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The Ohio State University Ph.D. 1985

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FINITE ELEMENT RESPONSE MODELING OF CRACK GEOMETRIES INDUCED BY HYDRAULIC FRACTURING

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

By

Hussien A. Khattab, B.S., M.Eng.

*****

The Ohio State University

1985

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TO
MY PARENTS
WITH ALL MY LOVE
ACKNOWLEDGEMENTS

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PUBLICATIONS (continued)


"Earthquake Response of a Dam-Reservoir System," Masters Thesis, Carleton University, Ottawa, Ontario, Canada, June 1980
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NOMENCLATURE

\[ A \quad = \quad \text{Fracture surface area} \]
\[ A_{-} \quad = \quad \text{Vertical cross-sectional area} \]
\[ C \quad = \quad \text{Fluid loss coefficient} \]
\[ E \quad = \quad \text{Young's modulus} \]
\[ f, g \quad = \quad \text{Even, odd components of the effective pressure, } \Delta p, \text{ respectively} \]
\[ G \quad = \quad \text{Elastic shear modulus} \]
\[ h \quad = \quad \text{Crack half-height} \quad (h = (h_u + h_L)/2) \]
\[ h_u \quad = \quad \text{Crack upper-height} \]
\[ h_L \quad = \quad \text{Crack lower-height} \]
\[ K' \quad = \quad \text{Fluid consistency index} \]
\[ L \quad = \quad \text{Fracture half-length} \]
\[ n' \quad = \quad \text{Flow behavior index} \]
\[ P \quad = \quad \text{Pressure of the fracturing fluid} \]
\[ \Delta P \quad = \quad \text{Effective crack pressure} \quad (\Delta P = P - \sigma) \]
\[ Q_x \quad = \quad \text{Integrated flow rate over any vertical cross-section height} \]
\[ Q_0 \quad = \quad \text{Bore hole flow rate} \]
\[ r \quad = \quad \text{Inverse of flow behavior index} \quad (r = 1/n') \]
\[ [S] \quad = \quad \text{Pressure-width compliance matrix} \]
\[ W \quad = \quad \text{Crack width} \]
\[ \sigma \quad = \quad \text{Minimum horizontal in situ stress} \]
\[ \nu \quad = \quad \text{Poisson's ratio} \]
NOMENCLATURE (Continued)

\( \eta \) = Normalized position along the vertical axis

\( \zeta \) = Normalized position along the horizontal axis
CHAPTER I
INTRODUCTION

1.1 General

Hydraulic fracturing is a stimulation technique used to increase well productivity via enhanced reservoir permeability. The process entails creation of a fracture system in the formation of interest by injecting a pressurized fluid through a borehole to cause material failure of the geological medium. The induced fracture is then propped open by proppant particles that are added to the fracturing fluid, so that the created fractures function as communication channels to recover crude oil or natural gas from the formation through the wellbore.

The hydraulic fracturing technique has been used to overcome some of the shortcomings of other well stimulation techniques that have been tried and also accomplishes other basic functions such as overcoming wellbore damage, creating deep penetrating reservoir fracture to improve the well productivity, aiding in secondary recovery operation and assisting in the injection or disposal of brine and industrial waste materials.

1.2 Literature Review

One of the earliest rigorous hydraulic fracture modeling efforts is due to Kristianovich and Zeltov [1] with the plane strain assumption of a vertical fracture induced by a highly viscous fluid and the frac
fluid flow models introduced by Carter [2]. Subsequent coupled flow-structural models by Perkins and Kern [3], Nordgren [4], Geertsma and DeKlerk [5], and Daneshy [6] have provided fundamental information on fracture width, length, and leak-off characteristics. Critiques of these models, pertinent assumptions, and numerical comparisons have been presented by Geertsma and Haafkens [7]. Related research on hydraulic fracture mechanisms is also due to Haimson and Fairhurst [8], Seth and Gray [9], Wong and Farmer [10], and Shuck and Advani [11].

Incorporation of the effects of elasto-diffusive coupling for investigating hydraulic fracture propagation is due to Rice and Cleary [12] and Ruina [13]. Numerical studies on models with discrete variations of in situ stress have been conducted by Simonsen et al. [14] and Advani et al. [15]. The effects of multi-layering have been investigated by Cleary [16], Advani and Lee [17], Daneshy [18], van Eekelen [19], and Hanson [20]. Recent research applicable to vertical hydraulic fracture design and configuration prediction has been reported by Clifton and Abou-Sayed [21], Cleary [22,23], Mastrojannis et al. [24], Palmer and Carroll [25,26], Warpinski et al. [27], Rubin [28], Settari and Cleary [29,30], Cleary et al. [31,32], McLeod [33], and Nilson and Griffiths [34]. Of particular note is a basic theory of two dimensional fracture propagation, using a Lagrangian formulation, developed by Biot et al. [35]. This theory has been successfully used for the design of fracture treatments in oil and gas reservoirs in Canada, California, the mid-continent and Rocky Mountains, the North Sea, Gulf Coast, and in northern Germany. A recent overview of current hydraulic fracturing design and treatment technology has been presented by Veatch [36].
Various field tests associated with the Eastern Gas Shales Project and Western Gas Sands Project [37,38] along with mine back experiments [39,40] have been performed. A Multi-Well Experiment (MWX) is currently being conducted by Sandia National Laboratories [41] to enhance technology and energy recovery related to the low permeability, lenticular gas sands in the West. A mine back program in the East is currently in the planning stages [42]. Results of several small scale experiments have been presented by Teufel and Clark [43], Anderson [44], Rubin [45], Ingraffea [46], Biot et al. [47], Kenner et al. [48], Schmidt [49] and Papadopoulos et al. [50]. These laboratory experiments have investigated the roles of fracture toughness, multilayering, in situ stresses, and friction across interfaces in fracture growth. Interpretations of field fracture parameters from pressure decline curves have been made by Nolte [51]. Research on proppant transport has been reported by Novotny [52] and a related summary of proppant transport models and settling velocity correlations has been presented by Clark and Quadir [53] and Clark and Guler [54]. Detailed critiques of various 2D and 3D hydraulic fracture models along with work on vertical fracture growth have been recently presented by Mendelsohn [55,56]. Subsequent research on the analysis of growth and interaction of multiple hydraulic fractures has been presented by Narendran and Cleary [57]. Numerical simulations of hydraulic fracture models have been conducted by Barree [58] and Roegiers and Ishijima [59]. Several related papers on hydraulic fracture geometry prediction were also presented by Ahmed [60], Keck et al. [61], Abousayed et al. [62,63], and Palmer and Craig [64] at the 1984 Unconventional Gas Recovery Symposium sponsored by SPE/DOE/GRI. A recent paper
by Warpinski and Teufel [65] has examined the role of geological discontinuities (joints, faults, and bedding planes) and in situ stress on hydraulic fracture propagation.

1.3 Objectives of the Present Research

The main objective of this research is to develop an advanced hydraulic fracture model which is consistent with the current state-of-the-art in computational mechanics.

Accordingly, the studies presented here are approached in the following stages:


2) Development of a computer code for predicting the fracture width, length, pressure profile, and stress intensity factors.

3) Formulation of a pseudo three-dimensional hydraulic fracture model describing the two-dimensional vertical fracture configuration and fracture width using finite element methodology.

4) Development of an associated computer program for predicting the three-dimensional fracture geometry, and pressure profiles for formations subjected to symmetric and asymmetric in situ stresses.
1.4 **Scope of the Research**

The two-dimensional hydraulic fracture model is presented in Chapter II. First, the governing fluid flow equations in the vertically induced fracture are reviewed. Then the associated finite element formulations are presented. The role of Newtonian and non-Newtonian fluids is considered in the formulations.

Chapter III presents the advanced pseudo three-dimensional hydraulic fracture model analysis, simulations and associated code development.

Numerical results for selected two- and three-dimensional cases in terms of transient fracture configuration and fluid flow pressure profiles are presented in Chapter IV.

Chapter V summarizes results and discusses the significance of the present work and research recommendations.

Detailed computational procedures and supporting materials are provided in the Appendices.
CHAPTER II
TWO-DIMENSIONAL FINITE ELEMENT MODEL

2.1 Introduction

This chapter reviews the governing fluid flow equations in the vertically induced fracture and presents the associated finite element formulations for the fluid pressure profiles, fracture width and fracture length, considering fluid leak-off in the formation. The role of Newtonian and non-Newtonian fluids in the formulations is included. The present idealized vertical fracture has the following assumptions (Figure 2.1):

1. The fracture height, H, is assumed to be constant.
2. For each vertical cross section perpendicular to the direction of propagation, the fluid pressure, P, is assumed to be constant.
3. The pressure drop, dP, in the horizontal direction is governed by the flow resistance in a narrow elliptically shaped flow channel.
4. The fluid pressure at the propagating edge of the fracture is equal to the tectonic stress, \( \sigma_{HMIN} \), i.e., at \( x=L \), the effective pressure, \( \Delta P \), is zero.

2.2 Governing Equations

Based on the previous assumptions, the elliptical width profile \( W \) can then be characterized, according to England and Green [70],
Figure 2.1 Vertical Fracture Model Idealization
\[ W(x,z,t) = \frac{(1-\nu) \Delta p(x,t)}{G} H(1 - \left(\frac{2z}{H}\right)^2)^{1/2} \]  
(2.1)

where \( G, \nu \) are the formation shear modulus and Poisson's ratio, respectively and \( \Delta p = p - \sigma_{\text{HMIN}} \) is the effective crack opening pressure. From the definition of the maximum width \( W(x,t) = W(x,0,t) \) and Eq.(2.1) we have

\[ \Delta p(x,t) = \frac{G}{H(1-\nu)} W(x,t) \]  
(2.2)

For a non-Newtonian, power-law fluid, the shear stress \( \tau_{ij} \)-shear rate \( \dot{\gamma}_{ij} \) relationship is given by

\[ \tau_{ij} = K' (\dot{\gamma}_{ij})^{n'} \]  
(2.3)

where \( K' \) is the consistency index and \( n' \) is the flow behavior index. For \( n'=1 \), the fluid is Newtonian and \( K'=\mu \) is the conventional fluid viscosity.

The pressure gradient for flow in an elliptical channel, from momentum considerations, is given by

\[ \frac{\partial}{\partial x} (\Delta p) = - \frac{64 K'}{3 \pi} \frac{Q}{W} \left[ \frac{2n'+1}{n'} \right] \]  
(2.4)

where \( Q \) is the flow rate.

For an incompressible fluid, the continuity equation is

\[ \frac{\partial Q}{\partial x} + \frac{\pi H \partial W}{4 \partial t} + Q_L = 0 \]  
(2.5)
where \( Q_L = \frac{2HC}{\sqrt{t-\tau(x)}} \) is the fluid leak-off rate,

\( C \) is the fluid loss coefficient, and

\( \tau(x) \) is the elapsed time after the frac fluid reaches a point \( x \).

The loss coefficient, for an incompressible fluid with constant crack pressure, from Darcy's law is given by

\[
C = \left[ \frac{K \Delta p}{2\mu} \right]^{1/2}
\]

(2.6)

where \( \phi \) is the formation porosity and \( K \) is the formation permeability.

Differentiating Eq.(2.2) with respect to \( x \) and introducing the result into Eq.(2.4), we get

\[
Q = \left( \frac{1}{M} \right) \frac{\partial}{\partial x} \left( \frac{1}{2} W^{2n'+2} \right)^r ,
\]

(2.7)

where

\[
M = - \frac{128K'(1-\nu)}{3\pi G} \left( \frac{1-n'}{n'+1} \right) (2+\nu) n' ,
\]

and

\[
r = \frac{1}{n'}
\]

Substituting Eq.(2.7) into Eq.(2.5) yields

\[
\left( \frac{1}{M} \right) \frac{\partial}{\partial x} \left[ \left( \frac{\partial}{\partial x} W^{2n'+2} \right)^r \right] + \frac{nH}{4} \frac{\partial W}{\partial t} + \frac{2HC}{(t-\tau)^{1/2}} = 0.
\]

(2.8)

Equation (2.8) is subjected to the initial condition \( W(x,0)=0 \) for \( 0 \leq x \leq L(t) \), and the boundary condition \( W(x,t)=0 \) for \( x \geq L(t) \).

For a two-sided fracture, and uniform flow rate \( Q(0,t) = Q_0 / 2 = Q_2 \), the appropriate boundary condition is

\[
\left( \frac{1}{M} \right) \left( \frac{\partial}{\partial x} W^{2n'+2} \right)^r = Q_2
\]

(2.9)
The length at a given instant, for the case of the fluid occupying the entire fracture, can be obtained from the volume balance associated with the pumped fluid, fracture volume, and leak-off volume. For a uniform flow rate \( Q_0 \) we have

\[
\frac{dV}{dt} + \int_0^t \frac{C}{(t-T)^{1/2}} \, dA = Q_0 
\]

where the fracture volume \( V(t) \) and the fracture surface area \( A(t) \) are given by

\[
V(t) = \frac{\pi H}{2} \int_0^L W(x,t) \, dx \quad (2.11)
\]

\[
A(t) = 4HW(t) \quad (2.12)
\]

Introducing Eqs. (2.11) and (2.12) into Eq. (2.10) gives

\[
\frac{d}{dt} \int_0^L W(x,t) \, dx + \frac{8C}{\pi} \int_0^t \frac{dL}{dt} \frac{d}{dt} (t-T)^{1/2} = \frac{4Q_0^2}{\pi H} \quad (2.13)
\]

2.3 Finite Element Formulations

Various finite element formulations can be developed for the set of governing equations (2.8) and (2.13). An attractive way is to introduce a simple transformation \( x = \xi L(t) \), with \( \xi \in [0,1] \), to minimize difficulties associated with the moving boundary in Eq. (2.8), which permits Eqs. (2.8) and (2.13) to be written as

\[
\left( \frac{1}{LH} \right)^r \frac{1}{L} \frac{\partial}{\partial \xi} \left[ \frac{3}{3r} W^{2n' + 2} \right] + \frac{\pi H}{4} \xi \frac{\partial}{\partial \xi} \frac{\partial W}{\partial t} + \frac{\pi H}{4} \frac{\partial W}{\partial t} + 2HC \cdot f(t,\tau) = 0 \quad (2.14)
\]
\[ \frac{d}{dt} \left[ L \int_0^1 W(\xi, t) d\xi \right] + \frac{8C}{\pi} \int_0^t f(t, \tau) \frac{dL}{d\tau} d\tau = \frac{4Q_2}{mH} \]  \hspace{1cm} (2.15)\

respectively, with \( f(t, \tau) = 1/(t-\tau)^2 \). The weak form of the width equation (2.14) is given by

\[ \int_0^1 \left( \frac{1}{LM} \right)^r \frac{1}{L} \frac{\partial}{\partial \xi} \left[ \left( \frac{\partial}{\partial \xi} W^{2n+2} \right)^r \right] d\xi + \int_0^1 \frac{mH}{4} \xi \left[ \frac{\partial W}{\partial \xi} \right] \right] d\xi + \right] d\xi + \int_0^1 2HC \cdot f(t, \tau) v \cdot d\xi = 0 \]  \hspace{1cm} (2.16)\

Integrating the first term by parts and using the boundary condition (2.9) gives

\[ \int_0^1 \left( \frac{1}{LM} \right)^r \frac{1}{L} \left( \frac{\partial}{\partial \xi} W^{2n+2} \right)^r \frac{\partial W}{\partial \xi} d\xi + \int_0^1 \frac{mH}{4} \xi \left[ \frac{\partial W}{\partial \xi} \right] \right] d\xi + \right] d\xi + \int_0^1 2HC \cdot f(t, \tau) v \cdot d\xi = 0 \]  \hspace{1cm} (2.17)\

Now Eq.(2.17) can be discretized on the fixed interval \([0,1]\) by taking, with the usual summation convention on repeated indices,

\[ W(\xi, t) = W_j^i(t) N_j(\xi) \]  \hspace{1cm} (2.18)\

and use of the Galerkin approach to yield

\[ K_{ij}(W,L)W_j^i(t) + C_{ij}W_j^i(t) + F_i = 0 \]  \hspace{1cm} (2.19)\

where
As for the prediction of the fracture length, we let
\[ g(t) = \int_0^1 W(\xi, t) d\xi \]
and rewrite Eq.(2.15) as
\[ L(t) \frac{dg}{dt} + g(t) \frac{dL}{dt} + \frac{8C}{\pi} \int_0^t f(t, \tau) \frac{dL}{d\tau} d\tau = \frac{4Q_2}{\pi H} \] (2.20)

The third term in Eq.(2.20) dominates and this results in numerical instability when the Galerkin approach is used. The weighted residual method can be applied to the integro-differential equation (2.20) by modifying the procedure outlined by Reddy and Murty [66]. The weak form of Eq.(2.20) can now be written as
\[
\int_0^T L(t) \frac{dg}{dt} \psi_i dt + \int_0^T g(t) \frac{dL}{dt} \psi_i dt \\
+ \int_0^T \left( \frac{8C}{\pi} \int_0^t f(t, \tau) \frac{dL}{d\tau} d\tau \right) \psi_i dt = \frac{4Q_2}{\pi H} \int_0^T \psi_i dt \] (2.21)
The upwinding functions $\psi_i$ are expressed in the form [67], Figure (2.2)

$$\psi_i(t) = \psi_i(t) + \alpha F(t)$$

$$\psi_i(t) = \psi_i(t) - \alpha F(t)$$

where $\psi_i(t)$ are the usual isoparametric shape functions and $F(t) = \frac{3}{4} (1 + t)(1 - t)$.

The length equation (2.21) can now be discretized in the fluid injection time interval $[0, T]$ by taking

$L(t) = L_{\text{inj}}(t)$, and

$g(t) = G_{\text{inj}}(t)$

to obtain

$$(S_{ij} + T_{ij})L_{ij} = R_i$$

where the elements are taken in the time domain and

$$S_{ij} = \int_0^T (G^{k_i}_j) \psi_i \frac{d\psi_j}{dt} dt + \int_0^T (G^{k_i}_j) \psi_i \psi_j dt$$

$$T_{ij} = \frac{8C}{\pi} \int_0^T \int_0^T f(t, \tau) \frac{d\psi_i}{dt} \psi_j \psi_i d\tau$$

$$R_i = \frac{4}{\pi H} \int_0^T Q_2 \psi_i dt$$

2.4 Solution Technique

The strongly coupled semi-discrete equations (2.19) and (2.23) must be solved simultaneously to obtain fracture width and length at
Figure 2.2 Upwinding Functions $\psi_i$
a given time. For convenience in computations, the number of time steps
used to solve (2.19) can be considered as the number of time elements
in (2.23) having nodal points at the same time stations. The usual two-
point time stepping scheme along with Newton-Raphson iteration will be
used to solve the width equation (2.19) when the fracture length \(L(t)\)
is known. In this scheme, for each time interval, the time rate of
change of \(\dot{W}\) at a particular time \(t^*\) is given by

\[
\dot{W}^* = \frac{1}{\Delta t} (W_{n+1} - W_n)
\]

with \(t^* = (1-\alpha) t_n + \alpha t_{n+1}\), \(\alpha \in (0,1)\).

Moreover a linearized behavior of \(W^*_t\) is assumed, i.e.

\[
W^*_t = (1-\alpha) W_n + \alpha W_{n+1}
\]

Rewriting Eq.(2.19) with the preceding approximations and \(\alpha=1\), we
get

\[
Y_i = (K_{ij}(W_{n+1}) + \frac{1}{\Delta t} C_{ij} W_{n+1}^j + (F_i - \frac{1}{\Delta t} C_{ij} W_n^j) = 0
\]

Using the curtailed Taylor series expansion for \(Y_i\) in the above, we
obtain

\[
Y_i^{m+1} = Y_i^m + [\frac{\partial Y_i}{\partial W}]^m \Delta W_i^m = 0
\]

(2.24)

where \(m\) designates the \(m\)th iteration and

\[
[\frac{\partial Y_i}{\partial W}]^m = [K_{ij}(W_{n+1}) + \frac{1}{\Delta t} C_{ij} + \frac{\partial K_{ij}}{\partial W} W_{n+1}^j]^m
\]

Solution of Eq.(2.24) yields \(\Delta W_i^m\), and hence \(W_{i}^{m+1}\) can be obtained from

\[
W_{i}^{m+1} = W_i^m + \Delta W_i^m
\]
The iterative process for Eq. (2.24) is continued until a satisfactory convergence is reached. Using the obtained nodal values of the crack width $W$, Eq. (2.23) can be solved for an improved value of the half fracture length $L(t)$. The following steps can be taken for a successive approximation:

i) set $m=0$ and initialize

ii) assume $L_m(t)$ according to Eq. (4.2) for each time node $t \in [0,T]$ (8)

iii) solve (2.19) with the assumed $L_m(t)$ for each time step to obtain $W^m_m$

iv) solve (2.23) using $W^m_m$ to obtain $L_{m+1}$ containing new time nodal values

v) if $\max |L_{m+1} - L_m| \geq \varepsilon_{tol}$, go to step iii) with $m=m+1$

vi) print and stop

The Newton-Raphson method used in step iii) requires 5-10 iterations while the successive approximation involving steps iii) and iv) converges within 2-5 iterations, indicating that the fracture length estimate (Eq. (4.2)) is fairly good. The flow chart for this iterative procedure is illustrated in Fig. 2.3.

2.5 Stress Intensity Factor Concept

The instantaneous crack opening mode stress intensity factor $K_I$ during fracture extension is given by [70]

$$K_I(t) = \frac{1}{(\pi L)^{1/2}} \int_{-L(t)}^{L(t)} \Delta p(x,t)(\frac{L+x_0}{L-x})^{1/2} dx$$

(2.25)
START

READ INPUT DATA

SET \( m = 0 \) AND ASSUME \( \ell_m(t_n) \) FOR EACH TIME NODE

\[ t_n [0,T] \ (n=1,2,...,NTIM) \]

NTIM = NUMBER OF TIME NODES

SOLVE

\[ K_{ij}(W,L)\dot{W}^j(t) + C_{ij}W^j(t) + F_i(t) = 0 \]

FOR \( \dot{W}(t) \) AT EACH TIME NODE

USE OBTAINED VALUES OF \( \dot{W}(t) \) TO SOLVE

\[ (S_{ij} + T_{ij})L^j = R_i \]

FOR \( \ell_{m+1}(t_n) \)

\[ |L^i_{m+1} - L^i_m| \leq \varepsilon_{TOL} \]

YES

STOP

Figure 2.3 Schematic Flow Chart for Two-Dimensional Model
Using the effective pressure width relation (2.2) for the horizontal plane, Eq. (2.25) yields

\[ K_1(t) = \frac{G}{2(1-\nu)L^2} \int_{-L(t)}^{L(t)} W(x,t) \left( \frac{L+x}{L-x} \right)^{1/2} dx \] (2.26)

Introducing the coordinate transformation \( \xi = x/L \) and the symmetry argument in Eq. (2.26), we get

\[ K_1(t) = \frac{G}{2(1-\nu)L^2} \int_0^1 \frac{W(\xi,t)}{(1-\xi^2)^{1/2}} d\xi \] (2.27)

Equation (2.27) can be numerically integrated using the Gauss-quadrature rule after solving for \( W(\xi,t) \) at each time step.

Numerical results and comparisons with other known solutions are presented in Ref. [ ].
3.1 Introduction

The first generation hydraulic fracture models assume a constant height corresponding to the payzone height. These models, which are classified as two-dimensional models, may adequately represent reservoirs where the fracture is contained in the payzone by material interfaces or in situ stress differentials. When these containment features do not exist, these models will lead to an inaccurate fracture geometry description.

In this chapter, formulations for a sophisticated pseudo three-dimensional finite element model are presented. This model employs a numerical approach to solve coupled non-linear partial differential equations for the fluid pressure and fracture dimensions. The fracture is first divided into a number of discrete vertical sections. Fracture width and pressure evaluations are conducted by assuming, for each vertical crack, one-dimensional flow in the vertical direction. The model then utilizes the fluid flow along the horizontal direction to predict the fracture height and length. An iteration process is designed until satisfactory convergence is attained.
3.2 Governing Equations

A schematic of the fracture geometry is illustrated in Figure 3.1. For an elastic isotropic medium, subjected to a non-uniform tectonic stress, the fracture width, \( W \), of a vertical line crack can be expressed as a function of effective crack pressure, \( \Delta P \), and the crack half-height, \( h \), in the form [70]

\[
W(\eta,t) = \frac{4(1-\nu)}{-G} h \left[ \int_{-1}^{1} \frac{ds}{|\eta|} \int_{0}^{S} \frac{f(u,t)}{\sqrt{S-u^2}} \, du \right. \\
+ \int_{-1}^{1} \frac{ds}{|\eta|} \int_{0}^{S} \frac{ug(u,t)}{\sqrt{S-u^2}} \, du \right] 
\]  
(3.1.a)

where \( G \) and \( \nu \) are the shear modulus and Poisson's ratio of the elastic medium, \( h = (h_u + h_L)/2 \) is the crack half-height with \( h_u \) and \( h_L \) as the crack upper and lower heights, \( \Delta P(u,t) = f(u,t) + g(u,t) \) is the effective crack pressure [\( \Delta P = P(u,t) - \sigma(u) \)] with \( f(u,t) \) and \( g(u,t) \) as its even and odd components with respect to the normalized coordinate \( \eta \), \( P(u,t) \) is the fluid pressure, and \( \sigma(u) \) is the tectonic stress, and \( \eta = y/h \) is the normalized position along the crack \((-1 \leq \eta \leq 1)\). Equation (3.1.a) is subjected to the boundary conditions

\[ W(\pm 1, t) = 0 \]  
(3.1.b)

and the initial condition

\[ W(\eta, 0) = 0 \]  
(3.1.c)

The fluid flow in the vertical direction is given by (Appendix E)

\[
\frac{\partial}{\partial y} \left[ \gamma r^2 + r \left( \frac{\partial P}{\partial y} \right) r \right] + q_L + \frac{\partial W}{\partial t} = 0 \]  
(3.2.a)

where \( q_L = 2C/\sqrt{t} \) is the fluid leak-off rate, \( C \) is the fluid loss
Figure 3.1 Schematic of Vertical Fracture Configuration

Figure 3.2 Vertical Fracture Width Profile
coefficient, and parameters $r$ and $\gamma$ are related to the conventional non-Newtonian frac fluid properties as follows:

\[
\frac{1}{\gamma} = -K'(2+r)^n'(2)^{n'+1},
\]

(3.2.b)

\[r = 1/n'.
\]

(3.2.c)

Equation (3.2.a) is subjected to the boundary condition

\[P(+1,t) = \sigma(+1).
\]

(3.2.d)

For an incompressible fluid, the frac fluid continuity equation along the horizontal direction is given by

\[
\frac{\partial Q_x}{\partial x} + \frac{4Ch}{\sqrt{t-\tau}} + \frac{3A_1}{3t} = 0
\]

(3.3.a)

where $Q_x(x,t)$ is the integrated flow rate over the height at any point $x$ on the horizontal direction, and $A_1(x,t)$ is the cross-sectional area at the same point $x$. In the above, $Q_x$ and $A_1$ are given by the integrals

\[
Q_x(x,t) = \int_{-h}^{h} [(\beta)^r W^{2+r} (\frac{\partial P}{\partial x})^{r}] dy
\]

(3.3.b)

with

\[
\frac{1}{\beta} = -\frac{16}{3\pi} K'(2+r)^n'(2)^{n'+1},
\]

and

\[
A_1(x,t) = \int_{-h}^{h} W(y,t) dy
\]

(3.3.c)

Equation (3.3.a) has boundary conditions in the form

\[Q_x(+L,t) = 0, \quad \text{and} \]

(3.3.d)

\[Q_x(0,t) = Q_0/2.
\]

(3.3.e)

For a uniform borehole flow rate $Q_0$, the volume balance associated with the injected fluid, fracture volume, and leak-off volume is given by the integral equation
\[
\frac{d}{dt} \left[ \int_0^L A_1(x,t) \, dx \right] + \int_0^t \frac{dA}{\sqrt{t-T}} \, dt = Q_2 \tag{3.4.a}
\]

where \( A(t) \) is the surface area of the fracture approximated by

\[
A(t) = \int_0^L 4h(x,t) \, dx, \tag{3.4.b}
\]

\( L(t) \) is the fracture half-length and \( Q_2 = Q_0/2 \).

3.3 Finite Element Formulations

3.3.1 Crack width-effective pressure formulations

The numerical formulations for the integral equation (3.1.a), relating the crack width and the effective pressure, are difficult. To simplify this formulation, the domain \( R \) over which the Eq. (3.1.a) is valid, can be split into two simple domains \( R_1 \) and \( R_2 \) as shown in Figures 3.3 and 3.4. Equation (3.1.a) can now be written as:

\[
W(n_i, t) = \frac{4(1-\nu)}{\pi G} \int_0^1 f(u, t) du \int_0^1 \frac{s ds}{\sqrt{(s^2-n_i^2)(s^2-u^2)}}
\]

\[
+ \frac{4(1-\nu)}{\pi G} \int_0^1 f(u, t) du \int_0^1 \frac{s ds}{\sqrt{(s^2-n_i^2)(s^2-u^2)}}
\]

\[
+ \frac{4(1-\nu)}{\pi G} \int_0^1 \left| u g(u, t) du \right| \frac{n_i ds}{\sqrt{(s^2-n_i^2)(s^2-u^2)}}
\]

\[
+ \frac{4(1-\nu)}{\pi G} \int_0^1 \left| u g(u, t) du \right| \frac{n_i ds}{s \sqrt{(s^2-n_i^2)(s^2-u^2)}} \tag{3.5}
\]

The normalized crack height is divided into \( N \)-elements with \((N+1)\) nodes, as shown in Figure 3.5. We let \( f(u, t) \) and \( g(u, t) \) to be
Figure 3.3 Schematic of Domain $R$ Over Which Eq. (3.1.a) is Applied

Figure 3.4 Schematic of Domains $R_1$ and $R_2$ for Eq. (3.5)
Numbers in circles refer to node numbers.

Numbers in squares refer to element numbers.

Figure 3.5 Finite Element Mesh for the Line Crack
approximated by

\[ f(u,t) = \phi_i^e F_i^e \]

\[ g(u,t) = \phi_i^e G_i^e \]

(3.6.a)

(3.6.b)

where \( \phi_i^e \) (i=1,2) are the interpolation functions and \( F_i^e \), \( G_i^e \) are the nodal values of the functions \( f \) and \( g \). Using the approximations (3.6.a) and (3.6.b), Eq.(3.5) yields

\[ W_i = \frac{4(1-\nu)}{\nu G} \left( I_{11} + I_{12} + I_{21} + I_{22} \right) \]

(3.7.a)

where

\[ I_{11} = \sum_{e=1}^{E} \int_e \left( \phi_1^e F_1^e + \phi_2^e F_2^e \right) A_1^e(\eta_i) \]

(3.7.b)

\[ I_{12} = \sum_{e=E_{i+1}}^{E} \int_e \left( \phi_1^e G_1^e + \phi_2^e G_2^e \right) g_1^e(\eta_i) \]

(3.7.c)

\[ I_{21} = \sum_{e=1}^{E} \int_e \left( \phi_1^e G_1^e + \phi_2^e G_2^e \right) u_2^e(\eta_i) \]

(3.7.d)

\[ I_{22} = \sum_{e=E_{i+1}}^{E} \int_e \left( \phi_1^e G_1^e + \phi_2^e G_2^e \right) u_2^e(\eta_i) \]

(3.7.e)

\[ A_1^e(\eta_i) = \int_{-1}^{1} \frac{s ds}{\sqrt{(s^2-\eta_i^2)(s^2-u^2)}} \]

(3.7.f)

\[ B_1^e(\eta_i) = \int_{-1}^{1} \frac{s ds}{\sqrt{(s^2-\eta_i^2)(s^2-u^2)}} \]

(3.7.g)
The nodal values $F_i$ and $G_i$ of the even and odd functions $f$, $g$ can be written in terms of the nodal values of the effective pressure $\Delta P$ by

$$F_i = \frac{1}{2}(\Delta P_i + \Delta P_j)$$  \hspace{1cm} (3.8.a)

$$G_i = \frac{1}{2}(\Delta P_i - \Delta P_j)$$  \hspace{1cm} (3.8.b)

where $j = N+2-i$

Using Eqs. (3.8.a) and (3.8.b) into Eq. (3.7.a), rearranging and expressing the results in a matrix form yields

$$\{W\} = h \{\tilde{W}\}$$  \hspace{1cm} (3.9)

where

$$\{\tilde{W}\} = [S]\{P\} - [S]\{\sigma\}$$  \hspace{1cm} (3.10)

$\{W\}$ is the crack width nodal value, $\{P\}$ is the fluid pressure nodal value, and $\{\sigma\}$ is the tectonic stress nodal value.

### 3.3.2 Vertical fluid flow formulations

Introducing the normalized coordinate $\eta$ into Eq. (3.2.a) gives

$$\frac{1}{h} \frac{a}{\eta} \left[ \gamma \frac{d^2 W}{\eta^2} - \frac{1}{h} \frac{\partial}{\partial \eta} \left( \frac{\partial W}{\partial \eta} \right) \right] + \frac{2C}{\sqrt{\epsilon - 1}} + \frac{\partial W}{\partial t} + \frac{\hat{h}}{h} \frac{\partial W}{\partial \eta} = 0$$  \hspace{1cm} (3.11)
The weak form of Eq.(3.11) can be written as

\[
\frac{1}{h^{1+r}} \int_{-1}^{1} \frac{\partial}{\partial n} \left[ \gamma r \frac{w^2}{v^2} \frac{\partial P}{\partial n} \right] v d\eta + \int_{-1}^{1} \frac{2C}{v^2 - r^2} v d\eta \\
+ \int_{-1}^{1} \frac{\partial W}{\partial t} v d\eta + \frac{h}{n} \int_{-1}^{1} n \frac{\partial W}{\partial n} v d\eta = 0
\]

(3.12)

Integrating the first term of Eq.(3.12) by parts and using the boundary conditions (3.2.d) yields

\[
- \frac{\gamma r}{h^{1+r}} \int_{-1}^{1} w^{2+r} \frac{\partial P}{\partial n} \frac{\partial v}{\partial n} d\eta + \int_{-1}^{1} \frac{2C}{v^2 - r^2} v d\eta + \int_{-1}^{1} \frac{\partial W}{\partial t} v d\eta \\
+ \frac{h}{n} \int_{-1}^{1} n \frac{\partial W}{\partial n} v d\eta = 0
\]

(3.13)

The finite element discretization of Eq.(3.13) can be conducted on the fixed interval \([-1,1]\) by taking

\[ W = \phi_j w^j \quad \text{and} \quad P = \phi_m p^m \]

and using the Galerkin procedure with \( v = \phi_i \) to get

\[
- \frac{\gamma r}{h^{1+r}} \int_{-1}^{1} (\phi_j w^j)^2 + r \frac{\partial \phi_m}{\partial n} p^m \frac{\partial \phi_i}{\partial n} d\eta + \int_{-1}^{1} \frac{2C}{v^2 - r^2} \phi_i d\eta \\
+ \int_{-1}^{1} \phi_j w^j \phi_i d\eta + \frac{h}{n} \int_{-1}^{1} n \left( \frac{\partial \phi_i}{\partial n} w^j \right) \phi_i d\eta = 0.
\]

(3.14)

Introducing Eq.(3.9) into Eq.(3.14) and rearranging in matrix form we obtain

\[ [A](\vec{w}) + [C](\dot{\vec{w}}) + \{F(\vec{w},P)\} = 0 \]

(3.15)

where
3.3.3 Horizontal fluid flow formulations

Introducing the coordinate transformation, \( x(t) = \xi L(t) \), the continuity equation along the horizontal direction, Eq.(3.3.a) can be rewritten as

\[
\frac{1}{L} \frac{\partial Q}{\partial \xi} + \frac{4Ch}{\sqrt{\tau - \tau}} + \frac{\partial A_1}{\partial t} + \frac{\xi}{L} \frac{\partial A_1}{\partial \xi} = 0
\]  

(3.16)

The weak form of Eq.(3.16) is given by

\[
\int_0^1 \frac{1}{L} \frac{\partial Q}{\partial \xi} v d\xi + \int_0^1 \frac{4Ch}{\sqrt{\tau - \tau}} v d\xi + \int_0^1 \frac{\partial A_1}{\partial t} v d\xi + \int_0^1 \frac{\xi}{L} \frac{\partial A_1}{\partial \xi} v d\xi = 0.
\]  

(3.17)

Integrating the first term of Eq.(3.17) by parts and using the boundary conditions Eqs.(3.3.d) and (3.3.e) yields

\[
- \int_0^1 \frac{1}{L} Q \frac{\partial v}{\partial \xi} d\xi + \int_0^1 \frac{4Ch}{\sqrt{\tau - \tau}} v d\xi + \int_0^1 \frac{\partial A_1}{\partial t} v d\xi
\]

\[
+ \int_0^1 \frac{\xi}{L} \frac{\partial A_1}{\partial \xi} v d\xi - \frac{Q}{L} v \bigg|_{\xi=0} = 0.
\]  

(3.18)

The normalized fracture half-length can now be divided into a number of nodes equal to the number of vertical sections, as shown in Figure 3.6. Using Eqs.(3.3.b) and (3.3.c) together with the introduced coordinate transformations yields
• Numbers in circles refer to node numbers
• Numbers in squares refer to element numbers

Figure 3.6 Finite Element Discretization Along the Horizontal Direction

$\xi_1 = 0$

$\xi_{K+1} = 1$
\[ Q = \left( \frac{\partial}{\partial L} \right)^2 \int h^{3+r} \left( \frac{\partial P_0}{\partial \xi} \right)^r \cdot Z_1 \quad , \]  \hspace{1cm} (3.19) \\
\[ A_1 = h^2 \cdot Z_2 \quad \]  \hspace{1cm} (3.20) \\
where \( Z_1 \) and \( Z_2 \) are functions of the crack width along each vertical crack and are given by \\
\[ Z_1 = \int_{-1}^{1} \hat{w}^{3+r}(p_1) \, d\eta \quad , \] \\
\[ Z_2 = \int_{-1}^{1} \hat{w} \, d\eta \quad . \] \\

Introducing Eqs. (3.19) and (3.20) into Eq. (3.18) yields \\
\[- \frac{1}{L} \int_{0}^{1} \left( \frac{\partial}{\partial \xi} \right)^2 h^{3+r} \left( \frac{\partial P_0}{\partial \xi} \right)^r Z_1 \frac{\partial v}{\partial \xi} \, d\xi + \int_{0}^{1} \frac{4Ch}{v^{1-t}} v \, d\xi \]
\[ + \int_{0}^{1} \frac{3}{v^{1-t}} (h^2 Z_2) v \, d\xi + \int_{0}^{1} \xi \frac{\partial}{\partial \xi} \left( \frac{\partial h^2 Z_2}{\partial \xi} \right) v \, d\xi - \frac{0}{L} v \bigg|_{\xi=0} = 0 \]  \hspace{1cm} (3.21) \\
Now Eq. (3.21) can be discretized on the fixed interval \([0,1]\) with the usual summation convention on repeated indices by assuming \\
\[ h(\xi, t) = H^j(t)N_j(\xi) \]  \hspace{1cm} (3.22) \\
and using the Galerkin approach to yield \\
\[ [A](H) + [C](H)[\dot{H}] + [F](H) = 0 \]  \hspace{1cm} (3.23) \\
where \\
\[ A_{ij} = \int_{0}^{1} \frac{4C}{v^{1-t}} N_i N_j \, d\xi \quad , \] \\
\[ C_{ij} = 2 \int_{0}^{1} \left( N_i^m h^m \right)(N_j Z_2^0)N_i N_j \, d\xi \]
\[ F_i = -\frac{R}{L} \int_0^1 (N_i H_m^i)^{3+r} \left( \frac{\partial N_i}{\partial \xi} \right) p_i \left( N_i Z_1^j \right) \frac{\partial N_i}{\partial \xi} d\xi \]
\[ + \int_0^1 (N_i H_m^i)^2 (N_i Z_2^j) N_i d\xi \]
\[ + 2 \int_0^1 \frac{\xi L}{N_i H_m^i} \left( \frac{\partial N_i}{\partial \xi} \right) H^i (N_i Z_2^j) N_i d\xi \]
\[ + \int_0^1 \frac{\xi L}{N_i H_m^i} (N_i Z_2^j) N_i d\xi - \frac{Q_2}{L} N_i \bigg|_{\xi=0} . \]

3.3.4 Volume balance formulations

Introducing the coordinate transformation \( x = \xi L \), Eq.(3.4.a) can be rewritten as

\[ \frac{d}{dt} \left( L(t) \int_0^1 A_1(\xi,t) d\xi \right) + \int_0^t \frac{dA}{\nu t-t} c \frac{d\xi}{\nu t-t} = Q_2 \]

where

\[ A_1(\xi,t) = h^2 \cdot Z_2 \]

\[ A(t) = 4 \int_0^1 h(\xi,t) d\xi . \]

Using Eqs.(3.25) and (3.26) into Eq.(3.24) yields

\[ L(t) \frac{df_1}{dt} + \frac{dL}{dt} f_1(t) + 4 \int_0^t \left[ \frac{dL}{dt} f_2(\tau) + \frac{dL}{dt} f_2(\tau) \right] \frac{c}{\nu t-t} d\tau = Q_2 \]

where

\[ f_1(t) = \int_0^1 h^2(\xi,t) \cdot Z_2(\xi,t) d\xi \]

\[ f_2(t) = \int_0^1 h(\xi,t) d\xi . \]
Equation (3.27) can now be discretized in the fluid injection time interval \([0,T]\) by taking

\[
L(t) = \zeta_j L^j(t) ,
\]

\[
f_1(t) = \phi_K F^K_1(t) , \quad \text{and}
\]

\[
f_2(t) = \phi_m F^m_2(t) .
\]

At any time, \(t_i\), Eq.(3.27) can be written as

\[
\begin{align*}
L^i(\phi_K F^K_1) + (\zeta_j L^j)F^j_1 + 4 \int_0^{t_i} \left[ (\phi_j L^j)(\phi_m F^m_2) + (\phi_j L^j)(\phi_m F^m_2) \right] \frac{C}{\sqrt{t_i - \tau}} \, d\tau - Q_2 = 0
\end{align*}
\]

\[ (3.28) \]

### 3.4 Solution Technique

The discretized Eqs.(3.10), (3.15), (3.23) and (3.28) are strongly coupled equations and they must be solved simultaneously to obtain fracture width, fluid pressure, fracture height profile, and fracture length at a given time. Assuming the fracture half-length \(L(t_i)\) and the fracture half-height profile \(h(\xi,t_i)\) at a given time \(t_i\), Eqs.(3.10) and (3.15) can be solved simultaneously to obtain the values of the fracture width, \(W\), and fluid pressure, \(P\), at each vertical section as shown in Appendix A. Using the obtained values of \(W\) and \(P\), Eqs.(3.23) and (3.28) can be solved to obtain new values for the height profile \(h(\xi,t_i)\) and the fracture half-length \(L(t_i)\). The iteration process is continued until satisfactory convergence is attained, and the solution process then marches with time. Figure 3.7 illustrates the flow chart for this computational procedure.
START
READ INPUT DATA
SET $t = t_0 + \Delta t$

ASSUME INITIAL VALUES FOR HEIGHT PROFILE $h^m(\xi,t)$ AND HALF FRACTURE LENGTH $L^m(t)$

SOLVE EQUATIONS (3.10) AND (3.15) FOR $p$ AND $w$
AT EACH VERTICAL SECTION

USE OBTAINED VALUES OF $p$ AND $w$ TO SOLVE EQUATIONS (3.23) AND (3.28) FOR NEW VALUES OF $h^{m+1}(\xi,t)$ AND $L^{m+1}(t)$

$|h^{m+1} - h^m| \leq \varepsilon_{tol1}$
$|L^{m+1} - L^m| \leq \varepsilon_{tol2}$

YES
$\rightarrow t = T$

NO
$t = t + \Delta t$

STOP

Figure 3.7 Flow Chart for Finite Element Model Computational Procedure
CHAPTER IV
NUMERICAL RESULTS

4.1 Introduction

In this chapter, numerical simulations for the prediction of hydraulic fracture geometry and associated response variables, based on the developed finite element formulations are presented.

4.2 Two-Dimensional Model Simulations

Prior to presenting the numerical results from the preceding formulations, previously deduced fracture width and half fracture length for simplified cases are presented for illustration and numerical comparison.

The Perkins-Kern[4]-Nordgren[5] formulations, with vertical plane dominating stiffness, of the continuity and momentum equations yield

\[ W(0,t) = 4 \left( \frac{2u(1-\nu)Q^2}{3\pi GH} \right)^{1/4} t^{1/8} \]  

(4.1)

\[ L(t) = \frac{Q^2 t^{1/2}}{\pi HC} \]  

(4.2)

for large elapsed times and/or leak-off coefficient, C.

The equations derived by Geertsma and DeKlerk [6] based on horizontal cross-sectional analysis and the concept of equilibrium fracture propagation are
\begin{align*}
W(0,t) &= \left( \frac{84\mu(1-\nu)Q_L^2}{\pi GH} \right)^{1/4} \\
L(t) &= \frac{Q_2 W(0,T)}{16\pi C^2} \left[ \frac{2\beta}{\pi} - 1 + e^{\beta^2} \text{erfc} \beta \right]
\end{align*}

where $W(0,T) = 1.5W(0,t)$, $\beta = 8C\sqrt{E}/\sqrt{\pi} W(0,T)$, and $W(0,T)$ designates the bore hole fracture width when the pump injection stops at time $T$.

An improved version of Perkins-Kern and Nordgren formulations, compatible with the presented numerical framework has been developed. The computational procedure and associated numerical comparisons are presented in Appendix C.

As a basis for numerical comparisons, the preceding two-dimensional model formulations presented in Chapter II have been applied to two cases. In the first case, the numerical results from the developed finite element formulations are compared to those obtained from Eqs. (4.1)-(4.4) using the following hydraulic fracture parameters:

- Flow rate, $Q = 1.58 \text{ m}^3/\text{min} (10 \text{ bpm})$,
- Flow behavior index, $n' = 1$ and 0.63,
- Consistency index, $K' = 0.006 \text{ Pa} \cdot \text{min} (36 \text{ cp})$,
- Fluid loss coefficient, $C = 0.000457 \text{ m/sqrt/min} (0.0015 \text{ ft/sqrt/min})$,
- Total height, $H = 30.48 \text{ m} (100 \text{ ft})$,
- Shear modulus, $G = 17.9 \text{ GPa} (2.6 \times 10^6 \text{ psi})$, and
- Poisson's ratio, $\nu = 0.15$.

Figure 4.1 illustrates the maximum width versus time comparisons at the bore hole for Newtonian and non-Newtonian frac fluid properties. Computed crack pressure profiles, transient fracture lengths, and stress intensity factors are revealed in Figs. 4.2, 4.3 and 4.4, respectively.
Figure 4.1 Bore Hole Width Versus Time Comparisons for Cases in References [4,5,6]
Figure 4.2 Crack Pressure Profile Comparisons for the Present Formulations
Figure 4.3 Transient Half Fracture Length Comparisons for Cases in References [4,5,6]
Figure 4.4 Stress Intensity Factors Versus Time for the Present Formulations
In the second case, comparisons of fracture geometry and fluid pressure profiles [71] are presented in Figs. 4.5, 4.6 and 4.7 for the following parameters:

Flow rate, \( Q = 6.32 \text{ m}^3/\text{min} \) (40 bpm),

Flow behavior index, \( n' = 0.34 \),

Consistency index, \( K' = 0.88 \text{ Pa} \cdot \text{min}^{n'} (0.074 \text{ lb} \cdot \text{s}^{n'}/\text{ft}^2) \),

Fluid loss coefficient, \( C = 2.1 \times 10^{-4} \text{ m}/\sqrt{\text{min}} (6.90 \times 10^{-4} \text{ ft}/\sqrt{\text{min}}) \),

Total height, \( H = 91.44 \text{ m} \) (300 ft),

Shear modulus, \( G = 17.90 \text{ GPa} \) (2.60 \times 10^6 \text{ psi} ), and

Poisson's ratio, \( \nu = 0.15 \).

Tabulated values for this case are compared in Table 4.1.

4.3 Pseudo Three-Dimensional Model Simulations

The developed pseudo three-dimensional finite element model formulations have been applied to several cases. Figures 4.8 through 4.16 illustrate the hydraulic responses for the benchmark case studied in Ref. [68] with modifications in the vertical tectonic stress.

Figure 4.8 reveals the bore hole fracture width versus time for the indicated frac fluid and in situ reservoir properties. The corresponding fracture half-length, bore hole half-height and bore hole fracture fluid effective pressure are illustrated in Figs. 4.9, 4.10 and 4.11, respectively. The evolution of the vertical fracture configuration is shown in Fig. 4.12. Figures 4.13 and 4.14 show the width profiles at three time instants along the horizontal and vertical axes, respectively, and the corresponding pressure plots are shown in Figs. 4.15 and 4.16.
Figure 4.5 Bore Hole Width Versus Time Comparisons for Case in Reference [71]
Figure 4.6 Transient Half Fracture Length Comparisons for Case in Reference [71]
Figure 4.7 Crack Pressure Profile Comparisons for Case in Reference [71]
Table 4.1 Comparison of Responses for Case in Reference [71]

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Present Formulation</th>
<th>Reference [71]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Max. pressure</td>
<td>528.30 psi (3.64 MPa)</td>
<td>493.40 psi (3.40 MPa)</td>
</tr>
<tr>
<td>Max. width</td>
<td>0.619 in (1.574 cm)</td>
<td>0.594 in (1.509 cm)</td>
</tr>
<tr>
<td>Max. half-length</td>
<td>1392 ft (424.30 m)</td>
<td>1800 ft (548.64 m)</td>
</tr>
</tbody>
</table>
Figure 4.8 Borehole Fracture Width Versus Time for Indicated Fracture Fluid and In Situ Reservoir Parameters

- $n' = 1.00$
- $k' = 6.00 \times 10^{-3}$ N·min/m$^2$
- $C = 4.57 \times 10^{-2}$ m$/\sqrt{\text{min}}$
- $G = 17.9$ GPa
- $\nu = 0.20$
- $Q = 10$ bpm
- $T = 60$ min
Figure 4.9 Fracture Half Length Versus Time for Case Presented in Figure 4.8
Figure 4.10 Fracture Half Height Versus Time for Case in Figure 4.8
Figure 4.11 Bore Hole Effective Pressure Versus Time for Case in Figure 4.8
Figure 4.12 Evolution of Vertical Fracture Configuration for Case in Figure 4.8
Figure 4.13 Fracture Width Versus Time Along Horizontal Axis for Case in Figure 4.8
Figure 4.14 Fracture Width Versus Time Along Vertical Axis Along Vertical Axis at Bore Hole for Case in Figure 4.8
Figure 4.15 Effective Crack Pressure Versus Time Along Horizontal Axis for Case in Figure 4.8
Figure 4.16 Effective Crack Pressure Versus Time Along Vertical Axis at Bore Hole for Case in Figure 4.8
Comparisons of fracture configurations with selected simulation results from two models are presented. The first model is the fully 3-D model of hydraulic fracture propagation developed by TERRA TEK [63], and the second model is a pseudo 3-D model developed at ORU [72]. Figures 4.17, 4.19, 4.21, 4.23 and 4.25 illustrate vertical fracture configuration comparisons between the TERRA TEK model and the present formulation for 5 cases with hydraulic fracture parameters given in Table 4.2. Figures 4.18, 4.20, 4.22, 4.24 and 4.26 reveal the computed vertical geometry at three time instants for these cases. Tabulated values for pressure, width, height and length comparisons corresponding to TERRA TEK, ORU and the present formulations are shown in Tables 4.3 and 4.4.
n' = 0.39
K' = 0.120 lb.s^n'/ft^2 (1.1637 N.min^n'/m^2)
C = 0.65x10^{-3} ft/\text{min} (0.19312x10^{-4} m/\text{min})
E = 7.50x10^5 psi (5.17985 GPa)
\nu = 0.20
Q = 40 bpm (6.336 m^3/\text{min})
T = 40 min

Figure 4.17 Vertical Fracture Configuration Comparisons for Case (I)
Figure 4.18 Vertical Fracture Configuration Evolution for Case (I)
\[ n' = 1.00 \]
\[ K' = 0.2 \times 10^{-4} \text{ lb.s/ft}^2 (0.1598 \times 10^{-4} \text{ N.min/m}^2) \]
\[ C = 0.65 \times 10^{-3} \text{ ft}/\sqrt{\text{min}} (0.19812 \times 10^{-3} \text{ m}/\sqrt{\text{min}}) \]
\[ E = 7.5 \times 10^5 \text{ psi} (5.17985 \text{ GPa}) \]
\[ \nu = 0.20 \]
\[ Q = 40 \text{ bpm} (6.336 \text{ m}^3/\text{min}) \]
\[ T = 38 \text{ min} \]

Figure 4.19 Vertical Fracture Configuration Comparisons for Case (K)
Figure 4.20 Vertical Fracture Configuration Evolution for Case (K)
$n' = 0.75$

$K' = 0.07 \text{ lb.s}^{n'}/\text{ft}^2 (0.1557 \text{ N.min}^{n'}/\text{m}^2)$

$C = 0.65 \times 10^{-3} \text{ ft}/\sqrt{\text{min}} (0.19812 \times 10^{-3} \text{ m}/\sqrt{\text{min}})$

$E = 7.5 \times 10^5 \text{ psi} (5.17985 \text{ GPa})$

$\nu = 0.29$

$Q = 40 \text{ bpm} (6.336 \text{ m}^3/\text{min})$

$T = 43 \text{ min}$

Figure 4.21 Vertical Fracture Configuration Comparisons for Case (L)
Figure 4.22 Vertical Fracture Configuration Evolution for Case (L)
$n' = 1.0$

\[ K' = 0.00157 \text{ lb}\cdot\text{s}/\text{ft}^2 = (1.253 \times 10^{-3} \text{ N}\cdot\text{min}/\text{m}^2) \]

\[ C = 0.17 \times 10^{-3} \text{ ft}/\sqrt{\text{min}} = (0.51816 \times 10^{-4} \text{ m}/\sqrt{\text{min}}) \]

\[ E = 5.1881 \times 10^6 \text{ psi} = (35.8311 \text{ GPa}) \]

\[ v = 0.29 \]

\[ Q = 25 \text{ bpm} = (3.960 \text{ m}^3/\text{min}) \]

\[ T = 5.30 \text{ min} \]

Figure 4.23 Vertical Fracture Configuration Comparisons for Case (M)
Figure 4.24 Vertical Fracture Configuration Evolution for Case (M)
$n' = 1.00$

$K' = 0.20 \times 10^{-4} \text{ lb \cdot s/ft}^2 (0.15987 \times 10^{-4} \text{ N\cdotmin/m}^2)$

$C = 0.17 \times 10^{-3} \text{ ft/\sqrt{min}} (0.51816 \times 10^{-4} \text{ m/\sqrt{min}})$

$E = 5.1881 \times 10^6 \text{ psi} (35.8311 \text{ GPa})$

$\nu = 0.29$

$Q = 25 \text{ bpm (3.96 m}^3/\text{min})$

$T = 2.40 \text{ min}$

---

Figure 4.25 Vertical Fracture Configuration Comparisons for Case (0)
Figure 4.26 Vertical Fracture Configuration Evolution for Case (0)
<table>
<thead>
<tr>
<th>Case</th>
<th>$n'$</th>
<th>$K'$</th>
<th>$C$</th>
<th>$E$</th>
<th>$\nu$</th>
<th>$Q$</th>
<th>$T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>0.39</td>
<td>0.120 &amp; (1.637)</td>
<td>0.650x10^{-3} &amp; (0.1981x10^{-3})</td>
<td>7.50x10^{5} &amp; (5.1798)</td>
<td>0.20 &amp; (6.336)</td>
<td>40 &amp; 40</td>
<td></td>
</tr>
<tr>
<td>K</td>
<td>1.00</td>
<td>0.20x10^{-4} &amp; (0.1599x10^{-4})</td>
<td>0.650x10^{-3} &amp; (0.1981x10^{-3})</td>
<td>7.50x10^{5} &amp; (5.1798)</td>
<td>0.20 &amp; (6.336)</td>
<td>40 &amp; 40</td>
<td></td>
</tr>
<tr>
<td>L</td>
<td>0.75</td>
<td>0.070 &amp; (0.1557)</td>
<td>0.650x10^{-3} &amp; (0.1981x10^{-3})</td>
<td>7.50x10^{5} &amp; (5.1798)</td>
<td>0.20 &amp; (6.336)</td>
<td>40 &amp; 43</td>
<td></td>
</tr>
<tr>
<td>M</td>
<td>1.00</td>
<td>0.00157 &amp; (0.001255)</td>
<td>0.170x10^{-3} &amp; (0.5182x10^{-4})</td>
<td>5.1881x10^{6} &amp; (35.8311)</td>
<td>0.29 &amp; (3.96)</td>
<td>25 &amp; 5.30</td>
<td></td>
</tr>
<tr>
<td>O</td>
<td>1.00</td>
<td>0.20x10^{-4} &amp; (0.1599x10^{-4})</td>
<td>0.170x10^{-3} &amp; (0.5182x10^{-4})</td>
<td>5.1881x10^{6} &amp; (35.8311)</td>
<td>0.29 &amp; (3.96)</td>
<td>25 &amp; 2.40</td>
<td></td>
</tr>
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</table>
Table 4.3 Pressure and Width Comparisons

<table>
<thead>
<tr>
<th>Case</th>
<th>Max. Pressure, psi (MPa)</th>
<th>Max. Width, in (cm)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>TT</td>
<td>ORU</td>
</tr>
<tr>
<td>I</td>
<td>92 (0.6354)</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>0.750</td>
<td>-</td>
</tr>
<tr>
<td>K</td>
<td>78 (0.5387)</td>
<td>66 (0.4558)</td>
</tr>
<tr>
<td></td>
<td>0.850</td>
<td>0.620 (1.5748)</td>
</tr>
<tr>
<td>L</td>
<td>138 (0.9531)</td>
<td>134.5 (0.9289)</td>
</tr>
<tr>
<td></td>
<td>0.900</td>
<td>0.940 (2.3876)</td>
</tr>
<tr>
<td>M</td>
<td>845 (5.8360)</td>
<td>716 (4.9447)</td>
</tr>
<tr>
<td></td>
<td>0.310</td>
<td>0.230 (0.5842)</td>
</tr>
<tr>
<td>O</td>
<td>423 (2.9214)</td>
<td>610 (4.2127)</td>
</tr>
<tr>
<td></td>
<td>0.110</td>
<td>0.120 (0.3048)</td>
</tr>
</tbody>
</table>
Table 4.4 Height and Length Comparisons

<table>
<thead>
<tr>
<th>Case</th>
<th>Wellbore Height, ft (m)</th>
<th>One Wing Length, ft (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>TT</td>
<td>ORU</td>
</tr>
<tr>
<td>I</td>
<td>501.3 (152.80)</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>(110.02)</td>
<td>(130.15)</td>
</tr>
<tr>
<td>K</td>
<td>330 (100.58)</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>(82.00)</td>
<td>(202.69)</td>
</tr>
<tr>
<td>L</td>
<td>241.4 (73.60)</td>
<td>239.7 (73.00)</td>
</tr>
<tr>
<td></td>
<td>(82.17)</td>
<td>(134.72)</td>
</tr>
<tr>
<td>M</td>
<td>105.1 (32.03)</td>
<td>100.1 (30.51)</td>
</tr>
<tr>
<td></td>
<td>(32.50)</td>
<td>(120.10)</td>
</tr>
<tr>
<td>O</td>
<td>81.8 (24.93)</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>(25.69)</td>
<td>(115.82)</td>
</tr>
</tbody>
</table>
5.1 Discussion

Many theories have been developed to predict hydraulic fracture geometry. However, applications of finite element methodology to this problem have been limited. Chapter II of this work reviews the governing fluid flow equations in the vertically induced crack and presents the finite element formulations associated with the fluid pressure profiles, fracture width, and length. Fluid leak-off effects in the formation and the role of both Newtonian and non-Newtonian fluids are included in the formulations.

The traditional two-dimensional models, including the work in Chapter II, assume a constant fracture height based on the reservoir characteristics and perforation intervals. As a further refinement, it was essential to replace the constant height assumption with a height profile variation, consistent with the advanced formulations in recent literature [21,22,23,25,26].

To achieve this refinement, a pseudo-three-dimensional hydraulic fracture model using finite element methodology has been developed. This model employs an innovative numerical approach to solve the coupled non-linear partial differential equations for the fracture fluid pressure and induced fracture geometry.
Among others, this model has the following advantages:

1. It considers the fluid flow to consist of a dominant flow $q_x(x,y,t)$, along the payzone direction, in addition to flow in the vertical direction, $q_y(x,y,t)$, of height growth.

2. Fluid leak-off, $q_L(x,y,t)$, in the formation is included for both vertical and horizontal cross-sectional characterizations.

3. An improved finite element methodology for solving linear and nonlinear integral equations of Fredholm and Volterra types has been developed [73]. This method is used to formulation the pressure-width integro-differential equation. This technique removes the need to assume the cross-sectional shape a priori as is done in traditional fracture models. In addition, this technique permits use of any arbitrary in situ stress distributions in the formation, making it possible to address important problems which were previously beyond reach. It also provides a framework for subsequent refinement using layered media with specified bi-material characteristics.

4. The CPU run times for the cases discussed in Chapter IV are reasonable for the present model; typical run requires 45 sec of computational time on an IBM370. This compares with the ORU model simulation time of about 2 min. on a DG Eclipse minicomputer and with a TERRA TEK run of around 2 hours on a DEC 20.

Numerical results for selected cases involving height containment due to discontinuous tectonic stress fields are presented in Chapter IV. The hydraulic fracture simulations employing the finite element methodology, exhibit reasonable overall agreement with available results on fracture length, height, and width for various cases. Case I presented
in Figure 4.17, however, reveals poor comparisons between fracture height and width magnitudes using the Terra Tek results and the presented formulations. The discrepancy may be attributed to the limitations of the presented finite element model for length/height aspect ratios below 2.5, and the lack of a fracture growth criterion in the formulations.

5.2 Conclusions and Recommendations

As a result of the present study, the following conclusions are drawn:

1. The finite element simulations for the constant height hydraulic fracture model yield accurate predictions for the fracture width, pressure and length responses. Reasonable comparisons with the available approximate width and length characterizations [4,5,6,71] are obtained.

2. The two-dimensional model is adequate for cases with very strong stress barriers on both sides of the payzone. However, if the barriers are not strong, and the fracture migration is anticipated, use of the pseudo-three-dimensional model is recommended.

3. The finite element approach has been used to develop a pseudo-three-dimensional model which has been successfully applied to predict fracture geometry and fluid pressure profiles.

4. The developed model is more advanced than the conventional two-dimensional model and simpler than the complex fully three-dimensional model. Therefore, the present model fills the gap that exists between both models.
5. The pseudo-three-dimensional model yields reasonable comparisons with available results on fracture length, height and width for vertically symmetric and unsymmetric tectonic stresses.

6. A modified two-dimensional model employing the fracture criterion, $K_t = K_{IC}$, where $K_t$ is the stress intensity factor and $K_{IC}$ the fracture toughness, has been developed in Appendix D. However, in view of its strong dependence on the formation fracture toughness $K_{IC}$, further work is recommended.

The following recommendations are proposed to enhance the versatility of the developed model:

1. The present model could be modified to employ a realistic fracture criteria as a constituent equation to monitor fracture growth.

2. In the present model the formation is modeled as a homogeneous, isotropic, elastic solid with no variations in elastic properties between zones. The analysis could be extended to include the effects of multi-layered formation with different elastic properties, by incorporating the finite element methodologies for layered media [74].

3. The effects of a bi-material interface with friction, stick-slip characterizations should also be similarly introduced in the model [75].

4. The formulations should be extended to consider the proppant transport and non-Newtonian fluid behavior, and compressible fluid behavior such as foam and other "exotic" fracture fluids.

5. Consideration of material anisotropy and non-linearity in the formation, including effects of natural joints are important and should be included in future work.
6. Interfacing of the developed model to a suitable reservoir production model by coupling of nodal points will yield a comprehensive simulation for potential commercial applications.
LIST OF REFERENCES


APPENDIX A

ITERATIVE SOLUTION FOR HYDRAULIC FRACTURE WIDTH AND PRESSURE EQUATIONS

In this Appendix, the iterative procedure for fracture width and pressure computations related to Section 3.4 is outlined.

For the solution technique in Section 3.4, a one-step integration scheme is considered. In this scheme, for each time interval, the time rate of change of \( W \) at a particular time \( t^* \) is given by:

\[
\dot{W}^* = \frac{1}{\Delta t} (W_{n+1} - W_n)
\]  
(A.1)

with \( t^* = (1-\alpha)t_n + \alpha t_{n+1} \quad \alpha \in (0,1) \)

Moreover, the dependent variables at \( t^* \) are expressed by:

\[
P^* = (1-\alpha)P_n + \alpha P_{n+1}
\]  
(A.2)

\[
W^* = (1-\alpha)W_n + \alpha W_{n+1}
\]  
(A.3)

Rewriting Eqs. (3.10) and (3.15) with the preceding approximations and \( \alpha=1 \) gives

\[
Y_i = \dot{W}_n + S_{ij}P_{n+1} + S_{ij}p^j = 0
\]  
(A.4)

\[
R_i = A_{ij}\dot{W}_n + \frac{1}{\Delta t} C_{ij}W_{n+1} + F_i + \frac{1}{\Delta t} \sum C_{ij}\dot{W}^j = 0
\]  
(A.5)

Using the curtailed Taylor series expansion for \( Y_i \) and \( R_i \) in the above, one obtains
Solution of Eqs. (A.6) gives $\Delta \bar{W}_k$ and $\Delta P_\xi$, and hence $\bar{W}_k$ and $P_\xi$ can be obtained from

$$\bar{W}_{m+1} = \bar{W}_m + \Delta \bar{W}_m \quad \text{and} \quad P_{m+1} = P_m + \Delta P_m$$

where $m$ denotes the iteration number. The iteration process is continued until convergence is reached.
APPENDIX B

ELLIPtical vertical fracture configuration

As a special case of the finite element formulations of the pseudo-three-dimensional model presented in Chapter III, an elliptical vertical fracture configuration is assumed with simultaneous critical stress intensity criteria posed at extreme locations of the major and minor axes of the elliptical crack, in a manner similar to Palmer [25,26].

For this case the approximate fracture width (w) - effective crack pressure (Δp) relation is given by:

\[ W(x,y,t) = c(x,t) \left\{ \frac{su(s,t)}{y/h(x,t), s^2 - (y/h(x,t))^2} \right\} \]  

where

\[ u(s,t) = \int_{s}^{\infty} \frac{P(f,t)}{\sqrt{s^2 - f^2}} \, df , \text{ and} \]

\[ c(x,t) = \frac{2(1-\nu)}{\pi G} h(x,t) \]  

The momentum equations for fluid flow in the crack are selected in the form:

\[ \frac{\partial}{\partial x} (\Delta p) + \frac{n^1}{W^{4n+3}} |q|^2 n_x q_x = 0 \]  

\[ \frac{\partial}{\partial y} (\Delta p) + \frac{n^1}{W^{4n+3}} |q|^2 n_y q_y = 0 \]
where \( q_x, q_y \) are flow rates in the x and y directions, respectively, 
\( |q| = (q_x^2 + q_y^2)^{1/2} \) is the resultant flow rate magnitude, and parameters \( n \) and \( n' \) are related to the conventional non-Newtonian frac fluid properties as follows:

\[
n' = k'[(2+r)2(1+r)]n',
\]
\[
n = (n'-1)/2 ,
\]
\[
r = 1/n'
\]

For an incompressible fluid, the continuity equations is

\[
\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + q_L + \frac{\partial W}{\partial t} = 0
\]

Suitable manipulation of Eqs. (B.4) and (B.5) yields

\[
|q|^{2n} = \left(\frac{\partial P}{\partial x}\right)^{2n+1} \left\{ \left(\frac{\partial P}{\partial x}\right)^2 + \left(\frac{\partial P}{\partial y}\right)^2 \right\}^{(\frac{n}{2n+1})}
\]

Substituting Eq. (B.7) into (B.4) and (B.5), and defining \( \xi = x/L \) and \( \eta = y/H \), we obtain the following expressions for \( q_\xi \) and \( q_\eta \):

\[
q_\xi = -\lambda W^{2+r} \left[ \frac{1}{L} \left( \frac{\partial P}{\partial \xi} \right)^2 + \frac{1}{H} \left( \frac{\partial P}{\partial \eta} \right)^2 \right] \left( \frac{r-1}{2} \right)
\]

\[
q_\eta = -\lambda W^{2+r} \left[ \frac{1}{H} \left( \frac{\partial P}{\partial \eta} \right)^2 + \frac{1}{L} \left( \frac{\partial P}{\partial \xi} \right)^2 \right] \left( \frac{r-1}{2} \right)
\]

with \( \lambda = \left( \frac{1}{n} \right)^r \)

The governing coupled equations for pressure \( (\Delta P) \) and width \( (W) \) now reduce to the following:

\[
\frac{1}{L} \frac{\partial q_\xi}{\partial \xi} + \frac{1}{H} \frac{\partial q_\eta}{\partial \eta} + q_L + \frac{\partial W}{\partial t} + \xi \frac{\partial W}{\partial \xi} + \eta \frac{\partial W}{\partial \eta} = 0
\]
\[ W(\xi, \eta, t) = c(\xi, t) \int_{\beta}^{1} \frac{sd_s}{\sqrt{s^2 - \beta^2}} \int_{0}^{s} \frac{\Delta P(f, t)}{\sqrt{s^2 - f^2}} df \]  

(B.11)

with \( \beta = \frac{hH}{h(\xi, t)} \)

For a symmetric vertical fracture, the following boundary conditions are assumed for Eqs. (B.10) and (B.11).

\[ q_\xi(0, \eta, t) = \frac{Q_0}{2} = Q_2 \]  

(B.12)

\[ \frac{\partial \Delta P}{\partial \eta}(\xi, 0, t) = 0 \]  

(B.13)

\[ W(\xi, \eta, t) = 0 \text{ outside the fracture domain} \]  

(B.14)

The weak form of Eq. (B.10) after use of the Green's theorem, boundary conditions and Eqs. (B.8) and (B.9) is

\[ \int_{\Omega} \lambda W^2 + r \left[ \frac{1}{L^2} \left( \frac{\partial \Delta P}{\partial \xi} \right)^2 + \frac{1}{H^2} \left( \frac{\partial \Delta P}{\partial \eta} \right)^2 \right] \frac{1}{2(r-1)} \left( \frac{1}{L^2} \frac{\partial \Delta P}{\partial \xi} \frac{\partial V}{\partial \xi} + \frac{1}{H^2} \frac{\partial \Delta P}{\partial \eta} \frac{\partial V}{\partial \eta} \right) d\xi d\eta \]

\[ + \int_{\Omega} \frac{2c}{\sqrt{t-t}} v d\xi d\eta + \int_{\Omega} \left( \xi \frac{\partial W}{\partial t} + H \frac{\partial W}{\partial \eta} \right) d\xi d\eta \]

\[ - \frac{Q_2}{L} \int_{S} v ds = 0 \]  

(B.15)

where \( v(\xi, \eta) \) is a pre-multiplying smooth function. Similarly, the weak form of Eq. (B.11) is

\[ \int_{\Omega} W \psi d\xi d\eta = \int_{\Omega} c(\xi, t) \left[ \frac{1}{\beta} \frac{sd_s}{\sqrt{s^2 - \beta^2}} \int_{0}^{s} \frac{P(f, t)}{\sqrt{s^2 - f^2}} df \right] v d\xi d\eta \]  

(B.16)

Equations (B.15) and (B.16) can now be discretized in the domain by expressing
ΔP(ξ,η,t) = ϕ₁(ξ,η)Δp^i(t)

\[ W(ξ,η,t) = ϕ₁(ξ,t)W^i(t) \]

and use of the Galerkin approach to obtain

\[ K_{ij} p^j + A_{ij} W^j + G_{ij} W^j + F_i = 0 \quad \text{(B.17)} \]

\[ G_{ij} W^j - C_{ij} p^j = 0 \quad \text{(B.18)} \]

where

\[ K_{ij} = \int_Ω \left[ (W_{m}^m)^2 + \frac{1}{2} \left( \frac{\partial P}{\partial ξ} \right)^2 + \frac{1}{H} \frac{\partial P}{\partial η} \right] \frac{1}{2(r-1)} \]

\[ \frac{1}{L} \frac{\partial φ_i}{\partial ξ} \frac{\partial φ_j}{\partial ξ} + \frac{1}{H} \frac{\partial φ_i}{\partial η} \frac{\partial φ_j}{\partial η} \frac{1}{2(r-1)} \int_Ω dξdη \]

\[ A_{ij} = \int_Ω \left[ \frac{c(ξ,t)}{L} \frac{\partial φ_i}{\partial ξ} + \frac{H}{H} \frac{\partial φ_i}{\partial η} \right] φ^i dξdη \]

\[ C_{ij} = \int_Ω \left[ \frac{c(ξ,t)}{L} \frac{\partial φ_i}{\partial ξ} + \frac{H}{H} \frac{\partial φ_i}{\partial η} \right] φ^j dξdη \]

\[ G_{ij} = \int_Ω φ_{ij} \phi^j dξdη \], and

\[ F_i = \int_Ω \frac{2c}{\sqrt{t-τ}} φ_i \phi^i dξdη - \frac{2c}{L} \int_Ω \phi_i \phi^i \]

Equations (B.17) and (B.18) can be solved simultaneously using the iterative procedure presented in Appendix A to obtain the nodal values of the pressure and width.

After obtaining the nodal values of the pressure and width with an initially assumed geometry, a least squares technique is used to fit the crack pressure P(x,y) to the cubic equation, in the form
\[ P(x,y) = A_{00} + A_{10}x + A_{01}y + A_{20}x^2 + A_{11}xy + A_{02}y^2 + A_{30}x^3 + A_{21}x^2y + A_{12}xy^2 + A_{03}y^3 \]  

(B.19)

This pressure distribution is supplied as an input for the analytical computations of the crack opening mode stress intensity factors at the intersection points of the major and minor axes with the ellipse. An iterative procedure with modified fracture geometry, using the governing field equations, is adopted until these stress intensity values are equal to the critical stress intensity factor.

The preceding formulations have been applied to the benchmark case studied in Reference [68] with a uniform vertical tectonic stress. In addition, a fracture toughness of 2.2 MPa \( \sqrt{m} \) (2000 psi\( \sqrt{in} \)) is employed.

Figures (7.1) and (7.2) illustrate the time dependent growth of the minor and major axes of the elliptical crack. The corresponding transient configurations are revealed in Fig. (7.3).

Figures (7.4) and (7.5) show the width profiles at three time instants along the minor and major axes, respectively, and the corresponding pressure plots are shown in Figs. (7.6) and (7.7).
Figure 6.1 Elliptical Crack Major Axis Versus Time
Figure 6.2 Elliptical Crack Minor Axis Versus Time
Figure 6.3 Elliptical Crack Transient Configuration
Figure 6.4 Fracture Width Versus Time Along Elliptical Crack Major Axis
Figure 6.5 Fracture Width Versus Time Along Elliptical Crack Minor Axis
Figure 6.6 Effective Crack Pressure Versus Time Along Elliptical Crack Minor Axis
Figure 6.7 Effective Crack Pressure Versus Time Along Elliptical Crack Major Axis
APPENDIX C

MODIFIED VERSION OF PERKINS-KERN FORMULATIONS

The Perkins-Kern [4]-Nordgren [5] formulations entailing the continuity and momentum equations, considering the effect of non-Newtonian fluid flow, yield

\[ W(0,t) = a \alpha(t) \] (C.1)

where \( a = \left( \frac{64}{3} (n' + 1) \right)^{\alpha} \left( \frac{6Q_2}{H} \right)^{\alpha/6} \left( \frac{K_a(1-\nu)H}{6} \right)^{\alpha} \),

\( \alpha = \frac{1}{2(n' + 1)} \),

\( \epsilon = \frac{n'}{2(n' + 1)} \), and

\( K_a = K' \left( \frac{2n' + 1}{3n'} \right)^n \)

The volume balance associated with the pumped fluid, fracture volume, and leak-off volume is given by

\[ \frac{dV}{dt} = Q_2 - C \int_0^t \frac{dA}{dt} \frac{d\tau}{\sqrt{t-\tau}} \] (C.2)

where

\[ V(t) = \frac{\pi H}{4} \int_0^x W(x,t)dx \] (C.3)

The shape of the fracture in the x-direction can be approximated as [7]

95
\[ W(x,t) = W(0,t)(1-x/\xi)^{1/4} \]  

(C.4)

Introducing Eqs. (C.3) and (C.4) into Eq. (C.2) yields

\[ \frac{\pi H}{5} \frac{d}{dt}(W(0,t) \cdot \xi(t)) + 2HC \int_0^t \frac{d\xi}{\sqrt{t-\tau}} = Q_2 \]  

(C.5)

Using Eq. (C.1) into Eq. (C.5) gives

\[ C_1 \frac{d}{dt}(\xi^{\alpha+1}) + C_2 \int_0^t \frac{d\xi}{\sqrt{t-\tau}} = Q_2 \]  

(C.6)

where

\[ C_1 = \frac{\pi H}{5} a, \]  

and

\[ C_2 = 2HC \]

Equation (C.6) is a nonlinear integro-differential equation, which can be solved by using the finite element methodology developed for solving integral equations [73], to obtain \( \xi(t) \) and then using Eq. (C.1) to get \( W(0,t) \). For the sake of comparison, the preceding modified formulations have been applied to two cases and the obtained results are compared to the results from Eqs. (4.1) and (4.2) and the finite element formulations presented in Chapter II. The hydraulic fracture parameters used in both cases are given in Tables (7-1) and (7-2). Figures (7.1) and (7.3) illustrate the maximum width versus time comparisons at the bore hole. The corresponding transient fracture half-lengths are revealed in Figs. (7.2) and (7.4).
Table 7-1 Hydraulic Fracture Parameters Used for Comparisons in Figures 7.1 and 7.2

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Flow Rate</td>
<td>$1.58 \times 10^3$ (bpm)</td>
</tr>
<tr>
<td>Flow Behavior Index</td>
<td>0.76</td>
</tr>
<tr>
<td>Consistency Index</td>
<td>$0.01535 \times 10^2$ (lb.s$^{n'}$/ft$^{2}$)</td>
</tr>
<tr>
<td>Fluid Loss Coefficient</td>
<td>$0.3931 \times 10^{-3}$ (0.1290x10$^{-2}$)</td>
</tr>
<tr>
<td>Total Height</td>
<td>27.43 (90)</td>
</tr>
<tr>
<td>Shear Modulus</td>
<td>13.586 (1.9672x10$^6$)</td>
</tr>
<tr>
<td>Poisson's Ratio</td>
<td>0.22</td>
</tr>
</tbody>
</table>
Table 7-2  Hydraulic Fracture Parameters Used for Comparisons in Figures 7.3 and 7.4

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Flow Rate</td>
<td>1.58 (10)</td>
</tr>
<tr>
<td>Flow Behavior Index</td>
<td>0.46</td>
</tr>
<tr>
<td>Consistency Index</td>
<td>0.1456 (0.020)</td>
</tr>
<tr>
<td>Fluid Loss Coefficient</td>
<td>0.2134x10^{-3} (0.70x10^{-3})</td>
</tr>
<tr>
<td>Total Height</td>
<td>45.72 (150)</td>
</tr>
<tr>
<td>Shear Modulus</td>
<td>13.586 (1.9672x10^{6})</td>
</tr>
<tr>
<td>Poisson's Ratio</td>
<td>0.22</td>
</tr>
</tbody>
</table>
Figure 7.1 Bore Hole Width Versus Time Comparisons for Data in Table (7.1)
Figure 7.2 Transient Fracture Half-Length Comparisons for Data in Table (7.1)
Figure 7.3 Bore Hole Width Versus Time Comparisons for Data in Table (7-2)
Figure 7.4 Transient Fracture Half-Length Comparisons for Data in Table (7-2)
APPENDIX D
MODIFIED TWO-DIMENSIONAL MODEL EMPLOYING FRACTURE TOUGHNESS CRITERIA

The pressure-width equation is given by:

\[ W(x,t) = \frac{4(1-v^2)}{E} \Delta p \left( 1 - \frac{x^2}{\lambda^2} \right)^{1/2} \]  

(D.1)

The crack opening mode stress intensity factor, \( K_I \), during fracture extension is given by [70]

\[ K_I(t) = \frac{1}{\sqrt{\pi \lambda}} \int_0^\lambda \Delta p(x,t) \left( \frac{\lambda + x}{\lambda - x} \right)^{1/2} dx \]  

(D.2)

Assuming that \( \Delta p \) is constant along the fracture length, and using the fracture criteria \( K_I = K_{IC} \), Eq. (D.2) yields

\[ \Delta p = K_{IC} \sqrt{\pi \lambda} \]  

(D.3)

The volume balance associated with the pumped fluid, fracture volume, and leak-off volume, for Geertsma-DeKlerk formulations, is given by

\[ \frac{dV}{dt} = Q - C \int_0^t \frac{dA}{d\tau} \frac{dT}{\sqrt{T-\tau}} \]  

(D.4)

where

\[ V(t) = 2H \int_0^\lambda W(x,t) dx \], and

\[ A(t) = 4H \lambda \]  

(D.5)  

(D.6)
Introducing Eqs. (D.1), (D.3), (D.5) and (D.6) into (D.4) yields

\[ C_1 \frac{d}{dt}(l^{3/2}) = Q - 4HC \int_0^\infty \frac{d\xi}{d\tau} \frac{d\tau}{\sqrt{\xi - \tau}} \]  

(D.7)

where

\[ C_1 = \frac{2H(1-v^2)K_{IC}}{E} \sqrt{\pi} \]

Equation (D.7) can now be solved by using the finite element methodology developed for solving integral equations [73], to obtain \( \lambda(t) \) and then using Eq. (D.1) to get \( W(x,t) \).

Figure 8.1 illustrates the maximum width versus time comparisons at the bore hole for the hydraulic fracture parameters given in Table 7.1. The corresponding transient fracture half-lengths are revealed in Figure 8.2.
Figure 8.1 Bore Hole Width Versus Time Comparisons for Data in Table 7.1

- $K_{IC} = 1000 \text{ psi/\text{in}}$
- $K_{IC} = 2000 \text{ psi/\text{in}}$
- Geertsma-DeKlerk [6]
Figure 8.2 Transient Fracture Half-Length Comparisons for Data in Table 7-1

- $K_{IC} = 1000$ psi/$\text{in}$
- $K_{IC} = 2000$ psi/$\text{in}$
- Geertsma-DeKlerk [6]
APPENDIX E

DISCUSSION OF ASSUMPTION RELATED TO VERTICAL FLOW

In this appendix, the assumption related to vertical fluid flow is discussed. The fluid flow in the vertical direction, given by Eq.(3.2.a) follows from the assumption of vertical flow mass conservation with horizontal flow gradient contribution neglected. This equation has been assumed in lieu of the fracture toughness criteria for vertical crack growth. A related assumption regarding the adoption of CGD (Christianovich-Geertsma-Dekkerk) model has been made by Cleary [31,32] for flow in the vertical direction since the horizontal pressure gradient is small compared to the vertical gradient for elongated fractures.

The integrated flow of the continuity equation in the vertical direction is

\[ \frac{\partial Q_y}{\partial y} + Q_{Ly} + \frac{\partial A_2}{\partial t} \]  

(E.1)

where

- \( Q_y \) is the integrated flow rate,
- \( q_y \) is the flow rate per unit length,
- \( Q_{Ly} \) is the total leak-off rate,
- \( q_L \) is the leak-off per unit length,
- \( A_2 \) is the horizontal cross-sectional area.

Equation (E.1) reduces to
\( \frac{\partial q_y}{\partial y} + q_L + \frac{\partial w}{\partial t} = 0 \)  

(E.2)

with the assumption that the product terms \( \frac{\partial L}{\partial y} q_y \) and \( w \frac{\partial L}{\partial t} \) can be ignored relative to other terms for elongated fractures. The numerical comparison for horizontal and vertical flow gradients for an elongated fracture justifies the above assumption.