. Critical states for conducting boundaries</td>
<td>29</td>
</tr>
<tr>
<td>4. Critical wave numbers for conducting boundaries</td>
<td>30</td>
</tr>
<tr>
<td>5. Normalized local energy transfer for stationary transverse cells for $Pr=0.05$ and $Pr=0.25$ in a cavity with conducting boundaries</td>
<td>31</td>
</tr>
<tr>
<td>6. Perturbation streamlines for stationary transverse cells for $Pr=0.05$, $0.15$, $0.25$, and $0.4$ in a cavity with conducting walls</td>
<td>33</td>
</tr>
<tr>
<td>7. Perturbation isotherms for stationary transverse cells for $Pr=0.05$, $0.15$, $0.25$, and $0.4$ in a cavity with conducting walls</td>
<td>34</td>
</tr>
<tr>
<td>8. Oscillation frequencies for longitudinal modes in a cavity with conducting walls</td>
<td>36</td>
</tr>
<tr>
<td>9. Normalized local energy transfer for oscillatory longitudinal cells for $Pr=0.1$ and $Pr=0.3$ in a cavity with conducting walls</td>
<td>37</td>
</tr>
<tr>
<td>10. Streamlines and isotherms of oscillatory longitudinal rolls for $Pr=0.1$ and $Pr=0.3$ in a cavity with conducting wall</td>
<td>38</td>
</tr>
<tr>
<td>11. Normalized local energy transfer for stationary longitudinal rolls for $Pr=1.0$ and $Pr=10$ in a cavity with conducting boundaries</td>
<td>39</td>
</tr>
<tr>
<td>12. Stream patterns and isotherms of stationary longitudinal rolls for $Pr=1.0$ and $Pr=10$ in a cavity with conducting walls</td>
<td>40</td>
</tr>
</tbody>
</table>
13. Normalized local energy transfer for travelling transverse waves for \( Pr=1.0 \) and \( Pr=50 \) in a cavity with conducting walls .... 43

14. Perturbation streamlines for travelling transverse waves for \( Pr=0.7, 1.0, 4.0, \) and \( 50 \) in a cavity with conducting walls ....... 44

15. Perturbation isotherms for travelling transverse waves for \( Pr=0.7, 1.0, 4.0, \) and \( 50 \) in a cavity with conducting walls ....... 45

16. Critical states for insulated boundaries ....... 48

17. Critical wave numbers for insulated boundaries ........................................... 49

18. Normalized local energy transfer for stationary transverse cells for \( Pr=0.02 \) in a cavity with insulated boundaries .......... 51

19. Perturbation streamlines for stationary transverse cells for \( Pr=0.02, 0.08, \) and \( 0.1 \) in a cavity with insulated boundaries ....... 52

20. Perturbation isotherms for stationary transverse cells for \( Pr=0.02, 0.08, \) and \( 0.1 \) in a cavity with insulated boundaries ....... 53

21. Dynamics of local thermal field for \( Pr=0.02 \) .... 56

22. Dynamics of local thermal field for \( Pr=0.1 \) .... 58

23. Oscillation frequencies for longitudinal modes in a cavity with insulated boundaries .... 59

24. Perturbation stream patterns of three different modes for travelling transverse waves for \( Pr=0.8 \) in a cavity with insulated boundaries ............................. 61

25. Neutral stability surface for air (\( Pr=0.7 \)) in a cavity with conducting boundaries ............ 63

26. Neutral stability surface for gallium (\( Pr=0.02 \)) in a cavity with insulated boundaries ................................. 64

27. Flow chart of the secondary flow simulation .... 75
28. The velocity and temperature time-histories of two selected nodes in a cavity with insulated walls ........................................ 77
29. Growth of random velocity disturbances for Pr=0.1 in a cavity with conducting boundaries ........................................ 79
30. Total flow streamlines from the linear stability prediction and the secondary flow simulation for Pr=0.1 in a cavity with conducting boundaries ........................................ 81
31. Stream patterns from the linear stability prediction and the secondary flow simulation for Pr=0.1 in a cavity with conducting boundaries ........................................ 83
32. Neutral stability curve form linear stability prediction and results from secondary flow simulation for Pr=0.02 in a cavity with insulated walls ........................................ 84
33. Horizontal averaged velocity of the total flow as a function of y for different wave numbers for Pr=0.02, Gr=30,000 in a cavity with insulated walls ........................................ 86
34. Horizontal averaged velocity of the total flow as a function of y for different Grashof numbers for Pr=0.02, α=2.7 in a cavity with insulated walls ........................................ 87
35. Streamlines and isotherms from secondary flow simulation for Pr=0.02 in a cavity with insulated walls ........................................ 88
36. Normalized local energy transfer from linear theory and secondary flow simulation for Pr=0.02 in a cavity with insulated walls ........................................ 89
37. Stream patterns of the total flow field from linear theory and secondary flow simulation for Pr=0.02 in a cavity with insulated walls ........................................ 91
38. Isotherms of the total temperature field from linear theory and secondary flow simulation for Pr=0.02 in a cavity with insulated walls ........................................ 92
39. Normalized local kinetic energy from linear theory and secondary flow simulation for Pr=0.02 in a cavity with insulated walls .................................. 93

40. Nusselt numbers from secondary flow simulations and from analytical results .......... 95
NOMENCLATURE

A General matrix in the eigenvalue problem
a_i Coefficients
B General matrix in the eigenvalue problem
b_{ij} Coefficients
c_j Constants
D Differentiation operator, \( \partial / \partial y \)
E_b Total energy transfer from the buoyancy field
e_b Local energy transfer from the buoyancy field
E_d Total viscous dissipation
e_d Local viscous dissipation
E_l Total kinetic energy production from mean-flow by Reynolds stress
e_i Local kinetic energy production from mean-flow by Reynolds stress
g Gravitational acceleration
G_i Green's function
Gr Grashof number, \( g \gamma \rho g H^4 / \nu^2 \)
Gr_c Critical Grashof number
H Height of cavity
L Length of cavity
M One half number of mesh divisions, x-direction
N Number of mesh divisions, y-direction
Nu Nusselt number
0 Order of error
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>P</td>
<td>Perturbation magnitude of pressure</td>
</tr>
<tr>
<td>p</td>
<td>Pressure</td>
</tr>
<tr>
<td>Pr</td>
<td>Prandtl number, $\nu/\kappa$</td>
</tr>
<tr>
<td>Q</td>
<td>General variable</td>
</tr>
<tr>
<td>q'</td>
<td>General disturbance</td>
</tr>
<tr>
<td>Ra</td>
<td>Rayleigh number, $Gr \cdot Pr$</td>
</tr>
<tr>
<td>Re</td>
<td>Real part</td>
</tr>
<tr>
<td>s</td>
<td>Horizontal temperature gradient</td>
</tr>
<tr>
<td>T</td>
<td>Temperature</td>
</tr>
<tr>
<td>$T_H$</td>
<td>Hot wall temperature</td>
</tr>
<tr>
<td>$T_C$</td>
<td>Cold wall temperature</td>
</tr>
<tr>
<td>$TH_d$</td>
<td>Thermal dissipation</td>
</tr>
<tr>
<td>$TH_x$</td>
<td>Thermal gradient production in x-direction</td>
</tr>
<tr>
<td>$TH_y$</td>
<td>Thermal gradient production in y-direction</td>
</tr>
<tr>
<td>t</td>
<td>Time</td>
</tr>
<tr>
<td>U</td>
<td>Perturbation magnitude of velocity, x-direction</td>
</tr>
<tr>
<td>$U_C$</td>
<td>Characteristic thermal velocity, $g\nu s H^3/\nu$</td>
</tr>
<tr>
<td>u</td>
<td>Velocity, x-direction</td>
</tr>
<tr>
<td>V</td>
<td>Perturbation magnitude of velocity, y-direction</td>
</tr>
<tr>
<td>$\vec{V}$</td>
<td>Velocity vector</td>
</tr>
<tr>
<td>v</td>
<td>Velocity, y-direction</td>
</tr>
<tr>
<td>W</td>
<td>Perturbation magnitude of velocity, z-direction</td>
</tr>
<tr>
<td>w</td>
<td>Velocity, z-direction</td>
</tr>
<tr>
<td>$X_1, X_2$</td>
<td>Two linearly independent solutions</td>
</tr>
<tr>
<td>x</td>
<td>Coordinate, horizontal direction of cavity</td>
</tr>
<tr>
<td>y</td>
<td>Coordinate, vertical direction of cavity</td>
</tr>
<tr>
<td>xi</td>
<td></td>
</tr>
</tbody>
</table>
\( z \) Coordinate, transverse direction of cavity
\( \alpha \) Wave number, \( x \)-direction
\( \beta \) Wave number, \( z \)-direction
\( \gamma \) Coefficient of volumetric expansion
\( \Delta t \) Time increment
\( \zeta, \zeta^* \) Common operators
\( \Theta \) Perturbation magnitude of temperature
\( \kappa \) Thermal diffusivity
\( \nu \) Kinematic viscosity
\( \Pi \) Total pressure
\( \rho \) Density
\( \sigma \) Complex eigenvalue
\( \sigma_r \) Real part of \( \sigma \), amplification factor
\( \sigma_i \) Imaginary part of \( \sigma \), oscillation frequency
\( \omega \) Vorticity

**Subscripts**
\( b \) Base flow value
\( c \) Correction quantity
\( p \) Partial quantity
\( s \) Secondary flow value

**Superscripts**
\( N \) Time level
\( ' \) Disturbance
\( * \) Complex conjugate (unless noted otherwise)
Chapter 1
INTRODUCTION AND OVERVIEW

1.1 INTRODUCTION

The subject matter of this dissertation is natural convection in a closed shallow cavity with one sidewall heated and the other cooled. The word shallow means that the aspect ratio of the cavity (defined as the height to width ratio) is much smaller than unity. Since the region occupied by the fluid is completely closed, the imposed horizontal temperature gradient induces a circulating convective flow in the cavity. That is, for a small temperature difference between the sidewalls the fluid will ascend near the hot wall and flow toward the other end via the top half of the cavity; there the fluid descends next to the cold wall and flows back to its starting point through the bottom half of the cavity. The flow is laminar and for a sufficiently small aspect ratio it is parallel far away from the ends. It has nearly odd symmetry about the center of the cavity if the properties of the fluid do
not depend strongly on temperature.

By increasing the imposed horizontal temperature gradient the base flow, just described, will become unstable and a new state marked by secondary flows emerges. The secondary flows are characterized either by transverse modes of stationary cells or travelling waves; or by stationary or oscillating longitudinal rolls. The particular outcome depends on the properties of the fluid.

Further increase of the temperature gradient can cause another transition to yet more complicated pattern. That is, the two-dimensional flow with secondary circulations can be unstable and change for example into an oscillating three dimensional pattern. But this is also an intermediate state because for very large temperature differences the flow in the end becomes turbulent. It is the first laminar transition and the resulting flow which are the object of this study.

1.2 LITERATURE REVIEW

The flow in a shallow cavity finds applications in such widely different fields as the growing of semiconductor crystals (Hurle et al. 1974), dispersion of pollutants in estuaries (Cormack et al. 1974),
meteorology (Hart 1972), and the analysis of heat transfer in buildings (Bejan and Rossie 1981). Since Drummond (1981) and Drummond and Korpela (1985) have discussed most of the past research in this area only those articles closely related to the present study and the newly published ones will be discussed here.

The first stability study of this flow was by Hart (1972). By using the linear theory he found that, depending on the Prandtl number, both transverse modes and longitudinal modes were possible. Transverse modes are ones whose cell axes are perpendicular to the base flow direction. The longitudinal modes, on the other hand, comprise of rolls with axes parallel to the direction of the base flow. In this first paper Hart considered the horizontal (upper and lower) boundaries to be highly conducting. Later (Hart 1983) he re-examined the flow, but now with insulated boundaries.

To summarize his results, he has found the transverse modes to be always less stable than the longitudinal modes for conducting boundaries regardless of the value of Prandtl number. For insulated horizontal boundaries he found the transverse stationary modes to be the most unstable if Pr<0.015; the longitudinal oscillatory modes taking over in the range 0.015<Pr<0.27; and longitudinal stationary modes dominating for the higher Prandtl
number fluids. He also calculated the energy transfer to the disturbance and in this way he explained the difference between a hydrodynamic instability, which occurs in low Prandtl number fluids, and a buoyant instability, which characterizes the large Prandtl number fluids.

The only other theoretical paper addressing the stability of this flow is by Gill (1974). Whereas Hart's interest stemmed from astrophysics, Gill wished to explain whether the oscillating longitudinal disturbances were the source of temperature fluctuations in metal and semiconductor melts. He studied the stability characteristics in a qualitative way in the small Prandtl number limit and concluded that his theory explains reasonably the experimental results reported by Skafel (1972), who used mercury as the working fluid, and Hurle et al. (1974), who used molten gallium.

One way to identify the base flow instability, besides observations in laboratory experiments, is to perform a numerical experiment by solving the nonlinear Navier-Stokes and energy equation. Drummond (1981) successfully used finite difference methods to calculate the transverse modes within finite aspect ratio cavities. He was not able to find the transverse travelling wave instabilities predicted by Hart in the
range $0.5<\text{Pr}<1.5$ for the conducting boundaries. Another numerical study by Roux et al. (1984) has also given somewhat different results from Hart's for low Prandtl number fluids with insulated boundary conditions.

For the instability of the base and secondary flows, Nagata and Busse (1983) have reported numerical results for three-dimensional tertiary motions. Their results have shed new light on the general problem of transition to turbulence in plane shear flow. However, by assuming the limit of vanishing Prandtl number, the thermal perturbations play no role in their analysis.

Mallinson and de Vahl Davis (1975) and Ozoe et al. (1983) have simulated the three-dimensional natural convection in a box using finite difference methods which provided a way of exploring the local fluid dynamics. The end regions influence the flow significantly in both of these studies and, therefore, they cannot really be compared with the present work.

1.3 RESEARCH OBJECTIVES

Although Hart has reported extensive numerical results for the base flow stability, part of his results are in conflict with the later papers. Thus a careful verification of these instabilities over all the possible ranges of Prandtl numbers and thermal boundary
conditions is valuable. This is the first objective of the present study.

Furthermore, in most of the former studies, the disturbances to the base flow were assumed to be two-dimensional, either transverse or longitudinal. Thus the numerical and experimental efforts in them were concentrated on finding the so-called plane instabilities. However, Squire's (1933) theorem which guarantees a less stable two-dimensional disturbance than a three-dimensional one does not apply here. Therefore, the second objective of this study is to investigate whether three-dimensional disturbances are, in fact, important.

The third objective is to investigate the finite amplitude motions by solving the nonlinear equations numerically. This simulation will support the calculations of the onset of instability, as well as allow a better idea to be had of the nonlinear effects which have been neglected in the linear studies.

The numerical studies were carried out using a pseudo-spectral method (Gottlieb and Orszag 1977), with the Chebyshev and Fourier expansions. This approach has been shown by Marcus (1984) to be well suited for problems involving secondary motions and to yield highly accurate results.
Chapter 2

BASE FLOW INSTABILITY

2.1 FORMULATION

2.1.1 GOVERNING EQUATIONS

Consider a cavity shown in Figure 1 with the right wall heated and the left one cooled. The top and bottom are either insulated or highly conducting solid boundaries. For the core region of a sufficiently shallow (aspect ratio $H/L \ll 1$) cavity, as has been shown by Hart (1972), the Boussinesq form of the Navier-Stokes equations

\[ \nabla \cdot \mathbf{V} = 0, \quad (2.1) \]

\[ \frac{\partial \mathbf{V}}{\partial t} + \text{Gr}(\nabla \cdot \mathbf{V})\mathbf{V} = -\text{Gr} \nabla p + \mathbf{T} + \nabla^2 \mathbf{V}, \quad (2.2) \]

\[ \frac{\partial T}{\partial t} + \text{Pr} \text{Gr} (\nabla \cdot \mathbf{V}) T = \nabla^2 T, \quad (2.3) \]

admit a parallel flow solution, which is governed by the
Figure 1: A sketch of shallow cavity.
For insulated top and bottom walls the solution of these equations is

\[ \frac{d^2u_b}{dy^2} = \text{GrPr} \frac{\partial p_b}{\partial x}, \quad (2.4) \]

\[ T_b = \text{GrPr} \frac{\partial p_b}{\partial y}, \quad (2.5) \]

\[ \text{GrPr} u_b \frac{\partial^2 T_b}{\partial x^2} = \frac{\partial^2 T_b}{\partial x^2} + \frac{\partial^2 T_b}{\partial y^2}. \quad (2.6) \]

For insulated top and bottom walls the solution of these equations is

\[ u_b = \frac{1}{y(y^2 - \frac{1}{4})}, \quad (2.7) \]

\[ T_b = x + \frac{\text{GrPr}}{120}y(y^4 - \frac{5}{6}y^2 + \frac{5}{16}), \quad (2.8) \]

and for highly conducting walls the temperature distribution is given by

\[ T_b = x + \frac{\text{GrPr}}{120}y(y^4 - \frac{5}{6}y^2 + \frac{7}{48}). \quad (2.9) \]

Here the height \( H \) has been used as a scale for lengths and the thermal velocity \( U_C = g\gamma sH^3/\nu \) as the scale for velocities. In the definition of thermal velocity \( g \) is
the gravitational acceleration, $s$ the horizontal temperature gradient, $\gamma$ the coefficient of volumetric expansion, and $\nu$ the kinematic viscosity. The non-dimensional temperature is measured above the mean temperature of the end walls and contains the product $sH$ as the temperature scale. Time and pressure were scaled using $H^2/\nu$ and $\rho U_c^2$ respectively.

2.1.2 LINEAR STABILITY ANALYSIS

The stability of the base flow can be established by investigating the evolution of a small disturbance in the flow. Thus by writing

$$u = u_b + u', \quad v = v', \quad w = w', \quad p = p_b + p', \quad T = T_b + T',$$

(2.10)

where the primed quantities pertain to the disturbance, substituting these into Equations (2.1)-(2.3), making use of Equations (2.4)-(2.6), and neglecting products of disturbance quantities, yields

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0,$$

(2.11)

$$\frac{\partial u'}{\partial t} + Gr \frac{\partial u'}{\partial x} + Gr' \frac{\partial u_b}{\partial y} = -Gr \frac{\partial p'}{\partial x} + \nabla^2 u',$$

(2.12)
\[
\frac{\partial v'}{\partial t} + \text{Gru}_b \frac{\partial v'}{\partial x} = - \text{Gr} \frac{\partial p'}{\partial y} + \nabla^2 v' + T', \quad (2.13)
\]
\[
\frac{\partial w'}{\partial t} + \text{Gru}_b \frac{\partial w'}{\partial x} = - \frac{\partial p'}{\partial z} + \nabla^2 w', \quad (2.14)
\]
\[
\frac{\partial T'}{\partial t} + \text{Pr} \frac{\partial T'}{\partial x} + u' \frac{\partial T}{\partial x} + v' \frac{\partial T}{\partial y} = \nabla^2 T'. \quad (2.15)
\]

Assuming next that the disturbances have the following \(x, z\) and \(t\) dependence

\[
q'(x,y,z,t) = Q(y)e^{\sigma t} + i\alpha x + i\beta z, \quad (2.16)
\]

results in the set

\[
i\alpha U + \text{D}U + i\beta \text{W} = 0, \quad (2.17)
\]
\[
\sigma U + i\alpha \text{Gru}_b U + \text{GrDu}_b V = - i\alpha \text{GrP} + (D^2-\alpha^2-\beta^2)U, \quad (2.18)
\]
\[
\sigma V + i\alpha \text{Gru}_b V = - \text{GrDP} + (D^2-\alpha^2-\beta^2)V + \theta, \quad (2.19)
\]
\[
\sigma \text{W} + i\alpha \text{Gru}_b \text{W} = -i\beta \text{GrP} + (D^2-\alpha^2-\beta^2)\text{W}, \quad (2.20)
\]
\[
\sigma \text{Pr}\theta + \text{GrPr}(i\alpha u_b \theta + \text{U} + \text{DT}_b \text{V}) = (D^2-\alpha^2-\beta^2)\theta. \quad (2.21)
\]

These are to be supplemented with the boundary conditions

\[
U = V = \text{W} = 0 \text{ at } y = \pm 1/2, \quad (2.22)
\]
and either

$$\theta = 0 \text{ or } D\theta = 0 \text{ at } y = \pm 1/2, \quad (2.23)$$

the first thermal condition applying for conducting boundaries and the second for insulated ones. The notation $D=\partial/\partial y$ is used in the above equations.

In the following we consider mostly either transverse or longitudinal modes, the former being independent of the $z$-coordinate, the latter independent of $x$. The transverse modes have $\beta=0$ so that the set of Equations (2.17)-(2.21) reduces to

$$i\alpha U + DV = 0, \quad (2.24)$$

$$\sigma U + i\alpha G_{rb}U + GrD_{rb}V = -i\alpha G\rho + (D^2-\alpha^2)U, \quad (2.25)$$

$$\sigma V + i\alpha G_{rb}V = -GrD\rho + (D^2-\alpha^2)V + \theta, \quad (2.26)$$

$$\sigma Pr\theta + GrPr(i\alpha_{rb}\theta + U + DT_{rb}) = (D^2-\alpha^2)\theta. \quad (2.27)$$

The $W$-equation decouples and its solution shows that these components decay independently.

For the longitudinal modes $\alpha=0$ and the Equations (2.17)-(2.21) become

$$DV + \beta W = 0, \quad (2.28)$$

$$\sigma U + GrD_{rb}V = (D^2-\beta^2)U, \quad (2.29)$$
\[ \delta V = - \text{GrDP} + (D^2 - \beta^2) V + \theta, \quad (2.30) \]
\[ \delta W = -i \beta \text{GrP} + (D^2 - \beta^2) W, \quad (2.31) \]
\[ \delta \text{Pr0} + \text{GrPr}(U + DT_b V) = (D^2 - \beta^2) \theta. \quad (2.32) \]

To gain some insight into the physical aspects we also calculated the kinetic energy balance as well as the local energy distributions. Following Hart (1972), the kinetic energy balance of the transverse modes at the neutral states was calculated by integrating the local energy distributions over one cell. This gives

\[ E_i + E_b = E_d, \quad (2.33) \]

where

\[ E_i = \int_{-1/2}^{1/2} e_i dy = -\text{Gr} \int_{-1/2}^{1/2} (U\theta^* + U^* V) Du_b dy, \quad (2.34) \]
\[ E_b = \int_{-1/2}^{1/2} e_b dy = \int_{-1/2}^{1/2} (V\theta^* + V^* \theta) dy, \quad (2.35) \]
\[ E_d = \int_{-1/2}^{1/2} e_d dy = \int_{-1/2}^{1/2} (U D U^* + V D V^*) dy + \alpha^2 (U U^* + V V^*) dy, \quad (2.36) \]

In these equations \( * \) denotes a complex conjugate and \( e_b \) represents the local energy transfer from the buoyancy.
field, $e_d$ is the local viscous dissipation, and $e_i$ the local energy production from mean-flow by Reynolds stress. If the results show that disturbances obtain most of their energy from the mean flow by Reynolds stress, the instability is called a shear mode instability. On the other hand, if $E_b$ dominates then the instability is a buoyant type.

Equations (2.33)-(2.36) also hold for longitudinal modes if $e_d$ is replaced by

$$e_d = [DUDU^* + \beta^2 UU^* + \frac{1}{\beta^2} (D^2 D^2 V^* + 2\beta^2 D V V^*) + \beta^4 WW^*)],$$

(2.37)

2.2 METHOD OF SOLUTION

2.2.1 GENERAL DESCRIPTION

Spectral methods involve seeking the numerical solution to a differential equation in terms of a truncated series of known, smooth functions. Partially as a result of the development of fast transform methods in recent years they have become increasingly popular in numerical studies for problems exhibiting complicated physical behavior, where high accuracy is desired. Indeed, for this class of problems they have emerged as a viable alternative to finite difference and finite
element methods.

Gottlieb and Orszag (1977) classify the spectral methods into Galerkin methods, tau methods and pseudo-spectral methods. The important difference between a Galerkin method and a tau method is that the expansion functions in the Galerkin approximation have to satisfy the given boundary constraints individually whereas in the tau method those boundary constraints are imposed as part of the conditions in determining the expansion coefficients of the chosen functions. In a pseudo-spectral method, also known as the collocation method, a set of collocation points is selected first, then the residual of the expansions is forced to zero at these collocation points. In addition to the collocation points corresponding to the interior domain, the boundary conditions give just the right number of extra conditions so that the boundary nodes are taken care of. Obviously, the accuracy of the results depends on both the location and the number of collocation points, as well as the expansion functions.

In this study a pseudo-spectral method with Chebyshev expansions is used. The Chebyshev expansions have been shown by Gottlieb & Orszag (1977) to converge rapidly, are free of Gibbs’ phenomenon (which is a non-uniform behavior of the approximation at non-
periodic boundaries), and give good resolution near the boundaries.

To solve the set of differential equations by a pseudo-spectral method one seeks an approximation to the solution, which is a continuous function, in a finite dimensional vector space. The method of solution hinges on one finding a suitable projection operator. This operator sets up a correspondence between the various derivatives in an infinite dimensional vector space and matrices in a finite dimensional space. In the next section details of how this is carried out are discussed.

2.2.2 CALCULATION OF EIGENVALUES

To find an approximate solution to the differential equations formulated in Section 2.1.2 one needs matrix representations for the spatial derivatives. Consider therefore the Chebyshev expansion in the y-direction for a function $u$ evaluated at points $y_j$,

$$u_j = u(y_j) = \sum_{k=0}^{N} a_k T_k(y_j) ,$$

(2.38)

where
\[ T_k(y_j) = T_k(\cos \frac{\pi j}{N}) = \cos \left( \frac{\pi k j}{N} \right), \quad (2.39) \]

is the Chebyshev polynomial of order \( k \) and

\[ y_j = \cos \left( \frac{\pi j}{N} \right); \quad j = 0, 1, 2, \ldots N, \quad (2.40) \]

are the collocation points. Thus one can also write Equation (2.38) as

\[ u_j = \sum_{k=0}^{N} a_k \cos \left( \frac{\pi k j}{N} \right). \quad (2.41) \]

One needs next to find a matrix \([D]\) such that when

\[ [U] = [u_0, u_1, u_2, \ldots, u_N]^T, \quad (2.42) \]

is multiplied with \([D]\) the resulting vector will be an approximation to \( dU/dy \). The matrix \([D]\) can be constructed as a product of three coefficient matrices, namely

\[ [D] = [P]^{-1}[S][P], \quad (2.43) \]

where \([P]^{-1}\) is given in Equation (2.41) as

\[ [P_k j]^{-1} = \cos \left( \frac{\pi k j}{N} \right), \quad (2.44) \]
and \([P]^{-1}\) is the inverse of \([P]\)

\[
[P]_{jk} = \frac{2}{N} \cos\left(\frac{\pi k j}{N}\right); \quad c_m = \begin{cases} 2, & m = 0 \text{ or } N \\ 1, & \text{otherwise.} \end{cases} \tag{2.45}
\]

The differential coefficient matrix \([S]\) is obtained from the properties of Chebyshev polynomial expansions (Gottlieb & Orszag 1977, p160)

\[
\begin{align*}
s_{kp} a_p &= b_p = \frac{2}{N} \sum_{p=k+1}^{N} c_k p a_p. \tag{2.46}
\end{align*}
\]

The matrix for the second derivative of \(u\) with respect to \(y\) can be shown to be the matrix product \([D][D]\). If a new \(y\)-domain is selected to be \([-0.5, 0.5]\) instead of \([-1, 1]\) it can be shown that each element of \([D]\) is to be multiplied by 2.

The eigenvalue problems formulated in Section 2.1.2 can now be cast into a form of an algebraic eigenvalue problem. For example, the problem of the stability of convection in a cavity with conducting horizontal boundaries, governed by Equations (2.17)-(2.23)) can be written in a matrix form as

\[
[A]X + \sigma[B]X = [0], \tag{2.47}
\]

or in more detail it is given by
<table>
<thead>
<tr>
<th></th>
<th>( \text{GrDu}_b )</th>
<th>( i\alpha \text{Gr} )</th>
<th>( \zeta )</th>
<th>( \xi )</th>
<th>( \text{GrD} )</th>
<th>( -I )</th>
<th>( 0 )</th>
<th>( \gamma )</th>
<th>( \beta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \xi )</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \zeta )</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \text{RaDT}_b )</td>
<td>1</td>
<td>0</td>
<td>( \xi^* )</td>
<td>( \gamma )</td>
<td>( \beta )</td>
<td>( \text{Ra} )</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

- \( V \), \( P \), \( \theta \), \( U \), \( W \)
where $Ra=Gr.Pr$ is the Raleigh number, and

$$\zeta = \alpha^2 + \beta^2 + i\alpha Gr_U - D^2, \quad (2.49)$$

$$\zeta^* = \alpha^2 + \beta^2 + i\alpha Ra_U - D^2, \quad (2.50)$$

are two common operators. The rows on which unity
appears as the only element arise from boundary conditions.

With the \( [D] \) operator in hand, matrices \([A]\) and \([B]\) of equation (2.48) can easily be set up. However, since there are \( N \) rows of zeros in the \([B]\) matrix, row operations were applied first to reduce this \( 5N \times 5N \) matrix system to a \( 4N \times 4N \) one. The resulting system was then solved using the EIGZC subroutine in the IMSL library.

In equations (2.16)-(2.21) the parameters \( \alpha \) and \( \beta \) are wave numbers in the \( x \) and \( y \) directions and \( \sigma = \sigma_r + i\sigma_i \) is a complex eigenvalue. If the linear theory is adequate to describe the stability of the flow, as is assumed here, the stability characteristics are independent of the initial conditions. A general initial disturbance can be represented by a Fourier integral in the two infinite space directions, so that \( \alpha \) and \( \beta \), which characterize the various modes present, can take on any positive real values. The flow is stable if, for given values of \( Pr \) and \( Gr \), the non-dimensional amplification factor \( \sigma_r < 0 \) for all values of \( \alpha \) and \( \beta \). Conversely the flow is unstable if \( \sigma_r > 0 \) for some value of \( \alpha \) and \( \beta \). The states for which \( \sigma_r = 0 \) are called marginal states, or states of neutral stability for a given Prandtl number; that for which the Grashof number
is a minimum as a function of $\alpha$ and $\beta$ is called the critical state. It is the critical state which is of most importance because for a Grashof number which is slightly supercritical a mode with wave numbers corresponding to this state grows. To give an indication of the amplification factor spectra, Figure 2 for a fluid with $Pr=0.25$ in a cavity with conducting walls is included. In this figure the first nine eigenvalues, the lowest one with least damping, and the one with largest magnitude and thus largest damping, were plotted for increasing Grashof number. For very low Grashof numbers all eigenvalues are real and negative which indicates the base flow is stable with regard to small disturbances. As the Grashof number increases some of the real eigenvalues combine into complex conjugate pairs indicating that these modes resulting in waves travelling in opposite directions in the flow. However, these modes continue to decay as long as the amplification factor associated with that mode is negative. At a value of Grashof number equal to 67753.4 one of the curves crosses the abscissa. This corresponds to a real eigenvalue and the flow will therefore become unstable through a stationary state. The resulting flow pattern will consist of stationary transverse cells. This is so because owing to the
Figure 2: Non-dimensional amplification factor for $\alpha=2.11$, $\beta=0$, $Pr=0.25$ in a cavity with conducting boundaries.
smallness of the disturbances there is no interaction between the various modes in the linearized problem. For this reason all modes present in the initial condition but one decay, leaving this pure mode as the equilibrium solution.

As noted if the imaginary part of the critical \( \sigma \) is zero the instability sets in as stationary convection. On the other hand when the imaginary part is nonzero oscillations or travelling waves appear in the flow.

It is conventional to study the stability of flows to two dimensional disturbances, either transverse or longitudinal. To justify this in the present case, two completely three dimensional calculations of neutral surfaces were carried out. In both cases two dimensional disturbances were found to be the most dangerous ones. Although this does not prove that two dimensional disturbances are the critical ones for every Prandtl number, in what follows, owing to large computational cost of producing one neutral surface, results for two-dimensional disturbances are mainly reported.
2.2.3 VERIFICATION STUDY

In order to test the accuracy and verify the correctness of the computer program the stability of natural convection in a vertical slot was first solved and results from this calculation compared with the published results of Korpela et al. (1973), who used a Galerkin method. The comparison is shown in Tables 1 & 2.

Table 1: Non-dimensional amplification factor, \( \sigma_r \), for \( Pr=0, \alpha=2.65 \)

<table>
<thead>
<tr>
<th>Grashof number</th>
<th>Galerkin method N=11</th>
<th>pseudo-spectral method N=11</th>
<th>pseudo-spectral method N=19</th>
</tr>
</thead>
<tbody>
<tr>
<td>4000</td>
<td>-21.788642</td>
<td>-21.7995</td>
<td>-21.7887</td>
</tr>
<tr>
<td>8000</td>
<td>0.360753</td>
<td>0.064855</td>
<td>0.360576</td>
</tr>
<tr>
<td>12000</td>
<td>20.97443</td>
<td>19.7723</td>
<td>20.9739</td>
</tr>
<tr>
<td>16000</td>
<td>40.4645</td>
<td>37.7161</td>
<td>40.4631</td>
</tr>
<tr>
<td>20000</td>
<td>59.5581</td>
<td>54.9530</td>
<td>59.5546</td>
</tr>
</tbody>
</table>

The size of the eigenvalue problem to be solved is \( 4N \times 4N \) whether Galerkin or pseudo-spectral method is used. The comparison shows that only about one half the number of terms are needed for comparable accuracy when Galerkin method is used. The computational cost in the Galerkin method has to include the calculation of a number of inner products and even if they are precomputed they consume a sizeable amount of computer memory. Thus the
Table 2: The critical Grashof no. as a function of the order of approximation and the Prandtl number for $\alpha=2.65$.

<table>
<thead>
<tr>
<th>Pr</th>
<th>Galerkin method</th>
<th>pseudo-spectral method</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>N=4  N=6  N=8</td>
<td>N=11  N=15  N=21</td>
</tr>
<tr>
<td>0.001</td>
<td>7913  7919  7920</td>
<td>7974  7920  7920</td>
</tr>
<tr>
<td>0.01</td>
<td>7805  7811  7811</td>
<td>7861  7811  7811</td>
</tr>
<tr>
<td>0.1</td>
<td>7349  7355  7355</td>
<td>7398  7355  7355</td>
</tr>
<tr>
<td>1.0</td>
<td>7983  7989  7989</td>
<td>7968  7989  7989</td>
</tr>
<tr>
<td>10.0</td>
<td>10686 7948  7898</td>
<td>4584  6426  7772</td>
</tr>
</tbody>
</table>

two methods, when advantages and disadvantages are weighed, do not differ much in efficiency.

To make sure that enough terms were used in the calculation of the neutral states a careful convergence study was conducted for the transverse mode stability in a shallow cavity with the conducting boundaries. These results are shown in Table 3.

Although 25 terms in the expansions is enough for most of the cases, as the flow becomes more stable and the critical Grashof number therefore very large (close to one million) even 33 terms cannot guarantee an accurate result. The case of a Pr=0.3 fluid with a critical $\alpha=4.0$ is an example of this poor convergence. Many previous investigations have noted this phenomenon. It is related to the development of thin critical layers.
Table 3: Convergence study of the stability solutions, (transverse mode, conducting boundaries).

<table>
<thead>
<tr>
<th>Gr</th>
<th>Pr=0.1</th>
<th>Pr=0.25</th>
<th>Pr=0.3</th>
<th>Pr=0.6</th>
<th>Pr=10.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>α=2.58</td>
<td>α=2.11</td>
<td>α=4.0</td>
<td>α=6.9</td>
<td>α=7.75</td>
</tr>
<tr>
<td>N=9</td>
<td>12035.98</td>
<td>58258.45</td>
<td>38258.45</td>
<td>47591.81</td>
<td>1499.63</td>
</tr>
<tr>
<td>N=11</td>
<td>13503.41</td>
<td>182968.35</td>
<td>118643.71</td>
<td>37198.40</td>
<td>1619.07</td>
</tr>
<tr>
<td>N=13</td>
<td>12739.30</td>
<td>52845.00</td>
<td>110454.69</td>
<td>38029.77</td>
<td>1523.99</td>
</tr>
<tr>
<td>N=15</td>
<td>12686.74</td>
<td>191026.83</td>
<td>140622.54</td>
<td>72337.38</td>
<td>1544.14</td>
</tr>
<tr>
<td>N=17</td>
<td>12690.74</td>
<td>90765.33</td>
<td>242418.64</td>
<td>96423.16</td>
<td>1544.89</td>
</tr>
<tr>
<td>N=19</td>
<td>12690.90</td>
<td>68828.34</td>
<td>245492.02</td>
<td>85512.62</td>
<td>1545.45</td>
</tr>
<tr>
<td>N=21</td>
<td>12690.85</td>
<td>67964.77</td>
<td>406455.44</td>
<td>75508.25</td>
<td>1545.55</td>
</tr>
<tr>
<td>N=23</td>
<td>12690.85</td>
<td>67858.12</td>
<td>432383.35</td>
<td>77243.99</td>
<td>1545.51</td>
</tr>
<tr>
<td>N=25</td>
<td>12690.85</td>
<td>67771.07</td>
<td>557132.23</td>
<td>79374.18</td>
<td>1545.50</td>
</tr>
<tr>
<td>N=27</td>
<td>12690.85</td>
<td>67755.50</td>
<td>694742.68</td>
<td>79758.42</td>
<td>1545.50</td>
</tr>
<tr>
<td>N=29</td>
<td>12690.85</td>
<td>67753.62</td>
<td>788915.73</td>
<td>79733.19</td>
<td>1545.36</td>
</tr>
<tr>
<td>N=31</td>
<td>12690.85</td>
<td>67753.72</td>
<td>1010960.46</td>
<td>79707.70</td>
<td>1545.26</td>
</tr>
<tr>
<td>N=33</td>
<td>12690.85</td>
<td>67753.44</td>
<td>1125322.00</td>
<td>79698.70</td>
<td>1545.27</td>
</tr>
<tr>
<td>N=35</td>
<td>12690.85</td>
<td>67753.45</td>
<td>1367629.71</td>
<td>79698.47</td>
<td>1545.26</td>
</tr>
</tbody>
</table>

in the flow as the Grashof number is increased. To resolve the structure through critical layers in which the eigenfunctions are peaked a large number of terms are required. For most of the calculations reported here the convergence does not become intolerably slow. Based on the tests shown in Table 3, 33 terms were used in most of the base flow stability calculations and this was increased to 51 whenever it seemed necessary.
2.3 RESULTS AND DISCUSSION

In the first part of this section the stability results for conducting boundaries are given. These are followed by the insulated boundary case, after which a discussion of the three-dimensional disturbances is given.

2.3.1 CONDUCTING BOUNDARIES

Figure 3 shows how the critical Grashof numbers vary in this case with the Prandtl number, for both the transverse and longitudinal modes. These data are tabulated in Appendix A, Table 7. For Pr<0.14 the instability leads to stationary transverse cells, not unlike those in a vertical cavity (Korpela 1974). The critical wave numbers, given in Figure 4 and also tabulated in Table 7, show that the wavelength of these cells is roughly twice the cavity width. The disturbances obtain their energy from the interaction of the Reynolds stress with the mean velocity field. The terms involved in the energetics are displayed in Figure 5, which shows that, whereas the maximum rate of production takes place near the centerline, the largest part of the dissipation is near the walls. Some energy gets drawn into the potential energy field of buoyancy,
Figure 3: Critical states for conducting boundaries, 
L = longitudinal, T = transverse, 
Sta. = stationary, Osc. = oscillatory, 
Trav. = travelling.
Figure 4: Critical wave numbers for conducting boundaries.
Figure 5: Normalized local energy transfer for stationary transverse cells for Pr=0.05 (top) and Pr=0.25 (bottom) in a cavity with conducting boundaries.
but for Pr=0.05 this is only a small amount. As the Prandtl number is increased to 0.25 the local energy transfer from the mean flow is even greater than before, the excess being transferred into the buoyancy field near the centerline. This is shown in the bottom half of Figure 5. Close to \( y=\pm 0.4 \) disturbance energy is fed back into the base flow. At this location \( u \) and \( v \) components are sufficiently well correlated, and the velocity gradient has an opposite sign to its value at the center, to cause this peak. The streamlines and isotherms evolve, apparently continuously, from the patterns shown for Pr=0.05 in the top parts of Figures 6 and 7 to those shown for a fluid with Pr=0.25. A travelling transverse mode, however, becomes more critical when the Prandtl number is greater than 0.4. The more complicated patterns of streamlines and isotherms corresponding to this case are shown in the bottom half of Figures 6 and 7. In these figures only one wavelength needs to be shown owing to the periodicity in the \( x \)-direction. The height of the cavity, \( (H) \) is held fixed regardless of the magnification of the figures.

The abrupt change at 0.3<Pr<0.4 of the transverse modes is actually not physically realizable, because the longitudinal oscillatory modes are the critical ones for
Figure 6: Perturbation streamlines for stationary transverse cells for $Pr=0.05$ (top), $Pr=0.15$ (second), $Pr=0.25$ (third), and $Pr=0.4$ (bottom) in a cavity with conducting walls.
Figure 7: Perturbation isotherms for stationary transverse cells for Pr=0.05 (top), Pr=0.15 (second), Pr=0.25 (third), and Pr=0.4 (bottom) in a cavity with conducting walls.
the range $0.14 < \text{Pr} < 0.45$. The curve in Figure 3 for the critical states of these longitudinal modes is seen to turn back resulting in two critical states near $\text{Pr} = 0.4$. The oscillation frequencies for each of these branches are shown in Figure 8. At the low end of this range of Prandtl numbers most of the disturbance energy is transferred from the base flow, but at $\text{Pr} = 0.3$ already $1/3$ of the energy comes from the potential energy stored in the buoyancy field. The local energy transfer of these cases are shown in Figure 9 and the stream patterns and isotherms are illustrated in Figure 10.

Beyond $\text{Pr} = 0.45$ stationary longitudinal modes are the critical ones. Their wave number is about 8.3, which corresponds to a wavelength equal to 0.76 of the cavity height. From the local energy plots in Figure 11, one sees that the buoyant conversion, which accounts for 60% of the dissipation at $\text{Pr} = 1.0$ and over 90% at $\text{Pr} = 10$, takes place near the the walls symmetrically about the centerline of the cavity. The stream patterns and isotherms for these longitudinal cells are shown in Figure 12.

The stationary longitudinal modes are just slightly less stable at large $\text{Pr}$, than travelling transverse modes. Thus the physically realizable flow at high Prandtl numbers depends on the growth rates of the
Figure 8: Oscillation frequencies for longitudinal modes in a cavity with conducting walls.
Figure 9: Normalized local energy transfer for oscillatory longitudinal cells for Pr=0.1 (top) and Pr=0.3 (bottom) in a cavity with conducting walls.
Figure 10: Streamlines and isotherms of oscillatory longitudinal rolls for Pr=0.1 (top two) and Pr=0.3 (bottom two) in a cavity with conducting walls.
Figure 11: Normalized local energy transfer for stationary longitudinal rolls for Pr=1.0 (top) and Pr=10 (bottom) in a cavity with conducting boundaries.
Figure 12: Stream patterns and isotherms of stationary longitudinal rolls for $Pr=1.0$ (top two) and $Pr=10$ (bottom two) in a cavity with conducting walls.
respective modes and the nonlinear interaction left out from a linear stability analysis. At the large Prandtl numbers the critical states depend only on the Rayleigh number, which is defined as Ra=GrPr. Indeed, the longitudinal modes begin to appear at Ra \( \sim \) 15400 irrespective of the Prandtl number, for the inertial terms in the momentum equation are unimportant. The wave speeds displayed in Table 4 when multiplied by Pr are likewise constant for large Pr. Both transverse travelling modes and the stationary longitudinal modes obtain majority of their energy from the buoyancy field. The local production rates for the travelling transverse modes are shown in Figure 13 for Pr=1.0 and Pr=50. The evolution of the stream patterns and isotherms are illustrated in Figures 14 and 15. The peak buoyant conversion and the location of the centers of the stream patterns and isotherms now coincide and are close to the boundaries, and since most of the dissipation takes place also next to the boundary, the center region is void of structure and passive.

In comparing our results with those of Hart (1972), the differences stem from the role of these transverse travelling modes. He found them to be the most critical for Pr>0.05. Our results indicate that the transverse modes, whether stationary or travelling, become very
Table 4: Wave speeds for the travelling transverse mode in a cavity with conducting walls.

<table>
<thead>
<tr>
<th>Pr</th>
<th>( \text{Pr}_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.4</td>
<td>4359.6</td>
</tr>
<tr>
<td>0.5</td>
<td>2785.5</td>
</tr>
<tr>
<td>0.6</td>
<td>1782.0</td>
</tr>
<tr>
<td>0.7</td>
<td>1176.0</td>
</tr>
<tr>
<td>0.8</td>
<td>1110.4</td>
</tr>
<tr>
<td>0.9</td>
<td>1046.7</td>
</tr>
<tr>
<td>1.0</td>
<td>1023.0</td>
</tr>
<tr>
<td>2.0</td>
<td>908.0</td>
</tr>
<tr>
<td>3.0</td>
<td>867.0</td>
</tr>
<tr>
<td>4.0</td>
<td>852.0</td>
</tr>
<tr>
<td>5.0</td>
<td>835.0</td>
</tr>
<tr>
<td>6.0</td>
<td>834.0</td>
</tr>
<tr>
<td>10.0</td>
<td>830.0</td>
</tr>
<tr>
<td>50.0</td>
<td>840.0</td>
</tr>
</tbody>
</table>
Figure 13: Normalized local energy transfer for travelling transverse waves for Pr=1.0 (top) and Pr=50 (bottom) in a cavity with conducting walls.
Figure 14: Perturbation streamlines for travelling transverse waves for Pr=0.7 (top left), Pr=1.0 (top right), Pr=4.0 (bottom left) and Pr=50 (bottom right) in a cavity with conducting walls.
Figure 15: Perturbation isotherms for travelling transverse waves for Pr=0.7 (top left), Pr=1.0 (top right), Pr=4.0 (bottom left) and Pr=50 (bottom right) in a cavity with conducting walls.
stable near Pr=0.4. For Gr<10^6 from Figure 3 one sees that the two branches of the transverse modes have not crossed and may not cross ever at higher values. The clear change in the mode of instability is also evident from Table 5 which shows the energetics of the flow.

2.3.2 INSULATED BOUNDARIES

In Figure 16 the critical states for various Prandtl numbers are shown and the critical wave numbers for the various modes are given in Figure 17. These data are also tabulated in Appendix A, Table 8. Although our results are similar to those published by Hart (1983), three points are worth noting.

First, the curve for the transverse modes becomes very steep for Pr \( \leq 0.1 \) so that these modes are strongly stabilized for large values of Pr; second, the curve for longitudinal oscillatory modes has a knee near Pr=0.2 allowing a reverse transition as the Grashof number is increased for fluids with Prandtl numbers in the range 0.1-0.2, roughly; third, our neutral states for longitudinal oscillatory modes occur at larger Gr than was found by Hart. These findings are in good agreement with the numerical results reported by Roux et al. (1984). Therefore, the crossover between the
Table 5: Energy balance for transverse modes in a cavity with conducting walls.

<table>
<thead>
<tr>
<th>Pr</th>
<th>E_b</th>
<th>E_i</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.001</td>
<td>-0.12</td>
<td>100.12</td>
</tr>
<tr>
<td>0.005</td>
<td>-0.66</td>
<td>100.66</td>
</tr>
<tr>
<td>0.01</td>
<td>-1.42</td>
<td>101.44</td>
</tr>
<tr>
<td>0.05</td>
<td>-12.47</td>
<td>112.47</td>
</tr>
<tr>
<td>0.07</td>
<td>-21.52</td>
<td>121.52</td>
</tr>
<tr>
<td>0.10</td>
<td>-37.04</td>
<td>137.04</td>
</tr>
<tr>
<td>0.15</td>
<td>-50.65</td>
<td>150.66</td>
</tr>
<tr>
<td>0.20</td>
<td>-51.41</td>
<td>151.41</td>
</tr>
<tr>
<td>0.25</td>
<td>-58.85</td>
<td>158.85</td>
</tr>
<tr>
<td>0.3</td>
<td>-85.21</td>
<td>185.05</td>
</tr>
<tr>
<td>0.4</td>
<td>134.56</td>
<td>-33.98</td>
</tr>
<tr>
<td>0.5</td>
<td>138.88</td>
<td>-38.82</td>
</tr>
<tr>
<td>0.6</td>
<td>133.77</td>
<td>-33.77</td>
</tr>
<tr>
<td>0.7</td>
<td>118.45</td>
<td>-18.45</td>
</tr>
<tr>
<td>0.8</td>
<td>111.57</td>
<td>-11.57</td>
</tr>
<tr>
<td>0.9</td>
<td>106.74</td>
<td>-6.74</td>
</tr>
<tr>
<td>1.0</td>
<td>100.43</td>
<td>-0.43</td>
</tr>
<tr>
<td>2.0</td>
<td>97.73</td>
<td>2.27</td>
</tr>
<tr>
<td>3.0</td>
<td>97.53</td>
<td>2.47</td>
</tr>
<tr>
<td>4.0</td>
<td>97.96</td>
<td>2.04</td>
</tr>
<tr>
<td>5.0</td>
<td>98.33</td>
<td>1.67</td>
</tr>
<tr>
<td>6.0</td>
<td>98.66</td>
<td>1.34</td>
</tr>
<tr>
<td>10.0</td>
<td>99.29</td>
<td>0.71</td>
</tr>
<tr>
<td>50.0</td>
<td>99.89</td>
<td>0.11</td>
</tr>
</tbody>
</table>
Figure 16: Critical states for insulated boundaries,
L = longitudinal, T = transverse,
Sta. = stationary, Osc. = oscillatory.
• represents data from Roux et al.
Figure 17: Critical wave numbers for insulated boundaries.
oscillatory longitudinal and stationary transverse cells shifts to a large Pr with the result that transverse cells are the critical ones for $Pr \approx 0.035$. This is close to the value of Prandtl numbers for both gallium and mercury with which experiments have been carried out. The critical Grashof numbers for the two kinds of modes for these fluids are quite close. Thus, since in the experiments longitudinal oscillatory cells were in fact found, their occurrence is likely to be the result of their faster growth rate. This appears also to be the view of Gill (1974), who ignored the transverse cells completely in his analysis.

For low Prandtl numbers the instability sets in as stationary transverse cells. The local energetics are shown in Figure 18 for $Pr=0.02$ and the evolution of stream patterns and isotherms are illustrated in Figures 19 and 20 for these modes. Their total energetics are tabulated in Table 6. Since the fluid has a very high thermal diffusivity, the thermal perturbations are less important than the hydrodynamic ones and accordingly the thermal boundary conditions do not influence the mechanism for instability. To study the effects of different thermal boundaries further, we have calculated the dynamics of the local thermal fields. By multiplying Equation (2.32) by $\theta^*$ and integrating over a
Figure 18: Normalized local energy transfer for stationary transverse cells for $Pr=0.02$ in a cavity with insulated boundaries.
Figure 19: Perturbation streamlines for stationary transverse cells for $Pr=0.02$ (top), $Pr=0.08$ (center), and $Pr=0.1$ (bottom) in a cavity with insulated boundaries.
Figure 20: Perturbation isotherms for stationary transverse cells for \( Pr=0.02 \) (top), \( Pr=0.08 \) (center), and \( Pr=0.1 \) (bottom) in a cavity with insulated boundaries.
Table 6: Energy balance for transverse modes in a cavity with insulated walls.

<table>
<thead>
<tr>
<th>Pr</th>
<th>$E_b$</th>
<th>$E_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.001</td>
<td>-0.13</td>
<td>100.13</td>
</tr>
<tr>
<td>0.005</td>
<td>-0.88</td>
<td>100.88</td>
</tr>
<tr>
<td>0.01</td>
<td>-2.35</td>
<td>102.36</td>
</tr>
<tr>
<td>0.02</td>
<td>-7.26</td>
<td>107.28</td>
</tr>
<tr>
<td>0.03</td>
<td>-15.12</td>
<td>115.13</td>
</tr>
<tr>
<td>0.04</td>
<td>-25.76</td>
<td>125.81</td>
</tr>
<tr>
<td>0.05</td>
<td>-37.07</td>
<td>137.07</td>
</tr>
<tr>
<td>0.06</td>
<td>-45.74</td>
<td>145.76</td>
</tr>
<tr>
<td>0.08</td>
<td>-52.61</td>
<td>152.61</td>
</tr>
<tr>
<td>0.10</td>
<td>-58.02</td>
<td>158.02</td>
</tr>
<tr>
<td>0.11</td>
<td>-73.26</td>
<td>173.26</td>
</tr>
<tr>
<td>0.12</td>
<td>-87.96</td>
<td>187.96</td>
</tr>
<tr>
<td>0.13</td>
<td>-109.85</td>
<td>209.79</td>
</tr>
</tbody>
</table>
cell yields

\[ TH_h + TH_v = TH_d, \quad (2.51) \]

where

\[ TH_h = -GrPr \int_{-0.5}^{0.5} (\mathbf{v}^* \mathbf{u}^* + \mathbf{u}^* \mathbf{v}^*) \partial T_b / \partial x \, dy, \]

\[ TH_v = -GrPr \int_{-0.5}^{0.5} (\mathbf{v}^* \mathbf{v}^* + \mathbf{v}^* \mathbf{v}^*) \mathbf{D}_b \, dy, \]

\[ TH_d = 2 \int_{-0.5}^{0.5} (\alpha^2 \mathbf{u}^* \mathbf{u}^* + \mathbf{D} \mathbf{D} \mathbf{u}^*) \, dy. \quad (2.52) \]

\( TH_h \) and \( TH_v \) represent the rates of production of the magnitude of the temperature disturbances by horizontal and vertical base temperature gradients; \( TH_d \) is then accounts for the dissipation of these disturbances by conduction. Figure 21 shows the balance for the thermal dynamics for a fluid of \( Pr=0.02 \) with two different thermal boundary conditions. \( TH_d \) has been plotted on the negative side so that the figure does not become cluttered. For conducting boundaries \( Gr_c=8272 \) was used in calculation whereas \( Gr_c=8642 \) was used for insulated boundaries. At these conditions the kinetic energy transfer from the buoyancy field is less than ten percent of the total kinetic energy production. Since the eigenfunction have been first normalized, it is
Figure 21: Dynamics of local thermal field for Pr=0.02, transverse stationary modes, in a cavity with conducting boundaries (solid) and with insulated boundaries (dash).
clear that in comparison to kinetic energy production terms the thermal dynamics are less important and particularly so for a flow in a cavity with insulated boundaries. Another set of results for \( \text{Pr}=0.1 \), where \( \text{Gr}_c \) is about three times larger for insulated boundaries than for conducting one, is shown in Figure 22. Although the amounts of production from the vertical field are almost the same for both kinds of boundaries, the horizontal production is substantially smaller for insulated boundaries. Since the horizontal temperature gradient is the same in both cases and the eigenfunctions are normalized, it must be that the conducting boundaries improve the correlation between the disturbance temperature and horizontal velocity component.

As the Prandtl number increases to 0.035 the oscillatory longitudinal modes become important. The oscillation frequencies are shown in Figure 23. For still higher Prandtl numbers longitudinal stationary modes are the most critical until \( \text{Pr} \lesssim 2 \) is reached, beyond which there are three transverse travelling modes close to one another. These occur at very high values of \( \text{Gr} \) (higher than one million) and consequently their investigation requires a great amount of computer time, owing to the slow convergence of the series.
Figure 22: Dynamics of local thermal field for Pr = 0.1, transverse stationary modes, in a cavity with conducting boundaries (solid) and with insulated boundaries (dash).
Figure 23: Oscillation frequencies for longitudinal modes in a cavity with insulated boundaries.
approximations used here. The transverse travelling modes at large Pr have a very complicated spatial structure. As an example for a fluid with Pr=0.8 the stream patterns of the three modes which lie close to one another are shown in Figure 24.

2.3.3 THREE DIMENSIONAL DISTURBANCES

As discussed in Section 1.3, most of the former studies have assumed two-dimensional disturbances to be less stable than three-dimensional ones and our second objective was to see whether three-dimensional disturbances are, in fact, important. This was done for two fluids, namely air (Pr=0.7) and gallium (Pr=0.02). One reason of selecting these two fluids is that the critical Grashof numbers for transverse modes and longitudinal modes are very close to one another for air in a cavity with conducting boundaries and for gallium in an insulated cavity. The computation time of the neutral surface for these three-dimensional disturbances is large. Roughly 800 minutes of CPU on an IBM 3081 is required in each case. The results were plotted in Figures 25 and 26. For air, shown in Figure 25, a stationary longitudinal mode at β=8.0 is the most unstable one with $\text{Gr}_c \approx 18125$. For a fluid with Pr=0.02 (molten gallium) in a cavity with insulated boundaries
Figure 24: Perturbation stream patterns of three different modes for travelling transverse waves for Pr=0.8 in a cavity with insulated boundaries.
the instability sets in as stationary transverse cells at $Gr=8642.1$ for $\alpha=2.70$ and $\beta=0$. The ridge in Figure 26 which occurs at $\alpha \approx 1.0-1.5$ separates the neutral states to stationary ones on the right of the ridge and to oscillatory states to the left of it.
Figure 25: Neutral stability surface for air (Pr=0.7) in a cavity with conducting boundaries.
Figure 26: Neutral stability surface for gallium (Pr=0.02) in a cavity with insulated boundaries.
Chapter 3
SECONdARY FLOW SIMULATION

3.1 FORMULATION

To keep the parametric study manageable we consider the two-dimensional secondary flows of only the low-Prandtl-number fluids which are characterized by the transverse stationary mode of instability. Using the vector identities

\[(\vec{\nabla} \cdot \vec{\nabla}) \vec{u} = \frac{1}{2} \vec{\nabla}(-\vec{\nabla} \cdot \vec{u}) - \vec{\nabla} x (\vec{\nabla} x \vec{u}), \quad (3.1)\]

\[(\vec{\nabla} \cdot \vec{\nabla}) T = \vec{\nabla} \cdot (\vec{\nabla} T) - T(\vec{\nabla} \cdot \vec{u}), \quad (3.2)\]

equations (2.2) and (2.3) can be written as

\[\frac{\partial \vec{u}}{\partial t} = \text{Gr} (\vec{\nabla} x \vec{\omega}) - \text{Gr} \vec{v}(p + \frac{\vec{\nabla} \cdot \vec{u}}{2}) + \vec{v}^2 \vec{u} + T_j, \quad (3.3)\]

\[\frac{\partial T}{\partial t} = - \text{Gr} \vec{\nabla} \cdot (\vec{\nabla} T) + \frac{1}{\text{Pr}} \vec{v}^2 T, \quad (3.4)\]

where \(\vec{\omega}\) is the vorticity vector defined by
\[ \mathbf{\omega} = \mathbf{\nabla} \times \mathbf{V}. \quad (3.5) \]

We can further set each variable to be the sum of a base flow contribution plus a secondary flow one, as

\[ \mathbf{V} = \mathbf{V}_b + \mathbf{V}_s', \]
\[ \mathbf{\omega} = \mathbf{\omega}_b + \mathbf{\omega}_s', \]
\[ T = T_b + T_s', \]
\[ P = P_b + P_s'. \quad (3.6) \]

Thus the Navier-Stokes and energy equations, written in terms of secondary flow variables, are

\[ \frac{\partial \mathbf{V}_s}{\partial t} = \text{Gr} (\mathbf{V}_b + \mathbf{V}_s) \times (\mathbf{\omega}_b + \mathbf{\omega}_s) - \text{Gr} \mathbf{\nabla} \Pi + \nabla^2 \mathbf{V}_s + T_s \mathbf{j}, \quad (3.7) \]
\[ \frac{\partial T_s}{\partial t} = - \text{Gr} [\mathbf{V} \cdot (\mathbf{V}_s (T_b + T_s) \mathbf{j})] + \mathbf{u}_b \frac{\partial T_s}{\partial x} + \frac{1}{\text{Pr}} \nabla^2 T_s. \quad (3.8) \]

Notice that Equations (2.4)-(2.6) have been used to eliminate some base flow terms and \( \Pi \) represents the total pressure

\[ \Pi = P_s + \frac{1}{2} (\mathbf{V}_s + \mathbf{V}_b)^2. \quad (3.9) \]

Periodicity in the \( x \)-direction can be imposed over a cell length of \( 2\pi/\alpha \), so that
Introducing the secondary flow variables introduces the advantage of making the boundary conditions in the y-direction homogeneous. For the highly conducting horizontal boundaries we have

\[ u_s(x = -\pi, \alpha) = u_s(x = \pi, \alpha), \]
\[ v_s(x = -\pi, \alpha) = v_s(x = \pi, \alpha), \]
\[ T_s(x = -\pi, \alpha) = T_s(x = \pi, \alpha). \]  \hspace{1cm} (3.10)

Introducing the secondary flow variables introduces the advantage of making the boundary conditions in the y-direction homogeneous. For the highly conducting horizontal boundaries we have

\[ u_s = v_s = T_s = 0 \text{ at } y = \pm \frac{1}{2}; \]  \hspace{1cm} (3.11)

for the insulated case \( \frac{\partial T_s}{\partial y} \) will be zero at the upper and lower walls. Equations (3.7) and (3.8) are solved subject to the kinematic condition

\[ \nabla \cdot \vec{V}_s = 0. \]  \hspace{1cm} (3.12)

3.2 TIME-SPLITTING METHOD

In the secondary flow simulations Chebyshev expansions are used in the y-direction and Fourier expansions in the periodic x-direction. Thus, each variable \( Q(x,y,t) \) can be written as a spectral sum
\[ Q(x,y,t) = \Re \left\{ \sum_{m=-M}^{M} \sum_{n=0}^{N} \hat{Q}(m,n,t) T_n(y) e^{i\alpha mx} \right\}, \quad (3.13) \]

where \( T_n(y) \) are Chebyshev polynomials. The spectral coefficients \( \hat{Q}(m,n,t) \) will be referred to as the solution in the spectral space. It is also convenient to define variables in a mixed physical-spectral space by the expansion

\[ Q(m,y,t) = \sum_{n=0}^{N} \hat{Q}(m,n,t) T_n(y). \quad (3.14) \]

The left hand side of this equation is the Fourier coefficient in the expansion (3.13). Gottlieb and Orszag (1977) discuss how to calculate derivatives and nonlinear terms using pseudo-spectral approximations. Briefly, \( x \)-derivatives are obtained by multiplying the Fourier coefficients by \( i\alpha m \) in the mixed space; \( y \)-derivatives are done spectrally in a manner discussed earlier in Section 2.2.2. Nonlinear products in the Equations (3.7) & (3.8) can be computed by multiplying together the two factors in a given product at the collocation points in the physical space. The transforms between the various spaces can be made efficiently with the use of fast Fourier transforms (FFT).
Following Marcus (1984), we used a method with no time-splitting error for velocity calculations. The original idea of advancing the velocity from time step $N$ to time step $N+1$ is to use a scheme of three fractional steps. Where in the first fractional step the nonlinear convective term and the buoyant term are computed using an Adams-Bashforth method

\[ \frac{\hat{v}_{N+1}^{1/3}}{s} = \frac{\hat{v}_{N}^{1/3}}{s} + \frac{3}{2} \Delta t \left[ -\nabla \cdot \nabla \times (\hat{v}_{N}^{1/3} + \hat{v}_{b}^{1/3}) \times (\omega_{N}^{1/3} + \omega_{b}^{1/3}) + \frac{3}{2} \frac{T_{N}^{1/3}}{s} \right] \]

\[ - \frac{1}{2} \Delta t \nabla \cdot (\hat{v}_{N-1}^{1/3} + \hat{v}_{b}^{1/3}) \times (\omega_{N-1}^{1/3} + \omega_{b}^{1/3}) + \frac{1}{2} \frac{T_{N-1}^{1/3}}{s} \right], \tag{3.15} \]

the second fractional step takes care of the contribution of the pressure

\[ \frac{\hat{v}_{N+2}^{2/3}}{s} = \frac{\hat{v}_{N+1}^{2/3}}{s} - \Delta t \nabla \cdot \nabla \frac{\Pi_{N+1}}{s}, \tag{3.16} \]

and the third fractional step is an implicit viscous step

\[ \frac{\hat{v}_{N+1}^{1/3}}{s} = \frac{\hat{v}_{N+2}^{2/3}}{s} + \Delta t \nabla^{2} \frac{\Pi_{N+1}}{s}. \tag{3.17} \]

While solving the second fractional step the total pressure $\Pi_{N+1}$ is obtained from Equation (3.16) by enforcing the kinematic constraint

\[ \nabla \cdot \frac{\hat{v}_{N+2}^{2/3}}{s} = 0. \tag{3.18} \]
so that Equation (3.16) reduces to

\[ \Delta t G r \nabla^2 \Pi_{N+1} = \nabla \cdot \nabla_{s}^{N+1/3}, \quad (3.19) \]

The correct boundary conditions at \( y = \pm 1/2 \) would be

\[ \frac{\partial \Pi_{N+1}}{\partial y} = j . (\nabla_{s}^{N+1/3} + \Delta t \nabla^2 \nabla_{s}^{N+1}), \quad (3.20) \]

but one still has \( \nabla_{s}^{N+1} \) as an unknown. By approximating it by \( \nabla_{s}^{N+1/3} \) a considerable time-splitting error would occur. Marcus (1984) overcomes this difficulty by introducing a Green's function. He defines

\[ \Pi_{N+1} = \Pi_{p}^{N+1} + \Pi_{c}^{N+1}, \]

\[ \nabla_{s}^{N+2/3} = \nabla_{p}^{N+2/3} + \nabla_{c}^{N+2/3}, \]

\[ \nabla_{s}^{N+1} = \nabla_{p}^{N+1} + \nabla_{c}^{N+1}, \quad (3.21) \]

with the terms with a subscript \( p \) denoting the partial solutions and the correction quantities being identified with a subscript \( c \). In this way one can get the partial total pressure \( \Pi_{p}^{N+1} \) from

\[ \nabla_{p}^2 \Pi_{p}^{N+1} = \frac{1}{\Delta t G r} [\nabla \cdot \nabla_{s}^{N+1/3}], \quad (3.22) \]

with the Dirichlet boundary conditions
\[ \Pi_{p}^{N+1} = 0 \text{ at } y = \pm 1/2. \] (3.23)

Thus one can calculate \( v_{p}^{N+2/3} \) and \( v_{p}^{N+1} \) from

\[ v_{p}^{N+2/3} = v_{s}^{N+1/3} - \Delta t \text{Gr} \Pi_{p}^{N+1}, \] (3.24)

\[ v_{p}^{N+1} = (1 - \Delta t v^2)^{-1} v_{p}^{N+2/3}. \] (3.25)

The correction to the pressure term obeys

\[ \nabla^2 \Pi_{c}^{N+1} = 0, \] (3.26)

with the inhomogeneous boundary conditions at \( y = \pm 1/2 \)

\[ \Delta t \text{Gr} \frac{\partial \Pi_{c}^{N+1}}{\partial y} = \hat{j} \cdot (v_{s}^{N+1/3} + \Delta t \nabla^2 v_{s}^{N+1}) - \Delta t \text{Gr} \frac{\partial \Pi_{p}^{N+1}}{\partial y}. \] (3.27)

The correction to the velocity is

\[ v_{c}^{N+2/3} = v_{s}^{N+2/3} - v_{p}^{N+2/3} \]

\[ = -\Delta t \text{Gr} \Pi_{c}^{N+1}. \] (3.28)

The original viscous step (Equation (3.17)) becomes now

\[ v_{s}^{N+1} = v_{p}^{N+1} + (1 - \Delta t v^2)^{-1} v_{c}^{N+1} \]

\[ = v_{p}^{N+1} - \Delta t \text{Gr} (1 - \Delta t v^2)^{-1} \Pi_{c}^{N+1}. \] (3.29)

It is clear now that if one can solve for the pressure correction term \( \Pi_{c}^{N+1} \), then the whole time step can be advanced without the time-splitting error.
Since the continuity condition is enforced in each time step, Equation (3.29) has to be divergence free and can be written as

$$\Delta t \nabla \cdot (1 - \Delta t \nabla^2)^{-1} \nabla \Pi_{c}^{N+1} = \nabla \cdot \nabla p^{N+1}. \quad (3.30)$$

One can now define a Green's function, which only has to be computed once

$$G_i(m,y) = -\Delta t \nabla (1 - \Delta t \nabla^2)^{-1} \nabla X_i(m,y), \quad (3.31)$$

and express $\Pi_{c}^{N+1}$ as

$$\Pi_{c}^{N+1}(m,y) = a_{1}^{N+1}(m)X_1(m,y) + a_{2}^{N+1}(m)X_2(m,y), \quad (3.32)$$

where $X_1$ and $X_2$ are two linearly independent solutions of

$$\nabla^2 [X_1(m,y)e^{i\alpha m x}] = \nabla^2 [X_2(m,y)e^{i\alpha m x}] = 0, \quad (3.33)$$

with the boundaries conditions

$$X_1 = 1, X_2 = 0 \text{ at } y = \frac{1}{2},$$

$$X_1 = 0, X_2 = 1 \text{ at } y = -\frac{1}{2}. \quad (3.34)$$

Thus Equation (3.29) is equivalent to
The coefficients \( \mathbf{a}_1^{N+1} \) are to be calculated in each time step from

\[
\mathbf{10} \mathbf{0} \mathbf{0} ( \mathbf{m} , \mathbf{y} ) = \mathbf{10} \mathbf{0} \mathbf{0} ( \mathbf{m} , \mathbf{y} ) + \mathbf{a}_1^{N+1} ( \mathbf{m} ) G_1 ( \mathbf{m} , \mathbf{y} ) + \mathbf{a}_2^{N+1} ( \mathbf{m} ) G_2 ( \mathbf{m} , \mathbf{y} ) .
\]  

(3.35)

The temperature calculation for each time step is divided into two fractional steps in a similar way. The first nonlinear fractional step is

\[
\Delta t \mathbf{G} \mathbf{v} \cdot ( \mathbf{1} - \Delta t \mathbf{v}^2 )^{-1/2} ( \mathbf{a}_1^{N+1} \mathbf{X}_1 + \mathbf{a}_2^{N+1} \mathbf{X}_2 ) = \mathbf{v} \cdot \mathbf{v}^{N+1} .
\]  

(3.36)

evaluated at \( y=\pm 1/2 \). If one calculates and stores four coefficients \( b_{11} , b_{12} , b_{21} \) and \( b_{22} \) in the beginning of a program where

\[
b_{11} = \Delta t \mathbf{G} \mathbf{v} \cdot [ ( \mathbf{1} - \Delta t \mathbf{v}^2 )^{-1/2} \mathbf{v} \mathbf{X}_i ] \text{ at } y = 1/2 ,
\]

\[
b_{21} = \Delta t \mathbf{G} \mathbf{v} \cdot [ ( \mathbf{1} - \Delta t \mathbf{v}^2 )^{-1/2} \mathbf{v} \mathbf{X}_i ] \text{ at } y = -1/2 ,
\]

(3.37)

then for each time step one can evaluate \( \mathbf{a}_1^{N+1} \) from

\[
\begin{bmatrix}
  b_{11} & b_{12} \\
  b_{21} & b_{22}
\end{bmatrix}
\begin{bmatrix}
  \mathbf{a}_1^{N+1} \\
  \mathbf{a}_2^{N+1}
\end{bmatrix}
= \begin{bmatrix}
  \mathbf{v} \cdot \mathbf{v}^{N+1} \text{ at } y = 1/2 \\
  \mathbf{v} \cdot \mathbf{v}^{N+1} \text{ at } y = -1/2
\end{bmatrix}
\]

(3.38)

and solve the true viscous step in Equation (3.35).

The temperature calculation for each time step is divided into two fractional steps in a similar way. The first nonlinear fractional step is
and the second implicit fractional step

\[ T_s^{N+1} = (1 - \frac{\Delta t}{Pr})^{-1} T_s^{N+1/2}, \quad (3.40) \]

can be calculated by a backward Euler method. A flow chart of the computational procedure is shown on Figure 27.

3.3 VALIDITY OF THE SIMULATIONS

The accuracy of the secondary flow simulations is assured by checking a normalized \( \nabla \cdot \vec{\nu} \) to be less than \( 10^{-6} \) everywhere in the fluid at all times. A second test of the solution accuracy is accomplished by computing curl of the Navier-Stokes equation

\[ R = \nabla \times \left[ \text{Gr}(\vec{V}_b + \vec{V}_s)(\omega_b + \omega_s) + \nu^2 \vec{V}_s + T_s \hat{\nu} \right], \quad (3.41) \]

which will decay to zero as the steady state is approached. For some values of the parameters steady state may not be possible because the actual solution is periodic in time. Such was the case in the study by
Figure 27: Flow chart of the secondary flow simulation.
Gresho & Upson (1983) on a closely related problem of liquid metal convection in a square cavity. We have examined the velocity and temperature time histories of two selected nodes using four different time steps ($\Delta t=0.003, 0.03, 0.3 & 3.0$) and have not found the kind of fast oscillations noted by them. The time histories are shown in Figure 28. The steady state solutions obtained using each of these time steps differ insignificantly from one another. According to Marcus' analysis (1984) the algorithm we use should have a temporal error of $O(\Delta t^2)$. This was verified by comparing the growth rates obtained from the linear stability analysis and the flow simulations. From these tests we conclude that the computational scheme gives a faithful representation of the steady state solution with the cellular structure.

A typical calculation using 33 terms in both $x$ and $y$ expansions takes roughly 15 seconds of CPU time per time step on an IBM 3081 computer. The most costly computation, accounting for about 60% of the total CPU time, is the implicit fractional step where the inverse of the Laplacian is calculated. For each time step this calculation is repeated twice, once for the velocity and once for the temperature. Normally the flow reaches steady state after 3000 to 5000 time steps. The exact
Figure 28: The velocity (left) and temperature (right) time-histories of two selected nodes in a cavity with insulated walls.
number largely depends on the initial guess of the velocity and temperature fields and the time step selected. In general, we took the initial guess to be the solution of the linear stability problem. For a weakly nonlinear problem this speeds the computation the most.

3.4 RESULTS AND DISCUSSION

Several numerical experiments have been carried out in the range of Grashof numbers for which, according to the linear stability theory, secondary flows are present. First, however, the decay of an initially random field was calculated for a fluid with Pr=0.1 in a cavity with conducting boundaries. This was done for a 17x17 grid in one cell domain. For a Grashof number slightly less than the critical one (Gr_c=12691) the initial field was indeed seen to decay to a quiescent condition. For a supercritical Grashof number (Gr=13000) the same initial random field begins to grow into a stream pattern which is quite similar to the one obtained from the linear stability theory. The results are shown in Figure 29 at three different instants during the development of the flow field. In order to see the flow patterns clearly the plots show the flow in a domain covering two cells. So the calculated pattern
Figure 29: Growth of random velocity disturbances when $Gr=13000 > Gr_\text{c} = 12691$ for $Pr=0.1$ in a cavity with conducting boundaries.
for one cell is duplicated in the plots. If the Grashof number is much higher than the critical one, these disturbances can grow to a new stream pattern which is qualitatively quite different from that given by the linear stability solution at the critical Gr. Indeed, for sufficiently large value of Grashof number a sequence of transitions will take place which eventually lead to three dimensional oscillatory motions in a laboratory experiment. It is for this reason that calculations for only slightly supercritical flows are reported.

In Figure 30 the total flow at Gr=13,000 is shown and compared with the linear stability prediction. Since the strength of the secondary flow from the linear stability study is unknown the eigenfunction has been normalized first such that the strength can be controlled at will. By adding the base flow and 0.4 times the secondary flow yields the pattern shown in Figure 30. At this condition the similarity between the two total flows is clear. However, the secondary stream patterns show that the cellular pattern loses some of its regularity. The counterclockwise rotating flow in the bottom part of Figure 31 has been squeezed to less space than the adjacent cells. In the study of Drummond the nonlinear effects were seen to cause an increase in
Figure 30: Total flow streamlines from the linear stability prediction (top) and the secondary flow simulation (bottom) for Pr=0.1 in a cavity with conducting boundaries.
the wavelength. However in his case the cells were free
to move laterally and all adjustment was taken up in the
end regions. Because periodic boundary conditions are
imposed in our calculation, such an adjustment of the
cells is not possible.

Because of its importance in growing crystals,
flows of gallium with Pr=0.02, were considered next.
Steady solutions were obtained for numerous values of \( \alpha \)
and, for Gr exceeding the critical value, \( Gr_c=8642 \),
determined by linear stability theory. The results
where solutions were found are shown in Figure 32. No
subcritical finite-amplitude solution was found. But
for small values of \( \alpha \), i.e. \( \alpha<1.5 \), the numerical
simulation could produce solutions with apparently 2\( \alpha \) or
3\( \alpha \) as the basic wave number. In some cases solutions
oscillate in time or posses stream patterns
qualitatively different from the slightly-supercritical
ones. These may correspond to Eckhaus instability, but
since the Eckhaus instability analysis was not carried
out the results of this part of the stability diagram
remain uncertain. Steady solutions were also obtained
beyond the right bound of the neutral stability curve
for large wave numbers. A similar phenomenon has been
observed and discussed by Nagata and Busse (1983).

The effect of different wave numbers and Grashof
Figure 31: Stream patterns from the linear stability prediction (top) and the secondary flow simulation use Gr=13000 (bottom) for Pr=0.1 in a cavity with conducting boundaries.
Figure 32: Neutral stability curve from linear stability prediction and results from secondary flow simulation. ◇ means that a steady solution was found, ○ means uncertain result was found, and △ means that no solution was found. The data are for a fluid with Pr=0.02 in a cavity with insulated walls.
numbers on the horizontal averaged velocity of the total flow as a function of \( y \) was also investigated. Figure 33 shows that this horizontal averaged velocity changes only slightly as a function of wave number. However, as shown in Figure 34, changes in Grashof number have marked influence on the average velocity. The flow at large values of \( Gr \) tends to shift the maximum of the average velocity closer to the upper and lower walls. These results are basically the same as those reported by Nagata and Busse. The shift in the maximum is caused by the stronger secondary flow which mixes the fast and slow moving fluid and thereby homogenizes the mean momentum in the interior of the cavity.

Typical steady state streamlines and isotherms of the secondary flow are depicted in Figure 35. The calculations were carried out on a 33x33 grid at \( Gr=8700 \), which is slightly supercritical (\( Gr_c=8642 \)). The local energy transfer terms are shown in Figure 36. Comparing these results with the ones obtained from the linear stability theory (Figures 18, 19 and 20) similarities in the stream patterns and in the local production rates are clear. However, the isotherms in the two cases are markedly different. Since the buoyant effects are largely unimportant for small Prandtl number fluids this difference in secondary temperature field
Figure 33: Horizontal averaged velocity of the total flow as a function of $y$ for different wave numbers for $Pr=0.02$, $Gr=30,000$ in a cavity with insulated walls.
Figure 34: Horizontal averaged velocity of the total flow as a function of y for different Grashof numbers for Pr=0.02, α=2.7 in a cavity with insulated walls.
Figure 35: Streamlines (top) and isotherms from secondary flow simulation for Pr=0.02 in a cavity with insulated walls.
Figure 36: Normalized local energy transfer from linear theory (dash) and secondary flow simulation (solid) for Pr=0.02 in a cavity with insulated walls.
will neither influence the resulting total flow nor temperature solutions. To give an indication what this corresponds to in practice we have calculated the physical quantities for gallium in a shallow cavity, 1 cm in height and 20 cm in width. For Gr=8700 the thermal velocity is 29.46 cm/sec, the maximum base flow velocity is thus 0.237 cm/sec and the total temperature difference \((T_H - T_L)\) is 16.75 degree in Kelvin. If there are eight cells produced at this condition and the base temperature field will have about 2 degree difference across one cell. The secondary temperature field varies only 3 percent of this amount, or roughly 0.05 degrees. The horizontal secondary velocity, depending on the location, can change the base flow velocity up to 20%. The total flow patterns and thermal fields are shown in Figures 37 and 38. A strength factor of 0.4 was used for the linear theory prediction. We have also calculated the local kinetic energy in the nonlinear case. It is seen in Figure 39 to be similarly distributed as in the linear case.

The Nusselt number defined as

\[
Nu = \int_{-0.5}^{0.5} \left( \frac{dT}{dx} - RaUT \right) dy,
\]

has been calculated to indicate the effect of secondary
Figure 37: Stream patterns of the total flow field from linear theory (top) and secondary flow simulation (bottom) for Pr=0.02 in a cavity with insulated walls.
Figure 38: Isotherms of the total temperature field from linear theory (top) and secondary flow simulation (bottom) for Pr=0.02 in a cavity with insulated walls.
Figure 39: Normalized local kinetic energy from linear theory (dash) and secondary flow simulation (solid) for Pr=0.02 in a cavity with insulated walls.
flow on horizontal heat transfer. These results are shown in Figure 40 and compared with the analytical results for laminar flows derived by Bejan and Tien (1978)

\[ Nu = 1.0 + \frac{Ra^2}{362880}. \]  \hspace{1cm} (3.43)

For low Grashof numbers the laminar base flows are stable and two curves join together, and the deviation for high Grashof numbers can be considered as the effect of secondary flows on horizontal heat transfer. That is, because of mixing, the horizontal velocities are reduced for a given driving potential, with the result that less heat is transported from the hot end to the cold end. It should be noted that since data plotted on this figure were all obtained for Pr=0.02 it might not be accurate for large Grashof numbers where the secondary flows could become unstable and lead to three-dimensional motions.
Figure 40: Nusselt numbers from secondary flow simulation (solid) for Pr=0.02 in a cavity with insulated walls and from analytical results for laminar base flow (dash) by Bejan and Tien (1978).
Chapter 4

CONCLUSIONS

4.1 SUMMARY

In this dissertation the stability of natural convection in a shallow cavity has been studied theoretically. The flow is driven by a horizontal temperature gradient between the isothermal vertical side walls. The top and bottom boundaries are considered to be either insulated or highly conducting.

The eigenvalue problem arising from the linear stability theory was solved pseudospectrally with Chebyshev expansions. The critical wavelength, wavespeed, and Grashof number were determined for a set of Prandtl numbers in a wide range. Many of the previous results calculated by Hart (1972) were corrected and new results were obtained. For the conducting boundaries the shear instability leads to stationary transverse cells for \( Pr < 0.14 \) and the instability is a buoyant type for higher values of \( Pr \). The instability sets in as oscillating longitudinal
rolls in the range $0.14<\text{Pr}<0.45$, and as stationary longitudinal rolls for larger Prandtl numbers. For the insulated boundaries the instability sets in as stationary transverse cells for $\text{Pr}<0.035$, but the longitudinal oscillatory modes are the most critical in the range $0.035<\text{Pr}<0.2$. Stationary longitudinal rolls are the least stable beyond $\text{Pr}=0.2$ until $\text{Pr}=2$ is reached. The perturbation streamlines and isotherms of some typical cases are presented to illustrate the difference between the different modes of instability. The energy flow from the base velocity and the buoyancy fields to the disturbance kinetic energy is also presented to obtain a physical interpretation of the reasons for the base flow instability.

The importance of the three-dimensional disturbances was investigated to assure that they are more stable than the two-dimensional ones. This was shown to be true for two fluids, namely air ($\text{Pr}=0.7$) in a cavity with conducting boundaries and gallium ($\text{Pr}=0.02$) in a cavity with insulated top and bottom. The neutral stability surfaces are presented and in both cases the two-dimensional disturbances were found to be more critical than the three-dimensional ones.

The finite amplitude motions were then simulated by solving the nonlinear equations numerically with the use
of pseudo-spectral methods. For slightly supercritical flows the secondary cells were shown to have stream patterns similar to those predicted by the linear stability theory. The secondary thermal fields on the other hand were influenced greatly by the nonlinearity left out in the stability analysis. Nevertheless, the total flow patterns and total thermal fields are quite similar in both cases. The effects of cell size and Grashof numbers on the mean flow velocity distribution in the vertical direction was determined. The results agree qualitatively with previous studies in a related problem. The local kinetic energy distributions were also calculated in addition to the energy flows and all compared with the linear predictions. This simulation has supported the calculations of the onset of the instability, as well as allowed a better understanding of the nonlinear effects which have been neglected in the linear stability studies.

4.2 FURTHER WORK

A two-dimensional flow simulation for the longitudinal rolls could be carried out next to see the secondary motions of the higher Prandtl number fluids.

With the two-dimensional numerical solution at hand the next transition, which is likely to lead to a
three-dimensional tertiary motion, could then be investigated. Finding the marginal states of the secondary flows would allow one to see the parameter range in which the two-dimensional flow simulations are valid. Next, a direct three-dimensional simulation would reveal the kind of flows present at yet higher values of Grashof numbers. Finally, three-dimensional flows in finite aspect ratio cavities could be calculated to see the influence of aspect ratio on the flow.
APPENDIX A

DATA FOR THE CRITICAL STATES
AND CRITICAL WAVE NUMBERS
APPENDIX A

DATA FOR THE CRITICAL STATES
AND CRITICAL WAVE NUMBERS

Table 7: Critical states and critical wave numbers in a cavity with conducting boundaries.

<table>
<thead>
<tr>
<th>Pr</th>
<th>α</th>
<th>β</th>
<th>Gr_C</th>
<th>Pr</th>
<th>α</th>
<th>β</th>
<th>Gr_C</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.001</td>
<td>2.7</td>
<td>0</td>
<td>7943.2</td>
<td>*0.01</td>
<td>0</td>
<td>2.0</td>
<td>167644.0</td>
</tr>
<tr>
<td>0.005</td>
<td>2.7</td>
<td>0</td>
<td>7998.4</td>
<td>*0.1</td>
<td>0</td>
<td>2.0</td>
<td>20442.6</td>
</tr>
<tr>
<td>0.01</td>
<td>2.7</td>
<td>0</td>
<td>8077.5</td>
<td>*0.2</td>
<td>0</td>
<td>2.0</td>
<td>13397.0</td>
</tr>
<tr>
<td>0.05</td>
<td>2.65</td>
<td>0</td>
<td>9219.8</td>
<td>*0.3</td>
<td>0</td>
<td>1.9</td>
<td>12140.0</td>
</tr>
<tr>
<td>0.07</td>
<td>2.62</td>
<td>0</td>
<td>10280.2</td>
<td>*0.4</td>
<td>0</td>
<td>1.7</td>
<td>15042.0</td>
</tr>
<tr>
<td>0.10</td>
<td>2.58</td>
<td>0</td>
<td>12690.9</td>
<td>*0.43</td>
<td>0</td>
<td>1.3</td>
<td>19037.0</td>
</tr>
<tr>
<td>0.15</td>
<td>2.5</td>
<td>0</td>
<td>17784.0</td>
<td>*0.43</td>
<td>0</td>
<td>1.3</td>
<td>40941.0</td>
</tr>
<tr>
<td>0.20</td>
<td>2.3</td>
<td>0</td>
<td>25918.1</td>
<td>*0.4</td>
<td>0</td>
<td>1.7</td>
<td>46546.0</td>
</tr>
<tr>
<td>0.25</td>
<td>2.11</td>
<td>0</td>
<td>67753.5</td>
<td>0.01</td>
<td>0</td>
<td>7.6</td>
<td>362494.0</td>
</tr>
<tr>
<td>0.3</td>
<td>2.35</td>
<td>0</td>
<td>437340.6</td>
<td>0.1</td>
<td>0</td>
<td>7.6</td>
<td>90788.0</td>
</tr>
<tr>
<td>*0.4</td>
<td>6.1</td>
<td>0</td>
<td>449851.1</td>
<td>0.2</td>
<td>0</td>
<td>7.9</td>
<td>53846.0</td>
</tr>
<tr>
<td>*0.5</td>
<td>6.5</td>
<td>0</td>
<td>182099.1</td>
<td>0.3</td>
<td>0</td>
<td>8.0</td>
<td>38544.0</td>
</tr>
<tr>
<td>*0.6</td>
<td>6.9</td>
<td>0</td>
<td>79698.8</td>
<td>0.5</td>
<td>0</td>
<td>8.2</td>
<td>24639.0</td>
</tr>
<tr>
<td>*0.7</td>
<td>7.8</td>
<td>0</td>
<td>36500.4</td>
<td>0.6</td>
<td>0</td>
<td>8.2</td>
<td>20879.0</td>
</tr>
<tr>
<td>*0.8</td>
<td>8.2</td>
<td>0</td>
<td>28095.3</td>
<td>1.0</td>
<td>0</td>
<td>8.3</td>
<td>12695.9</td>
</tr>
<tr>
<td>*0.9</td>
<td>8.2</td>
<td>0</td>
<td>23298.3</td>
<td>10.0</td>
<td>0</td>
<td>8.3</td>
<td>1358.6</td>
</tr>
<tr>
<td>*1.0</td>
<td>8.3</td>
<td>0</td>
<td>20024.8</td>
<td>100.0</td>
<td>0</td>
<td>8.2</td>
<td>135.6</td>
</tr>
<tr>
<td>*2.0</td>
<td>8.2</td>
<td>0</td>
<td>8537.4</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>*3.0</td>
<td>8.0</td>
<td>0</td>
<td>5445.7</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>*4.0</td>
<td>7.9</td>
<td>0</td>
<td>3998.8</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>*5.0</td>
<td>7.8</td>
<td>0</td>
<td>3159.0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>*6.0</td>
<td>7.8</td>
<td>0</td>
<td>2611.5</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>*10.0</td>
<td>7.8</td>
<td>0</td>
<td>1545.3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>*50.0</td>
<td>7.8</td>
<td>0</td>
<td>306.5</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

* means travelling or oscillatory mode.
Table 8: Critical states and critical wave numbers in a cavity with insulated boundaries.

<table>
<thead>
<tr>
<th>Pr</th>
<th>α</th>
<th>β</th>
<th>( \text{Gr}_C )</th>
<th>Pr</th>
<th>α</th>
<th>β</th>
<th>( \text{Gr}_C )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.001</td>
<td>2.7</td>
<td>0</td>
<td>7944.7</td>
<td>*0.001</td>
<td>0</td>
<td>0.15</td>
<td>50575.0</td>
</tr>
<tr>
<td>0.005</td>
<td>2.7</td>
<td>0</td>
<td>8021.8</td>
<td>*0.01</td>
<td>0</td>
<td>0.45</td>
<td>16458.0</td>
</tr>
<tr>
<td>0.01</td>
<td>2.7</td>
<td>0</td>
<td>8166.2</td>
<td>*0.02</td>
<td>0</td>
<td>0.60</td>
<td>12032.2</td>
</tr>
<tr>
<td>0.02</td>
<td>2.7</td>
<td>0</td>
<td>8642.1</td>
<td>*0.07</td>
<td>0</td>
<td>0.95</td>
<td>7706.0</td>
</tr>
<tr>
<td>0.03</td>
<td>2.7</td>
<td>0</td>
<td>9437.1</td>
<td>*0.1</td>
<td>0</td>
<td>1.0</td>
<td>7345.0</td>
</tr>
<tr>
<td>0.04</td>
<td>2.7</td>
<td>0</td>
<td>10660.7</td>
<td>*0.15</td>
<td>0</td>
<td>0.85</td>
<td>8309.0</td>
</tr>
<tr>
<td>0.05</td>
<td>2.65</td>
<td>0</td>
<td>12354.0</td>
<td>*0.2</td>
<td>0</td>
<td>0.4</td>
<td>19633.0</td>
</tr>
<tr>
<td>0.06</td>
<td>2.6</td>
<td>0</td>
<td>14385.7</td>
<td>*0.2</td>
<td>0</td>
<td>0.4</td>
<td>26400.0</td>
</tr>
<tr>
<td>0.08</td>
<td>2.45</td>
<td>0</td>
<td>19539.9</td>
<td>*0.15</td>
<td>0</td>
<td>0.85</td>
<td>27530.0</td>
</tr>
<tr>
<td>0.1</td>
<td>2.35</td>
<td>0</td>
<td>34259.9</td>
<td>*0.1</td>
<td>0</td>
<td>1.0</td>
<td>58076.0</td>
</tr>
<tr>
<td>0.11</td>
<td>2.55</td>
<td>0</td>
<td>58339.0</td>
<td>*0.07</td>
<td>0</td>
<td>0.95</td>
<td>139673.0</td>
</tr>
<tr>
<td>0.12</td>
<td>2.75</td>
<td>0</td>
<td>101816.4</td>
<td>*0.01</td>
<td>0</td>
<td>5.8</td>
<td>880000.0</td>
</tr>
<tr>
<td>0.13</td>
<td>3.0</td>
<td>0</td>
<td>236680.9</td>
<td>*0.01</td>
<td>0</td>
<td>5.7</td>
<td>289693.0</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.05</td>
<td>0</td>
<td>5.6</td>
<td>154974.0</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.1</td>
<td>0</td>
<td>5.4</td>
<td>134490.0</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.2</td>
<td>0</td>
<td>5.1</td>
<td>134809.0</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.3</td>
<td>0</td>
<td>4.7</td>
<td>152346.0</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.4</td>
<td>0</td>
<td>4.4</td>
<td>181085.0</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.6</td>
<td>0</td>
<td>4.1</td>
<td>266955.0</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.7</td>
<td>0</td>
<td>3.6</td>
<td>339340.4</td>
</tr>
</tbody>
</table>

* means oscillatory mode.
LIST OF REFERENCES


