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ZEROS OF P-ADIC L-FUNCTIONS

The Ohio State University

Ph.D. 1985

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ZEROS OF p-ADIC L-FUNCTIONS

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate School
of the Ohio State University

By

Nancy Ellen Childress, B.B., B.S.Ed., M.S.

***

The Ohio State University
1985

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To My Parents
ACKNOWLEDGEMENTS

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INTRODUCTION

"In the days of Dirichlet and Hermite, and even of Minkowski, the appeal to 'continuous variables' in arithmetical questions may well have seemed to come out of some magician's bag of tricks. In retrospect, we see now that the real numbers appear there as one of the infinitely many completions of the prime field, one which is neither more or less interesting to the arithmetician than its p-adic companions..."

- André Weil, [W]

The real numbers are but one of infinitely many completions of the rational numbers, Q. One may complete Q with respect to any absolute value defined on it, and in recent years many mathematicians have devoted their efforts to the study of the "p-adic" completions of Q. That is, if p is any fixed prime, one may assign to each integer a non-Archimedean absolute value, defined as |a| = p^{-ν}, where ν = ord_p(a) is the exact power of p dividing a. This absolute value may be extended to all the rationals by |a/b| = p^{-ν(a)+ν(b)}. If one completes the rationals with respect to this absolute value, one obtains Q_p, the p-adic rationals.

Topologically, this field has many interesting properties. It is totally disconnected, "open" disks are closed, any point in the interior of a disk may serve as its center, any triangle is isosceles. The ring of integers in this field is called Z_p, and consists of those p-adic rationals having absolute value no greater than 1. Z_p is, in fact, the completion of the rational integers Z, i.e. Z is dense in Z_p.
The Kubota-Leopoldt p-adic L-functions were invented in the early 1960's as analogues to classical Dirichlet L-series, [K-L]. As with classical L-series, the definition of the p-adic L-function may be made initially for rational integers only, and continued uniquely to all of \( \mathbb{Z}_p \), by the above remark concerning the denseness of \( \mathbb{Z} \) in \( \mathbb{Z}_p \).

For over one hundred years, classical L-series have been an important topic in number theory. They are related, (Dirichlet's Theorem on Arithmetic Progressions), to the distribution of primes in \( \mathbb{Z} \), and their zeros have long held an interest for mathematicians because of this, and other applications. For example, the famous Riemann hypothesis, still an open question, concerns the locations of the zeros of the most trivial of these L-series.

In 1958, the theory of \( \mathbb{Z}_p \)-extensions was introduced by K. Iwasawa, and in 1969, he published an interpretation of the p-adic L-functions in terms of \( \mathbb{Z}_p \)-extensions, [I2]. This interpretation has led to a very successful marriage of algebraic and analytic objects, which is still being exploited today. It is known, for example, that p-adic L-functions are related to the power of p dividing the class numbers of certain imaginary abelian fields. Recently, Mazur and Wiles, [M-W], have shown that p-adic L-functions are essentially the characteristic power series of particular Galois actions which arise in Iwasawa's theory.

This project investigates the relationship of the zeros of the p-adic L-functions to the zeros of certain power series introduced by Iwasawa as part of his theory of \( \mathbb{Z}_p \)-extensions. This relationship has
not heretofore been completely formalized. Little is presently known concerning the zeros themselves, although several authors have computed specific examples for quadratic fields, and small primes. The results of some of these computations, which are accomplished by exploiting the relationship between Iwasawa's series and the p-adic L-functions, have shown a discrepancy, which is explained here, so that the relationship between the zeros of these two functions is now completely described.
CHAPTER I
IWASAWA THEORY AND p-ADIC L-FUNCTIONS

Introduction

The purpose of this chapter is to fix notation, recall the classical theory, and state some recent results which will be needed. We begin with a basic discussion of Dirichlet characters, and their properties, and a description of the relationship between the group rings and power series rings associated to integral rings over $\mathbb{Z}_p$. This discussion includes a statement of the p-adic Weierstrass Preparation Theorem, which is fundamental to Iwasawa Theory.

Section 2 contains a definition of the Kubota-Leopoldt p-adic L-functions, a discussion of characters of the first and second kinds, and the introduction of a related power series over $\mathbb{Z}_p$ due to Iwasawa. This relationship, (along with the Main Conjecture, recently proved by Mazur and Wiles), is discussed in more detail in section 3.

Section 4 contains an explanation of the purpose of this research, and a general outline of its progress. In section 5, a statement of Krasner's lemma is given for future reference.
1: Preliminaries

Let $p$ be an odd prime. Let $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ denote the $p$-adic integers, the $p$-adic rationals, and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. We let $|a|$ denote the $p$-adic absolute value of $a$, which is defined for all $a \in \mathbb{C}_p$, through the (unique) extension of the absolute value on $\mathbb{Q}_p$. By $v_p$, we shall mean the associated valuation.

A (p-adic) Dirichlet character is a multiplicative homomorphism $\chi: (\mathbb{Z}/n\mathbb{Z})^\times \rightarrow \mathbb{C}_p$.

Note that the values of $\chi$ are necessarily $\phi(n)^{\mathbb{Z}}$ roots of unity, where $\phi$ is Euler's $\phi$-function. By $\mathbb{Z}_p[\chi]$ we will mean the ring obtained by adjoining the values of $\chi$ to $\mathbb{Z}_p$. We remark that $\mathbb{Z}_p[\zeta_d]$ is the ring of integers of $\mathbb{Q}_p(\zeta_d)$, where $\zeta_d$ is a primitive $d^{\text{th}}$ root of unity. Note also that if $n|m$, then $\chi$ induces a homomorphism from $(\mathbb{Z}/m\mathbb{Z})^\times$ to $\mathbb{C}_p$ by composition with the natural map. Thus $\chi$ may be considered as a homomorphism on $(\mathbb{Z}/m\mathbb{Z})^\times$. If we take $n$ to be the minimal such "modulus," then $n$ is called the conductor of $\chi$, denoted $f_\chi$. When a character is defined modulo its conductor, it is called primitive. The product of two Dirichlet characters $\chi$, $\psi$ is defined to be the primitive character associated to the character $\Theta: a \rightarrow \chi(a)\psi(a)$. It is well-known that for $(f_{\chi},f_{\psi}) = 1$, $f_{\chi\psi} = f_{\chi}f_{\psi}$. Dirichlet characters are often regarded as maps on $\mathbb{Z}$ by defining $\chi(a) = \chi(a \mod f)$ if $(a,f) = 1$, and $\chi(a) = 0$ otherwise.

For our purposes, it is convenient to consider $\chi$ as a character of the group $\text{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$, (where $\zeta_n$ is a primitive $n^{\text{th}}$ root of unity), by
making the identification $\text{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \cong (\mathbb{Z}/n\mathbb{Z})^\times$. Following this convention, for $X$ of conductor $n$, we let $K_X$ be the fixed field of the kernel of $X$. Then $K_X$ is a subfield of $\mathbb{Q}(\zeta_n)$. More generally, if $X$ is a group of Dirichlet characters and $n$ is the least common multiple of their conductors, [so that the characters of $X$ are all characters of $\text{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$], then let $K$ be the fixed field of the intersection of the kernels of the characters in $X$. We see that $X$ is precisely the set of characters of the Galois group $\text{Gal}(K/\mathbb{Q})$. Observing standard notation, we denote the group of characters of a given group $G$ by $G^\wedge$. For $G$ finite abelian, a well-known result of group theory gives $G^\wedge \cong G$.

Let $R$ be a ring of integral elements in a finite extension of $\mathbb{Q}_p$, e.g. $R = \mathbb{Z}_p[X]$. Let $\mathfrak{p}$ be the maximal ideal of $R$, and say $\mathfrak{p} = (w)$.

If $\Gamma$ is a multiplicative topological group isomorphic to the additive group $\mathbb{Z}_p$, we say $\zeta$ is a topological generator of $\Gamma$ if the cyclic subgroup generated by $\zeta$ is dense in $\Gamma$. [For example, $\zeta$ could be chosen to correspond to $1 \in \mathbb{Z}_p$ under this isomorphism, since $1$ generates $\mathbb{Z}$, which is dense in $\mathbb{Z}_p$.] Since the closed subgroups of $\mathbb{Z}_p$ are of the form $p^n\mathbb{Z}_p$, the closed subgroups of $\Gamma$ are of the form $\Gamma^p\zeta^n$. Let $\Gamma_n = \Gamma/\Gamma^n\zeta^n$. Then $\Gamma_n$ is cyclic of order $p^n$ generated by the coset of $\zeta$. We consider the group ring $R[\Gamma]$. If $n \geq 0$, then there is a natural projection $R[\Gamma_m] \twoheadrightarrow R[\Gamma_n]$ induced by the projection $\Gamma_m \twoheadrightarrow \Gamma_n$, and since $R[\Gamma_n] \cong R[T]/((1+T)^{p^n} - 1)$, [via $y \mod p^n \rightarrow 1 + T \mod ((1+T)^{p^n} - 1)$], we have the profinite group ring of $\Gamma$,

$$R[\Gamma] = \varprojlim R[\Gamma_n] \cong \varprojlim R[T]/((1+T)^{p^n} - 1) \cong R[[T]],$$

($R[[T]]$ denotes the ring of formal power series over $R$), the
isomorphism being given by \( \psi \to 1 + T \). Note that \( \mathbb{R}[\Gamma] \) is contained in \( \mathbb{R}[[T]] \), since an element of \( \mathbb{R}[\Gamma] \) gives a sequence of elements in the \( \mathbb{R}[\Gamma_n] \) which are related via the natural projection map, thus respected by the projective limit.

We find that a power series \( U(T) \in \mathbb{R}[[T]] \) is a unit whenever its constant term is a unit in \( \mathbb{R} \). We say a polynomial \( D(T) \) is distinguished if it is monic, and all of its other coefficients are divisible by \( \psi \).

Among the more useful results concerning the power series ring \( \mathbb{R}[[T]] \) is the (p-adic) Weierstrass Preparation Theorem:

\[ \text{WEIERSTRASS PREPARATION THEOREM:} \quad \text{If } f(T) \in \mathbb{R}[[T]] \text{ is non-zero, then we may uniquely write} \]
\[ f(T) = \psi^m D(T) U(T), \]

where \( \psi \) is a non-negative integer, \( D(T) \) is a distinguished polynomial, and \( U(T) \) is a unit power series.\( \blacksquare \)

(See Washington, [WN1], for a proof.)
Let $\omega$ be the $p$-adic Teichmüller character, defined as follows: for $a \in \mathbb{Z}_p$, let $\omega(a)$ be the unique $(p-1)^{th}$ root of unity in $\mathbb{Z}_p$ satisfying $\omega(a) \equiv a \pmod{p}$. [Then $\omega$ is a Dirichlet character on $\mathbb{Z}$ of order $p-1$ and conductor $p$.] Let $\langle a \rangle = \omega(a)^{-1}a$. [Then $\langle a \rangle$ is a one-unit, i.e. $\langle a \rangle \equiv 1 \pmod{p}$.]

Let $d$ be a positive integer prime to $p$. Assume $d \not\equiv 2 \pmod{4}$.

Let $q_n = p^{n-1}d$, $K_n = \mathbb{Q}(\zeta_{q_n})$, and $K_0 = \bigcup_{n=0}^\infty \mathbb{Q}(\zeta_{q_n})$, where $\zeta_{q_n}$ is a $q_n$-th root of unity. Then $\text{Gal}(K_n/\mathbb{Q}) \cong G \times \Gamma_n$, where $G = \text{Gal}(K_0/\mathbb{Q})$ and $\Gamma = \text{Gal}(K_0/K) \cong 1 + q_0 \mathbb{Z}_p = (1 + q_0)^\infty$. Let $\kappa$ be the isomorphism between $\Gamma$ and $1 + q_n \mathbb{Z}_p$ defined by $\kappa(\zeta_{p^n}) = \zeta_{p^n}^{p^n-1}q_n$ for every $n$, where $\zeta_{p^n}$ is a $p^n$-th root of unity, and where $\zeta_{p^n}$ is a topological generator for $\text{Gal}(K/K)$. [So $\zeta_{p^n}$, $\kappa$ may be chosen such that $1 + q_0$ is the image, under $\kappa$, of $\zeta_{p^n}$.] The elements of $\Gamma$ fixing $K_n$ are just the elements of $\Gamma_n$ \* $1 + q_n \mathbb{Z}_p$. So we have $\text{Gal}(K_n/K_0) = \Gamma_n/\Gamma_n^\infty = \Gamma_n$, and $\text{Gal}(K_n/\mathbb{Q}) \cong G \times \Gamma_n$.

Let $\Psi$ be a Dirichlet character of conductor $dp^j$, for some $j \geq 0$. Then $\Psi$ may be regarded as a character of $\text{Gal}(K_n/\mathbb{Q})$, (for $q_n \geq dp^j$), so that it may be written uniquely as $\Psi = \chi \Psi'$, with $\chi \in G$ and $\Psi' \in \Gamma_n$. $\chi$ is called a character of the first kind, and $\Psi'$ is called a character of the second kind. Note that characters of the first kind are associated with $K_n = \mathbb{Q}(\zeta_{q_n})$ and hence have conductors dividing $pd$, while characters of the second kind are associated with the subfield of $\mathbb{Q}(\zeta_{p^n})$ of degree $p^n$ over $\mathbb{Q}$, and hence are either trivial or have $p$-power order and conductor of the form $p^j$ with $j \geq 2$. 

2: p-adic L-functions
Before defining the $p$-adic $L$-functions, we recall some basic facts and definitions concerning the Bernoulli numbers and polynomials.

Let $B_j$ be the $j^{th}$ Bernoulli number. A theorem of von Staudt-Clausen gives $v_p(B_j) = -1$ if $p-1$ divides $j$, while J. C. Adams' theorem gives $v_p(B_j) \geq v_p(j)$ if $p-1$ does not divide $j$. These results will be needed in chapter III. Related to the Bernoulli numbers are the Bernoulli polynomials, given by

$$B_n(x) = \sum_{j=0}^{n} \frac{n!}{j!(n-j)!} B_j x^{n-j}$$

The generalized Bernoulli numbers can then be defined for any Dirichlet character $\chi$ as

$$B_{n,\chi} = \frac{F^{n-1}}{\chi(-1)} \sum_{a \equiv 1 \mod f} \frac{B_n(a/F)}{a^n}$$

where $F$ is any multiple of $f$. 

Let $\chi$ be any primitive Dirichlet character. Then the $p$-adic $L$-function $L_p(s,\chi)$ is the unique continuous $p$-adic function $\mathbb{Z}_p \to \mathbb{C}_p$ such that:

$$L_p(1-n,\chi) = -(1 - \chi(p)p^{n-1})B_{n,\chi}p^{-n}/n$$

for every positive integer $n$. Note that if $p-1$ divides $n$, then $L_p(1-n,\chi)$ agrees with the classical Dirichlet $L$-function except for the Euler factor at $p$, which does not appear. Also, if $\chi$ is odd, (i.e. $\chi(-1) = -1$), then $L_p(s,\chi)$ is identically zero.

Let $\chi$ be a non-trivial Dirichlet character. Let $q = \text{l.c.m.}(f,p)$, where $f$ is the conductor of the character $\chi$, and let $B_j$ be the $j^{th}$ Bernoulli number.

Then the $p$-adic $L$-function, $L_p(s,\chi)$, may be written, (according to a formula of Washington, [WN2]):
L_p(s,X) = q^{-1}(s-1)^{-1} \sum_{\omega = 1, \omega \neq 1} \chi(a) \langle a \rangle \langle a \rangle^{s-1} \sum_{j=1}^{\infty} (q/a)^j B_j.

In chapter III, we will use the binomial theorem to obtain an infinite series for \langle a \rangle^{s-1} so that Washington’s formula may be used to make approximations to the L-functions for specific choices of \chi.

Let \chi be a non-trivial even Dirichlet character of the first kind associated to \mathbb{K}, a real cyclic extension of \mathbb{Q}. Then Iwasawa, [II], has constructed a formal power series \( f_{\chi}(T) \in \mathbb{Z}_p[\chi][[T]] \) connected to \( L_p(s,X) \) via:

\[ f_{\chi}(\kappa(\chi)_0^{s} - 1) = L_p(s,X) \]

where \( \kappa \) is defined by \( \kappa(\chi_0^{n}) = \chi_0^{-n}\langle \chi_0 \rangle \) for every \( n \), where \( \chi_0 \) is a \( p^n \)-th root of unity, and where \( \chi_0 \) is a topological generator for \( \text{Gal}(\mathbb{K}_0/\mathbb{K}) \). We may take \( \kappa(\chi_0) = 1+q \), as above. With the proper definition for exponentiation, (see chapter II), we will find that \( L_p(s,X) \) is then defined and analytic for \( |s| < p^{1-1/(p-1)} \). Morita, [M], has shown that \( L_p(s,X) \) cannot be continued to a single-valued analytic function on any larger \( s \)-disk. Now the change of variable \( T = \kappa(\chi_0)^{s} - 1 \) relates zeros of \( f_{\chi}(T) \) with zeros of \( L_p(s,X) \). Under this relation, the \( s \)-disk, \( D_\chi = \{ s : |s| < p^{1-1/(p-1)} \} \), maps into the "open" unit \( T \)-disk. (A partial inverse is given by \( s = \log(1+T)/\log(1+q) \), where "log" denotes the \( p \)-adic logarithm, (see chapter II)). Because of the domains of convergence of the logarithm and exponential functions, we will find that if \( s_0 \in D_\chi \) is a zero of \( L_p(s,X) \), then we have a corresponding \( \alpha_0 = (1+q)^{s_0} - 1 \) in the open unit \( T \)-disk which is a zero of \( f_{\chi}(T) \). But, as we will show, the converse is not necessarily true.
The power series $f_x(T)$ is known to satisfy the hypothesis of the Weierstrass Preparation Theorem, i.e. it has coefficients in the ring $\mathbb{Z}_p[\chi]$, which is integral over $\mathbb{Z}_p$. Thus it may be written as a product:

$$f_x(T) = p^\mu D_x(T)U(T)$$

where $U(T)$ is a unit power series, $D_x(T)$ is a distinguished polynomial, and $\mu$ is a non-negative integer depending on $K$ and $\chi$. In particular, this means that if $f_x(T)$ is considered as a function on the open unit $T$-disk, then it will have only finitely many zeros. Also it is known that for abelian extensions of $\mathbb{Q}$, $\mu = 0$, (see section 3), so that for those fields,

$$f_x(T) = D_x(T)U(T).$$
3: Some Iwasawa Theory and the Main Conjecture

This section will allow us to interpret the series $f_\lambda(T)$ introduced above.

Suppose $K_0, K_1, \ldots$ is a sequence of number fields, each contained in its successor, with $\text{Gal}(K_\alpha/K_0) \cong \mathbb{Z}/p^n\mathbb{Z}$. Let $K_\alpha = U K_\alpha$. Then $\text{Gal}(K_\alpha/K_0) = \varprojlim(\mathbb{Z}/p^n\mathbb{Z}) = \mathbb{Z}_p$. This Galois group is often written multiplicatively, in which case it is denoted $\Gamma$. The extension $K_\alpha/K_0$ is called a $\mathbb{Z}_p$-extension or a $\Gamma$-extension. [The basic $\mathbb{Z}_p$-extension of $Q$ is obtained by taking $K_\alpha$ to be the unique subfield of $Q(\zeta_p)\infty$ which is cyclic of degree $p^n$ over $Q$. Given the discussion in the previous section, we see that this extension corresponds to the group of all characters of the second kind. If $K$ is any number field, then we may take $K_\alpha$ to be the compositum of $Q_\alpha$ and $K$. The fields $K_\alpha$ in the preceding section constitute an example of this. $K_\alpha$ is then called the cyclotomic $\mathbb{Z}_p$-extension of $K$. It is conjectured that the number of distinct $\mathbb{Z}_p$-extensions of a given field $K$ is exactly one more than the number of complex embeddings of $K$.]

Say $e_\alpha$ is the exponent of the exact power of $p$ dividing the class number $h_\alpha$ of $K_\alpha$. One of the principal results of Iwasawa theory states that there exist fixed integers $\lambda$, $\mu$, and $c$ such that

$$e_\alpha = \lambda n + \mu p^n + c$$

for all $n$ sufficiently large. [This is known as Iwasawa's Theorem. See Washington, [WN1], for a proof.] The integers $\lambda$, $\mu$ are called the Iwasawa invariants of the field $K_\alpha$. 

The proof of the following theorem has its origins in the work of Gold, [G3], who considered the case when $K$ is quadratic. The first proof of the general case is due to Ferrero-Washington, [F-W]. A recent proof using $p$-adic $L$-functions has been given by Sinnott, [S2].

**THEOREM (Ferrero-Washington):** Let $K$ be an abelian extension of $\mathbb{Q}$, and let $K_\infty/K$ be the cyclotomic $\mathbb{Z}_p$-extension. Then $\mu = 0$.

In particular, suppose $K$ is an abelian CM-field, (i.e. $K$ is a totally imaginary quadratic extension of a totally real field). Consider the cyclotomic $\mathbb{Z}_p$-extension of $K^-$, the totally real subfield of $K$ of index 2. Let $h_n^+$ denote the class number of the $n$th layer of the tower. As usual, let $h_n^- = h_n/h_n^+$. Let $e_n^+$, (resp. $e_n^-$) be the exact power of $p$ dividing $h_n^-$, (resp. $h_n^+$). Then $e_n = e_n^+ + e_n^-$, $\mu = \mu^+ + \mu^-$, and $\lambda = \lambda^+ + \lambda^-$. It is known that if all of the characters of $K$ are of the first kind, then $\lambda^- = \sum \deg(D_{\chi}(T))$, where the sum is taken over all odd characters, $\chi$, which are in $\text{Gal}(K/\mathbb{Q})^\wedge$. Furthermore, $\mu^- = \sum \mu_\chi$, where $\mu_\chi$ is the exponent for $\chi$ in the decomposition of $f_\infty(T)$ via the Weierstrass Preparation Theorem. Since we know that $\mu = 0$ for abelian fields $K$, we know that $\mu^- = 0$ for these fields, and hence that all the $\mu_\chi$ are zero.

Kida, [K], has proved a formula for $\lambda^-$ which is remarkably similar to the Riemann-Hurwitz genus formula from the theory of compact Riemann surfaces. One specific application of this formula will be needed in chapter III. To wit, let $E/K$ be an abelian extension of degree $p$, with
E and K CM-fields. Then $\lambda_{\kappa} - \delta_\kappa = p(\lambda_\kappa - \delta_\kappa) + (p-1)\tau$, where $\delta_\kappa$ is one if $K_\kappa$ contains all the $p$-power roots of unity, and zero otherwise, and $\tau$ is the number of primes of $K_\kappa$ which do not divide $p$, but which ramify in $E_{\kappa^-}/K_{\kappa^-}$. For Kida's formula, cf. also [S1], [G-M].

Now let $\Lambda = \mathbb{Z}[T]$. This power series ring is a Noetherian unique factorization domain, whose irreducible elements are $p$, and the irreducible distinguished polynomials.

For two $\Lambda$-modules $M$, $N$, we say $M$ is pseudo-isomorphic to $N$ if there is an exact sequence of $\Lambda$-modules

$$0 \to A \to M \to N \to B \to 0$$

where $A$ and $B$ are finite. We write $M \sim N$ in this case. If $M$ is a finitely generated $\Lambda$-module, then

$$M \sim \Lambda^r \otimes \bigoplus_{n=1}^{\infty} \Lambda/(p^{e_n}) \otimes \bigoplus_{n=1}^{\infty} \Lambda/(f_n(T)^{t_n})$$

where $e_n, t_n, m, \nu \in \mathbb{Z}$, and $f_n$ is an irreducible distinguished polynomial.

Assume $K$ is the field associated to a non-trivial odd Dirichlet character $\chi \neq \chi^{-1}$ of the first kind, (thus $\chi\omega$ is a non-trivial even Dirichlet character of the first kind, so in agreement with section 2). Let $K_0 = K(\zeta_p)$, $K_1 = K(\zeta_{p^2})$, etc. Let $\psi_0$ be the topological generator of $\text{Gal}(K_0/K)$ which corresponds to $1 + T$ under the isomorphism discussed in section 1. Let $f_\kappa \in \mathbb{Z}[X]$ be as in section 2. Let $A_n$ be the $p$-Sylow subgroup of the ideal class group of $K_n$, (so $p^{e_n} = \#A_n$).

Let $L_n$ be the maximal unramified abelian $p$-extension of $K_n$, so $X_n = \text{Gal}(L_n/K_n) \cong A_n$. Let $L = \bigcup_{n=0}^{\infty} L_n$, and $X = \text{Gal}(L/K)$. Then $X$ is a $\Lambda$-module, which is finitely generated and $\Lambda$-torsion, so is pseudo-
isomorphic to a direct sum

\[ \bigoplus_{1 \leq n \leq \infty} \Lambda / (p^{\infty}) \bigoplus \bigoplus_{n=1}^{m} \Lambda / (f_n(T)^{\infty}) \]

as above. The invariant \( \lambda \) is simply the sum of the degrees of
the distinguished polynomials \( f_n(T)^{\infty} \) in this decomposition, while \( \mu \) is
the sum of the \( s_n \). Since \( K \) is an abelian extension of \( \mathbb{Q} \), we know that
\( \mu = 0 \) in this case. Decompose the \( \mathbb{C}_p \) vector space \( V = \chi \otimes \mathbb{C}_p \)
according to the idempotents \( c_{\chi^*} = (\#G)^{-1} \sum_{\sigma \in G} \chi(\sigma) \sigma^{-1} \), the sum being taken
over all \( \sigma \in G = \text{Gal}(K/\mathbb{Q}) \). Let \( V_{\chi} \) be the direct summand corresponding
to \( c_{\chi^*} \), and let \( g_{\chi}(T) \) be the characteristic polynomial for \( V_{\chi} \) for \( \Gamma \) acting
on \( V_{\chi} \). [We see that \( V = \bigoplus \mathbb{C}_p[[T]]/(f_n(T)^{\infty}) \), so that the group \( \Gamma \) acts
on \( V \) via the injection \( \mathbb{C}_p[[T]] \to \mathbb{C}_p[[T]] \). Hence \( V_{\chi} \) acts as
\( 1 + T \).] Then the Main Conjecture states that \( D_{\chi}(T) = g_{\chi}(T) \). So the
decomposition of \( f_{\chi^*}(T) \) via the Weierstrass Preparation Theorem gives
us the characteristic polynomial \( g_{\chi}(T) \) of a vector space which in turn
is related to the \( p \)-parts of the ideal class groups in the tower of
fields constituting the cyclotomic \( \mathbb{Z}_p \)-extension of \( K \), the field
associated to the character \( \chi \). Mazur and Wiles, [M-W], have recently
given a proof of the Main Conjecture for \( K \) abelian over \( \mathbb{Q} \).
4. The Problem

This research began as an attempt to describe the characteristic polynomial of the $\Lambda$-module of section 3, above, for a specific class of number fields $K$. [These number fields are the subject of chapter III, section 3.] The degree of this polynomial, the Iwasawa invariant $\lambda$, can be computed for these fields using the formula of Kida, but the structure of the decomposition cannot be determined from this formula. After some initial algebraic work, computational examples were desired. This led to the study of a paper by Wagstaff, [WG3], concerning some computations he had done for real quadratic extensions of $\mathbb{Q}$. With the discovery that some of the so-called "zeros" of the $p$-adic $L$-functions for these fields did not lie within the domain of convergence of the $L$-function, came the need to formalize a description of the true relationship between the zeros of the series $f_\alpha$, and those of the $L$-function, $L_p(s,\chi)$.

In these examples, the zeros of $f_\alpha$ obviously did not always correspond to zeros of the $L$-function, even in the case when the related $s$-value for the $L$-function was within the correct domain, as was first seen for the special fields in section 3 of chapter III. A suitable description of the actual relationship of the zeros of $f_\alpha$ to the zeros of (several) $L$-functions was found for these fields, using characters of the second kind and a certain identity, (see chapter II, section 2), relating $f_\alpha$ to the function $L_p(s,\chi\psi)$, where $\chi$ is a character of the first kind, as usual, but $\psi$ is a character of the second kind. [This identity allows us to use the Iwasawa series to
treat the L-function of any Dirichlet character, since every character can be written as a product of a character of the second kind and a character of the first kind, (see section 2 above). Using the results for these special fields as a prototype, it seemed that the solution was closely tied to the domain of convergence of the p-adic exponential function, but also to the p-power roots of unity. Section 2 of Chapter II is devoted to the description of this relationship. The relationship established for $f_x$ and the L-functions seems incomplete, however, as it still allows the series $f_x$ to have a zero which does not correspond to a zero of any L-function. For example, the computations of Wagstaff give values of $s$ which are outside the domain of the L-function; "twisting" the L-function by a character of the second kind does not change this.

The third section of chapter II resolves this problem by considering the generalizations of the Kubota-Leopoldt L-functions defined by Deligne-Ribet, [D-R]. These L-functions are defined over the (totally real) fields $\mathbb{Q}_m$ that constitute the basic $\mathbb{Z}_p$-extension of $\mathbb{Q}$, and have progressively larger domains of convergence. Wagstaff's $s$-values fell into these domains, and in fact, happened to be zeros of these L-functions. This discovery led to an attempt to show that this sort of result was true in every case. In section 3 of chapter II we show that every zero of the Iwasawa series corresponds to a zero of some L-function, and thus the description is complete.
5. A Note From Field Theory

For future reference we state a result due to Krasner, which is useful in chapter III. A proof can be found in [WN1].

**KRASNER’S LEMMA:** Let $K$ be a complete field with respect to a non-Archimedean valuation. Let $\delta$, $\epsilon$ be elements of the algebraic closure of $K$, with $\delta$ separable over $K(\epsilon)$. Suppose that we have

$$|\epsilon - \epsilon'| < |\delta - \delta'|$$

for all conjugates $\delta' \neq \delta$ of $\delta$.

Then $K(\delta)$ is contained in $K(\epsilon)$. $\blacksquare$
CHAPTER II
THE MAIN RESULTS

Introduction

In this chapter, the zeros of the Kubota-Leopoldt $p$-adic L-functions are studied. As these functions are known to be related to the formal power series $f_x(T)$ over $\mathbb{Z}_p$ defined by Iwasawa, it is this relationship which will be exploited to study the behavior of zeros of the L-functions. The zeros of these Iwasawa series are better understood, (via the Main Conjecture), and are more readily computed. The assumption that a zero of the Iwasawa series $f_x(T)$ for a Dirichlet character $\chi$ will yield a corresponding zero of the L-function $L_p(s, \chi)$ is in general false, however. For instance, in [WG], Wagstaff has computed zeros of the Iwasawa series for various fields and has used the relationship with $L_p(s, \chi)$ to compute values for $s$ which in many cases are not within the domain of convergence of the L-function, so clearly cannot be zeros of it.

The relationship between zeros of the series $f_x(T)$ and those of $L_p(s, \chi)$ will be explored and made precise. We then will consider the more general case of $p$-adic L-functions defined over the fields $\mathbb{Q}_m$ constituting the layers of the basic $\mathbb{Z}_p$-extension of $\mathbb{Q}$, and the relationship of their zeros to those of the series $f_x(T)$ will be discovered.
1: The \( p \)-adic Exponential and Logarithmic Functions

We first state some general results concerning \( p \)-adic logarithms and exponentials. Over \( \mathbb{C}_p \), define:

\[
\exp(X) = \sum_{n=0}^{\infty} \frac{X^n}{n!}
\]

\[
\log(1+X) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}X^n}{n}
\]

Then:

1. \( \exp(X) \) has radius of convergence \( p^{-1/p - 1} \).
2. \( \log(1+X) \) may be extended to all of \( \mathbb{C}_p \), (where \( \mathbb{C}_p \) is the completion of the algebraic closure of \( \mathbb{Q}_p \)), such that \( \log p = 0 \), and \( \log(xy) = \log x + \log y \) for all \( x, y \in \mathbb{C}_p \).

3. If \( |x| < p^{-1/p - 1} \), then \( |\log(1+x)| = |x| \); if \( |x| \leq p^{-1/p - 1} \), then \( |\log(1+x)| \leq |x| \).

(For proofs, see Washington, [WN1].)

In fact, let \( D = \{x : |x| < p^{-1/p - 1}\} \). So \( D \) is an additive group, and \( 1 + D \) is a multiplicative group. Then it is well-known, [KZ], that \( \exp : D \to 1 + D \), and \( \log : 1 + D \to D \) are inverse isomorphisms.

Thus, we have the following:

**PROPOSITION 1**: Let \( |a| \leq 1 \), \( a \in \mathbb{C}_p \). Then for all positive \( n \in \mathbb{Z} \),

\( a^p^n \in D \) if and only if \( (1+a)^p^n - 1 \in D \).

**proof**: Now:

\[
(1+a)^p^n - 1 = p^n a + \ldots + (p^n)a^r + \ldots + a^{p^n}.
\]

Each of the terms on the right, except \( a^{p^n} \) is of absolute value at
most $|l\alpha| < 1$ as well, then $|(1+\alpha)p^n - 1| < p^{-1/\mu - 1}$. Conversely, if $(1+\alpha)p^n - 1 \in D$, then $\alpha_p^n$ clearly must also be. Hence the desired result is obtained. 

Now let $|x| < 1$, $x \in \mathbb{C}_p$, and suppose $\log(1+x) = 0$. Then we may choose an integer $n$ such that $x^n \in D$. By proposition 1, $(1+x)p^n \in 1 + D$. But $\log(1+x)p^n = p^n \log(1+x) = 0$, so $(1+x)p^n = 1$. (Since $\log$ is an isomorphism on $1 + D$, it will have trivial kernel there.) So $1 + x$ is a $p^n$-th root of unity.

**PROPOSITION 2**: Let $|\alpha| < 1$, $\alpha \in \mathbb{C}_p$, and suppose $|\log(1+\alpha)p^m| < p^{-1/\mu - 1}$. Let $\delta \in \mathbb{C}_p$ such that $1+\delta$ is a $p^m$-th root of $\exp(\log(1+\alpha)p^m)$. Then $1+\alpha = \zeta(1+\delta)$ for some $p$-power root of unity $\zeta$.

**proof**: Note that $\exp(\log(1+\alpha)p^m)$ is defined, and in fact, (by the above), is in $1 + D$, i.e. is a unit. Thus $\delta$ exists and, since $$(1+\delta)p^m \in 1 + D, \delta p^m \in D,$$ so that $|\delta| < 1$. Now:

\[
\exp(\log(1+\alpha)p^m) = (1+\delta)p^m
\]

\[
\log(\exp(\log(1+\alpha)p^m)) = \log(1+\delta)p^m
\]

\[
\log(1+\alpha)p^m = \log(1+\delta)p^m
\]

So $\log(1+\alpha) = \log(1+\delta)$, and thus $\log[(1+\alpha)/(1+\delta)] = 0$. Now $(1+\alpha)/(1+\delta)$ may be written as $1 + x$ for $|x| < 1$. Thus $1 + \alpha = \zeta(1 + \delta)$ for some $p$-power root of unity $\zeta$, by the remark above.
2. L-functions Over \( \mathbb{Q} \)

We turn to the study of the Kubota-Leopoldt L-functions, i.e. L-functions over \( \mathbb{Q} \). Throughout this section we will assume that \( \chi \) is a non-trivial even Dirichlet character of the first kind, and that \( f_\chi(T) \) is the Iwasawa series for \( \chi \), i.e.

\[
L_\chi(s, \chi) = f_\chi(\chi(\nu_0)^s - 1).
\]

Now let \( \nu \) be a Dirichlet character of the second kind of \( p \)-power order. Such characters are in bijective correspondence with the \( p \)-power roots of unity via:

\( \nu \) corresponds to \( \zeta \) if and only if \( \nu(\nu_0)^{-1} = \zeta \).

Write \( \nu_\zeta \) for the character of the second kind associated to \( \zeta \). Then:

\[
L_p(s, \chi \nu_\zeta) = f_\chi(\zeta \chi(\nu_0)^s - 1).
\]

Because of this relationship, it is natural to consider the family \( \{L_p(s, \chi \nu_\zeta)\} \) for \( \zeta \) ranging through the \( p \)-power roots of unity. We will therefore attempt to determine when a zero of \( f_\chi(T) \) corresponds to a zero of any one of these related L-series.

**Theorem 3:** Let \( a \) be a zero of \( f_\chi(T) \), and let \( s_\alpha = \log(1+a)/\log(1+q) \). If \( |s_\alpha| < p^{1/(p-1)} \), then \( L_p(s_\alpha, \chi \nu_\zeta) = 0 \) for some \( p \)-power root of unity \( \zeta \).

**Proof:** Since \( a \) is a zero of the distinguished polynomial \( D_\chi(T) \), we know \( |a| < 1 \). Now \( |\log(1+a)| = |s_\alpha| |\log(1+q)| < p^{1/(p-1)}p^{1/(p-1)} = p^{1/(p-1)} \). Choose \( \delta \in \mathbb{C}_p \) such that \( \exp(\log(1+a)) = 1 + \delta \). By proposition 2, \( 1 + a = \zeta(1 + \delta) \) for some \( p \)-power root of unity \( \zeta \). So:
\[
L_\rho(s_\alpha, \chi \psi) = f_\chi(\zeta(1+q)^{s_\alpha} - 1)
\]

\[
= f_\chi(\zeta \exp(s_\alpha \log(1+q)) - 1)
\]

\[
= f_\chi(\zeta \exp(\log(1+\alpha)) - 1)
\]

\[
= f_\chi(\zeta \zeta^{-1}(1+\alpha) - 1)
\]

\[
= f_\chi(\alpha)
\]

\[
= 0.\]

Of course, we also seek a means of determining which of these \(L\)-series has the zero.

**THEOREM 4**: Let \(\alpha\) be a zero of \(f_\chi(T)\), and let \(s_\alpha = \log(1+\alpha) / \log(1+q)\).

Then: \(s_\alpha\) is a zero of one (or more) of \(\{L_\rho(s, \chi \psi)\}\) if and only if there is a \(p\)-power root of unity \(\rho\) such that \(|\alpha + 1 - \rho| < p^{-1}/p^{-2}\).

Furthermore, in the case that \(\rho\) exists, it is unique, and gives \(L_\rho(s_\alpha, \chi \psi) = 0\).

**proof**: Say \(L_\rho(s_\alpha, \chi \psi) = 0\) for some \(p\)-power root of unity \(\zeta\).

Then, since \(L_\rho(s_\alpha, \chi \psi)\) is defined, we must have \(|s_\alpha| < p^{1-1/p^{-1}}\).

So \(|\log(1+\alpha)| = |s_\alpha \log(1+q)| < p^{-1}/p^{-1}\), and hence \(\exp(\log(1+\alpha))\) is defined. Then by proposition 2, \(\exp(\log(1+\alpha)) = p^{-1}(1+\alpha)\) for some \(p\), a \(p\)-power root of unity. Since \(\exp\) is an isomorphism from \(D\) to \(1 + D\), we have \(|\exp(\log(1+\alpha)) - 1| < p^{-1}/p^{-1}\). That is, \(|p^{-1}(1+\alpha) - 1| < p^{-1}/p^{-1}\). So \(|1 + \alpha - p| < p^{-1}/p^{-1}\).

Conversely, say there is a \(p\)-power root of unity \(\rho\), such that \(|\alpha + 1 - \rho| < p^{-1}/p^{-1}\). Then \(|p^{-1}(1+\alpha) - 1| < p^{-1}/p^{-1}\), so that \(|\log(p^{-1}(1+\alpha))| < p^{-1}/p^{-1}\), by 3, above. But \(\log(p^{-1}(1+\alpha)) = \)
\[
\log(1+a). \text{ Thus } |s_\alpha| = |\log(1+a)/\log(1+q)| < p^{p-1/(p+1)} = p^{1-1/(p+1)}, \text{ so by theorem 3, } s_\alpha \text{ is a zero of one of } \{L_p(s, \wp^r)\}.
\]

For such a \( p \), since \(|p^{-1}(1+a) - 1| < p^{-1/(p-1)}\), we have
\[
\exp(\log(p^{-1}(1+a)) = p^{-1}(1+a). \text{ Then:}
\]
\[
L_p(s_\alpha, \wp^r) = f_\alpha(p(1+q)^{s_\alpha} - 1)
\]
\[
= f_\alpha(p\exp(s_\alpha\log(1+q)) - 1)
\]
\[
= f_\alpha(p\exp(\log(1+a)) - 1)
\]
\[
= f_\alpha(p^{p^{-1}(1+a)} - 1)
\]
\[
= f_\alpha(1+a)
\]
\[
= 0.
\]

Suppose \( p_1, p_2 \) are \( p \)-power roots of unity, and \(|a + 1 - p_1| < p^{-1/(p-1)} \) for \( j=1,2 \). Then \(|p_2 - p_1| = |(a+1-p_1)-(a+1-p_2)| \leq \max_{j=1,2}(|a+1-p_j|) < p^{-1/(p-1)}\). So \(|p_1^{-1}p_2 - 1| < p^{-1/(p-1)}\). Now \( p_1^{-1}p_2 \) is a \( p \)-power root of unity, say of order \( p^n \). Thus
\[
|p_1^{-1}p_2 - 1| = p^{-p/p^n(p-1)} < p^{-1/(p-1)}\). So \( p^{-1}(p-1) < p-1, \text{ and hence } n=0. \text{ Hence } p_1 = p_2. \]

Note that the uniqueness of \( p \) does not imply that \( L_p(s_\alpha, \wp^r) \) is the only member of \( \{L_p(s, \wp^r)\} \) having \( s_\alpha \) as a zero. However, it does yield the following:

**COROLLARY 5:** Suppose \( \alpha \) is a zero of \( f_\alpha \) and \( p \) is a \( p \)-power root of unity such that \(|\alpha + 1 - p| < p^{-1/(p-1)}\). Let \( \zeta \) be any \( p \)-power root of unity, and \( s_\alpha = \log(1+a)/\log(1+q) \). Then:
\[
L_p(s_\alpha, \wp^r) = 0 \text{ if and only if } f_\alpha(\zeta p^{-1}(1+a) - 1) = 0.
\]
proof: Say $L_p(s, \chi^t) = 0$. Then $f_x(\zeta \exp(s \log(1+q) - 1)) = 0$, so
\[ f_x(\zeta \exp(\log(1+q) - 1)) = 0. \]
Now $|a + 1 - p| < p^{-1} < \delta^{-1}$, so
\[ |p^{-1}(1+a) - 1| < p^{-1} < \delta^{-1}, \]
which gives $\exp(\log(p^{-1}(1+a)) = p^{-1}(1+a)$. But $\log(p^{-1}(1+a)) = \log(1+a)$. So $0 = f_x(\zeta \exp(\log(1+a)) - 1) = f_x(p^{-1}(1+a) - 1)$.

Conversely, say $\zeta p^{-1}(1+a) - 1$ is a zero of $f_x$. Now
\[ |\zeta p^{-1}(1+a) - 1| < p^{-1} < \delta^{-1}, \]
so $|\zeta p^{-1}(1+a) - 1| < p^{-1} < \delta^{-1}$, which implies that $|\zeta p^{-1}(1+a) - 1 - \zeta + 1| < p^{-1} < \delta^{-1}$. By theorem 4, we conclude that $\zeta p^{-1}(1+a) - 1$ corresponds to a zero of $L_p(s, \chi^t)$. This zero is given by $\log(1 + \zeta p^{-1}(1+a) - 1)/\log(1+q) = \log(\zeta p^{-1}(1+a))/\log(1+q) = \log(1+a)/\log(1+q) = s_\alpha$. \[ \square \]
Suppose now that \( \alpha \) is a zero of \( f_\chi \) but that \( s_\alpha \) is outside the domain of the \( L \)-function, i.e. \( |s_\alpha| > 2 \, p^{1-1/2}(p-1) \). This seems to occur quite frequently. Using Washington's formula, Wagstaff, [WG], has computed approximations for the zeros of \( f_\chi(T) \) for \( \chi \) a real quadratic character and \( p = 3,5 \). For example, for \( \chi \) of conductor 101 and \( p = 3 \) we find \( D_\chi \) to be an irreducible (over \( \mathbb{Q}_3 \)) quadratic which has zeros \( \alpha \equiv 2055 \pm 647(3)^{1/2}(\text{mod } 3^7) \). He determines \( s_\alpha \equiv 7 \pm 1048(3)^{-1/2}(\text{mod } 3^7) \), each of which has absolute value \( 3^{1/2} = p^{1-1/2}(p-1) \). (For further such examples, see chapter III, or Kobayashi, [KY].) In this section we provide an interpretation for these values of \( s_\alpha \) which do not fall inside the domain of \( L_p(s,\chi) \).

Define \( B_n = \{ s : |s| < p^{n+1-1/2}(p-1) \} \) for each non-negative \( n \in \mathbb{Z} \). Let \( \zeta \) be a primitive \( p^{n+1} \)-th root of unity, and let \( \mathbb{Q}_m \) be the unique subfield of \( \mathbb{Q}(\zeta) \) which is cyclic of degree \( p^m \) over \( \mathbb{Q} \), i.e. the \( m \)-th field in the tower constituting the basic \( \mathbb{Z}_p \)-extension of \( \mathbb{Q} \). Then, for \( s_\alpha \in B_m \), we have \( L_p(s_\alpha,\chi,\mathbb{Q}_m) \), the \( p \)-adic \( L \)-function defined over \( \mathbb{Q}_m \), with \( \chi \) now considered as a character of \( \text{Gal}(K_m/Q_m) \). For a detailed definition of these functions, see Deligne-Ribet, [D-R], or Ribet, [R].

We note that, for \( s \in B_n \), \( L_p(s,\chi,\mathbb{Q}_m) = \Pi \zeta L_p(s,\chi\zeta) \), where \( \zeta \) runs through all \( \chi \)-power roots of unity of order dividing \( p^m \). Moreover, \( L_p(s,\chi,\mathbb{Q}_m) \) is Iwasawa analytic on \( B_m \), i.e. there is an associated Iwasawa power series, as with the \( L \)-functions defined over \( \mathbb{Q} \).

Corresponding to \( L_p(s,\chi\zeta) \) is the Iwasawa series \( f_\chi(\zeta(1+T)-1) \). So \( \Pi \zeta L_p(s,\chi\zeta) \) corresponds to \( \Pi \chi f_\chi(\zeta(1+T)-1) \), i.e.
Coleman, [C], has shown that $\prod f(x) (\zeta(1-T)^s - 1)$ can be written as a power series $g((1+T)^{spm} - 1)$.

Let $h_x(T)$ be the Iwasawa series associated to $L_p(s, \chi, Q_m)$, i.e.

$$L_p(s, \chi, Q_m) = h_x(\kappa(\chi_m)^s - 1),$$

where $\chi_m$ is a topological generator for $Gal(K_0/K_m)$. We may take $\chi_m = \chi_0^{\rho_m}$ and $\kappa(\chi_0) = 1+q$, so that

$$L_p(s, \chi, Q_m) = h_x((1+q)^{spm} - 1),$$

for $s \in B_m$.

Now for $s \notin B_0$, this gives

$$h_x((1+q)^{spm} - 1) = L_p(s, \chi, Q_m)$$

$$= \prod f(x) (\zeta(1-T)^s - 1)$$

$$= g((1+q)^{spm} - 1).$$

If $g(T) - h_x(T)$ is not identically zero then it satisfies the hypothesis of the Weierstrass Preparation Theorem, so has finitely many zeros is the open unit $T$-disk. But, for all $T \in ((1+q)^{spm} - 1; s \notin B_0) = S$, this function is zero, and $S$ is an infinite set. So $g(T) - h_x(T) = 0$ for all $T$ in the open unit $T$-disk, i.e.

$$g((1+q)^{spm} - 1) = h_x((1+q)^{spm} - 1)$$

for all $s \notin B_m$.

We can generalize the results of theorems 3, 4 to this case as follows.

**Theorem 6:** Let $a$ be a zero of $f_x(T)$, and $s_a = \log(1+a)/\log(1+q)$. Say $m = \min(n: s_a \notin B_n)$. Then $s_a$ is a zero of one of $\{L_p(s, \chi, Q_m)\}$, where $\zeta$ runs through all $p$-power roots of unity.
proof: \( f_x(a) = 0 \), so \( g((1+a)^p^n - 1) = h_x((1+a)^p^n - 1) = 0 \). In the set \( B_m \), \( a \) corresponds to:

\[
\log\left(\frac{(1+a)^p^n - 1}{1/(1+a)^p^n}\right) = \log(1+a)^p^n/p^n \log(1+q) = s_a.
\]

[We know \( p = (1+a)^p^n - 1 = (1+q)^s_a^p^n - 1 \), so \( s_a = \log(1+p)/p^n \log(1+q) \).] Thus \( s_a p^n \log(1+q) = p^n \log(1+a) \), and \( \exp(s_a p^n \log(1+q)) = \exp(p^n \log(1+a)) = \exp(p^n \log(1+q)). \) Now

\[|\log(1+a)^p^n| = |s_a p^n \log(1+q)| < p^{m-1-1/(p-1)} p^{-m-1} = p^{-1/(p-1)}.
\]

Hence, letting \( \delta \in \mathbb{C}_p \) such that \( \exp(p^n \log(1+a)) = (1+\delta)^p^n \), proposition 2 gives \( 1 + a = \zeta(1+\delta) \) for some \( p \)-power root of unity \( \zeta \). So:

\[
\exp(p^n s_a \log(1+q)) = p^{-1}(1+a)^p^n.
\]

Thus

\[
L_p(s_a, X, \chi, Q_m) = h_x(p(1+q)^p^n - 1) \quad \text{[see below]}
\]

\[
= h_x(p \exp(p^n s_a \log(1+q)) - 1)
\]

\[
= h_x(p p^{-1}(1+a)^p^n - 1)
\]

\[
= h_x((1+a)^p^n - 1)
\]

\[
= 0.
\]

THEOREM 7: Let \( X \) be an even character of the first kind, and let \( f_x(T) \) be the Iwasawa series associated to \( L_p(s, X) \). Suppose \( f_x(a) = 0 \), and let \( s_a = \log(1+a)/\log(1+q) \). Then: \( s_a \) is a zero of \( L_p(s, X, \chi, Q_m) \) for some \( p \)-power root of unity \( \zeta \) if and only if there is a \( p \)-power root of unity \( \zeta \) such that \( |(1+a)^p^n - \zeta| < p^{-1/(p-1)} \).

proof: Suppose \( L_p(s_a, X, \chi, Q_m) = 0 \). Then \( s_a \in B_m \), and hence \( |s_a| = p|\log(1+a)| < p^{m-1-1/(p-1)} \). Thus \( |\log(1+a)| < p^{m-1-1/(p-1)} \), so that

\[|\log(1+a)^p|^n < p^{-1/(p-1)} \]. Because \( \exp \) is an isomorphism from \( D \) to
1 + D, \left| \exp(\log(1+a)^p^m) - 1 \right| < p^{-1/p-1}, and by proposition 2, 
\exp(\log(1+a)^p^m) = p^{-1}(1+a)^p^m, for some p-power root of unity \( p \).
So \( |p^{-1}(1+a)^p^m - 1| < p^{-1/p-1} \), and \( |(1+a)^p^m - p| < p^{-1/p-1} \).

Conversely, suppose there is a p-power root of unity \( p \) such that \( |(1+a)^p^m - p| < p^{-1/p-1} \). Then \( |p^{-1}(1+a)^p^m - 1| < p^{-1/p-1} \). So \( |\log(p^{-1}(1+a)^p^m)| < p^{-1/p-1} \). But \( \log(p^{-1}(1+a)^p^m) = \log(p^m(1+a)^m) \), so \( |\log(p^{-1}(1+a)^p^m)| < p^{-1/p-1}, |\log(1+a)| < p^{-1/p-1} \), and thus \( s_a < p^{m-1/p-1} \). Hence \( s_a \in B_m \), so by theorem 6, there is a p-power root of unity \( \xi \) such that
\[ L_p(s_a, \chi^f \xi, Q_m) = 0. \]

For \( s \in B_a \), and \( \xi \) a p-power root of unity, we know that
\[ L_p(s, \chi^f \xi, Q_m) = \Pi f \chi^f \xi \left( (1+q)^s - 1 \right) \]
\[ = \Pi f \chi(\xi(1+q)^s - 1) \]
\[ = \Pi f \chi(\xi(1+T) - 1) \]
\[ = \Pi f \chi(\xi(1+U) - 1) \]
\[ = g((1+U)^p^m - 1) \]
\[ = g((\xi(1+T))^p^m - 1) \]
\[ = g(\xi^p^m(1+q)^5p^m - 1) \]
\[ = h\chi(\xi^p^m(1+q)^5p^m - 1), \]
where \( T = (1+q)^s - 1 \), and \( U = (1+T) - 1 \). Since \( L_p(s, \chi^f \xi, Q_m) \) and \( h\chi(\xi^p^m(1+q)^5p^m - 1) \) are both Iwasawa analytic on \( B_m \), they must agree on all of \( B_m \).

Note that for \( p \) as in theorem 7, we have \( |p^{-1}(1+a)^p^m - 1| < p^{-1/p-1} \), so that \( \exp(\log(p^{-1}(1+a)^p^m) = p^{-1}(1+a)^p^m = \exp(\log(1+a)^p^m) \).
If ζ is any $p^m$-th root of $p$, then $\exp(\log(1+\alpha)p^m) = \xi^{-p^m}(1+\alpha)p^m$.

Hence:

$$L_p(s_\alpha,\chi,\mathbb{Q}_m) = h_x(\xi^p(1+\alpha)p^m - 1)$$

$$= h_x(\xi p^m \exp(\log(1+\alpha)p^m) - 1)$$

$$= h_x(\xi p^m \xi^{-p^m}(1+\alpha)p^m - 1)$$

$$= h_x((1+\alpha)p^m - 1)$$

$$= 0.$$

By an argument identical to the uniqueness argument for theorem 4, we find that $p$ is unique. So the $\xi$ produced in theorem 7 is any of the $p^m$-th roots of the unique $p$-power root of unity $p$ satisfying

$$|(1+\alpha)p^m - p| < p^{-1/(p-1)}.$$  Furthermore, we have a result analogous to corollary 5.

**Corollary 8:** Say $\alpha$ is a zero of $f_x$, (so that $(1+\alpha)p^m - 1$ is a zero of $h_x$), and $p$ is a $p$-power root of unity such that $|(1+\alpha)p^m - p| < p^{-1/(p-1)}$. Let $\xi$ be any $p$-power root of unity, and $s_\alpha = \log(1+\alpha)/\log(1+\alpha)$. Then:

$$L_p(s_\alpha,\chi,\mathbb{Q}_m) = 0$$

if and only if $h_x(\xi p^m \xi^{-p^m}(1+\alpha)p^m - 1) = 0$.

At this point, we note a result of Morita, [M], which provides a "multi-valued analytic continuation" of $L_p(s,x)$ to any of the disks $B_n$, defined as follows:

There exists a polynomial

$$C(x) = x^{p^m} + a_1(s)x^{p^m-1} + \ldots + a_{p^m}(s)$$

which is irreducible over the quotient field of the ring of all Krasner
analytic functions that converge on $B_n$, with $a_j(s)$ Krasner analytic on $B_n$, and any root of $C(X) = 0$ for $s \in B_1$ has the form $L_p(s, x^{p^m})$ for some $p$-power root of unity $\zeta$ of order dividing $p^m$.

So $s_0$ is a zero of the multi-valued analytic continuation if and only if one of the zeros of $C(s, X) = X^{p^m} + a_1(s_0)X^{p^{m-1}} + \ldots + a_{p^m}(s_0)$ is $X = 0$, which occurs precisely when $a_{p^m}(s_0) = 0$. But, on $B_1$, $a_{p^m}(s) = \prod L_p(s, x^{p^m}) = L_p(s, x, Q_m)$, and since $a_{p^m}(s)$ and $L_p(s, x, Q_m)$ are both Krasner analytic on $B_m$, we must have $a_{p^m}(s) = L_p(s, x, Q_m)$ on $B_m$. So $s_0$ is a zero of the multi-valued analytic continuation if and only if $L_p(s_0, x, Q_m) = 0$.

Theorem 6 gives $L_p(s_0, x, Q_m) = 0$ for some $\zeta$, a $p$-power root of unity. Say the order of $\zeta$ is $p^n \geq p^m$. Then $L_p(s_0, x, Q_m) = h_x(x^{p^n}(1+q)^{n-p^n} - 1) = h_x((1+q)^{n-p^n} - 1) = L_p(s_0, x, Q_m)$. For $s \in B_1$, $L_p(s, x, Q_m)$ divides $L_p(s, x, Q_m)$, but each is analytic on $B_m$. Hence $L_p(s_0, x, Q_m) = 0$, and this gives $s_0$ as a zero of the multi-valued analytic continuation of $L_p(s, x)$ to $B_n$.

Returning to our example from Wagstaff, we see that $L_p(s_0, x, Q_1)$ is defined for both values of $\alpha$. So, for each $\alpha$, $L_p(s_0, x, Q_1) = 0$ for some $p$-power root of unity $\zeta$. Using theorem 7, we seek a $p$-power root of unity $\rho$ such that $|((1+\alpha)^p - \rho| < p^{-1/(p-1)}$. $\rho = 1$ is seen to satisfy this for either $\alpha$. Hence $L_p(s_0, x, Q_1) = 0$ for both values $s_0$. Alternatively, by the above, we may say that the multi-valued analytic continuation of $L_p(s, x)$, to the disk $B_1$, has a zero at $s_0$ for each $\alpha$. 

CHAPTER III
APPLICATIONS AND EXAMPLES

Introduction

In this chapter, results from chapter II are applied to specific cases. In order to compute the examples, a modification of Washington's formula is used. The first section describes a method for calculating an approximation for the zeros of the p-adic L-function associated to a specific non-trivial even character of the first kind, \( \chi \). The second section contains the estimates necessary to determine the accuracy of this method. Subsequent sections deal with a certain class of imaginary cyclic extensions \( K/Q \), and with quadratic extensions of \( Q \). Several computations were made by the author, using the algorithm of section 1, for a program written in a language suited to algebraic symbol manipulations, (so that roots of unity could be properly handled), for each of these types of fields. Results of some of these computations are included as well.
1: A Calculation Method

Recall that the p-adic L-function, \( L_p(s, \chi) \), may be written, (according to a formula of Washington):

\[
L_p(s, \chi) = q^{-1}(s-1)^{-1} \sum_{a=1, (a, p) = 1}^{\infty} \chi(a)\langle a \rangle^{s-1} \sum_{j=1}^{\infty} (q/a)^j B_j. \tag{1}
\]

Rearranging this and using the binomial theorem for \( \langle a \rangle^{s-1} \) we obtain:

\[
L_p(s, \chi) = -q^{-1} \sum_{a=1, (a, p) = 1}^{\infty} \chi(a) \left( \sum_{j=1}^{\infty} \left( q/a \right)^j B_j \right) -(s-1) \left( \sum_{j=1}^{\infty} \left( q/a \right)^j B_j \right). \tag{2}
\]

Letting \( \delta_1(a, s) = \sum_{j=1}^{\infty} \left( q/a \right)^j B_j \) and \( \delta_2(a, s) = \sum_{j=1}^{\infty} (-q/a)^j B_j \), (3)

(2) may be written:

\[
L_p(s, \chi) = -q^{-1} \sum_{a=1, (a, p) = 1}^{\infty} \chi(a) \left( \delta_1(a, s) + \delta_2(a, s) - (s-1) \delta_1(a, s) \delta_2(a, s) \right). \tag{4}
\]

Truncating the infinite series \( \delta_n(a, s) \) after \( j = m \), and summing over all \( a \), gives a polynomial in \( s \). The coefficients of this polynomial will serve as approximations for the corresponding coefficients in the expansion of \( L_p(s, \chi) \) about \( s = 0 \), the accuracy of this approximation being determined by the choice of \( m \). If

\[
f_X(T) = \sum_{j=0}^{\infty} e_j T^j \tag{5}
\]

is the Iwasawa series for \( \chi \), then \( T \) and \( s \) are related by

\[s = \log(1+T)/\log(1+q), \text{ for } |T| \text{ sufficiently small, (i.e. less than} \]
So the formal substitution $\log(1+T)/\log(1+q) \to s$ in $L_p(s,X)$ will produce, (using the expansion for the logarithm given in chapter II), a series in $T$ which must agree with $f_X(T)$. Hence, from the approximation of $L_p(s,X)$, an approximation for the first few coefficients of $f_X(T)$ is obtained. Because $f_X(T)$ satisfies the hypotheses of the Weierstrass Preparation Theorem, and since $\mu = 0$, its zeros may be approximated from these coefficients. For a given zero $a$ of $f_X(T)$, the corresponding $s_a = \log(1+a)/\log(1+q)$ may be approximated. By the results of chapter II, $s_a$ will be a zero of $L_p(s,\xi^k,\Omega_m)$ for some $p$-power root of unity $\xi$ and some integer $m \geq 0$. 
2. Some Estimates

The following discussion leads to a result providing an upper bound for the error introduced by the approximations described above. Similar estimates were made in [WG], although some of these seem to be in error.

**Lemma 9.** Let $p$ be an odd prime, and $k$ a positive integer. Use \( \ln \) to denote the real natural logarithm. Then
\[
\frac{\ln \left( \frac{x+k}{p^k} \right)}{\ln p} \leq \frac{\ln k - 1}{\ln p} \quad \text{for all } x \geq 2.
\]

**Proof.** Let $g_{p,k}(x) = kp^{-1}/(x+k)$. Then its derivative is
\[
k(\ln p)p^{-1}(x+k) - kp^{-1} = \frac{kp^{-1}[(x+k)(\ln p) - 1]}{(x+k)^2}
\]
which is positive for $x+k \geq 1/\ln p$, i.e. for all $x \geq 0$. Hence $g_{p,k}(x)$ is increasing on $[0,\infty)$. Now $g_{p,k}(2) = kp/(k+2) = p/(1+2/k) \geq p/3 \geq 1$. So $\ln \left( \frac{kp^{-1}/(x+k)}{3} \right) \geq 0$ for all $x \geq 2$.

This gives:
\[
\frac{\ln k + \ln p^{-1} - \ln(x+k)}{\ln p} \geq 0
\]
\[
\frac{\ln k - 1 - \ln \left( \frac{(x+k)/p^k}{\ln p} \right)}{\ln p} \geq 0
\]

Making the change of variables $s = \log(1+T)/\log(1+q)$ and using the series expansion for $\log(1+X)$ gives:
\[
s = \frac{\log(1+T)}{\log(1+q)} = \left[ \sum_{n=1}^{\infty} \frac{(-1)^{n-1}T^n}{n} \right] / \left[ \sum_{n=1}^{\infty} \frac{(-1)^{n-1}q^n}{n} \right]
\]
Let \( A = \sum_{n=1}^\infty (-q)^{n-1}/(1/n) \), (a unit), and let \( S(U) = (1/A) \sum_{n=1}^\infty (-q)^{n-1}U^n/n \).

Similarly, let \( A_k = \sum_{n=1}^k (-q)^{n-1}(1/n) \), (also a unit), and let

\[
S_k(U) = (1/A_k) \sum_{n=1}^k (-q)^{n-1}U^n/n.
\]

Suppose \( 1 \leq j \leq k \). The coefficient of \( U^j \) in \( S(U) \) is \( c_j = (-q)^{j-1}/(jA) \), while that of \( U^j \) in \( S_k(U) \) is \( c_j^* = (-q)^{j-1}/(jA_k) \).

Now

\[
|A - A_k| = \left| \frac{(-q)^n}{n+1} + \frac{(-q)^{k+1}}{(k+2)} + \ldots \right| \leq p^{-k} \max_{n \geq 1} (p^{1-n}/\ln k) \]

\[= p^{-k} \max \{ p^{1-n+y_{(n+1)}}, \max_{n \geq 2}(p^{1-n+y_{(n)}}, (n+1)!) \} \leq p^{-k} \max \{ p^{1-n+y_{(n+1)}}/\ln p, p^n/k/\ln p \} \]

since \( 1 - n + y_{p}(n+k) \leq 1 - n + \ln(n+k)/\ln p = 1 + \ln[(n+k)/p^n]/\ln p \leq \ln k/\ln p \) by the lemma above. So \( c_j^* \equiv c_j (\mod p^{k-1}(k+1)/\ln p) \).

Say \( j > k \). Then \( c_j^* = 0 \), and \( |c_j| = |(-q)^{j-1}/(jA)| = p^{1-j}/|j| = p^{1-j+y_j} \leq p^{1-j+y_j+1}/\ln p \). For an estimate independent of \( j \), note that

\[
-k + \ln(k+1)/\ln p - 1 + j - \ln j/\ln p = \ln[p^{j-k-1}(k+1)/j] / \ln p
\]

But the function \( p^n/x \) is increasing for \( x \geq 1 \), and since \( j \geq k + 1 \),

\[
[p^{j}/j][p^{k+1}/(k+1)] \geq 1 \text{ so that } \ln[p^{j}/j][p^{k+1}/(k+1)]/\ln p \geq 0 \text{ and }
\]

\[ -k + \ln(k+1)/\ln p \geq 1 - j + \ln j/\ln p. \] So

\[
|c_j^* - c_j| = |c_j| \leq p^{k-1}(k+1)/\ln p \text{ for all } j > k.
\]

This gives \( c_j^* \equiv c_j (\mod p^{k-1}(k+1)/\ln p) \) for all \( j > k \).
So in general the coefficients of $S_k(U)$ approximate those of $S(U)$ modulo $p^{k-1}\ln(k+1)/\ln p$.

Now, we turn to the $L$-function itself. Say

$$L_p(s, x) = a_0 + a_1 s + a_2 s^2 + \ldots$$  \hfill (6)

and that for some integer $n \leq k - \ln(k+1)/\ln p$, there is a sequence $(a_j^*)$ having $a_j^* \equiv a_j \pmod{p^n}$ for each $j$. Let $L_{p,k}(s, x) = a_0^* + a_1^* s + \ldots$. By the relationship between $L_p(s, x)$ and $f(T)$,

$$f(T) = L_p(S(T/q), x).$$  \hfill (7)

Let $L_p(S(U), x) = b_0 + b_1 U + \ldots$, and $L_{p,k}(S_k(U), x) = b_0^* + b_1^* U + \ldots$ Since $a_j^*$ approximates $a_j$ modulo $p^n$, and $c_j^*$ approximates $c_j$ modulo $p^{k-1}\ln(k+1)/\ln p$, it must be that $b_j^* \equiv b_j \pmod{p^n}$. By (7), $b_j = q^j e_j$.

Let $e_j^* = b_j^*/q^j$. Then $L_{p,k}(S(T/q), x) = e_0^* + e_1^* T + \ldots$, and:

$$e_j^* \equiv e_j \pmod{p^{n-j}} \quad \text{for each } j.$$  \hfill (8)

The following theorem provides the means for determining the sequence $(a_j^*)$ described above, and hence completes this discussion.

**Theorem 10**: Say $L_p(s, x) = a_0 + a_1 s + a_2 s^2 + \ldots$, and let $n$ be a fixed positive integer. Choose $m \in \mathbb{Z}$ such that

$$j - \nu_p(j!) + \min(0, \nu_p(B_j)) \geq n + 1 \quad \text{for all } j > m.$$  \hfill (9)

Let

$$L_{p,k}(s, x) = -q^{-1} \sum_{\ell=1}^{s} X(a) \left( \sum_{j=1}^{r_a^2 \ell_1} j^{-1} [(-q/a)^{j-1}]^{B_j} \right)$$

$$+ (-s-1) \sum_{j=1}^{r_a} \left( \sum_{j=1}^{r_a^2 \ell_1} j^{-1} [(-q/a)^{j-1}] \right) \left( \sum_{j=1}^{r_a^2 \ell_1} j^{-1} [-q/a]^{B_j} \right).$$  \hfill (9)

and suppose $L_{p,k}(s, x) = a_0^* + a_1^* s + a_2^* s^2 + \ldots$. Then $a_j^* \equiv a_j \pmod{p^n}$ for all $j \geq 0$.  \hfill (9)
proof: We compute a lower bound for the p-order of the jth terms of δ₁, δ₂. Say δ₁(a,s) = r₀ + r₁s + ..., δ₂(a,s) = t₀ + t₁s + ...

The jth term, (j>1), in the original expansion for δ₁(a,s) is

\[(j+1)! (a^\omega(a)^{-1} - 1)^j = \prod_{i=0}^{j}(a^\omega(a)^{-1} - 1) j/j! \]

\[ = [d_1s + d_2s^2 + ... + d_{j-1}s^{j-1}] (a^\omega(a)^{-1} - 1)^j/j! \]

with dᵢ ∈ Z.

So the contribution of this term to rₖ has p-order

\[ v_p (d_1s + d_2s^2 + ... + d_{j-1}s^{j-1}) \cdot (a^\omega(a)^{-1} - 1)^j/j! \]

letting dᵢ = 0 for k ≥ j).

Now a = ω(a) (mod p), so \( v_p(a^\omega(a)^{-1} - 1)^j ≥ j \). Also, dᵢ ∈ Z, so \( v_p(d_1s + d_2s^2 + ... + d_{j-1}s^{j-1}) = v_p(j!) \).

Hence the p-order of this contribution is not less than \( j - v_p(j!) \).

The jth term, (j>1), in the original expansion for δ₂(a,s) is

\[(j+1)! (-q/a)^j B_j = \prod_{i=0}^{j} (-q/a)^j B_j/j! \]

\[ = [d_1s + d_2s^2 + ... + d_{j-1}s^{j-1} (-q/a)^j B_j/j! \]

with dᵢ ∈ Z.

So the contribution of this term to tₖ has p-order

\[ v_p (d_1s + d_2s^2 + ... + d_{j-1}s^{j-1} (-q/a)^j B_j/j!) \]

\[ = v_p(d_1s + d_2s^2 + ... + d_{j-1}s^{j-1} (-q/a)^j B_j/j!) \]

\[ ≥ j + v_p(B_j) - v_p(j!) \]

(since \( v_p(q) = 1 \) and \( (p,a) = 1 \), \( v_p(-q/a)^j = j \)).

Now choose \( m ≥ 1 \) such that \( j - v_p(j!) + \min(0,v_p(B_j)) ≥ n + 1 \) for all \( j > m \). Let \( \tau_1(a,s), \tau_2(a,s) \) be the finite series gotten by truncating the original expressions for δ₁, δ₂, (respectively), after \( j = m \). Then the contributions of \( δ_1(a,s) - \tau_1(a,s) \) to \( r_k \) and \( δ_2(a,s) - \tau_2(a,s) \) to \( t_k \) (for any \( k \)), have p-order no less
than \( n + 1 \). Thus the coefficients of \( s^n \) in \( \delta_j(a,s) \) and \( \tau_j(a,s) \), (for \( j = 1,2 \)), are congruent modulo \( p^{n-j} \) for every \( k \).

Comparing coefficients in (4) and (6), we find that

\[
\begin{align*}
\alpha_k &= -q^{-1} \sum_{a=1, (a,p) = 1}^{q} \chi(a) (t_k + \text{rot}_k + \text{rot}_{k-1} + \ldots + r_{k0} - \text{rot}_{k-1} - \text{rot}_{k-2} - \ldots - r_{k-1}t_0) \\
&= -q^{-1} \sum_{a=1, (a,p) = 1}^{q} \chi(a) c_k, \text{ say.}
\end{align*}
\]

Similarly \( \alpha_k^* = -q^{-1} \sum_{a=1, (a,p) = 1}^{q} \chi(a) c_k^* \). By the above, \( \alpha_k^* \equiv c_k \pmod{p^{n+1}} \). So \( \alpha_k^* \equiv \alpha_k \pmod{p^n} \) as desired.

In summary, if one wishes to determine the coefficient of \( T^j \) modulo \( p^a \), it is necessary first to find an integer \( k \) such that \( k = j - \ln(k+1)/\ln p \geq a \). One then takes \( n = a + j \) in theorem 2.

After finding the corresponding \( m \) described in the hypothesis of the theorem, the infinite series in (2) should be truncated after the \( m^{th} \) term as in (9), producing \( L_p,\kappa(s,x) \). The coefficients \( a_j,^* \) of the powers of \( s \) in this series are approximations for the coefficients \( a_j \) in \( L_p(s,x) \), the approximation being accurate modulo \( p^n \). The coefficient of \( T^j \) in \( L_p,\kappa(S_k(T/q),x) \) is then accurate modulo \( p^{n-j} = p^a \) as desired. Note that by (8) the lower order coefficients in \( L_p,\kappa(S_k(T/q),x) \) are accurate modulo progressively higher powers of \( p \), e.g. if the coefficient of \( T^2 \) is accurate modulo \( p \), then the coefficient of \( T \) is accurate modulo \( p^2 \). The constant term is determined completely by the \( j = 1 \) terms in (9), and thus will be as accurate as the approximation for the values of \( \chi(a) \).
3: Certain Imaginary Cyclic Extensions of $\mathbb{Q}$ of Order $2p$

Fix an odd prime $p$. Choose $h$, $t$ such that $h > 0$, $h \neq 3$, $(p, h) = 1$, where $h$ is the class number of $E = \mathbb{Q}(\sqrt{-h})$, $-h$ is not a square modulo $p$, $t \equiv 1 \pmod{p}$, $t \equiv 1 \pmod{p^2}$, $t$ a prime, $-h$ is a non-zero square modulo $t$. Then $\mathbb{Q}(\zeta_t)$ contains a totally real subfield $F$ of degree $p$ over $\mathbb{Q}$. ($\zeta_t$ is a primitive $t^{th}$ root of unity.) Also $K = EF$ is cyclic, imaginary of degree $2p$ over $\mathbb{Q}$. Kida's formula gives $\lambda_k = p - 1$, where $\lambda_k$ is the Iwasawa invariant of the field $K$, (we know $\mu = 0$ since $K$ is abelian over $\mathbb{Q}$).

$\text{Gal}(K/\mathbb{Q})^\wedge$, the character group of $K/\mathbb{Q}$, (which is isomorphic to $\text{Gal}(K/\mathbb{Q})$), is cyclic of order $2p$, and therefore has $p-1$ generators. Let $\chi$ be such. Then $\chi$ is odd, so that $\chi u$ is even of order $p(p-1)$.

Now $\mathbb{Z}_p[\chi u] = \mathbb{Z}_p[\chi] = \mathbb{Z}_p[\zeta_{2p}] = \mathbb{Z}_p[\zeta_p]$, where "$\zeta_p" means a primitive $p^{th}$ root of unity. Let $L$ be the quotient field of the ring $\mathbb{Z}_p[\zeta_p]$, i.e. $L = \mathbb{Q}_p(\zeta_p)$. Now:

$\text{Aut}(\text{Gal}(K/\mathbb{Q})^\wedge) \cong (\mathbb{Z}/2p\mathbb{Z})^\wedge \cong \text{Gal}(L/\mathbb{Q}_p)$ canonically.

Let $\sigma_k \in \text{Gal}(L/\mathbb{Q}_p)$ be the image of $\sigma_k \in \text{Aut}(\text{Gal}(K/\mathbb{Q})^\wedge)$ where $\sigma_k : \chi \mapsto \chi^k$ for each $k \in (\mathbb{Z}/2p\mathbb{Z})^\wedge$, i.e. $\sigma_k(\chi(a)) = \chi^k(a)$ for all $a$ and all $k$. Note that $\chi^k$ is also a generator of $\text{Gal}(K/\mathbb{Q})^\wedge$ for each $k$.

Let $n$ be a positive integer. Then:

$L_p(1-n, \chi u) = -(1 - \chi u^{1-n}(p) p^{n-1}) (1/n) \sum_{\alpha=1}^{f-1} \sigma(\chi u(a)) B_n(a/f)$

where $f$ is the conductor of $\chi u$, and $B_n(x)$ is the $n^{th}$ Bernoulli polynomial.

Since $L_p(1-n, \chi u)$ is then in $L$, we may apply $\sigma = \sigma_k$:

$\sigma(L_p(1-n, \chi u)) = -(1 - \sigma(\chi u^{1-n}(p) p^{n-1}) (1/n) \sum_{\alpha=1}^{f-1} \sigma(\chi u(a)) B_n(a/f)$
since \( u(a) = \omega(a) \) for all \( a \), (because \( \omega(a) \in \mathbb{Z}_p \)), and since the conductor of \( X^u \) is also \( f \). The set \( \{1-n: n > 0, n \in \mathbb{Z}\} \) is dense in \( \mathbb{Z}_p \), so \( \sigma(L_p(s,X^u)) = L_p(s,X^u) \) for all \( s \in \mathbb{Z}_p \). \( L_p(s,X^u) \in L \) for all \( s \in \mathbb{Z}_p \), so this is defined.

In terms of the Iwasawa series, (10) becomes \( \sigma(f_{x^u}((1+q)^s - 1)) = f_{\sigma(x^u)((1+q)^s - 1)} \) for all \( s \in \mathbb{Z}_p \). Let \( f_{x^u}(T) = e_0 + e_1T + \ldots \) as before. Define \( f_{x^u,\sigma}(T) = \sigma(e_0) + \sigma(e_1)T + \ldots \). Now \( e_j \in \mathbb{Z}_p[\zeta_p] \), so \( \sigma(e_j) \in \mathbb{Z}_p[\zeta_p] \), and for \( s \in \mathbb{Z}_p \), we have

\[
\sigma(f_{x^u}((1+q)^s - 1)) = f_{x^u,\sigma}((1+q)^s - 1) = f_{\sigma(x^u)((1+q)^s - 1)}.
\]

So \( f_{x^u,\sigma}(T) = f_{\sigma(x^u)}(T) \) for infinitely many \( T \) in the open unit \( T \)-disk. But \( f_{x^u,\sigma}(T) \) and \( f_{\sigma(x^u)}(T) \) satisfy the hypotheses of the Weierstrass Preparation Theorem, hence so does their difference. Since it has infinitely many zeros, it must be identically zero, i.e. \( f_{x^u,\sigma}(T) = f_{\sigma(x^u)}(T) \) as formal power series.

Say \( f_{x^u}(T) = D_{x^u}(T)U_{x^u}(T) \) where \( D_{x^u}(T) \) is the distinguished polynomial for \( f_{x^u}(T) \), and \( U_{x^u}(T) \) is a unit power series. Then, if \( D_{x^u}(T) = d_0 + d_1T + \ldots + d_nT^n \), \( U_{x^u}(T) = u_0 + u_1T + \ldots \), we have:

\[
f_{x^u,\sigma}(T) = [\sigma(d_0) + \sigma(d_1)T + \ldots + \sigma(d_n)T^n][\sigma(u_0) + \sigma(u_1)T + \ldots]
\]

\[
= D_{x^u,\sigma}(T)U_{x^u,\sigma}(T).
\]

Since \( \nu_p(a) = \nu_p(\sigma(a)) \) for all \( a \in \mathbb{L} \), we know that \( D_{x^u,\sigma}(T) \) is distinguished, and that \( U_{x^u,\sigma}(T) \) is a unit power series. Now \( f_{x^u,\sigma}(T) = f_{\sigma(x^u)}(T) \), so \( D_{x^u,\sigma}(T) = D_{\sigma(x^u)}(T) \). In particular, \( \lambda = \deg(D_{x^u,\sigma}(T)) = \deg(D_{\sigma(x^u)}(T)) \).
By Kida's formula, $\lambda_{K^-} = p^{-1}$. But $\lambda_{K^-}$ is the sum over all odd characters $\neq 0$ in $\text{Gal}(K/Q)^\wedge$ of the degrees of $D_{\alpha}(T)$. Now $\text{Gal}(K/Q)^\wedge = (X, X^2, \ldots, X^{p^\ell-1})$. We may ignore all even powers of $X$, since they are even characters. Since $h_\alpha$ is prime to $p$, $\lambda_{K^-} = 0$, so that the character $X^p$ contributes nothing. [$X^p$ is the quadratic character associated to the field $E$.] Hence only the $p-1$ generators of $\text{Gal}(K/Q)^\wedge$ are left. The deg$(D_{\alpha}(X), \omega(T))$ are all equal to $\lambda$, and since their sum is $p-1$, we have $\lambda(p-1) = p-1$, i.e. $\lambda = 1$. So: $D_{\alpha}(X), \omega(T) = T - a_{\alpha}(X)$, for every automorphism $\alpha$ in $\text{aut}[\text{Gal}(K/Q)^\wedge]$. Note that $a_{\alpha}(X)$ is then in $\mathbb{Z}_p[\zeta_p]$, and that the $a_{\alpha}(X)$ are conjugate under $\text{Gal}(L/Q_p)$.

Fix a character $X$, with $\text{Gal}(K/Q)^\wedge = (X)$, and let $\alpha = \alpha_x$. Let $s_\alpha = \log(1+\alpha)/\log(1+q)$. Let $\zeta$ be a fixed primitive $p^n$ root of unity, and $\psi$ be the character of the second kind associated to $\zeta^2$. Let $\psi$, be the trivial character.

Let $\psi = \zeta - 1$. Then $(\psi)$ is the unique prime of $\mathbb{Z}_p[\zeta]$ above $p$. Since $D_{\alpha}$ is distinguished, $\psi(\omega) = 1$. Say $\alpha = r\psi$, $r \in \mathbb{Z}_p[\zeta]$. Now the residue class field degree $f((\psi)/p) = 1$, so the residue class field, $\mathbb{Z}_p[\zeta/(\psi)]$, is just $\{0, 1, \ldots, p-1\}$. Say $r \equiv r \mod (\psi)$. Let $s_\alpha = \zeta^n - 1 + \zeta^{-n} = (\zeta - 1)(\zeta^{p-1} - 1 + \zeta + \ldots + \zeta^{p-2})$. So $s_\alpha \equiv \psi(p - n + r) \mod (\psi)^2$. Note that $\beta_r \equiv \psi(p - r + r) \equiv \psi p \equiv 0 \mod (\psi)^2$, so $|\beta_r| < |\psi| = p^{-1}/p^{-1}$. But $|\beta_r| = |\zeta^r - 1 + \zeta^{-r}a| = |1 - \zeta^r + a|$, so we have $L_p(s_\alpha X \psi^r) = 0$. Hence we are assured of a zero of $L_p(s, X \psi^r)$, for $\psi$ trivial or of order $p$. Note that the character of the second kind which gives the zero for $X^k$ is just $\psi^k$, where $\psi$ gives the zero for $X$. 


Note also that the uniqueness of \( \zeta \), combined with the fact that 
\( f_{\mathbf{u}} \) has only a single zero \( a \) implies that 
\( L_p(s, \chi_{\mathbf{u}} \zeta) \) is the unique L-function having a zero.

To find the character of the second kind, \( \zeta \), which gives
\( L_p(s, \chi_{\mathbf{u}} \zeta) = 0 \), it is necessary only to find the integer \( r \) such that
\( a/\mathbf{v} \equiv r \pmod{\mathbf{w}} \). Then \( \zeta_r \) is the unique such character of the second
kind. To find \( a/\mathbf{v} \) modulo \( \mathbf{w} \), consider \( f(T) = e_0 + e_1 T + \ldots = (T - a)(u_0 + u_1 T + \ldots) \). From this we see that \( e_0 = -au_0 \) and \( e_1 = -au_1 + u_0 \equiv u_0 \pmod{\mathbf{w}} \). Hence \( a/\mathbf{v} = -e_0/(u_0 \mathbf{w}) \equiv -e_0/(e_1 \mathbf{w}) \pmod{\mathbf{w}} \).

Thus, if \( e_1 \) is approximated modulo \( p \), \( r \) may be determined. To assure
that \( e_1 \zeta \) is accurate modulo \( p \), we need \( a = 1 \), \( j = 1 \), with \( a \) and \( j \) as in
the discussion following theorem 10. Hence it suffices to take \( n = 2 \)
in theorem 10. To determine minimum values for \( k \) and \( m \), the prime \( p \)
must be taken into account:

For \( p = 3 \), \( k = 4 \), \( m = 4 \) are the minimal choices.

For \( p > 5 \), \( k = 3 \), \( m = 2 \) are the minimal choices.

Noted that since it is required that the coefficients \( a \), be determined
modulo \( p^2 \), the character values \( \chi_{\mathbf{u}}(a) \) must be known modulo \( p^2 \). The
values of \( \mathbf{u} \) are in \( \mathbb{Z}_p \), and hence may be approximated to any desired
accuracy by rational integers. The values of \( \mathbf{X} \), however, are not in \( \mathbb{Z}_p \)
but are in \( \mathbb{Z}_p[\zeta] \). Since \( \{1, \zeta_1, \ldots, \zeta^{p-2}\} \) is a basis for \( L/\mathbb{Q}_p \), the
values of \( \mathbf{X} \) can be represented as vectors, i.e. \( 1 = [1, 0, \ldots, 0] \), \( \zeta = [0, 1, 0, \ldots, 0] \), \( \zeta^{p-1} = [-1, -1, \ldots, -1] \), etc. With this type of
representation, values of \( \mathbf{X} \) are exact, and accuracy is only a concern
for values of \( \mathbf{u} \).
Using the methods described in sections 1 and 2, examples for various values of $p, t, n$ were approximated with a computer. Tabulated below are some of the results of these computations. Note that the observation regarding the character of the second kind which gives the zero for $X^*$ is evident in these results.

**Table 1**

($X'$ is a generating character for $\text{Gal}(F/Q)^{\wedge}$, and if $\Theta$ is the non-trivial quadratic character of $E/Q$, then $\chi = X'\Theta$. Thus, distinguishing $\chi$ from the other generators of $\text{Gal}(K/Q)^{\wedge}$ is possible simply by knowing $X'$. Of course, $X'$ is fully determined by its value on a generator of $(Z/tZ)^{\wedge}$. For $t = 7$, this generator was taken to be $3$. The values of the various $X'(3)$ are given in the table, thus identifying $\chi$. Similarly, for $t = 13$, the generator used was $2$, so that $X'(2)$ identifies $\chi$.)

**(p = 3)**

<table>
<thead>
<tr>
<th>$t$</th>
<th>$n$</th>
<th>$X'$</th>
<th>$a_0 = e_0$</th>
<th>$a_1 \mod p^2$</th>
<th>$e_1 \mod p$</th>
<th>$a/w \mod w$</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>19</td>
<td>$\zeta$</td>
<td>$2\zeta(\zeta-1)$</td>
<td>-6</td>
<td>$1$</td>
<td>$1$</td>
</tr>
<tr>
<td>7</td>
<td>19</td>
<td>$\zeta^2$</td>
<td>$-2\zeta(\zeta-1)$</td>
<td>-6</td>
<td>$1$</td>
<td>$2$</td>
</tr>
<tr>
<td>7</td>
<td>13</td>
<td>$\zeta$</td>
<td>$4(\zeta-1)$</td>
<td>$6$</td>
<td>$2$</td>
<td>$1$</td>
</tr>
<tr>
<td>7</td>
<td>13</td>
<td>$\zeta^2$</td>
<td>$4(\zeta-1)(\zeta+1)$</td>
<td>$6$</td>
<td>$2$</td>
<td>$2$</td>
</tr>
<tr>
<td>7</td>
<td>10</td>
<td>$\zeta$</td>
<td>$-6\zeta$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>10</td>
<td>$\zeta^2$</td>
<td>$6(\zeta+1)$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>1</td>
<td>$\zeta$</td>
<td>$-2(\zeta-1)(\zeta+1)$</td>
<td>$3$</td>
<td>$1$</td>
<td>$1$</td>
</tr>
<tr>
<td>13</td>
<td>1</td>
<td>$\zeta^2$</td>
<td>$-2(\zeta-1)$</td>
<td>$3$</td>
<td>$1$</td>
<td>$2$</td>
</tr>
<tr>
<td>13</td>
<td>10</td>
<td>$\zeta$</td>
<td>$4(\zeta-1)$</td>
<td>$3$</td>
<td>$1$</td>
<td>$1$</td>
</tr>
<tr>
<td>13</td>
<td>10</td>
<td>$\zeta^2$</td>
<td>$4(\zeta-1)(\zeta+1)$</td>
<td>$3$</td>
<td>$1$</td>
<td>$1$</td>
</tr>
</tbody>
</table>
Table 2

(For \( t = 11 \), the generator of \((\mathbb{Z}/11\mathbb{Z})^*\) was taken to be 2. The values of \( X'(2) \) are given in the table, thus determining \( X \). The values for \( a_1 \) modulo \( p^2 \) are given as sequences of coefficients for the basis consisting of 1, \( \zeta \), \( \zeta^2 \), and \( \zeta^3 \).)

\[ \begin{array}{|c|c|c|c|c|c|} \hline \eta & \chi' & a_0 = a_0 & a_1 \mod p^2 & a_1 \mod p & \frac{a_1}{p} \mod \pi \\ \hline 2 & \zeta & 2\zeta(\zeta+1)/20, 0, 5, 0 & 4 & 4 & \mathbb{L}_p(s, x^{2}u) = 0 \\ 2 & \zeta^2 & -2\zeta(\zeta+1)/20, 0, 5, 20 & 4 & 3 & \mathbb{L}_p(s, x^{2}u) = 0 \\ 2 & \zeta^3 & 2\zeta^3(\zeta+1)/20, 0, 20, 5 & 4 & 2 & \mathbb{L}_p(s, x^{2}u) = 0 \\ 2 & \zeta^4 & -2\zeta^2(\zeta+1)(\zeta+1)/20, 0, 20, 20 & 4 & 1 & \mathbb{L}_p(s, x^{2}u) = 0 \\ 7 & \zeta & -2\zeta(\zeta+1)(\zeta+1)/10, 15, 0, 20 & 4 & 2 & \mathbb{L}_p(s, x^{2}u) = 0 \\ 7 & \zeta^2 & 2\zeta^2(\zeta+1)(\zeta+1)/10, 20, 15, 0 & 4 & 4 & \mathbb{L}_p(s, x^{2}u) = 0 \\ 7 & \zeta^3 & 2\zeta^3(\zeta+1)(\zeta+1)/10, 5, 5, 20 & 4 & 1 & \mathbb{L}_p(s, x^{2}u) = 0 \\ 7 & \zeta^4 & 2\zeta(\zeta+1)(\zeta+1)/20, 10, 5, 10 & 4 & 3 & \mathbb{L}_p(s, x^{2}u) = 0 \\ \hline \end{array} \]
Several authors have made calculations for the coefficients and zeros of \( f_x(T) \) in this case. \( \chi = 0 \), (the quadratic character of \( E \)), if \( E \) is real, and \( \chi = 0 \omega \) if \( E \) is imaginary. The method discussed in sections 1 and 2 has been employed by this author in this case as well.

The purpose here is to display several examples of how the results of chapter II may be used to simplify matters once the zeros of a given \( f_x(T) \) have been approximated. Let \( \lambda = \deg(D_x(T)) \) and note that \( \mathbb{Z}_p[\chi] = \mathbb{Z}_p \).

A. Suppose \( \lambda = 1 \). Then \( D_x(T) = T - a \in \mathbb{Z}_p[T] \), so that \( a \in \mathbb{Z}_p \).
Since \( D_x \) is distinguished, \( |a| \leq p^{-1} < p^{-1/2} \). So \( s_a = \log(1+a)/\log(1+q) \) is a zero of \( L_p(s,\chi) \).

B. Suppose \( \lambda = 2 \). Then \( D_x(T) = (T - a)(T - a_1) \), the factorization being taken over the splitting field \( F \) of \( D_x \). Either \( a \) and \( a_1 \) are both in \( \mathbb{Z}_p \), or \( D_x \) is irreducible, in which case \( |a| = |a_1| \), since they are Galois conjugate. But \( a a_1 \in \mathbb{Z}_p \), so \( |a a_1| \leq p^{-1} \). Hence \( |a| \leq p^{-1/2} \) in either case. If \( s_a \) is not a zero of \( L_p(s,\chi) \), then it must be that \( |a| > p^{-1/2} \), so that \( p - 1 \leq 2 \), and hence \( p = 3 \). For any other odd prime \( p \), \( s_a \) will be a zero of \( L_p(s,\chi) \).

Say \( p = 3 \) and \( |a| \geq p^{-1/2} \). Since \( |a| \leq p^{-1/2} \), we must have \( |a| = p^{-1/2} \). If \( s_a \) is to be a zero of \( L_p(s,\chi) \) for any character of the second kind \( \chi \), then there must be a \( p \)-power root of unity \( \zeta \) such that \( |a + 1 - \zeta| < p^{-1/2} \). If \( p^t \) is the order of \( \zeta \), then \( \nu_p(1-\zeta) = p^{1-t}(p-1)^{-1} \geq 1/2 \). If \( |a| \neq |1 - \zeta| \), then \( |a + 1 - \zeta| = \max\{|a|, 1-|\zeta|\} \geq p^{-1/2} \). So we must have \( |a| = |1 - \zeta| = p^{-1/2} \). Hence
we must have \( t = 1 \), i.e. \( \zeta \) is a cube root of unity. By Krasner's lemma, \( \mathbb{Q}_p(\zeta) \) is contained in \( F \). But \([F: \mathbb{Q}_p] = 2\), so it must be \( \mathbb{Q}_p(\zeta) = F = \mathbb{Q}_p(\sqrt{-3}) \). So the only way that \( s_\alpha \) can be a zero of any of the \( L_p(s_1 x^2) \) is if the splitting field of \( D_\infty \) is \( \mathbb{Q}_p(\sqrt{-3}) \). Since the residue class field of \( F \) is then just \( \{0, 1, 2\} \), and since \(|\alpha| = p^{-1/2}\), we know that, \((\text{for } \nu = \zeta - 1), \alpha/\nu \equiv 1 \text{ or } 2 \pmod{\nu}\). As in section 3, it can be shown that \( \alpha/\nu \equiv 1 \pmod{\nu} \) gives \(|\alpha + 1 - \zeta| < p^{-1/\nu} \), and \( \alpha/\nu \equiv 2 \pmod{\nu} \) gives \(|\alpha + 1 - \zeta^2| < p^{-1/\nu} \). So \( s_\alpha \) is a zero of \( L_p(s_1 x^2) \) for some \( \nu \) if and only if \( F = \mathbb{Q}(\sqrt{-3}) \).

C. Suppose \( \chi = 3 \). Then \( D_\infty(T) = (T - \alpha_1)(T - \alpha_2)(T - \alpha_3) \), the factorization being taken over the splitting field \( F \) of \( D_\infty \). If all of the zeros of \( D_\infty \) are in \( \mathbb{Z}_p \), then all of the corresponding \( s \)-values are zeros of \( L_p(s_1 x) \).

Suppose \( D_\infty(T) \) is an irreducible cubic. Then none of its zeros is in \( \mathbb{Z}_p \). Two possibilities exist. First we suppose that the splitting field \( F \) is of degree 3 over \( \mathbb{Q}_p \). Then all of the \( \alpha_j \) are Galois conjugate, and hence have the same absolute value. Since they are in an extension of degree 3, we have \(|\alpha_j| \leq p^{-1/3} \). Next suppose that the splitting field \( F \) is of degree 6 over \( \mathbb{Q}_p \). Then \( \alpha_j \) is in a non-normal extension of degree 3 over \( \mathbb{Q}_p \). Hence \(|\alpha_j| \leq p^{-1/3} \). If \(|\alpha_j| < p^{-1/\nu} \), then the corresponding \( s_\alpha \) is a zero of \( L_p(s_1 x) \). If \(|\alpha_j| \geq p^{-1/\nu} \), then \( 3 \geq p^{-1} \), so that we must have \( p = 3 \). Let \( \zeta \) be a \( p \)-power root of unity. Then as before, if \( \zeta \) is to satisfy \(|\alpha_j + 1 - \zeta| < p^{-1/\nu} \), \( \zeta \) must be a cube root of unity, and \(|\alpha_j| = p^{-1/2} \). But Krasner's lemma then implies that \( \mathbb{Q}_p(\zeta) \) is contained in \( \mathbb{Q}_p(\alpha_j + 1) = \)}
$Q_p(a_j)$, since $|\xi - \xi^2| = p^{-1/(p-1)} > |a_j + 1 - \xi|$. But the degree of $Q_p(\xi)$ over $Q_p$ is 2, while the degree of $Q_p(a_j)$ over $Q_p$ is 3. Contradiction. So if $|a_j| > p^{-1/(p-1)}$, then $p = 3$ and none of the functions $L_p(s,\chi^j)$ has a zero.

The only remaining possibility is that $D_x(T)$ is the product of an irreducible quadratic and a linear. The zero of the linear is necessarily in $Z_p$, and hence will give a zero of $L_p(s,\chi)$. The zeros of the irreducible quadratic may be treated as in B, with $\lambda = 2$. In particular, if $p \geq 5$ they must give zeros of $L_p(s,\chi)$, and if $p = 3$ then we need only check $|\alpha|$, (and $F$, if $|\alpha| = p^{-1/2}$), in order to determine if any $L_p(s,\chi^j)$ has a zero corresponding to $a_j$.

Among the examples which have been computed for $E$ a real quadratic extension of $Q$, are those of Wagstaff, [WG]. We have already discussed one such. We turn to two of the others.

$p = 3$, $f_x = 281$, $\lambda = 3$.

$e_0 \equiv 3 \pmod{9}$, so that $\psi_p(e_0) = 1$, and $D_x(T)$ is Eisenstein, thus irreducible. From C, we find that this is only possible if each of the zeros has $p$-order 1/3. Since $1/3 < 1/2 = 1/(p-1)$, we know that no of zero of $f_x$ gives a zero of $L_p(s,\chi^j)$ for any $j$. Hence none of the $L_p(s,\chi^j)$ has a zero.

$p = 3$, $f_x = 733$, $\lambda = 3$.

According to Wagstaff, the zeros of $D_x$ are

$\alpha_1 \equiv 780 \pmod{3^7}$

$\alpha_2 \equiv 162 + 340\sqrt{-3} \pmod{3^7}$

$\alpha_2 \equiv 162 - 340\sqrt{-3} \pmod{3^7}$
and the corresponding values for $\log(1+\alpha)/\log(1+q)$ are

\[ s_1 \equiv 2020 \pmod{3^7} \]
\[ s_2 \equiv 1600 + 311\sqrt{-3} \pmod{3^7} \]
\[ s_2 \equiv 1600 - 311\sqrt{-3} \pmod{3^7} \]

All of the $s_i$ are in the domain of $L_p(s, \chi)$, but only $|\alpha_1| < p^{-1/(p-1)}$; $|\alpha_2| = |\alpha_3| = p^{-1/2} = p^{-1/(p-1)}$. Since $F = \mathbb{Q}_3(\zeta_3)$, we know by the above that $s_2$ and $s_3$ will be each be a zero of one of $L_p(s, \chi_\psi)$ for $\psi$ of order 3. To find which, let $\zeta = (-1+\sqrt{-3})/2 \equiv -1094(1 + \sqrt{-3}) \pmod{3^7}$, so that $\bar{\zeta}^2 = (-1-\sqrt{-3})/2 \equiv -1094(1 - \sqrt{-3}) \pmod{3^7}$. Then $\alpha_2 + 1 - \bar{\zeta}^2 \equiv 930 + 1434\sqrt{-3} \pmod{3^7}$, and $\alpha_3 + 1 - \zeta \equiv 930 - 1434\sqrt{-3} \pmod{3^7}$. So $s_2$ is a zero of $L_p(s, \chi)$, $s_2$ is a zero of $L_p(s, \chi_\psi_{c^2})$ and $s_3$ is a zero of $L_p(s, \chi_\psi_{c^3})$. By corollary 5, each of these three $L$-functions has exactly one zero.

Finally, the results of some calculations for imaginary quadratic fields $E = \mathbb{Q}(\sqrt{-n})$, with characters $\chi$ are given. The coefficients of $f_{\chi \omega}(T)$ were computed by the method discussed in sections 1 and 2, with $m = k = 6$. This is sufficient to give the coefficient $e_{\omega}$ of $T^2$ modulo $p$, $e_{\omega}$ modulo $p^2$, etc. The values of $\lambda$ are taken from the results of Gold, [G1], [G2]. For additional examples, (with $p = 3$), see Kobayashi, [KY].
(The coefficients of $L_p(s, Xu)$ are accurate modulo $p^4$, while the coefficients $e_j$ of $f_{Xu}(T)$ are accurate modulo $p^{4-j}$.)

$$f_{Xu}(T)$$

<table>
<thead>
<tr>
<th>$h$</th>
<th>$f_X$</th>
<th>$\lambda$</th>
<th>$L_p(s, Xu)$</th>
<th>$f_{Xu}(T)$</th>
</tr>
</thead>
<tbody>
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<td>$55T + 19T^2 + 4T^3$</td>
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BIBLIOGRAPHY


