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Bor, Sheau-Shong

PHASE CONJUGATION CHARACTERISTICS OF GAUSSIAN BEAM

The Ohio State University

Ph.D. 1985

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PHASE CONJUGATION CHARACTERISTICS
OF GAUSSIAN BEAM

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of the Ohio State University

By

Sheau-Shong Bor, B.S., M.S.

The Ohio State University

1985

Dissertation Committee:

Dr. H. Hsu
Dr. S. H. Koozekanani
Dr. P. Roblin

Approved by

Adviser

Department of

Electrical Engineering
ACKNOWLEDGEMENTS

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VITA

September 20, 1948....... Born, Tainan, Taiwan, R.O.C.

1970............................. B.S., Electrical Engineering, Chung-Cheng Institute of Technology, Tao-Yuan, Taiwan, R.O.C.

1970-1974...................... Teaching and Research Assistant, Chung-Cheng Institute of Technology, Tao-Yuan, Taiwan, R.O.C.

1974-1976...................... M.S., Applied Physics, National Tsing-Hwa University, Hsin-Chu, Taiwan, R.O.C

1976-1981...................... Instructor, Chung-Cheng Institute of Technology, Tao-Yuan, Taiwan, R.O.C.

1981-Present................... Graduate student, Electrical Engineering, The Ohio state University, Columbus, Ohio
PUBLICATIONS


FIELD OF STUDY

Major field: Electrical Engineering

Studies in Quantum Electronics: Professor H. Hsu

Studies In Electromagnetics: Professor R. G. Kouyoumjian

Studies In Communications: Professor D. T. Davis

Studies In Solid State Physics: Professor J. G. Gottling

Studies In Mathematics: Professor J. T. Scheick
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<td>$A_i(Z)$</td>
<td>Real amplitude of the interacting wave.</td>
</tr>
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<td>$w_i$</td>
<td>Radian frequency of the respective interacting wave.</td>
</tr>
<tr>
<td>$C_i$</td>
<td>The coupling coefficient between the wave at $w_i$ and the remaining waves.</td>
</tr>
<tr>
<td>$z$</td>
<td>One dimensional spatial variable.</td>
</tr>
<tr>
<td>$L$</td>
<td>Total interaction length.</td>
</tr>
<tr>
<td>$k^2$</td>
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<td>$K$</td>
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<td>$G_C$</td>
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<td>$G_A$</td>
<td>Signal gain.</td>
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<td>$E$</td>
<td>Photon conversion efficiency.</td>
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<td>$u$</td>
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\( v \) \hspace{1cm} \text{Signal excitation.}

\( R_{PCM} \) \hspace{1cm} \text{Reflection coefficient of PCM.}

\( r_0 \) \hspace{1cm} \text{Reflection coefficient of the end mirror in SBS.}

\( R_B \) \hspace{1cm} \text{Signal to pump excitation ratio.}
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Chapter 1
Introduction

Nonlinear Optical Phase Conjugation is receiving much attention at present. The reason for the interest in optical phase conjugation is the recognition of its extremely attractive potential applications to many important scientific and technological problems\(^1\), such as real-time holography, image processing, adaptive optics, optical computing, and so on.

The current investigations on optical phase conjugation are primarily concentrated on two approaches: one is the three-wave stimulated Brillouin scattering\(^2, 3\) and the other is the degenerate four-wave mixing process\(^4, 5, 6\). All of these scattering processes are typical cases of backward-traveling-wave parametric interactions\(^7, 8\).

Although a comprehensive theory for traveling wave parametric processes exists, only a limited amount of work has been done in taking into account the effects of pump depletion on the interactions. Most
analyses to date assume a constant pump excitation which is independent of the interaction length of the parametric process. This small signal analysis is certainly valid in low photon conversion efficiency regime where the pump depletion may be ignored, but it is of limited validity otherwise. However, H. Hsu\textsuperscript{9} presented a large signal analysis of phase conjugate process by taking the pump depletion into account. Moreover, the solutions can be expressed in terms of Jacobin elliptic functions for the periodic transfer of energy between the waves.

The purpose of the present work is to extend the analysis to a general and more exact solution on both three-wave and four-wave interactions. Furthermore, the effectiveness of all the interactions will be gaged in terms of reflection coefficient, photon conversion efficiency, signal excitation and required pump excitation. These characteristics will be presented graphically for physical interpretation. The dynamics of the interaction processes can then be readily described.

The large signal analysis in three-wave mixing and in degenerate four-wave mixing are briefly reviewed in chapters 2 and 3 respectively. Then the phase conjugation response of a Gaussian beam in stimulated Brillouin scattering will be discussed in chapter 4 and that in degenerate four-wave mixing is presented in chapter 5. Following that, the characteristics of the higher order modes will be analyzed in chapter 6. The experimental result is shown in chapter 7.
Some of the significant findings of our work are first of all a unified approach to the analysis of optical phase conjugation. For example, the self-starting property of the phase conjugation mirror with end mirror and the relationship between the reflection coefficient of the phase conjugation mirror and the incident pump excitation are described. Another very important finding is the scattering response of a Gaussian incident beam and their spot size ratio. Finally, the significance of the present work and possible improvements will be discussed in chapter 8.
Chapter 2
A brief review of the Three-Wave Mixing theory of phase conjugation in stimulated Brillouin scattering

When a strong monochromatic wave is incident on a nonlinear medium, an interaction occurs in which an optical wave and an acoustic wave are generated. The optical wave (of frequency slightly below the incident frequency) propagates in the direction opposite to that of the incident beam and therefore can be called a phase conjugate wave. The acoustic wave (of frequency on the order of $10^{10}$ Hz) propagates in the direction of the incident beam. The interaction which produces these two waves can be called a backward-traveling-wave parametric interaction or stimulated Brillouin scattering (SBS).

The general large signal theory of the three-wave mixing was given by Hsu and Yu and some of the results are summarized below as a brief review.
In ideal phase conjugation, the frequency and phase-matching conditions for collinear three-wave interaction are:

\[ w_{p^+} = w_{s^+} + w_{s^-} \]
\[ k_{p^+} = k_{s^+} - k_{s^-} \] (2.1)

where \( w_{s^+}, w_{s^-} \) and \( w_{p^+} \) are the idler, signal and pump wave frequency respectively, and \( k_{s^+}, k_{s^-} \) and \( k_{p^+} \) are their respective wave vectors.

In stimulated Brillouin scattering, we have a forward-traveling pump wave \( A_{p^+} \), a backscattered wave (or phase conjugate wave) \( A_{s^-} \) and a forward-traveling acoustic wave \( A_{s^+} \). Therefore, the coupled differential equations for the wave amplitudes are

\[ \frac{dA_{s^+}(z)}{dz} = C_{s^+} A_{s^-}(z) A_{p^+}(z) \]
\[ \frac{dA_{s^-}(z)}{dz} = -C_{s^-} A_{s^+}(z) A_{p^+}(z) \]
\[ \frac{dA_{p^+}(z)}{dz} = -C_{p^+} A_{s^+}(z) A_{s^-}(z) \] (2.2)

where \( C_i(i = p^+, s^+, s^-) \) are the nonlinear coupling coefficients, and the boundary condition is \( A_{s^+}(0) = 0 \).
It follows from equation (2.2) that

\[
\frac{d}{dz} \left[ \frac{A_{s+}^2(z)}{C_{s+}} + \frac{A_{s-}^2(z)}{C_{s-}} \right] = 0
\]

\[
\frac{d}{dz} \left[ \frac{A_{p+}^2(z)}{C_{s+}} + \frac{A_{p+}^2(z)}{C_{p+}} \right] = 0 \tag{2.3}
\]

After integration, the following conservation relations can be obtained.

\[
\frac{A_{s+}^2(z)}{C_{s+}} + \frac{A_{s-}^2(z)}{C_{s-}} = \frac{A_{s-}^2(0)}{C_{s-}}
\]

\[
= \frac{A_{s+}^2(L)}{C_{s+}} + \frac{A_{s-}^2(L)}{C_{s-}}
\]

\[
\frac{A_{s+}^2(z)}{C_{s+}} + \frac{A_{p+}^2(z)}{C_{p+}} = \frac{A_{p+}^2(0)}{C_{p+}}
\]

\[
= \frac{A_{s+}^2(L)}{C_{s+}} + \frac{A_{p+}^2(L)}{C_{p+}} \tag{2.4}
\]

where \( L \) is the total interaction length.
Substituting equation (2.4) into equation (2.2), we have

\[
\frac{dA_{s-}(z)}{dz} = -\{ C_{s+} C_{s-} [ A_{s+}^2(0) - A_{s-}^2(0) ]
\]

\[
\times \left[ \frac{C_{s-}}{C_{p+}} A_{p+}^2(0) - A_{s-}^2(0) + A_{s-}^2(z) \right]^{1/2}
\]

(2.5)

By transforming the variable \( A_{s-}(z) \), we can put equation (2.5) in the standard elliptic integral form. Thus, the solution can be expressed in terms of elliptic functions.

Case (1):

When \( A_{p+}^2(0)/C_{p+} \geq A_{s-}^2(0)/C_{s-} \), the elliptic function parameter \( k^2 \) is defined as

\[
k^2 = \frac{A_{s-}^2(0)/C_{s-}}{A_{p+}^2(0)/C_{p+}}
\]

(2.6)

and the wave amplitudes are:

\[
A_{s+}(z) = \sqrt{C_{s+}/C_{s-}} A_{s-}(0) Sn(\Gamma z)
\]

\[
A_{s-}(z) = A_{s-}(0) Cn(\Gamma z)
\]
\[ A_{p+}(z) = A_{p+}(0) Dn(\Gamma z) \]  

(2.7)

where the pump excitation is given by

\[ u = \sqrt{C_{s+} C_{s-}} \ A_{p+}(0) L = \Gamma L \]  

(2.8)

Case (2):

When \( A_{p+}^2(0)/C_{p+} \leq A_{s-}^2(0)/C_{s-} \), the elliptic function parameter \( k^2 \) is defined as

\[ k^2 = \frac{A_{p+}^2(0)/C_{p+}}{A_{s-}^2(0)/C_{s-}} \]  

(2.9)

and the wave amplitudes are:

\[ A_{s+}(z) = \sqrt{C_{s+}/C_{s-}} \ k A_{s-}(0) Sn(\Gamma z) \]

\[ A_{s-}(z) = A_{s-}(0) Dn(\Gamma z) \]

\[ A_{p+}(z) = A_{p+}(0) Cn(\Gamma z) \]  

(2.10)

where the pump excitation is given by
\[ u = \sqrt{C_{s+}C_{s-}} A_{p+}(0)L = k\Gamma L \]  \hspace{1cm} (2.11)

with \( k^{-2} \geq 1 \).
Chapter 3

A brief review of Large Signal Degenerate Four-Wave Mixing Theory

3.1 Large Signal Degenerate Four-Wave Mixing Theory

The Degenerate Four-Wave Mixing (DFWM) process is one in which two counterpropagating pump beams interact via a nonlinear medium with a signal beam, all at the same frequency (degenerate interaction). As a result of the nonlinear optical interaction, a conjugate wave is generated. In this process the phase matching conditions are automatically satisfied$^4$.

In our analysis, we assume that we have plane collinear traveling waves in a lossless medium. The frequency and phase matching conditions are:
The subscripts "+" and "-" represent respectively the waves traveling in the forward and backward directions. Following the procedure of earlier large signal analysis\textsuperscript{9, 10}, the coupled amplitude equations are:

\begin{align*}
\frac{dA_{s+}}{dz} &= C_{s+}A_{p+}A_{p-}A_{s-} \\
\frac{dA_{s-}}{dz} &= -C_{s-}A_{p+}A_{p-}A_{s+} \\
\frac{dA_{p+}}{dz} &= -C_{p+}A_{p-}A_{s+}A_{s-} \\
\frac{dA_{p-}}{dz} &= C_{p-}A_{p+}A_{s+}A_{s-}
\end{align*}

(3.3)

where \( C_i \) and \( A_i \) (\( i = p+, p-, s+, s- \)) are the nonlinear coupling coefficients and the wave amplitudes respectively; and \( L \) is the interaction length.

We consider a case where \( A_{s+} \) is an idler wave and \( A_{s-} \) a signal wave. From equation (3.3), we can obtain the Manley-Rowe relationship as follows:

\[
\frac{|dA_{s+}^2/dz|/C_{s+}}{C_{s+}} = -\frac{|dA_{s-}^2/dz|/C_{s-}}{C_{s-}} = -\frac{|dA_{p+}^2/dz|/C_{p+}}{C_{p+}} = \frac{|dA_{p-}^2/dz|/C_{p-}}{C_{p-}}
\]

(3.4)
If we set

\[ \sin^2 \theta = \frac{A_{s+}^2(z) / C_{s+}}{[A_{p+}^2(0) / C_{p+}]} \]  
(3.5)

\[ p^2 = \frac{A_{p+}^2(0) / C_{p+}}{[A_{p-}^2(0) / C_{p-}]} \]  
(3.6)

\[ q^2 = \frac{A_{p+}^2(0) / C_{p+}}{[A_{p-}^2(0) / C_{p-}]} \]  
(3.7)

the first equation of equations (3.3) becomes

\[ \frac{dA_{s+}(z)}{dz} = \sqrt{C_{s+}} \sqrt{C_{s-} C_{p+} C_{p-}} \left[ A_{p+}(0) / \sqrt{C_{p+}} \right]^3 \]  
(3.8)

\[ \times \sqrt{1 - \sin^2 \theta} \sqrt{1 / p^2 + \sin^2 \theta} \sqrt{1 / q^2 - \sin^2 \theta} \]

After some transformations and lengthy elementary manipulations, we obtain

\[ \sin \alpha = Sn\left[ \frac{A_{p+}^2(0)}{C_{p+}} \sqrt{C_{s+} C_{s-} C_{p+} C_{p-}} \frac{\sqrt{1 + p^2}}{pq} \right] \]  
(3.9)

if \( k < 1 \)

and
\[ k \sin \alpha = Sn \left( k \frac{A_{p+}^2(0)}{C_{p+}} \sqrt{C_{s+}C_{p+}C_{p-}} \frac{\sqrt{1+p^2}}{pq} z, \frac{1}{k} \right) \]

if \( k > 1 \) \hspace{1cm} (3.10)

where

\[ \sin \alpha = \sqrt{\frac{(1+p^2)\sin^2 \theta}{1+p^2 \sin^2 \theta}} \] \hspace{1cm} (3.11)

and \( k^2 \) is the Jacobian elliptic functions parameter defined as:

\[ k^2 = \frac{|p^2+q^2|}{1+p^2} \] \hspace{1cm} (3.12)

At the same time, the solutions of eqs.(3.3) can be expressed in the following cases:

(1) \( k < 1 \)

\[ \frac{A_{p+}(z)}{\sqrt{C_{p+}}} = \frac{A_{p+}(0)}{\sqrt{C_{p+}}} \frac{Sn(\Gamma z)}{\sqrt{1+p^2 Cn^2(\Gamma z)}} \]

\[ \frac{A_{p-}(z)}{\sqrt{C_{p-}}} = \frac{A_{p-}(0)}{\sqrt{C_{p-}}} \frac{Sn^2(\Gamma z)}{\sqrt{1-q^2} \frac{Sn^2(\Gamma z)}{1+p^2 Cn^2(\Gamma z)}} \]
\[
\frac{(z,j)z^u c^d + z^b}{(z,j)u^d} - \frac{\wedge c^+}{\wedge (0)^+} = \frac{\wedge c^+}{(z)^+V}
\]

\[
\frac{(z,j)z^u c^d + z^b}{(z,j)u^d} - \frac{\wedge c^+}{\wedge (0)^-} = \frac{\wedge c^+}{(z)^-V}
\]

\[
\frac{(z,j)z^u c^d + z^b}{(z,j)u^d} - \frac{\wedge c^+}{\wedge (0)^0} = \frac{\wedge c^+}{(z)^0V}
\]

\[
I < \gamma (2)
\]

\[
\frac{b^d}{(\varepsilon^{d+i})^+} \frac{c^+}{(0)^d} = \frac{c^+}{(z)^dV}
\]

where

\[
\frac{(z,j)z^u c^d + I}{(z,j)u^d} - \frac{\wedge c^+}{\wedge (0)^-} = \frac{\wedge c^+}{(z)^-V}
\]

\[
\frac{(z,j)z^u c^d + I}{(z,j)u^d} - \frac{\wedge c^+}{\wedge (0)^0} = \frac{\wedge c^+}{(z)^0V}
\]
\[
\frac{A_{p-}(z)}{\sqrt{C_{p-}}} = \frac{A_{p-}(0)}{\sqrt{C_{p-}}} \sqrt{\frac{p^2 + q^2}{q^2 + p^2 Cn^2(\Gamma z)}} \tag{3.15}
\]

where

\[
\Gamma = \frac{A_{p+}^2(0)}{C_{p+}} \sqrt{C_{s+} C_{s-} C_{p+} C_{p-}} \sqrt{\frac{q^2 + p^2}{pq}} \tag{3.16}
\]

Some of the important parameters will be defined here because all the interactions can be expressed by signal gain \( G_A \), conversion gain or reflection coefficient \( G_C \), photon conversion efficiency \( \varepsilon \), pump excitation \( u \) and signal excitation \( v \).

In this chapter, we will consider two different pumping cases. Section 3.2 is a scattering process with single pump excitation, and in section 3.3 we use two separate pump excitations. For simplicity, but without loss of generality, we assume \( C_{p+} = C_{p-} = C_p \) in this chapter.
3.2 Single pump excitation into end reflector

If a perfect reflector is used for pumping at \( z=L \) such that 
\[ A_{p+}(L) = A_{p-}(L) = A_p(L), \]
with the idler \( A_{s+}(0) = 0 \), the input signal is \( A_{s-}(L) \) (another choice is to repeat the analysis with \( A_{s-}(L) = 0 \) and the input signal \( A_{s+}(0) \)). The field amplitude distributions of all interacting waves can be calculated from eqs.(3.13) or eqs.(3.15) which depend on the elliptic function parameter \( k \).

Based upon these equations, the characteristics of all interaction can be expressed in terms of the following five system parameters.

Signal Gain

\[
G_A = \frac{A_{s-}^2(0)}{A_{s-}^2(L)} \quad (3.17)
\]

Conversion Gain or Reflection Coefficient

\[
G_C = \frac{A_{s+}^2(L)/C_{s+}}{A_{s-}^2(L)/C_{s-}} \quad (3.18)
\]

Photon Conversion Efficiency

\[
E = \frac{A_{p+}^2(0) - A_{p-}^2(0)}{A_{p+}^2(0)} \quad (3.19)
\]
Pump Excitation

\[ u = \frac{A_{p+}^2(0)}{C_{p+}} \sqrt{C_{s+} C_{s-} C_{p+} C_{p-}} \ L \]

\[ = A_{p+}^2(0) \sqrt{C_{s+} C_{s-}} \ L \]  \hspace{1cm} (3.20)

Signal Excitation

\[ v = \frac{A_{s-}^2(L)}{C_{s-}} \sqrt{C_{s+} C_{s-} C_{p+} C_{p-}} \ L \]

\[ = A_{s-}^2(L) C_{p} \sqrt{C_{s+}/C_{s-}} \ L \]  \hspace{1cm} (3.21)

(1) \( k < 1 \)

\[ \sqrt{E} = Sn[u \sqrt{\frac{(2-E)(E-1)}{k^2E - 2k^2 + 1}}], k] \]  \hspace{1cm} (3.22)

where

\[ k^2 = \frac{E + G_C(2-E)}{E(2-E)(1+G_C)} \]  \hspace{1cm} (3.23)

\[ u/v = 2G_C/E \]  \hspace{1cm} (3.24)

\[ G_C = G_A - 1 \]  \hspace{1cm} (3.25)

(2) \( k > 1 \)
\[ k\sqrt{E} = Sn[u \sqrt{\frac{E + G_C(2-E)}{2G_C}}, \frac{1}{k}] \]  \hspace{1cm} (3.26)

For the special case of oscillation, i.e., when the gain \( G_A \) becomes infinity and \( A_\pi(L) = 0 \), the general expressions can be written as:

\[ u = \frac{K}{\sqrt{1-E/2}} \]  \hspace{1cm} (3.27)

and

\[ k^2 = \frac{1}{E} = \frac{q^2}{2} \]  \hspace{1cm} (3.28)

where \( K \) is the real quarter period of the elliptic function.

For small-signal analysis, we make the assumption that the pump depletion may be ignored or the photon conversion efficiency equals to zero. From eq.(3.27), we find the threshold of oscillation to be \( u = \pi/2 \). This result is identical to earlier small-signal analysis\(^4, 5, 6\).

It is of interest to find the variation of the reflection coefficient \( G_C \) with input signal excitation \( v \) for various pump excitations \( u \). Fig.1 shows the results. When the pump excitation is below the threshold
(u ≤ π/2), the system is stable with finite reflection coefficient. But the system will oscillate as soon as the pump excitation u exceeds the threshold. Thus, \( G_C \) approaches infinity at \( v = 0 \).

If the input signal is not zero, the system will have finite reflection coefficient at any pump value, even above the oscillation threshold. Furthermore, at fixed pump excitation, the reflectivity decreases when the signal excitation increases due to pump depletion.

According to eq.(3.26) and eq.(3.27) complete pump conversion is impossible for any \( G_C \) under finite pump excitation. However, if the input signal is \( A_{s+}(0) \) and the idler is \( A_{s-}(L) = 0 \), complete pump conversion becomes possible for any \( G_C \) under finite pump excitation\(^9\).
Figure 1: Reflection Coefficient $G_C$ with input signal excitation $v$ for various pump excitation $u$
3.3 Two pump excitation

If the pump beam is equally split and serves as forward and backward pump beams, the boundary condition becomes $A_{p+}(0) = A_{p-}(L)$. We choose $A_{s+}(0) = 0$ and $A_{s-}(L)$ as the input signal (or we can choose $A_{s+}(0)$ as the signal and $A_{s-}(L)$ the idler, but the result is the same because of symmetry).

In this case, the field amplitude distributions of all the interacting waves are given by eqs.(3.13) or (3.15) because $p$, $q$, $\sin\theta$ and $k$ remain the same as before. But $E$ and $u$ must be redefined as:

$$E = \frac{A_{p+}^2(0) + A_{p-}^2(L) - A_{p+}^2(L) - A_{p+}^2(0)}{A_{p+}^2(0) + A_{p-}^2(L)}$$

$$= \frac{A_{p+}^2(0) - A_{p+}^2(L)}{A_{p+}^2(0)} = \frac{A_{p-}^2(L) - A_{p-}^2(0)}{A_{p-}^2(L)}$$

(3.29)

and

$$u = \frac{A_{p+}^2(0) + A_{p-}^2(L)}{C_{p+} + C_{p-}} \sqrt{C_{s+} C_{s-} C_{p+} C_{p-}} L$$

$$= 2A_{p+}^2(0) \sqrt{C_{s+} C_{s-}} L$$
\[ 2A_p^2(L) \sqrt{C_{p+}C_{p-}} L \] (3.30)

(1) \( k < 1 \)

\[ \sqrt{E(2-E)} = Sn[ \frac{u}{2} \sqrt{\frac{(2-E)(1+G_C)E}{G_C}}, \ k ] \] (3.31)

where

\[ k^2 = \frac{E+G_C}{E(2-E)(1+G_C)} \] (3.32)

and

\[ u/v = 2G_C/E \] (3.24)

(2) \( k > 1 \)

\[ \sqrt{\frac{E+G_C}{1+G_C}} = Sn[ \frac{u}{2} \sqrt{\frac{E+G_C}{G_C}}, \ \frac{1}{k} ] \] (3.33)

For the special case of oscillation, the general expressions can be written as:
\[ u = 2K \]  

(3.34)

and

\[ k^2 = \frac{1}{E(2-E)} \]  

(3.35)

From these equations, we find that complete pump conversion cannot be reached for any \( G_C \) under finite pump excitation.
Chapter 4
Phase conjugation characteristics of Gaussian beam in Three-Wave Mixing

4.1 Phase conjugation mirror with threshold

One of the important applications of phase conjugation is the phase conjugate mirror (PCM), an optical element that can be used to generate phase conjugate replicas of incident optical beams. Of the various nonlinear optical techniques that can be utilized as phase conjugators, stimulated Brillouin scattering (SBS) has received a great deal of attention since it is perhaps the simplest and most efficient interaction.

In this chapter, we will discuss the phase conjugation response of a Gaussian beam that uses a PCM formed by stimulated Brillouin scattering.
The stimulated Brillouin scattering PCM can be considered as an oscillator in a three-wave mixing with the boundary conditions \( A_{s+}(0) = 0 \) and \( A_{s-}(L) = 0 \). In this case, we have a forward-traveling pump wave \( A_{p+} \), a backscattered wave (or phase conjugate wave) \( A_{s-} \) and a forward-traveling acoustic wave \( A_{s+} \). Therefore, the only solution of the three-wave stimulated Brillouin scattering is equation (2.7) and the following condition must also be satisfied:

\[
\nu = \sqrt{C_{s+}C_{s-}} A_{p+}(0)L = K
\]

(4.1)

where \( A_{p+}(0) \) is the incident beam intensity and \( K \) is the real quarter period of the elliptic function. The threshold of oscillation in stimulated Brillouin scattering is obviously at \( \nu = K = \pi/2 \), as is well-known\(^{10} \).

For simplicity, but without loss of generality, the incident, reflected and transmitted beam amplitude can be defined as:

\[ A_{\text{inc}} = \text{incident beam amplitude} \]
\[ = A_{p+}(0) \]

\[ A_{\text{ref}} = \text{reflected beam amplitude} \]
\[ = A_{s-}(0) = kA_{p+}(0) \]

\[ A_{tr} = \text{transmitted beam amplitude} \]
Obviously, the reflection coefficient $R_{PCM}$ of the phase conjugation mirror can be defined as (since $C_e \approx C_p$)

$$R_{PCM} = k^2$$

(4.3)

where $k^2$ is the parameter (modulus) of the elliptic function.

Assume a Gaussian beam is incident on a stimulated Brillouin scattering medium. Then, the pump excitation $u$ due to the incident beam will have a profile in $r$ as:

$$u(r) = \sqrt{C_{s+} + C_{s-}} A_{p+}(0) L$$

$$= K(r) = K(0) \exp\left[-(r/w_i)^2\right]$$

(4.4)

where $r = \sqrt{x^2 + y^2}$ with the incident beam propagating along the $z$ axis. $w_i$ is the effective radius (or waist) of the Gaussian beam. $K(0)$ is the peak value corresponding to the maximum intensity of the incident beam. $K(0)$ has to exceed the threshold intensity for stimulated Brillouin scattering to occur.
We assume that the effective radius (or waist) of the incident Gaussian beam and the radius of curvature of the wave front are much larger than the wavelength, such that the plane wave analysis of chapters 2 and 3 can still be applied to incident Gaussian beam.

We normalize the beam amplitudes as:

\[
A'_{\text{inc}} = (\sqrt{C_{s+} + C_{s-} - L}) A_{\text{inc}} = u(r) = K(r)
\]

\[
A'_{\text{ref}} = (\sqrt{C_{s+} + C_{s-} - L}) A_{\text{ref}} = k(r)K(r)
\]

and

\[
A'_{\text{tr}} = (\sqrt{C_{s+} + C_{s-} - L}) A_{\text{tr}} = \sqrt{1-k^2(r)} K(r)
\] (4.5)

By applying the relationship between \( k \) and \( K \) derived in Appendix A, the reflected (or scattered) beam intensity and the transmitted beam intensity can be calculated very precisely from the incident beam intensity. The result is shown in Fig.2.

Fig.2(a) shows the incident Gaussian beam. The parameter \( w_1 \) plays the role of the effective radius of the beam or what we called the spot
size of the beam. Fig. 2(b) shows the reflected beam. Obviously, the reflected beam is not a Gaussian beam anymore, but the 1/e point can be defined as the spot size $w_s$ of the reflected beam. Fig. 2(c) shows the transmitted beam. As shown in these figures, the reflection can be very large and the transmitted beam will exhibit a ring. Moreover, the size of the ring will depend on the maximum value of the incident beam intensity. This ring structure has been observed in our laboratory and shown in chapter 7. If the intensity of the incident beam is below the threshold, we will find no reflected beam and all the incident beam will be transmitted.

Although the reflected beam is no longer a Gaussian beam, the definition of the 1/e spot size still can be used and it is still an important parameter.

Because eq.(4.4) can be rewritten as

$$\frac{r}{w_i} = \sqrt{\ln \frac{K(0)}{K(r)}} \tag{4.6}$$

Thus, the reflected beam intensity and the corresponding spot size $r = w_s$ can be obtained from the equations in Appendix A and eq.(4.6). Fig. 3 shows the relationship between the spot size ratio and the incident
beam excitation. Fig.3 is calculated by iteration with an accuracy of $10^{-10}$. We note that the ratio $w_8/w_i$ is less than unity, i.e., the reflected beam is always narrower than the incident pump beam and this result is consistent with energy conservation. If the incident pump intensity is below the threshold, all the incident beam will be transmitted and no reflection will be produced and the spot size ratio will be zero. Once the incident intensity exceeds the threshold value, the corresponding spot size ratio can be obtained.

The fractional spot size reduction with increasing pump excitation is shown in Fig.4. In fact, Fig.4 gives a different viewpoint of the characteristic in Fig.3. Obviously, the dotted line in Fig.4 can be considered as a good approximation of large incident beam excitation, but there is a sizable difference between the dotted and solid lines. However, a very useful asymptotic equation has been found from this figure and the procedure is described as follows.

First of all, a simple expression of the dotted line can be written as:

$$
\frac{w_s}{w_i} = 1 - e^{-1.12(n-1)}
$$

(4.7)

where $n = u/(\pi 2)$
Secondly, correction terms must be introduced into eq.(4.7). A reasonable asymptotic equation between the spot size ratio and the incident beam excitation is found to be

\[ \frac{w_s}{w_i} = 1 - e^{-1.12(n-1)} + e^{-1.35n} \left[ 1 - e^{-16(n-1)} \right] \]  

(4.8)

In general, this asymptotic equation is very useful because the maximum error between this equation and the curve of Fig.3 is only 2.4 percent.
Figure 2: (a) Incident Gaussian beam. (b) Reflected beam. (c) Transmitted beam.
Figure 3: Spot size ratio $w_s/w_i$ vs incident beam excitation $u/(\pi/2)$
**Figure 4:** Fractional spot size reduction \( \frac{w_i - w_o}{w_i} \) versus incident beam excitation \( \frac{u}{(\pi/2)} \)
4.2 Phase conjugation mirror with no threshold

If an end mirror is used for reflecting at \( z = L \) such that the signal \( A_s(L) \) at \( z = L \) depends on the \( A_{p+}(L) \), with the idler \( A_{s+}(0) = 0 \), the input pump is \( A_{p+}(0) \).

In this case, the reflection coefficient of the end mirror is defined as

\[
 r_0 = \frac{A_{s-}^2(L)/C_{s-}}{A_{p+}^2(L)/C_{p+}} \tag{4.9}
\]

However, the reflection coefficient of the PCM is still defined as equation(4.3), i.e., \( R_{PCM} = k^2 \).

Substituting Eq.(2.7) into Eq.(4.9), we have

\[
 A_{s-}^2(0) = r_0 A_{p+}^2(0) \left| \frac{Dn(\Gamma L)}{Cn(\Gamma L)} \right|^2 \frac{C_{s-}}{C_{p+}} \tag{4.10}
\]

Thus, the relationship between the \( R_{PCM} \) (or \( k^2 \)) and \( r_0 \) can be obtained.

\[
 k^2 = r_0 \frac{1-k^2 Sn^2(\Gamma L)}{1-Sn^2(\Gamma L)} \tag{4.11}
\]
where

\[ \Gamma L = u = \sqrt{C_{s+} C_{s-}} A_{p+}(0)L \]  

(2.8)

Actually, equation (4.11) can be rewritten as follows.

\[ S_n(\Gamma L, k^2) = \sqrt{\frac{k^2 - r_0}{k^2(1-r_0)}} \]  

(4.12)

or

\[ u = \Gamma L = \int_0^{s-1} \frac{d\Phi}{\sqrt{k^2(1-r_0) - \sqrt{1-k^2\sin^2\Phi}}} \]  

(4.13)

Figure 5 shows the results of our calculation for the reflection coefficient of the PCM \((k^2)\) versus the pump excitation \((u)\), when the reflection coefficient of the end mirror \((r_0)\) is kept at various values. The curve \(r_0 = 0\) is identical to the result we obtained in the last section, indicating an oscillation threshold with infinite gain at \(u = \pi/2\). In fact, this curve shows the relationship between the modulus \((k^2)\) and the
quarter period \((K)\) of the elliptic function because the incident pump excitation is equal to \(K\) in the case of \(r_0 = 0\).

On the other hand, if \(r_0 \neq 0\), a very important characteristic can be observed from this figure. In the absence of the pump, all curves (except \(r_0 = 0\)) begin at the value \(r_0\) along the ordinate, meaning that the reflectivity of the PCM is equal to the reflectivity of the end mirror. This is obvious since the system is assumed to be lossless. Furthermore, we note that there is no longer any threshold for the incident excitation. However, with increasing pump excitation, all the curves gradually increase and converge towards unity for large incident pump excitations. We may thus conclude that the reflectivity of the PCM with end mirror can be calculated for any incident pump excitation.

To proceed to calculate the effect on the use of the end mirror to a Gaussian beam, we follow the same procedure as before. However, the expressions are different because the boundary condition is changed.

\[
\begin{align*}
A'_{\text{inc}} &= u = \sqrt{C_{s+}C_{s-}} A_{p+}(0) L = \Gamma L \\
A'_{\text{ref}} &= \sqrt{C_{s+}C_{s-}} A_{s-}(0) L = k\Gamma L \\
A'_{\text{tr}} &= \sqrt{C_{s+}C_{s-}} A_{p+}(L) L = \Gamma LDn(\Gamma L) 
\end{align*}
\] (4.14)
where

$$Dn(\Gamma L) = \sqrt{\frac{1-k^2}{1-r_0}}$$

If \( r_0 = 0 \) (or \( A_{n,0}(L) = 0 \)), Eq.(4.14) becomes Eq.(4.2).

Assume that a Gaussian beam is incident on a SBS medium with a profile of \( r \) as

$$A_{\text{inc}}(r) = u = \Gamma L(r) = \Gamma L(0) \exp[1-(\frac{r}{w_1})^2]$$

(4.15)

where \( \Gamma L(0) \) is the peak value corresponding to the maximum intensity of the incident beam. If \( r_0 = 0 \), it has to exceed the threshold intensity for SBS to occur.

Usually, the reflectivity of the end mirror is known and, following the same procedure described in the last section, one can obtain the reflected and transmitted beam excitations. With \( r_0 = 0.2 \), Fig.6(a) shows the incident Gaussian beam while Fig.6(b) displays a reflected beam. We note that the reflected beam is still very close to a Gaussian beam but the beam waist is changed. Fig.6(c) shows the transmitted beam and it still exhibits a ring.
Although the threshold incident excitation is no longer needed, the structure of the ring still depends on the maximum value of the incident pump excitation and the reflectivity of the end mirror. The response of different incident pump excitations with the same reflectivity are shown in Fig.7, while those of the same incident excitation but with different reflectivities are shown in Fig.8. From Fig.7(c), we find that the ring structure with large incident excitation will be much clearer than that with low incident excitation. On the other hand, Fig.8(c) shows that the sharpness of the ring structure is increased when the reflectivity of the end mirror is decreased. However, the reflected beam does not exhibit any major difference in shape in both cases.

Eq.(4.15) can be rewritten as:

$$\frac{r}{w_i} = \sqrt{\ln \frac{\Gamma L(0)}{\Gamma L(r)}}$$

(4.16)

The reflected beam and the corresponding spot size $r = w_s$ can be obtained from the equations in Appendix A and equation(4.16). Fig.9 shows the relationships between the spot size ratio and the incident beam excitation, when the reflection coefficient of the end mirror ($r_0$) is kept at various values. The curve $r_0 = 0$ is identical to the Fig.7, indicating the incident threshold at $u = \pi/2$. However, if $r_0 \neq 0$, a very
different characteristic can be observed from this figure. In the absence of the pump, the spot size ratios are independent of $r_0$ (except $r_0 = 0$) and are equal to 1. When the incident pump is gradually increased, the spot size ratios decrease until the pump excitation reaches $\pi/2$. This is the threshold for the case of PCM without end mirror (i.e., $r_0 = 0$). At this point, the system has minimum values of the spot size ratio. In fact, the dip of these curves depend on the value of $r_0$. Once the incident pump excitation exceeds $\pi/2$, the spot size ratios will increase and will eventually approach unity. All these curves (include the one at $r_0 = 0$) closely converge towards unity spot size ratio for a large pump excitations. But the shapes of these curves at low pump excitation exhibit very distinct differences because the threshold for $r_0 = 0$ is removed by adding the end mirror. The use of the end mirror can be very important for application of PCM in a laser cavity.
Figure 5: PCM reflection coefficient $R_{PCM}$ versus incident beam excitation $u$, at fixed end mirror reflection coefficient $r_0$. 
Figure 6: (a) Incident Gaussian beam. (b) Reflected beam. (c) Transmitted beam.
Figure 7: The response of different incident pump excitations with the same reflectivity ($r_0 = 0.2$)
Figure 8: The response of same incident pump excitation ($\Gamma L = 1.8797$) with different reflectivities.
Figure 9: Spot size ratio $w_s/w_i$ versus incident beam excitation $u/(\pi/2)$ at fixed reflectivity $r_0$
4.3 Summary and conclusion of chapter 4

In this chapter, the phase conjugation response of a Gaussian beam in stimulated Brillouin scattering is discussed.

In section 4.1, a PCM without end mirror is considered and some important results have been found. First, we note that the reflection coefficient of PCM ($R_{PCM}$) can be expressed in terms of the modulus of the elliptic function ($k$), and the incident pump excitation ($u$) can be expressed by the real quarter period of the elliptic function ($K$). From the known relationship between $k$ and $K$ (derived in Appendix A), the reflection coefficient of the PCM can be found from the incident beam excitation very easily. Second, when a Gaussian beam with a beam waist $w_i$ is incident on a SBS medium, the reflected beam is not Gaussian anymore and the transmitted beam will exhibit a ring. This is due to the fact that the intensity of incident pump beam exceeded the threshold for generating a phase conjugated (or reflected) output. Finally, the spot size ratio is calculated by iteration with an accuracy of $10^{-10}$ and the procedure can be described as follows. From equation (4.5) and equation (A.6), every point of the reflected (and transmitted) beam can be calculated very precisely and the value of the $1/e$ point of the reflected beam can be found. But, for the $1/e$ point, the corresponding $k$ and $K$ are not so easy to separate. However, we are able to find the values corresponding
to the 1/e points of the reflected beam accurately by iteration. Once the value of K corresponding to the 1/e point of the reflected beam is known, the spot size ratio can be obtained from equation (4.6). In addition, a very useful asymptotic equation of the spot size ratio has been found and is presented in equation (4.8).

A PCM with end mirror is considered in section 4.2 and several important characteristics have been observed. The most important one which account for the major difference between the last section and this section is the self-starting property. From Figures 5 to 9, we note that there is no threshold for the incident beam excitation. Since the stimulated Brillouin scattering exhibit a threshold, the end mirror plays an important role in the wave interaction for small incident beam excitations. Once the incident beam excitation exceeds the threshold of SBS, the SBS PCM takes over the whole interaction and the end mirror is no longer needed. Thus the system behaves as self-starting with the end mirror and will automatically convert to a SBS PCM.

On the other hand, the effect of using the end mirror to a Gaussian beam is also very interesting. With a constant reflectivity of the end mirror, we found that the reflected beam is still very close to Gaussian but the beam waist is changed while the transmitted beam still exhibits a ring and the spot size ratio can be obtained following the same procedure as before.
Chapter 5
Phase conjugation characteristics of Gaussian beam in Degenerate Four-Wave Mixing

5.1 Phase conjugation mirror without end mirror

In practice phase-conjugate reflection is most often accomplished by means of degenerate four-wave mixing in a nonlinear optical medium, when this medium is actively pumped by two coherently related counterpropagating pump beams. Therefore, the phase conjugation response of a Gaussian beam that uses a PCM formed by degenerate four-wave mixing is also very interesting and the response of the two cases discussed in chapter 3 will be analyzed in this chapter.

Assume a Gaussian signal beam is incident on a nonlinear medium. The signal excitation $v$ due to the incident signal beam will have a profile in $r$ given by:
\[ v(r) = A_x^2(L) \, C_p \, \sqrt{\frac{C_s^+}{C_s^-}} \, L \]

\[ = v(0) \, \exp[-(r/w_i)^2] \quad (5.1) \]

where \( r = \sqrt{x^2 + y^2} \) with the incident signal beam propagating along
the negative \( z \) axis. \( w_i \) is the effective radius (or waist) of the Gaussian
beam. \( v(0) \) is the peak value corresponding to the maximum intensity of
the incident signal beam.

Since the reflection coefficient (or conversion gain) is \( G_C \), the
reflected signal excitation (or phase conjugated signal excitation) and the
transmitted signal excitation can be expressed as follows.

\[ v_{\text{ref}}(r) = G_C(r) \, v(r) \]

\[ = G_C(r) \, v(0) \, \exp[-(r/w_i)^2] \]

and
\[ v_{tr}(r) = G_A(r) \, v(r) \]

\[ = (G_C(r) + 1) \, v(0) \, \exp[-(r/w)^2] \quad (5.2) \]

We assume that the effective radius (or waist) of the incident Gaussian pump beam and the radius of curvature of the wave front are much larger than those of the incident Gaussian signal beam, the incident pump beam can be considered as a plane wave.

In the single pump excitation process (section 3.2), with boundary condition \( A_{p+}(L) = A_{p-}(L) = A_p(L) \), the idler is \( A_{s+}(0) = 0 \) while the input signal is \( A_{s-}(L) \).

If we choose the pump excitation \( u = 2 \), peak value of the signal excitation \( v(0) = 0.9262924766 \) and the corresponding reflection coefficient \( G_C(0) = 1 \) at \( r = 0 \), the reflected (or phase conjugated) signal excitation and the transmitted signal excitation can be obtained very precisely by iteration. The result (all values are normalized to the input signal excitation with normalization factor is \( v(0) \)) is shown in Fig.10.

Fig.10(a) shows the incident signal Gaussian beam. The parameter \( w_i \) is called the spot size of the beam. Fig.10(b) shows the reflected sig-
nal beam. Obviously, the reflected beam is not Gaussian anymore and the spot size (or waist) becomes very large. Fig.10(c) shows the transmitted beam. Following the Gaussian distribution, the value of the signal excitation \( v(r) \) will gradually decrease as \( r \) increases. However, the corresponding reflection coefficient \( G_C \) will increase very fast with decreasing signal excitation. This is primarily due to the pump depletion. In fact, for any given constant value of incident pump excitation \( u \), the reflection coefficient \( G_C \) increases rapidly with further decrease in signal excitation \( v^9 \). Therefore, as shown in these figures, the transmitted beam can be very large and the reflected beam becomes almost flat.

Fig.11 shows the same response for the incident signal excitation with pump excitation \( u = 1 \), peak value of the signal excitation \( v(0) = 0.6731867943 \) and the corresponding reflection coefficient \( G_C(0) = 0.5 \).

If we use two separate pump excitations (section 3.3), with boundary condition \( A_{p+}(0) = A_{p-}(L) \), the idler and input signals are still \( A_{s+}(0) = 0 \) and \( A_{s-}(L) \). We note that the result is similar to that of the first case.
Figure 10: (a) Incident Gaussian beam (b) Reflected beam (c) Transmitted beam
Figure 11: (a) Incident Gaussian beam  (b) Reflected beam  (c) Transmitted beam
5.2 Phase conjugation mirror with end mirror

If an end mirror is used for reflecting at \( z = L \) (as shown in Fig. 12), then the input signal \( A_{s-}(L) \) at \( z = L \) depends on the input pump \( A_{p+}(0) \). The idler is \( A_{s+}(0) = 0 \).

In Fig. 12, the signal to pump excitation ratio can be defined as

\[
R_B = \frac{v}{u} \tag{5.3}
\]

In fact, the value of this ratio depends on the reflectivities of beam splitter and the end mirror. In Fig. 12(a), for example, if the reflectivity of the beam splitter is 50\%, the signal to pump excitation ratio is actually equal to the reflectivity of the end mirror. On the other hand, in Fig. 12(b), the reflectivities of the beam splitter and the end mirror must be chosen appropriately such that the boundary condition \( A_{p-}(L) = A_{p+}(0) \) can be satisfied. However, the reflection coefficient of the PCM is still defined as equation (3.18), i.e., \( R_{PCM} = G_C \).

Substituting equation (5.3) into equation (3.18), we have

\[
G_C = \frac{E}{2R_B} \tag{5.4}
\]
Actually, this equation is consistent with the equation (3.24).

Fig. 13 shows the relationship between the reflection coefficient of the PCM $G_C$ and the signal to pump excitation ratio $R_B$ with the photon conversion efficiency $E$ as the parameter. When the signal to pump excitation ratio $R_B$ is equal to zero, i.e., $v = 0$, the system becomes an oscillator and the reflection coefficient of the PCM $G_C$ will remain at infinity. The state of oscillation can be quenched by the presence of any input signal, i.e., $R_B \neq 0$, and the reflection coefficient of the PCM $G_C$ will immediately drop from infinity to a finite value. In fact, the values of $G_C$ are always less than $1/2R_B$ because the photon conversion efficiency $E$ is always less than 1.

Fig. 14 gives a different viewpoint of Fig. 13. For any given value of $R_B$, the reflection coefficient $G_C$ increases with the photon conversion efficiency $E$. However, $E = 1$ corresponds to the limiting case of the reflection coefficient $G_C$ in both figures.

The single pump excitation process with the end mirror is shown in Fig. 12(a). In this case, equation (3.22) and equation (3.26) can be rewritten as follows:

\[(1) \quad k < 1\]
\[ u = \frac{1}{\sqrt{2R_B(1+G_C)(1-G_CR_B)}} \int_0^{\sin^{-1}(\sqrt{2G_CR_B})} \frac{d\Phi}{\sqrt{1-k^2\sin^2\Phi}} \]  

(5.5)

(2) \( k > 1 \)

\[ u = \frac{1}{\sqrt{1-R_B(G_C-1)}} \int_0^{\sin^{-1}(k\sqrt{2R_BG_C})} \frac{d\Phi}{\sqrt{1-k^2\sin^2\Phi}} \]  

(5.6)

where

\[ k^2 = \frac{1 + R_B(1 - G_C)}{2R_B(1 - R_BG_C)(1 + G_C)} \]  

(5.7)

Figure 15 shows the results of our calculation for the reflection coefficient of the PCM (\( G_C \)) versus the pump excitation (\( u \)), when the signal to pump excitation ratio (\( R_B \)) is kept at various values. The curve \( R_B = 0 \) (i.e., \( v = 0 \)) is identical to the result for small-signal theory, indicating an oscillation threshold with infinite reflection coefficient of the PCM at \( u = \pi/2 \). If a signal is injected into the system (i.e., \( R_B \neq \) \( 0 \))
0), the reflection coefficient of the PCM will immediately drop down. However, with increasing pump excitation, the reflection coefficient of the PCM is gradually increased until the limiting point \( G_C < \frac{1}{2} R_B \) is reached.

To proceed to calculate the effect of the use of the end mirror on a Gaussian beam, we follow the same procedure as before. With \( R_B = 0.02 \), if we choose the pump excitation \( u(0) = 1.15111374 \) such that the corresponding reflection coefficient \( G_C(0) = 3 \), the reflected signal excitation and transmitted signal excitation (all values are normalized to the input signal excitation and the normalization factor is \( v(0) = 0.022302275 \)) are shown in Fig. 16.

In Fig. 16(a) we show the incident signal Gaussian beam while in Fig. 16(b), the reflected signal beam. We note that the reflected beam is not Gaussian and the spot size is changed. Fig. 16(c) shows the transmitted beam. As shown in these figures, the peak values of the reflected and transmitted beam can be very large because the reflection coefficient of the PCM is usually larger than 1.

If we use two separate pump excitations with the end mirror (as shown in Fig. 12(b)), equation (3.31) and equation (3.33) can be rewritten as:
(1) \( k < 1 \)

\[
u = \frac{1}{\sqrt{R_B(1-G_C R_B)(1+G_C)}} \int_0^{\sin^{-1}\sqrt{E[2-E]}} \frac{d\Phi}{\sqrt{1-k^2 \sin^2 \Phi}} \quad (5.8)
\]

(2) \( k > 1 \)

\[
u = \frac{2}{\sqrt{1+2R_B}} \int_0^{\sin^{-1}\sqrt{\frac{G_C(1+2R_B)}{1+G_C}}} \frac{d\Phi}{\sqrt{1-\frac{1}{k^2} \sin^2 \Phi}} \quad (5.9)
\]

where

\[
k^2 = \frac{1 + 2R_B}{4R_B(1-G_C R_B)(1+G_C)}
\]

Fig. 17 shows the results of our calculation for the \( G_C \) versus \( u \) with parameter \( R_B \). We note that the result is similar to Fig. 15 except the magnitude of \( u \). Comparing equation (3.20) and equation (3.30), we find that the difference between these figures is predictable.

In fact, the phase conjugation response of a Gaussian beam is also
similar to that of the first case. For example, with $R_B = 0.02$, if we choose the pump excitation $u(0) = 2.230119335$ such that the corresponding reflection coefficient $G_C(0) = 3$, the reflected signal beam and the transmitted beam (all values are normalized to the input signal excitation with normalization factor is $v(0) = 0.0446023867$) will exhibit the same shape as shown in Fig.16.
Figure 12: Schematic diagram of a PCM with end mirror via degenerate four-wave mixing
(a) Single pump excitation (b) Two separate excitations
Figure 13: $G_C$ as a function of $R_B$ with $E$ as a parameter
Figure 14: $G_C$ as a function of $E$ with $R_B$ as a parameter
Figure 15: Reflection Coefficient $G_C$ versus pump excitation $u$ at different fixed $R_B$ (For single pump excitation)
Figure 16:  
(a) Incident Gaussian beam  
(b) Reflected beam  
(c) Transmitted beam
Figure 17: Reflection Coefficient $G_C$ versus pump excitation $u$ at different fixed $R_B$ (For two separate pump excitations)
5.3 Summary and conclusion of chapter 5

In this chapter, the phase conjugation response of a Gaussian beam in degenerate four-wave mixing is discussed.

Assuming a Gaussian signal beam and two constant pump beams incident on a nonlinear medium, the response of a PCM is considered in section 5.1. We note that the reflected beam is not Gaussian and the spot size becomes very large. This is due to the fact that pump depletion decreases with reduction in signal excitation. Therefore, the reflection coefficient of PCM corresponding to the outer region of the Gaussian signal beam will be much higher than that at the center of the beam. As is shown in the diagram, the reflected beam becomes almost flat. Furthermore, the transmitted beam is also very large because the signal gain $G_C$ is usually larger than 1.

The response of PCM considered in section 5.2 is for constant signal to pump excitation ratio. First, the reflection coefficient of the PCM is always less than $1/2R_B$ ( $R_B$ is the signal to pump excitation ratio ), because the photon conversion efficiency is always less than 1. In fact, if the input signal is zero (i.e., $R_B = 0$), the system becomes an oscillator and the reflection coefficient of the PCM will approach infinity. Second, the phase conjugation response of a Gaussian beam is shown in figure 16.
We note that the peak values of the reflected and transmitted beam can be very large and the reflected beam is also no longer Gaussian.
Chapter 6

The characteristic
of
the higher order modes

6.1 Higher order modes characteristics

The variation of the normalized amplitudes of interaction waves versus the normalized interaction lengths can be obtained from eqs.(3.13) and eqs.(3.15) and rewritten as follows:

(1) \( k < 1 \)

\[
S_+ = \frac{A_{s+}(z)/\sqrt{C_{s+}}}{A_{p+}(0)/\sqrt{C_{p+}}} = \frac{S_n(\Gamma z)}{\sqrt{1+p^2Cn^2(\Gamma z)}}
\]
\[ S_- = \frac{A_{s-}(z)/\sqrt{C_{s-}}}{A_{p+}(0)/\sqrt{C_{p+}}} = \frac{1}{q} \sqrt{1-q^2} \frac{Sn^2(\Gamma z)}{1+p^2Cn^2(\Gamma z)} = \frac{1}{q^2} - S_+^2 \]

\[ P_+ = \frac{A_{p+}(z)/\sqrt{C_{p+}}}{A_{p+}(0)/\sqrt{C_{p+}}} = \frac{\sqrt{1+p^2Cn(\Gamma z)}}{\sqrt{1+p^2Cn^2(\Gamma z)}} = \sqrt{1 - S_+^2} \]

\[ P_- = \frac{A_{p-}(z)/\sqrt{C_{p-}}}{A_{p+}(0)/\sqrt{C_{p+}}} = \frac{1}{p} \sqrt{\frac{1+p^2}{1+p^2Cn^2(\Gamma z)}} = \frac{1}{p^2} + S_+^2 \tag{6.1} \]

(2) \( k > 1 \)

\[ S_+ = \frac{A_{s+}(z)/\sqrt{C_{s+}}}{A_{p+}(0)/\sqrt{C_{p+}}} = \frac{Sn(\Gamma z)}{\sqrt{q^2+p^2Cn^2(\Gamma z)}} \]

\[ S_- = \frac{A_{s-}(z)/\sqrt{C_{s-}}}{A_{p+}(0)/\sqrt{C_{p+}}} = \frac{\sqrt{q^2+p^2Cn(\Gamma z)}}{q\sqrt{q^2+p^2Cn^2(\Gamma z)}} = \frac{1}{q^2} - S_+^2 \]

\[ P_+ = \frac{A_{p+}(z)/\sqrt{C_{p+}}}{A_{p+}(0)/\sqrt{C_{p+}}} = \frac{\sqrt{\frac{Sn^2(\Gamma z)}{q^2+p^2Cn^2(\Gamma z)}}}{q^2} = \sqrt{1 - S_+^2} \]
\[ P_- = \frac{A^-_p(z)/\sqrt{C^-_p}}{A^+_p(0)/\sqrt{C^+_p}} \]

\[ = \frac{1}{p} \sqrt{\frac{p^2 + q^2}{q^2 + p^2 C_n^2(\Gamma z)}} = \frac{1}{p^2} + S^2_+ \]  

(6.2)

In fact, these solutions are multi-valued because the Jacobian elliptic functions \( S_n(x) \) and \( C_n(x) \) are periodic functions of \( x \) with period \( 4K \). Usually, we choose \( x \) between zero and \( K \) which is the so-called fundamental region. We can also choose the value of \( x \) in the higher order harmonic regions which are defined as:

Fundamental region \[ -K < x_1 < K \]

Second order harmonic region \[ K < x_2 < 3K \]

Third order harmonic region \[ 3K < x_3 < 5K \]

.......

nth order harmonic region \[ (2n-3)K < x_n < (2n-1)K \]  

(6.3)

the relationship between these regions are:
the relationship between these regions are:

\[ x_n = 2(n-1)K + (-1)^{n-1}x_1 \quad n=1,2,3... \]  \hspace{1cm} (6.4)

For the special case of oscillation, equation (6.4) becomes

\[ x_n = (2n-1)x_1 \quad n=1,2,3... \]  \hspace{1cm} (6.5)

Two cases in chapter 3 will be considered here.
6.2 Single pump excitation into end reflector

From the definition of the elliptic function, equation (3.22) and equation (3.26) can be rewritten as follows:

(1) $k < 1$

$$u = \frac{1}{\sqrt{(2-E)(E-1)}} \int_0^{\sin^{-1}(\sqrt{E})} \frac{d\Phi}{\sqrt{1-k^2\sin^2\Phi}}$$

(6.6)

(2) $k > 1$

$$u = \frac{1}{\sqrt{E + G_C(2-E)}/2G_C}} \int_0^{\sin^{-1}(k\sqrt{E})} \frac{d\Phi}{\sqrt{1-k^2\sin^2\Phi}}$$

(6.7)

If we use the principal value of the inverse sine function, we can get the fundamental value from the above equations. Otherwise, the nth order result can be obtained when we use the nth order value of the inverse trigonometric function. The solutions of this inverse sine function can be written as:

$$\sin^{-1}\Phi_L = (n-1)\pi - (-1)^{n-1}\sin^{-1}\Phi_L \quad n=1,2,3...$$

(6.8)

This expression is completely equivalent to eq.(6.3) or eq.(6.4).
Solving eq.(6.6) or eq.(6.7), the pump excitation $u$ or the signal excitation $v$ of the fundamental mode can be obtained for any conversion efficiency and reflection coefficient. Then, from eq.(6.4), higher order pump excitation $u$ and signal excitation $v$ can be obtained. These higher order results are the same as the solutions of eq.(6.6) or eq.(6.7) except eq.(6.8) is used as the integration upper limit. The relationships between these parameters are shown in Fig.18, Fig.19 and Fig.20. Having all the results obtained here, the spatial field distributions can be calculated from eqs.(6.1) or eqs.(6.2) and are plotted in Fig.21 to Fig.29. Figures 30, 31 and 32 show the spatial field distributions of the oscillation case.
6.3 Two pump excitation

Eq.(3.31) and eq.(3.33) can be rewritten as follows:

(1) \( k < 1 \)

\[
    u = 2\sqrt{\frac{k^2 G_C}{E+G_C}} \int_0^{\sin^{-1}\sqrt{E(2-E)}} \frac{d\Phi}{\sqrt{1-k^2 \sin^2 \Phi}} 
\]  

\( (6.9) \)

(2) \( k > 1 \)

\[
    u = 2\sqrt{\frac{G_C}{E+G_C}} \int_0^{\sin^{-1}\left(\sqrt{\frac{E+G_C}{1+G_C}}\right)} \frac{d\Phi}{\sqrt{1-\frac{1}{k^2 \sin^2 \Phi}}} 
\]  

\( (6.10) \)

Following the same procedure described in section 3.2, we show the modes pattern in Fig.33; 34 and several field distributions in Fig.35 to Fig.38.
6.4 Physical interpretation of the characteristics of higher order modes

Figure 18 shows the results of our calculation for the photon conversion efficiency $E$ versus the pump excitation $u$ when the reflection coefficient $G_C = 1$. The small-signal theory corresponds to the limit $E = 0$. Dotted lines in the figure can be obtained from eq.(3.27) and eq.(6.5) which reveal the characteristics of the system as an oscillator. Usually, we use these dotted lines to separate the fundamental region from the higher order regions. Note that the reflectivity approaches infinity when the oscillation condition is satisfied. The threshold of oscillation for the fundamental mode, at $E = 0$, is at $u = \pi/2$, and the threshold for the higher order modes are at $u = 3\pi/2, 5\pi/2, ...$. But the higher order modes are usually suppressed by the oscillation in the fundamental modes because of pump depletion.

In Figure 18, the curves of different modes are of similar shape, but in Figures 19, 20, 33 and 34, they are completely different. The major difference between these two cases is due to the elliptic function parameter $k$. From eq.(3.23) or eq.(3.32), we find $k = 1$ when $E = 1$ and the pump excitation $u$ approaches infinity as the real quarter period $K(k = 1)$ approaches infinity. Therefore, we conclude that a complete pump conversion is not likely to occur for both cases of sections 3.2 and 3.3.
In the fundamental mode, except $E = 1$, the pump excitation will never reach infinity even if the elliptic function parameter $k$ is equal to one. However, in higher order modes, even if $E \neq 1$, the pump excitation will approach infinity when $k$ approaches one. Physically, this phenomena can be explained by means of the field distributions.

From eq.(3.23) of section 3.2, with $G_C \geq 1$, the value of $k$ decreases monotonically from infinity to one when the value of $E$ increases from zero to one. Thus, all the $E$-u curves are smooth irrespective of the fundamental mode or higher order modes. For $G_C < 1$, we find the following relationships:

$$k^2 > 1 \quad 0 < E < \frac{2G_C}{G_C + 1}$$

$$k^2 < 1 \quad \frac{2G_C}{G_C - 1} < E < 1$$

$$k^2 = 1 \quad E = \frac{2G_C}{G_C + 1}$$

Therefore, the pump excitation will approach infinity for all the higher order curves when $E$ approaches $2G_C/(1+G_C)$ and 1.
In section 3.3, we find similar relationships from eq. (3.32).

\[ k^2 > 1 \quad 0 < E < \frac{G_C}{G_C + 1} \]

\[ k^2 < 1 \quad \frac{G_C}{G_C - 1} < E < 1 \]

\[ k^2 = 1 \quad E = \frac{G_C}{G_C + 1} \quad (6.12) \]

Obviously, the pump excitation will approach infinity for all the higher order curves when \( E \) approaches \( G_C/(1+G_C) \) and 1.

Field distributions of the fundamental, second order and third order modes corresponding to the points \( E = 0.5 \) as indicated in Fig. 18 are shown in Figures 21, 22 and 23 respectively, and those field distributions corresponding to the same points in Fig. 19 are shown in Figures 24, 25 and 26. Comparing these figures, we note that the curves in Fig. 22 and Fig. 25 are physically impossible because an incident signal should not be attenuated at the input, as it enters an amplifying medium.

In Figures 27, 28 and 29, we show the field distributions of the
fundamental, second order and third order modes corresponding to the points \( E = 0.885 \) as indicated in Fig.19. In Figures 35, 36, 37 and 38, we show those field distributions corresponding to the points \( E = 0.499 \) in Fig.33. However, the field distributions corresponding to the points \( E = 0.501 \) in Figure 33 will not be presented. This is due to the fact that these curves are similar to those in Figures 35, 36, 37 and 38, except that \( P_+ \) and \( S_- \) are interchanged. We note that the field distributions corresponding to the points \( E = 0.5 \) (i.e., \( k = 1 \)) in Fig.33 can not be calculated because of the infinite pump excitation. The variations of field distributions inside the medium can be seen from those curves. We find that \( P_+ \) approaches \( S_- \) as \( k \) approaches 1, but there is a sizable difference between \( P_- \) and \( S_+ \) due to the boundary condition in agreement with equation(6.1) or equation(6.2). Actually, if the system is in the higher order mode, \( P_+ \) and \( S_- \) will never be exactly equal because the complete pump conversion can never be reached under the finite pump excitation. This is the reason why all the fundamental mode curves are smooth while most of the higher order curves display distinct characteristics of sharp discontinuity towards infinity pump excitation at some finite values of \( E \).

At the same time, we observe that these higher order modes display more than one complete period of the interaction fields. For each complete period of interaction fields, there is obviously no net exchange
of energy. Thus the net effective interaction length is reduced as the corresponding order of the mode is increased. For a given pump excitation \( u \), the required pump beam intensity should be much larger for higher order mode, because the effective interaction length is much reduced.

According to Figures 18, 19, 20, 33 and 34, the whole system always stays in the fundamental mode even if the pump excitation is increased, and never gets the chance to be switched to the higher order modes because the complete pump conversion can not be reached under the finite pump excitation. This conclusion is different from an earlier observation\(^9\) where the signal is \( A_{s+}(0) \) and the idler is \( A_{s-}(L) \). In that case (ref.9), complete pump conversion is possible under finite pump excitation. This means that the pump field will change sign whenever the system achieves complete pump conversion. Then the state of amplification will become attenuation because the signal will actually feed energy back to a completely depleted pump wave. Therefore, the system will automatically switch to the second order mode; but the third order mode will never be reached.

Fig.39 shows the reflection coefficient \( G_C \) versus the pump excitation \( u \) for a fixed signal excitation \( v \). The dotted lines in this figure represent even order modes which are experimentally inaccessible, and this situation can be confirmed by carefully examining the spatial field
distributions of all the interacting waves. We note that this figure does not reveal hysteresis and bistability characteristics because the reflection coefficients of the fundamental mode and the higher order modes will never approach each other for finite pump excitations. However, in practice when phase variation occurs due to additional nonlinear interaction, the system can be switched from the fundamental high conversion gain operation to higher order modes with much less effective interaction and lower conversion gain, thus causing hysteresis and bistability\textsuperscript{14, 15}.

In Figures 30, 31 and 32, we show the fundamental and higher order modes for oscillation at $E = 0.5$ respectively.
6.5 Ideal characteristics of phase conjugation

The ideal phase conjugation is caused by the interaction Hamiltonian $H^{'}_{16, 17}$,

$$H^{'} = d a^*_s a_s^* a^*_p a_p + d a^*_s a_s a_p^* a_p$$

$$= d a^*_s a_s a^*_p a_p + \text{complex conjugate}$$  \hspace{1cm} (6.13)

where $d$ is the effective nonlinear coupling coefficient$^{18}$ and $a_i$ are

$$a_i = A_i e^{-j(w_i t - k_i z)}$$

$$= |A_i| e^{jQ_i} e^{-j(w_i t - k_i z)} \hspace{1cm} i = s+, s-, p+, p-$$  \hspace{1cm} (6.14)

The general coupled mode equations can be written as:

$$\frac{d a_i}{dz} = j k_i \frac{\partial (H^{'})}{\partial (a_i^*)}$$

$$\frac{d a_i^*}{dz} = -j k_i \frac{\partial (H^{'})}{\partial (a_i)}$$  \hspace{1cm} (6.15)

Then, we get:
\[
\begin{align*}
\frac{dA_{s+}}{dz} &= k_{s+} dA_{s} + A_{p+} A_{p-} \sin \Phi \\
\frac{dA_{s-}}{dz} &= -k_{s-} dA_{s} + A_{p+} A_{p-} \sin \Phi \\
\frac{dA_{p+}}{dz} &= -k_{p+} dA_{s} + A_{s+} A_{p-} \sin \Phi \\
\frac{dA_{p-}}{dz} &= k_{p-} dA_{s} + A_{s-} A_{p+} \sin \Phi 
\end{align*}
\quad (6.16)
\]

\[
\begin{align*}
\frac{d\phi_{s+}}{dz} &= -k_{s+} d \frac{A_{p+} A_{p-} A_{s-}}{A_{s+}} \cos \Phi \\
\frac{d\phi_{s-}}{dz} &= k_{s-} d \frac{A_{p+} A_{p-} A_{s+}}{A_{s-}} \cos \Phi \\
\frac{d\phi_{p+}}{dz} &= k_{p+} d \frac{A_{s+} A_{s-} A_{p-}}{A_{p+}} \cos \Phi \\
\frac{d\phi_{p-}}{dz} &= -k_{p-} d \frac{A_{s+} A_{s-} A_{p+}}{A_{p-}} \cos \Phi 
\end{align*}
\quad (6.17)
\]

where \( k_{i}d = C_{i} \) and \( \Phi = \phi_{p+} + \phi_{p-} - \phi_{s+} - \phi_{s-} \).

Solving these equations, we can show that

\[
A_{p+}(z) A_{p-}(z) A_{s+}(z) A_{s-}(z) \cos \Phi = \text{constant}
\quad (6.18)
\]
The constant is identically equal to zero when one of the waves is assumed to be an idler so that \( A_i = 0 \) at the boundary. For equation (6.18) to hold for any \( z \), \( \cos \Phi = 0 \) i.e., \( \Phi(z) = \pi/2 \) or \( 3\pi/2 \). Thus, equations (6.16) become the same as equations (3.3). This is called the ideal phase conjugation because only the amplitude equations need be considered in the analysis and the phase relationship between the interacting waves will remain invariant. Obviously, the signs of the \( \sin \Phi \) then play the important role of determining whether we have amplification (i.e. signal waves get the power from the pump waves) or attenuation (i.e. power is transferred from the signal waves back to the pump waves). If the signal is \( A_{s+} \), then a state of amplification means a positive slope \( dA_{s+}/dz \) and a state of attenuation corresponds to the negative slope \( dA_{s+}/dz \). Therefore, it can be understood why all the even order mode curves in the last section are physically impossible because it indeed no longer exhibits amplification characteristics at the input.

In this chapter, only ideal phase conjugation is considered. However, if we introduce some perturbation terms into the system, the ideal phase conjugation may not hold anymore and some of the characteristics may change.
First of all, in a lossy medium, absorptive perturbation terms must be included in equation (6.16). Although analytical solution can be obtained for small signals, only numerical analysis can be applied to large signals since no analytical solution is available\textsuperscript{19}.

Secondly, the dispersive perturbation terms can be included in equation (6.17) due to additional nonlinear interaction. Then the hysteresis and bistability characteristics become possible\textsuperscript{14, 15, 20} because the higher order modes can be reached in this non-ideal phase conjugate case.
6.6 Summary and conclusion of chapter 6

In this chapter, two different cases are considered. The first case is a scattering process with single pump excitation, and the second case is a process with two separate pump excitations. Furthermore, we choose boundary conditions with idler at $z = 0$, i.e., $A_{s+}(0) = 0$ and the input signal is $A_{s+}(L)$ at $z = L$ for both cases.

In the chapter 3, the amplitude solutions of the DFWM process for both cases are obtained and several important parameters are also defined. At the same time, we find that complete pump conversion is impossible for any $G_C$ under finite pump excitation for both cases. However, in the first case, if the boundary condition is changed such that the input signal is $A_{s+}(0)$ at $z = 0$ and the idler is $A_{s-}(L) = 0$ at $z = L$, complete pump conversion becomes possible for any $G_C$ under finite pump excitation.

The definition and the corresponding expressions of the higher order modes are given in sec. 6.1 to sec. 6.3. Appropriate graphs are presented in sec. 6.4 for physical interpretation. We note that the net effective interaction length is reduced as the corresponding order of the mode is increased. In other words, a system in the fundamental mode will be more efficient than a system in the higher order modes.
In section 6.5, the coupled mode equations were derived from the interaction Hamiltonian $H'$. In fact, all elementary nonlinear systems can be described by the interaction Hamiltonian $H'$. Furthermore, the interaction Hamiltonian $H'$ can be chosen such that the corresponding coupled equations just have one particular term responsible for phase conjugation. This is called ideal phase conjugation for which only amplitude equations need to be considered and the phases of all the interacting waves are invariant.

In ideal phase conjugation, the exact analytic solutions can be obtained. Therefore, all the characteristics can be analyzed and interpreted exactly. On the other hand, if there are additional perturbing nonlinear interactions, the ideal phase conjugation may not be valid anymore. The exact analytic solutions will be difficult to find and numerical analysis becomes the only way in these special cases\textsuperscript{14, 15, 21}.

In general, the characteristics of the three- and four-wave scatterings are very similar. In fact, the small-signal analysis will not reveal any characteristic difference between the three- and four-wave processes. However, the large signal characteristic of the three-wave mixing is simpler than that of the DFWM. Therefore, the analysis for the higher order modes of three-wave mixing is omitted in our discussion.
Figure 18: Photon conversion efficiency $E$ versus pump excitation $u$ when the reflection coefficient $G_C = 1$. Dotted lines reveal the characteristics of the system as an oscillator.
Figure 19: Photon conversion efficiency $E$ versus pump excitation $u$ when the reflection coefficient $G_c = 0.8$. Dotted lines reveal the characteristics of the system as an oscillator.
Figure 20: Photon conversion efficiency $E$ versus pump excitation $u$ when the reflection coefficient $G_c = .5$. Dotted lines reveal the characteristics of the system as an oscillator.
Figure 21: Field distributions at $G_C = 1$, $E = 0.5$ and $u_1 = 1.078257796$
Figure 22: Field distributions at
\( G_C = 1, \ E = 0.5 \) and \( u_2 = 3.234773506 \)
Figure 29: Field distributions at $G_C = 1$, $E = 0.5$ and $u_3 = 9.7043204$
Figure 24: Field distributions at $G_C = 0.8$, $E = 0.5$ and $u_1 = 1.002110497$
Figure 25: Field distributions at $G_C = 0.8$, $E = 0.5$ and $u_2 = 3.351976018$
Figure 26: Field distributions at 
$G_C = 0.8$, $E = 0.5$ and $u_3 = 9.710283527$
Figure 27: Field distributions at $G_c = 0.8$, $E = 0.885$ and $u_1 = 1.656392391$
Figure 28: Field distributions at 
$G_C = 0.8$, $E = 0.885$ and $u_2 = 8.281477874$
Figure 29: Field distributions at $G_C = 0.8$, $E = 0.885$ and $v_3 = 21.53213292$
Figure 30: Field distributions at $G_C = \infty$, $E = 0.5$ and $u_1 = K$
Figure 31: Field distributions at $G_C = \infty$, $E = 0.5$ and $u_2 = 3K$
Figure 32: Field distributions at
$G_C = \infty$, $E = 0.5$ and $u_3 = 5K$
Figure 39: Photon conversion efficiency $E$ versus pump excitation $u$ when the reflection coefficient $G_C = 1$. Dotted lines reveal the characteristics of the system as an oscillator.
Figure 94: Photon conversion efficiency $E$ versus pump excitation $u$ when the reflection coefficient $G_C = .5$. Dotted lines reveal the characteristics of the system as an oscillator.
Figure S5: Field distributions at $G_C = 1$, $E = 0.499$ and $u_1 = 2.14891046$
Figure 86: Field distributions at $G_C = 1$, $E = 0.499$ and $u_2 = 14.3245722$
Figure 87: Field distributions at $G_C = 1$, $E = 0.499$ and $u_3 = 35.10022697$
Figure 38: Field distributions at
\( G_C = 1, E = 0.499 \) and \( u_4 = 47.27130224 \)
Figure 9: Reflection coefficient $G_C$ versus pump excitation $u$ when the signal excitation $v = 1$. 
Chapter 7
Experimental Result

The experimental arrangement for the generation and observation of the stimulated Brillouin scattering is shown in Fig.40. An intense (wavelength = 6943Å) radiation from a giant-pulse ruby laser with a peak power of ~500 megawatts and pulse duration of 14.5 nsec can be focused on different regions of the nonlinear crystal. At the focal point, we assume that the crystal is free of optical distortions. We used different kinds of crystal and liquid for the nonlinear medium, among those are: quartz, lithium niobate, acetone, ethanol and urea.

In this experiment we are able to observe the response of a Gaussian incident beam. The result with lithium niobate (LiNbO₃) as nonlinear medium and pump voltage at 8500 volts is shown in Fig.41.

Fig.41(a) shows the incident Gaussian beam, namely, the laser spot size of a passive Q switched pump beam transmitted without the non-
linear medium. Fig. 41(b) shows the same beam transmitted but with the optical system and the nonlinear medium present. Comparing these figures, we can see that the transmitted pump beam has a ring structure and this conclusion is consistent with the prediction of section 4.1.
Figure 40: Schematic of Experimental arrangement
Figure 41: Photographic demonstration of ring structure with SBS (a) incident beam (b) transmitted beam
Chapter 8
Summary and Conclusion

We have thus far seen that the phase conjugation response of an incident Gaussian beam is very interesting, especially in the three-wave stimulated Brillouin scattering.

Since the solutions of large signal phase conjugation are in terms of elliptic functions, it is natural to try to interpret the physical characteristics in terms of the function parameters. In fact, the reflection coefficient of PCM can be expressed in terms of the modulus of the elliptic function \( k \), and the incident pump excitation in terms of the real quarter period of the elliptic function \( K \). From the asymptotic equation between \( k \) and \( K \) which is derived in Appendix, the reflection coefficient of the PCM can be found from the incident beam excitation very easily.

Although stimulated Brillouin scattering requires only the incident wave, the intensity of the incident wave must exceed a threshold for a
phase conjugate output to be obtained. Therefore, the threshold is a very important parameter in three-wave mixing. Nevertheless, we note that there is no threshold to overcome by the incident beam excitation for a PCM with end mirror. This is due to the fact that the end mirror will self-start the wave interaction for small incident beam excitations and will eventually convert to a SBS PCM.

Ideally, the PCM pumping beam should be a perfect Gaussian beam in order to achieve Gaussian modes, so the response of the Gaussian beam is also very important. In this thesis, we find that the scattered (reflected) beam is not Gaussian and the beam waist is changed while a Gaussian beam is incident on a stimulated scattering medium. However, this result is different from the earlier papers^{22, 23}. In these papers, the scattered beam is still Gaussian but the beam waist is changed when a Gaussian beam is incident on a SBS medium.

Usually, the transmitted beam will exhibit a ring and this ring structure has been observed in our experiments. On the other hand, the spot size ratio of the incident beam waist and the reflected beam waist can also be calculated very precisely.

However, this ring structure disappeared in the case of degenerate four-wave mixing. The peak value of the reflected and transmitted beam
can be very large because the reflection coefficient of the PCM (in DFWM process) is usually larger than 1. Nevertheless, if an end mirror is used for a PCM, the reflection coefficient of the PCM is always less than $1/2R_B$.

According to the discussion of chapter 6, we note that the whole interaction system in the case of ideal phase conjugation always stays in the fundamental mode and never gets the chance to switch to the higher order modes. In practice, when phase variation occurs due to additional nonlinear interaction, the system can be switched from the fundamental to higher order modes, thus causing hysteresis and bistability.
Appendix A.
The asymptotic equation of $k$ and $K$

Since the characteristic of large signal phase conjugation can be interpreted by means of the elliptic function, the relationship between the modulus (or parameter) of the elliptic function $k$ and the real quarter period of the elliptic function $K$ becomes very important. Our objective in this appendix is to derive an asymptotic equation between these two parameters.

Actually, the real quarter period of the elliptic function $K$ is also called complete elliptic integral of the first kind and is defined as\(^{12}\)

$$
K = \int_0^{\pi/2} \frac{d\Phi}{\sqrt{1 - k^2 \sin^2 \Phi}} = \int_0^1 \frac{dr}{\sqrt{1 - r^2} \sqrt{1 - k^2 r^2}} \quad (A.1)
$$

This equation can also be expressed in terms of the product of series\(^{13}\)
Obviously, we can get $K$ very easily from these equations if $k$ is known. However, for any given $K$, the corresponding $k$ is difficult to find because no explicit formula is available expressing $k$ as a function of $K$. For phase conjugate mirror analysis, we often need to know $k$ from $K$. Although we have been able to calculate the results accurately by iteration, the process is nevertheless very long and tedious.

Fortunately, we can derive an asymptotic equation from the definition of the complete elliptic integral of the first kind and the procedure can be summarized as follows:

$$K = \ln \left( \frac{4}{k'} \right) + M(K)$$  \hspace{1cm} (A.3)$$

where $k^2 + (k')^2 = 1$, and $M(K) \to 0$ when $k \to 1$
Solving equation (A.3), we have

\[ k^2 = 1 - 16 \exp(-2K + M(K)) \quad (A.4) \]

and the boundary conditions are

\[ M(K) \to 0, \quad \text{when } k \to 1 \]
\[ K = \pi/2, \quad \text{when } k = 0 \]

If we set

\[ M(K) = M \exp[-N(K - \pi/2)] \]

\( M \) can be found to be \((\pi - \ln 16)\). Therefore, this equation can be written as

\[ k^2 = 1 - 16 \exp(-2K + (\pi \ln 16) \exp[-N(K - \pi/2)]) \quad (A.5) \]

where \( N \) is a correction factor.

A fairly precise (the accuracy is on the order of 10^{-4}) asymptotic equation is found to be:
\[ k^2 = 1 - 16 \exp\{-2K+(\pi - \ln 16)\exp\{-\sqrt{2.2(K \cdot \pi/2)}\}\} \]  (A.6)
References


