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DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By
Ákos Seress, M.Sc.

The Ohio State University
1985

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7) Quick gossiping by conference calls, submitted to SIAM Algebraic
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8) with D. Miklós, M. Newman, D. West, The number game, to appear
9) On pairwise relative prime numbers, submitted to the Periodica
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INTRODUCTION

The original version of the gossip problem was proposed by A. Boyd and popularized by P. Erdős. There are n persons, each knowing a different item of information. They communicate by phone calls, and whenever two persons talk they tell each gossip they know at that time. We seek the minimum number g(n) of calls required to transmit all the information to everyone. g(n)=2n-4 if n≥4, as it was proved independently by B. Baker and R. Shostak [1], R. Bumby [5], A. Hajnal, E. Milner and E. Szemerédi [13], J. Spencer (unpublished), and R. Tijdeman [27].

There are a lot of ways to generalize the problem. One possibility is to restrict the set of possible phone calls (sworn enemies do not talk to each other). Denote by G the graph of possible phone calls, i.e. the vertices of G are the persons, and two vertices are connected iff the phone call between the appropriate persons is allowed. Clearly, G must be connected, otherwise we cannot transmit all the information to everyone. In the case of connected graphs, g(n)=2n-4 if G contains a four-cycle, otherwise g(n)=2n-3 (see R. Bumby [5], D. Kleitman and J. Shearer [15]).
Instead of graphs, we can consider directed graphs, representing one-directional transfers of information. Defining $g_{dir}(n)$ as $g(n)$ above, it is easy to see that $g_{dir}(n) = 2n - 2$. In the case of restricted networks, the graph of possible calls must be strongly connected, and $g_{dir}(n) = 2n - 2$ for all strongly connected graphs (see F. Harary and A. Schwenk [14], M. Golumbic [12]).

A further possibility is the consideration of uniform hypergraphs, i.e. the persons communicate by $k$-conference calls for some fixed $k$. Denoting the minimum number of necessary calls by $g_k(n)$, we obtain

$$g_k(n) = \begin{cases} \left\lfloor \frac{n-k}{k-1} \right\rfloor + 1 & \text{if } k \leq n < k^2 \\ \left\lfloor \frac{n-k}{k-1} \right\rfloor & \text{if } n \geq k^2 \end{cases}$$

(see J. Bermond [3], K. Lebensold [17], D. Kleitman and J. Shearer [15]).

Another variation asks for the minimum time of transmission, when parallel calls are permitted. Each vertex can participate in at most one call per unit of time, and each call takes one unit of time. W. Knödel [16] and P. Schmitt [21] gave the answer in the case of complete graphs and uniform hypergraphs, resp., while R. Entringer and P. Slater [9] examined the time of transmission in complete digraphs. In each case, the behavior of the minimum time is logarithmic in the number of vertices.
The purpose of this work is the investigation of the gossip problem with the assumption that everybody hears each gossip exactly once. Calling schemes satisfying this assumption do not exist for all \( n \); in fact, I heard about the problem on the Sixth Hungarian Colloquium of the János Bolyai Mathematical Society, where M. Gerèb [11] asked (in the case of phone calls) whether there exist feasible \( n \)'s not of the form \( n=2^k \).

Chapter I deals with phone calls, i.e. with ordinary graphs. In §1 we determine the set \( F_2 \) of feasible \( n \)'s, i.e. \( n \in F_2 \) iff there exists a sequence of calls among \( n \) persons such that everybody hears each information exactly once. We prove that

\[
F_2 = \{1, 2, 4, 8, 12, 16\} \cup \{n: n \geq 20 \text{ and } 2 \mid n\}.
\]

For \( n \in F_2 \), we can define \( f(n) \) as the minimum number of necessary calls. D. B. West [31] proved that \( f(n) \leq 2.25n-6 \) if \( n=4k, k \geq 2 \), and \( f(n) \geq 2n-3 \) if \( n>8 \) (see [28]). §§2-4 contain the main result of the thesis, namely

\[
f(n) = 2.25n-6 \text{ if } n=4k, k \geq 2 \text{ and } \]

\[
2.25n-4.5 \leq f(n) \leq 2.25n-3.5 \text{ if } n=4k+2, k \geq 5.
\]

As a consequence of our method, we obtain a new (sixth) proof of the original gossip problem, too.
Chapter II gives a complete answer in the easy case of directed graphs. Each $n$ is feasible, and, defining $f_{\text{dir}}(n)$ as the minimum number of calls when everybody hears each of the others' gossips exactly once, $f_{\text{dir}}(n)=2n-2$. However, $n-1$ persons must hear back their own information.

In Chapter III, we deal with uniform hypergraphs, i.e. with the case when the people communicate by $k$-conference calls. Denoting the set of feasible $n$'s by $F_k$, in §6 we prove that $n \in F_k$, $n > 1$ implies $n$ must be of the form $n = k + x(k-1)$ for some integer $x$; on the other hand, for each $k$ there exists $x_0(k)$ such that $x > x_0(k)$ implies $k + x(k-1) \in F_k$. As before, we can define $f_k(n)$ as the minimal number of necessary calls. To give a non-trivial lower bound for $f_k(n)$ seems to be hopeless at this moment; in §7 we prove that

$$f_k(n) \geq \frac{3k^2-1}{(k-1)^2} \cdot n + k^3.$$

Finally, in §8, we solve the slow gossiping problem for graphs, directed graphs and hypergraphs. Denote $h_k(n)$ the maximal number of calls in calling schemes on $n$ points s.t. the persons communicate by $k$-conference calls and everybody hears each gossip exactly once. The constructions in §1 and §5 show that $h_k(n) \geq c n \log n$, where $c$ is a constant depending on $k$. Here we prove (joint result with D. Miklós, M.
Newman, and D. West (16)) that $h_k(n) = c' n \log n$ for some constant $c'$.

Furthermore, denote by $h_{\text{dir}}(n)$ the maximal number of calls in the case of directed graphs (again, everybody hears each gossip exactly once). We prove that $h_{\text{dir}}(n) = (n^2 + n - 2)/2$.

We use standard combinatorial notation; all undefined notions can be found for example in Berge (2).
CHAPTER I

GRAPHS

§1. The set of feasible n's

Let $F_2 = \{ n : \text{there exists a sequence of calls among } n \text{ persons such that everybody hears each gossip exactly once} \}$.

A calling scheme among $n$ persons can be represented by a graph on $n$ points. The edges of the graph are linearly ordered according to the ordering of calls. Using this representation, we can express each notion by usual graph theoretic terms: for example, "a hears b's gossip" means that there exists a path beginning at $b$ and terminating at $a$ s.t. the edges of the path are ordered increasingly. We shall develop this graph-theoretical language in §2, where we begin the proof of the lower bound for $f(n)$.

In this section, we will use recursive constructions. If we already have appropriate calling schemes for some small values of $n$, we can try to combine these schemes to obtain a good sequence of calls for a larger value. The simplest combining method is the "product-lemma":

6
Lemma 1.1: If $n_1 \in \mathbb{F}_2$ and $n_2 \in \mathbb{F}_2$ then $n_1 n_2 \in \mathbb{F}_2$.

Proof: In the first step, we divide the persons into $n_2$ groups of size $n_1$. In each group, we perform a sequence of calls s.t. everybody hears each gossip in his own group exactly once. (It is possible since $n_1 \in \mathbb{F}_2$.)

In the second step, we divide the persons into $n_1$ groups of size $n_2$: we put one person from each original group into each new group. Since $n_2 \in \mathbb{F}_2$, it is possible to perform an appropriate sequence of calls in each new group such that finally everybody hears each gossip exactly once. □

An immediate consequence of Lemma 1.1. is that the powers of 2 are in $\mathbb{F}_2$. Fig. 1. shows the sequences of calls for the values 2, 4, and 8. In each case, $a_1, \ldots, a_{2^k}$ denotes the persons, $n_1 = 2^{k-1}$, and $n_2 = 2$. In the first step, the groups are $\{ a_1, \ldots, a_{2^{k-1}} \}$ and $\{ a_{2^{k-1}+1}, \ldots, a_{2^k} \}$, while in the second step the groups are $\{ a_i, a_{2^k-1+1} \} \ (1 \leq i \leq 2^{k-1})$.

However, the product-lemma has serious limitations: for example, if we have constructions only for even $n$'s, we can obtain new constructions only for $n$'s divisible by 4. Hence, instead of the product-lemma, we shall use an "addition-lemma" (see Lemma 1.9.).
Theorem 1.2: $F_2 = \{ 1, 2, 4, 8, 12, 16 \} \cup \{ n: n \geq 20 \text{ and } 2 \mid n \}$. 

Remark 1.3: The theorem will be proven by a sequence of lemmas. To make the description easier, we introduce two notions. The phrase "to organize the conversations" means to give a sequence of calls s.t. everybody hears each gossip exactly once. The other notion is the joined gossip (JG). Suppose that after a certain call, say, the $i^{th}$ call, the following situation occurs. There is a subset of gossips $G = \{ g_1, \ldots, g_k \}$ satisfying the following property: if a person knows one gossip from $G$ then he knows each gossip from $G$. This means that after the $i^{th}$ call, the gossips in $G$ spread together: if a person hears one of them, he must hear all of them at the same time. Hence, after the $i^{th}$ call, we can consider $G$ as a single (joined) gossip. For example, after the first step in the proof of Lemma 1.1, we can say that we
obtained $n_2$ JG’s, each of them is known by $n_1$ persons. In what follows, mostly we shall use two-step constructions similar to Lemma 1.1. In the first step, the persons are partitioned into groups, and we organize the conversations in each group. After that, persons belonging to the same group know the same subset of gossips, and these subsets will be considered as JG’s in the second step. □

Lemma 1.4: Suppose $n \in F_2, n \geq 2$, and let us consider a sequence of calls proving this property. Then

(i) Associating each person to his last telephone-partner, we obtain disjoint pairs.

(ii) Associating each person to his first telephone-partner, we obtain disjoint pairs.

Proof: (i) After each phone call, the persons who have talked to each other know the same gossips; if one of them knows all of the gossips then so does his partner.

(ii) Suppose that there exist three persons, $a, b, c$, s.t. the call $(a, b)$ is the first conversation for $b$ but $a$’s first partner was $c$. Then $c$ knows $a$’s gossip before the call of $a$ and $b$. After the call $(a, b)$, $b$’s gossip is known only by $a$ and $b$, hence everybody hears $b$’s gossip together with $a$’s information. Thus, when $c$ hears $b$’s gossip, he hears $a$’s gossip a second time. This contradiction proves the statement. □

Proposition 1.5: If $n \in F_2, n \geq 2$ then $2 \mid n$.

Proof: It is a trivial consequence of Lemma 1.4. □
Lemma 1.6:  \( 12 \in \mathbb{F}_2 \).

Proof: Let us divide the twelve persons into three groups, \( A_1, A_2, A_3 \), \( |A_i| = 4 \) for \( i = 1, 2, 3 \) and let us organize the conversations in each group. We obtain three JG's, each is known by four persons. After that we make three new groups: the \( i \)th new group contains two persons from \( A_i \) and one from each of the other \( A_j \)'s. Clearly, it is possible to organize the conversations in each new group. (See Fig. 2. In Fig. 2, the rows correspond to the original groups. The numbers denote the JG's known after the first step, and the broken lines give the groups in the second step of the construction.)

![Fig. 2.](image_url)

Since it is the first non-trivial construction, we give the sequence of calls explicitly. Denote by \( a_1, a_2, \ldots, a_{12} \) the persons; in the first
In the first step, we make the groups $A_1 = \{a_1, a_2, a_3, a_4\}$, $A_2 = \{a_5, a_6, a_7, a_8\}$, $A_3 = \{a_9, a_{10}, a_{11}, a_{12}\}$, and the calls are $a_1a_2, a_3a_4, a_1a_3, a_2a_4, a_5a_6, a_7a_8, a_5a_7, a_6a_8, a_9a_{10}, a_{11}a_{12}, a_9a_{11}, a_{10}a_{12}$. In the second step, the groups are $\{a_1, a_2, a_5, a_9\}$, $\{a_3, a_6, a_7, a_{10}\}$, $\{a_4, a_8, a_{11}, a_{12}\}$, and we perform the calls $a_5a_9, a_1a_5, a_2a_9, a_3a_{10}, a_3a_6, a_7a_{10}, a_4a_8, a_4a_{11}, a_8a_{12}$. (See Fig. 3.)

![Fig. 3.](image-url)
Lemma 1.7: Suppose there are four disjoint JG's which are known by 3,3,2,2 persons respectively. Then it is possible to organize the conversations among these ten persons.

Proof: Denote by $a_1, a_2, \ldots, a_{10}$ the persons; the JG's are known by $a_1, a_2, a_3$ resp. $a_4, a_5, a_6$ resp. $a_7, a_8$ resp. $a_9, a_{10}$. A possible sequence of calls is $a_1 a_7, a_4 a_9, a_7 a_9, a_1 a_{10}, a_1 a_5, a_6 a_{10}, a_4 a_8, a_2 a_4, a_3 a_8$. (See Fig. 4.)

![Fig. 4.](image)

Lemma 1.8: $20 \in F_2$.

Proof: Let us divide the twenty persons into five groups, $A_1, A_2, A_3, A_4, A_5$, $|A_i| = 4$ for $i = 1, 2, \ldots, 5$ and let us organize the conversations in each group. We obtain five JG's, each is known by four persons. After that we make two new groups, the first one containing 3,3,2,1, and 1 persons resp. from $A_1, A_2, A_3, A_4,$ and $A_5$. In the first new group,
performing an additional call between the persons originating from $A_4$ and $A_5$, the situation of Lemma 1.7 occurs, so we can organize the conversations. Similarly, it is possible to organize the conversations in the second new group. (See Fig. 5.)

**Lemma 1.9:** If $4 \mid m$, $0 \leq 4k \leq m$, $m \in \mathbb{F}_2$, $m+4k \in \mathbb{F}_2$ then $3m+4k \in \mathbb{F}_2$.

**Proof:** Let us divide $3m+4k$ persons into three groups, $A, B, C$, $|A|=m+4k$, $|B|=|C|=m$, and let us organize the conversations in each group. After that we make $m/4 - k$ new groups each containing 12 persons, and $4k$ groups containing four persons: in the larger groups, there are four persons from each of $A, B, \text{ and C}$, while in the smaller ones there are 2, 1, and 1 persons from $A, B, \text{ and C}$ resp. The proof of
Lemma 1.6 implies that it is possible to organize the conversations in each new group. (See Fig. 6.)

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig6.png}
\caption{Fig. 6.}
\end{figure}

Lemma 1.10: If \(4, 8, 12, 16, 20, 24 \in F_2\) then, for all \(n\), \(4 \mid n\) implies \(n \notin F_2\).

Proof: Suppose, on the contrary. Let \(n\) be the smallest number s.t. \(4 \mid n\) and \(n \notin F_2\). Then \(n = 3m + 4k\), \(m \geq 8\), \(k = 0\) or \(1\) or \(2\). Since \(n\) is the smallest counterexample, \(m \notin F_2\) and \(m + 4k \notin F_2\), and this leads to contradiction by Lemma 1.9.

Proposition 1.11: If \(4 \mid n\) then \(n \notin F_2\).

Proof: \(4, 8, 16 \in F_2\) because each power of two is in \(F_2\). \(12 \in F_2\) and \(20 \in F_2\) by Lemmas 1.6, and 1.8. resp. \(24 \in F_2\) is a consequence of Lemma 1.9.
with \( m=8, k=0 \). Hence the proposition follows from Lemma 1.10. □

**Lemma 1.12:** (i) There is a sequence of calls among ten persons s.t. after these calls the persons can be divided into two groups satisfying the following conditions.

There are five persons in each group. Two of them knows six of the original gossips, and the other three persons know the other four original gossips. Hence these six-element and four-element subsets can be considered as JG’s.

(ii) Making an additional call either in one or in both of the groups described above, we can reach one of the following cases:

1) Two persons know all ten gossips; there are two groups containing 3 resp. 5 persons and two JG’s; these JG’s are known by 2 resp. 1 and 3 resp. 2 persons.

2) Four persons know all ten gossips; there are two groups containing 3 persons and two JG’s; these JG’s are known by 2 resp. 1 persons.

**Proof:** The persons are \( a_1, a_2, \ldots, a_{10} \). A possible sequence of calls is

\[ a_1a_2, a_3a_4, a_1a_3, a_2a_4, a_5a_6, a_7a_8, a_5a_7, a_6a_8, a_9a_{10}, a_1a_9, a_5a_{10}. \]

In the first group there will be \( a_1, a_5, a_7, a_8, a_9 \), the JG’s are \( \{1, 2, 3, 4, 9, 10\} \) and \( \{5, 6, 7, 8\} \); in the second group there will be \( a_2, a_3, a_4, a_5, a_{10} \), the JG’s are \( \{5, 6, 7, 8, 9, 10\} \) and \( \{1, 2, 3, 4\} \). (See Fig. 7.)
Lemma 1.13: If \( 22 \equiv n \equiv 46 \) (mod 4) then \( n \in F_2 \).

Proof: The organization of the conversations can be read from the tables of Fig. 6. First we divide the persons into three or four groups given by the rows, we organize the conversations by Proposition 1.11, or apply Lemma 1.12(ii) in each group, finally we make new groups according to the columns. Lemma 1.7 is applied in one of the new groups, while in the other groups the organization of the conversations is trivial.

Lemma 1.14: \( 50,54,58,62 \in F_2 \).

Proof: The organization of the conversations can be read from the tables of Fig. 9. First, we divide the persons into four groups given by the rows, we organize the conversations in each group by Proposition 1.11, and Lemma 1.13, after that we make new groups given by the columns.
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Fig. 8.
Proposition 1.15: If \( n \equiv 2 \pmod{4} \), \( n \geq 22 \) then \( n \in F_2 \).

Proof: Suppose that the proposition is not true, and let \( n \) be the smallest number s.t. \( n=4k+2, k \geq 5, n \not\in F_2 \). Because of Lemmas 1.13. and 1.14, \( n \geq 66 \), so \( n \) can be written in the form \( n=m+16s, m=50 \) or 54 or 58 or 62, \( s \geq 1 \). If, for example, \( m=50 \) then we divide the persons into four groups of size 8+4s, 8+4s, 12+4s, 22+4s, resp. We organize the conversations in each group (it is possible by Proposition 1.11. and the fact that \( n \) is the smallest counterexample), and after that we...
make two new groups: the first one contains 50 persons, 6, 8, 12, and 22 resp. from the original groups, while the second one contains 16s persons, 4s from each of the original groups. By the proof of Lemma 1.14, \( n \not\in F_2 \), we obtain contradiction.

**Proposition 1.16**: \( 6, 10, 14, 18 \not\in F_2 \).

**Proof**: Suppose, on the contrary, that the conversations can be organized. By Lemma 1.4(ii), after the first calls we obtain \( n/2 \) JG's, each of them is known by two persons. We call these JG's the level 1 JG's. By Lemma 1.4(i), the last conversations give a perfect matching, too. We examine how many of the level 1 JG's do the persons know before their last conversations. There cannot be a person knowing \( n/2 - 1 \) level 1 JG's before his last call: if there were any then his second from the last partner would know the same \( n/2 - 1 \) level 1 JG's. The two persons who know the only missing level 1 JG from the very beginning could speak only with them, so this missing level 1 JG would not be known by \( n - 4 \) persons.

Now we are done in the case \( n = 6 \): the last conversations could be only between persons knowing 2 and 1 level 1 JG's resp., contradicting the previous paragraph.

If \( 10 \not\in F_2 \) then there is no person knowing four level 1 JG's before his last call, so everybody knows 2 or 3 level 1 JG's. There are five persons knowing three level 1 JG's. Picking one of them, his second from the last partner must be among these five persons, too, and their
phone call must be second from the last conversation for both of them. So we obtain a perfect matching among five persons, and this contradiction proves \(10 \notin F_2\).

If \(14 \notin F_2\) then there is no person knowing 6 level 1 JG's before his last call. The idea described in the previous paragraph shows that it is impossible that everybody knows 3 or 4 level 1 JG's before the last calls, because the second from the last calls would give a perfect matching among the seven persons knowing 4 level 1 JG's. Using the same idea, there are even number of pairs knowing 2 resp. 5 level 1 JG's before their last call, therefore there are odd number of pairs knowing 3 resp. 4 level 1 JG's. Hence there is a person, \(a\), who knows 4 level 1 JG's, e.g. \(\{1,2,3,4\}\), and his second from the last partner, \(b\), speaks to \(c\) after \(a\), s.t. \(c\) knows only one level 1 JG before this call, e.g. 5. The last partners of \(b\) and \(c\) are \(d\) and \(e\), they know \(\{6,7\}\), and we can suppose that their second conversation was the call \((d,e)\). Let \(f,g,h\) be the three persons knowing originally 5,6,7, and satisfying \(\{c,d,e\} \cap \{f,g,h\} = \emptyset\). The last partner of \(a\) knows \(\{5,6,7\}\) before their conversation, and, since \(f,g,h\) play symmetric roles, we may suppose that \(f\) and \(g\) speak to each other, after that \(f\) talks with \(h\), and \(f\) is the last partner of \(a\). Then, when \(g\) hears \(h\)'s original level 1 JG, he hears \(f\)'s level 1 JG a second time. (See Fig. 10.)
The case $n=18$ is similar to the previous one. Supposing $18 \in F_2$, the same idea shows that there is no person knowing 8 level 1 JG's before his last conversation, and there are even number of pairs knowing 2 resp. 7 level 1 JG's. It can be easily seen that there are even number of pairs knowing 3 resp. 6 level 1 JG's, so there is a person who knows 5 level 1 JG's before his last call, and his second from the last partner's next conversation is a call with a person knowing 1 or 2 JG's. An argument similar to the case $n=14$ leads to a contradiction. Details are omitted. □

Proof of Theorem 1.2: Propositions 1.5, 1.11, 1.15, and 1.16. give the proof. □
§2. Terminology and preliminary results

For \( n \in F_2 \), denote \( f(n) \) the minimal number of calls among \( n \) persons when everybody hears each gossip exactly once. In this section, we introduce the necessary terminology, and prove some results to be used in the following sections.

A calling scheme among \( n \) persons can be represented by a graph \( G \) on \( n \) points whose edges are linearly ordered according to the order of calls. Let \( V \) and \( E \) denote the vertex-set and edge-set of \( G \), resp. Using the notation \( m = |E| \), \( T \) can be written as a sequence \( T = \langle e_1, e_2, \ldots, e_m \rangle \) where \( e_i \in E \) for each \( 1 \leq i \leq m \). In what follows, "vertex" and "person" as well as "edge" and "call" are interchangable, and sometimes we use expressions like "the vertex \( x \) hears a gossip". Throughout §§2-4, the letters \( a, b, c, d, x, y, z, u \) will always denote elements of \( V \), while \( e \) stands for edges of \( G \). The edge between the vertices \( x, y \) will be denoted by \( (x, y) \). \( g(z) \) is the gossip originally known by \( z \), and, for \( A \subset V \), \( g(A) = \{ g(z) : z \in A \} \). "\( g(x) \) is transmitted by \( e \)" means that one of the endpoints of \( e \) knows \( g(x) \) before the call \( e \). «...» denotes sequences.

The ordering on \( T \) can be used to introduce a relation \( a \Rightarrow b \). \( a \Rightarrow b \) holds iff there exists a path \( a = x_0, x_1, \ldots, x_j = b \) s.t. « \( (x_i, x_{i+1}) : 0 \leq i \leq j-1 \) » is an
increasing subsequence of $T$. We are interested in calling schemes where everybody hears each information exactly once; i.e. for every $a, b \in V$, there exists exactly one increasing path from $a$ to $b$. (It is easy to see that this condition implies that there are no increasing paths from $a$ to $a$, for each $a \in V$.) Let us call such a sequence $T$ to be a good scheme, and denote by $P(a, b)$ the unique increasing path from $a$ to $b$. Clearly, a graph generated by a good scheme does not contain loops or multiple edges.

We shall frequently use the following lemma:

**Lemma 2.1:** If $T = \langle e_1, e_2, \ldots, e_m \rangle$ is a good scheme then the reverse ordering $T' = \langle e_m, e_{m-1}, \ldots, e_1 \rangle$ is a good scheme.

**Proof:** For each $x, y \in V$, there exists a unique increasing path from $y$ to $x$ in $T$. The reverse of this path will be the only increasing path from $x$ to $y$ in $T'$. □

Although $T$ gives a linear ordering of calls, some of these calls are interchangeable without changing any of the increasing paths in $G$. More precisely, we can introduce a partial order $<_T$ generated by $T$, namely the transitive closure of the relation $\{ e_i, e_j \}: i < j$ and $e_i, e_j$ have a common endpoint.

**Lemma 2.2:** (i) $e_i <_T e_j$ if and only if there exists an increasing path $P = \langle e_i, e_{i_2}, \ldots, e_{i(k)} \rangle$ s.t. $e_i = e_{i_1}$ and $e_j = e_{i(k)}$.

(ii) If the sets of information transmitted by $e_i$ and $e_j$
are disjoint or equal then $e_i$ and $e_j$ are incomparable in $\prec_T$.

**Proof:** (i) Clearly, if $P$ exists then $e_i \prec_T e_j$. Conversely, if $e_i \prec_T e_j$ then there exists a sequence of pairs $P' = \langle (e_i, e_j), (e_j, e_j), \ldots, (e_j, e_j) \rangle$ s.t. the edges in each pair have a common endpoint, and we can choose $P$ as an appropriate subsequence of $P'$.

(ii) It is a trivial consequence of the first part of the lemma. □

**Lemma 2.3:** Let $T = \langle e_1, e_2, \ldots, e_m \rangle$ be a calling scheme, $\sigma$ a permutation of \{1, 2, ..., m\}, and $T(\sigma) = \langle h_1, \ldots, h_m \rangle$ the calling scheme generated by $\sigma$, i.e. $h_i = e_{\sigma(i)}$ for $i = 1, 2, \ldots, m$.

(i) If $\sigma$ satisfies the property

$$e_i \prec_T e_j \Rightarrow \sigma^{-1}(i) \prec \sigma^{-1}(j)$$

then the increasing paths are the same in $T$ and $T(\sigma)$. In particular, if $T$ is a good scheme then $T(\sigma)$ is a good scheme.

(ii) If $e_i$ and $e_j$ are incomparable in $\prec_T$ then there exists a permutation $\sigma$ satisfying (2.3) s.t. $|\sigma^{-1}(i) - \sigma^{-1}(j)| = 1$, i.e. $e_i$ and $e_j$ are consecutive edges in $T(\sigma)$.

**Proof:** (i) Let $P = \langle e_{i_1}, e_{i_2}, \ldots, e_{i_k} \rangle$ be an increasing path in $T$. Then $e_{i_1} \prec_T e_{i_2} \prec_T \ldots \prec_T e_{i_k}$, hence $\sigma^{-1}(i_1) < \sigma^{-1}(i_2) < \ldots < \sigma^{-1}(i_k)$. The edge $e_{i(j)}$ appears as $h_{\sigma^{-1}(i(j))}$ in $T(\sigma)$, so $P = \langle h_{\sigma^{-1}(i_1)}, \ldots, h_{\sigma^{-1}(i_k)} \rangle$ is an increasing path in $T(\sigma)$. Conversely, let $P' = \langle h_{i_1}, h_{i_2}, \ldots, h_{i_k} \rangle$ be
an increasing path in $T(\sigma)$. $h_{\sigma(i(j))} = e_{\sigma(i(j))}$, and, since $e_{\sigma(i(j))}$ and $e_{\sigma(i(j+1))}$ are adjacent, they are compared in $<_T$ for $j = 1, 2, \ldots, k-1$.

Thus the ordering of $\sigma(i(j))$ and $\sigma(i(j+1))$ is the same as the ordering of $\sigma^{-1}(\sigma(i(j))) = i(j)$ and $\sigma^{-1}(\sigma(i(j+1))) = i(j+1)$. $P' = \langle e_{\sigma(i(1))}, \ldots, e_{\sigma(i(k))} \rangle$ is an increasing path in $T$.

(ii) We can suppose that $i < j$. Denote by $S = \langle e_{s_1}, e_{s_2}, \ldots, e_{s(k)} \rangle$ the subsequence of edges from $T$ where $i < s_1 < s_2 < \ldots < s(k) < j$ and for all $1 < s < j$, $e_s S$ if and only if $e_i S$. Notice that $e_{s} S, e_{s} T e_{r}, r < j$ imply $e_{r} S$.

Denote by $Q = \langle e_{q_1}, e_{q_2}, \ldots, e_{q(j-1-k)} \rangle$ the subsequence of the remaining edges between $e_i$ and $e_j$.

Let

$$\sigma^{-1}(r) = \begin{cases} 
  r, & \text{if } r \leq i - 1 \\
  i - 1 + p, & \text{if } r = q_p \\
  (i - 1) + (j - i - k) + 1 = j - k - 1, & \text{if } r = j \\
  j - k, & \text{if } r = i \\
  j - k + p, & \text{if } r = s_p \\
  r, & \text{if } r > j
\end{cases}$$

This means that in the new scheme $T(\sigma) = \langle h_1, \ldots, h_m \rangle$ we perform the first $i - 1$ calls of $T$, then the calls from $Q, \{e_j\}, \{e_i\}, S$ in that order, and finally the last $m - j$ calls of $T$. $\sigma$ satisfies the property (2.3): $e_j$ is
Incomparable in \( \prec_T \) with all \( e_s \in S \), since \( e_i \) and \( e_j \) are incomparable; furthermore, \( e_s \prec_T e_q \) or \( e_l \prec_T e_q \) are impossible for any \( e_s \in S \) and \( e_q \in Q \). Since \( \sigma^{-1} \) changes only the ordering of the edges mentioned above, \( \sigma \) satisfies (2.3).

For the remaining part of the section, let \( T \) be a good scheme. If we consider all edges containing a fixed vertex, \( \prec_T \) gives a linear ordering of them. The edge \((x,y)\) is a first-edge, second-edge, last-edge, second-from-the-last-edge (or sftl-edge) for \( x \) if it is first, second, last, or second from the last in the linear ordering at \( x \). A quadruple \((x_1,x_2,x_3,x_4)\) is a beginning quadruple (BQ) if \((x_1,x_2)\) and \((x_3,x_4)\) are first-edges for \( x_1 \) and \( x_2 \), \( x_3 \) and \( x_4 \), resp., and \((x_1,x_3),(x_2,x_4)\) are second-edges for \( x_1 \) and \( x_3 \), \( x_2 \) and \( x_4 \), resp. A quadruple is a finishing quadruple (FQ) if it is a BQ in the reverse ordering of \( T \).

We proved in Lemma 1.4. that both the first-edges and the last-edges give a perfect matching of \( G \). Clearly, if \( n>2 \) then these matchings are disjoint, hence each vertex of \( G \) has degree at least two.

**Lemma 2.4:** If \( n>4 \) then each vertex has degree at least three.

**Proof:** Suppose, on the contrary, that \((x,y) \prec_T (x,z)\) are the only edges adjacent to \( x \). Let \((z,w)\) be the sftl-edge for \( z \). After the call \((z,w)\) \( w \) knows all of the gossips except \( g(x) \) and \( g(y) \), thus the only possibility is \((y,w)\) \( \in E \) and \( y \) is a vertex of degree two. Since \( n>4 \), there is a vertex \( u \) distinct from \( x,y,z,w \), and there cannot be an increasing path from \( x \)
For a fixed \( x \in V \), we can define the output of \( x \) as
\[
O(x) = \{ e \in E : g(x) \text{ is transmitted through } e \}.
\]
Equivalently, \( e \in O(x) \) iff there exists an increasing path from \( x \) whose last edge is \( e \). The input of \( x \) is defined as
\[
I(x) = \{ e \in E : \text{if } e \text{ is deleted from } T \text{ then } x \text{ does not hear all the items of information} \}.
\]
An alternative description of \( I(x) \) is that it consists of the edges belonging to \( O(x) \) in the reverse ordering; \( e \in I(x) \) iff there exists an increasing path to \( x \) whose first edge is \( e \).

**Lemma 2.5:**  
(i) \( I(x) \) and \( O(x) \) are trees  
(ii) \( I(x) \) contains all of the first-edges, \( O(x) \) contains all of the last-edges, and both trees have \( n/2 \) points of degree one  
(iii) \( I(x) \cap O(x) = \{ e \in E : e \text{ is adjacent to } x \} \)

**Proof:** We prove (i) and (ii) only for \( O(x) \), the statements concerning \( I(x) \) follow from Lemma 2.1.  
(i) Since \( x \leadsto y \) holds for all \( y \in V \), \( O(x) \) consists of one component. If the edges \( e_{i_1}e_{i_2}...e_{i(k)} \) form a circle in \( O(x) \), where \( i_1 < i_2 < ... < i(k) \), then there exist increasing paths from \( x \) to both endpoints of \( e_{i(k)} \) s.t. the last edges of these paths are \( e_{i(j)} \) for some \( j < k \). Hence at the call \( e_{i(k)} \) the endpoints of \( e_{i(k)} \) hear \( g(x) \) a second time.  
(ii) Clearly, \( g(x) \) is transmitted through all of the last-edges. The endpoint of each last-edge who did not know \( g(x) \) earlier is a vertex of degree one in \( O(x) \).
(iii) Suppose $e \in I(x) \cap O(x)$. Then $e$ is the last edge of some increasing path starting at $x$ and the first edge of a path terminating at $x$. Joining these two paths, and deleting $e$ if the two paths meet at the same endpoint of $e$, we obtain an increasing path from $x$ to $x$, a contradiction unless both paths contained only the edge $e$. In this case $e$ is adjacent to $x$. \hfill \Box

**Corollary 2.6:** \[ |I(x) \cup O(x)| = 2n - 2 \text{-degree}(x). \] \hfill \Box

**Remark 2.7:** The results obtained so far enable us to sketch a new proof for the original gossip problem; namely, if $S$ is a calling scheme satisfying $a \ast b$ for all $a, b \in V$ then $|S| \geq 2n - 4$. Suppose that $n$ is the least counterexample and $S$ is a calling scheme with $|S| < 2n - 4$. Then, by the Lemma of Baker and Shostak [1], nobody hears his own information (NOHO property). In NOHO graphs, spanning trees $O_1(x)$ and $I_1(x)$ can be defined in $O(x)$ and $I(x)$ (see West [28]) s.t. Lemma 1.4, Lemma 2.5, (ii), (iii) and Corollary 2.6 remain valid. There exists a point of $V$ with degree at most 3, otherwise $|S| \geq 2n$. If, for some $x$, degree$(x) = 2$ then $|S| \geq 2n - 4$ by Corollary 2.6. Finally, if each degree is at least 3, and degree$(x) = 3$, then $|O_1(x) \cup I_1(x)| = 2n - 5$, and there exists a call in $S \setminus (O_1(x) \cup I_1(x))$: by Lemma 2.5 (ii), there exists $y \in V$ of degree one in both trees so one of $y$'s calls is not in $O_1(x) \cup I_1(x)$. \hfill \Box
§3. Upper bound for $f(n)$

In this section, we describe good schemes using $2.25n-c$ calls where the constant $c$ depends on the mod 4 residue class of $n$. As it was mentioned in the introduction, the construction in the case $n=4k$ was given by D. B. West [31]. We repeat it not only for the convenience of the reader but as it serves an illustration for the main theorem (Theorem 4.1. in the next section).

**Construction 1:** If $n=4k$, $k\geq 2$ then there exists a good scheme using $2.25n-6$ calls.

Let us first consider the case $k\geq 3$.

**Step 1:** We begin by dividing the people into $k$ groups of size 4. In each group, we perform four calls in a square so that everybody knows the four gossips of his group. We arrange the people into a $4\times k$ matrix s.t. the members of the $i^{th}$ group, $\{x_{j,i}: 1 \leq j \leq 4\}$, are in the $i^{th}$ column. We refer to the information known in the $i^{th}$ column (it is the union of 4 original gossips) as the $i^{th}$ joint gossip (JG).

**Step 2:** $x_{2,1}$ calls (in that order) $x_1,2,x_1,3,\ldots,x_1,k-2$ and $x_{1,k}$ calls $x_{2,k-1},x_{2,k-2},\ldots,x_{2,3}$. (In the case $k=3$ there are no calls in Step 2.)
After these calls $x_{1,i}$ ($1 \leq i \leq k-2$) knows the first $i$ JG's and $x_{2,i}$ ($3 \leq i \leq k$) knows the $i^{th}$ and greater JG's; $x_{2,1}$ knows the JG's 1 to $k-2$ and $x_{1,k}$ knows the JG's 3 to $k$.

**Step 3:** Again, we divide the people into $k$ groups of size four, namely

$$(x_{1,1}, x_{2,3}, x_{3,2}, x_{4,2})$$

$$(x_{1,2}, x_{2,4}, x_{3,3}, x_{4,3})$$

... 

$$(x_{1,k-2}, x_{2,k}, x_{3,k-1}, x_{4,k-1})$$

$$(x_{1,k-1}, x_{2,1}, x_{3,k}, x_{4,k})$$

$$(x_{1,k}, x_{2,2}, x_{3,1}, x_{4,1})$$

In each group, two persons know a certain JG while the other two persons know different information whose union covers everything except the particular JG mentioned above. Clearly, we can perform three calls in each group so that everybody hears each gossip exactly once.

We used $4k$ calls in Step 1, $2(k-3)$ calls in Step 2, and $3k$ calls in Step 3, altogether $9k-6=2.25n-6$ calls.

In the case $k=2$ we perform four calls after Step 1, pairing the persons who stand in the same row.

**Construction 2:** If $n=4k+10$, $k \geq 3$ then there exists a good scheme using $2.25n-3.5$ calls.
Step 1: We divide the people into \( k+2 \) groups, \( A_1, A_2, \ldots, A_{k+2} \), s.t.

\[
|A_1| = |A_2| = \ldots = |A_{k-1}| = 4, \quad |A_k| = 2, \quad |A_{k+1}| = 4, \quad \text{and} \quad |A_{k+2}| = 8.
\]

We perform calls in each group so that everybody knows each gossip represented in the group. The members of \( A_1 \) are \( x_{j,1} \) (\( 1 \leq j \leq |A_1| \)). As before, we refer to their commonly known information as the \( i \)th JG.

Step 2: \( x_{2,1} \) calls (in that order) \( x_{1,2}, x_{1,3}, \ldots, x_{1,k} \). On the other hand, we perform \( (x_{2,k}, x_{2,k+1}) \), and after that \( x_{1,k+2} \) calls

\( x_{2,k}, x_{2,k-1}, \ldots, x_{2,3} \). After these calls \( x_{1,i} \) (\( 1 \leq i \leq k \)) knows the first \( i \) JG's, \( x_{2,i} \) (\( 3 \leq i \leq k \)) knows the \( i \)th and greater JG's, \( x_{2,k+1} \) knows the \( k \)th and \((k+1)\)st JG's, \( x_{2,1} \) knows the JG's 1 to \( k \), and \( x_{1,k+2} \) knows the JG's 3 to \( k+2 \).

Step 3: Again, we divide the people into \( k+2 \) groups, namely

\[
(x_{1,1}, x_{2,3}, x_{3,2}, x_{4,2})
\]
\[
(x_{1,2}, x_{2,4}, x_{3,3}, x_{4,3})
\]
\[
\ldots
\]
\[
(x_{1,k-3}, x_{2,k-1}, x_{3,k-2}, x_{4,k-2})
\]

(If \( k=3 \) then the groups above do not exist.)

\[
(x_{1,k-2}, x_{2,k+1}, x_{3,k-1}, x_{4,k-1}, x_{5,k+2}, x_{6,k+2}, x_{7,k+2}, x_{8,k+2})
\]
\[
(x_{1,k-1}, x_{2,k})
\]
In the groups of size four the situation is the same as in Step 3 of Construction 1. In the group of size eight first we perform the calls 

\[(x_1, k-2, x_2, k+1), (x_1, k-2, x_3, k-1), (x_2, k+1, x_4, k-1),\] 

and four additional calls pairing these four members with the persons originated from \(A_{k+2}\).

Step 1 requires \(4(k-1)+1+4+12=4k+13\) calls, Step 2 requires \((k-1)+1+(k-2)=2k-2\), while Step 3 requires \(3(k-3)+7+1+3+3+3=3k+8\), altogether \(9k+19=2.25n-3.5\) calls.
Construction 1 in the case \( n = 24 \). The table shows the JG's known after Step 2.

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Construction 2 in the case \( n = 26 \). The table shows the JG's known after Step 2.

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Fig. 11.
§4. Lower bound for f(n)

In this section we are going to prove that 2.25n - c calls are necessary in a good scheme. We use two different methods. One of them is the enumeration of calls: by Lemma 1.4, we have n/2 first-edges and n/2 last-edges, and all of these edges are different. Continuing this enumeration, we count the second-edges, sftl-edges, certain third-edges, etc., taking care not to count an edge twice. The other method is to choose a vertex x with small degree, and examine I(x) and O(x). By Corollary 2.6, there are approximately 2n edges in I(x) \cup O(x), and we try to find n/4 edges not belonging to I(x) \cup O(x). A combination of the two approaches will provide the final result.

Theorem 4.1: If T is a good scheme then \(|T| \geq 2.25n - 6\).

Proof: We proceed by induction on n. In the case n \leq 8, 2n - 4 \geq 2.25n - 6 so there is nothing to prove. Suppose that n is the smallest counterexample; we shall reach contradiction by a sequence of lemmas.

Lemma 4.2: (i) Suppose that \{a,b,c,d\} is a beginning quadruple (for the definition, see §2), and x \in V \setminus \{a,b,c,d\}. Then one of the calls among \{a,b,c,d\} is in E \setminus (I(x) \cup O(x)).
(ii) The same result holds if \( \{ a, b, c, d \} \) is a FQ.

**Proof:** (i) The edges of a BQ form a four-cycle \( C \). \( g(x) \) is not transmitted in \( C \), so \( O(x) \cap C = \emptyset \). Since \( I(x) \) is a tree, \( I(x) \) cannot contain \( C \), so one of the edges of \( C \) is not in \( I(x) \cup O(x) \).

(ii) By Lemma 2.1. \( \square \)

Because of the second method mentioned in the introduction of the section, we should like to have as many BQ's as possible. The next lemma enables us to change an "almost-quadraple" into a BQ:

**Lemma 4.3:** Suppose that \( (a, b) \) and \( (c, d) \) are first-edges, \( (a, c) \) is a second-edge for both \( a \) and \( c \), but \( \{ a, b, c, d \} \) is not a BQ. Then there exists a good scheme \( T' \) satisfying the following properties:

1. \( |T'| = |T| \)
2. If \( \{ x, y, z, u \} \) is a BQ in \( T \) then it is a BQ in \( T' \)
3. The number of FQ's is the same in \( T \) and \( T' \)
4. \( \{ a, b, c, d \} \) is a BQ in \( T' \).

**Proof:** Let the second-edge for \( b \) be \( (x, b) \) and \( g(X) \) denote the set of information which \( x \) knows before the call \( (x, b) \). Similarly, let the second-edge for \( d \) be \( (y, d) \) and \( g(Y) \) denote the set of information \( y \) knows before the call \( (y, d) \). \( T = \langle e_1, \ldots, e_m \rangle \), \( e_1 = (x, b) \) and \( e_j = (y, d) \) for some \( i \) and \( j \); w.l.o.g., we can suppose that \( i < j \). Our first goal is to prove that \( e_i \) and \( e_j \) are incomparable in \( <_T \).
Claim 1: Let \( v \in V \setminus \{a,c\} \), and \( P(a,v) = \langle e_{i_1}, e_{i_2}, \ldots, e_{i(k)} \rangle \), \( P(c,v) = \langle e_{j_1}, e_{j_2}, \ldots, e_{j(l)} \rangle \) the paths showing \( a \rightarrow v \) and \( c \rightarrow v \). Then \( e_{i_1} = (a,b) \) if and only if \( e_{j_1} = (c,d) \).

Proof of Claim 1: If, for example, \( e_{i_1} = (a,b) \) and \( e_{j_1} \neq (c,d) \) then \( (a,c) \leq_T e_{j_1} \) and \( (a,c), e_{j_1}, e_{j_2}, \ldots, e_{j(l)} \) is another increasing path from \( a \) to \( v \). (If \( e_{j_1} = (a,c) \) then \( e_{j_1}, e_{j_2}, \ldots, e_{j(l)} \) is the increasing path.) \( \Box \)

In particular, in the case \( v = b \), we obtain that \( e_{j_1} = (c,d) \) in \( P(c,b) = \langle e_{j_1}, e_{j_2}, \ldots, e_{j(l)} \rangle \). Let \( e_{j(1)} = (u,b) \). Since \( (y,d) \leq_T e_{j_2} \leq_T (u,b) \), \( g(Y \cup \{d\}) \) is transmitted in \( (u,b) \) from \( u \) to \( b \). On the other hand, \( (x,b) \leq_T (u,b) \), since \( i < j < j(1) \); \( j < j(1) \) is true because in the case \( y = b \) we would get two paths from \( b \) to \( x \), if we apply Claim 1 for \( x \). Hence \( g(X \cup \{b\}) \) is transmitted in \( (u,b) \) from \( b \) to \( u \). We can conclude that the sets \( X \cup \{b\} \) and \( Y \cup \{d\} \) are disjoint, in particular, \( x, y, b, d \) are four different vertices. By Lemma 2.2 (ii), \( e_i \) and \( e_j \) are incomparable in \( <_T \).

By Lemma 2.3, there exists a permutation \( \sigma \) of \( \{1,2,\ldots,m\} \) s.t. the increasing paths are the same in \( T \) and \( T(\sigma) = \langle h_1, \ldots, h_m \rangle \), moreover, \((y,d) = h_t, (x,b) = h_{t+1}\) for some \( t \).
Now we define $T' = \langle h'_1, \ldots, h'_m \rangle$. Let $h'_r = h_r$ if $r < t$. If $r \geq t$, let

$$
h'_r = \begin{cases} h_r, & \text{if } x \notin h_r \text{ and } d \notin h_r \\
(d,z), & \text{if } h_r = (x,z) \text{ for some } z \\
(x,z), & \text{if } h_r = (d,z) \text{ for some } z
\end{cases}
$$

This means that beginning from the $t$th call, we change the role of $x$ and $d$. In particular, $h'_t = (x,y)$ and $h'_{t+1} = (b,d)$.

We claim that $T'$ satisfies the requirements of the lemma. Clearly, (i) and (iv) are satisfied. If a BQ in $T$ contains neither $x$ nor $d$ then it remains a BQ in $T'$. Clearly, $d$ cannot be in a BQ in $T$, and if $x$ is in a BQ then the edges of the BQ adjacent to $x$ must precede $(x,b)$ in $T$. Hence the BQ's of $T$ remain BQ's in $T'$. Similarly, if a FQ in $T$ contains neither $x$ nor $d$ then it remains FQ in $T'$. Suppose that an FQ in $T$ contains $x$ (or $d$). By Lemma 2.4, $(x,b)$ and $(y,d)$ are not last-edges in $T$, hence both edges of the FQ adjacent to $x$ (or $d$) are changed; we obtain a FQ containing $d$ (or $x$). Thus the number of FQ's is the same in $T$ and $T'$.

All that remained to prove is that $T'$ is a good scheme. Denote by $P(u,v)$ the increasing path from $u$ to $v$ in $T(\sigma)$.

First we prove that $v \rightarrow x$, $v \rightarrow d$ holds in $T'$ for all $v \in V \setminus \{x,d\}$. Let

$P(v,x) = \langle h_{s_1}, \ldots, h_{s(v,x)} \rangle$ and $P(v,d) = \langle h_{r_1}, \ldots, h_{r(v,d)} \rangle$. If $s(v,x) < t$ then

$\langle h'_{s_1}, \ldots, h'_{s(v,x)} \rangle$ shows $v \rightarrow x$ in $T'$. If $s(v,x) \geq t$ and $r(v,d) \geq t$ then
« h'_{r_1},...,h'_{r(v,d)} » shows ν→x. Finally, if s(v,x) ≥ t and r(v,d) ≤ t then v=c. Let P(a,d) = « (a,b),h_{l_2},...,h_{l(a,d)} »; the path « (c,d),(d,b),
h_{l_2},...,h_{l(a,d)} » is non-decreasing, and shows c→x in T'. Similarly, if r(v,d) < t or both r(v,d) ≥ t and s(v,x) ≥ t then v≡d holds in T'. Let P(c,x) = « (c,d),h_{p_2},...,h_{p(c,x)} ». If s(v,x) < t and r(v,d) ≥ t then « h's_{l_1},...,h's_{(v,x)},h'_{p_2},...,h'_{p(c,x)} » shows v≡d in T'.

Now we prove that at most one path exists from ν to x,d in T'. Let « h'_{r_1},...,h'_{r(q)} » be a new path in T' (i.e. it did not exist in T(σ)) from ν to x for some ν≡c. Clearly r(q)≥t. « h_{r_1},...,h_{r(q)} » = P(v,d), since the only possibility that it is not P(v,d) is that « h_{r_1},...,h_{r(1)} » = P(v,d) with r(1) < t and « h_{r(1)+1},...,h_{r(q)} » = P(x,d) with r(1+1)≥t, and this case is excluded since ν≡c. Hence at most one new path from ν to x can occur. If s(v,x)≥t then the original path P(v,x) is destroyed; if s(v,x)<t then there cannot be a new path at all. It is true because in this case P(v,d)=« h_{s_1},...,h_{s(v,x)},(x,b),h_{l_2},...,h_{l(a,d)} » (if (x,b) occurs twice in this sequence then we delete it), so the only candidate for a new path is not a path in T'. If ν≡c, and « h'_{g_1},...,h'_{g(q)} » is a new path from c to x then h'_{g_1}=(c,d), and « h_{g_2},...,h_{g(q)} » = P(x,d), hence at most one new path occurs; furthermore, P(c,x) is destroyed.
Similarly, suppose that \( \langle h_1',...,h_f(q) \rangle \) is a new path from \( v \) to \( d \).

We distinguish two cases according to \( s(v,x) \geq t \) or \( s(v,x) < t \) in \( P(v,x) \). If \( s(v,x) \geq t \) then \( \langle h_1',...,h_f(q) \rangle = P(v,x) \), since the only possibility that it is not \( P(v,x) \) is that \( \langle h_1',...,h_f(q) \rangle = P(v,x) \) with some \( f(i) < t \) and 

\( \langle h_{f(i+1)},...,h_f(q) \rangle = P(d,x) \) with \( f(i+1) \geq t \), and this case is excluded since \( s(v,x) \geq t \). Hence at most one new path from \( v \) to \( d \) occurs. If \( v \neq c \) then \( P(v,d) \) is destroyed; if \( v = c \) then the only candidate \( \langle (c,d), h' \rangle \) is not a path in \( T' \). In the other case, when \( s(v,x) < t \), 

\( \langle h_1',...,h_f(q) \rangle \neq P(v,x) \) since \( s(v,x) < t \leq f(q) \), hence \( \langle h_1',...,h_f(q) \rangle \) is the union of \( P(v,x) \) and \( P(d,x) \) in \( T(\sigma) \), and so at most one new path occurs. Since \( s(v,x) < t \) implies \( v \neq c \), \( P(v,d) \) is destroyed.

Claim 2: Let \( v \in V \setminus \{x,d\} \), \( P(x,v) = \langle h_1',...,h_f(x,v) \rangle \) and 

\( P(d,v) = \langle h_j',...,h_j(d,v) \rangle \). Then \( i \geq t \) if and only if \( j \geq t \).

Proof of Claim 2: If \( i \geq t \) then \( \langle (a,b), (b,x), h_i',...,h_f(x,v) \rangle \) is a non-decreasing path from \( a \) to \( v \) (maybe \( (b,x) = h_i' \)). By Claim 1, \( P(c,v) \) begins with \( (c,d) \), so \( (c,d) \prec_T h_j', j \geq t \). Conversely, if \( j \geq t \) then 

\( P(c,v) = \langle (c,d), h_j',...,h_j(d,v) \rangle \), thus, from Claim 1, \( P(a,v) \) begins with \( (a,b) \). Let \( P(a,v) = \langle (a,b), h_k, ..., h_k(a,v) \rangle \). Then \( \langle (x,b), h_k, ..., h_k(a,v) \rangle \) is a non-decreasing path from \( x \) to \( v \); since \( T(\sigma) \) is good, \( h_i' = (x,b) \) or
Clearly, Claim 2 implies that « h'_{i_1} ; \ldots ; h'_{i}(x,v) » and « h'_{j_1} ; \ldots ; h'_{j}(d,v) » are paths showing x→v and d→v in T', and there cannot be other paths from x or d to v. (Claim 2 excludes the possibility that d is an inner point of P(x,v) s.t. one of the edges adjacent to d has index less than t, and the other has index ≥ t. So P(x,v) can be destroyed only at the first edge; the similar statement is true for P(d,v) and x as an inner point.)

The path « h'_{p_2} ; \ldots ; h'_{p(c,x)} » shows x→d in T' and the non-decreasing « (d,b),h'_{1_2} ; \ldots ; h'_{1(a,d)} » shows d→x; clearly, there are no other paths between x and d.

Finally, let u,v∈V\{x,d}. If P(u,v) = « h_{q_1} ; \ldots ; h_{q(u,v)} » does not go through x or d then it remains unchanged in T'. If x (or d) occurs in P(u,v) but both edges adjacent to x (or d) have indices at least t then « h'_{q_1} ; \ldots ; h'_{q(u,v)} » shows u→v in T'. If x (or d) occurs in h_{q(i)} and h_{q(i+1)} s.t. q(i)<t≤q(i+1) then, by Claim 2, « h'_{q_1} ; \ldots ; h'_{q(i)},h'_{j_1} ; \ldots ; h'_{j(d,v)} » (or « h'_{q_1} ; \ldots ; h'_{q(i)},h'_{i_1} ; \ldots ; h'_{i(x,v)} ») shows u→v. Conversely, if a new path from u to v occurs then it must have one of the types mentioned above; the different types give different conditions on the original paths, so at most one new path occurs, and the original path is destroyed in each case. That finishes the proof of Lemma 4.3. □
Definition 4.4: A good scheme $T$ is normal if

(i) for each $x \in V$, either $x$ is in a BQ or $(x,y)$, the second-edge for $x$, is not a second-edge for $y$

(ii) the condition (i) holds in the reverse ordering of $T$.

By the repeated application of Lemma 4.3, for $T$ and the reverse ordering of $T$, we can see that for all good scheme $T$ there exists a normal scheme $T^*$ satisfying $|T^*| = |T|$. Hence, from now on, we suppose that $T$ is a normal scheme.

Lemma 4.5: Suppose $(x,y)$ is a first-edge s.t. $x$ and $y$ are not in a BQ, and let $(x,a)$ and $(y,b)$ be the second-edges for $x$ and $y$, resp. Denote by $g(A)$ and $g(B)$ the set of gossips known by $a$ and $b$, before the calls $(x,a)$ and $(y,b)$. Then

(i) $g(A) \cap g(B) = \emptyset$

(ii) $(x,a)$ and $(y,b)$ are not sftl-edges.

Proof: (i) Suppose, on the contrary, that $g(z) \in [g(A) \cap g(B)]$. Wlog, we can suppose that $e_i = (a,x)$ and $e_j = (y,b)$ with $i < j$. Since $T$ is normal, $e_i$ is not second-edge for $a$, thus after $(i-1)$ calls there are at least three persons knowing $g(z)$. Let $u$ be one of them, satisfying $u \neq a$ and $u \neq b$. Then $P(x,u)$ carries $g(z)$ to $u$ for a second time, we obtain contradiction.

(ii) Suppose, for example, that $(x,a)$ is a sftl-edge. If $(x,a)$ is in the FQ \{ $x,a,u,v$ \} then, before the call $(u,v)$, exactly one of $u$ and $v$ knows $g(b)$. Since $T$ is normal, $(y,b)$ is at least the third conversation of $b$, and at
most one of the first two partners of \( b \) is in \( \{u,v\} \). Denote by \( d \) the other partner of \( b \), not being in \( \{u,v\} \). The path \( P(y,d) \) carries \( g(b) \) to \( d \) for a second time, we obtain contradiction.

If \( (x,a) \) is not in a FQ but it is a sftl-edge for, say, \( x \) then let \( (x,z) \) be the last-edge for \( x \) and \( z \), and \( (u,z) \) the sftl-edge for \( z \). After the call \( (u,z) \) \( u \) knows the set \( g(V\setminus(A \cup \{y,x\})) \) of gossips, and, since \( T \) is normal, he has at least two further conversations. Moreover, if \( (u,c) \) is the last edge in the path \( P(x,u) \) then \( (c,u) \>\ (u,z) \). If \( P(x,u) \) begins with the edge \( (x,y) \) then \( u \) hears the gossips in \( g(B) \) a second time from \( c \). If \( P(x,y) \) does not begin with \( (x,y) \) then \( u \) hears all the information missing after \( (u,z) \) at the call \( (c,u) \), hence he hears some gossips a second time at his other conversation following \( (u,z) \).

Denote \( \alpha \) the number of BQ's whose intersection with any of the FQ's contains at most one point, \( \beta \) the number of FQ's whose intersection with any of the BQ's contains at most one point, finally let \( \delta \) be the number of BQ-FQ pairs with two common points (and, consequently, with a common edge).

**Lemma 4.6:** \( \delta > 0 \).

**Proof:** The number of first-edges and last-edges are both \( n/2 \), the number of second-edges is \( 2\alpha+2\beta+(n-4\alpha-4\beta) \), and the number of sftl-edges is \( 2\beta+2\delta+(n-4\beta-4\delta) \). By the Lemmas 2.4 and 4.5, and by the assumption on \( \alpha \) and \( \beta \), there are only \( \delta \) edges in the above list which we counted twice; hence
\[ |T| = m \geq 3n - 2(\alpha + \beta) - 5\gamma \]  

(4.6.1)

On the other hand, there exists \( x \in V \) with \( \deg(x) \leq 4 \), otherwise 
\( m \geq \frac{5n}{2} \). By Corollary 2.6, \( |I(x) \cup O(x)| \geq 2n - 6 \), and, by Lemma 4.2, at least \( \alpha + \beta + \gamma - 2 \) edges are in \( E \setminus (I(x) \cup O(x)) \). Hence

\[ m \geq 2n + \alpha + \beta + \gamma - 8 \]  

(4.6.2)

Adding (4.6.1) and the double of (4.6.2), we obtain \( 3m \geq 7n - 3\gamma - 16 \), or

\[ m \geq 7n/3 - \gamma - 16/3 \]  

(4.6.3)

By our very first assumption, \( 9n/4 - 7 \geq m \), hence \( \gamma \geq n/12 + 5/3 > 0 \).

For later use (see the proof of Theorem 4.13), let us mention that the weaker assumption \( m \leq 2.25n - 5.5 \) implies \( \gamma \geq n/12 + 1/6 \), \( \gamma > 0 \). \( \square \)

We introduce some new notation. A BQ-FQ pair with a common edge is called a common hexagon (CH), and in the CH \( Y^* = \{ y_i : 1 \leq i \leq 6 \} \)

\( Y = \{ y_1, y_2, y_3, y_4 \} \) stands for the BQ, and \( \{ y_3, y_4, y_5, y_6 \} \) for the FQ.

We choose the enumeration s.t. the edges in the CH are \( (y_1, y_3) \), 
\( (y_2, y_4), (y_1, y_2), (y_3, y_4), (y_5, y_6), (y_3, y_5) \) and \( (y_4, y_6) \). (We shall see in Lemma 4.7(1) that degree(\( y_i \)) > 3 for \( i \neq 3, 4 \), hence a BQ or FQ can belong to at most one CH.) After the calls in \( Y \), the members of \( Y \) know the same set \( g(Y) \) of gossips, and we say that \( z \notin Y \) hears \( g(Y) \) from \( y_i \) if \( y_i \) is the last point of the path \( P(y_i, z) \) which is adjacent to \( Y \). In what follows, we shall mostly examine the third-edges for \( y_1, y_2 \) (i.e.
the third edge in the linear ordering when we restrict \( <_T \) to the edges
containing \( y_j \) as well as the third-from-the-last-edges (sftl-edges) of
\( y_5, y_6 \).

By Lemma 4.6, we can fix a vertex \( x \) s.t. \( x=y_3 \) for some CH. From
now on, each reference to \( I(x), O(x) \) refers to this particular \( x \). Since
degree(\( x \))=3, the proof of Lemma 4.6 shows that there are \( \alpha+\beta+\delta-1 \)
edges in \( E[I(x) \cup O(x)] \) among the second-edges in BQ's and sftl-edges
in FQ's, and
\[
m \geq (2n-5)+\alpha+\beta+\delta-1 = 2n+\alpha+\beta+\delta-6 \quad (4.6.2')
\]
Executing the same computation as in the lemma, we obtain
\[
m \geq 7n/3 - \delta - 4 \quad (4.6.3')
\]
Finally, adding (4.6.2') and (4.6.3'), we obtain \( m \geq 13n/6 - 5 \). In
particular, if \( n=12 \) then \( m \geq 21 \); if \( n=16 \) then \( m \geq 29 + 2/3 \), or, since \( m \)
is an integer, \( m \geq 30 \). Hence Theorem 4.1. holds for \( n=12 \) and \( n=16 \);
from now on, we can suppose that \( n \geq 20 \).

**Lemma 4.7:** Let \( Y^* = \{ y_i : 1 \leq i \leq 6 \} \) be a CH not containing \( x \), \((a, y_1), (b, y_2)\)
be the third-edges for \( y_1, y_2 \), finally let \( g(A) \) and \( g(B) \) be the gossips
known by \( a \) and \( b \) before these conversations. Then

(i) \( (a, y_1) \) and \( (b, y_2) \) are not last-edges

(ii) exactly one of \( (a, y_1) \) and \( (b, y_2) \) is in \( I(x) \)
(ii) Suppose that \( g(A) \cap g(B) = \emptyset \) or \( g(A) \cap g(B) = g(\{z_1, z_2\}) \), where \( (z_1, z_2) \) is a first-edge not contained in a BQ. Moreover, in the second case, \( \{y_1, y_2\} = \{y, y'\} \) and \( \{z_1, z_2\} = \{z, z'\} \) s.t. \( (y, z) \) is the third-edge for \( y \) and second-edge for \( z \); and, if \( (z', u') \) is the second-edge for \( z' \) then the third-edge for \( y' \) is \( (y', z') \) or \( (y', u') \), and this third-edge follows \( (z', u') \) immediately in the linear ordering at \( z' \) (or \( u' \)).

**Proof:** (i) Suppose that \( (y_1, a) \) is a last-edge, and let \( (a, a^*) \) be the sftl-edge for \( a \). After the call \( (a, a^*) \), \( a^* \) knows \( g(V \setminus Y) \), so \( a^* \notin \{y_5, y_6\} \).

Hence \( a^* \) hears \( g(Y) \) from \( y_2 \), and the only possibility is \( a^* = b \).

Altogether 8 persons hear \( g(Y) \), contradicting \( n > 8 \).

(ii) Obvious.

(iii) Suppose that \( g(z) \in g(A) \cap g(B) \). If \( z \in Z \), where \( Z \) is a BQ, then at least two elements of \( Z \) must hear \( g(Y) \) from \( y_3 \) and \( y_4 \), contradicting \( n > 8 \).

If \( z \in Z = \{z, z'\} \), and \( Z \) is a first-edge not contained in a BQ, then denote by \( (u, z) \) and \( (u', z') \) the second-edges for \( z, z' \). If \( (u, u') \cap \{y, y'\} = \emptyset \) then at least two of \( (u, u', z, z') \) must hear \( g(Y) \) from \( y_3 \) and \( y_4 \), a contradiction since \( y_5 \) and \( y_6 \) must speak to each other just before calling \( y_3 \) and \( y_4 \).

Hence we can assume \( u = y \) for some \( y \in \{y_1, y_2\} \). \( u' = y' \) is impossible by Lemma 4.5(i). By Lemma 4.5(ii), \( (z', u') \) is not a sftl-edge, hence at most one of \( u', z' \) can hear \( g(Y) \) from \( y_3 \) and \( y_4 \). The other element of
(u',z') as well as his all further partners can hear g(Y) only from y'.
This means that this other element must speak to y' immediately after
(z',u'), since y' knows g(z) after his third conversation. □

Lemma 4.8: For any normal scheme T, there exists a normal scheme
T' satisfying the following properties:

1. |T'| = |T|
2. For all CH Y* = (y_i: 1 ≤ i ≤ 6) in T',
   (i) (y_1,a), the third-edge for y_1, is sftl-edge for a or y_1
      (wlog we suppose that it is so for y_1)
   (ii) (y_1,a) is not contained in a FQ
   (iii) (y_1,b) is the last-edge for y_1
   (iv) (b,b*) is the sftl-edge for b
      then (v) b* ∈ {y_5,y_6} and (b,b*) is tftl-edge for b*
3. Condition (2) holds for the reverse ordering of T'.

Proof: Let T be any normal scheme, and suppose that the CH Y*
satisfies (i)-(iv) from (2). We describe a transformation how to
change T s.t. in the modified scheme the conclusion of (2) holds for
this particular CH. By the repeated application of the transformation
on T and the reverse ordering of T, we can obtain T' satisfying the
requirements of the lemma.
Let $g(A)$ be the set of gossips known by $a$ before $(y_1, a)$, and $g(B)$ the set of gossips known by $b$ before $(y_1, b)$. Then $A \cup B \cup Y = V$, and $A, B, Y$ are pairwise disjoint. Let $(c, y_2)$ be the third-edge for $y_2$ and suppose that $c$ knows the gossips $g(C)$ before this call. We distinguish three cases:

Case 1: $C \not\subseteq B$

By (ii), $b^*$ cannot speak to $a$, so $b^*$, and all of his partners after the call $(b, b^*)$ must hear $g(Y)$ from $y_3$ or $y_4$. This means exactly that (v) holds for $b^*$, we do not have to modify $T$ at all.

Case 2: $C = B$

We shall prove that this case is impossible. We have two subcases:

α) $b^* \neq c$. Then, as in Case 1, (v) holds for $b^*$. Let $(c, c^*)$ be the call of $c$ immediately preceeding $(c, y_2)$. After this call $c^*$ knows $g(B)$, and $c^*$ can hear $g(Y)$ only from $y_1$. Since after $(a, y_1)$ both $a$ and $y_1$ know $g(A \cup Y)$, $(c^*, a)$ must be a last-edge, immediately following $(a, y_1)$.

That is a contradiction, since $T$ is a normal scheme, so, by (ii), $(a, y_1)$ cannot be sft1-edge for both of its endpoints.

β) $b^* = c$. Then let $(a, a^*)$ be the call of $a$ immediately preceeding $(a, y_1)$.

Claim: $a^*$ cannot hear $g(Y)$ from $y_3$ or $y_4$.

Proof of the Claim: Indeed, after $(a, a^*)$, $a^*$ knows $g(A)$. So all of $a^*$'s
partners after the call \((a,a^*)\) should hear \(g(Y)\) from \(y_3\) or \(y_4\). This means that \((a,a^*)\) is a tftl-edge for \(a^*\), and \(a^*\)'s sftl-edge is \((a^*,b')\) with some \(b'\) who knows \(g(B)\) before \((a^*,b')\). Let \((b',b'')\) be the tftl-edge for \(b'\). After \((b',b'')\), \(b''\) knows \(g(B)\), and he must hear \(g(Y)\) from \(y_1\).

Hence \((a,b'')\) is a last-edge immediately following \((y_1,a)\), a contradiction since \(T\) is normal. (See Fig. 12.)

\[
\begin{tikzpicture}
  \node (a) at (0,0) {$a$};
  \node (b) at (-2,1) {$b$};
  \node (b*) at (-2,-1) {$b^*$};
  \node (b') at (-4,0) {$b'$};
  \node (b'') at (-4,1) {$b''$};
  \node (y1) at (-1,2) {$y_1^*$};
  \node (y2) at (-1,-2) {$y_2^*$};
  \node (y3) at (-3,0) {$y_3^*$};
  \node (y4) at (-3,-2) {$y_4^*$};

  \draw (a) -- (y1);
  \draw (b) -- (b*) -- (y2);
  \draw (b') -- (y3);
  \draw (b'') -- (y3);
  \draw (b) -- (y4);

  \node at (-0.5,1.5) {last};
  \node at (-0.5,-1.5) {last};
  \node at (-1.5,0) {last};
\end{tikzpicture}
\]

**Fig. 12.**

The only remaining possibility is that \(a^*\) hears \(g(Y)\) from \(y_2\). Then \((a,a^*)\) is a sftl-edge for \(a^*\), and the last-edge for \(a^*\) is \((a^*,b^*)\) or \((a^*,y_2)\). Suppose, for example, that \((a^*,y_2)\) is the last-edge for \(a^*\). We shall construct a good scheme on \(n-4\) points using \(m-9\) edges,
contradicting the fact that $n$ was the minimal counterexample for
Theorem 4.1.

We delete $V$ from $V$, and the edges $(y_1,y_3), (y_2,y_4), (y_1,y_2), (y_3,y_4),
(y_1,a), (y_1,b), (b^*,y_2), (a^*,y_2), (y_3,y_5), (y_4,y_6)$ from $T$. We add $(a^*,b)$ as
the very last edge of the new scheme. (See Fig. 13.) None of the
increasing paths changed in $V \setminus (Y \cup \{a^*,b\})$, and, clearly, $a^*$ and $b$ hear
each gossip exactly once.

In $T$:

In the new scheme:

Fig. 13.

Case 3: $C \cap A \neq \emptyset$

Since, by Lemma 4.5(ii), $(y_1,a)$ cannot be a second-edge, Lemma
4.7(iii) implies that there exists a first-edge $(z_1,z_2)$ not contained in
a BQ s.t. $c=z_2$, and $(z_2,y_2)$ is the second-edge for $z_2$; furthermore,
wlog we can assume that \( a^* = z_j \) and \((z_j, a)\) is the second-edge for \( z_j \). \( (a^* \) is defined as in the case 2P.) \( z_j \), and all of his partners after the call \((z_j, a)\) must hear \( g(y) \) from \( y_3 \) or \( y_4 \), so \( z_j \in \{y_5, y_6\} \), and \((z_j, a)\) is tft1-edge for \( z_j \). Let \((z_j, b')\) be the sft1-edge for \( z_j \); then \( b' \) knows \( g(B) \) before this conversation. If \( b' = b^* \) then (v) holds for \( b^* \), and we do not have to modify \( T \) at all. If \( b' \neq b^* \) then let \((b', b'')\) be the tft1-edge for \( b' \).

By Lemma 2.2(ii) and Lemma 2.3(ii), there exists a permutation \( \sigma \) s.t. \((b', b'') = h_t \) and \((b, b^*) = h_{t+1} \) in the new scheme \( T(\sigma) = \langle h_1, h_2, ..., h_m \rangle \).

Let \( T' = \langle h'_1, h'_2, ..., h'_m \rangle \) be defined by the following way: \( h'_r = h_r \) if \( r \leq t+1 \). If \( r > t+2 \) then let

\[
h'_r =
\begin{cases}
  h_r, & \text{if } b', b^* \neq h_r \\
  (u, b'), & \text{if } h_r = (u, b^*) \\
  (u, b*), & \text{if } h_r = (u, b')
\end{cases}
\]

Clearly, \( T' \) is a normal scheme and (v) holds for \( b^* \) in \( T' \).

Furthermore, the number of CH's for which (2) does not hold is less in \( T' \) than it was in \( T \). Hence, by the repeated application of the procedure described above on \( T \) and the reverse ordering of \( T \), we can obtain a normal scheme \( T' \) satisfying (2) and (3).

\( \square \)

**Definition 4.9:** A normal scheme \( T \) is regular if (2) and (3) of Lemma 4.8 hold for \( T \).
From now on, we suppose that T is a regular scheme.

**Lemma 4.10:** Suppose that \( V^* \) is a CH, \((y_1,a)\) is the third-edge for \( y_1 \), and \((y_1,a)\) is contained in a FQ. Then this FQ cannot be included in a CH.

Moreover, if \( V''^* \) and \((y'_1,a')\) are resp. another CH and third-edge with the same properties then the FQ's containing \((y_1,a)\) and \((y'_1,a')\) are different.

**Proof:** Let \((y_2,c)\) be the third-edge for \( y_2 \), and suppose that \( c \) knows the set of gossips \( g(C) \) before this call. We distinguish two cases:

**Case 1:** \( g(C) \) contains a gossip \( g(z) \) where \( z \) is an element of a BQ \( Z=\{z_1, z_2, z_3, z_4\} \). We prove that this case is impossible.

From the elements of \( Z \), at most one, say \( z_2 \), hears \( g(Y) \) from \( y_2 \) (since \( g(z) \notin g(C) \)); at most one, say \( z_1 \), hears \( g(Y) \) from \( y_3 \) or \( y_4 \) (since \( n>8 \)); hence at least two of them are in the FQ containing \( y_1 \) and \( a \). By Lemma 4.7(iii), \( a \notin Z \); moreover, \( c=z_2 \) otherwise \( z_2 \) could not hear \( g(Y) \) from \( y_2 \), and \( z_1 \)'s third-edge must be already in the FQ \( \{y_3, y_4, y_5, y_6\} \) otherwise \( z_1 \)'s third partner could not hear \( g(Y) \). Similarly, the third-edges for \( z_3, z_4 \) must be already their last-edges in the FQ \( \{y_1, a, z_3, z_4\} \).

Now we construct a good scheme on \( n-4 \) points, using \( m-9 \) edges, contradicting the fact that \( n \) was the smallest counterexample.
We delete the points $y_3, y_4, z_3, z_4$ from $V$; we delete the eight edges in the BQ's $Y$ and $Z$, and the four last-edges adjacent to $y_3, y_4, z_3, z_4$. We write $(y_1, y_2), (z_1, z_2), (y_1, z_1)$ as the very first edges of the new scheme, and after that follow the remaining $m-12$ edges of $T$, in the original order. (See Fig. 14.) The new edge $(y_1, z_1)$ ensures that $a$ hears $g(z_1), i=1,2$ as well as $y_6$ hears $g(y_1), i=1,2$; the new scheme trivially satisfies all other conditions to be a good scheme.

In $T$:

In the new scheme:

Fig. 14.

Case 2: For each gossip $g(z) \in g(C)$, $z$ is not an element of a BQ. Then $c=z_2$, where $(z_1, z_2)$ is a first-edge not contained in a BQ, and $(y_2, z_2)$ is
the second-edge for $z_2$. (It is true since $c$ cannot be in a BQ, and the normality of $T$ implies that after two conversations everybody knows a gossip contained in a BQ.) Let $(z_1, b)$ be the second-edge for $z_1$, $z_1$ and $b$ cannot hear $g(Y)$ from $y_2$, and, by Lemma 4.5(ii), they cannot be in the same FQ. Hence one of them, say $z_1$, hears $g(Y)$ from $y_1$, and the other hears $g(Y)$ from $y_3$ or $y_4$; moreover, $(z_1, b)$ is the tftl-edge for both endpoints. Now we prove that the FQ containing $y_1$ is not in a CH. Suppose, on the contrary, that this FQ is $F = \{y_j, z_j, d_3, d_4\}$ for some CH $D^* = \{d_i: 1 \leq i \leq 6\}$. Then $d_1$ or $d_2$, say $d_1$, must be the fourth member in the FQ containing $y_3$, $y_4$ and $b$, and the third-edge for $d_1$ is $(d_1, b)$. If $d_2$ hears a gossip contained in a BQ at his third conversation then we obtain contradiction by Case 1 of this lemma. If the third-edge for $d_2$ is $(d_2, v_2)$ where $(v_1, v_2)$ is a first-edge not contained in a BQ then $(v_1, v_2) = (z_1, z_2)$. Moreover, by the already proven part of Case 2, $v_1$ must be in the FQ containing $d_1$ or in the FQ containing $d_3$ and $d_4$. Hence $v_1 = b$, $T$ is a not normal scheme on 12 points, a contradiction. (See Fig. 15.)
Finally we prove that the FQ containing $y_1$ is not assigned to other CH's. Suppose that $F = \{ z_1, y_1, z'_1, y'_1 \}$ where $(y'_i: 1 \leq i \leq 6)$, $z'_1$, $z'_2$, $b'$ satisfy the same conditions as the vertices $y_i$, $z_i$, $b$. We construct a good scheme on $n-4$ points using $m-10$ edges, contradicting the fact that $n$ was the smallest counterexample. We delete $z_1$, $y_1$, $z'_1$, $y'_1$ from $V$, and the following 12 edges from $T$: $(y_1, y_3)$, $(y_1, y_2)$, $(y'_1, y'_3)$, $(y'_1, y'_2)$, the four edges in $F$, $(b, z_1)$, $(z_1, z_2)$, $(b', z'_1)$, $(z'_1, z'_2)$. We add the edges $(z'_2, y'_3)$, $(z_2, y_3)$ as the very first two edges of the new scheme, and after that follow the remaining $m-12$ edges of $T$, in the original order. Clearly, we obtain a good scheme. (See Fig. 16.)
Lemma 4.11: Let T be a regular scheme, and let $Y^*, a, b, b^*$ satisfy the conditions (i)-(v) from Lemma 4.8. Then

1) $(b, b^*)$ is not tftl-edge for $b$
2) $(b, b^*)$ is not a second-edge
3) It is impossible that $b \in (s_1, s_2)$, $b^* \in (u_1, u_2)$ for some CH's $S^*$, $U^*$ and $(b, b^*)$ is third-edge for both endpoints.

Proof: 1) $(b, b^*)$ is sftl-edge for $b$.
2) Since $(b, b^*)$ is sftl-edge for one of its endpoints, and $(b, b^*)$ is not contained in a FQ, it cannot be a second-edge by Lemma 4.5(ii).
3) Suppose that $b = u_1$, $b^* = s_1$, and $(u_1, s_1)$ is third-edge for both endpoints. After the call $(y_1, a)$, $a$ knows the gossips $g(V \setminus (U \cup S))$, and $a$ has at least two further conversations. Hence $a$ must hear one of
g(U) and g(S) at his sftl-call, and the other at his last call. We can
suppose that he hears g(U) at his last call. Then the sftl-edge for a
must be (a,s_2), since a cannot hear g(S) from s_1, s_3, or s_4. Moreover,
(a,s_2) is the third-edge for s_2. By Lemma 4.7(1), the last-edge for a
cannot be (u_2,a), hence a hears g(U) from u_3 or u_4. Thus (a,s_2,u_3,u_4) is
a FQ, contradicting Lemma 4.10. □

Lemma 4.12: Let Y*, Z*, P*, Q* be CH's, y_1=q_5, z_1=p_5, and (y_1,z_1)\in E.
Then it is impossible that (y_1,z_1) is third-edge for both endpoints and
tftl-edge for both endpoints in the same time.

Proof: As usually, denote Y, Z, P, Q the BQ of the appropriate CH.
Suppose that the statement of the lemma does not hold, and let (a,p_6),
(b,q_6) be the tftl-edges for p_6, q_6 resp. After these calls a knows the
gossips g(A)=g(V\setminus (P \cup Y \cup Z)) and b knows the gossips
g(B)=g(V\setminus (Q \cup Y \cup Z)). A \cap B\neq \emptyset since n\geq 20 (see the discussion before
Lemma 4.7).

We examine that who can be the third partner of y_2, z_2. By Lemma
4.7(III), y_2 hears a part of g(V\setminus Z) at his third conversation; hence at
most one of a, b can hear g(Y) from y_2. (It is true since A \cap B \neq \emptyset and
A \cup B=V\setminus (Y \cup Z).) Thus one of a, b must hear g(Y) from y_3 or y_4; wlog
we can assume that b is in the FQ of y₃ and y₄. If (b,q₆) is not tftl-edge for b then c, the fourth person in this FQ, must know only g(Z) or g(Q) after his tftl-call, and this fact contradicts Lemma 4.10. Hence (b,q₆) is the tftl-edge for b, and c knows g(Q ∪ Z) after his tftl-call.

This implies that the third-edge for z₂ is (z₂,q₁) or (z₂,q₂), and this edge is third-edge for both endpoints. Wlog we can assume that (z₂,q₂) is this third-edge, and c=z₂. Similarly, a must hear g(Z) from z₃ or z₄, (a,p₆) is tftl-edge for a, and we can assume that (p₂,y₂) is third-edge for both endpoints, y₂∈{z₅,z₆}.

q₁ cannot hear g(Z) from z₂, hence q₁=p₆ or q₁=a. By Lemma 4.10, (q₁,u), the third-edge for q₁, cannot be in a FQ. u cannot hear g(Z) from z₂ either, hence u=p₆ or u=a, and before the call (q₁,u) u knows g(V \ (P ∪ Q ∪ Y ∪ Z)). Similarly, the third-edge for p₁ is (p₁,v) where (p₁,v)=(q₆,b), and before the call (p₁,v) v knows g(V \ (P ∪ Q ∪ Y ∪ Z)).

We construct a new scheme on n−12 points, using m−28 edges, contradicting the fact that n was the smallest counterexample. We delete the points y₁, z₁, p₁, qᵢ (i=1,3,4) from V, and the following 31 edges from T: 28 edges in the CH's Y*, Z*, P*, Q*, and (y₁,z₁), (v,p₁),
\((u,q_1)\). The new scheme consists of the remaining \(m-31\) edges of \(T\) in the original order, and \((y_2,z_2),(v,y_2),(u,z_2)\) as the very last three edges. Clearly, we obtain a good scheme. (See Fig. 17.)

\[
\begin{aligned}
\text{In } T: & & \\
& & \\
\text{In the new scheme:} & & \\
& & \\
\end{aligned}
\]

(The different marks around the vertices show the FQ's.)

**Proof of theorem 4.1:** Let \(\Gamma\) be the set of CH's not containing \(x\); then \(|\Gamma| = 8 - 1\). Let \(Y \in \Gamma\), and consider the third-edges for \(y_1, y_2\) as well as the tftl-edges for \(y_5, y_6\). Using Lemma 4.7(11) in \(T\) and the reverse ordering of \(T\), we obtain that exactly one of these third-edges is not in \(I(x)\), and exactly one of these tftl-edges is not in \(O(x)\). Denote \(e(Y^*)\) this third-edge, \(e(Y^*) \notin I(x)\), and \(f(Y^*)\) the tftl-edge \(f(Y^*) \notin O(x)\). By Lemmas 2.4. and 4.7(i), \(e(Y^*)\) and \(f(Y^*)\) are not first-edges and last-
edges. Moreover, \( e(Y^*) \) is not second-edge in a BQ, and \( f(Y^*) \) is not sftl-edge in a FQ.

Let \( \Gamma_1 = \{ Y^* \in \Gamma : e(Y^*) \in O(x), e(Y^*) \) is a sftl-edge for one of its endpoints, \( e(Y^*) \) is not in a FQ \} \) and \( \Gamma'_1 = \{ Y^* \in \Gamma : f(Y^*) \in I(x), f(Y^*) \) is a second-edge for one of its endpoints, \( f(Y^*) \) is not in a BQ \} \). 

If \( Y^* \in \Gamma_1 \) and, using the notation of Lemmas 4.8. and 4.11, \( e(Y^*) = (y_1, a) \), then \( g(x) \) is not transmitted by \( (b, b^*) \), since the gossips transmitted by \( (y_1, a) \) and \( (b, b^*) \) are disjoint. Hence \( f(Y^*) = (b, b^*) \). By Lemma 4.11.2), \( f(Y^*) \) is not a second-edge, hence \( \Gamma_1 \cap \Gamma'_1 = \emptyset \). Moreover, if \( Y^* \in \Gamma_1 \) then \( f(Y^*) \notin I(x) \) (thus \( f(Y^*) \notin I(x) \cup O(x) \)), and, by Lemma 4.11.1.3), \( f(Y^*) = f(Z^*) \) for \( Z \in \Gamma, Z \neq Y^* \), and \( f(Y^*) = e(U^*) \) for at most one \( U^* \in \Gamma \). Similarly, applying Lemma 4.11 for the reverse ordering of \( T \), we obtain that \( Y^* \in \Gamma'_1 \) implies \( e(Y^*) \notin I(x) \cup O(x), e(Y^*) = e(Z^*) \) for \( Z^* \in \Gamma \setminus \{Y\} \), and \( e(Y^*) = f(U^*) \) for at most one \( U^* \in \Gamma \). \( 4.1.1 \)

Let \( |\Gamma_1 \cup \Gamma'_1| = 0 \). We define two partitions of \( \Gamma \setminus (\Gamma_1 \cup \Gamma'_1) \) as follows:

\( \Gamma_2 = \{ Y^* : e(Y^*) \in O(x), e(Y^*) \) is not a sftl-edge \} \)

\( \Gamma_3 = \{ Y^* : e(Y^*) \) is contained in a FQ \} \)
\[ \Gamma_4 = \{ Y* : e(Y*) \notin O(x) \text{ and } e(Y*) \text{ is not contained in a } FQ \} \]
\[ \Gamma'_2 = \{ Y* : f(Y*) \notin I(x), f(Y*) \text{ is not a second-edge} \} \]
\[ \Gamma'_3 = \{ Y* : f(Y*) \text{ is contained in a } BQ \} \]
\[ \Gamma'_4 = \{ Y* : f(Y*) \notin I(x) \text{ and } f(Y*) \text{ is not contained in a } BQ \} \]

Clearly, the sets \( \Gamma_2, \Gamma_3, \Gamma_4 \) as well as \( \Gamma'_2, \Gamma'_3, \Gamma'_4 \) are pairwise disjoint, and \( \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 = \Gamma'_2 \cup \Gamma'_3 \cup \Gamma'_4 = \Gamma \setminus (\Gamma_1 \cup \Gamma'_1) \). Let \( |\Gamma_1| = \beta_1, |\Gamma'_1| = \beta'_1 \).

If \( Y* \notin \Gamma_2 \) then \( e(Y*) \) is not a second-edge, otherwise \( g(x) \) could not be transmitted by \( e(Y*) \). Because of the same reason, \( e(Y*) \neq e(Z*) \) for \( Y*, Z* \notin \Gamma_2, Y* \neq Z* \). Similarly, \( f(Y*) \neq f(Z*) \) for \( Y*, Z* \notin \Gamma'_2, Y* \neq Z* \), and \( f(Y*) \) is not a sfl-edge. By the formula (4.6.1), there are \( 3n-2(\alpha+\beta)-5\delta \) first-, last-, second-, and sfl-edges in \( T \); hence the total number of edges is at least

\[ m \geq 3n-2(\alpha+\beta)-5\delta+\beta_2+\beta'_2 \]  

(4.1.2)

By Lemma 4.10, \( \beta \geq \delta_3 \) and \( \alpha \geq \delta'_3 \). For each \( Y* \notin \Gamma_4, e(Y*) \notin I(x) \cup O(x), \) and \( Y* \notin \Gamma'_4 \) implies \( f(Y*) \notin I(x) \cup O(x). \) By Lemma 4.12, the edges in \( E \setminus (I(x) \cup O(x)) \) occur at most three times as \( e(Y*), f(Y*) \) for some \( Y* \notin \Gamma_4 \) or \( Y* \notin \Gamma'_4 \). Moreover, by (4.1.1), if an edge occurs as \( f(Y*) \) for
some $Y^* \in \Gamma_4$ then it occurs at most once as $e(Z^*)$, $f(Z^*)$ for some $Z^* \in \Gamma_4$ or $Z^* \in \Gamma_4'$. Hence, using (4.6.2'),

$$3m \geq 3(2n+\alpha+\beta+\delta-6)+2\delta_1 + 3\delta + 3\delta_4 + 3\delta'$$

Adding (4.1.2) and (4.1.3), we obtain

$$4m \geq 9n-18+\alpha+\beta+2\delta+2\delta_1 + 3\delta + 3\delta_4 + 3\delta'$$

Since $\beta \geq \delta_3$ and $\alpha \geq \delta'_3$, $4m \geq 9n-18-2\delta+2(\delta-1)=9n-20$, or $m \geq 2.25n-5$; a contradiction with our original assumption $m \leq 2.25n-7$. □

**Theorem 4.13:** If $n=4k+2$, $k \geq 5$ then $f(n) \leq 2.25n-4.5$

**Proof:** By Theorem 4.1, $f(n) \leq 2.25n-6$, and, since $f(n)$ is integer, $f(n) \geq 2.25n-5.5$. Suppose that $n$ is the smallest counterexample, i.e. $f(n)=2.25n-5.5$, and let $T$ be a good scheme with $|T|=f(n)$. All of the previous lemmas remain true since
a) $n \geq 20$ is true in the case $n=4k+2$ automatically
b) the only point where we used $|T| \leq 2.25n-7$ was Lemma 4.6, and the current weaker assumption also implies $\delta > 0$ (see the remark at the end of the proof of Lemma 4.6).

Hence, as before, we obtain $|T| \geq 2.25n-5$, a contradiction. □

**Remark 4.14:** A more careful analysis of the proof of Theorem 4.1 shows that a normal scheme with $|T|=2.25n-6$ is "essentially unique".

Case 1 of Lemma 4.10 must occur (see the vertices $x_{1,1}$ and $x_{2,k}$ in Construction 1 in §3), and $T$ is an extension of a normal scheme $T'$ on
n-4 points with $|T'| = 2.25(n-4) - 6$. In particular, $|T| = 2.25n - 6$ implies $\delta = n/4$. □
CHAPTER II
DIRECTED GRAPHS

§5. A complete answer

In the majority of graph theoretical problems, the case of directed graphs used to be more complicated than the case of graphs. However, this principle is not true for the gossip problem: each question can be much easier answered for directed graphs. For example, considering the original version of the problem, denote $g_{dir}(n)$ the minimal number of directed calls s.t. everybody hears each gossip. $g_{dir}(n) \geq 2n-2$, since after $n-2$ calls nobody can know all of the gossips (the graph of calls is not connected), and, at each further call, at most one new person will know all information.

Theorem 5.1: (i) For each $n$, there exists a sequence of calls s.t. each person hears all of the others' gossips exactly once.

(ii) Denote $f_{dir}(n)$ the minimal number of calls when everybody hears each gossip exactly once. Then $f_{dir}(n)=2n-2$. 

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(iii) In a good calling scheme, at least \( n-1 \) persons must hear back their own gossip. (Note that in the case of ordinary phone calls, the condition "everybody hears each of the others' information exactly once" implies that nobody can hear back his own gossip.)

**Proof:** (i) Let us denote the persons by \( a_1, a_2, \ldots, a_n \). Then \( \overrightarrow{a_1a_n}, \overrightarrow{a_2a_n}, \ldots, \overrightarrow{a_{n-1}a_n}, \overrightarrow{a_n a_{n-1}}, \overrightarrow{a_{n-2}a_{n-1}}, \ldots, \overrightarrow{a_n a_1} \) is a good sequence of calls.

(ii) The construction in (i) shows \( f_{\text{dir}}(n) \leq 2n-2 \). On the other hand,

\[
f_{\text{dir}}(n) \geq g_{\text{dir}}(n) \geq 2n-2.
\]

(iii) Suppose, on the contrary, that \( n \) is the smallest number s.t. there exists a good scheme \( S \) on \( n \) points and, executing \( S \), at most \( n-2 \) persons hear back their own gossip. (Clearly, \( n>2 \).) Let \( \overrightarrow{ab} \) be the last call in \( S \). \( a \) knows each gossip before this call, in particular, he knows \( g(b) \). \( b \) knows only \( g(b) \) before the call \( \overrightarrow{ab} \), otherwise he would hear a gossip for the second time. Hence all edges adjacent to \( b \) and different from \( \overrightarrow{ab} \) are of the form \( \overrightarrow{bc} \) for some vertex \( c \). Deleting all edges of \( S \) adjacent to \( b \), we obtain a sequence of calls \( T \) among \( n-1 \) persons.

Executing \( T \),

1) everybody hears each gossip
2) nobody hears a gossip twice
3) at most \( n-3 \) persons hear their own information.

2) and 3) are true because nobody hears a gossip which he did not hear
at $S$, and $b$ heard his own gossip at $S$. 1) is true because $b$ did not participate in transmitting information different of $g(b)$ in $S$. $T$ shows that $n$ is not the least counterexample, we obtain contradiction. □
CHAPTER III
HYPERGRAPHS

§6. The set of feasible n's

Let \( F_k = \{ n : \text{there exists a sequence of } k \text{-conference calls among } n \text{ persons such that everybody hears each gossip exactly once} \} \).

A calling scheme among \( n \) persons can be represented by a \( k \)-uniform hypergraph \( H = (V, E) \) on \( n \) points. The edges of the hypergraph are linearly ordered according to the ordering of calls. Using this representation, the condition that everybody hears each gossip exactly once means that for all \( a, b \in V \), there exists exactly one increasing subsequence of edges \( e_{i_1}, e_{i_2}, \ldots, e_{i(j)} \) s.t. \( a e_{i_1}, b e_{i(j)} \), and \( e_{i(m)} \not\subset e_{i(m+1)} \) for all \( 1 \leq m < j - 1 \). However, we return to the terminology and methods of §1. We have the appropriate version of the product-lemma (Lemma 1.1.): if \( n_1 \in F_k \) and \( n_2 \in F_k \) then \( n_1 n_2 \in F_k \). In particular, \( k^m \in F_k \) for each \( m \).
Theorem 6.1: (i) If $n \in F_k$, $n > 1$ then $n = k + xk(k-1)$ for some non-negative integer $x$

(ii) For each $k$, there exists a bound $x_0(k)$ s.t. for each $x > x_0(k)$,
$k + xk(k-1) \not\in F_k$.

Remark 6.2: We will prove that $x_0(k) = O(k^2)$. The exact value of $x_0(k)$ is not known; in particular, the following conjecture is still open:
If $x \geq 2k+1$, $x \neq 3k$ then $k + xk(k-1) \not\in F_k$. (The conjecture is true in the cases $k = 3, 4$.)

The theorem will be proven by a series of lemmas. We return to the terminology of §1 (see Remark 1.3. for the definition of "JG" and the phrase "to organize the conversations").

Lemma 6.3: Suppose $n \in F_k$, $n > 1$ and let us consider a sequence of calls proving this property.

(i) Associating each person to his first partners, we obtain disjoint $k$-tuples

(ii) Associating each person to his last partners, we obtain disjoint $k$-tuples, too

(iii) If $n \in F_k$, $n > 1$ then $k | n$.

Proof: The proof is the same as in the case of graphs (see Lemma 1.4. and Proposition 1.5.).
Lemma 6.4: If \( n \not\equiv F_k \) then \( k-1 \mid n-1 \).

Proof: Let us consider a sequence of calls proving \( n \equiv F_k \), and a fixed gossip \( G \). At the beginning \( G \) is known by one person while at the end \( G \) is known by \( n \) persons. At each call, either 0 or \( (k-1) \) new persons hear \( G \), hence \( k-1 \mid n-1 \). □

Proof of Theorem 6.1.(i): The solution of the simultaneous congruence system

\[
\begin{align*}
\begin{cases}
 n \equiv 1 & \text{mod} \ (k-1) \\
 n \equiv 0 & \text{mod} \ k
\end{cases}
\end{align*}
\]

is \( n \equiv k \text{ mod} \ (k^2-k) \). □

Lemma 6.5: \( s=1+p(k-1) \) JG's \((1 \leq p \leq k)\) and \( mk^2 \) persons are given. The persons are divided into \( s \) groups, \( A_1, A_2, ..., A_s \), \( |A_i| = a_i \), and the members of \( A_i \) know the \( i \)th JG. Suppose that the following conditions hold for all \( 1 < i \leq s \):

a) \( m \leq a_i \)

b) \( a_i \leq mk \)

c) \( k-1 \mid a_i - m \).

Then it is possible to organize the conversations.

Proof: By induction on \( m \). If \( m=1 \) then, from b), c), \( a_i = 1 \) or \( a_i = k \) for all \( i \).

\( a_1 + a_2 + ... + a_s = k^2 \) implies that \( a_i = 1 \) for \( pk-k \) groups and \( a_i = k \) for \( k+1-p \) groups. Hence, by Lemma 6.3.(i), it is possible to organize the conversations. Suppose that we proved the lemma for \( m \) and let
m+1≤a_1≤a_2≤...≤a_s≤(m+1)k, \ a_1+a_2+...+a_s=(m+1)k^2. Let us choose one person from A_j for 1≤i≤pk-k and k persons from A_j for pk-k+1≤i≤s. Each JG is represented in this new group, and it is possible to organize the conversations among these k^2 persons. We prove that the conditions of the lemma are fulfilled in the remaining part.

If a) is violated in the remaining part then \( a_{pk-k+1} < m+k \) (for the original \( a_{pk-k+1} \)). By c), \( a_{pk-k+1} = m+1 \), hence

\[
a_1+a_2+...+a_s ≤ (pk-k+1)(m+1) + (k-p)((m+1)k) = (m+1)(k^2-k+1),
\]

a contradiction. Thus a) is true in the remaining group. Similarly, if b) is violated then \( a_{pk-k} = (m+1)k \), and we obtain contradiction:

\[
a_1+a_2+...+a_s ≥ (pk-k-1)(m+1) + (k-p+2)((m+1)k) = (m+1)(k^2+k-1).
\]

Hence b) is true; clearly, c) is also satisfied. By the inductive hypothesis, it is possible to organize the conversations in the remaining group, too. □

**Lemma 6.6**: If \( 0≤p≤k+1 \) then \( k^2+pk^2(k-1)∈F_k \).

**Proof**: The case \( p=0 \) and \( p=k+1 \) is trivial. If \( 1≤p≤k \) then we divide the persons into \( s=1+p(k-1) \) groups, \( A_1, A_2,..., A_s \), s.t. \( |A_i| = k^2 \) for all \( i \)'s and organize the conversations in each group. We obtain \( s \) JG's, and Lemma 6.5. can be applied. □
Lemma 6.7: If $1 \leq p \leq k-2$ then $k^4+pk^2(k-1)\in \mathbb{F}_k$.

Proof: Let us divide the persons into $1+(p+2)(k-1)$ groups, $A_1, A_2, \ldots, A_{k-1}, B_1, B_2, \ldots, B_{k-1}, C_1, C_2, \ldots, C_{k-1}, D_1, D_2, \ldots, D_{(p-1)(k-1)+1}$, s.t.

$|A_1|=k^3, |B_1|=|C_1|=|D_1|=k^2,$ and let us organize the conversations in each group. After that we make $k-1$ new groups, $Y_1, Y_2, \ldots, Y_{k-1}$, of size $k^3$: in $Y_j$ there are $k^2$ persons from $A_j$ for $1 \leq i \leq k-1$, $k$ persons from $B_{(j-1)(k-p)+1}, B_{(j-1)(k-p)+2}, \ldots, B_{(j-1)(k-p)+k-p+1}$, and one person from each of the other groups. (The index of $B_m$ is taken mod $(k-1)$.)

For all $1 \leq i \leq k-1$, we choose a number $j(i)$ s.t. in $Y_{j(i)}$ there are $k$ persons from $B_{j}$, we send $k-1$ persons from $Y_{j(i)}$ back to $B_{j}$, and put $k-1$ additional persons from $C_{j}$ into $Y_{j(i)}$. Hence we obtain the groups $Y'_{j(i)}$, $1 \leq i \leq k-1$. In $Y'_{j(i)}$, there are $k^2$ persons from $k-1$ groups, $k$ persons from $k-p$ groups, and one person from $kp$ groups; hence, in $Y'_{j(i)}$, it is possible to organize the conversations. In the remaining part, there are altogether $k^2(pk+k-p)$ persons; in the remaining part of $A_{j}$, there are $k^3-(k-1)k^2=k^2$ persons, there are $k^2-(k-p-1)k-p=pk+k-p$ persons in $B_{j}$, $k^2-k-(k-2)\leq k^2-2k+2$ persons in $C_{j}$, and $k^2-k+1$ persons in $D_{j}$. Since $pk+k-p\leq k^2-2k+2$, the conditions of Lemma 6.5. are satisfied in this
remaining group. Hence it is possible to organize the conversations.
(See Fig. 18.)

\[
\begin{array}{cccc}
Y_1' & Y_2' & Y_3' & \text{remainder} \\
A_1 & 16 & 16 & 16 & 16 \\
A_2 & 16 & 16 & 16 & 16 \\
A_2 & 16 & 16 & 16 & 16 \\
B_1 & 4 & 1 & 4 & 7 \\
B_2 & 4 & 1 & 4 & 7 \\
B_3 & 1 & 4 & 4 & 7 \\
C_1 & 1 & 4 & 1 & 10 \\
C_2 & 1 & 4 & 1 & 10 \\
C_3 & 4 & 1 & 1 & 10 \\
D_1 & 1 & 1 & 1 & 13 \\
\end{array}
\]

Lemma 6.7: In the case \(k=4, \ p=1, \ j(1)=j(2)=2, \ j(3)=1\)

Fig. 18.

Lemma 6.8: \(k^4 + (k-1)^2 k^2 \in F_k\).

Proof: We divide the persons into 3k-2 groups: there are \(k^3\) persons in \(A_1, A_2, \ldots, A_{2k-3}\), and \(k^2\) persons in \(B_1, B_2, \ldots, B_{k+1}\). We organize the conversations in each group. After that we make 2k-3 new groups, \(Y_1, Y_2, \ldots, Y_{2k-3}\), of size \(k^3\): in \(Y_1\) there are \(k^2\) persons from
\[ A(j-1)(k-1)+1, A(j-1)(k-1)+2, \ldots, A(j-1)(k-1)+k-1, \] 

and \( k \) persons from the other \( A_i \)'s. If \( 1 \leq j \leq k-2 \) then \( Y_j \) contains \( k \) persons from \( B_1 \) and one person from the other \( B_i \)'s; if \( k-1 \leq j \leq 2k-4 \) then \( Y_j \) contains \( k \) persons from \( B_2 \) and one person from the other \( B_i \)'s; finally, in the case \( j=2k-3 \),

<table>
<thead>
<tr>
<th>( Y_1 )</th>
<th>( Y_2 )</th>
<th>( Y_3 )</th>
<th>( Y_4 )</th>
<th>( Y_5 )</th>
<th>( Y_6 )</th>
<th>( Y_7 )</th>
<th>remainder</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_1 )</td>
<td>25</td>
<td>25</td>
<td>5</td>
<td>25</td>
<td>5</td>
<td>25</td>
<td>5</td>
</tr>
<tr>
<td>( A_2 )</td>
<td>25</td>
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<td>25</td>
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</tr>
<tr>
<td>( A_3 )</td>
<td>25</td>
<td>5</td>
<td>25</td>
<td>5</td>
<td>25</td>
<td>25</td>
<td>5</td>
</tr>
<tr>
<td>( A_4 )</td>
<td>25</td>
<td>5</td>
<td>25</td>
<td>5</td>
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</tr>
<tr>
<td>( A_5 )</td>
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<tr>
<td>( A_6 )</td>
<td>5</td>
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<tr>
<td>( A_7 )</td>
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<td>25</td>
<td>5</td>
<td>25</td>
<td>10</td>
</tr>
<tr>
<td>( B_1 )</td>
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<td>5</td>
<td>5</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( B_2 )</td>
<td>1</td>
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<td>1</td>
<td>5</td>
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<tr>
<td>( B_3 )</td>
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<td>( B_4 )</td>
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<tr>
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<td>1</td>
<td>1</td>
<td>1</td>
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<tr>
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<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Lemma 6.8. in the case \( k=5 \)

Fig. 19.
$Y_j$ contains $k$ persons from $B_3$ and one person from the other $B_i$'s.

Clearly, it is possible to organize the conversations in each $Y_j$. In the remaining part, there are $(k+1)k^2$ persons altogether; the remaining part of $A_{i}$ $(1 \leq i \leq 2k-3)$ is of size $k^3-(k-1)k^2-(k-2)k=2k$, there are $k^2-(k-2)k-(k-1)1=k+1$ persons in $B_1$ and $B_2$, $k^2-(2k-4)1=k^2-3k+4$ persons in $B_3$, and $k^2-2k+3$ persons in $B_5, \ldots, B_{k+1}$. Since $k^2-3k+4 \geq k+1$, Lemma 6.5. can be applied. (See Fig. 19.)

**Proposition 6.9**: For all $x \geq 0$, $k^2+x(k-1)k^2 \in F_k$.

**Proof**: We use induction on $x$. If $0 \leq x \leq 2k$ then we are done by Lemmas 6.6, 6.7, and 6.8. Suppose that the statement is true for all $x < x_0$, $x_0 \geq 2k+1$. Let us write $x_0$ in the form $x_0=2+(2k-1)y+m$, $y \geq 1$, $0 \leq m < 2k-1$.

Because of the inductive hypothesis, $k^2+y(k-1)k^2 \in F_k$ and $k^2+(y+1)(k-1)k^2 \in F_k$. Dividing the $k^2+x_0(k-1)k^2$ persons into $2k-1$ groups, $A_1, A_2, \ldots, A_{2k-1}$: $|A_1|=|A_2|=\ldots=|A_m|=k^2+(y+1)(k-1)k^2$, $|A_{m+1}|=|A_{m+2}|=\ldots=|A_{2k-1}|=k^2+y(k-1)k^2$, and organizing the conversations in each group, we can apply Lemma 6.5: a) of Lemma 6.5. is satisfied since $1+x_0(k-1) \leq k^2(1+y(k-1)) \Leftrightarrow m \leq k-1+(k-1)^2y$ is true; b) of Lemma 6.5. is satisfied since $k^2(1+(y+1)(k-1)) \leq k(1+x_0(k-1)) \Leftrightarrow$
Lemma 6.10: Let $1 \leq m \leq k-2$. Suppose that $(2m+1)k^2 - mk$ persons and $3k-2$ disjoint JG’s are given. (k-1) JG’s are known by $m+1$ persons, one JG is known by $mk+1$ persons, (k-1) JG’s by $(m-1)k^2$ persons, one JG by $m+k$ persons, and (k-2) JG’s by $(m+2)k-1$ persons. Then it is possible to organize the conversations.

Proof: The organization of the conversations can be read from the table of Fig. 20. The groups $A_i$ are as large as it was described above.

\[
\begin{array}{c|c|c|c|c}
|A_1| &= m+1 & 1 & 1 & \ldots & 1 \\
|A_{k-1}| &= m+1 & \vdots & \vdots & \vdots & \vdots \\
|A_k| &= mk+1 & 1 & & & k & k & \ldots & k \\
|A_{k+1}| &= (m-1)k+2 & 1 & & & & & & 1 \\
|A_{2k-1}| &= (m-1)k+2 & \vdots & \vdots & \vdots & \vdots & k & k & \ldots & k \\
|A_{2k}| &= m+k & 1 & & & & & & 1 & k \\
|A_{2k+1}| &= (m+2)k-1 & 1 & 1 & \ldots & 1 & k & k & \ldots & k \\
|A_{3k-2}| &= (m+2)k-1 & \vdots & \vdots & \vdots & \vdots & k & k & \ldots & k \\
\end{array}
\]

Fig. 20.
First we choose one person from each of $A_1, A_2, \ldots, A_k$, and $A_{k+1}$, $A_{k+2}, \ldots, A_{2k}$, and these persons tell their JG's to each other. After that we make new groups given by the columns. From those 2k persons who we have chosen first, we order two to the first $(k-m)$ columns and one to the other $2m$ columns. Each JG is represented in each column, and, clearly, it is possible to organize the conversations everywhere.

**Lemma 6.11:** Let $1 \leq m \leq k-2$. Then $n=(m+1)k^2-mk+2(k^3-k^2)eF_k$.

**Proof:** Let us divide the persons into $3k-2$ groups, $A_1, A_2, \ldots, A_{3k-2}$, $|A_1|=|A_2|=\ldots=|A_{k-m}|=k$, $|A_{k-m+1}|=|A_{k-m+2}|=\ldots=|A_{3k-2}|=k^2$, and let us organize the conversations in each group. After that we divide the persons into two groups given by the table of Fig. 21. By Lemma 6.10, it is possible to organize the conversations in the first column. In the second column, we make $k-m-2$ groups of size $k^2$ by the following method: we always choose one person from the first $k-m$ rows and $k$ persons from the last $m$ rows. We choose $k-m-2$ times $k$ persons and $k+m$ times one person from the other rows; $k$ persons are chosen from those rows where the actual size of the remaining part of $A_1$ is the largest. It can be easily seen that finally one person remains in each of the first $k-m$ and last $m$ groups, while the size of the other groups is between $k$ and $k^2$. (The precise proof is similar to the proof.
of Lemma 6.5; the method also works in the case \( m = k - 2 \), when \( k^2 - (m+2)k + 1 = 1 \). After the remaining persons from the first \( k-m \) and last \( m \) groups speak to each other, we obtain \( 2k-1 \) JG's and \( k^3 \) persons; each JG is known by at least \( k \) and at most \( k^2 \) persons, hence Lemma 6.5. can be applied.
Proposition 6.12: Let $1 \leq m \leq k-2$. If $x \geq 3k+2$ then $(m+1)k^2-mk+x(k^3-k^2) \in \mathbb{Z}$.

Proof: First, let $3k+2 \leq x \leq 3k^2+3k-1$. Let us write $x$ in the form $x=5+(3k-3)y+p$, $0 \leq p < 3k-3$. We divide the $(m+1)k^2-mk+x(k^3-k^2)$ persons into $3k-2$ groups, $A_1, A_2, ..., A_{3k-2}$, $|A_1| = (m+1)k^2-mk+2(k^3-k^2)$, $|A_2| = ... = |A_{p+1}| = k^2+(y+1)(k-1)k^2$, $|A_{p+2}| = ... = |A_{3k-2}| = k^2+y(k-1)k^2$, and we organize the conversations in each group. (It is possible by Lemma 6.11, and Proposition 6.9.) We choose $(2m+1)k^2-mk$ persons in the distribution given by Lemma 6.10, and organize the conversations among them. In the remaining part, the number of persons is divisible by $k^2$.

If $x \leq 2k^2+(m+1)k+1$ then Lemma 6.5 can be applied: all conditions of Lemma 6.5 are fulfilled trivially, except condition a) for the remaining part of $A_1$: $(m+1)k^2-mk+2(k^3-k^2)-(m+1) > k^2((m+1)k^2-mk+x(k^3-k^2)-((2m+1)k^2-mk))$ is equivalent with $x \leq 2k^2+(m+1)k+1$.

If $2k^2+(m+1)k+2 \leq x \leq 3k^2+3k-1$ then we make $k^2+2k-3$ groups of size $k^3$ by the following method: we always choose one person from $A_1$. We select $k-1$ times $k^2$ persons, $k-1$ times $k$ persons, and $k-1$ times one person from the other $A_i$'s; $k^2$ persons are chosen from those groups where the actual size of the remaining part is the
largest, and one person is chosen from those groups where the actual size of the remaining part is the smallest. Finally,

\[(m+1)k^2-mk+x(k^3-k^2) - ((2m+1)k^2-mk) - (k+3)(k-1)k^3 =\]

\[= (x-k^2-3k)(k^3-k^2)-mk^2 \text{ persons remained; the remaining part of } A_1 \text{ is of the size } (m+1)k^2-mk+2(k^3-k^2) - (m+1) - (k^2+2k-3) =\]

\[= 2(k^3-k^2)+mk^2-(m+2)k-m+2. \text{ Lemma 6.5. can be applied in this remaining part: } (x-k^2-3k)(k-1)-m \leq 2(k^3-k^2)+mk^2-(m+2)k-m+2 \leq k((x-k^2-3k)(k-1)-m) \text{ if } 2k^2+(m+1)k+2 \leq x < 3k^2+3k-1, \text{ and, again, all of the other conditions of Lemma 6.5. are fulfilled trivially, since the sizes of the remaining parts in the other } A_j \text{'s can differ from the average size by at most } k^2.\]

If \(3k^2+3k \leq x \leq 9k^2-1\) then let us write \(x\) in the form \(x=3k+5+(3k-3)y+p, 0 \leq p < 3k-3\). We divide the \((m+1)k^2-mk+x(k^3-k^2)\) persons into \(3k-2\) groups, \(A_1, A_2, ..., A_{3k-2},\)

\[|A_1| = (m+1)k^2-mk+(3k+2)(k^3-k^2), |A_2| = ... = |A_{p+1}| = k^2+(y+1)(k-1)k^2,\]

\[|A_{p+2}| = ... = |A_{3k-2}| = k^2+y(k-1)k^2, \text{ and we organize the conversations in each group. (It is possible by Proposition 6.9. and by the already proven part of Proposition 6.12.) We choose } (2m+1)k^2-mk \text{ persons in the distribution given by Lemma 6.10. After that Lemma 6.5. can be applied: } (m+1)k^2-mk+(3k+2)(k^3-k^2)-(m+1) \leq k^{-1}(x(k^3-k^2)-mk^2) \text{ if } x \geq 3k^2+3k, \text{ and the other conditions of Lemma 6.5. are fulfilled} \]
trivially.

Finally, let \( x_0 \geq 9k^2 \) and suppose that \((m+1)k^2 - mk + x_0(k^3 - k^2) \in F_k\) for all \(3k^2 + 2 \leq x < x_0\). Let us write \( x_0 \) in the form \( x_0 = 3 + (3k - 2)y + p\), \(0 \leq p < 3k - 2\).

We divide the \((m+1)k^2 - mk + x_0(k^3 - k^2)\) persons into \(3k - 2\) groups, \(A_1\), \(A_2\), ..., \(A_{3k-2}\)

\[
|A_1| = (m+1)k^2 - mk + y(k^3 - k^2), \\
|A_2| = ... = |A_{p+1}| = k^2 + (y+1)(k-1)k^2, \\
|A_{p+2}| = ... = |A_{3k-2}| = k^2 + y(k-1)k^2,
\]

and we organize the conversations in each group. Applying Lemmas 6.10 and 6.5, we obtain

\[(m+1)k^2 - mk + x_0(k^3 - k^2) \in F_k. \]

**Lemma 6.13:** Suppose that \(k + 2(k-1)k^2\) persons and \(3k - 2\) disjoint JG's are given. \(2k - 2\) JG's are known by \(k^2 - k + 1\) persons, two JG's by \(k\) persons, and \((k - 2)\) JG's by \(2k - 1\) persons. Then it is possible to organize the conversations.

**Proof:** The organization of the conversations can be read from the table of Fig. 22. The groups \(A_j\) are as large as it was described above.

First we choose one person from each of \(A_1, A_2, ..., A_k\), and \(A_{k+1}\), \(A_{k+2}, ..., A_{2k}\), and these persons tell their JG's to each other. After that we make new groups given by the columns. From those \(2k\) persons who we have chosen first, we order two to the last column and one
to the other 2k-2 columns. Each JG is represented in each column, and, clearly, it is possible to organize the conversations everywhere.

![Fig. 22.](image)

**Lemma 6.14:** \( k+4(k^3-k^2) \in F_k \).

**Proof:** Let us divide the persons into 4k-3 groups, \( A_1, A_2, ..., A_{4k-3} \):

\[
|A_1| = k, \quad |A_2| = |A_3| = ... = |A_{4k-3}| = k^2,
\]

and let us organize the conversations in each group. After that we divide the persons into two groups given by the table of Fig. 23. By Lemma 6.13, it is possible to organize the conversations in the first column. In the second column,
we organize the conversations among the persons from $A_1, A_{k+1}$, $A_{k+2}, \ldots, A_{2k-1}$. We obtain $3k-2$ JG’s and $3k-2$ groups, $B_1, B_2, \ldots, B_{3k-2}$.

$|B_1| = \ldots = |B_{k-1}| = k-1, |B_k| = \ldots = |B_{2k-3}| = k^2 - 2k + 1,$

$|B_{2k-2}| = |B_{2k-1}| = k^2 - k$, and $|B_{2k}| = \ldots = |B_{3k-2}| = k^2 - 1$. After that we

| $A_1$ | $= k$      | 1     | $k-1$ |
| $A_2$ | $= k^2$    | 1     | $k^2 - 1$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $A_k$ | $= k^2$    | 1     | $k^2 - 1$ |
| $A_{k+1}$ | $= k^2$ | $k^2 - k + 1$ | $k-1$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $A_{2k-1}$ | $= k^2$ | $k^2 - k + 1$ | $k-1$ |
| $A_{2k}$ | $= k^2$    | $k^2 - k + 1$ | $k-1$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $A_{3k-2}$ | $= k^2$ | $k^2 - k + 1$ | $k-1$ |
| $A_{3k-1}$ | $= k^2$ | $k$     | $k^2 - k$ |
| $A_{3k}$ | $= k^2$    | $2k-1$ | $k^2 - 2k + 1$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $A_{4k-3}$ | $= k^2$ | $2k-1$ | $k^2 - 2k + 1$ |

Fig. 23.
make \( k-2 \) groups of size \( k^2 \), always choosing one person from \( B_1, \ldots, B_{k-1} \), and \( k \) persons from \( B_k \). From the other groups, we select \( k-3 \) times \( k \) persons, and \( k+1 \) times one person; \( k \) persons are chosen from those groups where the actual size of the remaining part is the largest. Finally, one person remains in each of \( B_1, \ldots, B_k \), while the size of the remaining part in the other \( B_i \)'s is between \( k \) and \( k^2 \). After the persons from \( B_1, \ldots, B_k \) speak to each other, Lemma 6.5. can be applied.

\[ \square \]

**Proposition 6.15:** If \( x \geq 4k+2 \) then \( k+x(k^3-k^2) \notin F_k \).

**Proof:** The proof is similar to the proof of Proposition 6.12, and we omit it.

\[ \square \]

**Proof of Theorem 6.1(ii):** By Propositions 6.9, 6.12, and 6.15.

\[ \square \]
§7. Linear upper bound for $f_k(n)$

For $n \in F_k$, denote by $f_k(n)$ the minimal number of $k$-conference calls among $n$ persons, in calling schemes where everybody hears each gossip exactly once. Let $c(k) = \frac{3k^2 - k - 1}{k^3 - k^2}$; we will prove that $f_k(n) \leq c(k)n + d$, where the constant $d$ depends on the mod$(k^3 - k^2)$ residue class of $n$.

By Theorem 6.1, the feasible $n$'s can be written in the form $n = k^2 - m(k^2 - k) + x(k^3 - k^2)$ for some integer $x$ and $0 \leq m \leq k - 1$. We use four different constructions, according to the cases $m = 0$, $m = 1$, $m = k - 1$, and $2 \leq m \leq k - 2$. The constructions described in §3 can be considered as special cases for $m = 0$ and $m = k - 1$, $k = 2$.

Construction 1: For $n = k^2 + x(k^3 - k^2)$, $x \geq 3$, there exists a good calling scheme of cardinality $c(k)n - \frac{2k^2 - k}{k - 1}$.

\begin{equation}
\text{Step 1:} \quad \text{We divide the persons into } y = 1 + x(k - 1) \text{ groups, } A_1, A_2, \ldots, A_y; \quad |A_i| = k^2 \text{ for each } 1 \leq i \leq y. \quad \text{We perform } 2k \text{ calls in each group s.t. everybody hears each gossip in his own group (first, we form } k \text{ disjoint } k\text{-tuples; second, we pick one person from each original } k\text{-tuple, obtaining another set of disjoint } k\text{-tuples). After these}
\end{equation}
calls, persons being in the same group know the same set of gossips; we refer to the information known in \(A_i\) (it is the union of \(k^2\) original gossips) as the \(i\)th joint gossip (JG).

**Step 2:** We form a set \(Y=\{a_1, a_2, \ldots, a_{y-2(k-1)}, b_{2k-1}, b_{2k}, \ldots, b_y\}\), where \(a_i, b_i \in A_i\) for each \(i\). Clearly, \(|Y|=2y-4(k-1)\). In \(Y\), we perform the calls \((a_1, a_2, \ldots, a_k), (a_1, a_{k+1}, a_{k+2}, \ldots, a_{2k-1}), \ldots, (a_1, a_{y-3(k-1)+1}, \ldots, a_{y-2(k-1)}), (b_y, b_{y-1}, \ldots, b_{y-(k-1)}), (b_y, b_{y-k}, b_{y-k+1}, \ldots, b_{y-2(k-1)}), \ldots, (b_y, b_{3k-3}, \ldots, b_{2k-1})\), in that order. After these calls, \(a_1\) knows the JG's 1 to \(y-2(k-1)\), \(a_i\) (\(2\leq i\leq y-2(k-1)\)) knows the JG's 1 to \(1+(k-1)\cdot\left\lceil (i-1)/(k-1) \right\rceil\), \(b_y\) knows the JG's 2\(k-1\) to \(y\), and \(b_i\) (\(2k-1\leq i\leq y-1\)) knows the JG's \(y-(k-1)\cdot\left\lceil (y-i)/(k-1) \right\rceil\) to \(y\).

**Step 3:** We form groups \(B_1, B_2, \ldots, B_y\); \(|B_i|=k^2\) for each \(1\leq i\leq y\). From \(Y\), we send \(a_i\) (\(2\leq i\leq y-2(k-1)\)) into \(B_{i+2k-3}\), \(a_1\) into \(B_y\), \(b_i\) (\(2k-1\leq i\leq y-1\)) into \(B_{i-(2k-3)}\), and \(b_y\) into \(B_1\). Persons from the groups \(A_i \setminus Y\) are assigned to the new groups by a \(y\times y\) matrix \(Z\). The rows of \(Z\) are labelled by the groups \(B_i\), the columns by the groups \(A_j\), and the entry \(z(i,j)\) gives the number of those persons from \(A_j \setminus Y\) who are sent into \(B_i\). \(Z\) must satisfy the following properties:
\(65\)

\(\alpha)\ z(1,1)z(1,2)+...+z(1,y) = |B_1 \setminus Y| \quad \text{for each } 1 \leq i \leq y \)

\(\beta)\ z(1,j)z(2,j)+...+z(y,j) = |A_j \setminus Y| \quad \text{for each } 1 \leq j \leq y \)

In each \(B_i\), we should like to give a sequence of calls s.t., eventually, everybody hears each gossip exactly once. To this end, we require \(Z\) to satisfy the following properties:

\(\delta)\ z(i,j) = 0 \text{ if one of the persons in } B_i \cap Y \text{ knows the } j^{th} \text{ JG after Step 2} \)

\(\delta)\ z(i,j) \in \{1, k\} \text{ if none of the persons in } B_i \cap Y \text{ knows the } j^{th} \text{ JG after Step 2} \)

We claim that the conditions \(\alpha) - \delta)\) imply that we can finish the algorithm in each new group. In each \(B_i\), we can partition the persons into equivalence classes. Two persons are in the same class if they know the same set of JG's after Step 2. Each JG is known in exactly one equivalence class, and the size of each class is 1 or \(k\). If, say, there are \(p\) classes of size \(k\), then we can form \(k-p\) \(k\)-tuples from the classes of size 1. Performing \(k-p\) calls in these \(k\)-tuples, we obtain \(k\) groups of size \(k\), and \(k\) additional calls finish the algorithm.

A possible matrix \(Z\) satisfying \(\alpha) - \delta)\) is the following (we list only the non-zero entries):

\(a)\) If \(1 \leq i \leq k\) then \(z(1,1) = z(1,k+1) = z(1,k+2) = ... = z(1,2k-2) = 1\) and \(z(1,1) = ... = z(1,i-1) = z(1,i+1) = ... = z(1,k) = k\).

If \(y-(k-1) \leq i \leq y\) then \(z(i,1) = z(i,y-2k+3) = z(i,y-2k+4) = ... = z(i,y-k) = 1\) and
\[ z(i, y-k+1) = z(1, y-k+2) = \ldots = z(1, y-k+1) = z(i, y-k+1) = \ldots = z(i, y) = k. \]

b) \[ z(k+1, 1) = z(k+2, 1) = \ldots = z(2k-1, 1) = 1 \]

If \( y-2k+3 \leq i \leq y-k \) then \( z(1, y-2k+3) = z(1, y-2k+4) = \ldots = z(1, y-k) = k \) and \( z(1, y-3k+4) = z(1, y-3k+5) = \ldots = z(1, y-2k+2) = z(1, y-k+1) = \ldots = z(1, y) = 1 \)

c) If \( k+1 \leq i \leq y-2k+2 \) and \( k-1 \mid i-1 \) then \( z(i, i-k+2) = z(i, i-k+3) = \ldots = z(i, i+1) = \ldots = z(i, i+k-2) = 1 \)

\[ z(i, i-k+2) = z(i, i-k+3) = \ldots = z(i, i-1) = z(i, i+1) = \ldots = z(i, i+k-2) = 1. \]

(See Fig. 24.) Clearly, conditions (a), (b), (c) are satisfied, and it is easy to check that (d) holds.

Step 1 requires \( 2k(1+x(k-1)) \) calls, Step 2 requires \( 2(x-2) \) calls, while Step 3 requires \( (2k+x-3)(k+1) + [1+x(k-1)-(2k+x-3)](k+2) \) calls; adding, we obtain \( (3k+2)(1+x(k-1)) + x-2k-1 = (3k+2) \cdot n/k^2 + x-2k-1 = c(k) \cdot n - [2k^2-k]/[k-1]. \) (In Step 3, we need \( k+1 \) calls in the \( B_i \)'s for \( 1 \leq i \leq k, y-k+1 \leq i \leq y \), and for \( i: k+1 \leq i \leq y-k \) and \( k-1 \mid i-1 \). In the other groups, we use \( k+2 \) calls.)
### Construction 1 in the case \( k=4, x=6 \)

**Fig. 24.**

**Remark 7.1:** Instead of looking at the formal description, perhaps it is easier to visualize what is happening. In the case \( x=3 \) (i.e. \( y=3k-2 \)), only the entries described in a), b) are present. As \( x \) increases, we can obtain \( Z_x \) from the matrix \( Z_{x-1} \) by inserting the pattern of Fig. 25.
the lower right \((2k-2)\times(2k-2)\) principal submatrix of \(Z_{x-1}\). (Again, see Fig.24.)

This observation offers another way of counting the number of necessary calls. Suppose that the algorithm for \(n\) required \(c(k)n+d\) calls, for some constant \(d\). The increment of \(n\) by \((k^3-k^2)\) requires \(2k(k-1)\) additional calls in Step 1, 2 calls in Step 2, and \(1\cdot(k+1)+(k-2)(k+2)\) calls in Step 3. Altogether we use \(c(k)n+d+3k^2-k-1=c(k)[n+(k^3-k^2)]+d\) calls, hence \(d\) can be determined from the case \(x=3\). The same observation yields in the further three constructions.

Let us note that (7.1.1) also gives the number of calls in the cases \(x=1,2\), using the constructions described in §6.

\[
\begin{bmatrix}
1 & k & k & \ldots & k & k & 1 & 1 & \ldots & 1 \\
k & 1 & k & \ldots & k & k & 1 & 1 & \ldots & 1 \\
k & k & 1 & \ldots & k & k & 1 & 1 & \ldots & 1 \\
k & k & k & \ldots & 1 & k & 1 & 1 & \ldots & 1 \\
k & k & k & \ldots & k & k & 1 & 1 & \ldots & 1 \\
1 & 1 & 1 & \ldots & 1 & 1 \\
1 & 1 & 1 & \ldots & 1 & 1 \\
1 & 1 & 1 & \ldots & 1 & 1 \\
\end{bmatrix}
\]

Fig. 25.
**Construction 2:** For $n = k + x(k^3 - k^2)$, $x \geq 5$, there exists a good calling scheme using $c(k) \cdot n - \left[2k^3 - 2k^2 + 2k - 1\right]/\left[k^2 - k\right]$ calls. \hspace{1cm} (7.2.1)

**Step 1:** We divide the persons into $y = 1 + x(k-1)$ groups, $A_1, A_2, ..., A_y$:

$|A_{2k-1}| = k$, and $|A_i| = k^2$ for each $1 \leq i \leq y$, $i \neq 2k-1$. We perform one call in $A_{2k-1}$ and 2k calls in the other groups s.t. everybody hears each gossip in his own group. Again, we refer to the information known in the $i$th group as the $i$th JG.

**Step 2:** We form the set $Y$ as in Construction 1, but the sequence of calls is slightly different: among the $a_i$'s, we perform the calls

\[
(a_k, a_{k+1}, a_{k+2}, ..., a_{2k-1}), (a_1, a_2, ..., a_k), (a_1, a_{2k}, a_{2k+1}, ..., a_{3k-2}), ..., (a_1, a_{y-3(k-1)+1}, ..., a_{y-2(k-1)});
\]

among the $b_i$'s, we perform the same sequence of calls as before: $(b_y, b_{y-1}, ..., b_{y-(k-1)})$,

$(b_y, b_{y-k}, b_{y-k-1}, ..., b_{y-2(k-1)})$, ..., $(b_y, b_{3k-3}, ..., b_{2k-1})$, in that order.

After these calls, $a_1$ knows the JG's 1 to $y-2(k-1)$, $a_i$ ($2 \leq i \leq k$) knows the JG's 1 to 2k-1, $a_i$ ($k+1 \leq i \leq 2k-1$) knows the JG's k to 2k-1, and $a_i$ ($2k \leq i \leq y-2(k-1)$) knows the JG's 1 to 1 + $\left[(i-1)/(k-1)\right]$; $b_y$ knows the JG's 2k-1 to y, and $b_i$ ($2k-1 \leq i \leq y-1$) knows the JG's $y-(k-1) \cdot \left[(y-1)/(k-1)\right]$ to y.

**Step 3:** We form groups $B_1, B_2, ..., B_y$: $|B_{3k-2}| = k$, and $|B_i| = k^2$ for
each \(1 \leq i \leq 1 + 3k - 2\). From \(Y\), we send \(a_i\) (\(2 \leq i \leq k\)) into \(B_{1 + 3k - 4}\),

\(a_i\) (\(k + 1 \leq i \leq 2k - 1\)) into \(B_{1 + k - 2}\), \(a_i\) (\(2k \leq i \leq y - 2(k - 1)\)) into \(B_{1 + 2k - 3}\), and

\(a_1\) into \(B_Y\). On the other hand, we send \(b_i\) (for \(2k - 1 \leq i \leq 4k - 5\) or

\(5k - 4 \leq i \leq y - 1\)) into \(B_{1 - (2k - 3)}\), \(b_{4k - 4}\) into \(B_{3k - 2}\), \(b_i\) (\(4k - 3 \leq i \leq 5k - 5\)) into

\(B_{1 - (2k - 2)}\), and \(b_y\) into \(B_1\). (At the assignment of the \(b_i\)'s, the difference

from Construction 1 is that \(b_{4k - 4}\) was taken out of the original

ordering, and everybody in between was shifted up by one group.)

The matrix \(Z\) can be defined on the following way:

a) \(z(1, 1) = \ldots = z(1, k - 1) = k\) and \(z(1, k) = \ldots = z(1, 2k - 2) = 1\)

If \(2 \leq i \leq k\) then \(z(i, 1) = \ldots = z(i, k - 1) = 1\) and \(z(i, k) = \ldots = z(i, 2k - 2) = k\)

If \(k + 1 \leq i \leq 2k - 2\) then \(z(i, 1) = \ldots = z(i, 2k - 1) = 1\) and \(z(i, 2k) = \ldots = z(i, 3k - 3) = k\).

b) If \(2k - 1 \leq i \leq 3k - 3\) then \(z(i, 3k - 2) = k\) and, for all \(2k \leq j \leq 4k - 4\), \(j \neq 3k - 2\),

\(z(i, j) = 1\)

If \(2k - 1 \leq i \leq 3k - 4\) then \(z(i, i - 2k + 2) = z(i, i - 2k + 3) = 1\) and for all \(1 \leq j \leq k - 1\),

\(j = i - 2k + 2, j = i - 2k + 3, z(i, j) = k\)

\(z(3k - 3, 1) = z(3k - 3, k - 1) = 1\) and for all \(2 \leq j \leq k - 2\), \(z(3k - 3, j) = k\).

c) \(z(3k - 2, 2k) = \ldots = z(3k - 2, 3k - 3) = 1\)

If \(i \geq 3k - 1\) then the \(i\)th row is the same as the \(i\)th row in

Construction 1. (See Fig. 26.)

Again, it is easy to check that \(Z\) satisfies \(\alpha)-6\).

In the case \(x = 5\), Step 1 requires \(1 + 5(k - 1)2k\) calls, Step 2 requires 6
calls, while Step 3 requires \(2k(k+1)+1+(3k-5)(k+2)\) calls. Adding these numbers, we can determine the constant in (7.2.1). (See Remark 7.1.)

Construction 2 in the case \(k=4, x=6\)

Fig. 26.
We shall need the following two lemmas in Constructions 3 and 4.

**Lemma 7.2:** For each \( k \geq 4 \), there exists a \((2k-3) \times (2k-3)\) matrix \( A \) satisfying the following properties:

a) each entry of \( A \) is 0 or 1

b) the row-sums and column-sums in \( A \) are the following:

- \( 2k-4 \) in one row
- \( k-3 \) in \((k-2)\) columns
- \( k-4 \) in \((k-3)\) rows
- \( k-1 \) in one column
- \( k-2 \) in \((k-4)\) rows
- \( k-3 \) in two rows

**Proof:** We construct a bipartite graph \( G=(U \times V, E) \) s.t. \(| U | = | V | = 2k-3\), and the degree sequences in \( U \) resp. \( V \) are the given row-sums and column-sums. \( A \) will be the adjacency matrix of \( G \).

First, let \( U^*=(u_1, \ldots, u_{2k-4}), V^*=(v_1, \ldots, v_{2k-4}) \) be two disjoint sets. Let \( E_1 \) be a \((k-4)\)-regular bipartite graph on \( U^* \times V^* \), \( E_2 \) a perfect matching between \( U^*, V^* \) s.t. \( E_1 \cap E_2 = \emptyset \); finally let \( e \) be an edge in \( U^* \times V^* \) s.t. \( e \in E_1, E_2 \). We can choose the enumeration of the points s.t. \( E_2 = \{(u_i, v_i): 1 \leq i \leq 2k-4 \} \) and \( e = (u_2, v_1) \).

Let \( E^* \) be the edge set \( E_1 \cup E_2 \cup \{e\} \setminus \{(u_i, v_i): 1 \leq i \leq 2k-4 \} \cup \{(u_1, v_1)\} \cup \{(u_i, v_i): i \geq k-1 \} \).

Finally, let \( u_0, v_0 \) be two new points. \( E = E^* \cup \{(u_0, v_i): 1 \leq i \leq 2k-4 \} \cup \{(v_0, u_i): 1 \leq i \leq k-1 \} \) satisfies the requirements of the lemma. \( \square \)
Lemma 7.3: (i) For each $k \geq 4$, $2 \leq m \leq k/2$, there exists a $(2k-3) \times (2k-3)$ matrix $B$ satisfying the following properties:

a) each entry of $B$ is 0 or 1

b) the row-sums and column-sums in $B$ are the following:

- $2m-2$ in one row
- $k-3$ in $(k-2)$ columns
- $k-4$ in $(m-1)$ rows
- $k-1$ in one column
- $k-3$ in $(k-m-1)$ rows
- $k-2$ in $(2k-2m-2)$ columns
- $k-1$ in $(k-m)$ rows

(ii) For each $k \geq 5$, $k/2 \leq m \leq k-2$, there exists a $(2k-3) \times (2k-3)$ matrix $C$ satisfying the following properties:

a) each entry of $C$ is 0 or 1

b) the row-sums and column-sums in $C$ are the following:

- $k-4$ in $(m-1)$ rows
- $k-1$ in one column
- $k-3$ in $(k-m-1)$ rows
- $k-2$ in $(2k-2m-2)$ columns
- $k-3$ in $(2m-2)$ columns
- $k-1$ in $(k-m)$ rows

Proof: The proof is similar to the proof of Lemma 7.2, and we omit it. $\square$

Construction 3: For $n = k^2 - (k-1)(k^2-k) + x(k^3-k^2)$, $x \geq 2k+3$, there exists a good calling scheme using $c(k) \cdot n + k^3 - 2k^2 + k - 4 - 1/(k^2-k)$ calls. (7.3.1)

Construction 4: For $n = k^2 - m(k^2-k) + x(k^3-k^2)$, $2 \leq m \leq k-2$, $x \geq k+4+m$, there exists a good calling scheme using
\[c(k) \cdot n + k^3 - 2k^2 - k + m - 1 - (k + mk)/(k^2 - k) \text{ calls.} \quad (7.4.1)\]

Since Step 1 and Step 2, as well as the initial part of Step 3 are identical in these constructions, at the beginning of the description we allow the values \(2 \leq m \leq k - 1.\)

**Step 1:** We divide the persons into \(y = 1 + (x-k+1)(k-1)\) groups, \(A_1, A_2, \ldots, A_y; |A_1| = k^3, \text{ if } 1 \leq i \leq k - 1, \quad |A_i| = k, \text{ if } i = k + j(k-1) \text{ for some } 1 \leq j \leq m,\)

and \(|A_i| = k^2\) for the remaining \(A_i\)'s. We perform \(3k^2\) resp. 1 resp. \(2k\) calls in each group s.t. everybody hears each gossip in his own group.

**Step 2:** We form the set \(Y\) as before. Among the \(a_j\)'s, we perform the calls \((a_k, a_{k+1}, a_{k+2}, \ldots, a_{2k-1}), (a_k, a_{2k}, a_{2k+1}, \ldots, a_{3k-2}), \ldots,\)

\((a_k, a_2+m(k-1), a_3+m(k-1), \ldots, a_{k+m(k-1)}), (a_1, a_2, \ldots, a_k),\)

\((a_1, a_2+(m+1)(k-1), a_3+(m+1)(k-1), \ldots, a_{k+(m+1)(k-1)}), \ldots,\)

\((a_1, a_{y-3(k-1)+1}, \ldots, a_{y-2(k-1)}); \text{ among the } b_j\)'s, we perform the same sequence of calls as before: \((b_y, b_{y-1}, \ldots, b_{y-(k-1)}),\)

\((b_y, b_{y-k}, b_{y-k-1}, \ldots, b_{y-2(k-1)}), \ldots, (b_y, b_{3k-3}, \ldots, b_{2k-1}),\) in that order.

After these calls, \(a_1\) knows the CG's 1 to \(y - 2(k-1),\) \(a_i \quad (2 \leq i \leq k)\) knows the CG's 1 to \(k + m(k-1),\) \(a_i \quad (k + 1 \leq i \leq k + m(k-1))\) knows the CG's \(k \text{ to } 1+(k-1) \cdot \left\lfloor (i-1)/(k-1) \right\rfloor, \text{ and } a_i \quad (k+1+m(k-1) \leq i \leq y - 2(k-1))\) knows the CG's 1 to \(1+(k-1) \cdot \left\lfloor (i-1)/(k-1) \right\rfloor; \) \(b_y\) knows the CG's 2\(k-1\) to \(y, \text{ and } b_i\)
(2k-1≤i≤y-1) knows the CG's \( y-(k-1)\cdot \left\lfloor \frac{y-1}{k-1} \right\rfloor \) to \( y \).

**Step 3:** We form groups \( B_1, B_2, ..., B_y \): \( |B_i| = k^3 \) for \((m+1)(k-1)-m+2 \leq i \leq (m+1)(k-1)-m+k\), \( |B_i| = k \) for \((m+3)(k-1)-m+1 \leq i \leq (m+3)(k-1)\), and \( |B_i| = k^2 \) for all the remaining \( B_i \)'s. We send \( a_i \) (2 ≤ i ≤ k) into
\( B_{i+(k-2)+(m+1)(k-1)} \), \( a_i \) \((m+1)(k-1) \leq i \leq 2(k-1)\) into \( B_{i+k-2} \), \( a_i \) \((m+1)(k-1) \leq i \leq 2(k-1)\) into \( B_{i+2k-3} \), and \( a_1 \) into \( B_1 \). On the other hand, we send \( b_j \) \((2k-1 \leq i \leq (m+3)(k-1)-m \) or \((m+5)(k-1) \leq i \leq y-1\) into
\( B_{i-(2k-3)} \), \( b_i \) \((m+3)(k-1)-m+1 \leq i \leq (m+5)(k-1)-1\) into
\( B_{(m+6)(k-1)-m+1} \), and \( b_y \) into \( B_1 \). (At the assignment of the \( b_j \)'s, the difference from Construction 1 is that the ordering of the \( b_j \)'s for \((m+3)(k-1)-m+1 \leq j \leq (m+5)(k-1)-1\) is reversed.)

At the definition of \( Z \), we replace 6) by the following condition:
6') \( z(i,j) \in \{1,k,k^2\} \) if none of the persons in \( B_i \) \( Y \) knows the \( j \)th JG after Step 2.

Clearly, 6') also ensures that we can finish the algorithm in each new group. The definition of \( Z \) is different at the two constructions.

**Continuation of Construction 3:** In the case \( k=3 \), \( Z \) is given on Fig. 27. For each \( x \geq 9 \) (i.e. \( y \geq 7 \)), the non-zero entries in the first 10 rows of \( Z \) are the same as on Fig. 27, and, beginning from the 11th row, the non-zero entries are the same as in Construction 1.
In the case $k \geq 4$, we can define $Z$ on the following way:

a) The first $2k-2$ rows are the same as in Construction 2.

b) If $2k-1 \leq i \leq 1+(k-1)^2$ and $k-1 \mid i-1$ then $z(i,j)=1$ for all $2 \leq j \leq k-1$, and $z(1,1)=z(1,1+1)=z(1,1+2)=\ldots z(1,1+k-2)=k$.

If $2k-1 \leq i \leq 1+(k-1)^2$ and $k-1 \not\mid i-1$ then let $i'=(k-1) \left\lfloor \frac{(i-1)}{(k-1)} \right\rfloor +1$. 

Construction 3 in the case $k=3$, $x=10$

Fig. 27.
z(i,j) = 1 if 1 ≤ j ≤ k-1 or i'-k+2 ≤ j ≤ i', and z(i,i'+1) = z(i,i'+2) = ... = z(i,i'+k-2) = k.

c) In c), i always satisfies 2+(k-1)2 ≤ i ≤ (k-1)2.

For all 1 ≤ j ≤ k-1, we have z(i,j) = k2, except the entry z(2+(k-1)2, 1) = k.

z(1, 1+(k+2)(k-1)) = k.

For all 2+(k-1)2 ≤ j ≤ 1+k(k-1) and 2+(k+2)(k-1) ≤ j ≤ (k+3)(k-1), z(i,j) = 1.

d) z(1+k(k-1), j) = k2 for all 1 ≤ j ≤ k-1.

z(2+k(k-1), 1) = z(3+k(k-1), 1) = k, and z(i,j) = 1 for all other entries satisfying 2+k(k-1)2 ≤ i ≤ (k+1)(k-1), 1 ≤ j ≤ k-1.

e) z(i,j) = 1 for all 1+(k+1)(k-1) ≤ i ≤ (k+2)(k-1), 2+k(k-1) ≤ j ≤ (k+1)(k-1).

If i ≥ 1+(k+2)(k-1) then the i-th row is the same as the i-th row in Construction 1.

We have not defined Z in the (2k-3)x(2k-3) submatrix Z* given by the rows 2+(k-1)2 ≤ i ≤ (k+1)(k-1) and columns 2+k(k-1) ≤ j ≤ (k+2)(k-1). First, we choose each entry in Z* to be equal to 1. Then the row-sums and column-sums in these rows and columns will be less than |B_i \Y|, |A_j \Y| by the following values:

(2k-4)(k-1) in one row
(k-4)(k-1) in (k-3) rows
(k-1)(k-1) in one row
(k-2)(k-1) in (k-4) rows
(k-3)(k-1) in two rows

(k-3)(k-1) in (k-2) columns
(k-1)(k-1) in one column
(k-2)(k-1) in (k-2) columns

Hence, adding (k-1)A* to Z*, where A* is obtained by an appropriate permutation of rows and columns in the matrix A constructed in
Lemma 7.2, we can achieve that $\alpha), \beta)$ hold in $Z$. Clearly, $\gamma)$ and $\delta')$ are also satisfied. (See Fig. 28.) Again, the constant in (7.3.1) can be determined from the case $x=2k+3$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{construction3.png}
\caption{Construction 3 in the case $k=4$, $x=11$}
\end{figure}
Continuation of Construction 4: We define $Z$ on the following way:

a) The first $2k-2$ rows are the same as in Construction 2.

b) If $2k-1 \leq i \leq 1+m(k-1)$, and $1=k+s(k-1)$ for some $1 \leq s \leq m-1$ then $z(1,s)=z(1,l+1)=z(1,l+2)=\ldots z(1,1+k-2)=k$, and $z(1,j)=1$ for all $1 \leq j \leq k-1$, $j \neq s$.

If $2k-1 \leq i \leq 1+m(k-1)$, and $k-1 \nmid i-1$ then let $i'=(k-1)\left\lceil (i-1)/(k-1) \right\rceil + 1$. $z(i,j)=1$ if $1 \leq j \leq k-1$ or $i'-k+2 \leq j \leq i'$, and $z(i,i'+1)=z(i,i'+2)=\ldots z(i,i'+k-2)=k$.

c) If $2+m(k-1) \leq i \leq (m+1)(k-1)-m$ then let $z(i,i-1-m(k-1))=1$, $z(i,j)=k$ for all $1 \leq j \leq k-1$, $j \neq i-1-m(k-1)$, and $z(i,j)=1$ for $2+m(k-1) \leq j \leq (m+2)(k-1)$.

We define the row $i=(m+1)(k-1)-m+1$ later.

d) For $(m+1)(k-1)-m+2 \leq i \leq (m+1)(k-1)$, let $z(i,j)=k^2$ if $1 \leq j \leq k-1$, $z(i,j)=1$ if $2+m(k-1) \leq j \leq 1+(m+1)(k-1)$, $z(i,1+(m+3)(k-1))=k$, and $z(i,j)=1$ if $2+(m+3)(k-1) \leq j \leq (m+4)(k-1)$.

e) For $1+(m+1)(k-1)-m+1 \leq i \leq (m+2)(k-1)-m+1$, let $z(i,j)=k^2$ if $1 \leq j \leq k-1$.

We define the row $i=(m+2)(k-1)-m+2$ later.

f) For $(m+2)(k-1)-m+3 \leq i \leq (m+2)(k-1)$, let $z(i,j)=1$ if $1 \leq j \leq k-1$.

If $(m+3)(k-1)-m+1 \leq i \leq (m+3)(k-1)$ then let $z(i,j)=1$ for $2+(m+1)(k-1) \leq j \leq (m+2)(k-1)$.

g) If $i \geq 1+(m+3)(k-1)$ then the non-zero entries in the $i^{th}$ row are the same as in Construction 1.

We have not defined $Z$ in the rows $i_1=(m+1)(k-1)-m+1$ and $i_2=(m+2)(k-1)-m+2$ as well as in the $(2k-3) \times (2k-3)$ submatrix $Z^*$ given by the rows $(m+1)(k-1)-m+2 \leq i \leq (m+3)(k-1)-m$, and columns $2+(m+1)(k-1) \leq j \leq (m+3)(k-1)$. 

We distinguish two cases: 1) \( m-1 \leq k-1-m \) (i.e. \( m \leq k/2 \)) and
2) \( m-1 > k-1-m \).

In the first case, let \( z(i_1, k-1-m) = 1 \), \( z(i_1, j) = k \) if \( 1 \leq j \leq k-1 \), \( j \neq k-1-m \),
and \( z(i_1, j) = 1 \) for \( 2+m(k-1) \leq j \leq (m+2)(k-1) \). Furthermore, let \( z(i_2, j) = k \) if \( m \leq j \leq k-1-m \),
and \( z(i_2, j) = 1 \) for all other \( j \)'s satisfying \( 1 \leq j \leq k-1 \).

This definition ensures that \( \alpha \), \( \beta \) hold in all rows and columns except the
rows and columns determining \( Z^* \). If we choose each entry in \( Z^* \) to be
equal to 1 then the row-sums and column-sums will be less than
\( |B_i \setminus Y|, |A_j \setminus Y| \) by the following values:

- \( (k-1)(2m-2) \) in one row
- \( (k-1)(k-3) \) in \( (k-2) \) columns
- \( (k-1)(k-4) \) in \( (m-1) \) rows
- \( (k-1)(k-1) \) in one column
- \( (k-1)(k-3) \) in \( (k-m-1) \) rows
- \( (k-1)(k-2) \) in \( (k-2) \) columns
- \( (k-1)(k-2) \) in \( (m-2) \) rows
- \( (k-1)(k-1) \) in \( (k-m) \) rows.

Hence, adding \( (k-1)B^* \) to \( Z^* \), where \( B^* \) is obtained by an appropriate
permutation of rows and columns in the matrix \( B \) constructed in
Lemma 7.3, we can achieve that \( \alpha \), \( \beta \) hold in \( Z \). Clearly, \( \delta \) and \( \delta' \) are
also satisfied. (See Fig. 29.)
Construction 4 in the case $k=5$, $x=11$, $m=2$

Fig. 29.
In the second case, when \( m > k-m \), let 
\[
z(i_1, j) = 1 \quad \text{if} \quad k-m \leq j \leq m-1, \quad z(i_1, j) = k \quad \text{for all other values satisfying} \quad 1 \leq j \leq k-1, \quad z(i_1, j) = k \quad \text{if} \quad 2+ (m+1)(k-1) \leq j \leq 2m-k+1+(m+1)(k-1), \quad \text{and} \quad z(i_1, j) = 1 \quad \text{if} \quad 2m-k+2+(m+1)(k-1) \leq j \leq (m+2)(k-1). \]
Furthermore, let \( z(i_2, j) = 1 \) for all \( 1 \leq j \leq k-1 \). Again, this definition ensures that \( \alpha \), \( \beta \) hold in all rows and columns except the rows and columns determining \( Z^* \). If we choose each entry in \( Z^* \) to be equal to 1 then the row-sums and column-sums will be less than \( |B_i \setminus Y|, |A_j \setminus Y| \) by the following values:

- \((k-1)(k-1)\) in \((k-m)\) rows
- \((k-1)(k-1)\) in \((2m-2)\) columns
- \((k-1)(k-4)\) in \((m-1)\) rows
- \((k-1)(k-1)\) in one column
- \((k-1)(k-3)\) in \((k-m-1)\) rows
- \((k-1)(k-2)\) in \((2k-2m-2)\) columns
- \((k-1)(k-2)\) in \((m-1)\) rows.

Hence, adding \((k-1)C^*\) to \( Z^* \), where \( C^* \) is obtained by an appropriate permutation of rows and columns in the matrix \( C \) constructed in Lemma 7.3, we can achieve that \( \alpha \), \( \beta \) hold in \( Z \). Clearly, \( \delta' \) and \( \delta' \) are also satisfied. (See Fig. 30.)

The constant in (7.4.1) can be determined from the case \( x=k+m+4 \).
Construction 4 in the case $k=5$, $x=12$, $m=3$

Fig. 30.
§8. Slow gossiping

In this section, we deal with the slow gossiping problem for graphs, directed graphs, and hypergraphs. For $n \in \mathbb{F}_k$ ($k \geq 2$), denote by $h_k(n)$ the maximal number of calls in good schemes among $n$ persons, using $k$-conference calls ("good", as before, means that everybody hears each gossip exactly once). Moreover, denote by $h_{dir}(n)$ the maximal number of calls in good schemes when the persons communicate by phone calls, but, at each conversation, only one of the participants tells the gossips he knows at that time.

**Theorem 8.1:** (D. Miklós, M. Newman, A. Seress, D. West [18])

There exists a constant $d$ (depending on $k$) s.t. $h_k(n) \leq d \cdot n \log n$.

**Proof:** (G. Siegler) Let $T$ be a good calling scheme, $|T| = m$, and denote by $x_1, x_2, ..., x_n$ the persons. For each $0 \leq i \leq m$, $1 \leq j \leq n$, let $p(i,j)$ be the number of gossips known by $x_j$ after the $i^{th}$ call. Let $p(i)$ be the product of $p(i,j)$ for $1 \leq j \leq n$. Then $p(0) = 1$, $p(m) = n^n$.

**Claim:** For each $1 \leq i \leq m$, $p(i)/p(i-1) \geq k^i$.

**Proof of the Claim:** Suppose that the $i^{th}$ call is among the persons in
the set \( A = \{ x_1, x_2, \ldots, x_{i(k)} \} \). Clearly,

\[
p(i, j) = \begin{cases} 
p(i-1, j) & \text{if } x_j \notin A \\
p(i-1, i_1) + p(i-1, i_2) + \ldots + p(i-1, i(k)) & \text{if } x_j \in A.
\end{cases}
\]

Therefore

\[
p(i)/p(i-1) = \left[ p(i-1, i_1) + \ldots + p(i-1, i(k)) \right]^{1/k} / \left[ p(i-1, i_1) \times \ldots \times p(i-1, i(k)) \right] \geq k^k.
\]

(In the last step we used the inequality between the arithmetic and geometric mean.) \( \Box \)

The Claim easily implies Theorem 8.1: \( k^k m \leq n^n \), thus

\[
m \leq (k \cdot \log k)^{-1} \cdot n \cdot \log n.
\]

**Theorem 8.2:** There exists a constant \( c \) s.t. \( h_k(n) \geq c \cdot n \cdot \log n \).

**Proof:** Denote by \( H_k(n) \) the number of calls in the constructions described in §§ 1 and 6. We shall prove that \( H_k(n) \geq c \cdot n \cdot \log n \). First, let \( n \) be of the form \( n = k^2 + x(k-1)k^2 \). Let us choose \( c' \) s.t. \( H_k(n) \geq c' \cdot n \cdot \log n \) if \( x \leq 2k \), and

\[
c' \leq \frac{(k+1)}{[k^2 \cdot \log 2k^2]}
\]  

(8.2.1)

We prove by induction on \( x \) that \( H_k(n) \geq c' \cdot n \cdot \log n \). Suppose that we proved the statement for all \( x < x_0 \) for some \( x_0 \geq 2k+1 \).

For \( n = k^2 + x(k-1)k^2 \), \( H_k(n) = H_k(n_1) + H_k(n_2) + \ldots + H_k(n_{2k-1}) + n(k+1)/k^2 \), where \( n/(2k-1) \geq n_1 \geq n/(2k-1) - (k^3 - k^2) \), \( n_1 = k^2 \mod (k^3 - k^2) \), and \( n = n_1 + n_2 + \ldots + n_{2k-1} \). (See the proof of Lemma 1.9, Proposition 1.11, and
Proposition 6.9.) By the inductive hypothesis,
\[ H_k(n) \geq c' n_1 \cdot \log n_1 + c' n_2 \cdot \log n_2 + \ldots + c' n_{2k-1} \cdot \log n_{2k-1} + n(k+1)/k^2 \geq \]
\[ \geq c' n \cdot \log\left[ n/(2k-1) - (k^3 - k^2)/n \right] + n(k+1)/k^2 = \]
\[ = c' n \cdot \log n + c' n \cdot \log\left[ 1/(2k-1) - (k^3 - k^2)/n \right] + n(k+1)/k^2 \geq \]
\[ \geq c' n \cdot \log n + c' n \cdot \log\left[ 1/(2k-1) - 1/(2k+1) \right] + n(k+1)/k^2 > \]
\[ > c' n \cdot \log n + n[-c' \log 2k^2 + (k+1)/k^2] \geq c' n \cdot \log n. \] (In the last step we used (8.2.1).)

In the case of graphs, let us choose \( c > 0 \) s.t. \( c < c' \), and \( H_2(n) \geq c n \cdot \log n \) for all \( n \in \mathbb{N} \). Using a similar computation as before, we can prove that \( H_2(2+4x) \geq c \cdot (2+4x) \cdot \log(2+4x) \), by induction on \( x \). For \( n \geq 66 \),
\[ H_2(n) \geq H_2(n_1) + \ldots + H_2(n_4) + n - 6, \] where \( n_1 + \ldots + n_4 = n \), and, for all \( 1 \leq i \leq 4 \),
\[ n/4 + 9.5 \geq n_1 \approx n/4 - 5.5 \] (see the proof of Proposition 1.13.). Using the inductive hypothesis and (8.2.1), we can prove \( H_2(n) \geq c n \cdot \log n \).

In the case of hypergraphs, we choose \( c < c' \) s.t. \( H_k(n) \geq c n \cdot \log n \) for all \( n \) of the form \( n = (m+1)k^2 - mk + x(k^3 - k^2) \), \( 1 \leq m \leq k-2, x < 9k^2 \), and for all \( n \) of the form \( n = k + x(k^3 - k^2), x < 12k^2 - 2k \). (The value \( 12k^2 - 2k \) originates from the undetailed proof of Proposition 6.15.) Using induction and (8.2.1) as above, we can prove \( H_k(n) \geq c n \cdot \log n \) for all \( n \in \mathbb{N} \).

\[ \square \]

Theorem 8.3: \( h_{\text{dir}}(n) = (n^2 + n - 2)/2. \)

\[ \text{Proof:} \quad \text{We prove the theorem by induction on } n. \text{ The case } n=2 \text{ is trivial.} \]
Suppose that we proved the statement for some $n$, and denote by $x_1$, $x_2$, ..., $x_{n+1}$ the persons in the case $n+1$. By the inductional hypothesis, there exists a good scheme $T$ on \{$_1$, $x_2$, ..., $x_n$\}, $|T| = (n^2 + n - 2)/2$, s.t.

each person in \{$_1$, $x_2$, ..., $x_n$\} hears the first $n$ gossips exactly once.

Then $S = T \cup \langle \overrightarrow{x_n+1}, \overrightarrow{x_1}, \overrightarrow{x_{n+1}}, \overrightarrow{x_2}, ..., \overrightarrow{x_{n}}, \overrightarrow{x_n} \rangle$ is a good scheme on $n+1$ points, $|S| = (n^2 + n - 2)/2 + (n+1) = ((n+1)^2 + (n+1) - 2)/2$, hence

$$h_{dir}(n+1) \geq ((n+1)^2 + (n+1) - 2)/2.$$

On the other hand, let $S$ be a good scheme on $n+1$ points. We shall prove that $|S| \leq ((n+1)^2 + (n+1) - 2)/2$. Let $\overrightarrow{ab}$ be the last call in $S$. a knows each gossip before this call, in particular, he knows $g(b)$. b knows only $g(b)$ before the call $\overrightarrow{ab}$, otherwise he would hear a gossip for the second time. Hence all edges adjacent to $b$ and different from $\overrightarrow{ab}$ are of the form $\overrightarrow{bc}$ for some vertex $c$. Since no edge $\overrightarrow{bc}$ can occur twice in $S$, at most $n+1$ edges are adjacent to $b$ in $S$. Deleting all edges of $S$ adjacent to $b$, we obtain a sequence of calls $T$ among $n$ persons. As we saw in the proof of Theorem 5.1, $T$ is a good scheme, so, by the inductional hypothesis, $|T| \leq (n^2 + n - 2)/2$, and $|S| \leq (n^2 + n - 2)/2 + (n+1) = ((n+1)^2 + (n+1) - 2)/2$. \[\square\]
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