INFORMATION TO USERS

This reproduction was made from a copy of a document sent to us for microfilming. While the most advanced technology has been used to photograph and reproduce this document, the quality of the reproduction is heavily dependent upon the quality of the material submitted.

The following explanation of techniques is provided to help clarify markings or notations which may appear on this reproduction.

1. The sign or "target" for pages apparently lacking from the document photographed is "Missing Page(s)". If it was possible to obtain the missing page(s) or section, they are spliced into the film along with adjacent pages. This may have necessitated cutting through an image and duplicating adjacent pages to assure complete continuity.

2. When an image on the film is obliterated with a round black mark, it is an indication of either blurred copy because of movement during exposure, duplicate copy, or copyrighted materials that should not have been filmed. For blurred pages, a good image of the page can be found in the adjacent frame. If copyrighted materials were deleted, a target note will appear listing the pages in the adjacent frame.

3. When a map, drawing or chart, etc., is part of the material being photographed, a definite method of "sectioning" the material has been followed. It is customary to begin filming at the upper left hand corner of a large sheet and to continue from left to right in equal sections with small overlaps. If necessary, sectioning is continued again—beginning below the first row and continuing on until complete.

4. For illustrations that cannot be satisfactorily reproduced by xerographic means, photographic prints can be purchased at additional cost and inserted into your xerographic copy. These prints are available upon request from the Dissertations Customer Services Department.

5. Some pages in any document may have indistinct print. In all cases the best available copy has been filmed.

University Microfilms International
300 N. Zeeb Road
Ann Arbor, MI 48106
Choi, Sul-young

MAXIMAL (0,1,2,...T)-CLIQUE OF SOME ASSOCIATION SCHEMES

The Ohio State University

University Microfilms International 300 N. Zeeb Road, Ann Arbor, MI 48106
MAXIMAL \{0,1,2,...,t\}-CLIQUES OF
SOME ASSOCIATION SCHEMES

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By

Sul-young Choi, B.S., M.Ed., M.S.

*****

The Ohio State University

1985

Reading Committee:

Eiichi Bannai

Thomas Dowling

D. K. Ray-Chaudhuri

Approved By

Eiichi Bannai

Advisor
Department of Mathematics
To my husband
VITA

September 30, 1953
Born - Won-Ju, Korea.

1975
B.S., The Seoul National University.
Seoul, Korea.

1977
M.Ed., The Seoul National University.
Seoul, Korea.

1980
M.S., University of Santa Clara.
Santa Clara, California.

1980-1985
Graduate Teaching Associate, Department
of Mathematics, Ohio State University.
Columbus, Ohio

PUBLICATIONS

1. Maximal \{0,1,2,...,t\}-cliques of J(n,k) (Submitted to European Journal
   of Combinatorics)

FIELDS OF STUDY

Major Field: Mathematics
Speciality: Combinatorics
# Table of Contents

Dedication ii

VITA iii

1. introduction 1

2. Maximal t-cliques of \( J(n,k) \) 9
   2.1 Introduction 9
   2.2 Maximal 2-cliques of \( J(n,k) \) 11
   2.3 Maximal t-cliques of \( J(n,k) \) with size at least \( cn^{t-1} \) 51

3. Maximal t-cliques of \( H(n,q) \) 65
   3.1 Introduction 65
   3.2 Maximal 2-cliques of \( H(n,q) \) 66
   3.3 Maximal t-cliques of \( H(n,q) \) with size at least \( cq^{t-1} \) 68

4. Maximal t-cliques of \( J_q(n,k) \) 106
   4.1 Introduction 106
   4.2 Maximal t-cliques of \( J_q(n,k) \) with size at least \( cq^{(n-k)(t-1)} \) 108

Reference 129
Chapter 1

Introduction

In 1961, Erdős, Ko and Rado [8] stated a theorem which asserts that the largest possible families $\mathcal{F}$ of $k$-subsets of an $n$-set with the property

$$|x \cap y| \geq k-t$$

for all $x$ and $y$ in $\mathcal{F}$ ($0 < t \leq k \leq n$)

are the families of all the $k$-subsets containing some fixed $(k-t)$-subset of the $n$-set whenever $n$ is sufficiently large with respect to $k$ and $t$. That is, if $n \geq n_0$ for some function $n_0(n_0(k,t))$, then the size of a $t$-clique $\mathcal{F}$ is bounded above by $\binom{n}{t} - \binom{k-t}{k-t}$ and $\mathcal{F}$ attains the upper bound only if $\mathcal{F}$ contains all the $k$-subsets which contain some fixed $(k-t)$-subset. The original proof by Erdős, Ko and Rado [8] established that $n_0(k,t) \leq (k-t)^t t^k$ and also contained the result $n_0(k,k-1) = 2k$. In 1976, P. Frankl [9] improved that $n_0(k,t) = (k-t+1)(t+1)$ for $k-t \geq 15$. Recently, R.M. Wilson [21] shows that $n_0(k,t) = (k-t+1)(t+1)$ for the remaining cases, i.e., $k-t = 2, 3, ..., 14$, and characterizes the extremal configuration when $n > (k-t+1)(t+1)$. 

1
Also, the analog of the Erdös-Ko-Rado theorem for finite vector spaces (i.e., q-analog of Johnson graph) was studied in Hsieh [13], for integer sequences (i.e., Hamming graph) in Frankl and Füredi [11] and Moon [16], for bilinear forms (i.e., q-analog of Hamming graph) in Delsarte [5] and Huang [15], and for isotropic spaces (i.e., dual polar graphs) in Stanton [20].

In terms of graph theory, the Erdös-Ko-Rado theorem can be recognized as a characterization of the "largest" sets of vertices of a graph such that every pair of vertices is at distance at most t. In general, we can ask a question of classifying all the "maximal" sets of vertices of a graph such that every pair of vertices is at distance at most t.

In this dissertation, we study the maximal sets of vertices with the above property in the view point of (p-polynomial) association scheme, namely distance-regular graphs. In the following, we will define t-cliques and distance-regular graphs.

First, let us suppose that $\mathcal{G}$ is a finite, simple, connected graph with diameter d, and R a non-empty subset of \{0,1,2,\ldots,d\}. We define an R-clique $\mathcal{F}$ of $\mathcal{G}$ as a subset of the vertices of $\mathcal{G}$ which satisfies the property that, for any two vertices $x$ and $y$ in $\mathcal{F}$, the distance $d(x,y)$ between $x$ and $y$ in $\mathcal{G}$ belongs to R. (Delsarte [4] introduced the notion of
R-clique for association schemes.) $\mathcal{F}$ is said to be maximal if $\mathcal{F}\cup\{u\}$ is not an R-clique for any vertex $u$ not in $\mathcal{F}$. For convenience, we will call $\mathcal{F}$ is a t-clique if $R=\{0,1,...,t\}$ for some integer $t>0$.

If $u$ is a vertex of $\mathcal{G}$, then $\mathcal{G}_i(u)$ will denote the set of vertices at distance $i$ from $u$. For a vertex $v\in\mathcal{G}_i(u)$, define

$$a_i(u,v) = |\mathcal{G}_i(u)\cap\mathcal{G}_1(v)|,$$

$$b_i(u,v) = |\mathcal{G}_{i+1}(u)\cap\mathcal{G}_1(v)|,$$

$$c_i(u,v) = |\mathcal{G}_{i-1}(u)\cap\mathcal{G}_1(v)|.$$

$\mathcal{G}$ is called a distance-regular graph if, for any pair of vertices $u$ and $v$ of $\mathcal{G}$, $a_i(u,v)$, $b_i(u,v)$ and $c_i(u,v)$ depend only on the distance $i$ between $u$ and $v$. One of the important properties of distance-regular graphs is that if $v$ is a vertex in $\mathcal{G}_i(u)$, then $|\mathcal{G}_j(u)\cap\mathcal{G}_k(v)|$ depends only on $i$, $j$, and $k$. (As a family of association schemes, the family of distance-regular graphs is the p-polynomial association schemes. [1])

There are 22 known families of distance-regular graphs (see [1], [3]) and it seems that not much work has been done about geometric characterization of the distance-regular graphs. (Cf. [2], [6], [18] for Johnson graph; [17] for Odd graph; [7] for Hamming graphs; [19] for $q$-analog of Johnson graph; [14] for $q$-analog of Hamming graph.) Bose and Laskar [2] is one of the original papers dealing with the characterization of the Hamming graph $H(2,q)$ and the Johnson graph $J(n,2)$.
As in Bose and Larkar [2], one of the crucial steps to characterize the distance-regular graphs is to show the existence of maximal 1-cliques of the right size. Also, as is shown in Dowling [6], once the maximal 1-cliques of the right size are found, then they sometimes lead to the structure of the neighborhood of a vertex in the graph, and often this leads in turn to a characterization of the graph by geometric argument. (See p.368 [1].)

It seems that the classification of all the maximal t-cliques of the distance-regular graphs can help us to have a better understanding of the geometric structures of those graphs. Thus we consider the problem of classifying all the maximal t-cliques of the distance-regular graphs.

For most of the known important distance-regular graphs, the maximal 1-cliques are described in [12]. As we mentioned before, the upper bounds for the sizes of the maximal t-cliques and those attaining the upper bound are known for some distance-regular graphs: Johnson graph, Hamming graph, q-analog of Johnson graph and q-analog of Hamming graph.

We tried to classify the maximal t-cliques of Johnson graph, Hamming graph and q-analog of Johnson graph. We will define two t-cliques \( F_1 \) and \( F_2 \) of a graph \( G \) are isomorphic if there exists an
automorphism $\phi$ of $\mathcal{G}$ satisfying that $\phi(\mathcal{F}_1) = \mathcal{F}_2$. Although the number of non-isomorphic maximal $t$-cliques for distance-regular graphs is finite, the very complicated computation would be required to classify all the maximal $t$-cliques. For example, there are 17 non-isomorphic maximal 2-cliques in Johnson graph $J(n,k)$ if $n > k + 3$. (See section 2.2.)

Therefore, we narrow the scope to the maximal $t$-cliques whose sizes are asymptotically large. Using different methods from the previous studies, we get following results for the Johnson graph, Hamming graph and $q$-analog of Johnson graph:

(1). For the Johnson graph $J(n,k)$, let $k$ and $t$ be fixed integers satisfying $3 \leq t < k$. If $n$ is sufficiently large compared to $k$, $t$ and $\mathcal{F}$ is a maximal $t$-clique of $J(n,k)$ such that $|\mathcal{F}| > cn^{t-1}$ for some $c > 0$, then up to isomorphism $\mathcal{F}$ is either type (J.1), type (J.2), type (J.3), type (J.4), or type (J.5). (More precisely, for any fixed constant $c = c(k,t) > 0$, there exists $n_0 = n_0(k,t,c)$ such that if $n > n_0$ and $\mathcal{F}$ is a maximal $t$-clique of $J(n,k)$ such that $|\mathcal{F}| > cn^{t-1}$, then $\mathcal{F}$ is one of those types up to isomorphism.) These five types (J.1), (J.2), (J.3), (J.4) and (J.5) are fully described in theorem 2.3.1. and where all these types depend on

(i). the structure of the $(k-t \cdot 1)$-sets $R_1, R_2, \ldots, R_r$ $(1 \leq r \leq \binom{k}{t-1})$ such that, for each $j = 1,2,\ldots,r$, $|u \cap R_j| \geq k-t$ for every vertex $u$ in $\mathcal{F}$, and

(ii). the relation among the vertices $u$'s of $\mathcal{F}$ such that $|u \cap R_j| = k-t$ for some $j = 1,2,\ldots,r$. 
This makes us possible to characterize up to \((t^{-3})\)rd-largest (or \((t-2)^{nd}\)-largest) maximal \(t\)-cliques if \(3 \leq t < k, k > 2t-3\) (or \(k \leq 2t-3\)) and \(n\) is sufficiently large compared to \(k\) and \(t\). (Cf. remark (2) and (5) in section 2.3.)

(2). For the Hamming graph \(H(n,q)\), let \(n\) and \(t\) be fixed integers satisfying \(4 \leq t < n\). If \(q\) is sufficiently large compared to \(n\), \(t\) and \(F\) is a maximal \(t\)-clique of \(H(n,q)\) such that \(|F| > cq^{t-1}\) for some \(c > 0\), then up to isomorphism \(F\) is either type (II.1), type (II.2), type (II.3)' or type (II.4). (More precisely, for any fixed constant \(c = c(n,t) > 0\), there exists \(q_o = q_o(n,t,c)\) such that if \(q > q_o\) and \(F\) is a maximal \(t\)-clique of \(H(n,q)\) satisfying that \(|F| > cq^{t-1}\), then \(F\) is one of those types up to isomorphism.) These four types (II.1), (II.2), (II.3)' and (II.4) are fully described in theorem 3.3.1. and corollary 3.3.1. and where these types depend on

(i). the structure of the set \(\{R_1, R_2, ... , R_r\} (1 \leq r \leq \binom{k}{t-1})\) where \(R_j (j = 1, 2, ..., r)\) is a set of \(t-1\) entries such that \(F\) contains every vertex \(u = (u_k)\) such that \(u_k\)'s are fixed for all \(k\)'s in \(R_j\) for some \(j\) (here, we may assume the fixed value is 1), and

(ii). the relation among the vertices \(v\)'s of \(F\) such that \(v = (v_k)\) and \(v_k \neq 1\) for an entry \(k\) not in \(R_j\) for some \(j = 1, 2, ..., r\).

Also, if \(4 \leq t < n\), \(n \geq 2t-4\) (or \(n < 2t-4\)) and \(q\) is sufficiently large compared to \(n\) and \(t\), we can characterize up to sixth largest (or fifth largest) maximal \(t\)-cliques. (Cf. remark (5) and (6) in section 3.3.)
(3). For the q-analog of Johnson graph $J_q(n,k)$, let $k$ and $t$ be fixed integers satisfying $2 \leq t < k$ and $q$ be a prime power. If $n$ is sufficiently large compared to $q$, $k$ and $t$ and $F$ is a maximal $t$-clique of $J_q(n,k)$ such that $|F| > c q^{(n-k)(t-1)}$ for some $c > 0$, then up to isomorphism $F$ is either type $(J_q.1)$, type $(J_q.2)$, type $(J_q.3)$ or type $(J_q.4)$. (More precisely, for any fixed constant $c = c(q,k,t) > 0$, there exists $n_0 = n_0(q,k,t,c)$ such that if $n > n_0$ and $F$ is a maximal $t$-clique of $J_q(n,k)$ satisfying $|F| > c q^{(n-k)(t-1)}$, then $F$ is one of those types up to isomorphism.) These four types $(J_q.1)$, type $(J_q.2)$, type $(J_q.3)$ or type $(J_q.4)$ are fully described in theorem 4.2.1. and where all these types depend on

(i). the structure of the $(k-t+1)$-spaces $R_1$, $R_2$, ..., and $R_r$ $(1 \leq r \leq k-t+1)$ satisfying that, for each $j = 1,2,...,r$, $\dim(u \cap R_j) > k-t$ for every vertex $u$ in $F$, and

(ii). the relation among the vertices $u$'s of $F$ such that $\dim(u \cap R_j) = k-t$ for some $j = 1,2,...,r$.

Also, if $2 \leq t < k$, $k > 2t-3$ (or $k \leq 2t-3$) and $n$ is sufficiently large compared to $q$, $k$, and $t$, then we can characterize up to $(t-3)^{rd}$-largest (or $(t-2)^{nd}$-largest) maximal $t$-cliques when $q$ is sufficiently large compared to $n$, $k$ and $t$. (Cf. remark (1) and (2) in section 4.3.)

Maximal 2-cliques of Johnson graph and Hamming graph are com-
pletely classified, as well: In $J(n,k)$, if $n > k + 3$, then there exists 17 non-isomorphic 2-cliques. (For the complete description, refer to theorem 2.2.1.) In contrast to the Johnson graph $J(n,k)$, the Hamming graph $H(n,q)$ has exactly 3 non-isomorphic maximal 2-cliques if $n \geq 3$:

1. $\{(u_k): u_i = 1, 2, \ldots, q \text{ for } i \leq 2, u_j = 1 \text{ for all } j > 2\}$
   size: $q^2$

2. $\{(u_k): u_i \neq 1 \text{ at most one } i\}$
   size: $n(q-1)+1$

3. $\{(1,1,1,1,\ldots,1), (2,2,1,1,\ldots,1), (1,2,2,1,1,\ldots,1), (2,1,2,1,1,\ldots,1)\}$
   size: 4.

Though only three classes of association schemes are discussed in this dissertation, it is hoped that the methods used here work for the same question of the other classes of association schemes. However, for the case of bilinear forms, I tried the same method as for the Hamming graph, but it does not seem to work as the distance functions for those two are quite different. May be the characterization of bilinear forms by Huang '14 can be used in this particular case. I hope that I can settle this in the very near future.
Chapter 2
Maximal t-cliques of J(n,k)

2.1 Introduction

The Johnson graph J(n,k) has \( \binom{X}{k} \), the set of all k-subsets of a fixed set X of cardinality n, as its vertex set and two vertices x and y are adjacent if \( |x\cap y| = k-1 \). Obviously, for any \( i = 1,2,...,k \), \( d(x,y) = i \), if and only if \( |x\cap y| = k-i \). Also, the diameter of the Johnson graph J(n,k) is k.

Erdős, Ko and Rado \cite{8} proved that if \( n \geq n_o(k,t) \) for some function \( n_o = n_o(k,t) \), then the size of a t-clique \( \mathcal{F} \) in the Johnson graph J(n,k) is bounded above by \( \binom{n-(k-t)}{t} \). They also showed that if n is sufficiently large compared to k and t, \( \mathcal{F} \) attains the upper bound only if \( \mathcal{F} \) contains all the vertices which contain some fixed (k-t)-subset. The original proof by Erdős, Ko and Rado \cite{8} established that \( n_o(k,t) \leq (k-t)+t(k^3) \) and also
contained the result $n_0(k, k-1) = 2k$. In 1976, P. Frankl \cite{9}, improved that $n_0(k, t) = (k-t+1)(t-1)$ for $k-t \geq 15$. Recently, R.M. Wilson \cite{21}, showes that $n_0(k, t) = (k-t-1)(t-1)$ for the remaining cases, i.e., $k-t = 2, 3, \ldots, 14$, and characterizes the extremal configuration when $n > (k-t-1)(t-1)$.

It is known that there are two non-isomorphic maximal 1-cliques in $J(n, k)$, whose sizes are $k+1$ and $n-k+1$ (cf. \cite{12}). Section 2.2 shows that there are exactly 17 non-isomorphic maximal 2-cliques if $n \geq k-4$. (If $n = k-3$, 3 of the 17 maximal 2-cliques do not exist. Obviously, if $n \leq k-2$, there exits only one maximal 2-clique which is $J(n, k)$ itself.) For maximal 3-cliques, so far we were able to identify only 30 non-isomorphic ones, although we suspect that there are at least more than 50. Even though the number of non-isomorphic maximal $t$-cliques in $J(n, k)$ is finite, the very complicated computation would be required to classify all maximal $t$-cliques. Therefore, we narrow the scope to the maximal $t$-cliques with asymptotically large size.

If $k > t \geq 2$, then there are more than one non-isomorphic maximal $t$-clique with size at least $cn^{t-1}$ for some non-zero constant $c$ if $n$ is sufficiently larger than $k$. For example we find 7 non-isomorphic maximal 2-cliques with size $cn - o(n)$. (Here, as usual, $f(n) = o(g(n))$ (or $f(n) = O(g(n))$ ) denotes that $f(n)$ is a function of $n$ such that $\lim_{n \to \infty} \frac{f(n)}{g(n)}$ is 0 (or bounded).
Frankl characterized the second largest maximal t-clique in $|10|$:

Let us assume $\mathcal{F}$ is a second largest t-clique in $J(n,k)$. Then for $n > n_0(k)$

1. If $k \geq 2t-1$, then

$$\mathcal{F} = \{u \in \binom{X}{k}: |u \cap Q| \geq k-t+1\}$$

for a $(k-t+2)$-subset $Q$ of $X$. (In this case, $|\mathcal{F}| = (k-t-2)(\binom{n-k+t-2}{t-2})$.)

2. If $k < 2t-1$, or $k=3$, $t=2$, then

$$\mathcal{F} = \{u \in \binom{X}{k}: R \subseteq u, u \cap P \neq \emptyset\} \cup \{u \in \binom{X}{k}: P \subseteq u, |u \cap R| \geq k-t-1\}$$

for a $(k-t)$-subset $R$ and a $(t+1)$-subset $P$ of $X$ satisfying $P \cap R = \emptyset$. (In this case, $|\mathcal{F}| = (t+1)\binom{n-k-1}{t-1} - (t-1)\binom{n-k-1}{t-2} - \ldots - (t-1)\binom{n-k-1}{t-1} - (k-t)$.)

We will study maximal 2-cliques and t-cliques of the Johnson graph $J(n,k)$ in section 2.2 and 2.3, respectively.

2.2 Maximal 2-cliques of $J(n,k)$

This section shows that $J(n,k)$ has 17 (or 14) non-isomorphic maximal 2-cliques if $n \geq k+4$ (or $n = k-3$). If $n \leq k-2$, $J(n,k)$ has only one maximal 2-clique, which is the set of vertices of $J(n,k)$ itself. So, we may assume that $n \geq k+3$. 
For convenience, we denote some vertices using parentheses. For example, \((R.1,2,a)\) is a vertex containing a \((k-3)\) set \(R\) and three elements 1, 2 and a, not in \(R\).

Let \(\mathcal{F}\) be a maximal 2-clique in \(J(n,k)\) and \(P\) a set of maximum size such that \(\mathcal{F}\) contains all \(k\)-subsets of \(P\). Then \(|P| = k-2, k-1, \text{ or } k\). (We will denote elements contained in \(P\) by 1, 2, 3, ... and elements contained in \(\bar{P}\) by a, b, c, ..., where \(\bar{P}\) means the complement of \(P\). And they are all distinct.) Let us define a set of vertices, \(A_i\), by

\[
A_i = \{u \in \mathcal{F} : u \cap P = i\}
\]

for \(i = k-1, k-2, \ldots, k\).

**Lemma 2.2.1.** Assume \(P = k-s\) \((0 \leq s \leq t)\) for a maximal \(t\)-clique \(\mathcal{F}\). Then \(\min\{i : A_i \neq \emptyset\} = k-t-s\).

**Proof:** If \(A_i = \emptyset\), clearly \(i \geq k-t-s\). Suppose that \(u \cap P > k-t-s\) for every vertex \(u\) in \(\mathcal{F}\). Then \(\mathcal{F}\) contains all vertices which contain \((k-1)\) elements of \(P\), which implies that \(P\) can be enlarged, contradicting the definition of \(P\).

**Theorem 2.2.1.** If \(n \geq k+4\), \(J(n,k)\) has 17 non-isomorphic maximal 2-cliques;

(L.1). \(\{k\text{-sets containing some fixed } (k-2)\text{ set}\}\)

size: \(\binom{n-k-2}{2}\)
(L.2). \{k-sets contained in some fixed (k+2) set\}
size: \(\binom{k+2}{2}\)

(L.3). \{(P)\} \cup \{k-sets containing (k-1) elements of P\}: \(\forall P = k\)
size: \(k(n-k) + 1\)

(L.4). \{k-sets containing R\} \cup \{k-sets containing (k-2) elements of R and 2 elements of a,b,c \}:
\(R^\prime = k-1\) and a,b,c \(\notin R\)
size: \(n + 2k - 2\)

(L.5). \{k-subsets of P\} \cup \{k-sets containing some fixed (k-2) subset of R of P and (k-1) elements of P\}:
\(P = k+1\)
size: \(3n - 2k - 2\)

(L.6). \{(P),(R,a,b)\} \cup \{u: \forall u \cap P = k-1, R \subseteq u\ or\ a \subseteq u\ or\ b \subseteq u\}:
\(P = k, R\): a (k-2)-subset of P, and a, b \(\notin P\)
size: \(2n - 2\)

(L.7). \{k-sets contained in P\} \cup \{u: u \cap P = R \cup \{a\} \cup \{(R,b,d),(R,c,e),(S,a,b,d),(S,a,b,e),(S,a,c,d)\}:\)
\(R = k-2, P = R \cup \{a,b,c\}, S\): a (k-3)-subset of R, and d, e \(\notin P\)
size: \(n + 4\)

(L.8). \{u: u \subseteq R \cup \{a\}\} \cup \{(R,b,c),(R,d,e),(S,a,b,d),(S,a,b,e),(S,a,c,d)\}:
\(S.a.c.e)\}:
\(R = k-2, S\): a (k-3)-subset of R, and a,b,c.d.e \(\notin R\)
size: \(n-k+7\)
(L.9). \{u: u \supseteq U \cup \{a,b,e\} \cup \{R,a,b,c,d\}, (R,a,c,e,f), (R,b,d,e,f), \\
(R,a,b,f,h), (R.a,d,e,h) (R,b,c,e,h)\}:

\[ |R| = k-4 \text{ and } a,b,c,d,e,f,h \notin R \]

size: \( n-k+7 \)

(L.10). \{k-subsets of P\} \subseteq \{(R,a,b,e), (R,c,d,e), (R.a,c,g), (R,a.d.f).

(R.b.c,f), (R,b.d.g)\}:

\[ |R| = k-3, \ P = R \cup \{a,b,c,d\}, \text{ and } e,f,g \in P \]

size: \( k-7 \)

(L.11). \{k-subsets of P\} \subseteq \{(R,a,b,e), (R,c,d,e), (R.a,c,g), (R,a.d.e).

(R.b.c,e), (R.b,d,g)\}:

\[ |R| = k-3, \ P = R \cup \{a,b,c,d\}, \text{ and } e,g \in P \]

size: \( k-7 \)

(L.12). \{(R.a,b,e), (R.a.b,f), (R,a,c,e), (R,a.d.e), (R.b.c.e), (R,b.d.e).

(R.a,c,d), (R.b.c.d), (R.c.e.f), (R.d.e.f)\}:

\[ |R| = k-3, \text{ and } a,b,c,d,e,f \subseteq R \]

size: 10

(L.13). \{(R.a,b,e), (R.a.b.f), (R,a,c,e), (R,a.c,f), (R,b.c.e), (R.b,c,f).

(R.a,b.d), (R.a.c.d), (R.b,c.d), (R.d.e,f)\}:

\[ |R| = k-3, \text{ and } a,b,c,d,e,f \subseteq R \]

size: 10

(L.14). \{(R,a,b,c), (R.a.d,e), (R.a.d.f), (R,b.d.e), (R.b.d.f), (R.c.e.f).

(R.c.d.e), (R.a,c.f), (R,a,c,d), (R.a.b.e)\}:

\[ |R| = k-3 \text{ and } a,b,c,d,e,f \subseteq R \]
Proof: We will discuss by 3 cases according to \( |P| \), which is either \( k+2 \), \( k+1 \) or \( k \):

Case 1. \( |P| = k+2 \): Since \( \min \{i: A_i = \emptyset\} = k \) by lemma 2.2.1.

\[ F = \{u: i_u \cap P = k\} \]

= The set of all \( k \)-subsets of some fixed \( (k-2) \)-set which is a maximal 2-clique of size \( \binom{k-2}{k} \) and isomorphic to (L.2).

Case 2. \( |P| = k-1 \): Since \( |P| = k+1 \), \( \min\{i: A_i = \emptyset\} = k-1 \) by lemma
2.2.1. If every vertex in \( A_{k-1} \) contains an element \( a \) in \( \overline{P} \), then \( \mathcal{F} \) contains all the \( k \)-subsets of a \((k+2)\) set \( P \cup \{a\} \), contradicting the definition of \( P \). Therefore, vertices in \( A_{k-1} \) can not share an element in \( \overline{P} \).

We will divide this case 2 into three subcases according to \( \min\{ |u \cap v \cap P| : u, v \in A_{k-1} \} \) which is either \( k-1 \), \( k-2 \) or \( k-3 \):

**Subcase 2.1.** Suppose \( \min\{ |u \cap v \cap P| : u, v \in A_{k-1} \} = k-1 \). Then there is a \((k-1)\)-subset \( Q \) of \( P \) such that \( Q \subseteq u \) for every vertex \( u \) in \( A_{k-1} \), and

\[
\mathcal{F} \subseteq \{ u : |u \cap P| = k \} \cup \{ u : u \cap P = Q \}
\]

whose right hand-side is a 2-clique but not maximal, since any vertex \( v \) such that \( |v \cap Q| = k-2 \) and \( |v \cap P| = k-1 \) is 2-adjacent to every vertex of that. Thus \( \mathcal{F} \) can not be maximal.

**Subcase 2.2.** Suppose \( \min\{ |u \cap v \cap P| : u, v \in A_{k-1} \} = k-2 \). Then there are two vertices \( x \) and \( y \) in \( A_{k-1} \) such that \( x \cap y \cap P = R \) for a \((k-2)\)-subset \( R \) of \( P \). Then we may put \( P = R \cup \{1,2,3\} \), \( x = (R,1,a) \) and \( y = (R,2,b) \) where \( a \) and \( b \) are in \( \overline{P} \). (If \( x = (R,1,a) \) and \( y = (R,2,a) \) for any such pair of vertices \( x \) and \( y \), then every vertex in \( A_{k-1} \) contains a which is not in \( P \).) If \( u \) is a vertex in \( A_{k-1} \), then either (1) \( R \subseteq u \) or (2) \( |u \cap R| = k-3 \) and \( \{1,2\} \subseteq u \). Notice that any two vertices satisfying (1) (or (2)) are 2-adjacent. Therefore, we obtain 2 non-isomorphic maximal 2-cliques:
\( \mathcal{F} = \{ u : |u \cap P| = k \} \cup \{ u : |u \cap P| = k-1, \text{ } R \nsupseteq u \} \)

size: \( 3n-2k-2 \) \hfill (2.1)

\( \mathcal{F} = \{ u : |u \cap P| = k \} \cup \{ u : u \cap P = R \cup \{1\} \text{ or } R \cup \{2\} \} \cup \{ u : |u \cap R| = k-3, \{1,2\} \subseteq u \} \)

size: \( k(n-k)+1 \) \hfill (2.2)

which are isomorphic to (L.5) and (L.3), respectively.

**Subcase 2.3.** Suppose \( \min\{|x \cap y \cap P| : x, y \in A_{k-1} \} = k-3 \). Then there are two vertices \( x \) and \( y \) in \( A_{k-1} \) such that \( x \cap y \cap P = S \) for a \( (k-3) \)-subset \( S \) of \( P \). Then we may put \( P = S \cup \{1,2,3,4\} \), \( x = (S,1,2,a) \) and \( y = (S,3,4,a) \) where \( a \) is not in \( P \). If \( u \) is a vertex in \( A_{k-1} \) but different from \( x \) and \( y \), then \( u \) satisfies one of the following:

1. \( u \cap P = S \cup \{1,3\} \)
2. \( u \cap P = S \cup \{1,4\} \)
3. \( u \cap P = S \cup \{2,3\} \)
4. \( u \cap P = S \cup \{2,4\} \)
5. \( u \cap S = k-4, \{1,2,3,a\} \subseteq u \)
6. \( u \cap S = k-4, \{1,2,4,a\} \subseteq u \)
7. \( u \cap S = k-4, \{1,3,4,a\} \subseteq u \)
8. \( u \cap S = k-4, \{2,3,4,a\} \subseteq u \)
9. \( u \cap S = k-5, \{1,2,3,4,a\} \subseteq u \).

Notice that \( A_{k-1} \) has a vertex \( u \) such that \( S \subseteq u \cap P \) but \( a \notin u \) (i.e., a
vertex which satisfies either (1), (2), (3) or (4) but does not contain a.)
Otherwise, every vertex in $\mathcal{A}_{k-1}$ contains a which is not in P. Without
loss of generality, we may assume $\mathcal{I}$ has a vertex $(S,1,3,b)$.

Suppose $|\{u \in \mathcal{A}_{k-1}: u \cap P = S \cup \{1,3\}\}| = 1$. Then

$$\mathcal{I} \supseteq \{u: |u \cap P| = k\} \cup \{(S,1,2,a), (S,3,4,a), (S,1,3,b)\}.$$ 

If $u$ is a remaining vertex of $\mathcal{A}_{k-1}$, then $u$ is one of the following:

(2) $u \cap P = S \cup \{1,4\}$
(3) $u \cap P = S \cup \{2,3\}$
(4') $u \cap P = (S,2,4,b)$
(5) $u \cap S = k-4$, $\{1,2,3,a\} \subseteq u$
(7) $u \cap S = k-4$, $\{1,3,4,a\} \subseteq u$. \hspace{1cm} \ldots \hspace{0.5cm} \hspace{0.5cm} \hspace{0.5cm} \ldots \hspace{0.5cm} \hspace{0.5cm} \hspace{0.5cm} \hspace{0.5cm} \hspace{0.5cm} (2.3)$

Among the vertices satisfying one of (2,3), only $(S,2,4,b)$ is not 2-
adjacent to $(S,1,3,a)$ which is not in $\mathcal{I}$. Therefore, $\mathcal{I}$ contains $(S,2,4,b)$
and can not contain a vertex satisfying (5) or (7). So, we can say, up
to isomorphism, $\mathcal{I}$ contains either $\{u: u \cap P = S \cup \{1,4\}\}$, $\{(S,1,4,a),
(S,2,3,a)\}$ or $\{(S,1,4,c), (S,2,3,c)\}$ for some $c$ in $\bar{P}$. Therefore, we get 3
non-isomorphic maximal 2-cliques:

$$\mathcal{I} = \{u: |u \cap P| = k\} \setminus \{(S,1,2,a), (S,3,4,a), (S,1,3,b), (S,2,4,b)\}$$
$$\cup \{(S,1,4,a), (S,2,3,a)\} \hspace{1cm} \ldots \hspace{0.5cm} \hspace{0.5cm} \hspace{0.5cm} \hspace{0.5cm} \hspace{0.5cm} \hspace{0.5cm} \hspace{0.5cm} \hspace{0.5cm} (2.4)$$

Defined as:

$$\mathcal{I} = \{u: |u \cap P| = k\} \setminus \{(S,1,2,a), (S,3,4,a), (S,1,3,b), (S,2,4,b)\}$$
$$\cup \{(S,1,4,a), (S,2,3,a)\} \hspace{1cm} \ldots \hspace{0.5cm} \hspace{0.5cm} \hspace{0.5cm} \hspace{0.5cm} \hspace{0.5cm} \hspace{0.5cm} \hspace{0.5cm} \hspace{0.5cm} (2.4)$$
which are isomorphic to (L.7), (L.11) and (L.10), respectively

Note that: (i). If \( c = a \) in (2.6), then \( \mathcal{F} \) in (2.6) is isomorphic to the one in (2.5).

(ii). If \( n = k+3 \), there does not exist a maximal 2-clique of type (2.6).

Suppose \( |\{u \in \mathcal{A}_{k-1} : u \cap P = \mathbb{S} \cup \{1,3\}\}| \geq 2 \). Then

\[
\mathcal{F} = \{u : |u \cap P| = k\} \cup \{(S,1,2,a), (S,3,4,a)\} \cup \{u : u \cap P = \mathbb{S} \cup \{1,3\}\}.
\]

If \( u \) is a remaining vertex of \( \mathcal{A}_{k-1} \), then \( u \) satisfies one of the following:

1. \( u \cap P = \mathbb{S} \cup \{1,4\} \)
2. \( u \cap P = \mathbb{S} \cup \{2,3\} \)
3. \( |u \cap S| = k-4, \{1,2,3,a\} \subseteq u \)
4. \( |u \cap S| = k-1, \{1,3,4,a\} \subseteq u \).

Thus, one of the following holds up to isomorphism:

\[
\mathcal{F} = \{u : |u \cap P| = k\} \cup \{(S,1,2,a), (S,3,4,a)\} \cup \{u : u \cap P = \mathbb{S} \cup \{1,3\}\} \cup \{u : u \cap S| = k-4, \{1,3,4,a\} \subseteq u\}.
\]

size: \( 2n-2 \) (2.7)

\[
\mathcal{F} = \{u : |u \cap P| = k\} \cup \{(S,1,2,a), (S,3,4,a)\} \cup \{u : u \cap P = \mathbb{S} \cup \{1,3\}\} \cup \{(S,1,4,b), (S,2,3,b)\}
\]

for some \( b \) in \( \bar{P} \). (2.8)

\[
\mathcal{F} = \{u : |u \cap P| = k\} \cup \{(S,1,2,a), (S,3,4,a)\} \cup \{u : u \cap P = \mathbb{S} \cup \{1,3\}\}
\]
\[ \cup \{(S,1,4,a), (S,2,3,a)\} \cup \{u: |u \cap S| = k-4, \{1,2,3,a\} \subseteq u\} \]
\[ \cup \{u: |u \cap S| = k-4, \{1,3,4,a\} \subseteq u\} \]

size: \(n+2k-2\) \hspace{1cm} (2.9)

The maximal 2-clique \(\mathcal{F}\) in (2.8) is isomorphic to the one in (2.4). But we get two new types of maximal 2-cliques, (2.7) and (2.9), in this subcase, which are isomorphic to (L.6) and (L.4), respectively.

Case 3. \(P| = k\): Since \(|P| = k\), \(\min\{i: A_i = \emptyset\} = k-2\) by lemma 2.2.1. Observe that:

(i). Two vertices \(u\) and \(v\) are 2-adjacent if \(|u \cap P| \geq k-1\) and \(|v \cap P| \geq k-1\).

(ii). If every vertex of \(A_{k-2}\) contains an element \(a\) in \(\bar{P}\), then \(\mathcal{F}\) contains all \(k\)-subsets of a \((k-1)\) set \(P \cup \{a\}\), a contradiction. Therefore, vertices in \(A_{k-2}\) do not share an element in \(\bar{P}\).

We will discuss this case 3 by three subcases according to \(\min\{u \cap v \cap P: u,v \in A_{k-2}\}\) which is either \(k-2\), \(k-3\), or \(k-4\):

Subcase 3.1. Suppose \(\min\{u \cap v \cap P: u,v \in A_{k-2}\} = k-2\). Then there is a \((k-2)\)-subset \(Q\) of \(P\) such that \(Q \subseteq u\) for every vertex \(u\) in \(A_{k-2}\). Moreover, since the vertices in \(A_{k-2}\) do not share element in \(\bar{P}\), every vertex in \(A_{k-1}\) contains \(Q\). So, we have a maximal 2-clique

\[ \mathcal{F} = \{(P)\} \cup \{u: |u \cap P| = k-1, Q \subseteq u\} \cup \{u: u \cap P = Q\} \]
the set of all vertices which contain a fixed (k-2) set size: \( \binom{n-k+2}{2} \).

which is isomorphic to (L.1).

Subcase 3.2. Suppose \( \min\{u \cap v \cap P : u, v \in A_{k-2}\} = k-3 \). Then there are two vertices \( x \) and \( y \) in \( A_{k-2} \) such that \( x \cap y \cap P = S \) for a (k-3)-subset \( S \) of \( P \). We may put \( P = S \cup \{1,2,3\} \), \( x = (S,1,a,b) \) and \( y = (S,2,a,c) \) where \( a, b, c \) are in \( \bar{P} \). (If \( x \cap \bar{P} = y \cap \bar{P} \) for any such a pair of vertices \( x \) and \( y \) in \( A_{k-2} \), then the vertices in \( A_{k-2} \) share two elements in \( \bar{P} \), which is a contradiction.) Since vertices in \( A_{k-2} \) do not share an element in \( \bar{P} \), \( |A_{k-2}| > 2 \). Moreover, if \( u \) is a vertex in \( A_{k-2} \) such that \( |u \cap S| = k-4 \), then \( u \) must contain \( \{1,2\} \). Since \( \min\{u \cap v \cap P : u, v \in A_{k-2}\} = k-3 \).

We will discuss this subcase 3.2 by another 9 subcases according to \( \{u \cap A_{k-2} : S \cup \{1,a\} \subseteq u\} \) and \( \{u \cap A_{k-2} : S \cup \{2,a\} \subseteq u\} \):

Subcase 3.2.1. Suppose that \( \{u \cap A_{k-2} : S \cup \{1,a\} \subseteq u\} \geq 3 \) and \( \{u \cap A_{k-2} : S \cup \{2,a\} \subseteq u\} \geq 3 \). Then every vertex in \( A_{k-2} \) contains \( a \), which is a contradiction.

Subcase 3.2.2. Suppose that \( \{u \cap A_{k-2} : S \cup \{1,a\} \subseteq u\} \geq 3 \) and
\{u \in A_{k-2}: \text{su}\{2,a\} \subseteq u\} = 2. \text{ Let us assume } \mathcal{F} \text{ contains } (S,2,a,c) \text{ and } (S,2,a,d). \text{ Then,}

\mathcal{F} \supseteq \{(S,1,2,3)\} \cup \{u: u \cap P = S \cup \{1\}, a \in u\} \cup \{(S,2,a,c),(S,2,a,d)\}.

If \(u\) is a vertex of \(\mathcal{F}\) which is not one of the above, then \(u\) is one of the following:

(1) \(u = (S,1,c,d)\)
(2) \(u \cap P = S \cup \{3\}, a \in u\)
(3) \(|u \cap P| = k-2, |u \cap S| = k-4, \{1,2,a\} \subseteq u\)
(4) \(|u \cap P| = k-1, a \in u\)
(5) \(u \cap P = S \cup \{1,2\}\).

Among the vertices satisfying either (1), (2), or (3), only \((S,1,c,d)\) does not contain \(a\). Therefore, \(\mathcal{F}\) contains \((S,1,c,d)\), that is,

\[\mathcal{F} \supseteq \{(S,1,2,3)\} \cup \{u: u \cap P = S \cup \{1\}, a \in u\} \cup \{(S,2,a,c),(S,2,a,d)\}, \]

\((S,2,a,d), (S,1,c,d)\}.

If \(u\) is a remaining vertex of \(\mathcal{F}\), then \(u\) satisfies one of the following:

(2)' \(u = (S,3,a,c)\)
(2)'' \(u = (S,3,a,d)\)
(3)' \(u \cap P = k-2, u \cap S = k-4, \{1,2,a,c\} \subseteq u\)
(3)'' \(u \cap P = k-2, u \cap S = k-4, \{1,2,a,d\} \subseteq u\)
(4)' \(u = (S,1,3,a)\)
(5) \(u \cap P = S \cup \{1,2\}\).
If $\mathcal{F}$ contains \{u: |u \cap P| = k-2, |u \cap S| = k-4, \{1,2,a,c\} \subseteq u\}, then $\mathcal{F}$ can not contain (S,3,a,d). Therefore, $\mathcal{F}$ contains (S,1,2,a) and (S,1,2,c), which implies that $\mathcal{F}$ contains all k-subsets of a (k-1) set $S \cup \{1,2,a,c\}$, a contradiction. So, $\mathcal{F}$ can not contain \{u: |u \cap P| = k-2, |u \cap S| = k-4, \{1,2,a,c\} \subseteq u\}, but does contain (S,3,a,d). Similarly, $\mathcal{F}$ can not contain \{u: |u \cap P| = k-2, |u \cap S| = k-4, \{1,2,a,d\} \subseteq u\}, but does contain (S,3,a,c).

And we get a new type of maximal 2-clique:

$$\mathcal{F} = \{(S,1,2,3)\} \cup \{u: u \cap P = S \cup \{1\}, a \in u\} \cup \{(S,2,a,c)\}$$

$$\cup \{(S,2,a,d)\}, (S,1,a,d)\} \cup \{(S,1,3,a), (S,1,2,a)\}$$

size: n-k+7,

which is isomorphic to (L.8).

Subcase 3.2.3. Suppose that \{|\{u \in \mathcal{A}_{k-2} : S \cup \{1,a\} \subseteq u\}| \geq 3\} and \{|\{u \in \mathcal{A}_{k-2} : S \cup \{2,a\} \subseteq u\} = 1\}. If we assume that $\mathcal{F}$ contains (S,2,a,c), then

$$\mathcal{F} \supseteq \{(S,1,2,3)\} \cup \{u: u \cap P = S \cup \{1\}, a \in u\} \cup \{(S,2,a,c)\}.$$ If $u$ is a vertex in $\mathcal{F}$ which is different from above, then it is one of the following:

1. $u \cap P = S \cup \{1\}, c \in u$
2. $u \cap P = S \cup \{3\}, a \in u$
3. $|u \cap P| = k-2, |u \cap S| = k-4, \{1,2,a\} \subseteq u$
4. $|u \cap P| = k-1, a \in u$
5. $u \cap P = S \cup \{1,2\}$
(6) \( u = (S,1,3,c) \).

Since vertices of \( A_{k-2} \) do not share an element in \( \overline{P} \), \( F \) contains the vertex \((S,1,c,d)\) for some \( d \) in \( \overline{P} \), that is,

\[
F \supseteq \{(S,1,2,3)\} \cup \{ u : \cup P = S \cup \{1\}, a \in u \} \cup \{(S,2,a,c),(S,1,c,d)\}.
\]

Thus if \( u \) is a remaining vertex of \( F \), then \( u \) satisfies one of the following:

1. \( \cup P = S - \{1\}, c \in u \)
2. \( u = (S,3,a,c) \)
3. \( u = (S,3,a,d) \)
4. \( |\cup P| = k-2, |\cup S| = k-4, \{1,2,a,c\} \subseteq u \)
5. \( |\cup P| = k-2, |\cup S| = k-4, \{1,2,a,d\} \subseteq u \)
6. \( u = (S,1,3,a) \)
7. \( u = (S,1,3,c) \).

Using a similar argument as in subcase 3.2.2., we can easily show that \( F \) can not contain \( \{ u : |\cup P| = k-2, |\cup S| = k-4, \{1,2,a,c\} \subseteq u \} \), but does contain \((S,3,a,d)\). Therefore,

\[
F \supseteq \{(S,1,2,3)\} \cup \{ u : \cup P = S \cup \{1\}, a \in u \} \cup \{(S,2,a,c),(S,1,c,d),(S,3,a,d)\}
\]

and if \( u \) is a remaining vertex of \( F \), then \( u \) satisfies one of the following:

2. \( u = (S,3,a,c) \)
3. \( u = (S,3,a,d) \)
(4)' \ u = (S,1,3,a)
(5)' \ u = (S,1,2,a)
(5)'' \ u = (S,1,2,d)
(6) \ u = (S,1,3,c).

But $\mathcal{F}$ can not have any vertex satisfying $(3)''$ since $\min\{u \cap v \cap P : u,v \in A_{k-2}\} = k-3$. So, $\mathcal{F}$ contains either $(S,3,a,c)$ or $(S,1,3,c)$. But if $\mathcal{F}$ contains $(S,3,a,c)$, it is isomorphic to the subcase 3.2.2. So, without loss of generality, we may assume that $\mathcal{F}$ contains $(S,1,3,c)$ but does not contain $(S,3,a,c)$. Then we get

$$\mathcal{F} = \{(S,1,2,3)\} \cup \{u: u \cap P = S \cup \{1\}, a \in u\} \cup \{(S,2,a,c), (S,1,c,d), (S,3,a,d)\} \cup \{(S,1,3,c), (S,1,3,a), (S,1,2,a), (S,1,2,d)\}$$

which is isomorphic to (3.2).

Now we may assume that $\{|u \in A_{k-2}: S \cup \{1,a\} \subseteq u\| \leq 2$ and $\{|u \in A_{k-2}: S \cup \{2,a\} \subseteq u\| \leq 2$ (and we will discuss by 6 different cases):

**Subcase 3.2.4.** Suppose $\{u \in A_{k-2}: S \cup \{1,a\} \subseteq u\} = \{(S,1,a,b), (S,1,a,d)\}$ and $\{u \in A_{k-2}: S \cup \{2,a\} \subseteq u\} = \{(S,2,a,c), (S,2,a,e)\}$. Then

$$\mathcal{F} \supseteq \{(S,1,2,3), (S,1,a,b), (S,1,a,d), (S,2,a,c), (S,2,a,e)\}.$$

If $u$ is a vertex of $\mathcal{F}$ but different from above, then $u$ is one of the following:

(1) \ u = (S,1,c,e)
(2) \ u = (S,2,b,d)
Among the vertices satisfying one of (1) - (6), only (S,2,b,d) (or (S,1,c,e)) is not 2-adjacent to (S,1,a,e) (or (S,2,a,d)) which is not \( \mathcal{F} \). Therefore, \( \mathcal{F} \) contains (S,2,b,d) and (S,1,c,e) but they are not 2-adjacent.

**Subcase 3.2.5.** Suppose \( \{u \in A_{k-2}: S \cup \{1,a\} \subseteq u\} = \{(S,1,a,b), (S,1,a,d)\} \) and \( \{u \in A_{k-2}: S \cup \{2,a\} \subseteq u\} = \{(S,2,a,c), (S,2,a,b)\} \). Then

\[ \mathcal{F} \subseteq \{(S,1,2,3), (S,1,a,b), (S,1,a,d), (S,2,a,c), (S,2,a,b)\}. \]

If \( u \) is a vertex of \( \mathcal{F} \) but different from above, then \( u \) is one of the following:

1. \( u = (S,1,b,c) \)
2. \( u = (S,2,b,d) \)
3. \( u \cap P = S \cup \{3\}, a \in u \)
4. \( u \cap P = k-2, u \cap S = k-4, \{1,2,a\} \subseteq u \)
5. \( u \cap P = S \cup \{1,2\} \)
6. \( u \cap P = k-1, a \in u \).

Among the vertices satisfying one of (1) - (6), only (S,2,b,d) (or (S,1,b,c)) is not 2-adjacent to (S,1,a,c) (or (S,2,a,d)) which is not \( \mathcal{F} \). Therefore, \( \mathcal{F} \) contains (S,2,b,d) and (S,1,b,c). That is,
\[ \mathcal{F} \supseteq \{(S,1,2,3), (S,1,a,b), (S,1,a,d), (S,2,a,c), (S,2,a,b),
(S,1,b,c), (S,2,b,d)\}. \]

If \( u \) is a remaining vertex of \( \mathcal{F} \), then \( u \) is one of the following:

(3)' \( u = (S,3,a,b) \)

(4)' \( |u \cap P| = k-2, |u \cap S| = k-4, \{1,2,a,b\} \subseteq u \)

(5) \( u \cap P = S \cup \{1,2\} \).

It is easy to see that \( \mathcal{F} \) contains two vertices \((S,1,2,a), (S,1,2,b)\) and the set \( \{u: |u \cap P| = k-2, |u \cap S| = k-4, \{1,2,a,b\} \subseteq u\} \), which implies that \( \mathcal{F} \) contains all \( k \)-subsets of a \((k+1)\) set \( S \setminus \{1,2,a,b\} \), a contradiction.

Subcase 3.2.6. Suppose \( \{u \in A_{k-2}: S \cup \{1,a\} \subseteq u\} = \{(S,1,a,b), (S,1,a,c)\} \) and \( \{u \in A_{k-2}: S \cup \{2,a\} \subseteq u\} = \{(S,2,a,c), (S,2,a,b)\} \). Then

\[ \mathcal{F} \supseteq \{(S,1,2,3), (S,1,a,b), (S,1,a,c), (S,2,a,c), (S,2,a,b)\} \]

If \( u \) is a vertex of \( \mathcal{F} \) but different from above, then \( u \) is one of the following:

(1) \( u = (S,1,b,c) \)

(2) \( u = (S,2,b,c) \)

(3) \( u \cap P = S \cup \{3\}, a \in u \)

(4) \( u = (S,3,b,c) \)

(5) \( |u \cap P| = k-2, |u \cap S| = k-4, \{1,2,a\} \subseteq u \)

(6) \( |u \cap P| = k-2, |u \cap S| = k-4, \{1,2,b,c\} \subseteq u \)

(7) \( u \cap P = S \cup \{1,2\} \)
(8) $|u \cap P| = k-1, a \in u$

If $\mathcal{F}$ contains $\{u: |u \cap P| = k-2, |u \cap S;| = k-4, \{1,2,a,b\} \subseteq u\}$, then $\mathcal{F}$ can not contain any vertex satisfying (3) or (4), since $\min\{|u \cap v \cap P: u,v \in A_{k-2}\} = k-3$. But $\mathcal{F}$ does contain $(S,1,2,a)$ and $(S,1,2,b)$, which implies $\mathcal{F}$ contains all $k$-subsets of a $(k+1)$ set $SU\{1,2,a,b\}$, a contradiction. Therefore, $\mathcal{F}$ can not contain $\{u: |u \cap P| = k-2, |u \cap S| = k-4, \{1,2,a,b\} \subseteq u\}$, but contains a vertex satisfying (3) or (4), that is, either $(S,3,b,c)$, $(S,3,a,c)$, or $(S,3,a,b)$, isomorphically or $(S,3,a,e)$, for some $e$ in $\bar{P}$. Also, $\mathcal{F}$ can not have a vertex satisfying (5) or (6), since $\min\{|u \cap v \cap P: u,v \in A_{k-2}\} = k-3$.

Subcase 3.2.6.(i). Suppose $\mathcal{F}$ contains a vertex $(S,3,a,e)$. Then

$\mathcal{F} \supseteq \{(S,1,2,3), (S,1,a,b), (S,1,a,c), (S,2,a,c), (S,2,a,b), (S,3,a,e)\}$.

If $u$ is a vertex of $\mathcal{F}$ but different from above, then $u$ is one of the following:

(3) $u \cap P = S \cup \{3\}, a \in u$
(4) $u = (S,3,b,c)$
(7) $u = (S,1,2,e)$
(8) $|u \cap P| = k-1, a \in u$.

By comparing with a vertex $(S,1,a,e)$ which is not in $\mathcal{F}$, it can be easily seen that $\mathcal{F}$ contains $(S,3,b,c)$. Thus,

$\mathcal{F} \supseteq \{(S,1,2,3), (S,1,a,b), (S,1,a,c), (S,2,a,c), (S,2,a,b), (S,3,a,e)\}$
\[ \cup \{ (S,3,b,c) \} \cup \{ u : u \cap P = S \cdot \{ 3 \}, a \in u \} \cup \{ (S,1,3,a), (S,2,3,a) \} \]

which is isomorphic to (3.2).

**Subcase 3.2.6.(ii).** Let us assume that \( F \) does not contain any vertex \((S,3,a,e)\), where \( e \) is in \( P \setminus \{ a,b,c \} \). but contains \((S,3,b,c)\). Then

\[ F \supseteq \{ (S,1,2,3), (S,1,a,b), (S,1,a,c), (S,2,a,c), (S,2,a,b), (S,3,b,c) \}. \]

If \( u \) is a vertex of \( F \) but different from above, then \( u \) is one of the following:

1. \( u = (S,1,b,c) \)
2. \( u = (S,2,b,c) \)
3. \( u \subset P = S \cup \{ 3 \}, a \in u \) (Without loss of generality, we may assume that \( F \) contains at most two vertices of this type, either \((S,3,a,b)\) or \((S,3,a,c)\).)

4. (7)\: u = (S,1,2,b)
5. (7)\: \prime u = (S,1,2,c)
6. (8)\: u = (S,1,3,a)
7. (8)\: \prime u = (S,2,3,a).

Thus \( F \) is isomorphic to one of the following: (Notice that there is no distinction between 1 and 2 (or b and c).)

\[ F = \{ (S,1,2,3), (S,1,a,b), (S,1,a,c), (S,2,a,c), (S,2,a,b), (S,3,b,c) \} \]
\[ (S,1,b,c), (S,2,b,c), (S,1,2,b), (S,1,2,c) \} \quad (3.3) \]

\[ F = \{ (S,1,2,3), (S,1,a,b), (S,1,a,c), (S,2,a,c), (S,2,a,b), (S,3,b,c) \} \]
\[ (S,1,b,c), (S,2,b,c), (S,1,2,b), (S,3,a,b) \} \quad (3.4) \]
\[ \mathcal{F} = \{(S,1,2,3), (S,1.a.b), (S.1.a.c), (S.2.a.c), (S,2.a.b), (S,3.b,c) \}
\]
\[ (S,1,b,c), (S,1,2.b). (S,1.2.c). (S,1.3,a) \} \]  \hspace{1cm} (3.5)

\[ \mathcal{F} = \{(S,1,2,3), (S,1.a,b), (S.1.a.c), (S.2.a.c), (S,2.a.b), (S,3.b,c) \}
\]
\[ (S,1,b,c), (S,2.b,c). (S.3.a.b), (S,3.a,c) \} \]  \hspace{1cm} (3.6)

size: 10

\[ \mathcal{F} = \{(S,1,2,3), (S,1.a,b), (S.1.a.c), (S.2.a.c), (S,2.a.b), (S,3.b,c) \}
\]
\[ (S,1,b,c), (S,1,2,b). (S,1,3,a) \} \]  \hspace{1cm} (3.7)

size: 10

\[ \mathcal{F} = \{(S,1,2,3), (S,1.a,b), (S.1.a.c), (S.2.a.c), (S,2.a.b), (S,3.b,c) \}
\]
\[ (S,1,2,b). (S,1,2.c). (S,1.3.a) \} \]  \hspace{1cm} (3.8)

\[ \mathcal{F} = \{(S,1,2,3), (S,1,a,b). (S.1.a.c), (S.2.a,c), (S,2.a.b), (S,3.b,c) \}
\]
\[ (S,1,b,c), (S,3.a.b), (S.1.3,a) \} \]  \hspace{1cm} (3.9)

\[ \mathcal{F} = \{(S,1,2,3). (S,1,a,b). (S.1,a.c), (S.2.a,c), (S,2.a.b). (S,3.b,c) \}
\]
\[ (S,1,2.b). (S,3.a,b). (S,1.3.a) \} \]  \hspace{1cm} (3.10)

\[ \mathcal{F} = \{(S,1,2,3). (S,1.a,b). (S.1,a,c). (S.2.a,c), (S,2.a.b). (S,3.b,c) \}
\]
\[ (S,3.a.b). (S,3.a,c), (S,1.3,a), (S,2.3,a) \} \]  \hspace{1cm} (3.11)

But \( \mathcal{F} \) in (3.3) is not maximal because of the set \( \{u: u \in P; i = k-2, \}
\]
\[ u \cap S = k-4, \{1,2.b,c} \subseteq u \} \). The maximal 2-cliques in (3.4), (3.5), (3.9)
and (3.10) are isomorphic. And the maximal 2-cliques in (3.6) and (3.8)
are isomorphic, too. But \( \mathcal{F} \) in (3.11) is not maximal if \( n > k+3 \), be-
cause of \( (S,3,a,e) \) for some \( e \) in \( P \). (However, it is isomorphic to the
maximal 2-clique in (3.2) if \( n = k+3 \). Thus, in this subcase, we obtain three new types of maximal 2-cliques (3.4), (3.6) and (3.7) which are isomorphic to (L.12), (L.13) and (L.14), respectively.

Subcase 3.2.6.(iii). Finally let us assume \( \mathcal{F} \) does not contain \((S.3,a,e)\) and \((S.3,b,c)\), but does contain \((S.3,a,c)\). Then

\[ \mathcal{F} \supseteq \{(S.1.2.3), (S.1,a,b), (S.1,a,c), (S.2,a,c), (S.2,a,b), (S.3,a,c)\} \]

If \( u \) is a vertex of \( \mathcal{F} \) which is different from above, then \( u \) is one of the following:

1. \( u = (S.1,b,c) \)
2. \( u = (S.2,b,c) \)
3. \( u = (S.3,a,b) \)
4. \( u = (S.1.2,c) \)
5. \( u \cap P = k-1 \). \( a \in u \).

Since \( \mathcal{F} \) does not contain the set \( \{u: u \cap P = k-2, \cdot \cup S \cap \{1,2,a,c\} \leq u\} \), it contains \((S.3,a,b)\). Also, \( \mathcal{F} \) contains either \((S.1.b,c)\) or \((S.2,b,c)\) since the vertices of \( A_{k-2} \) do not share an element in \( P \).

Therefore, we get the following up to isomorphism:

\[ \mathcal{F} = \{(S.1.2,3), (S.1,a,b), (S.1,a,c). (S.2,a,c), (S.2,a,b), (S.3,a,c) \}
\]

\[ \mathcal{(S.3.a.b), (S.1.b.c), (S.2.b.c), (S.1.2.a)} \} \]

(3.12)

\[ \mathcal{F} = \{(S.1.2,3), (S.1,a,b), (S.1,a,c), (S.2,a,c), (S.2,a.b), (S.3,a.c) \}
\]

\[ \mathcal{(S.3.a.b), (S.1.b.c), (S.1.2.a), (S.1.3.a)} \} \]

(3.13)
But $\mathcal{F}$ in (3.12) (or (3.13)) is isomorphic to the one in (3.4) (or (3.11)).

**Subcase 3.2.7.** Suppose $\{u \in A_{k-2}: S \supseteq \{1.a\} \subseteq u\} = \{(S,1,a,b), (S,1,a,d)\}$ and $\{u \in A_{k-2}: S \cup \{2.a\} \subseteq u\} = \{(S,2,a.c)\}$. Then

$$\mathcal{F} \supseteq \{(S,1,2,3), (S,1,a,b), (S,1,a,d), (S,2,a,c)\}.$$

If $u$ is a vertex of $\mathcal{F}$ but different from above, then $u$ is one of the following:

1. $u = S \cap \{1\}, c \in u$ but $a \notin u$
2. $u = (S,2,b,d)$
3. $u = S \supseteq \{3\}, a \in u$ (Without loss of generality, we may assume that $\mathcal{F}$ has at most one element of this type.)
4. $u = k-2, u \supseteq S = k-4, \{1.2.a\} \subseteq u$
5. $u = S \supseteq \{1.2\}$
6. $u = (S,1,3,c)$
7. $u = k-1, a \in u$.

Among the vertices satisfying one of the (1) - (7), only $(S,2,b,d)$ is not 2-adjacent to $(S,1,a,c)$ which is not in $\mathcal{F}$. Therefore $\mathcal{F}$ contains $(S,2,b,d)$, and

$$\mathcal{F} \supseteq \{(S,1,2,3), (S,1,a,b), (S,1,a,d), (S,2,a,c), (S,2,b,d)\}.$$

If $u$ is a remaining vertex of $\mathcal{F}$, then $u$ is one of the following:

1. $u = (S,1,c,b)$
2. $u = (S,1,c,d)$
(3)' \( u = (S,3,a,b) \)

(3)'' \( u = (S,3,a,d) \)

(4)' \( |u \cap P| = k-2, |u \cap S| = k-4. \{1.2,a,b\} \subseteq u \)

(4)'' \( |u \cap P| = k-2, |u \cap S| = k-4. \{1.2,a,d\} \subseteq u \)

(5) \( u \cap P = S \cup \{1,2\} \)

(7)' \( u = (S,2,3,a) \). \hfill (3.14)

Among the vertices in the list of (3.14), only \((S,1,c,d)\) (or \((S,1,c,b)\)) is not 2-adjacent to \((S,2,a,b)\) (or \((S,2,a,d)\)) which is not in \( \mathcal{F} \). Therefore, \( \mathcal{F} \) contains \( \{(S,1,c,d), (S,1,c,b)\} \) and we get

\[
\mathcal{F} = \{(S,1,2,3), (S,1,a,b), (S,1,a,d), (S,2,a,c), (S,2,b,d), (S,1,c,b), (S,1,c,d)\} \cup \{u: u \cap P = S \cup \{1,2\}\}
\]

which is isomorphic to the maximal 2-clique in (3.2).

Subcase 3.2.8. Suppose \( \{u \in \mathcal{A}_{k-2}: S \cup \{1,a\} \subseteq u\} = \{(S,1,a,b), (S,1,a,c)\} \) and \( \{u \in \mathcal{A}_{k-2}: S \cup \{2,a\} \subseteq u\} = \{(S,2,a,c)\} \). Then

\[
\mathcal{F} \supseteq \{(S,1,2,3), (S,1,a,b), (S,1,a,c), (S,2,a,c)\}.
\]

If \( u \) is a vertex of \( \mathcal{F} \) but different from above, then \( u \) is one of the following:

(1) \( u \cap P = S \cup \{1\} \), \( c \in u \) but \( a \notin u \) (Without loss of generality, we may assume that \( \mathcal{F} \) contains at most one vertex of this type, either \((S,1,c,b)\) or \((S,1,c,d)\) for some \( d \in \tilde{P} \))

(2) \( u = (S,2,b,c) \)
(3) \( u \cap P = S \cup \{3\}, a \in u \) (Again, without loss of generality, we may assume that \( \mathcal{F} \) has at most one vertex of this type. either \((S,3,a,b)\) or \((S,3,a,c)\))

(4) \( u = (S,3,b,c) \)

(5) \( |u \cap P| = k-2, |u \cap S| = k-4, \{1,2,a\} \subseteq u \)

(6) \( |u \cap P| = k-2, |u \cap S| = k-4, \{1,2,b,c\} \subseteq u \)

(7) \( u \cap P = S \cup \{1,2\} \)

(8) \( |u \cap P| = k-1, a \in u \)

(9) \( u = (S,1,3,c) \).

By comparing with \((S,2,a,b)\) which is not in \( \mathcal{F} \), we can say that \( \mathcal{F} \) contains either (i) \((S,1,c,d)\), or (ii) \((S,1,3,c)\).

Subcase 3.2.8.(i). Suppose that \( \mathcal{F} \) contains \((S,1,c,d)\). Then

\[ \mathcal{F} \subseteq \{ (S,1,2,3), (S,1,a,b), (S,1,a,c), (S,1,a,d), (S,2,a,c), (S,1,c,d) \} \]

and if \( u \) is a remaining vertex in \( \mathcal{F} \), then \( u \) is one of the following:

(2) \( u = (S,2,b,c) \)

(3)' \( u = (S,3,a,c) \)

(4) \( u = (S,3,b,c) \)

(5)' \( |u \cap P| = k-2, |u \cap S| = k-4, \{1,2,a,c\} \subseteq u \)

(5)'' \( |u \cap P| = k-2, |u \cap S| = k-4, \{1,2,a,d\} \subseteq u \)

(6) \( |u \cap P| = k-2, |u \cap S| = k-4, \{1,2,b,c\} \subseteq u \)

(7) \( u \cap P = S \cup \{1,2\} \)

(8)' \( u = (S,1,3,a) \)
(9) $u = (S,1.3,c)$.

If $\mathcal{F}$ contains a vertex satisfying $(5)'$, then $\mathcal{F}$ does not contain $(S,3,b,c)$, but does contain the set $\{u: u \cap P = k-2, u \cap S = k-4, \{1,2,a,c\} \subseteq u\}$, $(S,1,2,a)$ and $(S,1,2,c)$ also. This implies that $\mathcal{F}$ contains all k-subsets of a $(k-1)$ set $S \cup \{1,2,a,c\}$, contradiction. Therefore, $\mathcal{F}$ can not contain any vertex satisfying $(5)'$, but does contain $(S,3,b,c)$. Then it is isomorphic to subcase 3.2.5. or subcase 3.2.7. (Note that $\mathcal{F}$ contains $\{(S,1,a,c), (S,1,c,d), (S,3,c,b)\}$.)

Subcase 3.2.8.(ii). Next let us suppose that $\mathcal{F}$ contains $(S,1.3,c)$, but does not contain $(S,1,c,d)$. Then

$$\mathcal{F} \supseteq \{(S,1,2,3), (S,1,a,b), (S,1,a,c), (S,2,a,c), (S,1,3,c)\}.$$

If $u$ is a vertex of $\mathcal{F}$ which is different from above, then $u$ is one of the following:

(1)' $u = (S,1,c,b)$
(2) $u = (S,2,b,c)$
(3)' $u = (S,3,a,c)$
(3)'' $u = (S,3,a,b)$
(4) $u = (S,3,b,c)$
(5)' $|u \cap P| = k-2, |u \cap S| = k-4, \{1,2,a,c\} \subseteq u$
(6) $u \cap P = k-2, u \cap S = k-4, \{1,2,b,c\} \subseteq u$
(7) $u \cap P = S \cup \{1,2\}$
If $\mathcal{F}$ contains a vertex satisfying $(5)'$, then $\mathcal{F}$ does not contain $(S,3,a,b)$ and $(S,3,b,c)$, but does contain \{u: |u\cap P| = k-2, |u\cap S| = k-4, \{1,2,a,c\}\subseteq u\}, (S,1.2.a) and (S,1.2.c). This implies $\mathcal{F}$ contains all $k$-subsets of a $(k+1)$ set $S\cup\{1.2,a,c\}$, contradicting the definition of $P$. Therefore, $\mathcal{F}$ does not contain any vertex satisfying $(5)'$, but does contain either $(S,3,a,b)$ or $(S,3,b,c)$.

Suppose that $\mathcal{F}$ contains $(S,3,a,b)$. Then

$\mathcal{F} \supseteq \{(S,1.2.3), (S,1.a.b), (S,1.a.c), (S,2.a.c), (S,1,3,c), (S,3,a,b)\}$.

If $u$ is a remaining vertex of $\mathcal{F}$, then $u$ satisfies one of the following:

1. $u = (S,1.c,b)$
2. $u = (S,2.b.c)$
3. $u = (S,3.a.c)$
4. $u = (S,3,b,c)$
5. $u = (S,1.2.b)$
6. $|u\cap P| = k-1, a\subseteq u$.

Without loss of generality, we may assume that $\mathcal{F}$ does not contain $(S,3,a,c)$. (If $\mathcal{F}$ does, it is isomorphic to subcase 3.2.6.) So, $\mathcal{F}$ contains $(S,1.2.b)$ and

$\mathcal{F} \supseteq \{(S,1.2.3), (S,1.a.b), (S,1.a.c), (S,2.a.c), (S,1,3,c), (S,3,a.b), (S,1,2.b)\}$. 

Since the vertices of \( A_{k-2} \) do not share an element in \( \bar{P} \) (a, in this case), \( \mathcal{F} \) contains either \((S,1,c,b)\), \((S,2,b,c)\), or \((S,3,b,c)\). Thus \( \mathcal{F} \) is one of the following:

\[
\mathcal{F} = \{(S,1,2,3), (S,1,a,b), (S,1,a,c), (S,1,3,c), (S,3,a,b), (S,1,2,b)\} \cup \{(S,1,c,b), (S,1,2,a), (S,1,3,a)\} \quad (3.15)
\]

\[
\mathcal{F} = \{(S,1,2,3), (S,1,a,b), (S,1,a,c), (S,2,a,c), (S,1,3,c), (S,3,a,b), (S,1,2,b)\} \cup \{(S,1,c,b), (S,2,b,c), (S,1,2,a)\} \quad (3.16)
\]

\[
\mathcal{F} = \{(S,1,2,3), (S,1,a,b), (S,1,a,c), (S,2,a,c), (S,1,3,c), (S,3,a,b), (S,1,2,b)\} \cup \{(S,1,c,b), (S,3,b,c), (S,1,3,a)\} \quad (3.17)
\]

\[
\mathcal{F} = \{(S,1,2,3), (S,1,a,b), (S,1,a,c), (S,2,a,c), (S,1,3,c), (S,3,a,b), (S,1,2,b)\} \cup \{(S,1,c,b), (S,2,b,c), (S,3,b,c)\} \quad (3.18)
\]

\[
\mathcal{F} = \{(S,1,2,3), (S,1,a,b), (S,1,a,c), (S,2,a,c), (S,1,3,c), (S,3,a,b), (S,1,2,b)\} \cup \{(S,2,b,c), (S,1,2,a)\} \quad (3.19)
\]

\[
\mathcal{F} = \{(S,1,2,3), (S,1,a,b), (S,1,a,c), (S,2,a,c), (S,1,3,c), (S,3,a,b), (S,1,2,b)\} \cup \{(S,2,c,b), (S,3,b,c), (S,2,3,a)\} \quad (3.20)
\]

size: 10

\[
\mathcal{F} = \{(S,1,2,3), (S,1,a,b), (S,1,a,c), (S,2,a,c), (S,1,3,c), (S,3,a,b), (S,1,2,b)\} \cup \{(S,2,b,c), (S,1,3,a), (S,2,3,a)\}. \quad (3.21)
\]

But \( \mathcal{F} \) in (3.15) is isomorphic to the one in (3.11). And \( \mathcal{F} \) in (3.16) or (3.17) is isomorphic to the \( \mathcal{F} \) in (3.4). Also, \( \mathcal{F} \) in (3.18), (3.19) or (3.21) is isomorphic to the one in (3.7). But we obtain a new type of maximal 2-clique in this subcase. (3.20), which is isomorphic to (L.15).
Suppose that $\mathcal{F}$ does not contain $(S,3,a,b)$ but does contain $(S,3,b,c)$. Then

$$\mathcal{F} \supseteq \{(S,1.2,3), (S,1,a,b), (S,1,a,c), (S,2,a,c), (S,1.3,c), (S,3,b,c)\}.$$ 

If $u$ is a remaining vertex of $\mathcal{F}$, then $u$ is one of the following:

1. $u = (S,1,c,b)$
2. $u = (S,2,b,c)$
3. $u = (S,3,a,c)$
4. $u = (S,2.3,a)$
5. $u = (S,1.2,b)$
6. $u = (S,1.2,c)$
7. $u = (S,1.3,c)$
8. $u = (S,2.3,a)$.  

(3.22)

From (3.22), it is obvious that $\mathcal{F}$ contains $(S,1.2,c)$. Since $\mathcal{F}$ does not contain any vertex $v$ which satisfies that $v^-P = k-2$, $v^-S = k-4$, and $\{1.3,a,c\} \subseteq v$, $\mathcal{F}$ contains either $(S,2,b,c)$ or $(S,1.2,b)$. Also, since $\mathcal{F}$ does not contain any vertex $v$ which satisfies that $v^-P = k-2$, $v^-S = k-4$ and $\{1.2,b,c\} \subseteq v$, $\mathcal{F}$ contains either $(S,3,a,c)$, $(S,1.3,a)$ or $(S,2.3,a)$. Thus $\mathcal{F}$ is one of the following:

$$\mathcal{F} = \{(S,1.2,3), (S,1,a,b), (S,1,a,c), (S,2,a,c), (S,1.3,c), (S,3,b,c), (S,1.2,c)\} \cup \{(S,2,b,c), (S,1.2,b), (S,2.3,a)\}$$  

(3.23)

$$\mathcal{F} = \{(S,1.2,3), (S,1,a,b), (S,1,a,c), (S,2,a,c), (S,1.3,c), (S,3,b,c), (S,1.2,c)\} \cup \{(S,2,b,c), (S,1.2,b), (S,3,a,c)\}$$  

(3.24)

$$\mathcal{F} = \{(S,1.2,3), (S,1,a,b), (S,1,a,c), (S,2,a,c), (S,1.3,c), (S,3,b,c)\}.$$
\[(S,1,2,c) \cup \{(S,2,b,c), (S,3,a,c), (S,2,3.a)\} \quad (3.25)\]

\[\mathcal{F} = \{(S,1,2,3), (S,1,a,b), (S,1,a,c), (S,2,a,c), (S,1,3,c), (S,3,b,c),\}
\]

\[\mathcal{F} = \{(S,1,2,3), (S,1,a,b), (S,1,a,c), (S,2,a,c), (S,1,3,c), (S,3,b,c).\}, 26\]

\[\mathcal{F} = \{(S,1,2,3), (S,1,a,b), (S,1,a,c), (S,2,a,c), (S,1,3,c), (S,3,b,c),
(S,1,2,c) \cup \{(S,1,2,b), (S,1,3,a), (S,2,3,a)\}. \quad (3.27)\]

But \(\mathcal{F}\) in (3.23) is isomorphic to the one in (3.7), and \(\mathcal{F}\) in (3.24) and (3.26) are isomorphic to the one in (3.11). Also, \(\mathcal{F}\) in (3.25) or (3.27) is isomorphic to the one in (3.4).

**Subcase 3.2.9.** Suppose \(\{u \in \mathcal{A}_{k-2}: S \cup \{1.a\} \subseteq u\} = \{(S,1,a,b)\}\) and \(\{u \in \mathcal{A}_{k-2}: S \cup \{2,a\} \subseteq u\} = \{(S,2,a,c)\}.\) Then

\[\mathcal{F} = \{(S,1,2,3), (S,1,a,b), (S,2,a,c)\}.\]

If \(u\) is a vertex of \(\mathcal{F}\) but different from above, then \(u\) is one of the following:

1. \(u \cap P = S \cup \{2\}. b \in u\) but \(a \notin u\) (Without loss of generality, we may assume that \(\mathcal{F}\) contains at most one vertex of this type, either (S.2,b,c) or (S.2,b,d) for some d in \(\bar{P}\setminus\{a,b,c\}\).)

2. \(u \cap P = S \cup \{1\}. c \in u\) but \(a \notin u\) (Without loss of generality, we may assume that \(\mathcal{F}\) contains at most one vertex of this type.)

3. \(u \cap P = S \cup \{3\}, a \in u\)

4. \(u = (S,3,b,c)\)

5. \(|u \cap P| = k-2, |u \cap S| = k-4, \{1,2.a\} \subseteq u\)

6. \(|u \cap P| = k-2, |u \cap S| = k-4, \{1,2.b,c\} \subseteq u\)
(7) $u \cap P = S \cup \{1,2\}$

(8) $|u \cap P| = k-1, a \in u$

(9) $u = (S,2,3,b)$

(10) $u = (S,1,3,c)$.

By comparing with $(S,1,a,c)$ which is not in $\mathcal{F}$, we get that $\mathcal{F}$ contains either $(S,2,b,d)$ or $(S,2,3,b)$.

Subcase 3.2.9.(i). Suppose that $\mathcal{F}$ has $(S,2,b,d)$. Then

$\mathcal{F} \supseteq \{(S,1,2,3), (S,1,a,b), (S,2,a,c), (S,2,b,d)\}$.

If $u$ is a remaining vertex of $\mathcal{F}$, then $u$ is one of the following:

(2)' $u = (S,1,c,b)$

(2)'' $u = (S,1,c,d)$

(3)' $u = (S,2,a,b)$

(3)'' $u = (S,2,a,c)$

(4) $u = (S,3,a,b)$

(5)' $|u \cap P| = k-2, |u \cap S| = k-4, \{1,2,a,b\} \subseteq u$

(5)'' $|u \cap P| = k-2, |u \cap S| = k-4, \{1,2,a,d\} \subseteq u$

(6) $|u \cap P| = k-2, |u \cap S| = k-4, \{1,2,b,c\} \subseteq u$

(7) $u \cap P = S \cup \{1,2\}$

(8)' $u = (S,2,3,a)$

(9) $u = (S,2,3,b)$.
Since $\mathcal{F}$ does not contain $(S,2,a,b)$, it contains $(S,1,c,d)$. Then

$$\mathcal{F} \supseteq \{(S,1,2,3), (S,1,a,b), (S,2,a,c), (S,2,b,d), (S,1,c,d)\}.$$ 

If $u$ is a remaining vertex of $\mathcal{F}$, then $u$ is one of the following:

1. $u = (S,3,a,d)$
2. $u = (S,3,b,c)$
3. $|u \cap P| = k-2, |u \cap S| = k-4, \{1,2,a,d\} \subseteq u$
4. $|u \cap P| = k-2, |u \cap S| = k-4, \{1,2,b,c\} \subseteq u$
5. $u \cap P = S \cup \{1,2\}$

Since $\mathcal{F}$ does not contain $(S,2,a,d)$, $\mathcal{F}$ contains either $(S,3,b,c)$ or a vertex satisfying (6). Also, $\mathcal{F}$ contains either $(S,3,a,d)$ or a vertex satisfying (5). Since $\mathcal{F}$ does not have $(S,1,c,b)$, thus we get two types of maximal 2-cliques:

$$\mathcal{F} = \{(S,1,2,3), (S,1,a,b), (S,2,a,c), (S,2,b,d), (S,1,c,d)\}$$

- $\cup\{(S,3,b,c), (S,3,a,d)\}$
  - size: 7

$$\mathcal{F} = \{(S,1,2,3), (S,1,a,b), (S,2,a,c), (S,2,b,d), (S,1,c,d)\}$$

- $\cup\{(T,1,2,b,c), (T,1,2,a,d)\} \cup \{u: u \cap P = S \cup \{1,2\}\}$
  - for some $(k-1)$-subset $T$ of $S$
  - size: $n-k-7$

which are isomorphic to (L.16) and (L.9), respectively. Note that if $n = k-3$, there is no maximal 2-clique of type (3.28) or type (3.29).
Subcase 3.2.9.(ii). Suppose that $\mathcal{F}$ contains $(S.2,3,b)$ but does not contain $(S.2,b,d)$. Then

$$\mathcal{F} \subseteq \{(S,1,2,3), (S,1,a,b), (S,2,a,c), (S,2,3,b)\}.$$

If $u$ is a remaining vertex of $\mathcal{F}$, then $u$ satisfies one of the following:

(1)' $u = (S,2,b,c)

(2)' $u = (S,1,c,b)

(3) $u^{-P} = S \cup \{3\}, a \in u$ (For this subcase 3.2.9.(ii), without loss of generality, we may assume that $\mathcal{F}$ contains at most one vertex of this type, either $(S,3,a,b)$, $(S,3,a,c)$ or $(S,3,a,e)$ for some $e$ in $\tilde{P} \{a,b,c\}$.)

(4) $u = (S,3,b,c)

(5)' $u^{-P} = k-2, u^{-S} = k-4, \{1,2,a,b\} \subseteq u

(6) $u^{-P} = k-2, u^{-S} = k-4, \{1,2,b,c\} \subseteq u

(7) $u^{-P} = S \setminus \{1,2\}

(8) $u^{-P} = k-1, a \in u

(10) $u = (S,1,3,c).

Since $\mathcal{F}$ does not contain $(S,2,a,b)$, $\mathcal{F}$ contains $(S,1,3,c)$. Then

$$\mathcal{F} \subseteq \{(S,1,2,3), (S,1,a,b), (S,2,a,c), (S,2,3,b), (S,1,3,c)\}.$$

So, $\mathcal{F}$ can not contain any vertex satisfying $(5)'$. Since the vertices of $A_{k-2}$ can not share an element in $\tilde{P}$ (a, in this case), $\mathcal{F}$ contains either $(S,2,b,c)$, $(S,1,c,b)$, $(S,3,b,c)$ or $(T,1,2,b,c)$ for a $(k-4)$-subset $T$ of $S$. First, let us suppose that $\mathcal{F}$ contains $(S,2,b,c)$. Then
\( \mathcal{F} \supseteq \{(S,1.2.3), (S,1.a.b), (S,2.a,c), (S,2.3.b), (S,1.3,c), (S,2.b,c)\}. \tag{3.30} \)

Without loss of generality, we may again assume that \( \mathcal{F} \) does not contain \((S,1.c,b), (S,3.a,c)\) and \((S,3.b.c)\). (Otherwise, it is isomorphic to either subcase 3.2.5., subcase 3.2.6. or subcase 3.2.8.) If \( u \) is a remaining vertex of \( \mathcal{F} \), then it is one of the following:

\(3)\ ' u = (S,3,a,b)\)

\(6)\ ' |u \cap P| = k-2, |u \cap S| = k-4, \{1,2,b,c\} \subseteq u\)

\(7)\ ' u \cap P = S \cup \{1,2\}\)

\(8)\ ' u = (S,2,3,a)\).

Thus, one of the following holds:

\[ \mathcal{F} = \{(S,1.2.3), (S,1.a.b), (S,2.a,c), (S,2.3.b), (S,1.3.c), (S,2.b,c)\} \]

\[ \cup \{(S,3.a.b), (S,1.2.a), (S,1.2.b), (S,2.3.a)\}\]  \( \tag{3.31} \)

\[ \mathcal{F} = \{(S,1.2.3), (S,1.a.b), (S,2.a,c), (S,2.3.b), (S,1.3.c), (S,2.b,c)\} \]

\[ \cup \{u: u \cap P = S \cup \{1,2\}\}\]  \( \tag{3.32} \)

\[ \mathcal{F} = \{(S,1.2.3), (S,1.a.b), (S,2.a,c), (S,2.3.b), (S,1.3.c), (S,2.b,c)\} \]

\[ \cup \{u: u \cap P = S \cup \{1,2\}\} \setminus \{(S,2.3.a)\}. \]  \( \tag{3.33} \)

But the maximal 2-clique \( \mathcal{F} \) in (3.31) (or 3.33) is isomorphic to the one in (3.4) (or (3.2)). And the one in (3.32) is not maximal because of \((S,1.c,b)\).
Next let us suppose that $\mathcal{F}$ contains $(S,1,b,c)$. Then

$$\mathcal{F} \supseteq \{(S,1.2.3), (S,1,a,b), (S,2,a,c), (S,2.3,b), (S,1.3,c), (S,1,b,c)\}$$

which is isomorphic to $(3.30)$.

Suppose $\mathcal{F}$ contains $(S,3,b,c)$ but does not contain $(S,2,b,c)$ and $(S,1,c,b)$. Then

$$\mathcal{F} \supseteq \{(S,1.2.3), (S,1,a,b), (S,2,a,c), (S,2.3,b), (S,1.3,c), (S,3,b,c)\}.$$ 

Without loss of generality, we may assume that $\mathcal{F}$ does not contain $(S,3,a,b)$ and $(S,3,a,c)$. (If $\mathcal{F}$ does, it is isomorphic to either subcase 3.2.5., subcase 3.2.6. or subcase 3.2.8.)

If $u$ is a remaining vertex of $\mathcal{F}$, then it is one of the following:

$(3)''$ $u = (S,3,a,e)$

$(7)'$ $u = (S,1.2,b)$

$(7)''$ $u = (S,1,c)$

$(8)'$ $u = (S,2.3,a)$

$(8)''$ $u = (S,1.3,a)$.

Since $\mathcal{F}$ does not have $(S,3,a,b)$, $\mathcal{F}$ contains $(S,1.2,c)$. So, $\mathcal{F}$ can not have $(S,3,a,c)$ and we get

$$\mathcal{F} = \{(S,1.2.3), (S,1,a,b), (S,2,a,c), (S,2.3,b), (S,1.3,c), (S,3,b,c)\}$$

$$\cup \{(S,1.2,b), (S,1.2,c), (S,2.3,a), (S,1.3,a)\}$$

which is isomorphic to $(3.7)$. 

Finally, let us assume that $\mathcal{F}$ contains $(T,1,2,b,c)$ but does not contain $(S,2,b,c)$, $(S,1,c,b)$ and $(S,3,b,c)$. Then

$$\mathcal{F} \supseteq \{(S,1,2,3), (S,1,a,b), (S,2,a,c), (S,2,3,b), (S,1,3,c), (T,1,2,b,c)\}.$$ 

If $u$ is a remaining vertex of $\mathcal{F}$, then $u$ is one of the following:

(6) $|u \cap P| = k-2$, $|u \cap S| = k-4$, $\{1,2,b,c\} \subseteq u$

(7) $u \cap P = S \cup \{1,2\}$

(8) $u = (T,1,2,3,a)$.

Since $\mathcal{F}$ does not contain $(S,1,b,c)$, $\mathcal{F}$ should contain $(T,1,2,3,a)$. Therefore, we get

$$\mathcal{F} = \{(S,1,2,3), (S,1,a,b), (S,2,a,c), (S,2,3,b), (S,1,3,c), (T,1,2,b,c)\}$$

which is isomorphic to $(3.29)$.

Finally, let us consider the case $\min\{|u \cap P|: u \in \mathcal{F}_{k-2}\} = k-4$.

Subcase 3.3. Suppose that $\min\{|u \cap P|: u \in \mathcal{F}_{k-2}\} = k-4$. Then, there are two vertices $x$ and $y$ such that $|x \cap y \cap P| = k-4$. So, we may put $x \cap y \cap P = T$, $P = T \cup \{1,2,3,4\}$, $x = (T,1,2,a,b)$ and $y = (T,3,4,a,b)$ for some $(k-4)$-subset $T$ of $P$. If $v$ is a vertex in $\mathcal{F}_{k-1}$, $v$ contains either $a$ or $b$.

And if $u$ is a vertex in $\mathcal{F}_{k-2}$, then $u$ is one of the following:

(i) $T \subseteq u$, $|u \cap \{1,2\}| = |u \cap \{3,4\}| = 1$ and $|u \cap \{a,b\}| \geq 1$
(ii). $|u \cap T| = k-5$, $|u \cap \{1,2,3,4\}| = 3$, $\{a,b\} \subseteq u$

(iii). $|u \cap T| = k-6$, $\{1,2,3,4,a,b\} \subseteq u$.

Notice that two vertices $u$ and $v$ are 2-adjacent whenever $u$ and $v$ satisfy either (ii) or (iii). And $\mathcal{A}_{k-2}$ has two vertices $w$ and $z$ such that $T \subseteq w$, $a \in w$, but $b \notin w$, and $T \subseteq z$, $a \notin z$ but $b \in z$. Otherwise, the vertices of $\mathcal{A}_{k-2}$ share an element in $\tilde{P}$ (a or $b$, in this case).

Let us fix $w = (T,1,3,a,c)$. Then, without loss of generality, we may assume that $z$ is either $(T,1,3,b,c)$, $(T,1,3,b,d)$ or $(T,1,4,b,c)$. We will discuss by three subcases according to $z$.

**Subcase 3.3.1.** Suppose that $z = (T,1,3,b,c)$. Then

$$\mathcal{F} \supseteq \{(T,1,2,3,4), (T,1,2,a,b), (T,3,4,a,b), (T,1,3,a,c), (T,1,3,b,c)\}.$$ 

Then if $u$ is a remaining vertex of $\mathcal{F}$, then $u$ is one of the following:

1. $u = (T,1,3,a,c)$
2. $u = (T,1,4,a,b)$
3. $u = (T,1,4,b,c)$
4. $u = (T,2,3,a,b)$
5. $u = (T,2,3,a,c)$
6. $u = (T,2,3,b,c)$
7. $u = (T,2,3,a,b) \subseteq u$
8. $u = (T,3,4,a,b) \subseteq u$
9. $u \subseteq T = k-5$, $\{1,3,2,a,b\} \subseteq u$
10. $|u \cap T| = k-5$, $\{1,3,4,a,b\} \subseteq u$
(11) \( u = (T, 1, 3, 2, a) \)

(12) \( u = (T, 1, 3, 4, a) \)

(13) \( u = (T, 1, 3, 2, b) \)

(14) \( u = (T, 1, 3, 4, b) \). \( (3.34) \)

If \( \mathcal{E} \) contains \( \{u: |u \cap T| = k-5, \{1,3,2,a\} \subseteq u\} \), then \( \mathcal{E} \) cannot contain \( (T, 1, 4, a, c) \) and \( (T, 1, 4, b, c) \). So, \( \mathcal{E} \) contains \( (T, 2, 3, a, b), (T, 1, 3, a, b), (T, 1, 3, 2, a) \) and \( (T, 1, 3, 2, b) \), which implies that \( \mathcal{E} \) contains all \( k \)-subsets of a \((k+1)\) set \( T \cup \{1,2,3,a,b\} \). Contradicting the definition of \( P \). Therefore, \( \mathcal{E} \) does not contain \( \{u: |u \cap T| = k-5, \{1,3,2,a\} \subseteq u\} \). Among the vertices satisfying one of \((3.34)\), only \( (T, 1, 4, a, c) \) and \( (T, 1, 4, b, c) \) are not 2-adjacent to any vertex of the set \( \{u: |u \cap T| = k-5, \{1,3,2,a\} \subseteq u\} \). So \( \mathcal{E} \) contains either \( (T, 1, 4, a, c) \) or \( (T, 1, 4, b, c) \).

Similarly, we can show that \( \mathcal{E} \) does not contain \( \{u: |u \cap T| = k-5, \{1,3,4,a\} \subseteq u\} \) but does contain either \( (T, 2, 3, a, c) \) or \( (T, 2, 3, b, c) \). Therefore, \( \mathcal{E} \) contains either \( \{(T, 1, 4, a, c), (T, 2, 3, a, c)\} \) or \( \{(T, 1, 4, b, c), (T, 2, 3, b, c)\} \). Without loss of generality, we may assume that \( \mathcal{E} \) contains \( \{(T, 1, 4, a, c), (T, 2, 3, a, c)\} \). Thus,

\[
\mathcal{E} = \{(T, 1, 2, 3, 4), (T, 1, 2, a, b), (T, 3, 4, a, b), (T, 1, 3, a, c), (T, 1, 3, b, c)\} \\
\cup \{(T, 1, 4, a, c), (T, 2, 3, a, c)\} \cup \{(T, 1, 3, 2, a), (T, 1, 3, 4, a)\}
\]

which is isomorphic to the 2-clique in \((3.29)\).
Subcase 3.3.2. Suppose that \( z = (T,1,3,b,d) \). Then
\[
\mathcal{F} \supseteq \{(T,1,2,3,4), (T,1,2,a,b), (T,3,4,a,b), (T,1,3,a,c), (T,1,3,b,d)\}.
\]

If \( u \) is a remaining vertex of \( \mathcal{F} \), then \( u \) is one of the following:

1. \( u \cap P = T \cup \{1,3\}, a \in u \)
2. \( u \cap P = T \cup \{1,3\}, b \in u \)
3. \( u = (T,1,4,a,b) \)
4. \( u = (T,1,4,a,d) \)
5. \( u = (T,1,4,b,c) \)
6. \( u = (T,2,3,a,b) \)
7. \( u = (T,2,3,a,d) \)
8. \( u = (T,2,3,b,c) \)
9. \( u \cap T = k-5, \{1,2,3,a,b\} \subseteq u \)
10. \( u \cap T = k-5, \{1,3,4,a,b\} \subseteq u \)
11. \( u = (T,1,3,2,a) \)
12. \( u = (T,1,3,4,a) \)
13. \( u = (T,1,3,2,b) \)
14. \( u = (T,1,3,4,b) \).

By a similar argument as in subcase 3.3.1., we can easily show that \( \mathcal{F} \) does not contain \( \{u: u \cap T = k-5, \{1,2,3,a,b\} \subseteq u\} \) and \( \{u: u \cap T = k-5, \{1,3,4,a,b\} \subseteq u\} \), but \( \mathcal{F} \) contains, without loss of generality, \( \{(T,1,4,a,d), (T,2,3,a,d)\} \). So, we get
\[ \mathcal{F} = \{(T,1,2,3,4), (T,1,2,a,b), (T,3,4,a,b), (T,1,3,a,c), (T,1,3,b,d)\} \]
\[ \cup \{(T,1,4,a,d), (T,2,3,a,d)\} \cup \{u: u \cap P = T \cup \{1,3\}, a \in u\} \]
\[ \cup \{(T,1,2,3,a), (T,1,3,4,a)\} \]

which is isomorphic to the one in (3.35).

Subcase 3.3.3. Finally, let us suppose that \( z = (T,1,4,b,c) \). Then

\[ \mathcal{F} \supseteq \{(T,1,2,3,4), (T,1,2,a,b), (T,3,4,a,b), (T,1,3,a,c), (T,1,4,b,c)\}. \]

If \( u \) is a remaining vertex of \( \mathcal{F} \), then \( u \) is one of the following:

1. \( u \cap P = T \cup \{1,3\}, b \in u \)
2. \( u \cap P = T \cup \{1,4\}, a \in u \)
3. \( u = (T,2,3,b,c) \)
4. \( u = (T,2,4,a,c) \)
5. \( iu \cap T = k-5, \{1,3,4,a,b\} \subseteq u \)
6. \( u = (T,1,2,4,a) \)
7. \( u = (T,1,3,4,a) \)
8. \( u = (T,1,2,3,b) \)
9. \( u = (T,1,3,4,b) \).

By a similar argument as in subcase 3.3.1., we can show that \( \mathcal{F} \) does not contain \( \{u: iu \cap T = k-5, \{1,3,4,a,b\} \subseteq u\} \) but \( \mathcal{F} \) contains either \( (T,2,3,b,c) \) or \( (T,2,4,a,c) \). Without loss of generality, we may assume that \( \mathcal{F} \) contains \( (T,2,3,b,c) \). Then

\[ \mathcal{F} \subseteq \{(T,1,2,3,4), (T,1,2,a,b), (T,3,4,a,b), (T,1,3,a,c), (T,1,4,b,c)\} \]
\[ \cup \{(T,2,3,b,c)\} \cup \{u: u \cap P = T \cup \{1,3\}, b \in u\} \]
\[ \cup \{(T,2,4,a,c), (T,1,2,3,b), (T,1,3,4,b)\}. \]

So, \(\mathcal{F}\) is one of the following:

\[
\mathcal{F} = \{(T,1,2,3,4), (T,1,2,a,b), (T,3,4,a,b), (T,1,3,a,c), (T,1,4,b,c)\}
\cup \{\{(T,2,3,b,c)\} \cup \{u: u \cap \mathcal{P} = T \cup \{1,3\}, b \in u\}
\cup \{(T,1,2,3,b), (T,1,3,4,b)\} \quad (3.36)
\]

\[
\mathcal{F} = \{(T,1,2,3,4), (T,1,2,a,b), (T,3,4,a,b), (T,1,3,a,c), (T,1,4,b,c)\}
\cup \{(T,2,3,b,c)\} \cup \{(T,2,4,a,c)\} \quad (3.37)
\]

size: 7.

The maximal 2-clique \(\mathcal{F}\) in (3.36) is isomorphic to the one in (3.35). But we get a new type of maximal 2-clique, (3.37), which is isomorphic to (L.17). []

Remark: (1). If \(e = f\) in (L.10), then (L.10) is isomorphic to (L.11).

(2). If \(n = k + 3\), the maximal 2-cliques in (L.9), (L.10) and (L.16) do not occur.
2.3 Maximal t-cliques of J(n,k) with size at least cn^{t-1}

**Theorem 2.3.1.** Let k and t be fixed integers satisfying 3 ≤ t < k. For a fixed constant c > 0, there exists \( n_0 = n_0(k,t,c) \) such that if \( n > n_0 \) and \( \mathcal{F} \) is a maximal t-clique of the Johnson graph J(n,k) satisfying \( |\mathcal{F}| > cn^{t-1} \), then \( \mathcal{F} \) is one of the following up to isomorphism:

(J.1). \( \mathcal{F} \) is the set of all vertices which contain a fixed (k-t)-set. This is the largest maximal t-clique and \( |\mathcal{F}| = \binom{n}{t} - \binom{k-t}{t} - \cdots - \binom{k-t}{t} + \theta(n^t) \).

(J.2). For each \( r = 2, \ldots, t-1 \),

\[
|\mathcal{F}| = \binom{r}{1} \binom{n-(k-t)}{t-1-r} - \binom{r}{2} \binom{n-(k-t)}{t-2-r} - \cdots
- \binom{r}{t} \binom{n-(k-t)}{t} - \binom{k-t}{t} \binom{n}{t}.
\]

\[
= \begin{cases} 
\frac{r}{(t-1)!} n^{t-1} + o(n^{t-1}) & \text{if } r > 2 \\
\frac{k-t}{(t-1)!} n^{t-1} + o(n^{t-1}) & \text{if } r = 2.
\end{cases}
\]

(J.3). For each \( r = 2, 3, \ldots, t \),

\[
\mathcal{F} = \{ u \in \binom{X}{k} : u \not\supseteq Q, u \cap P_r \neq \emptyset \} \cup \mathcal{D} \cup \mathcal{E}
\]

where \( Q \): a (k-t)-set,

\( P_r \): an r-set and \( Q \cap P_r = \emptyset \),

\( \emptyset \neq \mathcal{D} \subseteq \{ u \in \binom{X}{k} : u \supseteq Q, u \cap P_r = \emptyset \} \),

\( \emptyset \neq \mathcal{E} \subseteq \{ u \in \binom{X}{k} : |u \cap Q| = k-t-1, u \supseteq P_r \} \),

\( \cap \mathcal{E} = P_r \).
and $\mathcal{D} \cup \mathcal{E}$ is maximal among the sets of vertices of $J(n,k)$ which satisfy
the above condition. In this case,

$$|\mathcal{D}| \leq \binom{t}{1} \binom{k-t}{1}^{t-1} \binom{n-(k-t)}{t-2} - 2,$$

$$|\mathcal{E}| \leq \sum_{r=1}^{t-2} \binom{k-t}{r-1} \binom{n-(k-t)}{t-r} - 1,$$

$$|\mathcal{F}| = \binom{r}{1} \binom{n-(k-t)}{t-1-r} + \binom{r}{2} \binom{n-(k-t)}{t-2-r} + \ldots + \binom{r}{r} \binom{n-(k-t)}{t-r}$$

$$+ |\mathcal{D}| + |\mathcal{E}|$$

$$= \binom{r}{t-1} n^{t-1} + o(n^{t-1}).$$

\hspace{1cm} (J.4)

$\mathcal{F} = \{ u \in \binom{X}{k} : u \supseteq R \} \cup \{ u \in \binom{X}{k} : |u \cap R| = k-t, u \setminus R \in \mathcal{F}' \}$

where $R$ is a $(k-t+1)$-set, and $\mathcal{F}'$ is a maximal $(t-1)$-clique in $J(n-(k-t+1), t)$ which is not the largest one (that is $\cap \mathcal{F}' = \emptyset$ and $|\mathcal{F}'| = o(n^{t-1})$.)

In this case,

$$|\mathcal{F}| = \binom{n-(k-t+1)}{t-1} + \binom{k-t+1}{n-t} |\mathcal{F}'| = \frac{1}{|\mathcal{F}'|} n^{t-1} + o(n^{t-1}).$$

\hspace{1cm} (J.5)

Let $R$ be a $(k-t+1)$-subset of $X$, then

$$\mathcal{F} = \{ u \subset \binom{X}{k} : u \supseteq R \} \cup K$$

where $K = K^1 \cup \ldots \cup K^s$ for some $2 \leq s \leq k-t+1$ such that:

(i). for each $j = 1, 2, \ldots, s$, $|(\cap K^j) \cap R| = k-t,$

(ii). every pair of vertices $u$ and $v$ in $K$, $|u \cap v| \geq k-t,$

(iii). if $i \neq j$, $\cap K^i \neq \cap K^j,$

(iv). there is at least one $j$ such that $K^j$ has two vertices $u$ and $u'$ which satisfy $|u^\ast u'| = k-t,$

(v). for every $j$, $\cap \{ u \setminus R : u \in K^i \text{ for some } i \neq j \} = \emptyset,$

and $K$ is maximal among the sets of vertices of $J(n,k)$ which satisfy the above condition. In this case,
For the proof of the theorem 2.3.1., we need some definitions and a lemma. If we fix a vertex $z$ in a maximal $t$-clique $\mathcal{F}$, then

$$k-t \leq \mu_{z} ;$$

for every vertex $u$ in $\mathcal{F}$. Let us define $\mathcal{B}_{i}$ for $i = k-t, k-t+1, \ldots, k$ by

$$\mathcal{B}_{i} = \{ u \in \mathcal{F} : \mu_{z} = i \}.$$

**Lemma 2.3.1.**

1. $\mathcal{F} = \mathcal{B}_{k-t} \cup \mathcal{B}_{k-t-1} \cup \ldots \cup \mathcal{B}_{k}.$
2. $|\mathcal{B}_{i}| \leq \binom{k}{i} \binom{n-k}{k-i} = O(n^{k-i})$

for $i = k-t, k-t+1, \ldots, k$. In other words,

$$|\mathcal{B}_{k-t+j}| \leq \binom{k}{j} \binom{n-k}{j} = O(n^{j})$$

for $j = 0, 1, \ldots, t$.

**Proof of theorem 2.3.1.:** Let us fix a constant $c > 0$, a vertex $z$ in $\mathcal{F}$ and define $\mathcal{B}_{i}, k-t \leq i \leq k$, as before. By lemma 2.3.1. (2), $\sum_{j=t}^{\infty} |\mathcal{B}_{k-t-j}| = O(n^{t-2})$. Hence we can show there exists $n_{1} = n_{1}(k,t,c)$ such that if $n > n_{1}$,

$$2 \sum_{j=t}^{\infty} |\mathcal{B}_{k-t+j}| < 2n^{t-1}.$$
Assume that $n > n_1$. Then either $B_{k-t+1}$ or $B_{k-t}$ has size at least $\frac{\xi}{4}n^{t-1}$.
We will divide into two cases according to $|B_{k-t+1}|$ and $|B_{k-t}|$.

Case 1. Assume that $|B_{k-t+1}| > \frac{\xi}{4}n^{t-1}$. Then
(1.1). there exists a $(k-t-1)$-subset $R$ of $z$ such that
$$\{|u \in B_{k-t+1}: R \subseteq u\} > \frac{\xi}{4}(\frac{k}{k-t+1})^{-1}n^{t-1}$$
since there are only $(\frac{k}{k-t+1})$ distinct $(k-t+1)$-subsets of $z$.

(1.2). If $R$ is a $(k-t-1)$-subset of $z$ such that $|v \cap R| < k-t$ for some vertex $v$ in $T$, then
$$\{|u \in B_{k-t+1}: R \subseteq u\} \leq \binom{n}{t-2}(n-k-1)$$
by comparing with such a vertex $v$. Let us choose $n_2 = n_2(k,t,c) \geq n_1$ so that
$$\binom{n}{t-2}(n-k-1) < \frac{\xi}{4}(\frac{k}{k-t+1})^{-1}n^{t-1}$$
if $n > n_2$.

Now suppose $n > n_2$. By (1.2), if $R$ is a $(k-t-1)$-subset of $z$ as in (1.1), then $k-t \leq |v \cap R|$ for every vertex $v$ of $T$. Hence,
$$\{u \in (X): R \subseteq u\} \subseteq T \subseteq \{u \in (X): k-t \leq |u \cap R|\}.$$
Let $R = \{R_1, R_2, \ldots, R_r\}$ $(1 \leq r)$ be the set of all such $(k-t+1)$-subsets of $z$ as in (1.1). Then, $|R_i \cap R_j| = k-t$ for all $i$ and $j$ with $1 \leq i < j \leq r$. We will divide this case 1 into three subcases according to $|\cap R_j|$:
(i). $|\cap R_j| < k-t$, (ii). $|\cap R_j| = k-t$, (iii). $|\cap R_j| > k-t$. 
Subcase 1.1. Assume that $|nR_j| < k-t$. Then $t \geq 3$. Let us fix two $(k-t+1)$-subsets $R_1$ and $R_2$ in $R$. By assumption, $R$ has a $(k-t+1)$-subset of $R_3$ which does not contain $R_1 \cap R_2$. Since $|R_1 \cap R_2| = |R_1 \cap R_3| = |R_2 \cap R_3|$, $R_3 \subseteq (R_1 \cup R_2)$. Since $k-t \leq |u \cap R_j|$ for every vertex $u$ of $\mathcal{F}$ and for every $j = 1, 2, \ldots, r$, by comparing with $R_1$, $R_2$ and $R_3$,

$$k-t+1 \leq |u \cap (R_1 \cup R_2)|$$

for every vertex $u$ in $\mathcal{F}$. Hence,

$$\mathcal{F} \subseteq \{ u \in \binom{X}{k} : k-t+1 \leq |u \cap (R_1 \cup R_2)| \}. $$

Since $\mathcal{F}$ is maximal,

$$\mathcal{F} = \{ u \in \binom{X}{k} : k-t+1 \leq |u \cap (R_1 \cup R_2)| \} = \{ u \in \binom{X}{k} : k-t+1 \leq |u \cap Q | \}$$

for some $(k-t+2)$-set $Q$, and

$$\mathcal{F} = (k-t+2)(n-\binom{k-1+t}{1})+(n-\binom{k-1+t}{2})$$

$$= \frac{k-t+2}{t-1} n^{t-1} + o(n^{t-1}).$$

which is type $(J.2)$. (Notice that $R = \{k-t-1\}$-subsets of $Q = R_1 \cup R_2$ and $r = k-t+2$.)

Subcase 1.2. Assume that $|nR_j| = k-t$. Then, $2 \leq r \leq t$ and $\cup R_j = k-t+r$.

For every vertex $u$ of $\mathcal{F}$, $k-t \leq u \cap R_j$ for all $j = 1, 2, \ldots, r$, and so $u$ satisfies one of the following:

(i). $R_j \subseteq u$ for some $j$,

(ii). $u \cap (\cup R_j) = \cap R_j$,
(iii). $|u \cap (\cap R_j)| = k-t-1, ((\cup R_j)\setminus (\cap R_j)) \subseteq u.$

Let us partition $\mathcal{F}$ into three subsets $\mathcal{C}, \mathcal{D}$ and $\mathcal{E};$

\[
\mathcal{C} = \{u \in \binom{X}{k}: R_j \subseteq u \text{ for some } j\},
\]

\[
\mathcal{D} = \{u \in \mathcal{F}: u \cap (\cup R_j) = \cap R_j\},
\]

\[
\mathcal{E} = \{u \in \mathcal{F}: |u \cap (\cap R_j)| = k-t-1, ((\cup R_j)\setminus (\cap R_j)) \subseteq u\}.
\]

Note that if $\mathcal{E}$ contains a vertex $u,$ then $\mathcal{E}$ contains all the vertices $v$ such that $|v \cap (\cap R_j)| = k-t-1, ((\cup R_j)\setminus (\cap R_j)) \subseteq v$ and $v \setminus (\cup R_j) = u \setminus (\cup R_j).$ Therefore, $(\cap \mathcal{E}) \cap (\cap R_j) = \phi.$ Also, $(\cap \mathcal{E}) \cap (\cap R_j) = \phi$ by the definition of $\mathcal{R},$ and so $(\cap \mathcal{E}) \cap z = (\cup R_j)\setminus (\cap R_j).$

If $\mathcal{D} = \phi,$ then $\cap \{u \setminus (\cup R_j): u \in \mathcal{E}\} = \phi$ and

\[
\mathcal{F} = \{u \in \binom{X}{k}: R_j \subseteq u \text{ for some } j\}
\]

\[
\cup \{u \in \binom{X}{k}: |u \cap (\cap R_j)| = k-t-1, ((\cup R_j)\setminus (\cap R_j)) \subseteq u\}.
\]

In this case, if $r=2,$ $\mathcal{R}$ becomes the set of all $(k-t+1)$-subsets of $R_1 \cup R_2$ and $|\cap R_j| < k-t,$ a contradiction. So, $r \geq 3.$ We can also rephrase $\mathcal{F}$ as follows;

\[
\mathcal{F} = \{u \in \binom{X}{k}: Q \subseteq u, u \cap P_r \neq \phi\}
\]

\[
\cup \{u \in \binom{X}{k}: |u \cap Q| = k-t-1, P_r \subseteq u\}
\]

where $Q$ and $P_r$ are a $(k-t)$-subset and an $r$-subset of $z,$ respectively, and $Q \cap P_r = \phi.$ Hence,

\[
|\mathcal{F}| = \binom{n}{t} \left(\binom{n-(k-t)-r}{t-1} + \binom{n-(k-t)-r}{t-2} + \ldots \right) + \binom{r}{t} \left(\binom{n-(k-t)-r}{t-r} + \binom{k-t-1}{t-r} \right)
\]

\[
= \binom{n}{t-r} n(t-1) + o(n^{t-1}), \quad 3 \leq r \leq t.
\]
which is type (J.2).

If \( \mathcal{E} = \emptyset \),
\[
\mathcal{F} \subseteq \{ u \in (X^k) : R_j \subseteq u \text{ for some } j \} \cup \{ u \in (X^k) : u \cap (\cup R_j) = \cap R_j \}
\]
\[
= \{ u \in (X^k) : (\cap R_j) \subseteq u \}.
\]
Therefore, \( \mathcal{F} \) is nothing but the set of vertices which contain a \((k-t)\)-subset \((= \cap R_j)\) and \( |\mathcal{F} = (n^k)^t \), the largest one (type (J.1)).

From now on, let us assume \( \mathcal{D} \neq \emptyset \) and \( \mathcal{E} \neq \emptyset \). If we define \( H = \cap \{ u \setminus z : u \in \mathcal{E} \} \), then \( 0 \leq |H| \leq t+1-r \). Notice that if \( H \neq \emptyset \), then \( \mathcal{F} \) contains the set \( \{ u \in (X^k) : (\cap R_j) \subseteq u, u \cap H \neq \emptyset \} \).

If \( 1 \leq |H| < t+1-r \), we can show that \( \mathcal{F} \) is isomorphic to a maximal \( t \)-clique with \( \mathcal{R} = r+|H| \) by choosing a vertex \( w = (\cup R_j) \cup H \cup K \) instead of \( z \) where \( K \) is a \((t-r-H)\)-subset of \( z \setminus (\cup R_j) \). (In fact, if we choose \( w \) instead of \( z \), then
\[
\mathcal{F} \cong \{ u \in (X^k) : R_j \subseteq u \text{ for some } j=1,2,...,r-H \}
\]
where \( H = \{ a_1, a_2, ..., a_{|H|} \} \) and \( R_{r+1} = (R_1 \cup R_2) \cup \{ a_i \} \) for \( i = 1,2,...,|H| \).

Without loss of generality, we may assume that \( |H| = 0 \) or \( t-1-r \). If \( |H| = t-1-r \), then
\[
\mathcal{F} = \{ u \in (X^k) : R_j \subseteq u \text{ for some } j \}
\]
\[
\cup \{ u \in (X^k) : u \cap (\cup R_j) = \cap R_j, u \cap H \neq \emptyset \}.
\[ \{u \in \binom{X}{k} : |u \cap (\bigcup R)\cap (\cap R)\} \subseteq u \}
\]
\[ = \{u \in \binom{X}{k} : Q \subseteq u, u \cap P \neq \emptyset \} \cup \{u \in \binom{X}{k} : |u \cap Q| = k-t, P \subseteq u \}
\]
where \( Q \) and \( P \) are a \((k-t)\)-set and a \((t+1)\)-set, respectively, and \( Q \cap P = \emptyset \). Hence,
\[
|\mathcal{F}| = \binom{t+1}{1}(n-k-1) + \binom{t+1}{2}(n-k-2) + \ldots + \binom{t+1}{t}(n-k-t) + (k-t)
\]
\[ = \frac{(t+1)!}{(t-1)!}n^{t-1} + o(n^{t-1})
\]
and \( \mathcal{F} \) is type (J.2). (Note that this is of the same type as the ones with \( D = \emptyset \).)

Now, let us assume \(|H| = 0\). For \( Q \) and \( P_r \) which are a \((k-t)\)-set and an \( r \)-set, respectively,
\[
\mathcal{F} = \{u \in \binom{X}{k} : Q \subseteq u, u \cap P_r \neq \emptyset \} \cup \mathcal{D} \cup \mathcal{E}
\]
where \( \phi \neq \mathcal{D} \subseteq \{u \in \binom{X}{k} : Q \subseteq u, u \cap P_r = \emptyset \}, \phi \neq \mathcal{E} \subseteq \{u \in \binom{X}{k} : |u \cap Q| = k-t, P_r \subseteq u \}, \cap \mathcal{E} = P_r.
\]
Since \( \mathcal{F} \) is maximal, \( \mathcal{D} \cup \mathcal{E} \) is maximal among the sets of vertices in \( J(n,k) \)
satisfying above condition. In this case,
\[
\binom{r}{1}(n-(k-t)-r) + \binom{r}{2}(n-(k-t)-r) + \ldots + \binom{r}{t}(n-(k-t)-r)
\]
\[ < |\mathcal{F}| \leq \binom{r}{1}(n-(k-t)-r) + \binom{r}{2}(n-(k-t)-r) + \ldots + \binom{r}{t}(n-(k-t)-r)
\]
\[ + \binom{r+1}{1}(n-k+t-r-2) + (k-t)t(n-k+t-r-1).
\]
Hence, \( |\mathcal{F}| = \frac{r}{(t-1)!}n^{t-1} + o(n^{t-1}), 2 \leq r \leq t, \) and \( \mathcal{F} \) is type (J.3).
Subcase 1.3. Assume that $|\cap R_j| > k-t$. Then $r = 1$ and

$\{u \in (X_k^j): R \subseteq u\} \subseteq \mathcal{F} \subseteq \{u \in (X_k^j): R \subseteq u\} \cup \{u \in (X_k^j): |u \cap R| = k-t\}$

where $R = R_1$. Let us define

$K = \{u \in \mathcal{F}: |u \cap R| = k-t\},$

$\{Q_1, ..., Q_s\} = \{u \cap R: u \in K\},$

$K_i = \{u \in K: Q_i \subseteq u\}, \quad i = 1, 2, ..., s.$

Then, $1 < s \leq k-t+1$, $Q_i \cap Q_j = k-t-1$ for each pair $i$ and $j$ with $1 \leq i < j \leq s$, and

$\mathcal{F} = \{u \in (X_k^j): R \subseteq u\} \cup K_1 \cup K_2 \cup ... \cup K_s.$

(1). Without loss of generality, we may assume that for each $j = 1, 2, ..., s$

$\cap\{u \setminus R: u \in K_i \text{ for some } i \neq j\} = \emptyset.$

Assume that it is not true, i.e., there exists an $i_0 \leq s$ and $\alpha$ such that $\alpha \in \cap\{u \setminus R: u \in K_i \text{ for some } i \neq i_0\}$. Then $\mathcal{F}$ contains all the vertices which contain $Q_{i_0} \cup \{\alpha\}$. By the definition of $R$, $\alpha \notin z$. If we choose a vertex $w = R \cup \{\alpha\} \cup T$ instead of $z$ where $T$ is a $(t-2)$-subset of $z \setminus R$, then

$\{u \in (X_k^j): R \subseteq u \text{ or } Q_{i_0} \cup \{\alpha\} \subseteq u\} \subseteq \mathcal{F}$

which is isomorphic to a maximal $t$-clique in subcase 1.2 or subcase 1.3.

(2). For each $i = 1, 2, ..., s$, put

$L_i^i = \{u \setminus Q_i: u \in K_i\}.$
Then, we may consider $\mathcal{L}^i$'s as sets of vertices in $J(n-k+t-1,t)$. Here, $\partial(\mathcal{L}^i,\mathcal{L}^j) \leq t-1$ for each pair $i$ and $j$ with $1 \leq i < j \leq s$. If $u \in \mathcal{L}^j$ and $|u \cap v| \geq 1$ for every vertex $v$ in $\mathcal{L}^i$, then $u \in \mathcal{L}^j$ for all $i \leq s$. Hence, $u \in \cap\{\mathcal{L}^i: i = 1,2,\ldots,s\}$ and $s = k-t+1$.

(3). If $\mathcal{L}^1 \cup \ldots \cup \mathcal{L}^s$ is a $(t-1)$-clique of $J(n-k+t-1,t)$, then $\mathcal{L}^1 = \mathcal{L}^2 = \ldots = \mathcal{L}^s$. Since $\mathcal{F}$ is maximal in $J(n,k)$, $\mathcal{F}' = \mathcal{L}^1 = \ldots = \mathcal{L}^s$ is a maximal $(t-1)$-clique of $J(n-k+t-1,t)$. But the vertices of $\mathcal{L}^1 \cup \ldots \cup \mathcal{L}^s$ do not contain a fixed element by (1). In this case

$$\mathcal{F} = \{u \in \binom{X}{k}: R \subseteq u \cup \{u \in \binom{X}{k}: |u \cap R| = k-t, u \setminus R \in \mathcal{F}'\}$$

which is the type (J.4), and

$$|\mathcal{F}| = \binom{n-k+t-1}{t-1} + (k-t+1)|\mathcal{F}'|
= \frac{1}{(t-1)!}n^{t-1} - o(n^{t-1}).$$

(Note that $|\mathcal{F}'| = o(n^{t-1})$ by $[10]$.)

(4). If $\mathcal{L}^1 \cup \ldots \cup \mathcal{L}^s$ is a $t$-clique of $J(n-k+t-1,t)$, then

$$\mathcal{F} = \{u \in \binom{X}{k}: R \subseteq u \cup \mathcal{K}^1 \cup \ldots \cup \mathcal{K}^s, \quad 2 \leq s \leq k-t+1,$$

satisfying (i). for each $j = 1,2,\ldots,s$, $|\cap\mathcal{K}^j \cap R| = k-t$,

(ii). $\partial(\mathcal{K}^i,\mathcal{K}^j) \leq t-1$, for every pair $i$ and $j$ with $1 \leq i < j \leq s$.

(iii). $\partial(\mathcal{K}^i,\mathcal{K}^j) = t$ for at least one $j \leq s$.

(iv). for every $j = 1,2,\ldots,s$,

$$\cap\{u \setminus R: u \in \mathcal{K}^i \text{ for some } i \neq j\} = \phi.$$

$\mathcal{F}$ is the type (J.5) in this case and
\[ |K^1 \cup \ldots \cup K^n| \leq (k-t+1)(t-1)(n-k+t-3), \]
\[ |F| = (n-k+1-t-1) + |K^1 \cup \ldots \cup K^n| \]
\[ = \left(\frac{1}{t-1}\right)n^{t-1} + o(n^{t-1}). \]

Case 2. Let us assume \(|B_{k-t+1}| \leq \frac{c}{4}n^{t-1}\) and \(|B_{k-t}| \geq \frac{c}{4}n^{t-1}\). By a similar argument as in case 1, we can show that there exists \(n_3 = n_3(k,t,c) \geq n_1\) such that if \(n > n_3\), then \(z\) has a \((k-t)\)-subset of \(R\) satisfying
\[ \{|u \in B_{k-t-1}: R \subseteq u\}| \geq \frac{c}{4}(k-t)^{-1}n^{t-1} \]
and for every vertex \(v\) of \(F\), \(k-t-1 \leq |v \cap R|\). Hence for such a \((k-t)\)-subset \(R\) of \(z\), we get
\[ F \subseteq \{u \in (\chi^k): R \subseteq u\} \cup \{u \in (\chi^k): |u \cap R| = k-t-1\}. \]

If \(T = \{u \in F: |u \cap R| = k-t-1\} = \emptyset\), then \(F\) is nothing but the set of vertices containing \(R\). So, let us assume that \(\{u \in F: |u \cap R| = k-t-1\} \neq \emptyset\).
If \(T = \cap \{u \in \chi^k: |u \cap R| = k-t-1\} = \emptyset\), then for every \(n > n_4\)
\[ \{|u \in B_{k-t-1}: R \subseteq u\}| \leq (t-1)(t-2)(n-k-2). \]
Therefore, if we choose \(n_4 = n_4(k,t,c) \geq n_3\) so that for every \(n > n_4\)
\[ (t-1)(t-2)(n-k-2) < \frac{c}{4}(k-t)^{-1}n^{t-1}, \]
then \(T \neq \emptyset\) and
\[ F \supseteq \{u \in (\chi^k): R \subseteq u, T \cap u \neq \emptyset\}. \]
By choosing a vertex \(w = R \cup T \cup S\) instead of \(z\) for some \((t-1)|T|\)-subset \(S\) of \(z \setminus R\), we can show that \(F\) is isomorphic to the one in case 1.

\[\square\]
Remark: (1). In type (J.4), if $F'$ is the largest maximal $(t-1)$-clique in $J(n-k-t-1,t)$, then $F$ is isomorphic to the one of type (J.2) with $r=2$.

(2). We can describe maximal $t$-cliques of type (J.3) very explicitly if $r$ is close to $t$. For example, if $r=t$, there are exactly $(t-1)$ non-isomorphic maximal $t$-cliques which depend on $(\mathcal{E})/z$: Let $(\mathcal{E})/z = \{a_1, a_2, \ldots, a_m\}$. Then $2 \leq m \leq t$. For each $m = 2, \ldots, t$,

$$F = \{u \subseteq (X_k) : Q \subseteq u, u \cap P_t = \emptyset\}$$

$$\omega\{u \subseteq (X_k) : Q \subseteq u, u \cap P_t = \emptyset, \{a_1, a_2, \ldots, a_m\} \subseteq u\}$$

$$\omega\{u \subseteq (X_k) : u \cap Q = k-t-1, P_t \subseteq u, a_i \in u \text{ for some } i = 1, 2, \ldots, m\},$$

and

$$F = \left(\binom{n-k}{t-1}\right) - \left(\binom{n-k}{t-2}\right) - \cdots - \left(\binom{n-k}{t-1}\right) + \left(\binom{n-k}{t-1}\right) - (k-t)m.$$ (Notice that if $m=1$, $F$ is isomorphic to the one of type (J.2) with $r=t-1$.)

(3). For the maximal $t$-cliques of type (J.3), we can interpret $D$ and $\mathcal{E}$ as a $t$-clique in $J(n-k+t-1,t-1)$ as follows: Consider $J(n-k-t-1,t-1)$ with underlying set $X' = (X \cup R_j) \cup \{\alpha_1, \ldots, \alpha_{r-1}\}$ where $\alpha_i \in X$ for all $i = 1, 2, \ldots, r-1$. Set

$$D' = \{(u \cup R_j) \cup \{\alpha_1\} : u \in D\}.$$ 

$$\mathcal{E}' = \{(u \cup R_j) \cup \{\alpha_2, \ldots, \alpha_{r-1}\} : u \in \mathcal{E}\}.$$ 

Then, $D'$ and $\mathcal{E}'$ are sets of vertices of $J(n-k+t-1,t+1)$ satisfying that
(i). \( D' = \emptyset, \mathcal{E}' \neq \emptyset, \partial(D', \mathcal{E}') \leq t. \)

(ii). \( a_1 \in D', \{a_2, \ldots, a_{r-1}\} \cap \mathcal{E}' \).

(iii). if \( u' \in D' \), then \( a_i \in u' \) for all \( i \geq 2 \).

(iv). if \( u' \in \mathcal{E}' \), then \( a_1 \in u' \),

where \( \partial(D', \mathcal{E}') \) denotes the maximum distance between \( D' \) and \( \mathcal{E}' \). Since \( \mathcal{F} \) is maximal, \( D' \cup \mathcal{E}' \) is maximal among the sets of vertices of \( J(n, k+t-1, t-1) \) satisfying (i), (ii), (iii) and (iv). Conversely, if we have a set of vertices \( D' \cup \mathcal{E}' \) in \( J(n, k+t-1, t-1) \) which is maximal in the sense of (i), (ii), (iii) and (iv), then we can construct a maximal t-clique of \( J(n, k) \).

Also, for the maximal t-cliques of type (J.5), we can interpret \( \mathcal{K} = \mathcal{K}_1 \cup \ldots \cup \mathcal{K}_s \) as a t-clique in \( J(n, t-1) \): Consider \( J(n, t-1) \) with the underlying set \( X' = (X \setminus R) \cup \{a_1, a_2, \ldots, a_{k-t-1}\} \) where \( a_i \in X \) for all \( i = 1, 2, \ldots, k-t-1 \). Define

\[
\mathcal{M}_i = \{(u R) \cup \{a_i\}: u \in \mathcal{K}_i\}
\]

for each \( i = 1, 2, \ldots, s \). Then, \( \mathcal{M}_i \)'s are sets of vertices of \( J(n, t-1) \) satisfying that

(i). \( \partial(\mathcal{M}_i, \mathcal{M}_j) \leq t \) for each \( i \) and \( j \) with \( 1 \leq i < j \leq s \),

(ii). \( \partial(\mathcal{M}_i, \mathcal{M}_j) = t \) for at least one \( j \geq s \),

(iii). \( u' \cap \{a_1, \ldots, a_{k-t-1}\} = a_i \) for each vertex \( u' \) in \( \mathcal{M}_i \),

(iv). for each \( j \), \( \cap\{u' \in \mathcal{M}_i: i \neq j\} = \emptyset \).
Since \( \mathcal{F} \) is a maximal \( t \)-clique, \( M = M^1 \cup M^2 \cup \ldots \cup M^s \) is maximal among the sets of vertices of \( J(n,t+1) \) satisfying (i), (ii), (iii) and (iv). Conversely, if we have a set of vertices \( M = M^1 \cup M^2 \cup \ldots \cup M^s \) in \( J(n,t+1) \) which is maximal in the sense of (i), (ii), (iii) and (iv), then we can construct a maximal \( t \)-clique of \( J(n,k) \).

(5). From theorem 2.3.1. and remark (2), if \( k > t \geq 3, k > 2t - 3 \) (or \( k \leq 2t - 3 \)) and \( n \) is sufficiently large compared to \( k \) and \( t \), then we can characterize up to \( (t+3)^{rd} \)-largest (or \( (t+2)\text{nd-largest} \)) maximal \( t \)-cliques, which are type (J.1), type (J.2) with \( r=2, t, t+1 \), and type (J.3) with \( r=t \). (Note that type (J.2) with \( r=2 \) will not be included if \( k \leq 2t-3 \).)
Chapter 3
Maximal t-cliques of $H(n,q)$

3.1 Introduction

Let $X = \{1, 2, \ldots, q\}$ with $q \geq 2$. The Hamming graph $H(n,q)$ has vertex set $H = X^n = \{u = (u_1, u_2, \ldots, u_n) : u_i \in X\}$, the Cartesian product of $n$ copies of $X$. Two vertices $u$ and $v$ of $H$ are adjacent whenever they differ in precisely one entry. Obviously, for each $i = 1, 2, \ldots, n$, $d(u,v) = i$ if and only if $u$ and $v$ differ in precisely $i$ entries. Also, the diameter of the Hamming graph $H(n,q)$ is $n$.

It is known that there is one non-isomorphic maximal 1-clique in $H(n,q)$ whose size is $q$ ([12]). We study maximal 2-cliques and $t$-cliques of $H(n,q)$ in section 3.2. and section 3.3., respectively.
3.2 Maximal 2-cliques of $H(n,q)$

If $n=2$, $H(n,q)$ has only one maximal 2-clique, which is the set of vertices of $H(n,q)$ itself. So, let us assume $n \geq 3$.

**Theorem 3.2.1.** In $H(n,q)$, if $n \geq 3$, then there exist exactly 3 non-isomorphic maximal 2-cliques:

1. $\{u_k: u_i = 1, 2, ..., q$ for $i < 2, u_j = 1$ for all $j \geq 2\}$
   - size: $q^2$

2. $\{u_k: u_i = 1$ at most one $i\}$
   - size: $n(q-1)-1$

3. $\{(1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1), (2,2,1,1,1,1,1,1,1,1,1,1,1,1,1,1), (1,2,2,1,1,1,1,1,1,1,1,1,1,1,1,1), (2,1,2,1,1,1,1,1,1,1,1,1,1,1,1,1)\}$
   - size: 4.

**Proof:** Let $F$ be a maximal 2-clique of $H(n,q)$. Without loss of generality, we may assume that $F$ contains $x=(1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1)$; $y=(2,2,1,1,1,1,1,1,1,1,1,1,1,1,1)$. Let $M$ be the set of vertices which are at most distance 2-apart from $x$ and $y$, i.e.,

$$M = \{u: d(u,x) \leq 2, d(u,y) \leq 2\}.$$  

Then $F \subseteq M$. Consider a partition of $M$:

$$M = \{x,y,w,z\} - \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D} \cup \mathcal{E} \cup \mathcal{K}_3 \cup \mathcal{K}_4 \cup \cdots \cup \mathcal{K}_n \cup \mathcal{L}_3 \cup \mathcal{L}_4 \cup \cdots \cup \mathcal{L}_n,$$

where $w=(1,2,1,1,1,1,1,1,1,1,1,1,1,1,1,1)$

$$z=(2,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1)$$
We will have two cases according to $\mathcal{F} A=\emptyset$ or $\mathcal{F} A=\mathcal{O}$:

Case 1. Assume $\mathcal{F} A=\emptyset$. Then $\mathcal{F} K_j=\emptyset$ for all $j \geq 2$. If $\mathcal{F} \mathcal{B}=\emptyset$, then $\mathcal{F} K_j=\emptyset$ for all $j \geq 2$ and so,

$$\mathcal{F} \subseteq \{x,y,w,z\} \quad \mathcal{A} \subseteq \mathcal{Z} \quad \mathcal{D} \subseteq \mathcal{E}.$$ 

Since $\mathcal{F}$ is maximal,

$$\mathcal{F} \subseteq \{x,y,w,z\} \quad \mathcal{A} \subseteq \mathcal{Z} \quad \mathcal{D} \subseteq \mathcal{E} \\
\quad = \{(u_k): u_1=1,2,...,q \text{ for } i \geq 2 \text{ and } u_i = 1 \text{ for all } j \geq 2\},$$

whose size is $q^2$.

If $\mathcal{F} \mathcal{B}=\emptyset$, then $\mathcal{F} K_j=\emptyset$ for some $j \geq 2$. Without loss of generality, we may assume that $\mathcal{F} K_j=\emptyset$. Then $\mathcal{F} \mathcal{D}=\mathcal{F} \mathcal{E}=\emptyset$, w.l.o.g. and so,

$$\mathcal{F} \subseteq \{x,y,z\} \quad \mathcal{A} \subseteq \mathcal{C} \quad \mathcal{K}_j \subseteq \cdots \subseteq \mathcal{K}_n.$$ 

From the maximality of $\mathcal{F}$,

$$\mathcal{F} = \{x,y,z\} \quad \mathcal{A} \subseteq \mathcal{C} \quad \mathcal{K}_1 \subseteq \cdots \subseteq \mathcal{K}_n.$$
\( \Xi = \{(u_k): u_i = 1 \text{ at most one } i\} \)

and its size is \( n(q-1)-1 \).

**Case 2.** Assume that \( \mathcal{F} \cdot A = \emptyset \). If either \( \mathcal{F} \cdot B \neq \emptyset \), \( \mathcal{F} \cdot C \neq \emptyset \) or \( \mathcal{F} \cdot D \neq \emptyset \), we can show that \( \mathcal{F} \) is isomorphic to the one in case 1. So, we may assume that \( \mathcal{F} \cdot B = \mathcal{F} \cdot C = \mathcal{F} \cdot D = \emptyset \).

Since \( \mathcal{F} \cdot A = \emptyset \), \( \mathcal{F} \cdot L_j \neq \emptyset \) for some \( j > 2 \). So \( \mathcal{F} \cdot L = \emptyset \) and \( z \in \mathcal{F} \). Also, since \( \mathcal{F} \cdot B = \emptyset \), \( \mathcal{F} \cdot K_i \neq \emptyset \) for some \( i > 2 \). So \( w \in \mathcal{F} \). If \( u = (u_k) \in \mathcal{F} \cdot L_j \) and \( v = (v_k) \in \mathcal{F} \cdot K_i \), then \( i = j \) and \( u_j = v_j \). Then we may assume that \( i = j = 3 \) and \( u_j = v_j = 2 \). Hence,

\[ \mathcal{F} = \{(1,1,1,1,\ldots,1), (2,2,1,1,\ldots,1), (2,1,2,1,\ldots,1), (1,2,2,1,\ldots,1)\} \]

and its size is 4.

[ ]

**3.3 Maximal t-cliques of \( H(n,q) \) with size at least \( cq^{t-1} \)**

Let \( \mathcal{F} \) be a maximal t-clique of \( H(n,q) \). If \( t = n \), there exists only one type of maximal t-clique, which is the set of vertices of \( H(n,q) \) itself, whose size is \( q^t \). So, let us assume \( t : n \).

Moon, Frankl and Füredi (11, 16) showed that the size of \( \mathcal{F} \) is
bounded above by \( q^t \) if either (i). \( q \geq n-t-2 \), or (ii). \( q \geq n-t-1 \) and \( n-t \geq 14 \). Moon also showed that when \( q \geq n-t+2 \), \( \mathcal{F} \) attains the upper bound only if \( \mathcal{F} \) contains all the vertices which have fixed values on \((n-t)\) entries.

For the maximal \( t \)-cliques of \( H(n,q) \) with asymptotically large size, we have the following theorem and corollary:

**Theorem 3.3.1.** Let \( n \) and \( t \) be fixed integers satisfying \( 4 \leq t < n \). For any fixed constant \( c > 0 \), there exists \( q_o = q_o(n,t,c) \) such that if \( q > q_o \) and \( \mathcal{F} \) is a maximal \( t \)-clique of the Hamming graph \( H(n,q) \) such that (i). there is no entry where every vertex of \( \mathcal{F} \) has a fixed value and (ii). \( |\mathcal{F}| > cq^{t-1} \), then \( \mathcal{F} \) is one of the following up to isomorphism;

(II.1) For each \( r = 2, 3, \ldots, t \),
\[
\mathcal{F} = \{(u_1, u_2, \ldots, u_n) : u_i = 1 \text{ for at least one } i \text{ for } t-r < i < t-1, \quad u_j = 1 \text{ for } j \geq t-1\}
\]
\[
|\mathcal{F}| = q^{t-1} - r(\binom{t}{1})(q-1)^{r-1} - (\binom{t}{2})(q-1)^{r-2} - \ldots - (\binom{t}{t})(q-1)^0 - (n-t)q^{t-r}(q-1)
\]
\[
= \begin{cases} 
\frac{1}{r}q^{t-1} - o(q^{t-1}) & \text{if } r = 2 \\
 rq^{t-1} + o(q^{t-1}) & \text{if } 2 < r < t
\end{cases}
\]

(II.2) For each \( r = 2, 3, \ldots, t-2 \),
\[
\mathcal{F} = \{(u_1, u_2, \ldots, u_n) : u_i = 1 \text{ for at least one } i \text{ such that } t-r < i < t-1, \quad u_j = 1 \text{ for all } j \geq t-1\}
\]
\[ \bigcup \{ (u_1, u_2, \ldots, u_n) : (u_1, u_2, \ldots, u_{t-r}) \in D_1, u_i \neq 1 \text{ for all } i \text{ such that } t-r < i < t+1, u_j = 1 \text{ for all } j \geq t+1 \} \]

\[ \bigcup \{ (u_1, u_2, \ldots, u_n) : (u_1, u_2, \ldots, u_{t-r}) \in D_2, u_i = 1 \text{ for all } i \text{ such that } t-r < i < t+1, u_j \neq 1 \text{ for only one } j \geq t+1 \} \]

where \( D_1 \cup D_2 \) is a set of vertices in \( H(t-r,q) \) which is maximal in the sense that

(i). \( D_1 \neq \emptyset, D_2 \neq \emptyset \) and \( \partial(D_1, D_2) = t-r-1 \),

(ii). there does not exist an entry where every vertex of \( D_2 \) has a fixed value.

For this case,

\[ |D_1| \leq (t-r)(t-r-1)q^{t-r-2}, \]

\[ |D_2| \leq (t-r)q^{t-r-1}, \]

\[ |\mathcal{F}| = q^{t-r} \left( \sum_{i=1}^{r} (q-1)^{i-1} + \sum_{i=1}^{r} (q-1)^{i-2} + \cdots + (q-1)^0 \right) \]

\[ \geq (q-1)^r |D_1| + \left( n-t(q-1) \right) |D_2| \]

\[ = rqt^{-1} + o(qt^{-1}). \]

(H.3) \( \mathcal{F} = \{ (u_1, u_2, \ldots, u_n) : u_i = 1 \text{ for all } i \geq t \} \)

\[ \bigcup_{t \leq j \leq n} \{ (u_1, u_2, \ldots, u_n) : (u_1, u_2, \ldots, u_{t-1}) \in D_j, u_j \neq 1, u_i = 1 \text{ for all } i \geq t \text{ except } j \} \]

where \( D = \bigcup \{ D_j : t \leq j \leq n \} \) is a set of vertices in \( H(t-1,q) \) which is maximal in the sense that

(i). \( D_j \neq \emptyset \) for all \( j \),

(ii). \( \partial(D_i, D_j) \leq t-2 \) for all \( i \) and \( j \) with \( t \leq i < j \leq n \),
(iii). for each \( j=t, \ldots, n \), there does not exist an entry where every vertex of \( \cup \{ D_i : i \neq j \} \) has a fixed value.

In this case,

\[ |D_j| \leq (t-1)(t-2)q^{t-3} \quad \text{for all } j \geq t, \]

\[ |\mathcal{F}| = q^{t-1} + (q-1) \sum_{j \geq t} |D_j| \]

\[ = q^{t-1} + o(q^{t-1}). \]

Corollary 3.3.1. Let \( n \) and \( t \) be fixed integers satisfying \( 4 \leq t < n \). For any fixed constant \( c > 0 \), there exists \( q_0 = q_0(n, t, c) \) such that if \( q > q_0 \) and \( \mathcal{F} \) is a maximal \( t \)-clique of the Hamming graph \( H(n, q) \) satisfying \( |\mathcal{F}| > cq^{t-1} \), then \( \mathcal{F} \) is either type (H.1), type (H.2) or one of the following up to isomorphism;

(H.3)' For some \( r \) with \( t-1 \leq r \leq n \),

\[ \mathcal{F} = \{ (u_1, u_2, \ldots, u_n) : u_i = 1 \text{ for all } i \geq t \} \]

\[ \cup \{ (u_1, u_2, \ldots, u_n) : (u_1, u_2, \ldots, u_{t-1}) \in D_j, u_j \neq 1, u_i = 1 \}
\]

for all \( i \geq t \) except \( j \)

where \( D = \cup \{ D_j : t \leq j \leq r \} \) is a set of vertices in \( H(t-1, q) \) which is maximal in the sense that

(i). \( D_j \neq \emptyset \) for all \( j \).

(ii). \( \partial(D_i \cap D_j) \leq t-2 \) for all \( i \) and \( j \) with \( t \leq i < j \leq r \).

(iii). if \( r < n \), then \( D_i \cap D_j = \emptyset \) for all \( i \) and \( j \) with \( t \leq i < j < r \).

(iv). for each \( j \geq t \), there does not exist an entry where every vertex of \( \cup \{ D_i : i \neq j \} \) has a fixed value.
In this case, for all $j \geq t$

\[ D_j \leq (t-1)(t-2)q^{t-3} \]

\[ F = q^{t-1} - (q-1) \sum_{j \geq t} D_j \leq q^{t-1} - (q-1)(t-1)(t-3)q^{t-3} \]

\[ = q^{t-1} + o(q^{t-1}). \]

(11.4) $F = \{(u_1, u_2, ..., u_n): u_i = 1 \text{ for all } i > t\}$ and $F = q^t$.

To prove the theorem 3.3.1., we need lemmas and definitions.

Lemma 3.3.1. If $F$ is a maximal t-clique in $H(n,q)$ with $n > t$, then either (i). there exists an entry where every vertex of $F$ has a fixed value, or (ii). there exists at least one entry such that $F'$ is still a t-clique of $H(n-1,q)$ where $F'$ is obtained by deleting that entry from $F$.

Proof: Suppose that, for each entry, we get a (t-1)-clique of $H(n-1,q)$ after deleting that entry from $F$. If $F'$ which is obtained by deleting an entry from $F$ is a (t-1)-clique of $H(n-1,q)$, then $F'$ is a maximal (t-1)-clique in $H(n-1,q)$ and $F \equiv 1,2, ..., q \times F'$. So, after deleting $t$ entries from $F$, we have only one vertex of $H(n-t,q)$. This implies that there are $n-t(\geq 1)$ entries such that every vertex of $F$ has a fixed value on each of them.

If there is an entry such that every vertex of $F$ has a fixed-value on it, then we may consider $F$ as a maximal t-clique of $H(n-1,q)$ by disregarding that entry. Conversely, if we have a maximal t-clique of
$H(n-1, q)$, we may consider it as a maximal $t$-clique of $H(n, q)$ by adding one more entry with a fixed value.

From now on, let us assume that there is no entry such that every vertex of $H$ has a fixed value on it. Also, we may assume that such an entry as in lemma 3.3.1(ii) is the last entry, $n^{th}$-entry, of $H$. Let $H'$ be the $t$-clique in $H(n-1, q)$ obtained by deleting the last entry from $H$. For each $j = 1, 2, ..., q$, put

$$A_j = \{(u_1, u_2, ..., u_{n-1}) : (u_1, u_2, ..., u_{n-1}, j) \notin H'\}.$$

Lemma 3.3.2. (1). $H' = A_1 \cup A_2 \cup ... \cup A_q$.

(2). $H = A_1 \cup A_2 \cup ... \cup A_q$.

(3). $A_j = \emptyset$ for at least two $j$'s.

(4). For some $j$, $A_j$ has two vertices $u$ and $v$ satisfying $d(u,v) = t$.

(5). $\partial(A_i, A_j) \geq t-1$ for every pair $i$ and $j$ with $1 \leq i \leq j \leq q$.

(6). $H: A_1 \cup A_2 \cup ... \cup A_q$ is a $(t-1)$-clique.

(7). $\partial(H, A_j) \geq t-1$ for every $j$.

Let us assume that $A_1$ has two vertices $x = (x_i)$ and $y = (y_i)$ of distance $t$ such that $x_i \cdot y_i = 1$ for all $i$ and $j$ with $1 \leq i, j \leq n-1$. Let $m$ be $1$ if $j = n-1$. For each $i = t-1, t-2, ..., n-1$, define

$$H_i = \{(u_k) \in H' : u_i \neq 1, u_k = 1$ for all $k > t$ except $i\}$$

and

$$H_i = \{(u_k) \in H' : u_k = 1$ for all $k > t\}.$$
Also, for every pair $i$ and $j$ with $t < i < j < n$, let us define

$$\mathcal{H}_{i,j} = \{(u_k) \in \mathcal{F}^t : u_i \neq 1, u_j \neq 1, u_k = 1 \text{ for all } k > t \text{ except } i \text{ and } j\}.$$  

Lemma 3.3.3. (1). Any vertex contained in $A_1 \setminus \mathcal{H}_t$ or $\mathcal{H}_t \cap (A_2 \cup \ldots \cup A_q)$ has at least one 1 and one 2 on its first $t$ entries.

(2). Any vertex contained in $(\cup \{A_i; t < i\}) \cap (A_2 \cup \ldots \cup A_q)$ has at least two 1's and two 2's on its first $t$ entries.

(3). Also, for each $j > 1$

$$|A_j| \leq \sum_{r=0}^t \binom{n-t-r}{r} (q-1)^{r+1} (t-(r-1))q^{-2(r-1)} \leq O(q^{t-2}).$$

Proof: By comparing with the vertices $x$ and $y$ in $A_1$.

Lemma 3.3.4. Assume that $c = c(k,t)$ is a fixed positive constant. If $q > q_1$ for some $q_1 = q_1(t,c) > \frac{2}{3}((\frac{1}{2}) + (\frac{1}{2}))$ and $|A_1 \setminus \mathcal{H}_i| > cq^{t-1}$, then

(1). $\mathcal{H}_i \cap (A_2 \cup \ldots \cup A_q) = \emptyset$ for all $j > t$ except $i$.

(2). Every vertex $z = (z_k)$ in $\mathcal{F}^t$ satisfies that $z_k \neq 1$ for at most two $k$'s with $k > t$.

(3). If $\mathcal{H}_{j,j'} \neq \emptyset$, then either $j$ or $j'$ is $i$ and $\mathcal{H}_{j,j'} \cap (A_2 \cup \ldots \cup A_q) = \emptyset$.

(4). There exists a pair of entries $(k_1, k_2)$ with $k_1 \leq t$ and $k_2 \leq t$ such that

$$|\{(u_k) \in \mathcal{H}_i \cap \mathcal{H}_j : u_{k_1} = 1, u_{k_2} = 2\}| > c_1 q^{t-1}$$
for some \( c_1 = c_1(t, c) > c\left\{2\left(\frac{t}{2}\right)\right\}^{-1} \).

For such a pair \((k_1, k_2)\),

(5). If \((u_k)\) is a vertex in \(\cup \{H_j: t < j, t \neq i\}, \cup \{H_{j,i}: j = i \text{ or } j = i\} \text{ or } A_2 \cup \ldots \cup A_q\), then \(u_{k_1} = 1\) and \(u_{k_2} = 2\).

(6). If \((u_k)\) is a vertex in \(A_1 \cap (H_i \cup H_t)\), then \(u_{k_1} = 1\) or \(u_{k_2} = 2\).

(7). \(A_1 \cap H_i \supseteq \{(u_k): u_{k_1} = 1, u_{k_2} = 2, u_{k_i} > 1, u_{k} = 1, \text{ for all } k > t \text{ except } i}\)

(8). \(B = A_2 = \ldots = A_q\) is a \((t-1)\)-clique.

Proof: (1). Otherwise, \(|A_1 \cap H_i| \leq (q-1)\left(\frac{t}{3}\right)q^{t-3} < cq^{t-1}\), too small.

(2). Let \(R\) be the set of values \(z_i\)'s of the vertex \(z = (z_k)\) with \(z_i \neq 1\) and \(z_k \neq 1\) for more than two \(k\)'s with \(k > t\). If \(|R| > 1\) or \(|R| = 0\), then

\[|A_1 \cap H_i| \leq (q-1)\left(\frac{t}{3}\right)q^{t-3} < cq^{t-1}.\]

If \(|R| = 1\), then for \(z_i\) in \(R\),

\[|A_1 \cap H_i| = \{|(u_k) \in A_1 \cap H_i: u_i = z_i\} = |\{(u_k) \in A_1 \cap H_i: u_i = z_i\}| \leq \left(\frac{t}{2}\right)q^{t-2} - (q-2)\left(\frac{t}{3}\right)q^{t-3} < cq^{t-1}.\]

(3). By a similar argument as in (2).

(4). Assume the contrary. Then for every pair of entries \((k_3, k_4)\) with \(k_3, k_4 < t\),

\[|\{(u_k) \in A_1 \cap H_i: u_{k_3} = 1, u_{k_4} = 2\}| < c\left\{2\left(\frac{t}{2}\right)\right\}^{-1}q^{t-1}.\]

Then, since there are \(2\left(\frac{t}{2}\right)\) choices for such pairs,

\[|A_1 \cap H_i| < 2\left(\frac{t}{2}\right)q^{t-1} < cq^{t-1}.

(5). If \(\cup \{H_j: t < j, j \neq i\}\) or \(H_t \cap (A_2 \cup \ldots \cup A_q)\) has a vertex \(v = (v_k)\) with either \(v_{k_1} \neq 1\) or \(v_{k_2} \neq 2\), then
\[ |\{(u_k) \in A_i \cap H_i: u_{k_1} = 1, u_{k_2} = 2\}| \leq (q-1)(1^{t-2})q^{t-3} < cq^{t-1}. \]

If \( \cup \{H_{ij}: j = i \text{ or } j' = i\} \) or \( H_i \cap (A_2 \cup \ldots \cup A_q) \) has a vertex \( v=(v_k) \) with either \( v_{k_1} \neq 1 \) or \( v_{k_2} \neq 2 \), then define \( R \) as the set of values \( v_i \)'s of such vertices \( v_i \)'s. If \(|R| > 1\), then

\[ |\{(u_k) \in A_i \cap H_i: u_{k_1} = 1, u_{k_2} = 2\}| \leq (q-1)(1^{t-2})q^{t-3} < cq^{t-1}. \]

If \(|R| = 1\), then for \( v_i \) in \( R \),

\[ |\{(u_k) \in A_i \cap H_i: u_{k_1} = 1, u_{k_2} = 2\}| = |\{(u_k) \in A_i \cap H_i: u_{k_1} = 1, u_{k_2} = 2, u_i = v_i\}| \]

\[ + |\{(u_k) \in A_i \cap H_i: u_{k_1} = 1, u_{k_2} = 2, u_i \neq v_i\}| \leq q^{t-2} + (q-2)(1^{t-2})q^{t-3} < cq^{t-1}. \]

(6). By a similar argument as in (5).

(7) and (8). From (1), (2), ..., (6). \[

\text{Corollary 3.3.2. If there is another pair of such entries (}k_3, k_4\text{) as in lemma 3.3.4.(4), then either }k_1 = k_3\text{ or }k_2 = k_4.\]

\textbf{Proof of theorem 3.3.1.} Let us fix a constant \( c > 0 \). Define \( A_i, H_i, \)

\( H_{i,j}, \) \( x \) and \( y \) as before. We will prove the theorem by two cases:

(i). \( \mathcal{F}' \) has a vertex \( z=(z_k) \) such that \( z_k \neq 1 \) for more than one \( k \) with \( k > t \),

(ii). if \( z=(z_k) \) is a vertex in \( \mathcal{F}' \), then \( z_k \neq 1 \) for at most one \( k > t \).
Case 1. Assume that $\mathcal{F}'$ has a vertex $z=(z_k)$ such that $z_k \neq 1$ for more than one $k$ with $k>t$. (Note that this case can not happen if (i) $n-t-1<2$, i.e., $n-t<3$ or (ii) $t<4$.) If $\mathcal{A}_m$ contains such a vertex $z$, then for each $i > 1$ with $i \neq m$,

$$|\mathcal{A}_i| = |\mathcal{A}_i \setminus \mathcal{H}_t| + |\mathcal{A}_i \cap \mathcal{H}_t|$$

$$\leq \sum_{k=t}^{n-t-1} (\frac{n-t-1}{r} (q-1) \left(\begin{array}{c} r+1 \end{array}\right) q^{t-2(r+1)} + \left(\begin{array}{c} t \end{array}\right) q^{t-3}$$

$$= O(q^{t-3})$$

by comparing with $x$ and $y$ in $\mathcal{A}_1$, and $z$ in $\mathcal{A}_m$. Hence, if $q > q_1$ for some $q_1 = q_1(n,t,c)$, then $|\mathcal{A}_i| < \frac{\xi}{4} q^{t-2}$ for all $i \neq m$, and so either $|\mathcal{A}_1|$ or $|\mathcal{A}_m|$ is greater than $\frac{\xi}{4} q^{t-1}$.

Subcase 1.1. If $m \neq 1$, then by comparing with two vertices $x$ and $y$ in $\mathcal{A}_1$,

$$|\mathcal{A}_m| = |\mathcal{A}_m \setminus \mathcal{H}_t| + |\mathcal{A}_m \cap \mathcal{H}_t|$$

$$\leq \sum_{k=t}^{n-t-1} (\frac{n-t-1}{r} (q-1) \left(\begin{array}{c} r+1 \end{array}\right) q^{t-2(r+1)} + \left(\begin{array}{c} t \end{array}\right) q^{t-2}$$

Hence we can find $q_2 = q_2(n,t,c) > q_1$ such that if $q > q_2$, then $|\mathcal{A}_m| < \frac{\xi}{4} q^{t-1}$. Also,

$$|\mathcal{A}_1 \setminus \mathcal{H}_t| \leq \left(\begin{array}{c} t \end{array}\right) q^{t-3},$$

$$|\mathcal{A}_1 \setminus \cup \{ \mathcal{H}_i : t \leq i \}| \leq \sum_{k=t}^{n-t-1} (\frac{n-t-1}{r} (q-1) \left(\begin{array}{c} r+1 \end{array}\right) q^{t-2(r+1)}$$

by comparing with $z$ in $\mathcal{A}_m$ and two vertices $x$ and $y$ in $\mathcal{A}_1$. 


Now, let us count $|\cup \{ A_i \cap H_i : t < i \}|$. First, let us assume that $z_k \neq 1$ for more than two $k$'s with $k > t$. By a similar argument as in lemma 3.3.4., if $q > q_3$ for some $q_3 = q_3(n,t,c)$, then $|A_i \cap H_i| < \frac{c}{n-t-1}q^{t-1}$ for each $i > t$. Hence if $q > q_4$ for some $q_4 = q_4(n,t,c)$, then $|A_i| < \frac{c}{3}q^{t-1}$.

If $z_k \neq 1$ for only two $k$'s with $k > t$, we may assume that $z_t + 1 \neq 1$ and $z_{t+2} \neq 1$. Then
\[ |A_i \cap H_i| \leq (q-1)(\frac{1}{3})q^{t-3} \]
for each $i > t+2$. Also,
\[
|A_i \cap H_{t+1}| = |\{(u_k) \in A_i \cap H_{t+1}: u_{t+1} = z_{t+1}\}| \\
+ |\{(u_k) \in A_i \cap H_{t+1}: u_{t+1} \neq z_{t+1}\}| \\
\leq (\frac{1}{2})q^{t-2} - (q-2)(\frac{1}{3})q^{t-3}
\]
and same is true for $A_i \cap H_{t+2}$. Therefore, if $q > q_5$ for some $q_5 = q_5(n,t,c) \geq q_4$, then $\cup \{ A_i \cap H_i : t < i \} \leq \frac{c}{6}q^{t-1}$ and $|F_i| \leq cq^{t-1}$.

Subcase 1.2. Assume that $m = 1$. Then $|A_1| > \frac{c}{2}q^{t-1}$. We may also assume that every vertex $u-(u_k)$ of $A_1$ for $i > 1$ has at most one $k > t$ such that $u_k \neq 1$. If $z=(z_k)$ has more than three $k$'s with $k > t$ such that $z_k \neq 1$, then
\[ |A_i \cap H_i| \leq (q-1)(\frac{1}{3})q^{t-3} \]
for each $i > t$. Also, by comparing with $z$, $x$ and $y$, we get
\[ |A_i \cap H_i| \leq (\frac{1}{4})q^{t-4} \]
and
\[ \sum_{i=1}^{t-1} \sum_{u \in \cup \{ A_i \cap H_i : t < i \}} \Sigma (n-t-1)(q-1)q^{t-5} \leq (\frac{c}{7})q^{t-2r} \]
Hence, there exists \( q_0 = q_0(n,t,c) \) such that \(|A_1| < \frac{\xi}{2} q^{t-1} \) if \( q > q_0 \).

If \( z = (z_k) \) has three \( k \)'s for \( k > t \) with \( z_k \neq 1 \), then

\[
|A_1 \cap \mathcal{H}_i| \leq (q-1)\binom{i}{4} q^{t-4}
\]

for each \( i > t \) such that \( z_i = 1 \). Also, for each \( i > t \) such that \( z_i / 1 \),

\[
|A_1 \cap \mathcal{H}_i| = |\{(u_k) \in A_1 \cap \mathcal{H}_i : u_i \neq z_i\}| + |\{(u_k) \in A_1 \cap \mathcal{H}_i : u_i = z_i\}|
\leq \left(\frac{6}{2}\right)q^{t-2} + (q-2)\binom{i}{3} q^{t-3}
\]

By comparing with the vertices \( z, x \) and \( y \), we get

\[
|A_1 \cap \mathcal{H}_1| \leq \binom{4}{i} q^{t-3}
\]

\[
|A_1 \cap \{\mathcal{H}_i : t \leq i\}| \leq 2 \sum \left( n \cdot \binom{t-1}{r} \binom{q^t}{t-1} \right) \binom{r}{t} q^{t-2r}
\]

Hence, there exists \( q_7 = q_7(n,t,c) \) such that \(|A_1| < \frac{\xi}{2} q^{t-1} \) if \( q > q_7 \).

Therefore, \( z = (z_k) \) has only two \( k \)'s with \( k > t \) such that \( z_k \neq 1 \). Let us suppose \( z_{t+1} \neq 1, z_{t+2} \neq 1 \) and \( A_1 \) has no vertex \( u \cdot (u_k) \) such that \( u_k \neq 1 \) for more than two \( k \)'s with \( k > t \). Using a similar computation as before, we can show that

\[
|A_1 \cap \mathcal{H}_i| \leq \binom{n-t-2}{2} \cdot \binom{t-1}{2} \cdot (n-t-1)! q^{t-1},
\]

\[
|A_1 \cap \mathcal{H}_i| \leq \binom{n-t-2}{2} \cdot \binom{t-1}{2} \cdot (n-t-1)! q^{t-1}, \quad \text{if } i > t - 2,
\]

\[
|A_1 \cap \{\mathcal{H}_{i,j} : t < i < j < n-1\}| \leq \binom{n-t-2}{2} \cdot \binom{t-1}{2} \cdot (n-t-1)! q^{t-1}
\]

if \( q > q_8 \) for some \( q_8 = q_8(n,t,c) > q_1 \). Now let us assume \( q > q_8 \). Since \( A_1 \)

has size greater than \( \frac{\xi}{2} q^{t-1} \) either \( A_1 \cap \mathcal{H}_{t-1} \) or \( A_1 \cap \mathcal{H}_{t-2} \) has size greater than \( \frac{\xi}{4} q^{t-1} \).
We will discuss this subcase 1.2. by another two subcases according to the size of $\mathcal{A}_1 \cap \mathcal{H}_{t+1}$ and $\mathcal{A}_1 \cap \mathcal{H}_{t+2}$:

Subcase 1.2.1. Assume that both $\mathcal{A}_1 \cap \mathcal{H}_{t+1}$ and $\mathcal{A}_1 \cap \mathcal{H}_{t+2}$ have size greater than $\frac{4}{5}q^t$. By lemma 3.3.4., if $q > q_0$ for some $q_0 = q_0(n, t, c) > q_2$, then

(1). $\mathcal{A}_2 = \ldots \cup \mathcal{A}_q \subseteq \mathcal{H}_t$.

(2). If $\mathcal{H}_{j, j'} \neq \emptyset$, then $j = t+1$, $j = t+2$ and $\mathcal{H}_{t+1, t+2} \cap \mathcal{A}_1 = \emptyset$ for all $i > 1$.

(3). There exists only one pair of entries $(k_1, k_2)$ with $k_1, k_2 < t$ such that

$$\mathcal{H}_{t+1} = \mathcal{A}_1 \cap \mathcal{H}_{t+1} \cup \{(u_k): u_{k_1} = 1, u_{k_2} = 2, u_{t+1} \neq 1, u_k = 1 \text{ for all } k > t \text{ except } t-1\},$$

$$\mathcal{H}_{t+2} = \mathcal{A}_1 \cap \mathcal{H}_{t+2} \cup \{(u_k): u_{k_1} = 1, u_{k_2} = 2, u_{t+2} \neq 1, u_k = 1 \text{ for all } k > t \text{ except } t-2\}.$$

(4). If $(u_k)$ is a vertex in $\cup \{k_i: t \leq i\}$, $\mathcal{H}_{t-1, t+2}$ or $\mathcal{A}_2 \cup \mathcal{A}_q$, then $u_{k_1} = 1$ and $u_{k_2} = 2$.

(5). If $(u_k)$ is a vertex in $\mathcal{A}_1 \cap \mathcal{H}_t$, then either $u_{k_1} = 1$ or $u_{k_2} = 2$.

We may put $k_1 = 1$ and $k_2 = 2$. If we define

$$D_1 = \{(u_3, u_4, \ldots, u_t): (u_1, u_2, \ldots, u_{n-1}) \text{ is in } \mathcal{H}_{t-1, t+2}\},$$

$$D_2 = \{(u_3, u_4, \ldots, u_t): (u_1, u_2, \ldots, u_{n-1}) \text{ is in } \mathcal{B}\},$$

then for $i > t+2$,

$$\mathcal{H}_i = \{(u_k): u_1 = 1, u_2 = 2, (u_3, u_4, \ldots, u_t) \in D_2, u_{i-1}, u_k = 1 \text{ for all } k \text{ except } i\}.$$
\( \mathcal{Y}_{t+1} = \{ (u_k) : u_1 = 1, u_2 = 2, u_{t-1} = 1, u_k = 1 \text{ for all } k > t \text{ except } t+1 \} \)

\( \mathcal{Y}_{t+2} = \{ (u_k) : u_1 = 1, u_2 = 2, u_{t-2} = 1, u_k = 1 \text{ for all } k > t \text{ except } t+2 \} \)

\( \mathcal{A}_1 \cap \mathcal{Y}_t = \{ (u_k) : \text{either } u_1 = 1 \text{ or } u_2 = 2, (u_3, u_4, \ldots, u_t) \in \mathcal{D}_2, u_k = 1 \text{ for all } k > t \} \)

\( \cup \{ (u_k) : u_1 = 1, u_2 = 2, u_k = 1 \text{ for all } k > t \} \)

\( \mathcal{Y}_{t-1,t+2} = \{ (u_k) : u_1 = 1, u_2 = 2, (u_3, u_4, \ldots, u_t) \in \mathcal{D}_1, u_{t-1} = 1, u_{t-2} = 1, u_k = 1 \text{ for all } k > t-2 \} \),

\[ B : \mathcal{A}_2 = \ldots = \mathcal{A}_q = \{ (u_k) : u_1 = 1, u_2 = 2, (u_3, u_4, \ldots, u_t) \in \mathcal{D}_2, u_k = 1 \text{ for all } k > t \} \]

We may regard \( \mathcal{D}_1 \) and \( \mathcal{D}_2 \) as sets of vertices of \( \Pi(t-2,q) \) satisfying

(i). \( \mathcal{D}_1 \cap \mathcal{D}_2 = \emptyset \), \( \partial(\mathcal{D}_1, \mathcal{D}_2) \leq t-3 \).

(ii). \( \partial(\mathcal{D}_1, \mathcal{D}_2) = t-2 \).

Since \( \mathcal{F} \) is maximal, \( \mathcal{D}_1, \mathcal{D}_2 \) is a set of vertices in \( \Pi(t-2,q) \) which is maximal in the sense of (i) and (ii). In this case,

\[ |\mathcal{D}_1| \leq (t-2)^3 \cdot q^{t-1} \]

\[ |\mathcal{D}_2| \leq (t-2)^3 \cdot q^{t-3} \]

\[ |\mathcal{F}| = |\mathcal{Y}_{t-1} - \mathcal{Y}_{t-2} \cap \ldots \cap \mathcal{Y}_{t-1,t-2} \cap |\mathcal{A}_1 \cap \mathcal{Y}_t = (q-1)|B \]

\[ = 2(q-1) q^{t-2} \cdot (n-t-3)(q-1)|\mathcal{D}_1| - (q-1)^2 \mathcal{D}_1 \]

\[ + (q^{t-2} - 2(q-1)|\mathcal{D}_2| - (q-1)|\mathcal{D}_2| \]

\[ = 2q^{t-1} - q^{t-2} \cdot (n-t)(q-1)|\mathcal{D}_2| - (q-1)^2 \mathcal{D}_1 \]

\[ = 2q^{t-1} + o(q^{t-1}) \]

and \( \mathcal{F} \) is type (II.2) with \( r = 2 \).
Subcase 1.2.2. Assume that $A_1 \cap \mathcal{H}_{t+1}$ (or $A_1 \cap \mathcal{H}_{t+2}$, isomorphically) has size greater than $\varepsilon q^{t-1}$, but $A_1 \cap \mathcal{H}_{t+2}$ (or $A_1 \cap \mathcal{H}_{t+1}$, respectively) does not.

By lemma 3.3.4., if $q > q_{10}$ for some $q_{10} = q_{10}(n,t,c) > q_j$, then

1. $\mathcal{H}_t \cap (A_2 \cup \ldots \cup A_q) = \emptyset$ for all $i > t+1$.
2. If $\mathcal{X}_{j,j'} \neq \emptyset$, then $j = t-1$. and $\mathcal{H}_{t-1,j'} \cap A_i = \emptyset$ for all $i > 1$.
3. There exists a pair of entries $(k_1, k_2)$ with $k_1, k_2 \leq t$ such that

$$A_1 \cap \mathcal{H}_{t+1} \cap \left( \{ (u_k) : u_{k_1} = 1, u_{k_2} = 2, u_{t+1} = 1, u_k = 1 \text{ for all } k \geq t+1 \} \right).$$

4. If $(u_k)$ is a vertex in $\{ \mathcal{X}_i : t+1 < i \}$, $\cup \{ \mathcal{X}_{t-1,j} : t+1 < j \}$ or $A_2 \cup \ldots \cup A_q$, then $u_{k_1} = 1$ and $u_{k_2} = 2$.
5. If $(u_k)$ is a vertex in $A_1 \cap \left( \mathcal{H}_{t-1} \cup \mathcal{H}_t \right)$, then either $u_{k_1} = 1$ or $u_{k_2} = 2$.
6. $B. A_2 \cup \ldots \cup A_q$ is a $(t-1)$-clique.

If there exists another pair of such entries $(k_1, k_2)$ as in (3), we may also assume that $k_2 = k_1$. Let $R$ be the collection of all entries, $k'$s such that $(k_1, k')$ is such a pair of entries as in (3). If $|R| = r$, $1 \leq r \leq t-2$ because the vertex $z$ has at least two 1's on its first $t$-entries. We may assume that $k_1 = 1$ and $R = \{2, 3, \ldots, r-1\}$.

(7). By the definition of $R$, there does not exist an entry $k_5$ with $r+1 < k_5 < t-1$ such that $u_{k_5}$ is fixed for every vertex $u_k$ in $\cup \{ \mathcal{X}_i : i > t+1 \} \cup (\cup \{ \mathcal{X}_{t+1,j} : j > t+1 \}) \cup B \cup \{ (u_k) \in A_1 : u_k = 1 \}$. Otherwise, $u_{k_5} = 2$ since $A_1$ contains the vertex $y$. Hence
\[ A_{t} \cap H_{t+1} = \{(u_k): u_1 = 1, u_{k_2} = 2, u_{t+1} \neq 1, u_k = 1 \text{ for all } k > t+1} \]

which implies \(|R| > 1\).

**Subcase 1.2.2.1.** Assume \(1 < r\). Let us define

\[ D_1 = \{(u_{r+2}, \ldots, u_t, u_{t+1}): (u_1, u_2, \ldots, u_{n-1}) \text{ is a vertex in} \]

\[ (H_{t+1} \cup H_t) - A_{t+1} \text{ with } u_1 = 1 \text{ and } u_j \neq 2 \text{ for all } j \text{ in } R\} \]

\[ D_2 = \{(u_{r+2}, \ldots, u_t, u_{t+1}): (u_1, u_2, \ldots, u_{n-1}) \text{ is in } B\} \]

Then for each \(i > t+1\),

\[ H_i \cup H_{t+1} = \{(u_k): u_1 = 1, u_j = 2 \text{ for all } j \in R, (u_{r+2}, \ldots, u_t, u_{t+1}) \in D_2, u_i \neq 1, u_k = 1 \text{ for all } k > t+1 \text{ except } i} \}

Also,

\[ \{(u_k) \in A_{t}: u_1 = 1, u_j = 2 \text{ for all } j \in R} \]

\[ = \{(u_k): u_1 \neq 2 \text{ for all } j \in R, (u_{r+2}, \ldots, u_t, u_{t+1}) \in D_2, u_k = 1 \text{ for all } k > t-1\} \}

We may regard \(D_1\) and \(D_2\) as sets of vertices of \(H(t-r,q)\) satisfying

(i). \(D_1 \neq \emptyset\), \(D_2 \neq \emptyset\), \(\partial(D_1, D_2) = t-r-1\).

(ii). there does not exist an entry where every vertex of \(D_2\) has a fixed value (by (7) in subcase 1.2.2.).

Since \(F\) is maximal, \(D_1 \cup D_2\) is a set of vertices in \(H(t-r,q)\) which is maximal in the sense of (i) and (ii). In this case,

\[ |D_1| \leq \binom{t-r}{1} \binom{t-1}{r} q^{t-r-2} \]

\[ |D_2| \leq \binom{t-r}{1} q^{t-r-1} \]

\[ |F| = |H_{t+1} \cup H_t| \cap A_{t+1} \cup \bigcup \{H_{t+1} \cup H_t: i > t+1\}; \mathcal{E}; \]

...
\[(q^t - r)((q-1)^r - 1) + \cdots + ((q-1)^r - r)\]
\[= q^t - (q-1)^r + \cdots + ((q-1)^r - r)\]
\[= rq^t - o(q^t) \quad 1 < r < t-1.\]

and \(\mathcal{F}\) is type \((H.2)\).

**Subcase 1.2.2.2.** Let us assume \(r = 1\). Define

\(D_1 = \{ (u_3, \ldots, u_t, u_{t-1}) : (u_1, u_2, \ldots, u_{n-1}) \) is in \(A_1\) and \(u_2 \neq 2\}\),

\(D_2 = \{ (u_3, \ldots, u_t, u_{t-1}) : (u_1, u_2, \ldots, u_{n-1}) \) is in \(A_1\) and \(u_1 \neq 1\}\),

\(D_3 = \{ (u_3, \ldots, u_t, u_{t-1}) : (u_1, u_2, \ldots, u_{n-1}) \) is in \(B\}\),

\(D_j = \{ (u_3, \ldots, u_t, u_{t-1}) : (u_1, u_2, \ldots, u_{n-1}) \) is in \(A_j \cup A_{t+1,j}\}\),

for each \(j > t + 1\). Then may regard \(D_1, D_2, D_3\) and \(D_j\)'s with \(j > t + 1\) as sets of vertices in \(\Pi(t-1,q)\) satisfying that

(i). \(D_i \cap \emptyset\) for all \(i\),

(ii). \(\partial(D_i, D_j) < t-2\) for all \(i\) and \(j\) such that \(i < j\) and \(i, j = 1, 2, 3, t, 2t - 3, \ldots, n-1\),

(iii). \(\partial(D_1, D_2) = t-2\),

(iv). there is no entry where every vertex of \(D_2 \cup D_3 \cup \bigcup\{D_j : j > t + 1\}\)

has a fixed value (by (7) in subcase 1.2.2.).

We may also assume that
(v). there is no entry where every vertex of \( D_1 \cup D_2 \cup (\cup \{ D_j : j > t+1 \}) \) has a fixed value: Suppose that it has such an entry. Since \( A_1 \) contains \( x, y \) and \( z \), such an entry cannot be the last one (the \((t+1)^{th}\)-entry of \( \mathcal{F} \)). Without loss of generality, we may assume that such an entry is the very first one of \( D_1 \cup D_2 \cup (\cup \{ D_j : j > t+1 \}) \) which corresponds to the third entry of \( \mathcal{F} \). Since \( A_1 \) contains \( x \), the fixed value on the third entry must be 1. Therefore,

\[
A_1 \cap \mathcal{H}_t \supseteq \{(u_k) : u_2 = 2, u_3 = 1, u_{t+1} \neq 1; u_k = 1 \text{ for all } k > t+1\}.
\]

By rearranging entries, we can show that this subcase is isomorphic to subcase 1.2.2.1.

Since \( A_1 \) contains the three vertices \( x, y \) and \( z \),

(vi). for each \( j = 3, t+3, t+4, \ldots, n-1 \), there is no entry where every vertex of \( \cup \{ D_i : i \neq j \} \) has a fixed value.

(vii). We may assume that there is no entry where every vertex of \( \cup \{ D_i : i \neq t+2 \} \) has a fixed value. (Otherwise, it is isomorphic to subcase 1.2.1.)

Since \( \mathcal{F} \) is maximal in \( \Pi(n,q) \), the set of vertices \( D_1 \cup D_2 \cup D_3 \cup (\cup \{ D_j : j > t+1 \}) \) is also maximal in \( \Pi(t-1,q) \) among those satisfying (i)-(vii). In this case,

\[
|D_i| \leq (t-1)^2 q^{t-3} \quad \text{for } i = 1, 2, 3, t+2, t+3, \ldots, n-1,
\]

\[
|\mathcal{F}| = (|\mathcal{Y}_{t+1} \cup \mathcal{Y}_t \cap A_1| + \sum_{t+1 < j} |(\mathcal{Y}_j \cup \mathcal{Y}_{t+1,j})|) + (q-1)|B|
\]

\[
= (q^{t-1} + (q-1)|D_1| + (q-1)|D_2|) + \sum_{t+1 < j} (q-1)|D_j| + (q-1)|D_3|
\]
\[= q^{t-1} + (q-1)(D_1 D_2 | D_3 + D_{t+1} | D_{t-3} + \ldots + D_{n-1})
\]
\[= q^{t-1} + o(q^{t-1})
\]

and \(\mathcal{F}\) is type (II.3).

Case 2. Assume that every vertex \(u=(u_k)\) in \(\mathcal{F}^0\) has at most one entry \(k>t\) such that \(u_k \neq 1\). We will have four subcases:

(i). \(n-1=1\),

(ii). \(n-1>1\) and \((A_2 \cup \ldots \cup A_q) \cap (\cup \{H_i : i>t\}) \neq \emptyset\),

(iii). \(n-1>1\) and \((A_2 \cup \ldots \cup A_q) \cap (\cup \{H_i : i>t\}) = \emptyset\),

(iv). \(n-1=0\).

Subcase 2.1. Assume \(n-1=1\). By comparing with the vertices \(x\) and \(y\) in \(A_1\),

\[|A_j \cap H_{t+1}| \leq (q-1)(t^1_2)^{(t^2_2)q^{t-4}}\]

for each \(j>1\). Hence, if \(q>q_1\) for some \(q_1=q_1(t)\), then

\[\forall_{j>1} |A_j \cap H_{t+1}| < q^{t-2} \text{ and either } |A_1 \cap H_{t+1}|, |A_1 \cap H_{t+1}| \text{ or } j \geq 1 \text{ is greater than } \frac{q^t}{q} q^{t-1} \text{. Now let us assume } q>q_1.\]

Subcase 2.1.1. Assume that \(A_1 \cap H_{t+1}\) has size greater than \(\frac{q^t}{q} q^{t-1}\). By lemma 3.3.4., if \(q>q_2\) for some \(q_2=q_2(t,c) \geq q_1\), then

(1) There exists a pair of entries \((k_1, k_2)\) with \(k_1, k_2 \leq t\) such that

\[A_1 \cap H_{t+1} \supseteq \{(u_k) : u_{k_1}=1, u_{k_2}=2, u_{t+1} \neq 1\}.
\]

(2) If \((u_k)\) is a vertex in \(A_2 \cup \ldots \cup A_q\), then \(u_{k_1}=1\) and \(u_{k_2}=2\).
(3). If \((u_k)\) is a vertex in \(A_t\), then either \(u_{k_1} = 1\) or \(u_{k_2} = 2\).

(4). \(B = A_2 \ldots = A_q\) is a \((t-1)\)-clique.

If there exists another pair of such entries \((k_3, k_4)\) as in (1), we may also assume that \(k_3 = k_1\). Let \(R\) be the collection of all entries \(k_0\)'s such that \((k_1, k_0)\) is such a pair of entries as in (1). If \(|R| = r\), then \(1 \leq r \leq t-1\). Let us put \(k_1 = 1\) and \(R = \{2, 3, \ldots, r-1\}\).

(5). By the definition of \(R\), there does not exist an entry \(k_5\) with \(r+1 < k_5 < t-1\) such that \(u_{k_5}\) is fixed for every vertex \(u = (u_k)\) in \(B \cup \{(u_k) \in A_1; u_j \neq 1\}\) by a similar argument as in (7) of subcase 1.2.2.

If \(u_{t-1} = 1\) for every vertex \(u = (u_k)\) in \(\{(u_k) \in A_t; u_j \neq 1\text{ or } u_j = 2\text{ for all } j \in R\}\). \(\mathcal{B}\), then

\[ \mathcal{B}_t-1 = \{(u_k); u_1 = 1, u_j = 2\text{ for some } j \text{ in } R, u_{t-1} = 1\} \]

\[ \mathcal{A}_t \cdot \mathcal{A}_t = \{(u_k); u_j = 1\text{ or } u_j = 2\text{ for all } j \in R, u_{t-1} = 1\} \]

\[ \mathcal{B} = \{(u_k); u_1 = 1, u_j = 2\text{ for all } j \in R, u_{t-1} = 1\}. \]

Hence, \(\mathcal{F}\) is type (II.1) and

\[ |\mathcal{F}| = |\mathcal{A}_1 - (q-1) \mathcal{B}| \]

\[ = q^{t-r}((\binom{t}{1})(q-1)^{r-1}-\binom{t}{2}(q-1)^{r-2}-\ldots+\binom{t}{r}(q-1)^{r-1}) \]

\[ - q^{t-r-1}((q-1)^r-(q-1))- (q-1)q^{t-r-1} \]

\[ = \binom{t}{r}4q^{t-1}3q^{t-2} \quad \text{if } r = 1 \]

\[ = (r-1)q^{t-1} - o(q^{t-1}) \quad \text{if } r = 2, 3, \ldots, t-1. \]
Now let us assume that \( u_{t-1} \neq 1 \) for some vertex \( u = (u_k) \) in \( \{(u_k) \in A_1: u_1 \neq 1 \text{ or } u_j \neq 2 \text{ for all } j \in R\} \cup B \). If \( r = t-1 \), it is isomorphic to the case with \( u_{t+1} = 1 \) for every vertex \( u = (u_k) \) in \( \{(u_k) \in A_1: u_1 \neq 1 \text{ or } u_j \neq 2 \text{ for all } j \in R\} \cup B \). Hence, \( r < t-1 \). We will have two subcases according to \( r \).

**Subcase 2.1.1.1.** Assume \( 1 < r < t-1 \). Let us define

\[
D_1 = \{(u_{r+2}, \ldots, u_t, u_{t+1}) : (u_1, u_2, \ldots, u_{t+1}) \text{ is a vertex in } (\mathcal{H}_{t+1} \cup \mathcal{H}_{t}) \cap A_1 \text{ with } u_1 = 1 \text{ and } u_j \neq 2 \text{ for all } j \in R\},
\]

\[
D_2 = \{(u_{r+2}, \ldots, u_t, u_{t+1}) : (u_1, u_2, \ldots, u_{t+1}) \text{ is in } B\}.
\]

From the maximality of \( \mathcal{F} \),

\[
\{(u_k) \in A_1 : u_1 \neq 1, u_j = 2 \text{ for all } j \in R\} = \{(u_k) : u_1 \neq 1, u_j = 2 \text{ for all } j \in R, (u_{r+2}, \ldots, u_{t+1}) \in D_2\}.
\]

We may regard \( D_1 \) and \( D_2 \) as sets of vertices of \( H(t-r,q) \) satisfying

(i). \( D_1 \neq \emptyset, D_2 \neq \emptyset, \partial(D_1, D_2) = t-r-1 \).

(ii). Except the last entry, there does not exist an entry where every vertex of \( D_2 \) has a fixed value.

By assumption, \( u_{t+1} \neq 1 \) for some vertex \( u = (u_{r+2}, \ldots, u_{t+1}) \) in \( D_1 \cup D_2 \).

Also, we may assume that \( D_2 \) has such a vertex \( u \). Suppose that \( D_2 \) does not have such a vertex. Then, \( D_1 \supseteq \{(u_{r+2}, \ldots, u_{t+1}) : u_{t-1} = 1\} \).

Since \( \partial(D_1, D_2) = t-r-1 \), \( D_1 \) and \( D_2 \) have two vertices \( w = (w_{r+2}, \ldots, w_{t-1}) \) and \( z = (z_{r+2}, \ldots, z_{t+1}) \), respectively, such that \( d(w, z) = t-r-1 \) and \( w_{t-1} \neq 1 \).
Then, \( w_j = z_j \) for some \( j < t+1 \) and \( w_k \neq z_k \) for all \( k \) except \( j \). By rearranging the entries (in fact, by exchanging \( j \)-th entry and \((t+1)\)-st entry in \( F \)), we can show that this case is isomorphic to the case with \( |R| = t+1 \).

Therefore, we may assume that

(ii)' there does not exist an entry where every vertex of \( D_2 \) has a fixed value.

(Note that this case cannot happen if \( t < 4 \)).

Since \( F \) is maximal, \( D_1 \cup D_2 \) is a set of vertices in \( H(t-r, q) \) which is maximal in the sense of (i) and (ii)'. In this case,

\[
\begin{align*}
D'_1 & \leq \binom{t-r}{1}(t-r-1)q^{t-r-1}, \\
D'_2 & \leq \binom{t-r}{1}q^{t-r-1}.
\end{align*}
\]

\[
F = (q^{l-1} - 1) - (q-1) B
\]

\[
= (q^l - (l)^\cdot(q-1)^{l-1} - \cdots - (l)^{q-1})
\]

\[
- (q-1)^{l}D'_1 - (q-1)^{l}D'_2
\]

\[
= q^{l-1} - 1 - (q-1)^{l}D'_2
\]

\[
= rq^{l-1} - 1 - (q-1)^{l}D'_2
\]

and \( F \) is type \((k, l, q)\) with \( 1 < r < t-1 \).

Subcase 2.1.1.2. Let us assume \( r = 1 \). Define

\[
D'_1 = \{(u_3, \ldots, u_t, u_{t-1}) : (u_1, u_2, \ldots, u_{t-1}) \text{ is in } A_1 \text{ and } u_2 \neq 2\},
\]

\[
D'_2 = \{(u_3, \ldots, u_t, u_{t-1}) : (u_1, u_2, \ldots, u_{t-1}) \text{ is in } A_1 \text{ and } u_1 = 1\},
\]
\[ D_3 = \{(u_3, \ldots, u_t, u_{t+1}) : (u_1, u_2, \ldots, u_{t-1}) \text{ is in } B\}. \]

Then we may regard \( D_1, D_2, \) and \( D_3 \) as sets of vertices in \( H(t-1,q) \) satisfying that

(i). \( D_i \neq \emptyset \) for \( i=1,2,3. \)

(ii). \( \partial(D_i, D_j) \leq t-2 \) for all \( i \) and \( j \) such that \( 1 \leq i < j \leq 3. \)

(iii). \( \partial(D_1, D_2) = t-2. \)

(iv). Except the last entry, there is no entry where every vertex of \( D_2 \cup D_3 \) has a fixed value (by (5) in subcase 2.1.1.).

(v). Except the last entry, there is no entry where every vertex of \( D_1 \cup D_2 \) has a fixed value since \( A_1 \) has the vertices \( x \) and \( y. \)

We may also assume that

(vi). except the last entry, there is no entry where every vertex of \( D_1 \cup D_3 \) has a fixed value (by a similar argument as in (v) of subcase 1.2.2.2.).

By assumption, \( u_{t-1} \neq 1 \) for some vertices in \( D_1 \cup D_2 \cup D_3 \). If \( u_{t-1} = 1 \) for every vertex \( u = (u_k) \) in \( D_1 \cup D_2 \), then \( D_1 = D_2 \) and

\[ D_3 = \{(u_1, \ldots, u_{t-1}) : u_{t-1} = 1\}. \]

(Note that this case cannot happen if \( t \neq 1. \) Hence,

\[ |D_1| = |D_2| \leq \left(\begin{array}{c} t-1 \\ 1 \end{array}\right) q^{t-3}, \]

\[ q^{t-2} \leq |D_3| \leq q^{t-2} + (q-1)\left(\begin{array}{c} t-2 \\ 1 \end{array}\right) q^{t-4}. \]

\[ |\mathcal{F}| = |A_1 \cap \mathcal{H}_t + 1| + |A_1 \cap \mathcal{H}_t| + (q-1)|B| = q^{t-1} - 2(q-1)|D_1| - (q-1)|D_3| = 2q^{t-1} - o(q^{t-1}). \]
and $\mathcal{F}$ is type (II.2) with $r=2$.

If $u_{t+1}=1$ for every vertex $u=(u_k)$ in $D_2\cup D_3$, (or $D_1\cup D_3$, isomorphically), then $D_2=D_3$ and $D_1 \supseteq \{(u_3, \ldots, u_{t-1}) : u_{t-1}=1\}$. (Notice that this cannot happen if $t<3$. Also, if $t=3$, then $\mathcal{F}$ is type (II.1) with $r=t=3$.) Hence,

$$|D_2|=|D_3| \leq (t^3_1) q^{t-3},$$

$$q^{t-2} \leq |D_1| \leq q^{t-2} + (q-1)(t^3_1 - t^3_1) q^{t-4},$$

$$|\mathcal{F}| = |A_1 \cap K_{i+1}| + |A_1 \cap K_i + (q-1)B|$$

$$= q^{t-1} - (q-1)(|D_1| + |D_2| - (q-1)|D_3|)$$

$$= 2q^{t-1} - o(q^{t-1}),$$

and $\mathcal{F}$ is type (II.2) with $r=2$.

Now we may assume that

(vii). there is no entry where every vertex of $D_i \cup D_j$ for any $i$ and $j$ with $1 \leq i < j \leq 3$ has a fixed value.

Since $\mathcal{F}$ is maximal in $H(n,q)$, the set of vertices $D_1 \cup D_2 \cup D_3$ is also maximal in $H(t-1,q)$ among those satisfying (i), (ii), (iii) and (vii). In this case,

$$D_i \leq (t^3_1 - t^3_1) q^{t-3} \quad \text{for } i=1,2,3,$$

$$\mathcal{F} = |(K_{i-1} \cap K_i) : A_1 \cap (q-1)B|$$

$$= (q^{t-1} - (q-1)(|D_1| + (q-1)|D_2| - (q-1)|D_3|)$$

$$= q^{t-1} - o(q^{t-1}).$$
and $\mathcal{F}$ is type (H.3).

Subcase 2.1.2. Assume that $|A_1 \cap \mathcal{H}_t| > \frac{1}{4} q^{t-1}$. Then $\mathcal{H}_{t+1} \cap (A_2 \cup \ldots \cup A_q) = \emptyset$. Let us fix a vertex $z_k = (z_k)$ in $A_2 \cup \ldots \cup A_q$. Since $\partial(A_1 \cap \mathcal{H}_t, z) \leq t-1$, there exists an entry $s \leq t$ such that $|\{(u_k) \in A_1 \cap \mathcal{H}_t : u_s = z_s\}| > 2^{t-1} q^{t-1}$. Then

(i). If $(u_k)$ is a vertex in $\mathcal{H}_{t+1} \cup A_2 \cup \ldots \cup A_q$, then $u_s = z_s$. Hence $B = A_2 = A_3 = \ldots = A_q$.

(ii). $A_1 \cap \mathcal{H}_t = \{(u_k) : u_s = z_s, u_{t+1} = 1\}$.

Notice that $\partial(B, \mathcal{H}_{t-1} \cup \{(u_k) \in A_1 : u_s \neq z_s, u_{t+1} = 1\}) = t-1$. Let $w = (w_k)$ and $w' = (w'_k)$ be the two vertices of $\mathcal{H}_{t-1} \cup \{(u_k) \in A_1 : u_s \neq z_s, u_{t+1} = 1\}$ and $B$, respectively, such that $d(w, w') = t$. Then $w_i = w'_i$ for some $i \leq t$ with $i \neq s$, and we can show that this is isomorphic to subcase 2.1.1. by rearranging the entries. (More precisely, if $w$ is in $\mathcal{H}_{t-1}$, exchange $n$-th-entry (of $\mathcal{F}$) with $s$-th-entry, and $i$-th-entry with $(t+1)$-th-entry. If $w$ is in $\{(u_k) \in A_1 : u_s \neq z_s, u_{t+1} = 1\}$, then move $(t-1)$-th-entry, $n$-th-entry (of $\mathcal{F}$) and $i$-th-entry to $n$-th-entry (of $\mathcal{F}$), $i$-th-entry and $(t-1)$-th-entry, respectively.)

Subcase 2.1.3. Assume that $\sum_{j>1} |A_j \cap \mathcal{H}_t| > \frac{1}{4} q^{t-1}$. By a similar argument as in lemma 3.3.4., if $q > q_3$ for some $q_3 = q_3(n, t, c) \geq q_1$, then there exists a pair of entries $(k_1, k_2)$ such that
Without loss of generality, we may assume $k_1=1$, $k_2=2$. Then

(i). if $(u_k)$ is a vertex in $\mathcal{H}_{t+1}$, then $u_1=1$ and $u_2=2$,

(ii). if $(u_k)$ is a vertex in $\mathcal{H}_t$, either $u_1=1$ or $u_2=2$.

Hence, $A_j \cap \mathcal{H}_t \supseteq \{(u_k): u_1=1, u_2=2, u_{t+1}=1\}$ for all $j \geq 1$. By exchanging $n^{th}$-entry with second entry (of $\mathcal{F}$), we can show that this case is isomorphic to the subcase 2.1.2.

Subcase 2.2. Let us assume that $n-t-1>1$ and $(A_2 \cup \ldots \cup A_q) \cap (\cup \{A_i: i>t\}) \neq \emptyset$, i.e., $A_m \cap \mathcal{H}_j \neq \emptyset$ for some $m>1$ and $j>t$. Then we can show that this case is isomorphic to case 1 by exchanging the $n^{th}$-entry with $j^{th}$-entry (of $\mathcal{F}$) where $j_1>t$ and $j_1 \neq j$.

Subcase 2.3. Assume that $n-t-1>1$, and $(A_2 \cup \ldots \cup A_q) \cap (\cup \{A_i: i>t\}) = \emptyset$. If $u$ is a vertex in $\cup \{A_i: i>t\}$, then $A_1$ contains at least one vertex which is at distance $t$ from $u$. (Otherwise, $A_j$ should contain such a vertex $u$ for all $j$, which implies $A_j \cap \mathcal{H}_j \neq \emptyset$ for some $j>1$ and $i>t$.) We will have another two subcases:

(i). $\partial(A_i, A_j)=t$ for some $i$ and $j$ with $t<i<j<n-1$.

(ii). $\partial(A_i, A_j)<t$ for every $i$ and $j$ with $t<i<j<n-1$.

Subcase 2.3.1. Let us assume that $n-t-1>1$ and $A_1$ has two vertices
\(w = (w_k)\) and \(z = (z_k)\) of distance \(t\)-apart with \(w \in \mathcal{H}_{i_1}\) and \(z \in \mathcal{H}_{i_2}\) \((i_1, i_2 > t\) and \(i_1 \neq i_2)\). Then for some \(k_1, k_2 \leq t\), \(w_{k_1} = z_{k_1}, w_{k_2} = z_{k_2}\), and \(w_k \neq z_k\) for all \(k \leq t\) except \(k_1\) and \(k_2\). We may put \(k_1 = 1, k_2 = 2, i_1 = t - 1\) and \(i_2 = t + 2\).

If \(\{w_1, w_2\} \neq \{1,2\}\), then we can show that this subcase is isomorphic to case 1 by exchanging first entry with \((t + 1)\)st-entry and second entry with \((t - 2)\)nd-entry (since \(\mathcal{A}_1\) contains the two vertices \(x\) and \(y\)).

So, let us assume \(w_1 = 1\) and \(w_2 = 2\).

Using a similar argument we can assume that

(i). if \((u_k)\) is a vertex in \(\mathcal{H}_{i_1} \cup \mathcal{H}_{i_2} \cup \cdots \cup \mathcal{H}_{i_2}\), then \(u_1 = 1\) and \(u_2 = 2\). Hence, \(\mathcal{B} \cdot \mathcal{A}_1 \cdots \mathcal{A}_q = \mathcal{N}_q\).

(ii). If \((u_k)\) is a vertex in \(\mathcal{A}_1 \cup (\mathcal{H}_{i_1} \cup \mathcal{H}_{i_2} \cup \cdots \cup \mathcal{H}_{i_2})\), then \(u_1 = 1\) or \(u_2 = 2\).

If every vertex \(u \cup (u_k)\) in \(\mathcal{H}_{i_1} \cup \mathcal{H}_{i_2}\) satisfies \(u_1 = 1\) and \(u_2 = 2\), then for each \(i > t\),

\(\mathcal{H}_i = \{(u_k): u_1 = 1, u_2 = 2, u_i = 1\ \text{for all} \ k > t\ \text{except} \ i\}\),

\(\mathcal{B} = \{(u_k): u_1 = 1, u_2 = 2, u_k = 1\ \text{for all} \ k > t\}\),

\(\mathcal{A}_1 \cap \mathcal{H}_i = \{(u_k): u_1 = 1\ \text{or} \ u_2 = 2, u_k = 1\ \text{for all} \ k > t\}\).

Hence,

\[
|F| = \prod_{i > t} |H_i| - |\mathcal{A}_1 \cap H_i| - (q - 1) |B|.
\]

\[
= (n - t - 1)(q - 1)q^{t - 2} + (q^{t - 2} + 2(q - 1)q^{t - 2}) - (q - 1)q^{t - 2}
\]

\[
= (n - t - 2)q^{t - 1} - (n - t - 1)q^{t - 2}
\]
and $\mathcal{F}$ is type (H.1) with $r=2$.

Assume that $\mathcal{H}_{t+1} \cup \mathcal{H}_{t+2}$ has a vertex $u=(u_k)$ such that either $u_1 \neq 1$ or $u_2 \neq 2$. Then by comparing with such a vertex $u$, $|B| \leq (t-2)q^{t-3}$. Also, $|\mathcal{H}_i| \leq (q-1)(t-2)q^{t-3}$ for all $i > t+2$. Therefore, we can choose $q_4 = q_4(n,t,c)$ so that if $q > q_4$, then $\sum_{j>1}|A_j| + \sum_{i>t+2}|\mathcal{H}_i| < \frac{3}{4}q^{t-1}$ and so either $\mathcal{H}_{t+1}$, $\mathcal{H}_{t+2}$ or $A_1 \cap \mathcal{H}_t$ has size greater than $\frac{3}{4}q^{t-1}$. Now let us assume that $q > q_4$.

Subcase 2.3.1.1. Assume that $\mathcal{H}_{t-1}$ (or $\mathcal{H}_{t+2}$, isomorphically) has size greater than $\frac{3}{4}q^{t-1}$. By lemma 3.3.4., if $q > q_5$ for some $q_5 = q_5(k,t,c) \geq q_4$, then

1. there exists a pair of entries $(k_1, k_2)$ with $k_1, k_2 \leq t$ such that

$$\mathcal{H}_{t+1} \supseteq \{(u_k): u_{k_1} = 1, u_{k_2} = 2, u_{t-1} \neq 1\}.$$

2. If $(u_k)$ is a vertex in $\cup\{\mathcal{H}_j: j > t+1\}$ or $A_2 \cup \ldots \cup A_q$, then $u_{k_1} = 1$ and $u_{k_2} = 2$.

3. If $(u_k)$ is a vertex in $A_1 \cap (\mathcal{H}_t \cup \mathcal{H}_{t+1})$, then either $u_{k_1} = 1$ or $u_{k_2} = 2$.

4. $B = A_2 = \ldots = A_q$ is a $(t-2)$-clique.

If there exists another pair of such entries $(k_3, k_4)$ as in (1), we may also assume that $k_3 = k_1$. Let $R$ be the collection of all entries $k'$ such that $(k_1, k')$ is such a pair of entries as in (1). If $|R| = r$, then $1 \leq r \leq t-1$. Let us put $k_1 = 1$ and $R = \{2, 3, \ldots, r+1\}$. 
(5). By the definition of $R$, there does not exist an entry $k_5$ with $r+1 < k_5 < t+1$ such that $u_{k_5}$ is fixed for every vertex $u=(u_k)$ in $(\cup \{j: j>t+1\}) \cup B \cup \{(u_k) \in A_1: u_1 \neq 1\}$ (by a similar argument as in (7) of subcase 1.2.2.).

If $u_{t+1} = 1$ for all vertices $u=(u_k)$ in $\{(u_k) \in A_1: u_1 \neq 1$ or $u_j \neq 2$ for all $j$ in $R\}$, then for each $i \geq t+1$,

$\mathcal{H}_i = \{(u_k): u_1 = 1, u_j = 2$ for all $j$ in $R, u_i \neq 1, u_k = 1$ for all $k > t$ except $i\}$

$\mathcal{H}_{t-1} = \{(u_k): u_1 = 1, u_j = 2$ for some $j$ in $R, u_{t-1} \neq 1, u_k = 1$ for all $k > t-1\}$

$A_1 \cap \mathcal{H}_t = \{(u_k): u_1 = 1$ or $u_j = 2$ for all $j$ in $R, u_k = 1$ for all $k > t\}$

$B_t = \{(u_k): u_{t-1} = 1, u_j = 2$ for all $j$ in $R, u_k = 1$ for all $k > t\}$

Hence, $\mathcal{F}$ is type (II.1) and

$$|\mathcal{F}| = |A_1| = (q-1)^r B$$

$$= q^{t-r}(\binom{t}{r})(q-1)^{r-1} + (\binom{t}{r})(q-1)^{r-2} + ... + (\binom{t}{r})(q-1)^{-r} - (n-t-2)(q-1)q^{t-r-1} + q^{t-r-1}(q-1)^{r} + (q-1)q^{t-r-1}$$

$$= \begin{cases} 
(q-1)^r + (q-1)q^{t-r-1} & \text{if } r = 1 \\
(n-t+2)q^{t-r-1} - (n-t+1)q^{t-2} & \text{if } r = 2, 3, ..., t-1.
\end{cases}$$

Now let us assume that $u_{t+1} \neq 1$ for some vertex $u=(u_k)$ in $\{(u_k) \in A_1$:
u_j ≠ 1 or u_j ≠ 2 for all j in \( R \). Then, as in subcase 2.1.1., \( r < t - 1 \). We will have two subcases according to \( r \).

**Subcase 2.3.1.1.(i).** Assume \( 1 < r < t - 1 \). Define

\[
D_1 = \{ (u_r, \ldots, u_t, u_{t+1}) : (u_1, u_2, \ldots, u_{n-1}) \text{ is a vertex in } A_1 \text{ with } u_1 = 1 \text{ and } u_j ≠ 2 \text{ for all } j \in R \},
\]

\[
D_2 = \{ (u_r, \ldots, u_t, u_{t+1}) : (u_1, u_2, \ldots, u_{n-1}) \text{ is a vertex in } A_1 \text{ with } u_1 ≠ 1 \text{ and } u_j = 2 \text{ for all } j \in R \},
\]

From the maximality of \( \mathcal{F} \),

\[
\mathcal{A}_j = \{ (u_1, \ldots, u_{n-1}) : u_1 = 1, u_j = 2 \text{ for all } j \in R, \quad (u_r, \ldots, u_{t+1}) \in D_2, u_j ≠ 1, u_k = 1 \text{ for all } k > t \text{ except } j \}
\]

for all \( j > t - 1 \), and

\[
\mathcal{B} = \{ (u_1, \ldots, u_{n-1}) : u_1 = 1, u_j = 2 \text{ for all } j \in R, \quad (u_r, \ldots, u_{t+1}) \in D_2, u_k = 1 \text{ for all } k > t \}.
\]

Since \( \mathcal{A}_i \cap \mathcal{A}_j = \emptyset \) for all \( i > 1 \) and \( j > t + 1 \), \( u_{t+1} = 1 \) for every vertex \( (u_r, \ldots, u_{t-1}) \) in \( D_2 \). By assumption \( u_{t-1} ≠ 1 \) for some vertex \( u = (u_r, \ldots, u_{t+1}) \) in \( D_1 \cup D_2 \), and so \( D_1 \cup \{ (u_r, \ldots, u_{t+1}) : u_{t-1} = 1 \} \). Since \( \partial(D_1 \cup D_2) = t-r-1 \), \( D_1 \) and \( D_2 \) have two vertices \( w = (w_r, \ldots, w_{t+1}) \) and \( z = (z_r, \ldots, z_{t+1}) \), respectively, such that \( d(w, z) = t-r-1 \) and \( w_{t+1} ≠ 1 \). Hence, \( w_j = z_j \) for some \( j < t + 1 \) and \( w_k ≠ z_k \) for all \( k \) except \( j \). By rearranging the entries (in fact, by exchanging \( j^{th}\)-entry and \( (t+1)^{st}\)-entry of
we can show that this case is isomorphic to the case with \(|R|=r+1\) or case 1.

**Subcase 2.3.1.1.(ii).** Let us assume \(r=1\). Define

\[ D_1 = \{(u_3, ..., u_t, u_{t-1}): (u_1, u_2, ..., u_{n-1}) \text{ is in } A_1 \text{ and } u_2 \neq 2\}, \]
\[ D_2 = \{(u_3, ..., u_t, u_{t-1}): (u_1, u_2, ..., u_{n-1}) \text{ is in } A_1 \text{ and } u_1 \neq 1\}, \]
\[ D_3 = \{(u_3, ..., u_t, u_{t-1}): (u_1, u_2, ..., u_{n-1}) \text{ is in } B\}. \]

Since \(S\) is maximal,

\[ H_j = \{(u_1, u_2, ..., u_{n-1}): u_1 = 1, u_2 = 2, (u_3, ..., u_{t-1}) \in D_3, u_j \neq 1, \]
\[ u_k = 1 \text{ for all } k > t \text{ except } j \}

for all \(j > t+1\). Hence \(u_{t+1} = 1\) for every \((u_3, ..., u_{t+1})\) in \(D_3\) and \(D_1 \cup D_2\) has a vertex \(v = (v_3, ..., v_{t-1})\) such that \(v_{t+1} \neq 1\). (Notice that this cannot happen if \(t < 3\).)

Then we may regard \(D_1, D_2, D_3\) as sets of vertices in \(H(t-1, q)\) satisfying that

(i). \(D_i \neq \emptyset\) for \(i = 1, 2, 3\).

(ii). \(\partial(D_i, D_j) \leq t - 2\) for all \(i\) and \(j\) such that \(i \leq i < j \leq 3\)

(iii). \(\partial(D_1, D_2) = t - 2\).

(iv). Except the last entry, there is no entry where every vertex of \(D_2 - D_3\) has a fixed value (by (5) in subcase 2.3.1.1.).

(v). There is no entry where every vertex of \(D_1 \cup D_2\) has a fixed value since \(A_1\) has the vertices \(x\) and \(y\), and \(D_1 \cup D_2\) has a vertex \(u = (u_3, ..., u_{t+1})\) such that \(u_{t+1} \neq 1\).
We may also assume that

(vi). except the last entry, there is no entry where every vertex of
\( \mathcal{P}_1 \cup \mathcal{P}_3 \) has a fixed value (by similar argument as in (v) of subcase 1.2.2.2.).

If \( u_{t+1} = 1 \) for every vertex \( u = (u_3, \ldots, u_{t+1}) \) in \( \mathcal{P}_2 \cup \mathcal{P}_3 \), (or \( \mathcal{P}_1 \cup \mathcal{P}_3 \), isomorphically), then \( \mathcal{P}_2 = \mathcal{P}_3 \), \( \mathcal{P}_1 \supseteq \{(u_3, \ldots, u_{t+1}): u_{t+1} = 1\} \) and \( \mathcal{P}_1 \) has a vertex \( v = (v_3, \ldots, v_{t-1}) \) such that \( v_{t-1} \neq 1 \). Hence,

\[
|\mathcal{P}_2| = |\mathcal{P}_3| \leq (t^{-2}) q^{t-3},
\]

\[
q^{t-2} < |\mathcal{P}_1| \leq q^{t-2} + (q-1)(t^{-1}) (t^{-3}) q^{t-4},
\]

\[
|\mathcal{F}| = |A_1 \cap K_{t+1}| + |A_1 \setminus K_t| + \sum_{j \geq t-1} |K_j| - (q-1)^3 B
\]

\[
= q^{t-1} + (q-1)|D_1| - |D_2| - (n-t)(q-1)|D_3| + (q-1)|D_3|
\]

\[
= 2q^{t-1} - o(q^{t-1}).
\]

and \( \mathcal{F} \) is type (II.2) with \( r = 2 \). (Note that if \( t=3 \), then \( \mathcal{F} \) is type (II.1) with \( r = t = 3 \).)

Now we may assume that

(vii). there is no entry where every vertex of \( \mathcal{P}_1 \cup \mathcal{P}_j \) for any \( i \) and \( j \) with \( i \neq j \) has a fixed value.

Since \( \mathcal{F} \) is maximal in \( H(n,q) \), the set of vertices \( \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3 \) is also maximal in \( H(t-1,q) \) among those satisfying (i), (ii), (iii) and (vii). In this case.
\[ |D_i| \leq (t_i^{t-1})(t_i^{t-2})q^{t-1} \quad \text{for } i = 1, 2, 3, \]

\[ |\mathcal{F}| = |(\mathcal{X}_t +_1 \cup \mathcal{X}_t) \cap A_1| + \sum_{j > t + 1} |\mathcal{X}_j| + (q-1)|B| \]

\[ = (q^{t-1} + (q-1)|D_1| + (q-1)|D_2|) + (n-t-2)(q-1)|D_3| - (q-1)|D_3| \]

\[ = q^{t-1} + o(q^{t-1}) \]

and \( \mathcal{F} \) is type (H.3).

Subcase 2.3.1.2. If \( A_1 \cap \mathcal{X}_t \) has size greater than \( \frac{q}{2} q^{t-1} \), then we can also show that this is isomorphic to case 1, subcase 2.2 or subcase 2.3.1.1. using the same argument as in subcase 2.1.2.

Subcase 2.3.2. Let us assume that \( n-t-1 > 1 \) and \( \partial(\mathcal{X}_i, \mathcal{X}_j) < t \) for every \( i \) and \( j \) with \( t < i < j \leq n-1 \). Then, we may also assume that for every vertex \( u \) in \( \cup \{ \mathcal{X}_i: i > t \} \), \( \partial(u, A_j) \leq t-2 \) for all \( j > 1 \). (Otherwise \( d(u, v) = t-1 \) for a vertex \( u \) in \( \mathcal{X}_{i_0} \) with \( i_0 > t \) and a vertex \( v \) in \( A_2 \cup \cdots \cup A_q \). By exchanging \( n^{th} \)-entry with \( i_1^{th} \)-entry (of \( \mathcal{F} \)) for some \( i_1 > t \) and \( i_1 \neq i_0 \), we can show that this case is isomorphic to case 1, subcase 2.3.1.)

Therefore, \( |\mathcal{X}_i| \leq (q-1)(\frac{t}{2})q^{t-3} \) for all \( i > t \), and \( A_j \leq (\frac{t}{3})q^{t-3} \) for all \( j > 1 \). Hence we can choose \( q_6 = q_6(n, t, c) \) so that if \( q > q_6 \), then \( \sum_{i > t} |\mathcal{X}_i| + \sum_{j > 1} |A_j| < \frac{q}{2} q^{t-1} \) and so \( A_1 \cap \mathcal{X}_t \) is of size greater than \( \frac{q}{2} q^{t-1} \). If \( q > q_6 \), then using the same argument as in subcase 2.1.2., we can show that this case is isomorphic to case 1, subcase 2.2. or 2.3.1.
Subcase 2.4. Assume that \( n-t-1=0 \). Then there exists \( q_7=q_7(t,c) \) such that either \( |A_1| \) or \( \sum_{j>1}|A_j| \) is greater than \( \frac{3}{2}q^{t-1} \). Suppose that \( q > q_7 \).

Subcase 2.4.1. Assume that \( A_1 \) has size greater than \( \frac{3}{2}q^{t-1} \). Let us fix a vertex \( z=(z_k) \) in \( A_2 \cup \cdots \cup A_q \). Since \( \partial(A_1, z) \leq t-1 \), there exists an entry \( s \leq t \) such that

\[
|\{(u_k) \in A_1: u_s = z_s\}| > \frac{3}{2}q^{t-1}.
\]

Then, for every vertex \( u=(u_k) \) in \( A_2 \cup \cdots \cup A_q \),

\[ u_s = z_s \]

and so, \( A_1 \supseteq \{(u_k): u_s = z_s\} \).

Let \( R \) be the set of all such entries and put \( R = \{1, 2, \ldots, r\} \) \((1 \leq r \leq t)\). If \( r = t \), then

\[ A_1 = \{(u_k): u_i = z_i \text{ for some } i\}. \]

\[ \mathcal{B} = A_2 \cdots \cup A_q = \{(z_k)\}. \]

Hence,

\[
\mathcal{F} = A_1 \setminus (q-1)\mathcal{B} = (q-1)^{t-1} - (q-1)^{t-2} - \cdots - (q-1)^{2} - (q-1)
\]

\[ = tq^r - o(q^{t-1}) \]

and \( \mathcal{F} \) is type (H.1).

If \( r < t \), define

\[ \mathcal{D}_1 = \{(u_{r-1}, \ldots, u_1): (u_1, u_2, \ldots, u_{n-1}) \text{ is a vertex of } A_1 \}. \]
with \( u_i \neq \varepsilon_i \) for all \( i \leq r \),

\[
D_2 = \{(u_{r+1}, \ldots, u_t): (u_1, u_2, \ldots, u_{n-1}) \text{ is a vertex of } B\}.
\]

Then we may regard \( D_1 \) and \( D_2 \) as sets of vertices in \( H(t-r,q) \) satisfying

(i). \( D_2 \neq \emptyset \), \( \partial(D_1, D_2) \leq t-r-1 \),

(ii). \( D_2 \neq \emptyset \), \( \partial(D_1, D_2) \leq t-r-1 \),

(iii). there does not exist an entry where every vertex of \( D_2 \) has a fixed value (by the definition of \( R \)).

Also,

(iv). if \( r = 1 \), we may assume that there does not exist an entry where every vertex of \( D_1 \) has a fixed value: Suppose that \( u_m = \alpha \) for every vertex \((u_{r+1}, \ldots, u_t) \) in \( D_1 \). Then \( B \supseteq \{(u_k): u_m = \alpha\} \). We can show that this case is isomorphic to the case with \( r \geq 2 \) by exchanging \( n^{th} \)-entry with first entry (of \( F \)).

(v). If \( r = t-1 \), we may assume \( D_1 = \emptyset \). (Otherwise, it is isomorphic to the case \( r = t \) since \( \partial(D_1, D_2) \leq t-r-1 \).)

Since \( F \) is maximal in \( H(n,q) (= H(t+1,q)) \), \( D_1 \cup D_2 \) is maximal among the sets of vertices in \( H(t-r,q) \) satisfying (i), (ii), (iii), (iv) and (v). For this case,

\[
|D_1| \leq \binom{t-r}{1} q^{t-r-2},
\]

\[
|D_2| = q^{t-r},
\]

\[
|D_2| \leq \binom{t-r}{1} q^{t-r-1},
\]

if \( D_1 = \emptyset \) (and \( r \geq 2 \)),

\[
|D_2| = q^{t-r},
\]

if \( D_1 \neq \emptyset \) and \( r \geq 2 \),
\[ \left| D_2 \right| \leq \left( \frac{t}{1} - 1 \right) \left( \frac{t}{1} - 2 \right) q^{t-3}, \text{ if } D_1 \neq \phi \text{ and } r = 1, \]

and

\[ |\mathcal{F}| = |A_1| + (q-1)|B| = (q^t - t) ((q-1)^{t-1} + (q-1)^{t-2} + \ldots + (q-1)^{t-r}) - (q-1)^r |D_1| + (q-1)|D_2| \]

\[ = \begin{cases} 
3q^{t-1} - 2q^{t-2} & \text{if } D_1 = \phi \text{ and } r = 2, \\
rt^{t-1} + o(q^{t-1}) & \text{otherwise.}
\end{cases} \]

In this case, \( \mathcal{F} \) is type (II.3) if \( r = 1; \) type (II.2) if \( D_1 \neq \phi \) and \( 1 < r < t - 1; \) and type (II.1) if \( D_1 = \phi \) and \( 1 < r < t. \) (Note that if \( D_1 \neq \phi, \) then \( t \geq 3. \) Also, if \( t = 3, \) then \( \mathcal{F} \) is type (II.1).)

**Subcase 2.4.2.** Assume that \( \sum_{j \geq 1} |A_j| \) is greater than \( \frac{3}{2} q^{t-1}. \) Using a similar argument as in lemma 3.3.4., we can choose \( q_8 = q_8(t,c) \) so that if \( q > q_8, \) then there exists a pair of entries \( (k_3, k_4) \) such that \( k_3, k_4 \leq t \) and \( \sum_{j \geq 1} \{(u_k) \in A_j; u_{k_3} = 1, u_{k_4} = 2\} \) is greater than \( c(\frac{3}{2})^{-1} q^{t-1}. \) Then \( u_{k_3} = 1 \) or \( u_{k_4} = 2 \) for every vertex \( u \cdot (u_k) \) in \( A_1. \) Also, if \( q > q_9 \) for some \( q_9 = q_9(t,c) \geq q_8, \) then

\[ A_j \supseteq \{(u_k); u_{k_3} = 1, u_{k_4} = 2\} \]

for all \( j \geq 1. \) If \( q > q_9, \) then we can show that this case is isomorphic to subcase 2.4.1. by exchanging \( n^{th} \)-entry with \( k_3^{th} \)-entry (of \( \mathcal{F} \)) (since \( A_1 \) contains the vertex \( x). \)
Remark: (1). If $t=3$, type (H.2) can not occur. So, there are three different families of maximal 3-cliques in $H(n,q)$ with size greater than $cq^{t-1}$ for some $c>0$.

(2). If $t=2$, type (H.2) and type (H.3)' can not occur. (cf. Theorem 3.2.1.)

(3). Type (H.1) and type (H.4) are extreme cases of type (H.2) with $P_1=\varnothing$ and $P_2=\varnothing$, respectively.

(4). Type (H.4) is an extreme case of type (H.3)' with $P_j=\varnothing$ for all $j\geq t-1$, also.

(5). In type (H.2). if r is close to t-2, we can express $\pi$ very explicitly. For example, if $r=t-2$, then there are only two non-isomorphic maximal t-cliques of this type: $D_1$ and $D_2$ are either

(i). $D_1=\{(1,1), (2,2)\}$, $D_2=\{(2,1), (1,2)\}$

or

(ii). $D_1=\{(1,1)\}$, $D_2=\{(u_1,1), (1,v_2): u_1, v_2=1,2,\ldots,q\}$

up to isomorphism. For each case.
\[ |\mathcal{F}| = q^2 \left( \binom{t-3}{1} (q-1)^{t-3} - \binom{t-5}{2} (q-1)^{t-4} - \ldots - \binom{t-2}{t-2} (q-1)^0 \right) \]
\[ + 2(q-1)^{t-2} - 2(n-t)(q-1) \]

and

\[ |\mathcal{F}| = q^2 \left( \binom{t-3}{1} (q-1)^{t-3} - \binom{t-5}{2} (q-1)^{t-4} - \ldots - \binom{t-2}{t-2} (q-1)^0 \right) \]
\[ + (q-1)^{t-2} - (2q-1)(n-t)(q-1) \].

(6). From Corollary 3.3.1. and remark (5), if \( n > t \geq 4 \), \( n \geq 2t-4 \) (or \( n < 2t-1 \)) and \( q \) is sufficiently large compared to \( n \) and \( t \), we can characterize up to the sixth largest (or the fifth largest) maximal \( t \)-cliques, which are type \((\text{II.1}), \text{ (II.1)} \) with \( r = 2 \), \( t-1 \), \( t-2 \) and type \((\text{II.2}) \) with \( r = t-2 \). (If \( n < 2t-4 \), type \((\text{II.1}) \) with \( r = 2 \) will not be included.)
Chapter 4
Maximal t-cliques of Jq(n,k)

4.1 Introduction

Let V be an n-dimensional vector space over GF(q). The q-analog of the Johnson graph, denoted J_q(n,k), on V has vertex set \([V]_k\), the collection of linear subspaces of dimension k. Two vertices u and v of J_q(n,k) are adjacent whenever \(\dim(u \cap v) = k-1\). Obviously, for any \(i = 1, 2, \ldots, k\), \(d(u, v) = i\) if and only if \(\dim(u \cap v) = k-i\). Also, the diameter of the q-analog of Johnson graph J_q(n,k) is k.

A linear subspace of V of dimension m is called an m-space (of V). Write \([n]_m\) for the number of m-spaces of V. Then,

Lemma 4.1.1.

\[ [n]_m = (q^n-1)(q^{n-1}-1) \ldots (q^{n-m+1}-1)/(q^m-1)(q^{m-1}-1) \ldots (q-1) \]
(2). \( \binom{n}{m} = \binom{n}{n-m} \).

(3). \( q^{(n-m)m} < \binom{n}{m} < q^{(n-m+1)m} \), if \( 0 < m < n \).

**Proof:** (3). If \( 0 < m < j \), then \( q^{j-m} < \frac{(q^{j-1})}{(q^{m-1})} < q^{j-m+1} \). []

**Lemma 4.1.2.** [3, p.7] Let \( V \) be an \( n \)-dimensional vector space over \( \text{GF}(q) \). Then,

(1). If \( X \) is a \( j \)-space of \( V \), then
\[
|\{Y \subseteq V : Y \text{ is a } k \text{-space of } V \text{ and } X \cap Y = O\}| = \binom{n-j}{k-j} q^k j
\]
where \( O \) is the 0-space of \( V \).

(2). If \( X \) is a \( j \)-space of \( V \) and \( Z \) is an \( m \)-subspace of \( X \), then
\[
|\{Y \subseteq V : Y \text{ is a } k \text{-space of } V \text{ and } X \cap Y = Z\}| = \binom{n-j}{k-m} q^{(k-m)(j-m)}.
\]
Especially,
\[
|\{Y \subseteq V : Y \text{ is a } k \text{-space of } V \text{ and } Y \supseteq Z\}| = \binom{n-m}{k-m} q^k.
\]

(3). If \( X \) is \( j \)-space of \( V \), then
\[
|\{Y \subseteq V : Y \text{ is a } k \text{-space of } V \text{ and } X \cap Y \text{ is an } m \text{-space}\}|
\leq \binom{n-j}{k-m} q^{(k-m)(j-m)}.
\]

If \( X \) is a subspace of \( V \) and \( a \) is an element of \( V \), we will denote \(<X, a>\) as the space spanned by \( X \) and \( a \).
4.2 Maximal t-cliques of $J_q(n,k)$ with size at least $cq^{(n-k)(t-1)}$

Suppose $\mathcal{F}$ is a maximal t-clique of $J_q(n,k)$. If $k=t$, $\mathcal{F}$ is the set of vertices of $J_q(n,k)$ itself. So let us assume $k>t$. Hsieh [13] shows that if either (i). $n \geq 2k+2$ or (ii). $n \geq 2k+1$ and $q \geq 3$, then the size of $\mathcal{F}$ is bounded above by $\binom{n-(k-t)}{t}$. And $\mathcal{F}$ attains the upper bound only if $\mathcal{F}$ contains all the vertices which contain a fixed (k-t)-space of V.

For the maximal t-cliques of $J_q(n,k)$ with asymptotically large size, we have the following theorem:

**Theorem 4.2.1.** Let $k$ and $t$ be fixed integers satisfying $2 \leq t < k$ and $q$ be a prime power. For any fixed constant $c > 0$, there exists $n_0 = n_0(q,k,t,c)$ such that if $n > n_0$ and $\mathcal{F}$ is a maximal t-clique of the q-analog Johnson graph $J_q(n,k)$ satisfying $|\mathcal{F}| > cq^{(n-k)(t-1)}$, then $\mathcal{F}$ is one of the following up to isomorphism:

1. $\mathcal{F}$ is the set of all vertices of $\binom{V}{k}$ which contain a fixed (k-t)-space and $|\mathcal{F}| = \binom{n-k-t}{t} = q^{(n-k)t-o(q(n-k)t)}$.
2. For each $s = 2, 3, ..., t-1$,
   
   $\mathcal{F} = \{ u \subseteq \binom{V}{k} : Q \subseteq u, \dim(u \cap P) \geq k-t+1 \}$
   
   $\cup \{ u \subseteq \binom{V}{k} : \dim(u \cap Q) = k-t-1, \dim(u \cap P) = k-t+s-1 \}$
   
   where $P$ is a $(k-t-s)$-space of $V$ and $Q$ is a (k-t)-space of $P$. In this case,
\[ |\mathcal{F}| = \sum_{1 \leq s \leq t} \binom{n-(k-t+s)}{t-s} q^{(t-1)(s-1)} + \binom{n-(k-t+s+2)}{t-2} q^{s(t-2)} + \cdots + \binom{n-(k-t+s)}{t-s} q^{t(2s-s)} + |\mathcal{D}| - |\mathcal{E}|. \]

For \( s=2,3,\ldots,t, \)
\[ \mathcal{F} = \{ u \in \mathcal{V} : \text{Q} \subseteq u, \dim(u \cap P) \geq k-t+1 \} \cup \mathcal{D} \cup \mathcal{E} \]
where \( P : \) a \((k-t+s)\)-space of \( V \),
\( Q : \) a \((k-t)\)-space of \( P \),
\( \phi \neq \mathcal{D} \subseteq \{ u \in \mathcal{V} : u \cap P = Q \} \).
\( \phi \neq \mathcal{E} \subseteq \{ u \in \mathcal{V} : \dim(u \cap Q) = k-t-1, \dim(u \cap P) = k-t+s-1 \} \),
\[ \cap \{ u + Q : u \in \mathcal{E} \} = P, \]
and \( \mathcal{D} \cup \mathcal{E} \) is maximal among the sets of vertices of \( J_q(n,k) \) satisfying the above conditions. In this case, \( |\mathcal{D}| \leq \sum_{1 \leq s \leq t} \binom{n-(k-t+s)}{t-2} q^{s(t-2)} \)
\[ = q^{2t}q^{(n-k)(t-2)} + o(q^{2t}q^{(n-k)(t-2)}) \]
\[ |\mathcal{E}| \leq \sum_{1 \leq s \leq t} \binom{n-(k-t+s+1)}{t-s} q^{(t-s)(s-1)} \]
\[ = q^{k-s-1}q^{(n-k)(t-s)} + o(q^{k-s-1}q^{(n-k)(t-s)}). \]

Therefore,
\[ |\mathcal{F}| = \binom{n-(k-t+s)}{t-1} q^{(t-1)(s-1)} + \binom{n-(k-t+s)}{t-2} q^{s(t-2)} + \cdots + \binom{n-(k-t+s)}{t-s} q^{t(2s-s)} + |\mathcal{D}| - |\mathcal{E}|. \]

\[ = q^{s-1}q^{(n-k)(t-1)} + o(q^{s-1}q^{(n-k)(t-1)}), \quad 2 \leq s \leq t. \]

\((J_q.4)\) \[ \mathcal{F} = \{ u \in \mathcal{V} : R \subseteq u \} \cup \mathcal{K} \]
where $R$ is a $(k-t+1)$-space of $V$ and $K = K^1 \cup K^2 \cup \ldots \cup K^s$ for some $s$ with $2 \leq s \leq \frac{k-t+1}{k-t}$ satisfying

(i). \( \dim(\cap K^i) \cap R = k-t \) for each $i$,

(ii). $K$ is a $t$-clique,

(iii). if $i \neq j$, $\cap K^i \neq \cap K^j$,

(iv). $\cap \{u + R : u \in K\} = R$,

(v). for each $j$, there exists a vertex $u$ in $\cup \{K^i : i \neq j\}$ satisfying $\dim((\cap K^i) \cap R, a \cap u) = k-t-1$ for any $(k-t+1)$-space $(\cap K^i) \cap R, a > \cap u$ which is different from $R$,

and $K$ is maximal among the sets of vertices of $J_q(n,k)$ satisfying the above conditions.

In this case,

\[
|K| \leq \frac{k-t+1}{1}
\]

\[
= q^{k-t} q^{(n-k)(t-2)} o(q^{(n-k)(t-2)}),
\]

\[
|\mathcal{F}| \leq \frac{n-(k-t-1)}{t-1} + |K|
\]

\[
= q^{(n-k)(t-1)} - o(q^{(n-k)(t-1)}).
\]

Before we prove theorem 4.2.1., we need some definitions and lemmas. If we fix a vertex $z$ in a maximal $t$-clique $\mathcal{F}$, then

\[
\dim(u \cap z) \geq k-t
\]

for every vertex $u$ in $\mathcal{F}$. Let us define $\mathcal{B}_i$

\[
\mathcal{B}_i = \{u \in \mathcal{F} : \dim(u \cap z) = i\}
\]
for \( i = k-t, k-t+1, \ldots, k \). Then,

**Lemma 4.2.1.**

1. \( \mathcal{F} = B_{k-t} \cup B_{k-t+1} \cup \ldots \cup B_k \).
2. \( |B_i| \leq \binom{n-k}{k-i} q(k-i)^2 \).
3. In other words, for each \( j = 0, 1, 2, \ldots, t \),
   \[ |B_{k-t+j}| \leq \binom{n-k}{k-j}q(t-j)^2 < q(t-j)(n-t+j+2). \]

**Proof:** (2). By lemma 4.1.2., (3).

(3). By lemma 4.1.1., (3). \[ \square \]

**Lemma 4.2.2.** For fixed integers \( k \) and \( t \) with \( 2 < t < k \), there exists \( n_o = n_o(k, t) \) such that
\[ \sum_{2 \leq j \leq t} |B_{k-t+j}| < q \frac{n}{2} q(n-k)(t-1) \]
if \( n > n_o \).

**Proof:** From lemma 4.2.1. (3), \( |B_{k-t+j}| < q(t-j)(n-t+j+2) \). Set \( f(x) = (t-x)(n-t+x-2) = -x^2 - x(2t-n-2) + t(n-t+2) \). Then we can choose \( n_o = n_o(k, t) \) so that
(i). \( f(x) \) decreases as \( x \) increases from 2 to \( t \) and
(ii). \( f(2) < (n-k)(t-1)-\frac{n}{2} - 1 \).

for every \( n > n_o \). Therefore,
\[ \sum_{2 \leq j \leq t} |B_{k-t+j}| < \frac{n}{2} q(n-k)(t-1) \]
if \( n > n_o \). \[ \square \]
Lemma 4.2.3. For fixed integers \( k \) and \( t \) with \( 2 \leq t < k \), there exists \( n_0 = n_0(k, t) \) such that if \( n > n_0 \) and \( |B_{k-t+1}| > q^{-\frac{a}{2}} q^{(n-k)(t-1)} \), then \( z \) contains a \((k-t+1)\)-subspace \( R \) such that \( \dim(u \cap R) \geq k-t \) for every vertex \( u \) in \( \mathcal{F} \). Therefore,

\[
\{ u \in [V] : R \subseteq u \} \subseteq \mathcal{F}.
\]

Proof: Suppose that for every \((k-t+1)\)-subspace \( R \) of \( z \), there exists a vertex \( y \) in \( \mathcal{F} \) such that \( \dim(y \cap R) = k-t-j \) for some \( j > 0 \). Then for every vertex \( u \) in \( B_{k-t+1} \), there exists a vertex \( v \) in \( \mathcal{F} \) such that \( \dim(u \cap v \cap z) = k-t-j \) for some \( j > 0 \).

Fix a vertex \( x \) in \( B_{k-t+1} \). Then

\[
|\{ u \in B_{k-t+1} : \dim(u \cap x \cap z) \leq k-t-2 \} | \\
\leq \sum_{2 \leq \alpha} \sum_{|k-t-\alpha|, t-1} \sum_{1 \leq a} q(\alpha+1)(\alpha+1) \\
\times \sum_{\int_{j=0}^{t-1} (t-1) \cdot (\alpha+1)(t-1) \cdot (\alpha+1)(t-1-\alpha-j) \cdot (t-1-\alpha-j) \cdot (t-1-\alpha-j) \cdot (t-1-\alpha-j) \cdot (t-1-\alpha-j)} \\
< q(n-k)(t-1) \sum_{a \geq 2} \sum_{j \geq 0} (q-2a^2-\alpha(-n-2k+2t-4)+(2k-1)q-j^2-j(2\alpha-2t+n-k+2))
\]

by lemma 4.1.2. and lemma 4.2.1. (Note that if \( \{ u \in B_{k-t+1} : \dim(u \cap x \cap z) \leq k-t-\alpha \} \neq \emptyset \) for some \( \alpha > 0 \), then \( \alpha + 1 \leq t-1 \).) Set \( g(x) = -2x^2 + x(-n-2k+2t-4)+(2k-1) \). Then we can choose \( n_1 = n_1(k, t) \) so that the following holds for every \( n > n_1 \):

(i). \( g(x) \) decreases as \( x \) increases from 2,

(ii). \( g(2) < -\frac{n}{2}-3 \)
(iii). \(-j^2-j(2\alpha-2t-n-k-2)<0\) for all \(j>0\) and \(\alpha\geq2\).

Hence,

\[
|\{u \in B_{k-t+1}: \dim(u\cap x\cap z) \leq k-t-2\}| < q^{-\frac{n}{2}-1}q^{(n-k)(t-1)}.
\]

Also,

\[
|\{u \in B_{k-t+1}: \dim(u\cap x\cap z) \geq k-t-1\}|
\leq (1+|^{k-t+1}|_{k-t}^{-1}q-|^{k-t+1}|_{k-t-1}^{-1}|^{t-j}|q^4)
\sum_{j \geq 0}q^{j}q^{(1-j)(t-1)n-k-t}q^{(t-2-j)(2t-2-j)}
\]

\[
< (q^{k+1}+q^{2k})q^{(n-k)(t-1)}q^{-n+k+3t-3}q^{\sum_{j \geq 0}j^{2}-j(n-k-2t+3)}
\]

by lemma 4.1.2., lemma 4.2.1. Again, for some \(n_2=n_2(k,t)\),

\[
|\{u \in B_{k-t+1}: \dim(u\cap x\cap z) \geq k-t-1\}| < q^{-\frac{n}{2}-1}q^{(n-k)(t-1)}
\]

if \(n>n_2\). Hence, if \(n>n_o\) where \(n_o=\max(n_1,n_2)\), \(|B_{k-t+1}|<q^{-\frac{n}{2}}q^{(n-k)(t-1)}\).

Lemma 4.2.4. For fixed integers \(k\) and \(t\) with \(2\leq t<k\), there exists \(n_o=n_o(k,t)\) such that if \(n>n_o\) and \(|B_{k-t}|>q^{-\frac{n}{2}}q^{(n-k)(t-1)}\), then \(z\) contains a \((k-t)\)-subspace \(R\) such that \(\dim(u\cap R) \geq k-t-1\) for every vertex \(u\) in \(\mathcal{F}\).

Proof: Suppose that for every \((k-t)\)-subspace \(R\) of \(z\), there exists a vertex \(y\) in \(\mathcal{F}\) such that \(\dim(y\cap R)=k-t-j\) for some \(j>1\). Then for every vertex \(u\) in \(B_{k-t}\), there exists a vertex \(v\) in \(\mathcal{F}\) such that \(\dim(u\cap v\cap z)=k-t-j\) for some \(j>1\).
Fix a vertex $x$ in $\mathcal{B}_{k-t}$. Then

$$\left|\{u \in \mathcal{B}_{k-t}: \dim(u \cap x \cap z) \leq k-t-2\}\right|$$

$$\leq 2 \sum_{\alpha \geq 2} q \alpha^2 \left( \sum_{j \geq 0} q \cdot \frac{q^{j+1} \cdot \left( n-(k+t) \cdot (t-\alpha-j) \cdot (2t-\alpha-j) \right)}{t-\alpha-j} \right)$$

$$< q(n-k)(t-1) \sum_{\alpha \geq 2} \sum_{j \geq 0} q \alpha^2 q^{j+1} \left( \frac{q^{j+1} \cdot \left( n-(k+t) \cdot (t-\alpha-j) \cdot (2t-\alpha-j) \right)}{t-\alpha-j} \right)$$

by lemma 4.1.2. and lemma 4.2.1.. (Note that if $\{u \in \mathcal{B}_{k-t}: \dim(u \cap x \cap z) \leq k-t-\alpha\} \neq \emptyset$ for some $\alpha > 0$, then $\alpha < t$.) Set $g(x) = -2x^2 + x(-n-2k+2t+2) + (n-k+t)$. Then we can choose $n_1 = n_1(k,t)$ so that the following holds for every $n > n_1$:

(i). $g(x)$ decreases as $x$ increases from $2$.

(ii). $g(2) < -\frac{1}{2} - 3$.

(iii). $-j^2 - j(2\alpha - 2t - n-k) < 0$ for all $j > 0$ and $\alpha \geq 2$.

Hence,

$$\left|\{u \in \mathcal{B}_{k-t}: \dim(u \cap x \cap z) \leq k-t-2\}\right| < q^{-\frac{n}{2} - 1} q^{(n-k)(t-1)}$$

Also,

$$\left|\{u \in \mathcal{B}_{k-t}: \dim(u \cap x \cap z) \geq k-t-1\}\right|$$

$$\leq (1 + \sum_{\alpha \geq 2} q \alpha^2) \sum_{j \geq 0} \frac{q(n-k \cdot (t-\alpha-j) \cdot (2t-\alpha-j))}{t-\alpha-j}$$

$$< q(n-k)(t-1) q^{-n+2k+5t-3} \sum_{j \geq 0} q^{-j - j(n-k-2t+4)}$$

by lemma 4.1.2., lemma 4.2.1.. Again, for some $n_2 = n_2(k,t)$,

$$\left|\{u \in \mathcal{B}_{k-t}: \dim(u \cap x \cap z) \geq k-t-1\}\right| < q^{-\frac{n}{2} - 1} q^{(n-k)(t-1)}$$

if $n > n_2$. Hence, if $n > n_0$ where $n_0 = \max(n_1, n_2)$, $\mathcal{H}_{k-t} < q^{-\frac{n}{2}} q^{(n-k)(t-1)}$. []
Proof of Theorem 4.2.1.: Let us fix a constant $c > 0$, a vertex $z$ in $\mathcal{F}$ and define $B_{t}$, $k - t \leq i \leq k$, as before. If $n > n_{1}$ for some $n_{1} = n_{1}(q, k, t, c)$, then $q^{-\frac{n}{2}} < \frac{c}{3}$ and either $|B_{k-t+1}|$ or $|B_{k-t}|$ has size greater than $\frac{c}{3}q^{(n-k)(t-1)}$ ($> q^{-\frac{n}{2}}q^{(n-k)(t-1)}$) by lemma 4.2.2. Suppose $n > n_{1}$.

Case 1. Assume that $|B_{k-t+1}| > \frac{c}{3}q^{(n-k)(t-1)}$. By lemma 4.2.3., if $n > n_{2}$ for some $n_{2} = n_{2}(q, k, t, c) > n_{1}$, there exists a $(k-t-1)$-subspace of $z$ such that $\dim(u \cap R) \geq k-t$ for every vertex $u$ in $\mathcal{F}$. So let us suppose $n > n_{2}$ and $\mathcal{R} = \{R_{1}, \ldots, R_{r}\}$ is the collection of all such $(k-t-1)$-subspaces of $z$. Then

1. $1 \leq r \leq \frac{k}{k-t+1}$.
2. For every vertex $u$ in $\mathcal{F}$, $\dim(u \cap R_{i}) \geq k-t$ for all $i \leq r$.
3. $\{u \subseteq V : R_{i} \subseteq u$ for some $i\} \subseteq \mathcal{F}$.
4. $\dim(R_{i} \cap R_{j}) = k-t$ and $\dim(R_{i} + R_{j}) = k-t + 2$ for any $i$ and $j$ with $1 \leq i < j \leq r$.

We will discuss this case by three subcases:

(i). $\dim(\cap R_{j}) < k-t$,
(ii). $\dim(\cap R_{j}) = k-t$,
(iii). $\dim(\cap R_{j}) = k-t + 1$.

Subcase 1.1. Let us assume that $\dim(\cap R_{j}) < k-t$. Then, $r \geq 3$. If we fix
R_1 and R_2, then R_1 \cap R_2 \not\subseteq R_i for some i \leq r. For such a (k-t+1)-space R_i,

\[ k-t+1 = \dim R_i \geq \dim(R_i \cap R_1 + R_i \cap R_2) \]
\[ \geq \dim(R_i \cap R_1) + \dim(R_i \cap R_2) - \dim(R_i \cap R_1 \cap R_2) \]
\[ = (k-t) + (k-t) - \dim(R_i \cap R_1 \cap R_2). \]

Hence, \(\dim(R_i \cap R_1 \cap R_2) = k-t-1\) and \(\dim R_i = \dim(R_i \cap R_1 + R_i \cap R_2)\), which implies
\[ R_i = R_i \cap R_1 + R_i \cap R_2 \]
\[ \subseteq R_1 + R_2. \]

If \(R_1 \cap R_2 \subseteq R_j\) for some \(j \neq 1, 2\), then \(R_j \cap R_1 \cap R_2 = R_1 \cap R_2 = R_j \cap R_1 = R_j \cap R_2\) by comparing dimensions. Also, since \(R_j \cap R_1 \cap R_i = R_1 \cap R_2 \cap R_i\) for some \(R_i \in \mathcal{R}\) with \((R_1 \cap R_2) \not\subseteq R_i\),
\[ k-t+1 = \dim(R_j) \geq \dim(R_j \setminus R_1 - R_j \setminus R_i) \]
\[ = \dim(R_j \setminus R_1) + \dim(R_j \setminus R_i) - \dim(R_j \setminus R_1 \setminus R_i) \]
\[ = k-t-1. \]

This implies that \(R_j = R_j \setminus R_1 - R_j \setminus R_i \subseteq R_j \setminus (R_1 - R_i) \subseteq (R_1 - R_i) \subseteq (R_1 + R_2).\)

From above, we get that if \(R \not\subseteq \mathcal{R}\), then \(R\) is a \((k-t+1)\)-subspace of a \((k-t+2)\)-space \((R_1 + R_2)\), and \(R_i + R_j = R_1 + R_2\) for any \(i\) and \(j\) with \(1 \leq i < j \leq r\).

Also, if \(u\) is a vertex of \(\mathcal{F}\), then
\[ \dim(u \cap (R_1 + R_2)) \geq k-t+1 \]
since \(\dim(\cap R_i) < k-t\). Hence,
\[ \mathcal{F} \subseteq \{u \subseteq [V^k] : \dim(u \cap (R_1 + R_2)) \geq k-t+1\}. \]
Since $\mathcal{F}$ is maximal,

$$\mathcal{F} = \{u \in \mathcal{V}^k : \dim(u \cap (R_i + R_j)) \geq k-t+1\}$$

which is type ($Jq.2$) with $s=2$. In fact, $\mathcal{R}$ is the set of all $(k-t-1)$-subspaces of $R_1 + R_2$. For this case,

$$|\mathcal{F}| = \frac{(k-t+2)!}{(k-t+1)!} \frac{n-(k-t+2)!}{t-1!} q(t-1) \cdot \frac{n-(k-t+2)!}{t-2!}$$

$$= q(k-t+1) q(n-k)(t-1) - o(q(k-t+1) q(n-k)(t-1)).$$

Subcase 1.2. Let us assume $\dim(\cap R_j) = k-t$ and set $Q = \cap R_j$, $P = \Sigma\{R_i : i=1,2,...,r\}$ and $\dim P = k-t-s$. Then, $2 \leq s \leq t$ and $2 \leq r \leq \frac{t}{s}$.

If every vertex in $\mathcal{F}$ contains $Q$, then

$$\mathcal{F} = \{u \in \mathcal{V}^k : Q \subseteq u\}$$

which is the maximum one (type ($Jq.1$)). In this case, $s=t$, $r=\frac{t}{s}$ and

$$|\mathcal{F}| = q(n-k)^t - o(q(n-k)^t).$$

Now, let us assume that $\mathcal{F}$ has a vertex which does not contain $Q$. Then

(1.2.1). If $u$ is a vertex of $\mathcal{F}$ which does not contain $Q$, then $\dim(u \cap Q) = k-t-1$ and $\dim(u \ominus (R_i + R_j)) = k-t-1$ for any $i$ and $j$ with $1 \leq i < j \leq r$. If $u$ is a vertex of $\mathcal{F}$ such that $u \not\supseteq Q$, then $\dim(u \cap Q) \leq k-t-1$.

For any $i$ and $j$ with $1 \leq i < j \leq r$,

$$k-t+2 = \dim(R_i - R_j)$$

$$> \dim(u \cap (R_i + R_j))$$
\[
\geq \dim(u \cap R_i) + \dim(u \cap R_j) - \dim(u \cap R_i \cap R_j)
\]
\[
= 2(k-t) - \dim(u \cap R_i \cap R_j)
\]
which implies that \(\dim(u \cap R_i \cap R_j) = \dim(u \cap R_j) = k-t-1\) and \(\dim(u \cap (R_i - R_j)) = k-t+1\).

(1.2.2). For any \(i\) and \(j\) with \(1 \leq i < j \leq r\), \(R\) contains all \((k-t+1)\)-subspaces of \(R_i + R_j\) which contain \(Q\): Let \(R\) be a \((k-t+1)\)-subspace of \(R_i - R_j\) which contains \(Q\). It suffices to show that \(\dim(u \cap R) \geq k-t\) for each vertex \(u\) in \(\mathcal{F}\) which does not contain \(Q\). Let \(u\) be such a vertex of \(\mathcal{F}\). Since \(\dim(u \cap (R_i - R_j)) = k-t+1\) by (1.2.1),
\[
k-t+2 = \dim(R_i - R_j) \geq \dim((u \cap (R_i + R_j)) - R)
\]
\[
= \dim(u \cap (R_i + R_j)) + \dim(R) - \dim(u \cap (R_i + R_j) \cap R)
\]
\[
= (k-t+1) - (k-t-1) - \dim(u \cap R)
\]
which implies \(\dim(u \cap R) \geq k-t\).

(1.2.3). In general, if \(u\) is a vertex of \(\mathcal{F}\) satisfying \(\dim(u \cap Q) = k-t-1\), then
\[
\dim(u \cap (\Sigma\{R_i: i \in J\})) = k-t+m-1
\]
where \(\dim\Sigma\{R_i: i \in J\} = k-t-m, J \subseteq \{1,2,\ldots,r\}\) and \(2 \leq m \leq s\): Let us show by induction on \(m\). If \(m=2\), the assertion holds by (1.2.1). Assume that it is true for all \(m_1 < m\). If \(\dim\Sigma\{R_i: i \in J\} = k-t+m\) for some \(J \subseteq \{1,2,\ldots,r\}\), then \(\dim(u \cap \Sigma\{R_i: i \in J, i \notin j\}) = k-t+m-1\) for some \(j \in J\), and
\[
k-t+m > \dim(u \cap \Sigma\{R_i: i \in J\})
\]
\[
\geq \dim(u \cap \Sigma\{R_i: i \in J, i \notin j\}) + \dim(u \cap R_j)
\]
\[-\dim(u\cap\bigcup\{R_i: i\in J, i\neq j\}) \geq (k-t+m-2)+(k-t)-(k-t-1) = k-t+m-1\]

since \(\bigcup\{R_i: i\in J, i\neq j\}\cap R_j = Q\). That implies \(\dim(u\cap\bigcup\{R_i: i\in J\}) = k-t-m-1\).

(1.2.4). \(K\) contains all \((k-t+1)\)-subspace \(R\) of \(P\) which contains \(Q\): Let \(R\) be a \((k-t+1)\)-subspace of \(P\). Then it suffices to show that \(\dim(u\cap R) \geq k-t\) for every vertex \(u\) in \(\mathcal{F}\) such that \(u\cap Q\). For such a vertex \(u\),

\[
k-t+s = \dim P \\
\geq \dim(u\cap P + R) \\
= \dim(u\cap P) - \dim(R) - \dim(u\cap P \cap R) \\
= (k-t-s-1) - (k-t-1) - \dim(u\cap R) \quad \text{by (1.2.3)},
\]

which implies that \(\dim(u\cap R) \geq k-t\).

By (1.2.1) and (1.2.4), we can partition \(\mathcal{F}\) into three subsets as follows;

\[
\mathcal{F} = \mathcal{C} \cup \mathcal{D} \cup \mathcal{E}
\]

where \(\mathcal{C} = \{u\in V_k: Q \subseteq u, \dim(u\cap P) \geq k-t+1\}\)

\(\mathcal{D} = \{u\in \mathcal{F}: u\cap P = Q\}\)

\(\mathcal{E} = \{u\in \mathcal{F}: \dim(u\cap Q) = k-t-1, \dim(u\cap P) = k-t+s-1\} \quad (\neq 0).\)

Then, \(\partial(\mathcal{E}, \mathcal{E}) \leq t\). So, it is enough to consider the relation \(\partial(\mathcal{D}, \mathcal{E}) \leq t\). Note that if \(u\) is a vertex of \(\mathcal{E}\), then \(\mathcal{E}\) contains every vertex \(u'\) such
that \( \dim(u' \cap Q) = k-t-1 \) and \( T \subseteq u' \) where \( T \) is the complement of \( u \cap Q \) in \( u \). (For each vertex \( v \) in \( D \),
\[
    v \cap u = v \cap (u \cap Q + T)
    = v \cap u \cap Q + v \cap T
    = u \cap Q + v \cap T,
\]
and so, \( \dim(v \cap u) = \dim(u \cap Q) + \dim(v \cap T) \geq k-t \) since \( Q \cap T = 0 \). Similarly, we can show that
\[
    \dim(v \cap u') = \dim(u' \cap Q + v \cap T) = \dim(u' \cap Q) + \dim(v \cap T) \geq k-t.
\]

Subcase 1.2.1. If \( D = \emptyset \), then
\[
    \mathcal{F} = \{ u \in V_k^q \mid u \subseteq Q, \dim(u \cap P) \geq k-t-1 \}
    \cup \{ u \in V_k^q \mid \dim(u \cap Q) = k-t-1, \dim(u \cap P) = k-t-s-1 \}
\]
and \( s > 2 \). (If \( s = 2 \), \( \mathcal{R} \) is the set of all \((k-t+1)\)-subspace of \( P \) and so \( \dim(R_j) = 0 \).) In this case, \( \mathcal{F} \) is type \(( \text{Jq.2} )\) and
\[
    |\mathcal{F}| = \sum_{1 \leq j \leq s} q^{n-(k-t+s)}q^{(t-j)(s-j)}q^{(k-t)}q^{(t-s+1)}
    = q^{s-1}q^{(n-k)(t-1)} + o(q^{s-1}q^{(n-k)(t-1)}) \quad (2 \leq s \leq t).
\]

Subcase 1.2.2. Let us assume \( D \neq \emptyset \) (and \( \mathcal{E} \neq \emptyset \)). Define
\[
    E = \cap\{ u-q : u \in \mathcal{E} \}.
\]
Since \( P \subseteq u + Q \) for every vertex \( u \) in \( \mathcal{E} \), \( P \subseteq E \). Put \( E = P - T \) where \( P \cap T = \emptyset \). Then \( 0 \leq \dim T \leq t+1-s \), since \( \dim E \leq k+1 \).
If \(0<\dim(T)<t+1-s\), i.e., \(<a>\subseteq T\) for an 1-space \(<a>\) of \(V\), then \(\mathcal{F}\) contains every vertex containing \(<Q,a>\), a \((k-t+1)\)-space which is not contained in \(\mathcal{R}\). By choosing a vertex \(w\) instead of \(z\) where \(w\supseteq <P,a>\), we can show that this case is isomorphic to the case with \(\dim P=k-t+s+1\) \((\leq k-1)\).

So, without loss of generality, we may assume \(\dim T=0\) or \(\dim T=t+1-s\). If \(\dim T=t+1-s\), then \(\dim E=k+1\) and, for every vertex \(u\) in \(\mathcal{E}\), \(u+Q=P+T\) and so \(u\subseteq P+T\). Also, for every vertex \(v\) in \(\mathcal{D}\), \(\dim(v\cap(P-T))\geq k-t+1\). (Otherwise, \(v\cap(P+T)=Q\) and \(\dim(v\cap u) = \dim(v\cap u\cap(P+T)) = \dim(Q\cap u) = k-t-1\) for a vertex \(u\) in \(\mathcal{E}\), a contradiction.). If we put \(W=P+T\), then

\[
\mathcal{E} \subseteq \{u \in \mathcal{V}_k: Q \subseteq u, \dim(u\cap W) \geq k-t+1\},
\]

\[
\mathcal{E} \subseteq \{u \in \mathcal{V}_k: \dim(u\cap Q)=k-t-1, \dim(u\cap P)=k-t-s-1, u \subseteq W\}
\]

\[
= \{u \in \mathcal{V}_k: \dim(u\cap Q)=k-t-1, u \subseteq W\}.
\]

Also, since \(\mathcal{F}\) is maximal,

\[
\mathcal{F} = \{u \in \mathcal{V}_k: Q \subseteq u, \dim(u\cap W) \geq k-t+1\}
\]

\[
\cup \{u \in \mathcal{V}_k: \dim(u\cap Q)=k-t-1, u \subseteq W\}
\]

for a \((k+1)\)-space \(W\) and

\[
|F| = \frac{\binom{n-k}{t}}{k} + \frac{\binom{n-k-1}{t+1}}{t-1} q^{(t-1)} + \frac{\binom{n-k+1}{t+2}}{t-2} q^{(t-2)}(t-1) + \cdots
\]

\[
+ \frac{\binom{n-k+1}{t+1}}{t-1} q^{2} q^{(t-k)}(t-1) + o(q^{(n-k)}(t-1)).
\]

Hence, \(\mathcal{F}\) is type \((Jq.2)\) with \(s=t+1\).
Now let us assume that dim T=0 and count the size of \( \mathcal{F} \). For a fixed vertex \( x \) in \( \mathcal{E} \), \( Q \subseteq u \cap (x+Q) \) and \( k-t+1 \leq \dim(u \cap (x+Q)) \) for every vertex \( u \) in \( \mathcal{D} \). Also, if \( U \) is a \((k-t+1)\)-space of \( x+Q \) containing \( Q \), then there exists a vertex \( y \) in \( \mathcal{E} \) such that \( \dim((y+Q) \cap U) = k-t \) (since \( \dim T=0 \)). Hence, by comparing with such vertices \( x \) and \( y \) in \( \mathcal{E} \),

\[
|\mathcal{D}| \leq \left[ t-s+1 \right]_1 q^{s(t-s+1)} q^{n-(k-t+s+2)} q^{s(t-2)} = q^{2t} q^{(n-k)(t-2)} + o(q^{2t} q^{(n-k)(t-2)}).
\]

Also, by comparing with a vertex in \( U \) in \( \mathcal{D} \),

\[
|\mathcal{E}| \leq \left[ k-t-1 \right]_s q^{s(t-s)} q^{1} q^{n-(k-t+s+1)} q^{(t-s)} = q^{k+s-1} q^{(n-k)(t-s)} + o(q^{k+s-1} q^{(n-k)(t-s)}).
\]

Therefore,

\[
|\mathcal{F}| = \left[ s \right]_1 q^{(n-(k-t+s))} q^{(t-1)(s-1)} + \left[ s \right]_1 q^{n-(k-t+s)} q^{(t-2)(s-2)} + ... + \left[ s \right]_1 q^{(n-k)(t-s)} + |\mathcal{D}| + |\mathcal{E}| = q^{s-1} q^{(n-k)(t-1)} + o(q^{s-1} q^{(n-k)(t-1)}),
\]

and \( \mathcal{F} \) is type \((Jq.3)\) in this case.

**Subcase 1.3.** Assume that \( \dim(\cap R_j) = k-t+1 \). Then \( r=1 \). Set \( R = R_1 \).

Then \( \mathcal{F} \) has at least two vertices \( u \) and \( v \) such that \( \dim(u \cap R) = \dim(v \cap R) = k-t \) but \( u \cap R \neq v \cap R \). (Otherwise, \( \mathcal{F} \) contains all the vertices containing some fixed \((k-t)\)-subspace of \( R \) and \( \dim(\cap R_j) = k-t \).) Define

\[
Q = \{ u \cap R: u \in \mathcal{F}, \dim(u \cap R) = k-t \}
\]

\[
= \{ Q_1, ..., Q_s \},
\]

\[
K^i = \{ u \in \mathcal{F}: u \cap R = Q_i \}
\]
for each $i=1,2,...,s$. Then we get the following:

(i). $2 \leq s \leq \lfloor \frac{k-t+1}{k-t} \rfloor$.

(ii). $\mathcal{F} = \{u: R \subseteq u \cap K^1 \cup ... \cup K^s\}$.

(iii). Without loss of generality, we may assume that $\cap\{u+R: u \in K^i \text{ for some } i=1,2,...,s\} = R$. Otherwise, there exists a $(k-t+2)$-space $<R,a>$ such that

$$<R,a> \subset u+R$$

for every vertex $u$ in $K^1 \cup ... \cup K^s$. And so

$$<R,a> = <R,a> \cap (u+R) = (<R,a> \cap u) + (<R,a> \cap R)$$

$$= (<R,a> \cap u) - R.$$ 

By comparing dimensions, we get $\dim(<R,a> \cap u) = k-t+1$. Therefore,

$$K^1 \cup ... \cup K^s \subseteq \{u \in \mathcal{V}_i: \dim(u \cap <R,a>) = k-t-1\}$$

and

$$\mathcal{F} = \{u \in \mathcal{V}_i: \dim(u \cap <R,a>) \geq k-t+1\},$$

which is isomorphic to the one in subcase 1.1.

Now, let us count $|K^i|$ for each $i$. If $v$ is a vertex of $K^i$ for some $i$, then by comparing with a fixed vertex $x$ in $K^j$ ($j \neq i$) we get

$$<Q_i \cap Q_j,a> \subseteq v \cap x$$

where $<Q_i \cap Q_j,a>$ is a $(k-t)$-space. Since $v \cap R = Q_i$ and $x \cap R = Q_j$, $<Q_j,a> \cap R = Q_j$ and $\dim<Q_j,a> = k-t+1$.

Also, for each $(k-t+1)$-space $<Q_i,a>$, without loss of generality, we
may assume that there exists a vertex \( y \) in \( K^m \) for some \( m \neq i \) such that \( \dim(<Q_i,a> \cap y) = k-t-1 \). (If \( \dim(<Q_i,a> \cap u) \geq k-t \) for every vertex \( u \) in \( U \), then
\[ F \supseteq \{ u \in [V]_k : u \supseteq <Q_i,a> \}. \]
But \( <Q_i,a> \subseteq z \) since \( r=1 \). By choosing \( z' \) instead of \( z \) where \( z' \) is a vertex which contains \( R \cup <Q_i,a> \), we can show that this is isomorphic to subcase 1.1. or 1.2.

Therefore,
\[ |K^i| \leq \binom{n}{t} q_i^{n-(k-t+3)} q^{t-2} \]
for each \( i=1,2,...,s \), and
\[ |F| = \binom{n-(k-t+1)}{t-1} - |K^1| + |K^2| + ... - |K^s| \]
\[ = q(n-k)(t-1) + o(q(n-k)(t-1)), \]
which is type \((Jq.4)\).

**Case 2.** Assume that \( |\mathcal{B}_{k-t}| > \frac{5}{6} q(n-k)(t-1) \). If there is a \((k-t)\)-space \( R \) of \( z \) such that \( \dim(u \cap R) = k-t \) for every vertex \( u \) in \( F \), then
\[ F = \{ u \in [V]_k : R \subseteq u \}, \]
which is the largest maximal \( t \)-clique with size \( \binom{n-K-t}{t} \) (type \((Jq.1)\)).

Now, let us assume that there does not exist such a \((k-t)\)-space \( R \).

By lemma 4.2.4., if \( n > n_3 \) for some \( n_3 = n_3(q,k,t,c) \geq n_1 \), then there ex-
ists a \((k-t)\)-subspace \(R\) of \(z\) such that \(\dim(u \cap R) \geq k-t-1\) for every vertex \(u\) in \(\mathcal{F}\). So let us suppose that \(n > n_3\) and \(\mathcal{R} = \{R_1, ..., R_r\}\) be the collection of all such \((k-t)\)-subspaces of \(z\). Then, \(1 \leq r\). Also, by comparing with \(R_1, r \leq \lceil \frac{k-t}{k-t-1} \rceil q\) since \(\dim(R_i \cap R_j) = k-t-1\) for every \(i\) and \(j\) such that \(1 \leq i < j \leq r\). Notice that

\[ R_i \subseteq \cap \{u + R_i; u \in \mathcal{F}, \dim(u \cap R_i) = k-t-1\} \]

for all \(i \leq r\).

If \(\dim(\cap \{u + R_i; u \in \mathcal{F}, \dim(u \cap R_i) = k-t-1\}) > k-t\) for some \(R_i\), then for a vertex \(u\) in \(\mathcal{F}\) with \(\dim(u \cap R_i) = k-t-1\),

\[
\begin{align*}
k-t+1 &= \dim(<R_i,a> \cap (u + R_i)) \\
&= \dim(<R_i,a> \cap u + <R_i,a> \cap R_i) \\
&= \dim(<R_i,a> \cap u) + \dim R_i - \dim(<R_i,a> \cap u \cap R_i) \\
&= \dim(<R_i,a> \cap u) + (k-t) - (k-t-1)
\end{align*}
\]

where \(<R_i,a>\) is a \((k-t+1)\)-subspace of \(\cap \{u + R_i; u \in \mathcal{F}, \dim(u \cap R_i) = k-t-1\}\).

This implies \(\dim(<R_i,a> \cap u) = k-t\). Therefore,

\[
\{u \in \mathcal{V} \mid <R_i,a> \subseteq u\} \subseteq \mathcal{F}
\]

and this case is isomorphic to case 1.

Now let us assume that for every \((k-t)\)-space \(R_i\) in \(\mathcal{R}\),

\[ R_i = \cap \{u + R_i; u \in \mathcal{F}, \dim(u \cap R_i) = k-t-1\}. \]

For each \(R_i\) \((i=1,2, ..., r)\), consider a vertex \(y\) in \(\mathcal{F}\) such that \(\dim(R_i \cap y) = k-t-1\). If \(v\) is a vertex of \(B_{k-t}\) containing \(R_i\), then

\[ <R_i,a> \subseteq v \cap (y + R_i) \]
for some \((k-t+1)\)-space \(<R_i,a>\). (Otherwise, \(v \cap (y+R_i) = R_i\) and \(\dim(v \cap y) < \dim(v \cap (y+R_i)) = k-t\), a contradiction.) Note that \(<R_i,a> \cap v = R_i\) since \(v \cap z = R_i\).

Since \(R_i = \cap\{u+R_i : u \in \mathcal{I}, \dim(u \cap R_i) = k-t\-1\}\), there is a vertex \(w\) in \(\mathcal{I}\) such that \(\dim(w \cap R_i) = k-t-1\) and \((w+R_i) \cap <R_i,a> = R_i\). Then,

\[
\begin{align*}
k-t & = \dim R_i = \dim(w-R_i) \cap <R_i,a> \\
& = \dim(w \cap <R_i,a> + R_i \cap <R_i,a>) \\
& = \dim(w \cap <R_i,a>) + \dim R_i - \dim(w \cap <R_i,a> \cap R_i) \\
& = \dim(w \cap <R_i,a>) + (k-t) - (k-t-1)
\end{align*}
\]

and \(\dim(w \cap <R_i,a>) = k-t-1\).

By comparing with such vertices \(y\) and \(w\) in \(\mathcal{I}\), we get

\[
\{u \in B_{k-t} : R_i \supset u\} \leq \binom{t}{t-1}^t \binom{n-k}{t-2}^t < q^{(n-k)(t-1)-n+k+3t}.
\]

But this is true for all \(R_i, i = 1,2,\ldots,r\). Hence,

\[
\begin{align*}
B_{k-t} & < (\binom{k-t}{k-t-1})^t q^{(n-k)(t-1)-n+k+3t} < q^{-\frac{t}{2}} q^{(n-k)(t-1)} < \frac{1}{2} q^{(n-k)(t-1)},
\end{align*}
\]

if \(n > n_4\) for some \(n_4 = n_4(q,k,t,c) \geq n_3\). \(\square\)

Remark : (1). In type \((Jq,3)\), if \(s=t\), there exist only \((t-1)\) non-isomorphic maximal \(t\)-cliques which depend on \(Z = \cup\{u \supset P : u \in \mathcal{E}\}\): Let us put \(\dim Z = k+m\). Then, \(2 \leq m\) and

\[
\mathcal{E} \subseteq \{u \in V_k : \dim(u \cap Q) = k-t-1, \dim(u \cap P) = k-1, u \subseteq Z\}.
\]

(A)
If \( v \) is a vertex in \( D \), then for every vertex \( u \) in \( E \)
\[
\dim(v+P) \cap (u+P) \geq \dim(v \cap u + v \cap P + u \cap P + P \cap P)
\]
\[
= \dim(v \cap u + P)
\]
\[
= \dim(v \cap u) + \dim P \cdot \dim(v \cap u \cap P)
\]
\[
\geq (k-t) + k-(k-t-1)
\]
\[
= k+1.
\]
Hence, \( v+P \supseteq u+P \) since \( \dim(u+P) = k+1 \). By the definition of \( Z \),
\( v+P \supseteq Z \) for every vertex \( v \) in \( D \). Since for every vertex \( v \) in \( D \),
\[
k+m = \dim Z = \dim((v+P) \cap Z)
\]
\[
= \dim(v \cap Z + P \cap Z)
\]
\[
= \dim(v \cap Z) + \dim(P \cap Z) - \dim(v \cap Z \cap P \cap Z)
\]
\[
dim(v \cap Z) = k-t+m \quad \text{(which also implies } t \geq m \text{), and}
\]
\[
\mathcal{D} \subseteq \{ u \in \mathcal{V}_k : u \cap P = Q, \ dim(u \cap Z) = k-t+m \}. \quad \text{(B)}
\]
Since \( \mathcal{F} \) is maximal, equalities in (A) and (B) hold. That is,
\[
\mathcal{F} = \{ u \in \mathcal{V}_k : Q \subseteq u, \ k-t-1 \leq \dim(u \cap P) \}
\]
\[
\cup \{ u \in \mathcal{V}_k : u \cap P = Q, \ dim(u \cap Z) = k-t+m \}
\]
\[
\cup \{ u \in \mathcal{V}_k : \ dim(u \cap Q) = k-t-1, \ dim(u \cap P) = k-1, \ u \subseteq Z \}
\]
and
\[
\mathcal{F} = \sum_{1 \leq j \leq t} [n-k q(t-j)(t-1) + m t^{m-t} q^{k-m} q^{k-m} q^{k-t-1-t} t \cdot t^{t} q^{t-1} q^{n-k}]
\]
\[
= q^{t-1} q^{n-k} (t-1) + o(q^{t-1} q^{n-k} (t-1)), \quad \text{if } 2 \leq m \leq t.
\]
(Here, if \( m = 1 \), then \( \mathcal{F} \) is isomorphic to the one of type (Jq.2).)
(2). From theorem 4.2.1. and remark (1), if \( k > t \), \( k \geq 2t-2 \) (or \( k < 2t-2 \)) and \( n \) is sufficiently large compared to \( k \) and \( t \), then we can characterize up to \((t+3)\)rd-largest (or \((t+2)\)nd-largest) maximal \( t \)-cliques of \( J_q(n,k) \) \( (t \geq 2) \), which are type \((J_q.1)\), type \((J_q.2)\) with \( s=2, t, t+1 \), and type \((J_q.3)\) with \( s=t \). (If \( k < 2t-2 \), type \((J_q.2)\) with \( s=2 \) will not be included.)

(3). We can also show that the following "Let \( n, k \) and \( t \) be fixed integers satisfying \( 2 < t < k \), \( n > k(t-1)-(t^2-6t+8) \) and \( n > 3k + 5t - 4 \). For every fixed constant \( c > 0 \), there exists \( q_0 = q_0(c) \) such that if \( q \) is a prime power bigger than \( q_0 \) and \( \mathcal{F} \) is a maximal \( t \)-clique of \( J_q(n,k) \) satisfying that \( \mathcal{F} >_{cq^{(n-k)(t-1)}} \mathcal{F} \) is one of the four types, \((J_q.1)\), \((J_q.2)\), \((J_q.3)\), and \((J_q.4)\), up to isomorphism." using a similar argument as for the theorem 4.2.1."
Reference


[10]. __________, On intersecting families of finite sets, Journal of


[14]. Tayuan Huang, A Characterization of the association scheme of bilinear forms, to be published.

[15]. __________, An analogue of the Erdős-Ko-Rado theorem for the distance-regular graphs of bilinear forms, to be published.


