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SOME RESULTS ON THE ASSOCIATION SCHEMES OF BILINEAR FORMS

The Ohio State University

Ph.D. 1985

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SOME RESULTS ON
THE ASSOCIATION SCHEMES OF BILINEAR FORMS

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By
Tayuan Huang, B.S., M.S.

****

The Ohio State University
1985

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I express sincere appreciation to Professor Eiichi Bannai for suggesting the problems attacked in this dissertation, and for his advice and encouragement. Thanks go to S. Choi, J. Hemmeter, Y. Hong, N. Manickam and S. Song for their helpful discussions. I would like also to thank my family for the inspiration and sharing the hardship during the preparation of this dissertation.
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   (To appear in European Journal of Combinatorics)

FIELDS OF STUDY

Major Field: Mathematics
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Introduction

There are several interesting ways to treat finite groups, codes and designs in the context of association schemes. An association scheme is a finite set $X$ with a set of relations $\{R_i\}_{0 \leq i \leq d}$ on $X \times X$ satisfying certain axioms of regularity. Each distance-regular graph is a symmetric association scheme with a metric property. E. Bannai [1, Chapter 3] has compiled a list of infinite families of distance-regular graphs with large diameters, believing that he has an essentially complete list at this time.

Two of these infinite families will be presented in this dissertation:

1. The $q$-analog of Hamming graph $H_q(d,n)$, $d \leq n$. This graph has the set of all $dxn$ matrices over $GF(q)$ as its vertex set, and two vertices $A$ and $B$ are adjacent if the rank of $A - B$ is 1.

2. The alternating graph $Alt(n,q)$. This graph has the set of all $nxn$ antisymmetric matrices over $GF(q)$ as its vertex set, and two vertices $A$ and $B$ are adjacent if the rank of $A - B$ is 2.
It is worth noting that the vertex set of each of these infinite families may also be interpreted as the set of all bilinear forms and the set of all alternating bilinear forms (of suitable vector spaces) respectively.

The maximal clique structures of distance-regular graphs, closely related to the structures of the graphs themselves, are interesting objects for the following two reasons:

(I). Let $\Gamma$ be a distance-regular graph of diameter $d$. We define graphs $\Gamma_r$ on the vertex set $V(\Gamma)$ of $\Gamma$, and two vertices $x$ and $y$ are adjacent if the distance between them in $\Gamma$ is at most $r$, where $1 \leq r \leq d$. A classical result in extremal set theory is the theorem of Erdos, Ko and Rado [14] which characterizes the cliques of $\Gamma_r$ with maximum size, where $\Gamma$ is the first class of the Johnson scheme $J(n,d)$. This theorem had a great impact on combinatorics. A survey of known and of some new generalizations and analogous results of this theorem can be found in [11].

We prove a result of this type for $H_q(d,n)$ in Chapter III.

**Theorem 3.1.1** Let $\mathcal{F} \subseteq M_{d \times n}(q)$, and $\text{rank}(A-B) \leq d-r$ for all $A, B \in \mathcal{F}$, where $0 \leq r \leq d$. Assume that $n \geq d+1$ and $(n, q) \neq (d+1, 2)$. Then
(1). \( |\mathcal{F}| \leq q^{n(d-r)} \), and

(2). \( |\mathcal{F}| = q^{n(d-r)} \), if and only if, up to isomorphism, \( \mathcal{F} = \{ A \mid A \in M_{d \times n}(q) \text{ with zero entries on the last } r \text{ rows} \} \).

As an application of the above theorem, we prove that

**Theorem 3.3.1** \( H_q(d,n) \) has no perfect \( e \)-codes where \( n \geq d+1 \), \( q \geq 2 \), \( (n, q) \neq (d+1, 2) \) and \( e \geq 1 \).

(II). By investigating the clique structures of \( \Gamma \), we may attach an incidence structure to \( \Gamma \). Among the others, this is a starting point to characterize the distance-regular graphs by their parameters. One of the crucial steps is to show the existence of maximal cliques of the right size. Bose and Laskar [2] is one of the original papers dealing with the characterization of this type. In order to use the Bose - Laskar argument, we usually need some restriction on the parameters. A survey of the characterization problem of known distance-regular graphs by the parameters can be found in [1, Section 3.8].

Following a result of Ray-Chaudhuri and Sprague [23], we briefly review how to attach an incidence structure to \( H_q(d,n) \) in terms of its maximal cliques in Chapter II. And then we characterize the distance-regular graph \( H_q(d,n) \) by its parameters, together with one additional condition in Chapter IV. The Bose - Laskar argument is used.
Theorem 4.1.1 Let $\Gamma$ be a distance-regular graph of diameter $d$ with parameters \{$b_0$, $b_1$, ..., $b_{d-1}$; $c_1$, ..., $c_d$\} such that

1. the number of edges of the induced subgraph on the common neighborhood of any two vertices $x$ and $y$ depends only on the distance between them.

2. $c_2 = q^2 + q$ and $b_i = q^{2i}(q^{d-i} - 1)(q^{n-i} - 1)/(q-1)$ for $0 \leq i \leq d-1$ (i.e., coincide with those of $H_q(d,n)$) where $n \geq 2d \geq 6$, and $q \geq 4$.

Then $q$ is a prime power and $\Gamma$ is isomorphic to $H_q(d,n)$.

As an application of the above theorem, we prove that

Theorem 4.6.1 $H_q(d,n)$ has no antipodal covering if $n \geq 2d \geq 6$ and $q \geq 4$.

In Chapter V, instead of an incidence structure, we propose a diagram for $\text{Alt}(n,q)$ (Theorem 5.3.1). It seems to be a starting point to characterize $\text{Alt}(n,q)$ in terms of its geometric structure.
Chapter 1
Preliminary

The definitions and notations used in this dissertation generally follow those used by Bannai, Ito [1] and Ray-Chaudhuri, Sprague [23, 24].

1.1 Distance-regular graphs and association schemes

A graph $\Gamma$ is an ordered pair $(V(\Gamma), E(\Gamma))$ consisting of a nonempty finite set $V(\Gamma)$ of vertices, and a subset $E(\Gamma)$, the edge set, of the set of unordered pairs of vertices in $V(\Gamma)$. We say vertices $x$ and $y$ adjacent if \{x, y\} $\in E(\Gamma)$. $[x_0(=x), x_1, ..., x_{n-1}, x_n(=y)]$ is an $x$ - $y$ path of length $n$ if $x_{i-1}$, $x_i$ are adjacent for all $1 \leq i \leq n$. The distance $\partial(x,y)$ between $x$ and $y$ is the minimum length among all $x$ - $y$ paths. The maximum of \{ $\partial(x,y)$ | $x$, $y \in V(\Gamma)$ \} is called the diameter of the graph $\Gamma$. For a vertex $x$ and a subset $A \subseteq V(\Gamma)$, $\partial(x,A)$ is defined to be the minimum of \{ $\partial(x,y)$ | $y \in A$ \}. For $x$ in $V(\Gamma)$, let $\Gamma_i(x) = \{ y | y \in V(\Gamma), d(x,y) = i \}$.
V(\Gamma) and \partial(x,y) = i \} . A distance-regular graph \Gamma of diameter d is one for which the parameters \( c_i = |\Gamma_{i-1}(x) \cap \Gamma_1(y)|, \ a_i = |\Gamma_j(x) \cap \Gamma_1(y)| \) and \( b_i = |\Gamma_{i+1}(x) \cap \Gamma_1(y)| \) depend not on the particular vertices we choose, but only on the distance \( i = \partial(x,y) \) between \( x \) and \( y \). It is clear that \( a_i = b_0 - b_i - c_i \). An important property of distance-regular graphs is that if \( x, y \in V(\Gamma) \) with \( \partial(x,y) = k \), then \( |\Gamma_j(x) \cap \Gamma_j(y)| \) depends only on \( i, j, k \), and \( \partial(x,y) = k \), denoted by \( p_{ij}^k \).

A (commutative) association scheme of d classes is a finite set \( X \) with a set of relations \( \{\mathcal{R}_i\}_{0 \leq i \leq d} \) on \( X \times X \) satisfying the following properties:

1. \( \mathcal{R}_0 = \{(x,x)| x \in X\} \), and \( \{\mathcal{R}_i\}_{0 \leq i \leq d} \) is a partition of \( X \times X \).
2. For any \( \mathcal{R}_i \), its transpose \( \mathcal{R}_i^t = \{(y,x)|(x,y) \in \mathcal{R}_i\} \) is equal to some \( \mathcal{R}_j, 0 \leq j \leq d \).
3. For any pair \( (x,y) \in \mathcal{R}_k \), the number of \( z \in X \) such that \( (x,z) \in \mathcal{R}_i \) and \( (z,y) \in \mathcal{R}_j \) is a constant \( p_{ij}^k \) depending only on \( i, j, k \).
4. \( p_{ij}^k = p_{ji}^k \) for all \( i, j, \) and \( k \).

An association scheme is called symmetric if it satisfies the additional property:

5. \( \mathcal{R}_i^t = \mathcal{R}_i \) for all \( i \).
For a distance-regular graph $\Gamma$ with vertex set $V(\Gamma)$, let $R_i = \{(x,y) | x, y \in V(\Gamma) \text{ and } \partial(x, y) = i\}, i = 0, 1, ..., d.$ Then $(V(\Gamma), \{R_i\}_{0 \leq i \leq d})$ is a symmetric association scheme. Since $\partial(x,y) \leq \partial(x,z) + \partial(z,y)$ for all vertices $x, y$ and $z$, we have $p_{ji}^k = 0$ if $|k-i| \geq 2$. There are a few infinite families of association schemes attached to classical groups and classical forms (see [1] for the details), all of these turn out to be distance-regular graphs. Two of these infinite families, which we present in this dissertation, are associated with bilinear forms and alternating bilinear forms respectively.

1.2 Incidence structures and $d$-net

An incidence structure is a triple $\Pi = (\mathcal{P}, \mathcal{L}, I)$, where $\mathcal{P}$ and $\mathcal{L}$ are nonempty, disjoint finite sets and $I \subseteq \mathcal{P} \times \mathcal{L}$. Elements of $\mathcal{P} = \{u, v, w, x, y, z, \ldots\}$ and $\mathcal{L} = \{\lambda, \eta, \xi, \ldots\}$ are called points and lines respectively. If $p \in \mathcal{P}$, $\lambda \in \mathcal{L}$ with $(p, \lambda) \in I$, we write $p \in \lambda$ and say that $p$ is a point of $\lambda$ or $\lambda$ is a line of $p$. Two points $p_1$ and $p_2$ are collinear if there is a line $\lambda \in \mathcal{L}$ such that $\lambda$ contains both. Two lines $\lambda_1$ and $\lambda_2$ meet at a point $p$ if $p \in \lambda_1$ and $p \in \lambda_2$, denoted by $\lambda_1 \neq \lambda_2$ at $p$. If $A \subseteq \mathcal{P}$ and $\lambda \in \mathcal{L}$, we say $\lambda$ meet $A$ if $\lambda \not\subseteq A$ and $\lambda \cap A \neq \emptyset$. $\Pi$ is said to be semilinear (resp. linear) if any two points are contained in at most (resp. exactly) one line. The adjacency graph of $\Pi$
is the graph having vertex set $P$, with any two points adjacent if they are collinear.

Let $S \subseteq P$. We associate with $S$, in a natural way, an incidence structure as follows: $\mathcal{L}(S)$ is the set of all lines of $\Pi$ that contain at least two points of $S$, and $I(S)$ is the restriction of $I$ to $S \times \mathcal{L}(S)$. The restriction of $\Pi$ to $S$ is the incidence structure $\Pi_S = (S, \mathcal{L}(S), I(S))$. $S$ is called connected if the adjacency graph of $\Pi_S$ is connected. $S$ is called line-closed if $S$ contains all points of a line whenever $S$ contains at least two points of it. $S \subseteq P$ is a subspace if $S$ is connected and line-closed. For any subset $T \subseteq P$, $<T>$ denotes the intersection of all line-closed sets which contain $T$. Clearly $<T>$ is line-closed. If $<T>$ is also connected, then $<T>$ is a subspace. Let $S \subseteq P$ be a subspace of $\Pi$. A set $T \subseteq S$ generates $S$ if $T$ is connected and $S = <T>$. The dimension of a subspace of $\Pi$ is the number $d$ such that $d + 1$ is the minimum among the cardinalities of all generating set for $S$. $S \subseteq P$ is called an $i$-space if $S$ is a subspace of dimension $i$.

The following two connected semilinear incidence structures are of particular interest in this dissertation. We will show in Chapter II how to attach the structure of a $d$-net to $H_q(d,n)$ in terms of its maximal cliques.
A connected semilinear incidence structure $\Pi = (P, L, I)$ is called a net, if

(B1) $|P| > 1$, and

(B2) $L$ is partitioned into at least three nonempty classes such that

a) the lines of each class partition $P$, and

b) the lines of different classes intersect.

A connected semilinear incidence structure $\Pi = (P, L, I)$ is called a $d$-net, where $d$ is the dimension of $\Pi$, if

(D1) each 2-space of $\Pi$ is a net,

(D2) the intersection of any two 2-spaces in a 3-space is either $\emptyset$ or a line,

(D3) the intersection of any two subspaces of $\Pi$ is a subspace.
Chapter 2
\( H_q(d,n) \) and Attenuated spaces

2.1 Distance-regular graph \( H_q(d,n) \)

One of the infinite families of distance-regular graphs is defined on the set \( M_{dxn}(q) \) of all \( d \times n \) matrices over \( GF(q) \), \( n \geq d \), and two vertices \( A \) and \( B \) are adjacent if the rank of \( A - B \) is 1. We denote this graph by \( H_q(d,n) \). The parameters of \( H_q(d,n) \) are \( b_i = q^{2i}(q^{d-i}-1)(q^{n-i}-1)/(q-1), 0 \leq i \leq d-1 \), and \( c_i = q^{i-1}(q^i-1)/(q-1), 1 \leq i \leq d \). Let \( \mathcal{R}_i = \{ (A,B) \mid A, B \in M_{dxn}(q) \text{ with rank}(A - B) = i \}, i = 0, 1, ..., d \). It is known that \( (M_{dxn}(q), \{\mathcal{R}_i\}_{0 \leq i \leq d}) \) is a symmetric association scheme and \( (M_{dxn}(q), \mathcal{R}_i) \) turns out to be the distance-regular graph \( H_q(d,n) \).

The vertex set \( M_{dxn}(q) \) is a vector space of dimension \( dxn \) over \( GF(q) \). Translations of the vector space are automorphisms of the
graph, as are left- (right-) multiplication by invertible $d \times d$ - matrices (invertible $n \times n$ - matrices). Any field automorphism applied to all coefficients of a matrix leads to a graph automorphism. This explains the existence of a subgroup of $\text{Aut}(H_q(d,n))$ of the form $GF(q)^{d(d+n)}(GL(d,q) \times GL(n,q))/GF^*(q)$. Notice that $d(A,B) = i$, if and only if the rank of $A-B$ is $i$. It follows easily that $H_q(d,n)$ is distance-transitive and of diameter $d$, see [8]. As distance-transitive, $H_q(d,n)$ satisfies the following condition, first introduced by D. Higman for strongly regular graphs.

**The weak 4-vertex condition:** The number of edges of the induced subgraph on the common neighborhood of any two vertices $x$ and $y$ depends only on the distance between them.

As mentioned above, the distance-regular graph $H_q(d,n)$ is defined on matrices. It may also be viewed as a distance-regular graph defined on the set of all bilinear forms on $U \times W$, where $U$ and $W$ are vector spaces of dimensions $d$ and $n$ respectively over $GF(q)$. In the next section, we will show the third representation for $H_q(d,n)$, namely the adjacency graph of certain semilinear incidence structure.
2.2 **Attenuated spaces**

Let $V$ be a vector space of dimension $n+d$ over $\mathrm{GF}(q)$, $U \subseteq V$ be a subspace with base $\{e_1, \ldots, e_d\}$, and $W$ be a complement of $U$ in $V$. Let

$$A_i = \{ A \mid A \subseteq V \text{ is an } i\text{-space and } \dim(A \cap W) = 0 \}$$

$i = d, \ d-1$. Then $(A_d, A_{d-1}, \supseteq)$ is a semilinear incidence structure. Any incidence structure isomorphic to $\Pi = (A_d, A_{d-1}, \supseteq)$ is called an $(n,q,d)$-attenuated space, or a $d$-attenuated space for short.

The correspondence between the adjacency graph of $\Pi$ and $H_q(d,n)$ is known in [23], we prove it in Theorem 2.2.1 for completeness.

**Theorem 2.2.1** With respect to a fixed base of $W$, $H_q(d,n)$ is isomorphic to the adjacency graph of $\Pi = (A_d, A_{d-1}, \supseteq)$.

**Proof:** Let $\{w_1, \ldots, w_n\}$ be a base of $W$ and $W^d$ be the Cartesian product of $d$ copies of $W$. Each element in $M_{dxn}(q)$

$$\begin{bmatrix}
    a_{11}, a_{12}, \ldots, a_{1n} \\
    a_{21}, a_{22}, \ldots, a_{2n} \\
    \vdots \\
    a_{d1}, a_{d2}, \ldots, a_{dn}
\end{bmatrix}$$

corresponds to a unique element $(x_1, \ldots, x_d)$ in $W^d$ where $x_i = \sum_{j=1}^{n} a_{ij}w_j$. It is clear that this is an one to one correspondence between $M_{dxn}(q)$ and $W^d$ and we denote it by $\phi$. 
Consider the natural projection
\[ \pi: V \to V/W \cong U \]
The kernel of \( \pi \) is \( W \). If \( A \in A_d \), i.e., a \( d \)-subspace of \( V \) and \( A \cap W = 0 \), the zero space, then \( A + W = U + W = V \). Hence \( \pi(A) \subseteq U \) has dimension \( d \) and so \( \pi(A) = U \). Let \( v_1, \ldots, v_d \in A \) such that \( \pi(v_i) = e_i \) for all \( i \). Since \( v_i = x_i + u_i \) for a unique choice \( x_i \in W \) and \( u_i \in U \), and \( \pi(v_i) = u_i = e_i \), we have \( v_i = e_i + x_i \). Routine arguments show that \( \{e_1+x_1, \ldots, e_d+x_d\} \) is a base of \( A \), and the choice of \( x_1, \ldots, x_d \) is unique. For if \( \{e_1+y_1, \ldots, e_d+y_d\} \) is another base of \( A \) where \( y_1, \ldots, y_d \in W \). Then \( e_i+y_i = \sum_{1 \leq j \leq n} \alpha_{ij}(e_j+x_j) \), and so \( \sum_{1 \leq j \leq n} \alpha_{ij}e_j - e_i = y_i - \sum_{1 \leq j \leq n} \alpha_{ij}x_j \in W \cap U \), a zero space, i.e., \( x_i = \sum_{1 \leq j \leq n} \alpha_{ij}e_j \). This implies that \( \alpha_{ij} = \delta_{ij} \). Hence \( e_i+y_i = e_i+x_i \) and so \( x_i = y_i \) for all \( i \).

Following this observation, we define the mapping
\[ \sigma^*: A_d \to W^d \]
by \( \sigma^*(A) = (x_1, \ldots, x_d) \), where \( A = <e_1+x_1, \ldots, e_d+x_d> \). Clearly, \( \sigma^* \) is well defined and bijective. Let \( \sigma: W^d \to A_d \) be the inverse of \( \sigma^* \), i.e.,
\[ \sigma((x_1, \ldots, x_d)) = <e_1+x_1, \ldots, e_d+x_d> \]
If \( X = (x_1, \ldots, x_d) \) and \( Y = (y_1, \ldots, y_d) \) are in \( W^d \), we define \( X - Y = (x_1-y_1, \ldots, x_d-y_d) \). Then \( \text{rank}(\phi^{-1}(X) - \phi^{-1}(Y)) = 1 \) if and
only if the subspace \(<x_1\cdot y_1, \ldots, x_d\cdot y_d>\) has dimension 1. From now on, we assume that \(X\) and \(Y\) are in \(W^d\) with \(\text{dim}(<x_1\cdot y_1, \ldots, x_d\cdot y_d>) = 1\). Without loss of generality, we may assume that \(x_1 - y_1 \neq 0\). Then \(x_i - y_i = \alpha_i(x_1 - y_1)\), for some \(\alpha_i \in \text{GF}(q)\), i.e., \(y_i = x_i - \alpha_i(x_1 - y_1)\), so \((y_1, \ldots, y_d) = (x_1, \ldots, x_d) + (\alpha_1, \ldots, \alpha_d)(x_1 - y_1)\).

In order to show that \(\sigma\phi\) is an isomorphism between \(H_q(d,n)\) and the adjacency graph of \(\Pi = (A_d, A_{d-1}, \emptyset)\), following the above notation, it suffices to show that \(\sigma(X)\) and \(\sigma(Y)\) are collinear. Let
\[
\mathcal{T} = (x_1, \ldots, x_d) + (\alpha_1, \ldots, \alpha_d)W
\]
\[
= \{(x_1 + \alpha_1 w, \ldots, x_d + \alpha_d w) | w \in W\}
\]
where \(x_i \in W\), and \(\alpha_i \in \text{GF}(q), 1 \leq i \leq d\), some \(\alpha_i\) are non-zero. So
\[
\sigma(\mathcal{T}) = \{<e_1 + x_1 + \alpha_1 w, \ldots, e_d + x_d + \alpha_d w> | w \in W\}
\]
Since \(W\) is a subspace,
\[
(x_1, \ldots, x_d) + (\alpha_1, \ldots, \alpha_d)W
\]
\[
= (x_1, \ldots, x_d) + \alpha(\alpha_1, \ldots, \alpha_d)W
\]
where \(\alpha \in \text{GF}(q)\) is non zero. Without loss of generality, we may assume the first non-zero entry in \((\alpha_1, \ldots, \alpha_d)\) is \(\alpha_j = 1\). Then
\[
<e_1 + x_1 + \alpha_1 w, \ldots, e_j + x_j + \alpha_j w, \ldots, e_d + x_d + \alpha_d w>
\]
\[
= <e_1 + x_1 + \alpha_1 w - \alpha_1(e_j + x_j + \alpha_j w), \ldots,
\]
\[
e_j - 1 + x_j - 1 + \alpha_j - 1 w - \alpha_j - 1(e_j + x_j + \alpha_j w),
\]
\[
e_j + x_j + \alpha_j w,
\]
\[
e_j + 1 + x_j + 1 + \alpha_j + 1 w - \alpha_j + 1(e_j + x_j + \alpha_j w), \ldots,
\]
\[
e_d + x_d + \alpha_d w + \alpha_d(e_j + x_j + \alpha_j w)>
\]
where \( f_i = e_i \cdot x_i \cdot \alpha_i - \alpha_i (e_j \cdot x_j + \alpha_j \cdot w) = e_i \cdot x_i \cdot \alpha_j \cdot x_j \text{ for } j \neq i \). Let

\[ B = \langle f_1, \ldots, f_{j-1}, f_{j+1}, \ldots, f_d \rangle. \]

Then \( B \subseteq \sigma(Z) \) for all \( Z \in T \). Because

\[ T = \langle x_1, \ldots, x_d \rangle + (\alpha_1, \ldots, \alpha_d) \cdot W, \]

\( \sigma(T) = \{<e_1 + x_1 + \alpha_1 \cdot w, \ldots, e_d + x_d + \alpha_d \cdot w>| w \in W\} \) consists of \( q^n \) elements and each element in \( \sigma(T) \) (\( \subseteq A_d \)) contains the \( (d-1) \)-subspace \( B \). Clearly \( B \cap W = \emptyset \), and \( B \subseteq \sigma(X) \cap \sigma(Y) \), i.e., \( \sigma(X) \) and \( \sigma(Y) \) are collinear in \( \Pi \).

This proves that the correspondence

\[ M = [a_{ij}] \rightarrow \phi(M) = (x_1, \ldots, x_d), \]

where \( x_i = \sum_{1 \leq j \leq n} a_{ij} \cdot w_j \)

\[ \rightarrow \sigma(\phi(M)) = <e_1 + x_1, \ldots, e_d + x_d> \]

establishes an isomorphism between \( H_q(d,n) \) and the adjacency graph of \( \Pi = (A_d, A_{d-1}, \varnothing) \).

Those points in \( H_q(d,n) \) corresponding to a line in \( \Pi = (A_d, A_{d-1}, \varnothing) \) form a clique. In terms of the isomorphism above, we may describe the structures of the maximal cliques of \( \Gamma \) explicitly. Let \( \Gamma \) be the adjacency graph of \( \Pi = (A_d, A_{d-1}, \varnothing) \), which is isomorphic to \( H_q(d,n) \), and let \( A, U \in A_d \) be adjacent.

**Lemma 2.2.2** If \( B \in \Gamma_1(A) \cap \Gamma_1(U) \), then either \( B \subseteq A + U \), or \( A \cap U \subseteq B \).
Proof: Suppose \( A \cap U \not\subseteq B \). Since \( \dim(A \cap B \cap U) \leq d-2 \) and
\[
\dim(U \cap B) + \dim(A \cap B) \\
\leq \dim((A+U) \cap B) + \dim(A \cap B \cap U),
\]
we have \( \dim((A+U) \cap B) \geq d \). Therefore, \( B \subseteq A+U \) as required. \[\]

Lemma 2.2.3 If \( C \) is a clique containing \( A, B, \) and \( U \), where \( A \cap U \subseteq B \) but \( B \not\subseteq A+U \), then \( A \cap U \subseteq \cap C \).

Proof: Suppose \( A \cap U \not\subseteq C \) for some \( C \in C \). Then \( C \subseteq A+U \) by Lemma 2.2.2. Since \( B \cap C \subseteq B \cap (A+U) \) and \( B \not\subseteq A+U \), we have \( B \cap C = B \cap (A+U) \) by comparing their dimensions. On the other hand, \( A \cap U \subseteq B \cap (A+U) \), therefore we have \( A \cap U \subseteq B \cap C \subseteq C \), a contradiction. \[\]

Corollary 2.2.3.1 If \( C \) is a clique containing \( A, B, \) and \( U \), where \( B \subseteq A+U \), but \( A \cap U \not\subseteq B \), then \( U \cap C \subseteq A+U \).

Proof: If there is an element \( C \in C \) with \( C \not\subseteq A+U \), then \( A \cap U \subseteq C \), and the previous lemma shows that \( A \cap U \subseteq \cap C \). This contradicts \( A \cap U \not\subseteq B \). \[\]

Let \( C \) be a clique in \( H_q(d,n) \), as \( H_q(d,n) \) is distance-transitive, without loss of generality, we may assume that \( U \in C \). Lemma 2.2.3 and Corollary 2.2.3.1 show that either
(1) $A \cap U \subseteq \cap C$, or

(2) $\cup C \subseteq A+U$.

for all $A \in C$. Here, we notice that $\dim(A \cap U) = d-1$, and $\dim(A+U) = d+1$. For a fixed $A$ in $C$, let

$C_1 = \{ B | B \in A_d$ and $A \cap U \subseteq B \}$, and

$C_2 = \{ C | C \in A_d$ and $C \subseteq A+U \}$

Clearly both $C_1$ and $C_2$ are maximal cliques of sizes $q^n$, $q^d$ respectively.

The above remark shows that each point has exactly these two types of maximal cliques. Those maximal cliques of the form $C_1$ are called grand cliques or lines, and those of the form $C_2$ are called assemblies.

The matrix representations of the maximal cliques of $U$ are as follows:

(i). For $C_1$. Without loss of generality, we may assume that $A \cap U = \langle e_2, ..., e_d \rangle$. If $B = \langle e_1+x_1, ..., e_d+x_d \rangle \in C_1$, since $\langle e_2, ..., e_d \rangle \subseteq \langle e_1+x_1, ..., e_d+x_d \rangle$, we have $x_2 = ... = x_d = 0$, i.e., $B = \langle e_1+x_1, e_2, ..., e_d \rangle$ and its corresponding matrix is

$$
\begin{bmatrix}
    a_{11}, \ldots, a_{1n} \\
    0, \ldots, 0 \\
    \vdots \\
    0, \ldots, 0
\end{bmatrix}
$$

where $x_1 = \sum_{1 \leq j \leq n} a_{1j}w_j$. There are $(q^d-1)/(q-1)$ $(d-1)$-subspaces of $U$, and each of them corresponds to a maximal clique of $U$ of this type. Those $(q^d-1)/(q-1)$ grand cliques of the zero matrix (i.e., $U$) are of the forms $\{(\beta_1x, ..., \beta_dx) | x \in W \}$ where $\beta_1, ..., \beta_d \in \text{GF}(q)$ are not all zero and the first non-zero entry is 1.
(II). For assembly $C_2$. Since $\dim((A+U)\cap W) = \dim(A+U) + \dim(W) - \dim((A+U)+W) = 1$, without loss of generality, we may assume $w_1 \in A+U$, where $W = \langle w_1, \ldots, w_n \rangle$. Suppose $B = \langle e_1+x_1, \ldots, e_d+x_d \rangle \in C_2$. Since $e_j + x_j \in B \subseteq A+U = \langle e_1, \ldots, e_d, w_1 \rangle$. It follows that

$$x_i \in \langle e_1, \ldots, e_d, w_1 \rangle$$

$$\Rightarrow x_i = \beta_i w_1 + \sum_{1 \leq j \leq n} \alpha_{ij} e_j$$

$$\Rightarrow \sum_{1 \leq j \leq n} \alpha_{ij} e_j = x_i - \beta_i w_1 \in U \cap W$$

$$\Rightarrow x_i = \beta_i w_1 \text{ and } \alpha_{ij} = 0$$

Hence $B = \langle e_1+\beta_1 w_1, \ldots, e_d+\beta_d w_1 \rangle$ and the corresponding matrix is

$$\begin{bmatrix}
\beta_1, 0, \ldots, 0 \\
\beta_2, 0, \ldots, 0 \\
\vdots \\
\beta_d, 0, \ldots, 0
\end{bmatrix}$$

The following lemma follows immediately.

**Lemma 2.2.4** Each point has $q^{d-1}/(q-1)$ lines, and $(q^n-1)/(q-1)$ assemblies.

**Lemma 2.2.5** Let $C_1, C_2$ be maximal cliques of types I and II respectively. If $|C_1 \cap C_2| \geq 2$, then $|C_1 \cap C_2| = q$. 
Proof: Let \( A, U \subseteq C_1 \cap C_2 \). Then \( \cap C_1 \) consists of a \((d-1)\)-subspace \( A \cap U \) and \( \cup C_2 \) is a \((d+1)\)-subspace \( A + U \). Then \( C_1 \cap C_2 \) consists of all those \( d \)-subspaces in \( A_d \) which contain \( A \cap U \) and are contained in \( A + U \), as required. 

The following properties about the relationship among points, lines and assemblies, which are interesting in their own, will be used in the proof of Proposition 2.2.8 and Proposition 2.2.9.

Lemma 2.2.6 If \( \lambda \) is a line and \( x \) is a point with \( \partial(x, \lambda) = i \), then \(|\Gamma_i(x) \cap \lambda| = q^i\).

Proof: Without loss of generality, we may assume that \( \lambda = \{(w, 0, \ldots, 0) | w \in W\} \) and \( x = (x_1, \ldots, x_d) \) with \( \partial(x, 0) = i \), where \( 0 \) is the zero matrix.

If \( i \) vectors of \( \{x_2, \ldots, x_d\} \), say \( x_2, \ldots, x_{i+1} \), are linearly independent, then \( x_1 \in <x_2, \ldots, x_{i+1}>. \) And \( y = (w, 0, \ldots, 0) \in \Gamma_i(x) \cap \lambda \) if and only if \( \partial(x, y) = \dim <x_1 - w, x_2, \ldots, x_d> = i \), i.e., \( x_1 - w \in <x_2, \ldots, x_{i+1}>. \) Hence \( w \in <x_2, \ldots, x_{i+1}>\), an \( i \)-subspace over \( GF(q) \), and so \(|\Gamma_i \cap \lambda| = q^i\).

If \( x_1, \ldots, x_i \) are linearly independent, then \( x_{i+1}, \ldots, x_d \in <x_1, \ldots, x_i>. \) \( y = (w, 0, \ldots, 0) \in \Gamma_i(x) \cap \lambda \) if and only if \( \partial(x, y) = \dim <x_1 - w, x_2, \ldots, x_d> = i \). If \( x_1 \) appears in the linear combination of one of \( x_{i+1}, \ldots, x_d \) (with respect to \( x_1, \ldots, x_i \), say \( x_{i+1} \), i.e.,
\[ x_{i+1} = \alpha_1 x_1 + \ldots + \alpha_i x_i \quad \alpha_i \neq 0. \]

Then
\[ x_1 \in <x_1^{-w}, x_2, \ldots, x_{i+1}> \]
and so
\[ w \in <x_1, \ldots, x_i>, \]
as required. Otherwise, we may assume \( x_{i+1}, \ldots, x_d \in <x_2, \ldots, x_i>. \) In this case, we choose \( w = x_1 + x_2, \) then
\[
\dim <x_1^{-w}, x_2, \ldots, x_d> \\
= \dim <x_1^{-w}, x_2, \ldots, x_i> \\
= \dim <x_2, \ldots, x_i> \\
= i-1,
\]
contradicting \( \partial(x, \lambda) = i. \) []

By a similar argument, replacing rows by columns, we may prove that

**Lemma 2.2.7** If \( A \) is an assembly and \( x \) is a point with \( \partial(x, A) = i, \) then \( |\Gamma_i(x) \cap A| = q^i. \)

Combining Lemma 2.2.6 and Lemma 2.2.7, we have

**Proposition 2.2.8** If \( \lambda \) is a line, \( x \) is a point with \( \partial(x, \lambda) = i, \) and \( y, z \in \Gamma_i(x) \cap \lambda, \) then \( |\Gamma_{i-1}(x) \cap \Gamma_1(y) \cap \Gamma_1(z)| = q^{i-1}. \)
Proof: Let \( T = \Gamma_i(x) \cap \lambda \), \( |T| = q^i \) by Lemma 2.2.6, and \( S = \{ u | u \in \Gamma_i(x) \} \) and \( \partial(u, \lambda) = 1 \). If \( u \in S \), then \( u \notin \lambda \), and \( \Gamma_1(u) \cap \lambda \subseteq T \). Counting the set \( \{(u, b) | u \in S \text{ and } b \in T \text{ are adjacent}\} \), we have \( |S|q = |T|c_i \).

For any pair \( z_1, z_2 \in T \), if \( u \in S \) and \( z_1, z_2 \in \Gamma_1(u) \), then \( u \in \Gamma_i^{-1}(x) \cap \Gamma_1(z_1) \cap \Gamma_1(z_2) \subseteq \Gamma_i^{-1}(x) \cap A_{z_1, z_2} \). Since \( \partial(x, A_{z_1, z_2}) = i-1 \), \( |\Gamma_i^{-1}(x) \cap A_{z_1, z_2}| = q^{i-1} \) by Lemma 2.2.7.

By counting the set \( B = \{(u, \{z_1, z_2\}) | u \in S, z_1, z_2 \in \Gamma_1(u)\} \) in two ways, we have \( |B| = |S|(|T|) \leq (|T|^i)q^{i-1} \). If there is a pair \( z_1, z_2 \in T \) such that \( |\Gamma_i^{-1}(x) \cap \Gamma_1(z_1) \cap \Gamma_1(z_2)| < q^{i-1} \), then \( |S|(|T|) < (|T|^i)q^{i-1} \), a contradiction. \( \square \)

One important property in Euclidean geometry is the axiom of parallelism. A similar situation holds in attenuated spaces. Two lines \( \lambda \) and \( \eta \) with \( \partial(\lambda, \eta) = i \) are called parallel if \( \partial(x, \eta) = \partial(y, \lambda) = i \) for any \( x \in \lambda \) and \( y \in \eta \).

Proposition 2.2.9 If \( \lambda \) is a line and \( x \) is a point with \( \partial(x, \lambda) = i \), then there is a unique line \( \eta \) of \( x \) with \( \partial(\lambda, \eta) = i \) which is parallel to \( \lambda \).

Proof: By translation, \( x + \lambda \) is a line of \( x \) with the required property. Now, we prove the uniqueness as follows:
Without loss of generality, we may assume \( \lambda = \{(w, 0, \ldots, 0)\mid w \in W\} \) and \( x = (x_1, \ldots, x_d) \) such that \( \vartheta(x,0) = i \). Also, we may assume that the dimension of \( \langle x_2, \ldots, x_d \rangle \) is \( i \) and \( x_2, \ldots, x_{i+1} \) are linearly independent. For if not, \( \text{dim} \langle x_2, \ldots, x_d \rangle \leq i-1 \), choose \( y = (x_1, 0, \ldots, 0) \) in \( \lambda \), then \( \vartheta(x,y) \leq i-1 \), contradicting \( \vartheta(x,\lambda) = i \).

Consider any line of \( x \), say \( \eta = \{(x_1 + \alpha_1 w, \ldots, x_d + \alpha_d w)\mid w \in W\} \), where \( \alpha_1, \ldots, \alpha_d \in \text{GF}(q) \). If any one of \( \{\alpha_2, \ldots, \alpha_d\} \) is non-zero, say \( \alpha_2 = 1 \), we want to show that \( \eta \) is not parallel to \( \lambda \). For any \( w^* \in W \), let

\[
A = \langle x_1 + \alpha_1 w - w^*, x_2 + w, x_3 + \alpha_3 w, \ldots, x_d + \alpha_d w \rangle \\
= \langle x_1 + \alpha_1 w - w^*, x_2 + w, \alpha_3 x_2 - x_3, \ldots, \alpha_d x_2 - x_d \rangle.
\]

If \( i = d-1 \), we may choose \( w, w^* \) from \( W \) such that both \( x_1 + \alpha_1 w - w^* \) and \( x_2 + w \) are zero vectors, then \( \text{dim}(A) = i-1 \). Now we assume \( i \leq d-2 \). Let

\[
x_r = \sum_{2 \leq j \leq i+1} \beta_j x_j \quad \text{for} \quad i+2 \leq r \leq d.
\]

Then

\[
\alpha_r x_2 - x_r \\
= \alpha_r x_2 - \sum_{2 \leq j \leq i+1} \beta_j x_j \\
= (\alpha_r \beta_{2r} \sum_{2 \leq j \leq i+1} \alpha_j \beta_{jr}) x_2 + \sum_{3 \leq j \leq i+1} \beta_{jr} (\alpha_j x_2 - x_j).
\]
If $\alpha_r - \beta_{2r} - \sum_{3 \leq j \leq i+1} \alpha_j \beta_{jr} \neq 0$ for some $r$, then

$$x_2 = 1/(\alpha_r - \beta_{2r} - \sum_{3 \leq j \leq i+1} \alpha_j \beta_{jr}) \{\alpha_r x_2 - x_r - \sum_{3 \leq j \leq i+1} \beta_{jr} (\alpha_j x_2 - x_j)\}$$

It follows that $x_2 \in A$ and so

$$A = <x_1 + \alpha_1 w - w^*, w, x_2, ..., x_{i+1}>.$$  

We may choose $w, w^*$ from $W$ such that $x_1 + \alpha_1 w - w^*, w, x_2, ..., x_{i+1}$ are linearly independent. Otherwise,

$$\alpha_r x_2 - x_r = \sum_{3 \leq j \leq i+1} \beta_{jr} (\alpha_j x_2 - x_j)$$

for all $r = i+2, ..., d$. Then

$$A = <x_1 + \alpha_1 w - w^*, x_2 + w, \alpha_3 x_2 - x_3, ..., \alpha_d x_2 - x_d>$$

$$= <x_1 + \alpha_1 w - w^*, x_2 + w, \alpha_3 x_2 - x_3, ..., \alpha_{i+1} x_2 - x_{i+1}>$$

We may choose $w, w^*$ from $W$ such that both $x_1 + \alpha_1 w - w^*$ and $x_2 + w$ are zero vectors. Then $\dim(A) = i-1$. In both cases, we have either $\dim(A) = i-1$ or $\dim(A) = i+2$ for a suitable choice of $A$. i.e., we can find two points $u \in \eta$ and $v \in \lambda$ such that $\partial(u, v) = i-1$ or $i+2$. Hence $\eta$ is not parallel to $\lambda$, as required.  

[ ]
Chapter 3
An analogue of
the Erdos-Ko-Rado theorem for $H_q(d,n)$

3.1 Introduction

Erdos, Ko and Rado [14] proved the following theorem for subsets of a finite set:

Let $\mathcal{F}$ be a collection of $d$-subsets of an $n$-set $X$ and $n \geq n_0(r,d)$, where $0 \leq r \leq d$. If $|A \cap B| \geq r$ for all $A, B \in \mathcal{F}$, then $|\mathcal{F}| \leq \left( \begin{array}{c} n-r \cr d-r \end{array} \right)$. Equality occurs if and only if $\mathcal{F}$ consists of all $d$-subsets which contain a fixed $r$-subset of $X$ (i.e., $\cap \mathcal{F} = \bigcap_{A \in \mathcal{F}} A$ consists of $r$ elements).

The original proof in [14] established that $n_0(r,d) \leq r+(d-r)\left( \begin{array}{c} d \cr r \end{array} \right)^3$. In 1976, P. Frankl [15] improved that $n_0(r,d) = (r+1)(d-r+1)$ for $r \geq 15$. Recently, Wilson [29] showed that $n_0(r,d) = (r+1)(d-r+1)$ in the remaining cases, i.e., $r = 2, 3, ..., 14$, and characterized the extremal configurations when $n > (r+1)(d-r+1)$. 24
In the language of graphs, the Erdos-Ko-Rado theorem gives the upper bound on the sizes of subsets of vertices with maximum distance $d-r$ in the Johnson graph $J(n,d)$ and characterizes the extremal cases, where $J(n,d)$ is the graph with the set of all $d$-subsets of an $n$-set as vertex set and two vertices adjacent if their intersection consists of $d-1$ points. We note that $J(n,d)$ is a distance-regular graph and the distance between any two vertices $A$ and $B$ is $d-r$ if and only if $|A \cap B| = r$. Analogous results are known for some other distance-regular graphs, e.g., [20] for Hamming graphs, [18] for q-analog Johnson graphs, and [27] for dual polar spaces. In this chapter, by modifying the technique used by Hsieh [18], we prove an analogous theorem for the distance-regular graph $H_q(d,n)$

**Theorem 3.1.1** Let $\mathcal{F} \subseteq M_{d\times n}(q)$, and $\text{rank}(A-B) \leq d-r$ for all $A, B \in \mathcal{F}$, where $0 \leq r \leq d$. Assume that $n \geq d+1$ and $(n,q) \neq (d+1, 2)$. Then

1. $|\mathcal{F}| \leq q^{n(d-r)}$, and
2. $|\mathcal{F}| = q^{n(d-r)}$, if and only if, up to isomorphism, $\mathcal{F} = \{ A | A \in M_{d\times n}(q) \text{ with zero on the last } r \text{ rows} \}$.

**Remark** (1). According to the results of Delsarte, [9, Theorem 3.9] and [10, p.237], it is true that $|\mathcal{F}| \leq q^{n(d-r)}$ in (1) for all $n \geq d$ and $q \geq 2$. However, we prove both (1) and (2) simultaneously under the given numerical conditions.
(2). Using different approach, A. Moon [20] proved the same result under the conditions that \( n > d + 1 \) and \( q \geq 3 \). Moreover if \( n > r + 2 \), assume \( r \leq (q-1)q^{n-r-3} \).

(3). We expect that (2) is also true for the case \((n,q) = (d+1, 2)\), although the inequality in Lemma 3.2.2 is no longer available. For the case \( n = d \), there are at least two types, namely either fixed \( r \) rows or fixed \( r \) columns.

As an easy corollary of Theorem 3.1.1, although not new, we prove in Section 3.4 that \( H_q(d,n) \) has no perfect e-codes for \( n \geq d + 1, q \geq 2, (n,q) \neq (d+1, 2) \) and \( e \geq 1 \) (Theorem 3.3.1). L. Chihara [7] proved the same by using generalized Lloyd's theorem.

The proof of this theorem depends on the fact that \( H_q(d,n) \) is isomorphic to the adjacency graph of the attenuated spaces \((\mathcal{A}_d, \mathcal{A}_{d-1}, \supseteq)\), as stated in Theorem 2.2.1. Since \( H_q(d,n) \) is distance transitive, without loss of generality, we may assume that the zero matrix is in \( \mathcal{F} \). In terms of the isomorphism, we may assume \( \mathcal{F} \subseteq \mathcal{A}_d \) with the property that \( U \in \mathcal{F} \), and \( \dim(A \cap B) \geq r \), for all \( A, B \in \mathcal{F} \). Under this convention, the statement (2) of the theorem can be restated as follows:

\[(2^*)\). If \(|\mathcal{F}| = q^{n(d-r)}\), then \( \mathcal{F} \) consists of all elements in \( \mathcal{A}_d \) which contain a fixed \( r \)-subspace of \( V \), i.e., \( \bigcap_{A \in \mathcal{F}} A \subseteq V \) is a subspace of dimension \( r \).
Remark: For the case \( n = d \), we have \(|\mathcal{F}| = q^{n(d-r)}\) for those \( \mathcal{F} \) which satisfy either \( \dim(\cap \mathcal{F}) = r \) or \( \dim(<\cup \mathcal{F}> ) = 2d-r \), where \(<\cup \mathcal{F}>\) denotes the subspace spanned by \( \cup \mathcal{F} \).

3.2 Proof of Theorem 3.1.1

For the proof, we need some lemmas. We denote the number of all \( r \)-subspaces of an \( n \)-dimensional vector space over \( \text{GF}(q) \) by \([r^n]\), the Gaussian integer, i.e.,

\[ [r^n] = (q^{n-1})(q^n-q)\cdots(q^n-q^{r-1})/(q^{r-1})(q^r-q)\cdots(q^r-q^{r-1}) \]

Lemma 3.2.1 \([r^n] = q^{r-1}(q^n-1)/(q^{n-1}-q^{r-1})[n_{r-1}^n] \].

Proof: Straightforward.  

The proof of Theorem 3.1.1 is based on the following inequality (Lemma 3.2.2) together with Lemma 3.2.3 (for counting purpose).

Lemma 3.2.2 If \( n \geq d+1 \), \((n,q) \neq (d+1,2)\), and \( 0 \leq r \leq d \), then \([r+p]d-r+1|p q^{n(d-r-p)} < q^{n(d-r)} 1 \leq p \leq d-r \).
Proof: We prove by induction on $p$. For the case $p = 1$, it suffices to show that $q^n \cdot [r+1][d-r+1] > 0$. Since

$$[r+1][d-r+1]$$

$$= (q^{r+1}-1)(q^{d-r+1}-1)/(q-1)^2$$

$$= q^{r-1}((q^{d-r+1}-1)/(q-1))(q^{r+1}-1)/(q^r-q^{r-1})$$

$$< q^{r-1}(q+2)(q^{d-r+1}-1)/(q-1)$$

and so

$$q^n \cdot [r+1][d-r+1]$$

$$> q^n \cdot q^{r-1}(q+2)(q^{d-r+1}-1)/(q-1)$$

$$= q^{r-1}(q^{n-r+1}-(q^{d-r+1}-1)(1+3/(q-1)))$$

$$= q^{r-1}(q^{d-r+1}(q^n-d-1-3/(q-1)) + 1 + 3/(q-1))$$

$$> 0$$

under the given condition. Hence $q^n \cdot [r+1][d-r+1] > 0$.

Now, we assume the statement is true up to $p-1$ where $2 \leq p \leq d-r$. For the case $p$, it suffices to show that

$q^{np} \cdot [r+p][d-r+1]^p > 0$. Since

$$[r+p][d-r+1]^p$$

$$= \{q^{r-1}(q^{r+p-1}-1)/(q^{r+p-1}-q^{r-1})\} \cdot (q^{d-r+1}-1)/(q-1)[r+(p-1)][d-r+1][r+1][d-r+1]^{p-1}$$

(by Lemma 3.2.1)

$$< \{q^{r-1}(q^{r+p-1}-1)/(q^{r+p-1}-q^{r-1})(q-1)\} q^{n(p-1)}$$

(by induction hypothesis)
and so

\[ q^{np} - [r^p][r_1^{d-r^1}] \]

\[ > q^{np-n+r-1} \]

\[ \left\{ q^{n+r+1} - \frac{(q^{r+p}-1)}{(q^{r+p-1}-q^{r-1})} \left( \frac{(q^{d-r^1}+1)}{(q-1)} \right) \right\} \]

\[ = q^{np-n+r-1} \]

\[ \left\{ q^{n+r+1} - \frac{q+1}{(q^{r+p-1}-q^{r-1})} \left( \frac{(q^{d-r^1}+1)}{(q-1)} \right) \right\} \]

\[ \geq q^{np-n+r-1} \left\{ q^{r+1} (q^{d-r^1}+1) \left( \frac{2}{(q-1)} \right) + 1 + \frac{2}{(q-1)} \right\} \]

\[ > 0. \]

under the given conditions. This completes the proof. \[ \]

**Lemma 3.2.3** Let V be an (n+d)-dimensional vector space over GF(q), W \( \subseteq \) V be an n-subspace, X \( \subseteq \) V be a s-subspace and X \( \cap \) W = \{0\}, where s \( \leq \) d. Then

1. The size of the set \{ Z \mid Z \subseteq V \text{ is an i-subspace and } Z \cap W = \{0\} \} is q^{ni[d]}.

2. The size of the set \{ Z \mid Z \subseteq V \text{ is a d-subspace with } X \subseteq Z \text{ and } Z \cap W = \{0\} \} is q^{n(d-s)}.

**Proof:** (1). Given an ordered i-tuple of linearly independent vectors in the quotient space V/W, there are q^{ni} ways (namely \( |\text{Hom}_{GF(q)}(V/W,W)| \)) to lift this i-tuple to an i-tuple of linearly independent vectors in V.
(2). Apply (1) to the quotient spaces $V/X$, $Z/X$ and $W/X (=W)$. 

From now on, we assume the same notation and conventions as mentioned in Section 2.2. For vectors $x_1, x_2, \ldots, x_s$, let $\mathcal{F}_{x_1, \ldots, x_s}$ be the set of all $A \in \mathcal{F}$ which contain $<x_1, \ldots, x_s>$, the subspace spanned by $x_1, \ldots, x_s$.

Suppose $\mathcal{F} \subseteq \mathcal{A}_d$, with the property that $\dim(A \cap B) \geq r$ for all $A, B \in \mathcal{F}$, and $\dim(\cap \mathcal{F}) \leq r-1$. We want to show that $|\mathcal{F}| < q^{n(d-r)}$.

Let $<x_1, \ldots, x_r> \subseteq V$ be a $r$-subspace with $<x_1, \ldots, x_r> \cap W = O$, the zero space, and $|\mathcal{F}_{x_1, \ldots, x_r}|$ be the maximum of $|\mathcal{F}_C|$ for all $r$-subspaces $C$ of $V$.

Since $\dim(\cap \mathcal{F}) \leq r-1$, there is an element $A_1 \in \mathcal{F}$ such that $\dim(<x_1, \ldots, x_r> \cap A_1) \leq r-1$, and so there is a $(d-r+1)$-subspace $B_1$ of $A_1$ with $B_1 \cap <x_1, \ldots, x_r> = O$, the zero space. Hence $A \cap B_1 \neq O$ for all $A \in \mathcal{F}$ by comparing their dimensions, and so $\mathcal{F}_{x_1, \ldots, x_r} \subseteq <y> \cup B_1 \mathcal{F}_{x_1, \ldots, x_r, y}$ (over all 1-subspaces $<y>$ of $B_1$). Therefore, it follows that

$$|\mathcal{F}_{x_1, \ldots, x_r}| \leq |d-r+1|q^{n(d-r-1)} \quad (1)$$

by Lemma 3.2.3(2) (for $X=<x_1, \ldots, x_r, y>$, any $y \in B_1$).
If there is a vector \( y^* \in B_1 \) with the property that 
\[
\dim(<x_1, \ldots, x_r, y^*> \cap A) \geq r \quad \text{for all } A \in \mathcal{F},
\]
then
\[
|\mathcal{F}| \leq \sum_{C \subseteq <x_1, \ldots, x_r, y^*> \text{ is a } r\text{-subspace}} |\mathcal{F}_c| \leq \left[ \begin{array}{c} r+1 \\ 1 \end{array} \right] [d-r+1] q^{n(d-r-1)} \quad \text{by (1)}
\]
\[
< q^{n(d-r)} \quad \text{by Lemma 3.2.2.}
\]

Otherwise, we may assume that for any \( y \in B_1 \), there is an element \( A_y \in \mathcal{F} \) such that \( \dim(<x_1, \ldots, x_r, y> \cap A_y) \leq r-1 \). Hence there is a (d-r+1)-subspace \( B_y \) of \( A_y \) such that \( <x_1, \ldots, x_r, y> \cap B_y = \emptyset \), and so \( A \cap B_y \neq \emptyset \) for all \( A \in \mathcal{F} \). Hence \( \mathcal{F}_{x_1, \ldots, x_r, y} \subseteq \bigcup_{<z> \subseteq B_y} \mathcal{F}_{x_1, \ldots, x_r, y, z} \) (over all 1-subspaces \( <z> \) of \( B_y \)). It follows that
\[
|\mathcal{F}_{x_1, \ldots, x_r, y}| \leq \sum_{<z> \subseteq B_y} |\mathcal{F}_{x_1, \ldots, x_r, y, z}| \leq \left[ \begin{array}{c} d-r+1 \\ 1 \end{array} \right] q^{n(d-(r+2))}
\]
for all \( y \in B_1 \) by Lemma 3.2.3(2) (for \( X = <x_1, \ldots, x_r, y> \)). Since \( A \cap B_1 \neq \emptyset \) for all \( A \in \mathcal{F} \), so
\[
|\mathcal{F}_{x_1, \ldots, x_r}| \leq \sum_{<y> \subseteq B_1} |\mathcal{F}_{x_1, \ldots, x_r, y}|
\]
where the summation is over all 1-subspaces \( \langle y \rangle \) of \( B_1 \).

Fix an element \( y_1 \in B_1 \), and denote the corresponding \((d-r+1)\)-subspace \( B_{y_1} \) by \( B_2 \). If there is an element \( y^* \in B_2 \) with the property that \( \dim(\langle x_1, \ldots, x_r, y_1, y^* \rangle \cap A) \geq r \) for all \( A \in \mathcal{F} \), then

\[
|\mathcal{F}| \leq \sum_{\mathcal{C} \subseteq \langle x_1, \ldots, x_r, y_1, y^* \rangle \text{ is a } r\text{-subspace}} |\mathcal{F}_C|
\]

\[
\leq \frac{r^2}{2} [d-r+1]^2 q^{n(d-r-2)}
\]

by (2)

\[
< q^{n(d-r)}
\]

by Lemma 3.2.2

Otherwise, we may assume that for any \( y \in B_2 \), there is an element \( A_y \in \mathcal{F} \) such that \( \dim(\langle x_1, \ldots, x_r, y_1, y \rangle \cap A_y) \leq r-1 \). Hence there is a \((d-r+1)\)-subspace \( B_y \) of \( A_y \) such that \( \langle x_1, \ldots, x_r, y_1, y \rangle \cap B_y = \emptyset \), and so \( A \cap B_y \neq \emptyset \) for all \( A \in \mathcal{F} \). It follows that

\[
|\mathcal{F}_{x_1, \ldots, x_r, y_1, y}|
\]

\[
\leq \sum_{\langle z \rangle \subseteq B_y} |\mathcal{F}_{x_1, \ldots, x_r, y_1, y, z}|
\]

\[
\leq [d-r+1] q^{n(d-(r+3))}
\]

for all \( y \in B_2 \) by Lemma 3.2.3(2) (for \( X = \langle x_1, \ldots, x_r, y_1, y, z \rangle \); and so
So far, we have either $|F| < q^{n(d-r)}$ or $|F_{x_1, \ldots, x_r, y_1}| \leq [d-r+1]_1 q^{n(d-r-3)}$ for all $y \in B_1$. Hence, in the latter case, we have

$$|F_{x_1, \ldots, x_r}| \leq \sum_{y \in B_1} |F_{x_1, \ldots, x_r, y}| \leq [d-r+1]_1 q^{n(d-r-3)}$$

(3)

Repeating this process, we may conclude that either (1) $|F| < q^{n(d-r)}$, or (2) $|F_{x_1, \ldots, x_r}| \leq [d-r+1]_1 p q^{n(d-r-p)}$ for all $p$, $1 \leq p \leq d-r$. If case (1) does not occur, then $|F_{x_1, \ldots, x_r}| \leq [d-r+1]_1 q^{n(d-r)}$. Let $A \in F$ be fixed. Then

$$|F| \leq \sum_{C \subseteq A} |F_C|$$

is a $r$-subspace

$$\leq [d]_1 [d-r+1] q^{n(d-r)}$$

by Lemma 3.2.2.

i.e., in both cases, we have $|F| < q^{n(d-r)}$. This completes the proof. \( \square \)
3.3 \( H_q(d,n) \) has no perfect \( e \)-codes

Let \( \Gamma \) be a distance-regular graph of diameter \( d \). For \( x \in V(\Gamma) \), the vertex set of \( \Gamma \), and an integer \( e \) with \( 1 \leq e < d/2 \), \( S(x,e) \) is defined to be the set \( \{ y \mid y \in V(\Gamma), \partial(x,y) \leq e \} \). Then \( \partial(y,z) \leq 2e \) for all \( y, z \in S(x,e) \), and the size \( |S(x,e)| \) is a function of \( e \) only, independent of \( x \). A non-empty subset \( C \subseteq V(\Gamma) \) is called a perfect \( e \)-code of \( \Gamma \) if \( \{S(x,e)\mid x \in C\} \) is a partition of \( V(\Gamma) \). If there is a perfect \( e \)-code \( C \) in \( \Gamma \), then \( |S(x,e)||C| = |V(\Gamma)| \), the so-called sphere-packing condition, and \( \partial(x,y) \geq 2e+1 \) for any two distinct vertices \( x \) and \( y \) in \( C \).

We shall show the non-existence of any perfect \( e \)-code in \( H_q(d,n) \) by using Theorem 3.1.1 together with the following result of Delsarte [9, p. 32]:

**Theorem** Let \( B \subseteq V(\Gamma) \) with the property that \( \partial(y,z) \geq r+1 \) for all distinct \( y, z \in B \), and \( T \subseteq V(\Gamma) \) with the property that \( \partial(y,z) \leq r \) for all \( y, z \in T \). Then \( |B||T| \leq |V(\Gamma)| \).

**Theorem 3.3.1** \( H_q(d,n) \) has no perfect \( e \)-codes where \( n \geq d+1 \), \((n, q) \neq (d+1, 2)\), and \( 1 \leq e < d/2 \).

**Proof:** Suppose, to the contrary, that there is a perfect \( e \)-code \( C \) in \( H_q(d,n) \), and \( A \in C \) be fixed. If \( T \subseteq V(\Gamma) \), here \( \Gamma = H_q(d,n) \), with the
property that \( \vartheta(B, C) \leq 2e \) for all \( B, C \in \mathcal{F} \), the sphere-packing condition and the theorem of Delsarte show that

\[
|C||\mathcal{F}| \leq |V(\Gamma)| = |C||S(A, e)|
\]
i.e., \( |\mathcal{F}| \leq |S(A, e)| \). Under the isomorphism in Theorem 2.2.1, Theorem 3.1.1 shows that \( |S(A, e)| = q^{n(2e)} \) and consequently \( \cap S(A, e) \subseteq V \) is a \((d-2e)\)-subspace. Without loss of generality, we may assume \( S(A, e) = \{ A + E \mid E \text{ is a } d \times n \text{ matrix with zero entries in the last } d-2e \text{ rows} \} \). Let \( B = A + F \), where \( F \) be a \( d \times n \) matrix with all entries in the first column non-zero, and all others zero. Then \( B \in S(A, e) \), contradicting the structure of \( S(A, e) \) above. \([\] \)
Chapter 4

A characterization of $H_q(d,n)$

4.1 Introduction

The classification of all infinite families of distance-regular graphs is a major problem. E. Bannai [1, Chapter 3] has compiled a list of distance-regular graphs with large diameters, believing that he has an essentially complete list at this time. The characterization problems of known important classes of distance-regular graphs by their parameters have a long history in combinatorics, e.g., [2, 12, 22, 28] for Johnson graphs, [21] for Hamming graphs, [25] for q-analogue Johnson graphs, and [6] for dual polar spaces. In this chapter, we prove an analogous result for $H_q(d,n)$. 
Theorem 4.1.1 Let $\Gamma$ be a distance-regular graph of diameter $d$ with parameters $\{b_0, b_1, ..., b_{d-1}; c_1, ..., c_d\}$ such that

1. the number of edges of the induced subgraph on the common neighborhood of any two vertices $x$ and $y$ depends only on the distance between them, i.e., the weak 4-vertex condition holds in $\Gamma$, and

2. $c_2 = q^2 + q$ and $b_i = q^{2i}(q^{d-i-1})(q^{n-i-1})/(q-1)$ for $0 \leq i \leq d-1$ (i.e., coincide with those of $H_q(d,n)$) where $n \geq 2d \geq 6$, and $q \geq 4$.

Then $q$ is a prime power and $\Gamma$ is isomorphic to $H_q(d,n)$.

Remark: An incidence structure is derived from $\Gamma$ by assuming $n \geq 2d \geq 6$ and $q \geq 4$. Also we need the condition that $q \geq 4$ for using a theorem of F. Buekenhout [3]. So far, we have no counterexamples for the remaining cases.

Sprague [24] characterized $H_q(3,n)$, for $n \geq 6$, $q \geq 2$ and $(q,n) \neq (2,6)$, by assuming the parameters and the weak 4-vertex condition. Later Ray-Chaudhuri and Sprague [23] gave a combinatorial characterization of 3-attenuated spaces by using the arguments similar to those used in [24]. Recently, Sprague [25] characterized $d$-attenuated spaces in terms of the structure of their 2-spaces.
Theorem A (Sprague [25]) Every finite d-net, where \( d \geq 3 \) is an integer, is an (\( n,q,d \))-attenuated space for some prime power \( q \) and positive integer \( n \). (The original theorem covers the infinite case also.)

In this chapter, based on the above theorem, we set out an extension of the techniques used in [23, 24] to prove Theorem 4.1.1. In section 2, by using the Bose-Laskar argument [2], we derive a semilinear incidence structure \( \Pi \) from \( \Gamma \) (as in the main theorem). After investigating the structure of 2-spaces of \( \Pi \) in Sections 4.3 and 4.4, we show in Section 4.5 that \( \Pi \) is a d-net. The proof of Theorem 4.1.1 will then be completed by theorem A. The geometric analysis of the 2-spaces of \( \Pi \) in Section 4.3 and part of Section 4.4 largely stems from [24].

As a corollary, we prove in Section 4.6 that \( H_q(d,n) \) has no antipodal covering if \( n \geq 2d \geq 6 \) and \( q \geq 4 \) (Theorem 4.6.1).

4.2 The incidence structure derived from \( \Gamma \)

Throughout Sections 4.2 - 4.5, we assume that \( \Gamma \) is a fixed distance-regular graph of diameter \( d \) satisfying conditions (1) and (2) of Theorem 4.1.1.

A semilinear incidence structure can be derived from \( \Gamma \) by the following theorem, due to Bose and Laskar.
Theorem B (Bose - Laskar [2]) Let G be a graph satisfying the following conditions:

(a) \( \deg(x) = r(k-1) \) for all \( x \in V(G) \).

(b) \( |G_1(x) \cap G_1(y)| = k - 2 + \alpha \) if \( x \) and \( y \in V(G) \) are adjacent,

(c) \( |G_1(x) \cap G_1(y)| \leq 1 + \beta \) if \( x \) and \( y \in V(G) \) are not adjacent,

where \( r \geq 1, k \geq 2, \alpha \geq 0 \) and \( \beta \geq 0 \) are fixed integers. Define a grand clique as a maximal clique with at least \( k-(r-1)\alpha \) vertices. Let \( k \) be strictly larger than the maximum of \( p(\alpha, \beta, r) \) and \( \rho(\alpha, \beta, r) \), where

\[
p(\alpha, \beta, r) = 1 + (r+1)(r\beta - 2\alpha)/2 \quad \text{and} \quad \rho(\alpha, \beta, r) = 1 + \beta + (2r - 1)\alpha.
\]

Then

(1) each vertex of \( G \) is contained in exactly \( r \) grand cliques, and

(2) each pair of adjacent vertices is contained in exactly one grand clique.

For the given graph \( \Gamma \), \( b_0 = (q^{n-1})(q^{d-1})/(q-1) \), \( a_1 = q^n+q^d-q-2 \) and \( c_2 = q(q+1) \). Let \( k = q^n, \ r = (q^{d-1})/(q-1), \ \alpha = q^d-q \) and \( \beta = q^2 + q + 1 \). Our assumptions on \( n, d, \) and \( q \) imply that \( k \) is larger than the maximum of \( p(\alpha, \beta, r) \) and \( \rho(\alpha, \beta, r) \) and that each grand clique contains at least \( k - (r-1)\alpha = q^n - ((q^d-q)^2/(q-1)) \) vertices. By theorem B, \( \Pi = (V(\Gamma), L, \in) \) is a semilinear incidence structure where the line set \( L \) is the set of all grand cliques of \( \Gamma \). Each point of \( \Gamma \) is contained in exactly \( r = (q^{d-1})/(q-1) \) lines and each pair of adjacent points \( x, y \) is contained in exactly one line, denoted by \( <x,y> \). Moreover, \( \Gamma \) is the adjacency graph of \( \Pi \).
Lemma 4.2.1  If \( x \) and \( y \in V(\Gamma) \) with \( \partial(x,y) = i \), then there are at most \( (q^i-1)/(q-1) \) lines of \( y \) at distance \( i-1 \) from \( x \).

Proof. Let \( \lambda_1, ..., \lambda_s \) be the lines of \( y \) which are at distance \( i-1 \) from \( x \). Since \( \partial(x,a) = i-1 \) or \( i \) for all \( a \in \lambda_j, 1 \leq j \leq s \), we have \( b_i \leq (r-s)(a_1+2) \) and so \( s \leq r - (b_y/(a_1+2)) \). Since \( s \) is an integer, \( s \leq (q^i-1)/(q-1) \), as required. \( \Box \)

Until Proposition 4.5.2, we only need the case \( i = 2 \). Let \( \lambda \) be a line and \( x \) be a point with \( \partial(x,\lambda) = 1 \). Then \( x \) is adjacent to at most \( q+1 \) points of \( \lambda \) by Lemma 4.2.1.

As usual, for sets \( A \) and \( B \), \( |A| \) is the cardinality of \( A \), and \( A \setminus B \) is the set \( \{x | x \in A \text{ but } x \notin B\} \). Let \( \lambda_0 \) be a line of the minimum size, \( x_0 \in \lambda_0 \) be fixed, and \( \lambda_1, ..., \lambda_{r-1} \) be the remaining lines of \( x_0 \). By counting the number of edges between \( \lambda_0 \setminus \{x_0\} \) and \( \bigcup_{1 \leq i \leq r-1}(\lambda_i \setminus \{x_0\}) \), we have \( (q^n+q^d-q-|\lambda_0|)(|\lambda_0|-1) \leq q(r-1)(|\lambda_0|-1) \). It follows that \( |\lambda_0| \geq q^n-((q^d-q)/(q-1)) \). This improves the lower bound on the size of lines given by Theorem B.
The technique used by Sprague in [24] can be generalized to prove Lemma 4.2.2 and Lemma 4.2.4 where we need the weak 4-vertex condition on \( \Gamma \). Let \( x, y \) be points on a line \( \lambda \) and let 

\[ S = (\Gamma_1(x) \cap \Gamma_1(y)) - \lambda. \]

The edge set \( T(x,y) \) of the induced subgraph on \( \Gamma_1(x) \cap \Gamma_1(y) \) can be partitioned into three parts: the edges with 0, 1, or 2 end-vertices on \( S \). Let \( |<x,y>| = \alpha \). Then

\[
(*) \quad (\frac{\alpha}{2} - 2) \leq |T(x,y)| \leq (\frac{\alpha}{2} - 1) + (a_1 + 2 - \alpha) + (a_1 + 2 - \alpha)(q-1).
\]

The average size of the lines of \( x \) is \( 1 + (b_0/r) = q^n \). We shall show in the next lemma that \( q^n \) is the exact size of each line in \( L \) by using the inequality (*) above.

**Lemma 4.2.2** Each line consists of \( q^n \) points.

Proof. Suppose, to the contrary, that there exist a point \( u \) and a line of \( u \) with size different from \( q^n \). Without loss of generality, we may assume \( \lambda_1, \lambda_2 \) are lines of \( u \) with \( |\lambda_1| \geq q^n + 1 \) and \( |\lambda_2| \leq q^n - 1 \). Let 

\[ f(x) = (x - 2) \quad \text{and} \quad g(x) = (x - 2) + (a_1 + 2 - x) + (a_1 + 2 - x)(q-1). \]

Since \( g(x) \) is concave upward and \( q^n - r + 1 \leq |\lambda_2| \leq q^n - 1 \), we have \( g(|\lambda_2|) \leq \) the maximum of \( g(q^n - r + 1) \) and \( g(q^n - 1) \). Straightforward computation shows that \( f(q^n + 1) > \) the maximum of \( g(q^n - r + 1) \) and \( g(q^n - 1) \). Hence \( g(|\lambda_2|) < f(|\lambda_1|) \). Choose \( y_i \) from \( \lambda_i, i = 1, 2 \) (distinct from \( u \)). Then

\[ |T(u,y_1)| \geq f(|\lambda_1|) > g(|\lambda_2|) \geq |T(u,y_2)| \]

by (*) , this contradicts the weak 4-vertex condition of \( \Gamma \). []
Next, we shall show that $|\Gamma_i(x) \cap \lambda| = q^i$ for any point $x$ and any line $\lambda$ with $\partial(x,\lambda) = i$, $i = 1, 2$, in Lemma 4.2.3 and Corollary 4.3.1.2 respectively by applying the Principle of weighted average (also used by Sprague in [24]), which is stated as follows without proof.

Let $X$ be a finite set and $g: X \to \mathbb{N} \cup \{0\}$ be a nonzero function. For any function $f: X \to \mathbb{R}$ we define $\text{AVE}(f,X) = \frac{\sum_{x \in X} f(x)}{|X|}$ and $\text{AVE}(f,X;g) = \frac{\sum_{x \in X} f(x)g(x)}{\sum_{x \in X} g(x)}$.

**Principle of weighted averages:**

1. If $f(x) > f(y)$ implies $g(x) > g(y)$ for all $x, y \in X$, then $\text{AVE}(f,X;g) \geq \text{AVE}(f,X)$.
2. If $f(x) > f(y)$ implies $g(x) < g(y)$ for all $x, y \in X$, then $\text{AVE}(f,X;g) \leq \text{AVE}(f,X)$.

In both cases, if $\text{AVE}(f,X) = \text{AVE}(f,X;g)$, then $f$ is a constant function.

**Lemma 4.2.3**

1. If $\lambda$ is a line and $x$ is a point with $\partial(x,\lambda) = 1$, then $|\Gamma_1(x) \cap \lambda| = q$.
2. If $x$ and $y$ are points with $\partial(x,y) = 2$, then there are exactly $q+1$ lines of $y$ at distance 1 from $x$.

**Proof:** Let $\mathcal{X} = \{\lambda | \lambda \in \mathcal{L} \text{ and } \partial(x,\lambda) = 1\}$, and $f: \mathcal{X} \to \mathbb{N} \cup \{0\}$ by $f(\lambda) = |\Gamma_1(x) \cap \lambda|$. We want to show that $f(\lambda) = q$ for all $\lambda \in \mathcal{X}$. 
Let \( y \in \Gamma_1(x) \) and \( \mathcal{X}_y = \{ \lambda \mid \lambda \in \mathcal{X} \text{ and } y \in \lambda \} \). Since there is a unique line containing \( x \) and \( y \), \( |\mathcal{X}_y| = r - 1 \), we have

\[
b_1 = \sum_{\lambda \in \mathcal{X}_y} (q^n - f(\lambda)) \quad \text{because } |\lambda| = q^n.
\]

Hence \( \sum_{\lambda \in \mathcal{X}_y} f(\lambda) = q(q^d - q)/(q - 1) \) and so \( \text{AVE}(f, \mathcal{X}_y) = q \) for all \( y \in \Gamma_1(x) \). Let \( \alpha_y = \sum_{\lambda \in \mathcal{X}_y} f(\lambda) \) where \( y \in \Gamma_1(x) \). Then \( \alpha_y = q|\mathcal{X}_y| \), and it follows that

\[
q = \frac{\sum_{y \in \Gamma_1(x)} \alpha_y}{\sum_{y \in \Gamma_1(x)} |\mathcal{X}_y|}
= \frac{\sum_{\lambda \in \mathcal{X}} (\sum_{y \in \Gamma_1(x) \cap \lambda} f(\lambda))}{\sum_{\lambda \in \mathcal{X}} f(\lambda)}
= \frac{\sum_{\lambda \in \mathcal{X}} f(\lambda)^2}{\sum_{\lambda \in \mathcal{X}} f(\lambda)}
= \text{AVE}(f, \mathcal{X}; f).
\]

By the Principle of weighted averages, \( \text{AVE}(f, \mathcal{X}) \leq \text{AVE}(f, \mathcal{X}; f) = q \).

On the other hand, if \( z \in \Gamma_2(x) \), then there are at most \( q + 1 \) lines of \( z \) at distance 1 from \( x \) by Lemma 4.2.1. Hence, \( \text{AVE}(f, \mathcal{X}_z) \geq c_2/(q + 1) = q \), where \( \mathcal{X}_z = \{ \lambda \mid \lambda \in \mathcal{X} \text{ and } z \in \lambda \} \). Let \( \alpha_z = \sum_{\lambda \in \mathcal{X}_z} f(\lambda) \) where \( z \in \Gamma_2(x) \). Then \( \alpha_z \geq q|\mathcal{X}_z| \) and so

\[
q \leq \frac{\sum_{z \in \Gamma_2(x)} \alpha_z}{\sum_{z \in \Gamma_2(x)} |\mathcal{X}_z|}
= \frac{\sum_{\lambda \in \mathcal{X}} (q^n - f(\lambda))f(\lambda)}{\sum_{\lambda \in \mathcal{X}} (q^n - f(\lambda))}
= \text{AVE}(f, \mathcal{X}; q^n - f).
\]

Again, by the Principle of weighted averages, \( \text{AVE}(f, \mathcal{X}) \geq \text{AVE}(f, \mathcal{X}; q^n - f) \geq q \). Combining these two inequalities, we have \( q = \text{AVE}(f, \mathcal{X}) = \text{AVE}(f, \mathcal{X}; f) \), and so \( f(\lambda) = q \) for all \( \lambda \in \mathcal{X} \). This proves (1).

To prove (2), we observe that \( |\Gamma_1(x) \cap \lambda| = q \) by (1) for all lines \( \lambda \) of \( y \) with \( \partial(x, \lambda) = 1 \). So there are at least \( c_2/q = q + 1 \) lines of \( y \) at distance 1 from \( x \), and the proof is completed by Lemma 4.2.1. \( \square \)
Corollary 4.2.3.1 Let $x$ be a point, $\mathcal{X} = \{\lambda \mid \lambda \in \mathcal{L} \}$ with $\partial(x, \lambda) = 2$ and $f(\lambda) = |\Gamma_2(x) \cap \lambda|$ for all $\lambda \in \mathcal{X}$. Then $\text{AVE}(f, \mathcal{X}) \leq \text{AVE}(f, \mathcal{X}; f) = q^2$.

Arguments similar to those used in Lemma 4.2.3 work for the above corollary because, by Lemma 4.2.3, for each pair of points $x$ and $y$ with $\partial(x, y) = 2$, there are exactly $q + 1$ lines of $y$ at distance 1 from $x$. Later, we will show that $\text{AVE}(f, X) = \text{AVE}(f, X; f) = q^2$ in Corollary 4.3.1.2. Consequently $|\Gamma_2(x) \cap \lambda| = q^2$ for any line $\lambda$ and any point $x$ with $\partial(x, \lambda) = 2$.

If $\lambda$ and $\eta$ are two distinct lines and $\lambda \cap \eta \neq \emptyset$, denoted by $\lambda \neq \eta$, then $\lambda \cap \eta$ consists of exactly one point by (2) of Theorem B.

Corollary 4.2.3.2 Let $x$, $y$ be points with $\partial(x, y) = 2$, $\lambda_0$, $\lambda_1$, ..., $\lambda_{r-1}$ and $\eta_0$, $\eta_1$, ..., $\eta_{r-1}$ be lines of $x$ and $y$ respectively. Then (relabelling if necessary) $\lambda_i \neq \eta_j$ if and only if $0 \leq i, j \leq q$ are distinct.

Proof: Let $\lambda_0$, $\lambda_1$, ..., $\lambda_s$ be lines of $x$ which meet at least one line of $y$. Then $c_2 = (s+1)q$ and so $s = q$ by Lemma 4.2.3. The same argument works if $x$ is replaced by $y$. 

Remark. Corollary 4.2.3.2 can be improved after we prove Proposition 4.3.4, which shows the relation between lines $\lambda_i$ and $\eta_i$, $0 \leq i \leq q$. 
Let \( \alpha \) (resp. \( \beta \)) be the number of edges of the induced subgraph of \( \Gamma \) on \( \Gamma_1(x) \cap \Gamma_1(y) \) where \( x, y \in V(\Gamma) \) with \( \partial(x,y) = 1 \) (resp. \( \partial(x,y) = 2 \)). By the weak 4-vertex condition on \( \Gamma \), \( \alpha \) and \( \beta \) depend only on the distance between \( x \) and \( y \). The next proposition is a first step in attaching an affine structure to a subset of \( (\Gamma_1(x) \cap \Gamma_1(y)) \cup \{x,y\} \) for adjacent pair \( x \) and \( y \), which is essential in determining the structure of the subspaces of \( \Pi \).

**Proposition 4.2.4** \((\Gamma_1(x) \cap \Gamma_1(y)) - \langle x,y \rangle\) is a clique for any adjacent pair \( x \) and \( y \).

Proof: Let \( x \) and \( y \in V(\Gamma) \) with \( \partial(x,y) = 1 \) and \( s \) be the number of non-adjacent unordered pair of points in \((\Gamma_1(x) \cap \Gamma_1(y)) - \langle x,y \rangle\). Then

\[
s = (q^{n-2}) + (q^{d-q}) + (q-2)(q^{d-q}) - \alpha \leq (q^{d-q})
\]

by an argument similar to the one used to derive (*)\(^{(*)}\). It suffices to show that \( s = 0 \).

Let \( u, v \in V(\Gamma) \) with \( \partial(u,v) = 2 \) and \( t \) be the number of edges with end-vertices \( a, b \) in \( \Gamma_1(u) \cap \Gamma_1(v) \) but \( \langle a,b \rangle \cap \{u,v\} = \emptyset \). Let \( B = \{\{a,b\} | a,b \in \Gamma_1(u) \cap \Gamma_1(v) \text{ are adjacent but } \langle a,b \rangle \cap \{u,v\} \neq \emptyset \} \). Then \( |B| = c_2(q-1) \) by Lemma 4.2.3, and \( t = \beta - |B| = \beta - q(q^2-1) \). Both \( t \) and \( s \) are independent of the choice of \( x, y, u \) and \( v \).
Let \( X = \{(a,b,u,v)\mid a, b, u, v \in V(\Gamma) \text{ are pairwise adjacent except } \partial(u,v) = 2, \text{ no three are collinear} \} \). Let \( k_i = |\Gamma_i(x)|, i=1, 2, \text{ i.e., } k_1 = b_0, k_2 = b_0b_1/c_2 \). Count \(|X|\) in two ways: (1) There are \(|V(\Gamma)|k_1\) ways to choose adjacent pair \(a\) and \(b\), and there are \(2s\) ways to choose \(u, v \in V(\Gamma)\) such that \((a,b,u,v) \in X\), \(|X| = 2|V(\Gamma)|k_1s\). (2) Similarly, there are \(|V(\Gamma)|k_2\) ways to choose pair \((u,v)\) with \(\partial(u,v) = 2\), and there are \(2t\) ways to choose \(a, b \in V(\Gamma)\) such that \((a,b,u,v) \in X\), \(|X| = 2|V(\Gamma)|k_2t\). Hence \(s = tk_2/k_1\). Because \(n \geq 2d \geq 6\) and \(s \leq (q^d - q)\), \(s = tk_2/k_1\) implies that \(t = 0\) and consequently \(s = 0\), as required. \(\Box\)

**Corollary 4.2.4.1** Let \(x, y \in V(\Gamma)\) with \(\partial(x,y) = 2\). If \(u, v \in \Gamma_1(x) \cap \Gamma_1(y)\) are adjacent, then \(\langle u, v \rangle \cap \{x, y\} \neq \emptyset\).

Proof: Suppose, to the contrary, that \(\langle u, v \rangle \cap \{x, y\} = \emptyset\), then \(x, y \in (\Gamma_1(u) \cap \Gamma_1(v)) - \langle u, v \rangle\) and so \(x, y\) are adjacent by Proposition 4.2.4, a contradiction. \(\Box\)

**Corollary 4.2.4.2** Let \(\lambda, \eta\) be lines with \(\partial(\lambda, \eta) = 1\), and \(x_1 \in \lambda, y_1 \in \eta\) be adjacent. Let \(\Gamma_1(x_1) \cap \eta = \{y_1, y_2, \ldots, y_q\}\) and \(\Gamma_1(y_1) \cap \lambda = \{x_1, x_2, \ldots, x_q\}\). Then \(\Gamma_1(x_i) \cap \eta = \{y_1, y_2, \ldots, y_q\}\) and \(\Gamma_1(y_i) \cap \lambda = \{x_1, x_2, \ldots, x_q\}\) for all \(i = 2, 3, \ldots, q\).
Proof: Since \( \{x_2, \ldots, x_q\} \cup \{y_2, \ldots, y_q\} \subseteq (\Gamma_1(x_1) \cap \Gamma_1(y_1)) - <x_1, y_1> \), they are pairwise adjacent by Proposition 4.2.4. Hence \( x_i \) is adjacent to each of \( y_1, \ldots, y_q \) and similarly \( y_j \) is adjacent to each of \( x_1, \ldots, x_q \) for any \( i, j \leq q \). \( \square \)

4.3 The structure of assemblies

Let \( x, y \in V(\Gamma) \) be adjacent and \( A_{x,y} = S_1 \cup S_2 \) where \( S_1 = (\Gamma_1(x) \cap \Gamma_1(y)) - <x,y> \) and \( S_2 = \{ z \mid z \in <x,y> \text{ is adjacent to each point of } S_1 \} \). Then \( A_{x,y} \), the assembly determined by the adjacent pair \( x \) and \( y \), is a clique by Proposition 4.2.4. The purpose of this section is to show that the restriction of \( \Pi \) to \( A_{x,y} \) is an affine space of dimension \( d \) over \( GF(q) \) (Proposition 4.3.7), which is a first step in determining the structure of the 2-spaces of \( \Pi \). The equivalent statements of Lemmas 4.3.1, 4.3.2, 4.3.3 Proposition 4.3.4 and their corollaries can be found in [24].

Lemma 4.3.1 \( A_{x,y} \) is a clique of \( q^d \) points for any adjacent pair \( x \) and \( y \) in \( V(\Gamma) \).
Proof: Let \( \lambda \) be the line \( <x,y> \). It only remains to show that \( \Gamma_1(u) \cap \lambda = \Gamma_1(v) \cap \lambda \) for any \( u, v \in S_1 \). Suppose, to the contrary, that there are \( u, v \in S_1 \) with \( \Gamma_1(u) \cap \lambda \neq \Gamma_1(v) \cap \lambda \), then \( <u,v> \cap \lambda \neq \emptyset \) (otherwise, \( \Gamma_1(u) \cap \lambda = \Gamma_1(v) \cap \lambda \) by Corollary 4.2.4.2). We may assume \( y \notin <u,v> \cap \lambda \). Since \( |\Gamma_1(u) \cap \lambda| = |\Gamma_1(v) \cap \lambda| = q \) by Lemma 4.2.3, there is a point \( z \in \Gamma_2(u) \cap \Gamma_1(v) \cap \lambda \), and so \( v, y \in \Gamma_1(z) \cap \Gamma_1(u) \) are adjacent but \( <v,y> \cap \{u,z\} = \emptyset \), contradicting Corollary 4.2.4.1. It follows that \( |S_2| = q \), and so \( |A_{x,y}| = q^d \) as required. \( \square \)

**Corollary 4.3.1.1** Let \( x \) be a point, \( \lambda \) be a line and \( A, A^* \) be assemblies,

1. \( |\lambda \cap A| = 0 \) or \( q \).
2. If \( |A \cap A^*| \geq 2 \), then \( A = A^* \).
3. If \( x \notin A \), then at most one line of \( x \) contains points of \( A \). Furthermore, if \( \partial(x,A) = 1 \), then \( |\Gamma_1(x) \cap A| = q \).

Proof: (1). Let \( z \in A \) be fixed and \( \lambda_0, \lambda_1, ..., \lambda_{r-1} \) be lines of \( z \). Since the points of \( A \) are not collinear, \( A \not\subseteq \lambda_i \) for all \( i \). It follows that \( A - \{z\} = \bigcup_{0 \leq i \leq r-1} (A \cap (\lambda_i \setminus \{z\})) \). Because \( A \cap \lambda_i \subseteq \Gamma_1(w) \cap \lambda_i \) for all \( w \in A - \lambda_i \), we have \( |A \cap \lambda_i| \leq q \) for all \( i \leq r-1 \). If \( |A \cap \lambda_i| \leq q-1 \) for some \( i \leq r-1 \), then \( |A - \{z\}| \leq q^{d-2} \), contradicting \( |A| = q^d \). Hence \( |A \cap \lambda_i| = q \) for all \( i \leq r-1 \).
(2). It suffices to show \( A = A_{a,b} \) for all distinct \( a, b \in A \). Let \( \eta = \langle a, b \rangle \). Then \( A - \eta \subseteq A_{a,b} - \eta \) and so \( A - \eta = A_{a,b} - \eta \) by comparing their sizes. On the other hand, \( A \cap \eta \) is a set of \( q \) points which are adjacent to all points of \( A_{a,b} - \lambda \), so we have \( A \cap \eta = A_{a,b} \cap \eta \) and consequently \( A = A_{a,b} \), as required.

(3). Suppose, to the contrary, that there is a point \( x \) and an assembly \( A \) with \( x \notin A \) such that \( \lambda_i \cap A \neq \emptyset \) for distinct lines \( \lambda_1, \lambda_2 \) of \( x \). Let \( u_i \in \lambda_i \cap A, i = 1, 2 \). Then \( x \in A_{u_1,u_2} = A \) by (2), a contradiction. The second part is clear from (1).

Corollary 4.3.1.2 Let \( \lambda \) be a line and \( x \) be a point with \( \partial(x, \lambda) = 2 \). Then

1. \( |\Gamma_2(x) \cap \lambda| = q^2 \), and
2. \( |\Gamma_1(y) \cap \Gamma_1(z) \cap \Gamma_1(x)| = q \) for all distinct points \( y, z \in \Gamma_2(x) \cap \lambda \).

Proof: Let \( T = \Gamma_2(x) \cap \lambda, S = \{u \mid u \in \Gamma_1(x) \text{ and } \partial(u, \lambda) = 1\} \). Counting in two ways the pairs \((u,b)\), where \( u \in S \) and \( b \in T \) are adjacent, we have \( |T|c_2 = |S|q \). Let \( B = \{(u,\{z_1,z_2\})\mid u \in S \text{ and } z_1, z_2 \in T \cap \Gamma_1(u) \text{ are distinct}\} \). For any pair \( z_1, z_2 \in T \), \( \Gamma_1(x) \cap \Gamma_1(z_1) \cap \Gamma_1(z_2) \subseteq \Gamma_1(x) \cap A_{z_1,z_2} \) has at most \( q \) points by Corollary 4.3.1.1. Hence \((|T|_2)^q \geq |B| = |S|(q)\) by Lemma 4.2.3.

1. Since \( |T|c_2 = |S|q \) and \((|T|_2)^q \geq |S|(q)\), we have \( |T| \geq q^2 \). Following the notation used in Corollary 4.2.3.1, we have
q^2 \leq \text{AVE}(f,X) \leq \text{AVE}(f,X:f) = q^2 \text{ and so } f(\eta) = q^2 \text{ for all lines } \eta \text{ with } \partial(x,\eta) = 2 \text{ by the Principle of weighted averages. Hence } T = \Gamma_2(x) \cap \lambda \text{ consists of } q^2 \text{ points, this proves (1).}

(2). Suppose, to the contrary, that there is a pair \( z_1, z_2 \) in \( T \) with 
\[ |\Gamma_1(x) \cap \Gamma_1(z_1) \cap \Gamma_1(z_2)| < q. \]
Then 
\[ |\mathcal{B}| = |\mathcal{S}(\mathcal{G})| < (|\mathcal{T}|/2)^q. \]
This is impossible because 
\[ |T|c_2 = |S|q \text{ and } |T| = q^2. \]

We will show that the structure of an assembly is determined by its relationship to external points. If \( Z \subseteq V(\Gamma) \), \( \mathcal{L}(Z) \) is defined to be the set \( \{ \lambda \mid \lambda \in \mathcal{L} \text{ and } |\lambda \cap Z| \geq 2 \} \). We begin with the following lemma.

Lemma 4.3.2 Let \( A \) be an assembly, \( u \in V(\Gamma) \) with \( \partial(u,A) = 2 \) and \( B = \Gamma_2(u) \cap A \). Then

(1) For any line \( \lambda \) with \( \lambda \cap A \neq \emptyset \), \( |\lambda \cap B| \geq 2 \) if and only if \( \lambda \cap \Gamma_1(u) \neq \emptyset \). Furthermore, if \( |\lambda \cap B| \geq 2 \), then \( \lambda \cap A = \lambda \cap B \).

(2) For each point \( x \) in \( B \), there are \( q+1 \) lines of \( x \) which meet at least at one other point of \( B \). Consequently, \( |B| = q^2 \).

(3) If \( \lambda, \eta \in \mathcal{L}(B) \) are distinct and \( \lambda \cap \eta \cap B = \emptyset \), then \( \lambda \cap \eta = \emptyset \).

Proof: (1). Let \( \lambda \) be a line which contains two points \( x \) and \( y \) of \( B \). Suppose \( \lambda \cap \Gamma_1(u) = \emptyset \), then \( \partial(u,\lambda) = 2 \) and so \( \Gamma_1(x) \cap \Gamma_1(y) \cap \Gamma_1(u) \neq \emptyset \) by Corollary 4.3.1.2(2). Let \( z \in \Gamma_1(x) \cap \Gamma_1(y) \cap \Gamma_1(u) \). Then \( z \in A_{x,y} = A \) and so \( \partial(u,A) \leq \partial(u,z) = 1 \), a contradiction. Conversely, if \( \lambda \) is a
line with \( \lambda \cap A \neq \emptyset \) and \( \lambda \cap \Gamma_1(u) \neq \emptyset \), then \( \Gamma_1(u) \cap \lambda \subseteq A^c \), the complement of \( A \), and so \( \lambda \cap A \subseteq \Gamma_2(u) \), hence \( \lambda \cap A = \lambda \cap B \) consists of \( q \) points. This proves (1).

(2). Since \( \partial(x,u) = 2 \), by Lemma 4.2.3, there are exactly \( q+1 \) lines of \( x \), say \( \lambda_0, \lambda_1, \ldots, \lambda_q \), such that \( \Gamma_1(u) \cap \lambda_i \neq \emptyset \), and these \( q+1 \) lines are exactly the lines of \( x \) which meet at least at two points of \( B \) by (1). \( |B| = q^2 \) follows from \( B = \bigcup_{0 \leq i \leq q} (B \cap \lambda_i) \).

(3). Since \( \lambda \cap A = \lambda \cap B, \eta \cap A = \eta \cap B \) by (1), \( \lambda \cap \eta \cap A = \emptyset \). Also \( \lambda \cap \eta \cap A^c = \emptyset \) by Corollary 4.3.1.1(3). This proves (3). []

It can be summarized that Lemma 4.3.2 is a first step to show that each assembly is isomorphic to an affine space.

**Corollary 4.3.2.1** The incidence structure \( \Pi_B = (B, \mathcal{L}(B), \in) \), as a subspace of \( \Pi_A = (A, \mathcal{L}(A), \in) \), is an affine plane of order \( q \).

Let \( \lambda, \eta \) be lines with \( \partial(\lambda, \eta) = 1 \). We define \( \lambda \) is to be parallel to \( \eta \), denoted by \( \lambda \parallel \eta \), if \( \partial(x, \eta) = 1 \) for each \( x \in \lambda \). Clearly \( \lambda \cap \eta = \emptyset \) when \( \lambda \parallel \eta \). By Corollary 4.2.4.2, \( \lambda \parallel \eta \) if and only if \( \eta \parallel \lambda \). The most important property between a point \( x \) and a line \( \lambda \) with \( \partial(x, \lambda) = 1 \) is the unique existence of a line of \( x \) which is parallel to \( \lambda \) (Proposition 4.3.4). To prove this property, we begin with the following technical lemma.
Lemma 4.3.3  Let $A$ be an assembly, $\lambda$ be a line with $\partial(\lambda,A) = 1$. If $S_\lambda = \{x| x \in \lambda \text{ and } \partial(x,A) = 1\}$ and $S_A = \{y| y \in A \text{ and } \partial(y,\lambda) = 1\}$, then $|S_A| = |S_\lambda| = q^2$.

Proof: Let $\mathcal{E} = \{(x,y)| x \in S_\lambda \text{ and } y \in S_A \text{ are adjacent}\}$. Then $|\mathcal{E}| = |S_A|q = |S_\lambda|q$ by Lemma 4.2.3 and Corollary 4.3.1.1(1). It follows that $|S_A| = |S_\lambda|$. Since $|S_A| \leq q^d$, there exists a point $u \in \lambda - S_\lambda$ such that $S_A \subseteq A \cap \Gamma_2(u)$, and so $|S_\lambda| = |S_A| \leq q^2$ by Lemma 4.3.2(2).

Let $x \in S_\lambda$, and $\eta$ be the unique line of $x$ which meets $A$. Let

- $T = \{z| z \in \Gamma_1(x) - \eta \text{ and } \partial(z,A) = 1\}$,
- $R = \{(y,z)| y \in A, z \in T \text{ are adjacent}\}$,
- $R_1 = \{(y,z)| (y,z) \in R \text{ and } y \in \eta\}$, and
- $R_2 = \{(y,z)| (y,z) \in R \text{ and } y \notin \eta\}$.

Then $|R| = |R_1| + |R_2| = q|T|$ by Corollary 4.3.1.1. Since $|A \cap \eta| = q$ and each $y \in A \cap \eta$ is adjacent to exactly $q^d-q$ points of $T$ (namely $(\Gamma_1(x) \cap \Gamma_1(y)) - \eta$), $|R_1| = q(q^d-q)$. On the other hand, $|A - \eta| = q^d-q$. For each $y \in A - \eta$, $\partial(x,y) = 2$ and $y$ is adjacent to exactly $q^2$ points of $T$ (namely $(\Gamma_1(x) \cap \Gamma_1(y)) - (\eta \cap \Lambda)$). Hence $|R_2| = (q^d-q)q^2$ and so $|T| = (|R_1| + |R_2|)/q = (r-1)(q^2-1)$ (note $r = (q^d-1)/(q-1)$).

The $r-1$ lines of $x$ distinct from $\eta$ partition $T$ and each such line contains at most $q^2-1$ points of $T$ because $|S_\lambda| \leq q^2$ for each line $\lambda$ of $x$ distinct from $\eta$ as mentioned at the beginning of the proof. Hence each such line of $x$ contains $q^2-1$ points of $T$ and so $|S_A| = |S_\lambda| = q^2$, as required. $\blacksquare$
Corollary 4.3.3.1 Let $A$ be an assembly, $u \in V(\Gamma)$ with $\partial(u,A) = 2$. Then there are $q+1$ lines of $u$ at distance 1 from $A$.

Proof: Let $B = \Gamma_2(u) \cap A$, $T = \{y| \partial(u,y) = 1 \text{ and } \partial(y,B) = 1\}$ and $E = \{(y,z)| y \in T \text{ and } z \in B \text{ are adjacent}\}$. Then $|B| = q^2$ by Lemma 4.3.2, $|\Gamma_1(z) \cap T| = c_2$ for any $z \in B$, and $|\Gamma_1(y) \cap B| = q$ for any $y \in T$ by Corollary 4.3.1.1. Hence $|E| = |B|(q^2+q) = q|T|$ and so $|T| = q^2(q+1)$ because $|B| = q^2$. Since $T$ contains exactly $q^2$ points of every line of $u$ at distance 1 from $A$ by Lemma 4.3.3, there are exactly $q+1$ lines of $u$ at distance 1 from $A$. []

Proposition 4.3.4 Let $x$ be a point, $\lambda$ be a line with $\partial(x,\lambda) = 1$. Then

1. There is a unique line $\eta$ of $x$ such that $\eta \parallel \lambda$.

2. If $\xi$ is a line of $x$ which is not parallel to $\lambda$ and $\lambda \cap \xi = \emptyset$, then there are exactly $q$ points of $\lambda$ at distance 1 from $\xi$.

Proof: (1). Let $A$ be an assembly with $x \in A$ and $\lambda \cap A = \emptyset$ (Let $w \in \lambda \cap \Gamma_1(x)$. Then any assembly which contains $x$ and distinct from $A_{x,w}$ must be disjoint from $\lambda$ by Corollary 4.3.1.1). Then $\partial(\lambda,A) = 1$. Let $S_A = \{z \in A| \partial(z,\lambda) = 1\}$. Then $|S_A| = q^2$ by Lemma 4.3.3. If $u \in \lambda$ with $\partial(u,A) = 2$. Then $S_A \subseteq \Gamma_2(u) \cap A$ and so $S_A = \Gamma_2(u) \cap A$ by comparing their sizes. Hence $S_A$ is an affine plane of order $q$ by Corollary 4.3.2.1.
First we prove the existence of a parallel line. For each point \( z \) of \( S_A \), there are \( q+1 \) lines of \( z \) in \( S_A \) and exactly \( q \) among them meet \( \lambda \) by Corollary 4.2.3.2. Then the set \( J \) of lines of \( S_A \) which do not meet \( \lambda \) is a parallel class of lines of the affine plane \( S_A \) of order \( q \).

Let \( \eta \in J \) with \( x \in \eta \), we want to show \( \partial(u,\eta) = 1 \) for all \( u \in \lambda \). If \( u \in \lambda \) with \( \partial(u,A) = 2 \), then \( \partial(u,\eta) = 1 \) by Lemma 4.3.2(1). On the other hand, if \( z \in \lambda \) with \( \partial(z,A) = 1 \), then \( \partial(z,\eta) \geq 1 \) because \( \eta \cap \lambda = \emptyset \). Let \( \xi \) be the line of \( z \) such that \( \xi \cap A \neq \emptyset \). Then \( \xi \in L(S_A) \) by Lemma 4.3.2(1) and \( \xi \notin J \) because \( \xi \neq \lambda \). Hence \( \xi \) meets all lines of \( J \) and so \( \partial(z,\eta) = 1 \). This shows that \( \eta \) is a line of \( x \) which is parallel to \( \lambda \).

We now prove the uniqueness. Suppose, to the contrary, there are two distinct lines \( \eta \) and \( \eta^* \) of \( x \) which are parallel to \( \lambda \). Let \( y \in \Gamma_2(x) \cap \lambda \). Since \( \partial(y,\eta) = \partial(y,\eta^*) = 1 \), \( \lambda \) meets at least one of \( \eta, \eta^* \) by Corollary 4.2.3.2, a contradiction.

(2) Let \( \Gamma_1(x) \cap \lambda = \{ y_1, y_2, \ldots, y_q \} \). Then \( \partial(y_i, \xi) = 1 \) for all \( i \). If \( z \in \lambda \cap \Gamma_2(x) \), then \( \partial(z, \eta) = 1 \) and \( \partial(z, \langle x, y_i \rangle) = 1 \), \( i = 1, 2, \ldots, q \). Hence \( \partial(z, \xi) = 2 \) by Lemma 4.2.3(2). This proves (2). [ ]

**Corollary 4.3.4.1** Let \( x, y \) be two points with \( \partial(x,y) = 2 \). The lines of \( x \) (resp. \( y \)) can be labelled as \( \lambda_0, \lambda_1, \ldots, \lambda_{r-1} \) (resp. \( \eta_0, \eta_1, \ldots, \eta_{r-1} \)) such that \( \lambda_i \neq \eta_j \) if and only if \( 0 \leq i, j \leq q \) are distinct and \( \lambda_i \parallel \eta_i \) for all \( i \leq q \).
Proof: This is clear from Corollary 4.2.3.2 and Proposition 4.3.4. 

Lemma 4.3.5 Let \( A \) be an assembly, \( u \) be a point with \( \partial(u, A) = 2 \) and \( B = \Gamma_2(u) \cap A \). If \( \lambda, \eta \in \mathcal{L}(B) \) and \( \lambda \cap \eta \cap B = \emptyset \), then \( \lambda \parallel \eta \).

Proof: By Lemma 4.3.2(3), \( \lambda \cap \eta = \emptyset \). By Proposition 4.3.4(2), it suffices to find \( q+1 \) points of \( \lambda \) which are at distance 1 from \( \eta \). There are exactly \( q+1 \) lines of \( u \), say \( \eta_0, \eta_1, ..., \eta_q \) which are at distance 1 from \( A \) by Corollary 4.3.3.1. Since \( \lambda, \eta \in \mathcal{L}(B) \), \( \partial(u, \lambda) = \partial(u, \eta) = 1 \) by Lemma 4.3.2(1). Therefore, \( q \) lines of \( \{\eta_0, \eta_1, ..., \eta_q\} \) meet \( \lambda \), similarly \( q \) lines of them meet \( \eta \). Hence at least one of these, say \( \eta_0 \), meet both. Let \( \eta_0 \cap \lambda = \{y\} \). Then \( \partial(u, y) = 1 \) and so \( y \notin A \). Hence \( \{y\} \cup (\lambda \cap A) \) is a set of \( q+1 \) points of \( \lambda \) which are at distance 1 from \( \eta \), as required. 

Let \( A \) be an assembly, \( u \) be a point with \( \partial(u, A) = 2 \), and \( B = \Gamma_2(u) \cap A \). It has been shown in Corollary 4.3.2.1 that \( B \) is an affine plane of order \( q \). We shall show that each 2-space of \( \Pi_A = (A, \mathcal{L}(A), \in) \) is of this form, determined by external points.

Lemma 4.3.6 Any three noncollinear points of an assembly \( A \) are contained in a unique affine plane \( B = \Gamma_2(u) \cap A \) for some point \( u \) with \( \partial(u, A) = 2 \).
Proof: Let \( x, y, z \in A \) be noncollinear, \( \lambda = \langle x, y \rangle, \eta = \langle y, z \rangle \) and \( w \in \lambda - A \). In addition to \( y, w \) is adjacent to \( q-1 \) points of \( \eta - A \) by Corollary 4.3.1.1. Let \( v \in \Gamma_1(w) \cap (\eta - A) \) and \( \xi = \langle v, w \rangle \). Since \( \lambda \) is the unique line of \( w \) meeting \( A \), \( \partial(\xi, A) = 1 \). There are exactly \( q^2 \) points of \( \xi \) at distance 1 from \( A \) by Lemma 4.3.3 and so there is a point \( u \in \xi \) with \( \partial(u, A) = 2 \) and \( B = \Gamma_2(u) \cap A \) is the required one.

Now, we prove the uniqueness. Suppose \( x, y, z \in B^* = \Gamma_2(u^*) \cap A \) for another point \( u^* \) with \( \partial(u^*, A) = 2 \). Let \( \xi \in \mathcal{L}(B) \), \( \xi^* \in \mathcal{L}(B^*) \) be lines of \( z \) with \( \xi \cap A \cap B = \emptyset \) and \( \xi^* \cap A \cap B^* = \emptyset \) respectively. Then \( \xi \parallel \lambda \) and \( \xi^* \parallel \lambda \) by Lemma 4.3.5. Consequently \( \xi = \xi^* \) by Proposition 4.3.4. Let \( \lambda \cap A = \{w_1(=x), w_2, \ldots, w_q(=y)\} \) and \( \xi_i = \langle z, w_i \rangle \), \( i = 1, 2, \ldots, q \). Then \( B = B^* = ((1 \leq i \leq q) \cup \xi) \cap A \), as required. \[ \]

By Lemma 4.3.2(1) and the construction above, we have

**Corollary 4.3.6.1** Each 2-space of \( \Pi_A = (A, \mathcal{L}(A), \in) \) is of the form \( \Gamma_2(u) \cap A \) for some point \( u \) with \( \partial(u, A) = 2 \), which is also an affine plane of order \( q \).

Before proceeding further, we recall the following theorem, due to Buekenhout [3].

**Theorem C** Let \( \Pi^* \) be a linear incidence structure and every plane (i.e., 2-space) of \( \Pi^* \) be an affine plane of order \( q \geq 4 \). Then \( \Pi^* \) is an affine space.
By Corollary 4.3.6.1 together with the above theorem, Proposition 4.3.7 follows easily and it provides a starting point in determining the structure of subspaces of \( \Pi \).

Proposition 4.3.7  
\( q \) is a prime power and, for any assembly \( A \), \( \Pi_A = (A, \mathcal{L}(A), \in) \) is isomorphic to the affine space \( AG(d,q) \) of dimension \( d \) over \( GF(q) \).

Proof: Since \( |A| = q^d \), \( d \geq 3 \) by assumption, and each line consists of \( q \) points, \( A \) is an affine space of dimension \( d \). Therefore, the order \( q \) is a prime power and \( \Pi_A \) is isomorphic to \( AG(d,q) \). []

4.4 The structure of 2-spaces of \( \Pi \)

We shall make use of the structure of assemblies obtained in the last section to determine the structure of 2-spaces of \( \Pi \). The next two lemmas, equivalent statements also found in [24], provide further properties about parallel lines, which are essential to the subsequent development of the structure of 2-spaces.

Lemma 4.4.1  
Let the line \( \eta \) meet a pair of parallel lines \( \lambda \) and \( \lambda^* \). Then

(1). If \( \eta \parallel \eta^* \), then \( \eta^* \neq \lambda \) if and only if \( \eta^* \neq \lambda^* \).

(2). If \( z \in \eta - (\lambda \cup \lambda^*) \) and \( \xi \) is a line of \( z \), then \( \xi \parallel \lambda \) if and only if \( \xi \parallel \lambda^* \), and \( \xi \neq \lambda \) if and only if \( \xi \neq \lambda^* \).
Proof: Let $\eta \cap \lambda = \{x\}$ and $\eta \cap \lambda^* = \{y\}$. (1) Suppose $\eta^* \cap \lambda = \{u\}$. If $\partial(y,u) = 1$, then $\lambda$, $\lambda^*$, $\eta$ and $\eta^*$ are lines of the affine plane $B$ determined by $x$, $y$ and $u$ by Lemma 4.3.6. Since $\lambda^*$, $\eta^*$ are not parallel in $B$, $\lambda^* \neq \eta^*$. Otherwise, we may assume $\partial(y,u) = 2$. Let the lines of $u$ (resp. $y$) be $\lambda_i$ (resp. $\eta_j$), $0 \leq i \leq r-1$, such that $\lambda_i \neq \eta_j$ if and only if $i, j \leq q$ are distinct, and $\lambda_i \parallel \eta_i$ for all $i \leq q$ as in Corollary 4.3.4.1. Since $\lambda = \lambda_i$, $\eta = \eta_j$ for some distinct $i, j \leq q$ and $\lambda \parallel \lambda^*$, $\eta \parallel \eta^*$, we have $\lambda^* = \eta_i$ and $\eta^* = \lambda_j$. It follows that $\lambda^* \neq \eta^*$, as required.

(2). It certainly can not be the case that $\xi$ is parallel to one of $\lambda$, $\lambda^*$ and meets the other. (If $\xi \parallel \lambda$ and $\xi \neq \lambda^*$ at some point $u$, then $\xi$, $\lambda^*$ are lines of $u$ which are parallel to $\lambda$, a contradiction.) It suffices to show that if $\xi$ and $\lambda$ are either parallel or intersecting, then $\xi$ and $\lambda^*$ are either parallel or intersecting.

**case a.** Assume $\xi \parallel \lambda$. Let $u \in \lambda^* \cap \Gamma_2(z)$ and $\xi^*$ be a line of $u$ with $\xi^* \parallel \eta$. Then $\xi^* \neq \lambda$ and so $\xi^* \neq \xi$ by (1). Hence $\xi \neq \lambda^*$ or $\xi \parallel \lambda^*$ by Corollary 4.3.4.1.

**case b.** Assume $\xi$ meets $\lambda$ at a point $w$. If $\partial(w,y) = 2$, then $\xi \neq \lambda^*$ or $\xi \parallel \lambda^*$ by Corollary 4.3.4.1. If $\partial(w,y) = 1$, then $x$, $z \in A_{y,w}$. Since $\lambda$, $\lambda^*$, $\eta$ and $\xi$ are coplanar lines of the affine space $A_{y,w}$, $\xi$ and $\lambda^*$ meet at some point, as required.
Lemma 4.4.2 Let $\lambda, \eta$ and $\eta^*$ be three distinct lines such that $\partial(\lambda, \eta) = \partial(\lambda, \eta^*) = 1$ with $\lambda \parallel \eta$ and $\lambda \parallel \eta^*$. If $\partial(\eta, \eta^*) = 1$, then $\eta \parallel \eta^*$.

Proof: Let $x \in \eta$, $y \in \eta^*$ with $\partial(x, y) = 1$. Since $\partial(x, \lambda) = \partial(y, \lambda)$ = 1, $|\Gamma_1(x) \cap \lambda| = |\Gamma_1(y) \cap \lambda| = q$.

Suppose $\Gamma_1(x) \cap \lambda = \Gamma_1(y) \cap \lambda$. Let $u, v \in \Gamma_1(x) \cap \lambda$ be distinct. Then $\lambda \in \mathcal{L}(A_{u,v})$ and $x, y \in A_{u,v}$. Since $\eta, \eta^*$ are parallel to $\lambda$, so $\eta, \eta^* \in \mathcal{L}(A_{u,v})$ by Proposition 4.3.4. By the transitivity of parallelism among lines of the affine space $A_{u,v}$, it follows that $\eta \parallel \eta^*$ by Lemma 4.3.5.

Otherwise, we may assume $\Gamma_1(x) \cap \lambda \neq \Gamma_1(y) \cap \lambda$. Pick a point $v$ from $(\Gamma_1(y) \cap \lambda) - \Gamma_1(x)$. Then $\partial(x, v) = 2$. Since $\partial(x, m) = \partial(v, <x, y>) = 1$ and $<x, y> \lambda$ are not parallel, they meet at some point by Corollary 4.3.4.1. Therefore, $\eta \parallel \eta^*$ by Lemma 4.4.1(2). \[
\]

Now, we are able to describe the structure of 2-spaces of $\Pi$ explicitly. Let $\lambda, \eta$ be lines which meet at a point $x_0$. For any $x \in \lambda$, there is a unique line $\eta_x$ of $x$ which is parallel to $\eta$ by Proposition 4.3.4 (we choose $\eta_{x_0} = \eta$). By Lemma 4.4.2, $\eta_x \parallel \eta_y$ for any distinct points $x$ and $y$ on $\lambda$. Let $\mathcal{H} = \mathcal{H}(\lambda, \eta) = \bigcup_{x \in \lambda} \eta_x$. $\mathcal{H}$ consists of exactly $q^{2n}$ points.
Lemma 4.4.3 Any 2-space of $\Pi = (V(\Gamma), L, \in)$ is of the form $\mathcal{H}(\lambda, \eta)$ for some intersecting lines $\lambda$ and $\eta$.

Proof: It suffices to show that $\mathcal{H} = \mathcal{H}(\lambda, \eta)$ is a line-closed set since the connectedness is clear. Let $S = \{\eta_u | u \in \lambda$ and $\eta_u \parallel \eta\}$ and $\xi$ be a line such that $|\xi \cap \mathcal{H}| \geq 2$. If $\xi \parallel \eta$, then $\xi \in S$ and so $\xi \subseteq \mathcal{H}$. If $\xi \parallel \lambda$, then $\xi \neq \eta_u$ for all $\eta_u \in S$ by Lemma 4.4.1(1) and so $|\xi \cap \mathcal{H}| \geq q^n$. Hence $\xi = \xi \cap \mathcal{H} \subseteq \mathcal{H}$. Without loss of generality, we may assume that $\xi$ is parallel to neither $\lambda$ nor $\eta$. Let $x, y \in \mathcal{H} \cap \xi$. If $\lambda^*$ is the line of $y$ which is parallel to $\lambda$, then $\lambda^*$ meets a line in $S$ which contains $x$ by Lemma 4.4.1(1), and consequently $\xi \neq \lambda$ by Lemma 4.4.1(2). It follows that $\xi \neq \eta_u$ for all $\eta_u \in S$ by Lemma 4.4.1(2) and so $\xi \subseteq \mathcal{H}$. This shows that $\mathcal{H} \subseteq V(\Gamma)$ is the smallest line-closed set which contains $\lambda$ and $\eta$. \]

Corollary 4.4.3.1 Let $\mathcal{H}$ be a 2-space of $\Pi$. Then

(1). Every point $x$ in $\mathcal{H}$ has exactly $q+1$ lines in $\mathcal{H}$.

(2). For any $x, y$ in $\mathcal{H}$, $\partial(x, y)$ is at most 2. If $\partial(x, y) = 2$, then $\Gamma_1(x) \cap \Gamma_1(y) \subseteq \mathcal{H}$.

Proof: Let $\mathcal{H} = \mathcal{H}(\lambda, \eta)$ where $\lambda$, $\eta$ are intersecting lines. We may assume $x \notin \eta$, since $\mathcal{H}(\lambda, \eta) = \mathcal{H}(\lambda, \eta^*)$ for any $\eta^* \in S = \{\eta^* | \eta^*$ is a line of $u \in \lambda$ which is parallel to $\eta\}$. Let $x \in \eta^*$ for some $\eta^* \in S$. Then
\( \partial(x, \eta) = 1 \) and so \( \Gamma_1(x) \cap \eta = \{x_1, \ldots, x_q\} \) is a set of \( q \) points. So \( \mathcal{H} \) contains at least \( q+1 \) lines of \( x \), namely \( \eta^* \) and \( \langle x, x_i \rangle \), \( i = 1, 2, \ldots, q \). If \( \xi \neq \eta^* \) is a line of \( x \) in \( \mathcal{H} \), then \( \xi \) meets \( \eta \) as shown in the proof of Lemma 4.4.3, i.e. \( \xi \) is one of \( \langle x, x_i \rangle \) for some \( i \leq q \). This proves (1). The second statement is clear from (1) and Corollary 4.3.4.1. \[ \square \]

\textbf{Lemma 4.4.4} Let \( \lambda^*, \eta^* \) be intersecting lines of \( \mathcal{H} = \mathcal{H}(\lambda, \eta) \). Then \( \mathcal{H} = \mathcal{H}(\lambda^*, \eta^*) \).

\textbf{Proof:} It suffices to show that \( \mathcal{H} \subseteq \mathcal{H}(\lambda^*, \eta^*) \). Let \( x \in \mathcal{H} \). If \( x \in \lambda^* \cup \eta^* \), then \( x \in \mathcal{H}(\lambda^*, \eta^*) \) is trivial. Suppose \( x \notin \lambda^* \cup \eta^* \), then \( \partial(x, \lambda^*) = \partial(x, \eta^*) = 1 \). There are \( q+1 \) lines of \( x \) in \( \mathcal{H} \) by Corollary 4.4.3.1, and at least \( q \) of them meet \( \lambda^* \). Similarly at least \( q \) of them meet \( \eta^* \). Since \( q \geq 4 \), at least one line of \( x \) in \( \mathcal{H} \), say \( \xi \), meets \( \lambda^* \) and \( \eta^* \) both at distinct points. This shows that \( \xi \subseteq \mathcal{H}(\lambda^*, \eta^*) \), and so \( x \in \mathcal{H}(\lambda^*, \eta^*) \) as required. \[ \square \]

Lemma 4.4.4 shows that the expression of a 2-space \( \mathcal{H} = \mathcal{H}(\lambda, \eta) \) of \( \Pi \) is independent of the intersecting lines we choose. The following corollary follows immediately from Corollary 4.4.3.1 and Lemma 4.4.4.

\textbf{Corollary 4.4.4.1} If \( \lambda, \eta \) are two lines of a 2-space of \( \Pi \), then either \( \lambda \parallel \eta \) or \( \lambda \neq \eta \).
We summarize the results we have obtained in the following proposition, which is the condition (D1) for a d-net.

**Proposition 4.4.5** Every 2-space of \( \Pi = (V(\Gamma), \mathcal{L}, \in) \) is a net.

**Proof:** Let \( \mathcal{H} = \mathcal{H}(\lambda, \eta) \) be a 2-space of \( \Pi \), \( \lambda \) and \( \eta \) meet at \( x \), and \( \lambda_1(=\lambda), \lambda_2, \ldots, \lambda_q, \lambda_{q+1}(=\eta) \) be lines of \( x \) in \( \mathcal{H} \). Let \( \mathcal{L}_i \) be the set of all lines in \( \mathcal{H} \) which are parallel to \( \lambda_i, 1 \leq i \leq q+1 \). Then

1. the lines of each class \( \mathcal{L}_i \) partition points of \( \mathcal{H} \) by Lemma 4.4.1 and by the construction of \( \mathcal{H} \).

2. any two lines from distinct classes are not parallel, hence they meet at some point by Corollary 4.4.4.1. It follows that \( \mathcal{H} \) is a net of \( q^{2n} \) points with \( q+1 ( \geq 5 ) \) parallel classes. [ ]

**4.5** \( \Pi = (V(\Gamma), \mathcal{L}, \in) \) is a d-net.

In the previous section, we have shown that any 2-space of \( \Pi \) is a net, i.e., the condition (D1) holds in \( \Pi \). Once we verify that condition (D2) holds in \( \Pi \) in Proposition 4.5.2, we shall complete the proof of condition (D3) by using the induction argument. Theorem 4.1.1 then follows immediately.
We require some further information concerning the subspaces of $\Pi$. Let $X$ be an $i$-space of $\Pi$, and $<x_0, x_1, ..., x_i>$ be a generating set of $X$. Since the structure of 2-spaces of $X$ coincides with that of $\Pi$ and the generating set is connected, without loss of generality, we may assume that $x_0, x_1$ are adjacent and $x_2, ..., x_i$ are contained in $A_{x_0,x_1}$, the assembly determined by $x_0, x_1$. Similar argument shows that each point in $X$ is in a generating set of $X$ with $i+1$ points. Let $Y$ be a proper subspace of $X$ and $y \in Y$. By the connectedness of $X$, the number of lines of $y$ in $Y$ is strictly less than the number of lines of $y$ in $X$. Hence the restriction of $\Pi_Y$ to $A$ is a proper subspace of the restriction of $\Pi_X$ to $A$, where $A = A_{x_0,x_1}$ is isomorphic to $AG(d,q)$.

Lemma 4.5.1  (1). The dimension of $\Pi$ is $d$.

(2). Every point $x$ of an $i$-space $X$ of $\Pi$ has exactly $(q^i-1)/(q-1)$ lines in $X$.

Proof: Suppose the dimension of $\Pi$ is $i_0$ and $X_j$ is a $j$-space of $\Pi$. Then, as a subspace of $AG(d,q)$, $j \leq \dim(X_j \cap A) \leq d-(i_0-j)$ for $j = 1, 2, ..., i_0$, where $A$ is as mentioned above. In particular, we $\dim(X_2 \cap A) = 2 = d-(i_0-2)$ for the case $j = 2$. It follows that $i_0 = d$ and $\dim(X_j \cap A) = j$ for $3 \leq j \leq d$. This proves (1). The second statement follows immediately from the fact that $X \cap A$ is an $i$-space of $AG(d,q)$.  [1]
Corollary 4.5.1.1 If $Y$ is a proper subspace of an $i$-subspace $X$ of $\Pi$, then the dimension of $Y$ is at most $i-1$.

Proof: By Lemma 4.5.1, the dimension of $Y$ is at most $i$. Suppose, to the contrary, that the dimension of $Y$ is $i$. Each point of $y$ in $Y$ has exactly $(q^{i-1})/(q-1)$ lines in $Y$, those are all lines of $y$ in $X$. By the connectedness of $X$, $X = Y$, a contradiction. 

We shall show in the following proposition that condition (D2) for a $d$-net holds in $\Pi$.

Proposition 4.5.2 Let $X$ be a 3-space of $\Pi$, $\mathcal{H}_1$ and $\mathcal{H}_2$ be distinct 2-spaces of $\Pi$ contained in $X$. Then $\mathcal{H}_1 \cap \mathcal{H}_2$ is either empty or a line.

Proof: If $\lambda$ is a line such that $\lambda \subseteq \mathcal{H}_1 \cap \mathcal{H}_2$, then $\lambda = \mathcal{H}_1 \cap \mathcal{H}_2$. (Otherwise, there exists $x \notin \lambda$ with $\lambda \cup \{x\} \subseteq \mathcal{H}_1 \cap \mathcal{H}_2$. Since $\partial(x,\lambda) = 1$, $\mathcal{H}_1 = \mathcal{H}_2$ is the 2-space determined by $\lambda \cup \{x\}$.)

All points, lines and 2-spaces (called planes) in the following arguments are restricted to $X$. Let $\lambda$ be a line and $x \in \lambda$, the planes containing $\lambda$ partition those lines of $x$ distinct from $\lambda$, and each contains $q$ lines of them by Corollary 4.4.3.1(1). Hence there are exactly $q+1$ planes containing $\lambda$ because $x$ has exactly $q^2 + q + 1$ lines in $X$ by Lemma 4.5.1.

Let $y \in X$. Counting the set $\{(\lambda, \eta, \mathcal{H})| \lambda, \eta \text{ are lines with } \lambda \cap \eta = \{y\} \text{ and } \mathcal{H} = \mathcal{H}(\lambda,\eta)\}$ in two ways shows that
\[(q^2 + q + 1) = \binom{q+1}{2}\text{(the number of planes containing } y)\.

Therefore, } y \text{ is contained in } q^2 + q + 1 \text{ planes.}

Let } \mathcal{H} \text{ be a fixed plane with } y \in \mathcal{H}. \text{ Then there are } q+1 \text{ lines of } y \text{ in } \mathcal{H}, \text{ each of which is contained in } q \text{ planes distinct from } \mathcal{H}. \text{ Therefore, there are } q^2 + q \text{ planes } \mathcal{H}^* \text{ such that } y \in \mathcal{H}^* \text{ and } \mathcal{H}^* \cap \mathcal{H} \text{ is a line. Since } y \text{ is contained in } q^2 + q + 1 \text{ planes, one for } \mathcal{H} \text{ itself, } \mathcal{H} \cap \mathcal{H}^* \text{ is a line for every plane } \mathcal{H}^* (\neq \mathcal{H}) \text{ which contains } y. \text{ Since } \mathcal{H} \text{ and } y \in \mathcal{H} \text{ are arbitrary, the proof is completed.} \]

Proposition 4.5.2 shows that any 3-space of } \Pi \text{ is a } (n,q,3)\text{-attenuated space which provides a starting point of the induction argument.}

**Corollary 4.5.2.1** Let } X \text{ be a 3-space of } \Pi \text{ and } x, y \in X. \text{ Then }

(1). } X \text{ is an } (n,q,3)\text{-attenuated space, and }

(2). \text{ The distance function on } X \text{ coincides with that of } \Pi. \text{ Furthermore, if } \vartheta(x,y) = t, \text{ then } t \leq 3. \text{ Therefore, all lines of } y \text{ at distance } t-1 \text{ from } x \text{ are in } X, \text{ and consequently } \Gamma_{t-1}(x) \cap \Gamma_1(y) \subseteq X.

**Proof:** (1) follows from Lemma 4.5.2 and Theorem A in section 4.1. To prove (2). Let } x \in X \text{ be fixed, } \mathcal{H} = \mathcal{H}(\eta_1, \eta_2) \text{ be a 2-space contained in } X \text{ and } \eta_1, \eta_2 \text{ meet at } x, \lambda \text{ be a line of } x \text{ in } X \text{ and meet } \mathcal{H} \text{ at } x \text{ only. Since } X \text{ is a 3-attenuated space, } X = \bigcup_{z \in \mathcal{H}}^\lambda x \text{ by}
Proposition 2.2.9, where $\lambda_z$ is the unique line of $z$ in $X$ parallel to $\lambda$.

By the structure of $X$ and Proposition 2.2.8, every point $y$ in $X$ can be joined with $x$ by a path of length $\partial(x,y)$ in $X$. It is clear that $\partial(x,y) \leq 3$. This proves the first part of (2).

Let $y \in X$ with $\partial(x,y) = 3$. Those $q^2+q+1$ lines of $y$ in $X$ are all at distance 2 from $x$, and there are no more such lines by Lemma 4.2.1. Hence $\Gamma_2(x) \cap \Gamma_1(y) \subseteq X$. If $\partial(x,y) = 2$, then $\Gamma_1(x) \cap \Gamma_1(y) \subseteq X$ follows from the structure of $X$ and Corollary 4.4.3.1. 

The above arguments can be generalized to show similar results for higher dimensional subspaces of $\Pi$. And Theorem 4.1.1 follows from the special case $i = d$.

**Proposition 4.5.3** Let $X$ be an $i$-space of $\Pi$, and $x, y \in X$.

Then

1. $X$ is an $(n,q,i)$-attenuated space.

2. The distance function on $X$ coincides with that of $\Pi$. Furthermore, if $\partial(x,y) = t$, then $t \leq i$. Therefore, all lines of $y$ at distance $t-1$ from $x$ are in $X$ and consequently $\Gamma_{t-1}(x) \cap \Gamma_1(y) \subseteq X$.

**Proof:** We prove by induction on the dimension of $X$. The statements are true for the case $i = 3$ by Corollary 4.5.2.1. Now, we assume that both statements hold for any $j$-spaces of $\Pi$ where $j \leq i$ and $X$ is an $i+1$ space of $\Pi$. 
In order to prove (1), by Theorem A, it suffices to verify condition (D3) for X. Let \( Y, Z \) be proper subspaces of \( X \), and \( y, z \in Y \cap Z \) with \( d(y, z) = t \). By Corollary 4.5.1.1, both \( Y \) and \( Z \) are attenuated spaces of dimensions at most \( i \), it follows from the induction hypothesis that \( \Gamma_{t-1}(y) \cap \Gamma_{1}(z) \subseteq Y \cap Z \). Repeating this process, it shows that \( y, z \) can be joined by a path of length \( t \) in \( Y \cap Z \). Hence, \( Y \cap Z \) is connected and so is a subspace of \( X \). This shows that \( X \) is an \((n, q, i+1)\)-attenuated space.

To prove (2). Let \( x \) be a fixed point in \( X \), \( Z \subseteq X \) be an \( i \)-space containing \( x \), and \( \lambda \) be a line of \( x \) in \( X \) which meet \( Z \) at \( x \) only. Since \( X \) is an \( i+1 \) attenuated space, \( X = \bigcup_{z \in Z} \lambda_z \) by Proposition 2.2.9, where \( \lambda_z \) is the unique line of \( z \) in \( X \) parallel to \( \lambda \). By the induction hypothesis and Proposition 2.2.8, the distance function on \( X \) coincides with that of \( \Pi \). Let \( y \in X \) with \( d(x, y) = t \) and \( x_0(=x), x_1, \ldots, x_{t-1}, x_t(=y) \) be a path in \( X \), then \( t \leq i+1 \) and \( E = \langle x_0, x_1, \ldots, x_t \rangle \subseteq X \) is a \( t \)-space of \( \Pi \). All those \((q^t-1)/(q-1)\) lines of \( y \) in \( E \) are at distance \( t-1 \) from \( x \), and there is no other such line by Lemma 4.2.1. It follows that \( \Gamma_{t-1}(x) \cap \Gamma_{1}(y) \subseteq E \subseteq X \), as required. 

This completes the proof of Theorem 4.1.1.
4.6 \( H_q(d,n) \) has no antipodal covering

A distance-regular graph \( \Gamma \) of diameter \( s \) is called antipodal if being opposite is an equivalent relation, i.e., for any \( x \in V(\Gamma) \), \( \partial(y,z) = s \) if \( y, z \in \Gamma_s(x) \) are distinct. We construct the derived graph \( \bar{\Gamma} \) of \( \Gamma \) by taking the vertices of \( \bar{\Gamma} \) to be the distinct blocks \( \{x\} \cup \Gamma_s(x) \) where \( x \in V(\Gamma) \), and two blocks being adjacent in \( \bar{\Gamma} \) whenever they contain adjacent vertices in \( \Gamma \). \( \Gamma \) is called an antipodal covering of \( \bar{\Gamma} \). Let \( \{b_0, b_1, \ldots, b_{s-1}; c_1, \ldots, c_s\} \) be the intersection array of \( \Gamma \), the following result is known, due to Gardiner ([16] Prop. 4.2)

**Theorem**

1. \( \bar{\Gamma} \) is a distance-regular graph with diameter \( \lceil s/2 \rceil \),
2. Let \( d = \lfloor s/2 \rfloor \) and \( \{\bar{b}_0, \bar{b}_1, \ldots, \bar{b}_{d-1}; \bar{c}_1, \ldots, \bar{c}_d\} \) be the intersection array of \( \Gamma \). Then \( \bar{b}_i = b_i \), \( \bar{c}_i = c_i \) for all \( i \leq d-1 \) and \( \bar{c}_d = c_d \), \( \bar{a}_d = a_d + b_d \) if \( d = (s-1)/2 \); \( \bar{a}_d = a_d \), \( \bar{c}_d = c_d + b_d \) if \( d = s/2 \). 

As a corollary of Theorem 4.1.1, we prove that:

**Theorem 4.6.1** \( H_q(d,n) \) has no antipodal covering if \( n \geq 2d \geq 6 \) and \( q \geq 4 \).
Proof: Suppose, to the contrary, that there is an antipodal distance-regular graph $\Gamma$ with intersection array $\{b_0, b_1, \ldots, b_{s-1}, c_1, \ldots, c_s\}$ such that its derived graph $\overline{\Gamma}$ is isomorphic to $H_q(d,n)$. Then $d = \lfloor s/2 \rfloor$, $\Gamma$ satisfies the weak 4-vertex condition, and the parameters $a_1, c_2$ and $b_1, \ldots, b_{d-1}$ of $\Gamma$ coincide with those of $H_q(d,n)$. As shown in section 4.2, an incidence structure $\Pi = (V(\Gamma), \mathcal{L}, \in)$ can be derived from $\Gamma$, and its adjacency graph is isomorphic to $\Gamma$. The same arguments in sections 4.3, 4.4 and 4.5 show that any d-space $X$ of $\Pi$ is a d-net. For any $x \in X$, all lines of $x$ in $\Pi$ are contained in $X$. This contradicts the connectedness of $\Pi = (V(\Gamma), \mathcal{L}, \in)$ since $X$ is a proper subset of $V(\Gamma)$.  \[\square\]
Chapter 5

A geometric interpretation of the association scheme of alternating bilinear forms

5.1 Introduction

Another infinite family of distance-regular graphs is defined on the set of alternating bilinear forms. Let $V$ be a vector space of dimension $n$ over $\text{GF}(q)$, $\text{Alt}(n,q)$ be the set of all alternating bilinear forms of $V$, and $\mathcal{X}_i = \{(f,g) | f, g \in \text{Alt}(n,q) \text{ and rank}(f-g) \text{ is } 2i\}$ for $0 \leq i \leq \lfloor n/2 \rfloor$ ($=d$). Then $(\text{Alt}(n,q), (\mathcal{X}_i)_{0 \leq i \leq d})$ is a symmetric association scheme, and $(\text{Alt}(n,q), \mathcal{X}_i)$ turns out to be a distance-regular graph of diameter $d$ with parameters $b_i = q^{4i-1}(q-1)(q^{n-2i}q^{n-2i})/(q^2-1)$, $0 \leq i \leq d-1$, and $c_i = q^{2i-2}(q^{2i-1})/(q^2-1)$, $1 \leq i \leq d$. Furthermore, this graph, denoted by $\text{Alt}(n,q)$, is distance-transitive, see [8].
Just as we may associate a geometric structure, $d$-net, to $H_q(d,n)$, we may also associate a suitable geometric structure to $\text{Alt}(n,q)$. In this chapter, first we study the local structure of $\text{Alt}(n,q)$ in Section 5.2. Then we propose a diagram corresponding to $\text{Alt}(n,q)$ in Section 5.3. Hopefully, this is a starting point to characterize $\text{Alt}(n,q)$ in terms of its geometric structure. It is worth noting here that there is another infinite family of distance-regular graphs (defined on the set of all quadratic forms) with the same intersection array as that of $\text{Alt}(n,q)$, see [1, 8].

5.2 The local structure of $\text{Alt}(n,q)$

As $\text{Alt}(n,q)$ is distance-transitive, it suffices to consider the local structure at $\mathcal{O}$, the zero alternating bilinear form. The structures of maximal cliques of $\mathcal{O}$ are as follows.

**Lemma 5.2.1** If $\mathcal{C}$ is a maximal clique containing $\mathcal{O}$, then either (i) Radical of $A$ is contained in a fixed hyperplane of $V$ for all $A \in \mathcal{C}$, $A \neq \mathcal{O}$, or (ii) $\bigcap_{A \in \mathcal{C}} \text{Rad}(A)$ is a $(n-3)$-dimensional subspace of $V$. Each maximal clique of $\mathcal{O}$ of type (i) (resp. type (ii)) consists of $q^{n-1}$ (resp. $q^3$) vertices.

**Proof:** see [17, p.43].
Without loss of generality, we may assume that \( V \) is an inner product space with orthonormal basis \( \{v_1, \ldots, v_n\} \). Consider the nest of subspaces

\[
(O=) \quad V_0 < V_1 < \ldots < V_{n-1} < V_n \quad (= V)
\]

where \( V_i = \langle v_1, \ldots, v_i \rangle, \; i = 1, 2, \ldots, n. \) and \( O \) is the zero space. Since \( \dim(V_i) = i \), \( V_i \) has \( (q-1)/(q-1) \) one dimensional subspaces, they are \( \langle \sum_{1 \leq j \leq r} \alpha_{rj} v_j \rangle \), where \( \alpha_{rj} \in \text{GF}(q) \) with \( \alpha_{rr} = 1 \), and \( 1 \leq r \leq i \).

Each of these one dimensional subspaces \( \langle \sum_{1 \leq j \leq r} \alpha_{rj} v_j \rangle \) corresponds to its perpendicular subspace (a hyperplane), i.e., \( \langle \sum_{1 \leq j \leq r} \alpha_{rj} v_j \rangle^\perp = \langle v_j - \alpha_{rj} v_r, v_s | 1 \leq j \leq r-1, \ r+1 \leq s \leq n \rangle \). Moreover, each of these hyperplanes uniquely determines a maximal clique of \( O \) of type (i). With respect to the base \( \{v_1, \ldots, v_n\} \), their expressions in terms of matrices are described in Lemma 5.2.2. For each matrix in \( \lambda_\alpha \) or \( \lambda_\infty \), the first two rows are given explicitly (so the first two columns follow immediately), and all other entries are zero. Similar convention holds for matrices in \( \lambda_{\alpha\beta} \) and the rest of this chapter.

**Lemma 5.2.2** The lines of \( O \) corresponding to one-dimensional subspaces of \( V_1, V_2 \) and \( V_3 \) are \( \{\lambda_\infty\} \), \( \{\lambda_\infty, \lambda_\alpha \mid \alpha \in \text{GF}(q)\} \) and \( \{\lambda_\infty, \lambda_\alpha, \lambda_{\alpha\beta} \mid \alpha, \beta \in \text{GF}(q)\} \) respectively, where \( \lambda_\infty = \)

\[
\begin{bmatrix}
0 & x_2 & x_3 & \cdots & x_n \\
- & 0 & 0 & \cdots & 0
\end{bmatrix}
\quad \mid x_i \in \text{GF}(q)
\]
\[\lambda_\alpha = \ \left\{ \begin{array}{ccc|c|c|c} 0 & x_2 & \alpha x_1 & \cdots & \alpha x_n \\ \hline 0 & x_3 & \cdots & x_n & x_1 \in \text{GF}(q) \end{array} \right\}\]

and \[\lambda_{\alpha,\beta} = \ \left\{ \begin{array}{ccc|c|c|c} 0 & \beta x_2 & \alpha x_3 & \cdots & \alpha x_n \\ \hline 0 & x_1 & \beta x_3 & \cdots & \beta x_n \end{array} \right\} \quad x_i \in \text{GF}(q)\]

Remark: The proof of Lemma 5.2.2 is straightforward, the details are omitted.

Let \( P = \text{Alt}(n,q) \), \( L \) be the set of all maximal cliques of type (i). Then \( \Pi = (P, L, \in) \) is an incidence structure (neither linear nor semilinear). From now on, elements in \( P \) (resp. in \( L \)) are called points (resp. lines).

Lemma 5.2.3 The incidence structure \( \Pi = (P, L, \in) \) satisfies the following properties:

1. Each line consists of \( q^{n-1} \) points, and there are \( (q^n-1)/(q-1) \) lines containing a fixed point.

2. \(|\lambda_0 \cap \lambda_1| \in \{0, q\} \) for any distinct lines \( \lambda_0 \) and \( \lambda_1 \in L \). If \( \lambda_0 \cap \lambda_1 \) consists of \( q \) points, then there are another \( q-1 \) lines, say \( \lambda_2 \), \ldots, \( \lambda_q \), such that \( \bigcup_{0 \leq i \leq q} \lambda_i = \lambda_0 \cap \lambda_1 \), and \( \bigcup_{0 \leq i \leq q} \lambda_i = \Gamma_1(x) \cap \Gamma_1(y) \) for all distinct \( x \)
and \( y \) in \( \lambda_0 \cap \lambda_1 \) (where \( \Gamma_1(x) = \{ y \mid y \in \text{Alt}(n,q) \text{ with rank}(x-y) = 2 \} \)).

(3) If \( \lambda \in \mathcal{L} \) and \( x \in \mathcal{P} \) with \( \partial(x, \lambda) = 1 \), then \( |\Gamma_1(x) \cap \lambda| = q^2 \).

Proof: (1) follows from Lemma 5.2.1, and the distance-transitivity of \( \text{Alt}(n,q) \). (2) follows from Lemma 5.2.2 and \( a_1 = q^n + q^{n-1} - q^2 - 2 \). For (3), as \( \text{Alt}(n,q) \) is distance-transitive, we may assume that \( \lambda = \lambda_\infty \), \( \partial(x, \mathcal{O}) = 1 \) and \( x \in (\bigcup_{0 \leq i \leq n} \lambda_i) \lambda_\infty \). Then (3) follows from (2) and Lemma 5.2.2. \[ \]

**Lemma 5.2.4** With respect to the induced incidence structure, each line is isomorphic to an affine space \( \text{AG}(n-1,q) \).

Proof: As distance-transitive, without loss generality, we may assume the line is \( \lambda_\infty = \)

\[
\left\{ \begin{array}{c|c}
0 & x_1 \ x_2 \ \cdots \ \cdots \ x_n \\
0 & 0 & 0 & \cdots & 0 \\
\end{array} \right\} \quad x_i \in \text{GF}(q)
\]

Each subline is of the form

\[
\left\{ \begin{array}{c|c}
0 & \alpha x_1 \ \cdots \ \cdots \ \cdots \\
0 & 0 & \cdots & 0 \\
\end{array} \right\} \quad \alpha \in \text{GF}(q)
\]

for fixed \( x_2, \ldots, x_n \in \text{GF}(q) \), consists of \( q \) points, and each 2-subspace of the induced incidence structure is isomorphic to

\[
\left\{ \begin{array}{c|c|c|c|c}
0 & 0 & \cdots & 0 & \alpha x_{n-1} \ \\
0 & 0 & \cdots & 0 & \beta x_n \\
\end{array} \right\} \quad \alpha, \ \beta \in \text{GF}(q)
\]

which consists of \( q^2 \) points, and each pair of points is contained in a unique subline, i.e., it is a \((q^2, q, 1)\) - design, an affine plane of order \( q \).
By Theorem C in section 4.3, the whole induced incidence structure is isomorphic to $AG(n-1,q)$. \[ \]

Now we consider the structure over the set $B = \{ (0, x_{12}, x_{13}, \ldots, x_{1n}) \mid x_{ij} \in GF(q) \}$

For any $x$ and $y \in B$, we define $x \sim y$ if $x-y \in A$, where $A = \{ (0, \alpha, 0, \ldots, 0) \mid \alpha \in GF(q) \}$

It is easy to check that "$\sim$" is an equivalence relation on $B$. In particular, it is also an equivalence relation on $\lambda_\infty$. So $\lambda_\infty$ is the disjoint union of $x_i + A$, $i = 0, 1, \ldots, q^{n-2} - 1$, $x_0 = \emptyset$. Then $B = \bigcup_{0 \leq i \leq q^{n-2} - 1} (x_i + \lambda_0)$ consists of $q^{2n-3}$ points. Furthermore, $\lambda_\alpha \cap (x_i + \lambda_0)$ consists of $q$ points for all nonzero $\alpha \in GF(q)$, and $\lambda_0 \parallel (x_i + \lambda_0)$ in the sense that $\text{rank}(x-y)$ is 2 or 4 for all $x \in \lambda_0$ and $y \in x_i + \lambda_0$. We may conclude that

**Lemma 5.2.5** \[ B/\sim \] is a net consisting of $q^{2n-4}$ points and each point has $q+1$ lines.
5.3 \textbf{A diagram for Alt(n,q)}

For any subspace $U$ of $V$, let $\text{Alt}(U)$ be the set of all alternating bilinear forms on $U$ and

$$A_i = \{(f,U) | U \subseteq V \text{ is an } (n-i)\text{-subspace and } f \in \text{Alt}(U)\}$$

$0 \leq i \leq n-1$. For $(f,U) \in A_i$ and $(g,W) \in A_j$, we define $(f,U) \leq (g,W)$ if $W \subseteq U$ and $f = g$ on $W$ (since $n-j = \dim(W) \leq \dim(U) = n-i$, this forces that $i \leq j$). Any two elements $(f,U)$ and $(g,W) \in \bigcup_{0 \leq i \leq n-1} A_i$ are called adjacent if they are comparable, i.e., either $(f,U) \leq (g,W)$, or $(g,W) \leq (f,U)$. With respect to this incidence relation, $\mathcal{G} = (A_0, A_1, ..., A_{n-1})$ forms a \textit{geometry of rank n}, see [5]. Let $(f,U) \in A_i$, and $j \leq i$, we define

$$\sigma_j(f,U) = \{(g,W) | (g,W) \in A_j \text{ and } (g,W) \leq (f,W)\}$$

and call it the $j$-th shadow of $(f,U)$. Instead of $\sigma_0$, we denote it by $\sigma$ for the case $j = 0$.

In particular, for $(f=0,U) \in A_1$, $\sigma(f,U) =
\begin{bmatrix}
0 & x_1 & x_2 & \cdots & x_{n-1} \\
0 & 0 & 0 & \cdots & 0
\end{bmatrix}
\begin{bmatrix}
x_i
\end{bmatrix} \in \text{GF}(q)
$$
which is a line of $O$ by Lemma 5.2. For $(f=0,W) \in A_2$, $\sigma(f,W) =
\begin{bmatrix}
0 & x_{i_2} & x_{i_3} & \cdots & x_{i_{n-1}} \\
0 & x_{a_2} & x_{a_3} & \cdots & x_{a_{n-1}}
\end{bmatrix}
\begin{bmatrix}
x_{i_j}
\end{bmatrix} \in \text{GF}(q)$
As shown in Lemma 5.2.5, $\sigma(f=\mathcal{O},W)/\sim$ is a net of $q^{2n-4}$ points and each point has $q+1$ lines.

A subset $F$ of $A$ is called a **flag** if all elements in $F$ are pairwise adjacent. We note that $|F \cap A_i| \in \{0, 1\}$ for any flag $F$ and $i$. $\text{Type}(F)$ is defined to be the set $\{i| |F \cap A_i| = 1\}$. The **residue** $\mathcal{R}(F)$ of $F$ is defined to be the set $\{(f,U)|(f,U) \in A \text{ is adjacent to all elements in } F\}$. $|\text{Type}(F)|$ is called the **rank** of $F$ and the number of elements in $\{0, 1, ..., n-1\}$ - $\text{Type}(F)$ is called the **corank** of $F$.

Suppose $\{(f,V), (f_1, U_1), ..., (f_{n-1}, U_{n-1})\}$ is a maximal flag (call it a chamber), where $(f_i, U_i) \in A_i$. Then $O < U_{n-1} < ... < U_1 < V$ is a nest of subspaces of $V$, and $f_i = f$ on $U_i$ for all $i$. By the Gram-Schmidt process, there is an orthogonal basis $\{u_1, ..., u_n\}$ of $V$ such that $U_i = \langle u_{i+1}, ..., u_n \rangle$. We consider the corresponding nest of perpendicular subspaces

$$0 < U_1 \perp U_2 \perp ... \perp U_{n-1} \perp V$$

here $U_i \perp = \langle u_1, ..., u_i \rangle$ is of dimension $i$. For the case $f = \mathcal{O}$, let

$$\mathcal{L}_1 = \{ \text{ all lines correspond to } \langle x \rangle \perp | \langle x \rangle \subseteq U_1 \perp \}$$

Then all points of lines $\langle x \rangle \perp$ in $\mathcal{L}_1$ have their radicals contained in $\langle x \rangle \perp$. Also all points of lines in $\mathcal{L}_1$ are contained in $\sigma(f=\mathcal{O},U_i)$, $i = 1, 2, 3$, by Lemma 5.2.2. The following lemma follows immediately from the structure of $\mathcal{G}$. 
Lemma 5.3.1  The geometry $\mathcal{G} = (A_0, A_1, ..., A_{n-1})$ satisfies the following properties:

1. **residual connectedness**, i.e., the residue of every flag of corank 2 (resp. corank 1) is connected, (resp. nonempty).

2. **transversal property**, i.e., every flag lies in a maximal flag, (indeed at least 2.)

A diagram will be a set $\Delta = \{0, 1, ..., n-1\}$ provided with a mapping which assigns to every pair of distinct elements $i, j \in \Delta$ a class $\Delta_{ij}$ of rank 2 geometries. Let $\Delta$ be a diagram. Then a geometry $\mathcal{G}$ of rank $n$ is said to have $\Delta$ for diagram if for every distinct $i, j \in \Delta$ and every flag $F$ of $\mathcal{G}$ of type $\Delta - \{i, j\}$, the residue $R(F) \in \Delta_{ij}$. The picture of a diagram on $\{i, j\}, i < j$, will consists of two nodes, one at the left standing for $i$ and the other at the right standing for $j$. These nodes are joined with a weighted stroke symbolizing the diagram, e.g.,

- $\circ \circ$ (empty edge) for generalized digons (i.e., complete bipartite graphs) and
- $\circ --- \circ$ for generalized triangles (i.e., projective planes), see [4, 5].

In order to find a suitable diagram for the geometry $\mathcal{G} = (A_0, A_1, ..., A_{n-1})$, we consider all residues $R(F)$ of flags of corank 2.

(1). If $F$ is a flag of corank 2 and $0 \in \text{Type}(F)$, then $R(F)$ is uniquely determined by the subspaces of $V$. Hence $R(F)$ is either a
projective plane or a complete bipartite graph, and the corresponding pictures are $\overline{\bullet_{i}^0 \bullet_{i+1}^0}$, $1 \leq i \leq n-2$, or $\overline{\bullet_i^0}$ (empty edge), $i \geq 1$ and $j-i \geq 2$.

(2). If $F$ is a flag of corank 2 with $1 \in \text{Type}(F)$ and $0 \notin \text{Type}(F)$, then $R(F)$ is a complete bipartite graph. We have $\overline{\bullet_0^0}$ (empty edge), for all $i \geq 2$.

(3). If $F$ is a flag with $\text{Type}(F) = \{2, 3, \ldots, n-1\}$. Without loss of generality, we may assume $F \cap A_2 = \{f = \emptyset, U\}$, where $U \subseteq V$ is a $(n-2)$ dimensional subspace. Hence $R(F) = \left\{ \begin{array}{c|ccc|c} 0 & x_{12} & x_{13} & \cdots & x_{1n} \\ \hline 0 & x_{22} & \cdots & x_{2n} \end{array} \right| x_{ij} \in \mathbb{GF}(q) \right\}$

As shown in Lemma 5.3.1, $R(F)/\simeq$ is a net of $q^{2n-4}$ points and each point has $q+1$ lines. Combining (1), (2), and (3), we conclude that

**Theorem 5.3.1** The geometry $\mathcal{G} = (A_0, A_1, \ldots, A_{n-1})$ belongs to the diagram

```
0 1 2 \ldots n-2 n-1
```

where $\overline{\bullet_0^0 \bullet_1^0}$ /\$\simeq$ is a net.
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