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SUPERMASSIVE STAR FORMATION -- THE EARLY PHASES

The Ohio State University

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SUPERMASSIVE STAR FORMATION -- THE EARLY PHASES

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

By
Michael Preston Merilan, B.S., M.S.

The Ohio State University
1985

Reading Committee:
Dr. Eugene R. Capriotti
Dr. Geoffrey Keller
Dr. Gerald H. Newsom

Approved By

Dr. Eugene R. Capriotti
Advisor
Department of Astronomy
DEDICATION

To my parents in recognition of their constant support and encouragement throughout my education.
ACKNOWLEDGMENTS

It is a pleasure to be able to thank my adviser, Dr. Eugene R. Capriotti, for originating the idea to undertake this investigation and for his invaluable aid and encouragement in its completion. Certainly a key factor in this work has been his keen physical insight. It is also a pleasure to thank Dr. George W. Collins and Dr. Gerald H. Newsom for helpful comments and encouragement along the way. I particularly wish to thank the Department of Astronomy and especially Dr. Collins for their assistance in securing the computing funds necessary for the completion of this numerical study. I would also like to thank the members of the Reading Committee for their time and efforts. Finally, I must express my thanks to the Department of Astronomy, especially Dr. William M. Protheroe in charge of astronomical teaching at the branch campuses, for the constant financial support during my studies at The Ohio State University.
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CHAPTER 1 - INTRODUCTION

In the past, supermassive stars (hydrogen nuclear burning objects with masses between \(10^3\) and \(10^9\) solar masses inclusive) have proven to be an intriguing and potentially quite significant topic in modern astronomy and astrophysics. However, at the same time, the controversy over their existence and the possible mechanisms apparently likely to act to suppress their formation have generally caused most authors to doubt their physical reality. Nevertheless, if supermassive stars were able to form at some epoch during the evolution of the Universe, they may hold the answers to some of the most perplexing current questions regarding fundamental elemental abundances. Furthermore, they have been applied as the potential "engines" at the centers of active galactic nuclei and quasars. Moreover, it has been known for some time now that basic qualitative calculations of cloud fragmentation scenarios due to hydrogen molecule formation in the early Universe seem to lead to an inescapable era in which supermassive stars should have been born. It is thus rather surprising that a review of the literature reveals that no researcher has ever calculated a realistic model of the formation of a supermassive object from plausible initial conditions. Nor have they ever attempted to semirigorously examine the physical constraints governing their potential formation through numerical solution of the
appropriate hydrodynamics equations of Newtonian physics and the post-
Newtonian equations arising from general relativity. It is to begin to
address this fundamental omission in the astrophysical literature that
this Ph.D. dissertation is directed.

The history of supermassive stars dates back to the early 1960s
when Hoyle and Fowler (1963 a,b) and Fowler (1964) pointed to them as
the potential powerhouses for the recently discovered quasars. Many of
the fundamental properties of supermassive stars were computed and it
was observed that the resultant luminosity was proportional to the
object's mass. In particular, for objects in the supermassive regime

\[ L = 2 \times 10^{38} \left( \frac{M}{M_\odot} \right) \text{ ergs/sec} \]  

(1-1)

where \( L \) is the luminosity, \( M \) is the mass, and \( M_\odot \) is the mass of the Sun,
approximately \( 1.989 \times 10^{33} \) grams (Allen 1976). To meet the energy
requirements of a quasar, typically around \( 10^{47} \) ergs/sec (Allen 1976),
therefore required an object in the neighborhood of \( 5 \times 10^8 M_\odot \). This
is well above the conventional stellar mass range of around 0.1 to 60 \( M_\odot \)
but still substantially less than the mass of a normal galaxy. The
supermassive stars subsequently studied were simply hydrogen burning,
high temperature configurations, principally supported by radiation
pressure, and with masses between \( 10^3 \) and \( 10^9 \) solar masses. An added
plus to the supermassive star as the model quasar engine lay in its
size. At that time light travel time considerations arising from
quasar variability implied a diameter well under a light year and the
computed supermassive star (SMS) diameters were well within this limit.
Furthermore, straightforward estimates of the SMS main sequence
lifetimes obtained by simply dividing the energy available through nuclear burning by the object's luminosity indicated a value independent of the mass of the star and equal to approximately $3 \times 10^{13}$ seconds or around $10^6$ years. This figure was noted to be consistent with the number statistics arising from quasar counts.

However, it is one thing to compute the equilibrium properties of a proposed object and quite another to confirm its probable existence; the natural question of stability arose. Classically, indefinitely high mass stars are dynamically stable against adiabatic perturbations from the equilibrium structure. The fundamental approach due to Eddington (1918, 1919, 1926) is to apply the continuity equation of hydrodynamics in its linearized form along with Euler's equation, and the standard adiabatic relations of thermodynamics to obtain a wave equation for adiabatic radial pulsations. The frequency $\sigma_0$ associated with the fundamental mode ($i = 0$) admits solutions where

$$\left(3\Gamma_1 - 4\right) \frac{4\pi G \rho}{3} \leq \sigma_0^2 \leq - \left(3\Gamma_1 - 4\right) \frac{\Omega}{I} \tag{1-2}$$

with $\Omega$ being the gravitational potential energy, $I$ being the moment of inertia about the origin, and all other symbols having their standard meanings (Ledoux and Pekeris 1941 and Ledoux 1945). In the case of stability one is obviously concerned with the higher frequency limit. Applying the form (Lang 1980) of $\Gamma_1$, and noting that as the stellar mass increases the ratio of gas to total pressure, $\beta$, decreases one gets

$$\sigma_0^2 \leq -K\beta \frac{\Omega}{I} \tag{1-3}$$

where only the gravitational potential energy is negative. Since the
result is positive, arbitrarily high mass structures are dynamically stable in the classical formalism. However, the result does not incorporate the effects of rotation or magnetic fields. In the case of supermassive objects, the natural extension was to consider the effects arising from a general relativistic analysis upon the pulsational stability. Chandrasekhar (1964a,b) and Feynman (1964) established the necessary general relativistic pulsational theory to address this very point. The total energy of the star may be written as the difference between the volume integral of the mass-energy density and the rest mass density. Namely

\[
E = c^2 \left( f \rho dv - f \frac{\rho dv}{\sqrt{1 - 2GM(r)/rc^2}} \right) \tag{1-4}
\]

where \( \rho_o \) is the rest mass density and \( \rho \) is the total mass-energy density including the contribution of the internal energy density due to gas and the radiation field. We also note that the denominator in the second term on the right hand side of equation (1-4) arises from integrating over the proper volume instead of the coordinate volume of the configuration.

Manipulation of the above expression allows one to ultimately express the total energy (the negative of the binding energy) as the sum of two terms. One is readily identifiable as the internal energy and the other as the gravitational energy. If rotation is introduced and the integrands are expanded in powers of \( GM(r)/rc^2 \) and a polytropic structure (\( n = 3 \)) is adopted for the SMS thereby allowing the integrations to be carried out then the ultimate result for \( E \) is

\[
E = Mc^2 \left[ -\frac{3}{8} \beta \frac{2GM}{Rc^2} + 1.265 \left( \frac{2GM}{Rc^2} \right)^2 - \frac{1}{2} \frac{R^2 \omega^2 k^2}{c^2} \right] \tag{1-5}
\]
where \( k \) is the radius of gyration. Examining each term in order we note that the first term corresponds to the Newtonian contribution and is negative thereby promoting stability. The second term arises from the post-Newtonian level and is positive thereby inducing instability. Indeed, when it exceeds the Newtonian contribution, as it obviously will for large masses since it is proportional to \( M^2 \) whereas the Newtonian term is only proportional to \( M \), then instability will dominate. Finally, the third term is due to the effects of rotation. Since it is negative it is apparent that in this case rotation acts to temporarily stave off the general relativistic instability for at least some masses. It was Feynman (1964) who first demonstrated that a general relativistic instability was present in spite of the fact that the structural properties of supermassive stars are essentially Newtonian. Fowler (1966a,b) includes a more detailed discussion and derivation of the situation. Thus, at some point the positive relativistic terms overcome the classical term(s) and configurations with masses above this point are dynamically unstable. Appenzeller (1973) and Reynolds (1979) found this boundary situation to occur at \( 4.1 \times 10^5 \) solar masses. Related work was also done by Zel'dovich and Novikov (1971). It is thus apparent that a nonrotating, nonmagnetic supermassive star of sizeable mass is subject to fragmentation and/or collapse.

However, Ozernoy (1966) observed that if rotation and magnetic field effects are allowed then significant alterations arise with regard to stability and observable properties. He referred to such a supermassive body, supported by rotation, magnetic fields, and conventional radiation pressure, as a magnetoid. In particular, two limiting
situations result (Ozernoy 1972). The first is the hot, high-entropy case in which the structure is roughly spherical. The second is the cold, low-entropy case in which the structure becomes a very thin disk. Both limits have been studied actively. The high temperature limit without magnetic fields but with rotation corresponds to the work of Fowler (1964, 1966). Incorporation of rotation and toroidal magnetic fields was done by Ozernoy (1966) and Ozernoy and Usov (1971). Supermassive stars with rotation and poloidal magnetic fields were considered by Ozernoy and Usov (1971). Finally, the low temperature limit without magnetic fields corresponding to a cold disk was considered by Bardeen and Wagoner (1969, 1971) and Salpeter and Wagoner (1971). It is in the case of a rapidly rotating, compact, supermassive disk that general relativity has its only influence on the configuration's structure. Otherwise, Newtonian effects dominate the structure while general relativistic effects determine, along with the rotation and magnetic field, the question of stability. Ozernoy (1971, 1972) viewed this range of magnetoids as a potential evolutionary sequence. Formation was felt to occur in an approximately spherical fashion with evolution to a disk stage during the final phases of the object's life. Ozernoy (1972) points to the Grewing and Lamla (1968) result of a quasi-Planckian quasar optical component arising in the spectra after subtraction of the synchrotron power law component as evidence for the presence of a hot magnetoid's thermal radiation in some quasars. Of course, the thermal spectrum arising from a classical, nonrotating, nonmagnetic supermassive star with a surface temperature $7 \times 10^{44} K$ is in poor agreement with the power law nature widely noted for quasars.
However, better agreement between the spectra may arise when it is noted that a hot magnetoid is capable of producing magnetodipole radiation as well as the primary thermal blackbody radiation. This is due to the rotation and the poloidal magnetic field as observed by Morrison (1969), Cavaliere et al. (1969,1971), Fowler (1971), Woltjer (1972), and Ozernoy and Usov (1971,1973). Gunn and Ostriker (1969) note that particle acceleration to relativistic energies via the MD radiation may take place. Ozernoy (1972) notes that the dual constraints imposed by the related quasar observations of the MD radiation being much stronger than the thermal radiation and the source lifetime being greater than $10^5$ years seem to require that the rotation be uniform and the magnetic field be quasi-dipole with magnetic and gravitational energy approximately equal. However, some hedging is required, particularly in the case of active galactic nuclei. Indeed, many problems remained in the case of quasars as well as active galactic nuclei (Kourganoff 1978 and Burbidge 1974). As a result, the supermassive star model for the central power source failed to gain any strong, widespread acceptance with ideas generally along the lines of gravitational energy sources like supermassive black holes remaining more viable. However, even if this is the correct model then we are left with the fundamental question of the origin of these SMBHs. Certainly an attractive qualitative precursor would be a conventional SMS. If this is the case then an investigation of the constraints on SMS formation would still be of major significance.

Moreover, it is interesting to note that although supermassive star models are not currently particularly popular as AGN and quasar
"engines" they have been used for other purposes. For example, for some
time it was recognized that the observed helium abundances required
either synthesis in a single, universal event ("Big Bang") or were due
to numerous explosions of supermassive stars during the initial stages
of galaxy evolution. Another early application arose with Bird (1964,
1968) who examined the explosive fragmentation of a massive protostellar
cloud. For masses around $10^4$ to $10^6$ solar masses, with the central
regions cooled via shielding due to the surrounding opacity, application
of the Virial Theorem resulted in a central condensation of $10^2$ to $10^3$
solar masses with subsequent contraction to stellar density. The
ensuing pulsational instability led to explosion with the fragments and
the cloud being regarded as the protostars associated with the observed
expanding associations of O and B stars. A particularly compelling SMS
situation arises in the case of Reddish (1978) in which a contracting
protogalactic atomic hydrogen cloud, contaminated with a small amount
of grains, is considered. Molecular formation onto grain surfaces takes
place in a nonuniform fashion which in turn leads to development of
pressure instabilities. These cause cloud fragmentation and star
formation with the formation rate controlled by the $H_2$ formation rate.
Although a considerable amount of idealization takes place along with
analytic simplifications, it appears inevitable that in certain cases,
supermassive objects form while the galactic contraction phase occurs.
Indeed, one might expect this intuitively on the basis of simple Jeans
mass considerations. In the case of Reddish's work, he finds the most
massive objects formed to be given by $2.2 \times 10^{-15} (n_g/n_H)^{-5/3}$ solar
masses where $n_g$ is the grain number density and $n_H$ is the hydrogen
number density. For $n_H/n_\text{H} = 10^{-12}$ and $M_{\text{cloud}} = 10^8 M_\odot$, the resulting upper limit is $2.2 \times 10^5 M_\odot$. Moreover, different formation site scenarios are possible. As the molecular formation rate is faster than the free fall contraction time, molecular formation, cloud fragmentation, supermassive star formation take place before significant alteration of the galaxy's structure. A uniform density would presumably give rise to supermassive stars throughout the galaxy while density enhancement toward the center would cause them to be born there first. After formation, the resulting photon field prevents further molecule formation until the supermassive objects have fully evolved and ceased to emit light, something on the order of $10^7$ years. A series of such supermassive formation outbursts were envisioned with the resulting explosions returning much of the mass into the interstellar medium and giving rise to significant helium, heavy element, and grain enrichment of the ISM. Ingenious negative feedback loops largely limit the observable effects of the resulting enrichment by the time a galactic age of $10^{10}$ years is achieved (comparable to the earlier Milky Way) but significant effects would be notable at earlier epochs. Furthermore, such a generation of supermassive objects would have direct consequences upon the fundamental rate at which the parent galaxy collapses since the gas would be heated by the resulting radiation. An entire sequence of molecular hydrogen formation on grains, cloud fragmentation, formation of supermassive objects, supernovae, formation of Population II dwarfs, and formation of Population I dwarfs would result. For typical parameter values it would be during the first $10^7$ years of galactic evolution that the generation of supermassive stars formed. The minimum
and maximum mass stars capable of formation changes after this time through the supernovae phase eventually achieving a range of about 0.2 to 100 solar masses for the current Milky Way.

Finally, there are compelling elemental abundance reasons to think that the existence of supermassive stars, at least at some time in the past, may not simply be an exercise of the imagination. Silk and Siluk (1972) have noted that the observed elemental abundances in the oldest generation stars appear to require elemental synthesis in supermassive stars at some prior epoch. However, the effect of supernova explosions of stars with masses of ten solar masses or greater seem to explain most of the relative heavy element abundances (Arnett 1971). Apparently however, the problem has yet to be fully resolved and if anything, current evidence appears to be turning more and more in favor of the high mass stars. El Eid, Fricke, and Ober (1981) have recently conducted some evolutionary and nucleosynthesis calculations for 80 to 500 solar mass, metal-lacking stars in an attempt to determine the range of explosive end phase and the resulting interstellar enrichment. Carr, Bond, and Arnett (1981) and Bond, Arnett, and Carr (1984) have considered what they refer to as very massive objects (VMOs) which go pair unstable during oxygen burning. As a result, the possible initial mass range is \(10^2\) to \(10^4\) solar masses. VMOs of around 100 solar masses or greater are found to form black holes while the lower mass objects apparently explode. These VMOs are shown to have potentially highly significant ramifications for elemental abundances as well as the classical "missing mass" problem of astronomy.
From a theoretical point of view, the interest in supermassive stars has never totally submerged. Indeed, Fry and Fuller (1984) studied the potential evolution of monopole-burning supermassive stars in which radiation pressure effects are countered by the monopole annihilation and the configuration is additionally stabilized by rotation. The motivation of course was to determine whether such an early epoch of specialized SMSs could obliterate all the postulated monopoles in a monopole-dominated universe and avoid the necessity of resorting to inflationary cosmologies as have become popular in line with grand unification efforts. These workers found such burning to be completed within a finite timescale, contrary to previous investigations, but did not attempt to seriously address the question of how such postulated objects might originally form.

From the observational standpoint, perhaps the only manner in which the theoretical supermassive star can ever achieve a totally "respected" position in astrophysics would be if "the monster could be found in its lair." While the high mass range which is the subject of this work has yet to be observed, probably for good reasons as we shall ascertain later, it now appears that one of the low mass "cousins" may have finally been located. IUE observations of the central star in the Tarantula Nebula (30 Doradus) in the Large Magellanic Cloud (LMC) imply the presence of a supermassive star. Initial mass estimates ranged between 2500 and 3500 solar masses. More recent studies (Savage et al. 1983 and Mathis, Savage, and Cassinelli 1984) have brought this value down to a maximum of about 1000 solar masses. Schmidt-Kaler and Feitzinger (1981) and Savage et al. (1983) have summarized the salient optical and
ultraviolet properties of this object (R136a). With $V = 10.77$, $B-V = -0.03$, $E_{B-V} = 0.46$, and $A_V = 2.16$, the total luminosity would be around $9 \times 10^7$ solar luminosities and the surface temperature would correspond to $6 \times 10^6\text{K}$. Early speckle photometry gave a diameter of below $0.08''$ which translated to below $0.02\text{pc}$ for a linear diameter at the LMC's distance. A tight cluster of hot, young stars is not absolutely ruled out. Schmidt-Kaler and Feitzinger concluded however that a mass of $3200$ solar masses, a radius of $90$ solar radii, a $v\cdot\sin(i) = 2200\text{ km/sec}$, and an optically thick stellar wind of $3600\text{ km/sec}$ which gives rise to a mass loss of $3 \times 10^{-3}$ solar masses per year provided the best fit to the fundamental observations. An age of around $10^6$ years would be implied. The values of Savage et al. (1983) tend to be somewhat more moderate but still within the lower end of the supermassive regime. Alternative models, particularly that of R136a as a supergiant accretion disk are suspect because of the low x-ray flux observed. Fanagia, Tanzi, and Tarenghi (1981,1983) have given preliminary infrared photometry and further speckle observations have been conducted by Meaburn et al. (1982) with the 3.9-m Anglo-Australian telescope. These speckle studies yielded an upper limit of $0.02''$ at which R136a is possibly barely resolved. This would correspond to a linear diameter of $5 \times 10^{-3}\text{pc}$. Of course the apparent isolation of such an object led quite naturally to a reexamination of the relevant vibrational stability properties. Ledoux, Noels, and Boury (1982) have performed linear, radial calculations of the pulsations of a $3 \times 10^3$ solar mass star. The expected characteristic time for amplification of the oscillation is only around $145$ years with escape velocity of the surface layers achieved
after $2.2 \times 10^3$ years. The result is therefore disturbingly short compared to the characteristic evolutionary timescale for such an object. Other potential abnormally massive stellar candidates do apparently exist however. As noted by Mathis, Savage, and Cassinelli (1984) the central object of the HII region NGC 3603 in the Milky Way may be a scaled down version of R136a. Moreover, additional giant HII regions in nearby galaxies may harbor similar objects. A particularly intriguing case arises with the BL Lacerta object B2 1308+326 as reported by Shipman (1984). Variability studies spanning at least a seven year period imply semiregular outbursts. Apparently equally viable models in a rather qualitative sense for the central energy source are either a $10 \times 10^9$ solar mass supermassive black hole or equally massive supermassive star.

In light of the extensive use of supermassive objects in the astrophysical literature and with the discovery of at least one rather good candidate, albeit at the lower mass end of the regime, it would be expected that the protostellar phase of a supermassive star's life would have been computed in a realistic fashion some time ago. Indeed, Cassinelli (1984) (private communication) reveals that such work is currently being done at the University of Wisconsin for the lower mass regime in a time independent formalism in hopes of application to R136a with emphasis on the role of dust in hampering the formation process much in the style of Kahn (1974) who treated conventional high mass cocoon stars. However, a survey of the astrophysical literature reveals that no investigator has yet performed a semirigorous, time dependent calculation of a supermassive protostar in the spirit of the conventional
protostellar calculations for approximately solar mass stars first done over a decade ago by Larson (1969) and for models up to ten solar masses as treated by Larson (1974).

Indeed, conventional hydrodynamic calculations have been performed for a 60 solar mass protostellar cloud by Appenzeller and Tscharnuter (1974), which gave rise to a 17 solar mass star, and for 50 and 150 solar mass clouds including the dynamical effects of dust by Yorke and Krugel (1977), but corresponding higher mass computations in the SMS range are lacking. Demaret and Perdang (1977) have considered the question of vibrational stability of pre-main sequence supermassive protostars under the approximation that general relativistic effects are negligible as are nuclear energy sources or sinks, that the radiation component dominates over the gas component, and that the pulsations are homologous. They find a vibrational instability arising during the Hayashi contraction phase but the analysis appears as almost an aside within Demaret's series of papers on stars in thermal imbalance, leaving fundamental questions of timescales unaddressed. An approximate treatment of the formation of high mass supermassive stars was performed by Appenzeller (1972a,b). Two different mass objects were considered: $5.2 \times 10^5$ and $7.5 \times 10^5$ solar masses. In the first case the dynamical collapse was halted by the energy generation in the core with outflow ensuing to disrupt the protostar. In the second case, the collapse took place but inverse $\beta$-decay limited the energy generated so that the collapse was not halted. The corresponding conclusion was that such supermassive objects could not form. However, the approach used in generating the dynamical models was to initially compute a supermassive structure in hydrostatic
equilibrium with the given mass, and a radius increased artificially by arbitrarily increasing the energy generation rate by a factor of ten. Collapse was simulated by decreasing the associated scaling factor at a rate faster than the dynamical time characteristic of the object. The question of how closely this corresponded to a physically realistic approximation to the true hydrodynamic situation was left completely open. An alternative view of the formation process, with similarities in the methods employed, was taken by Reynolds (1979) in his Ph.D. dissertation. In this work, supermassive objects were considered as forming via mass accretion as might be most appropriate in the center of a galaxy. This was done in an attempt to avoid the fundamental problems to the existence of a supermassive object posed by the Appenzeller work earlier. While the results were encouraging, they did not yield objects with masses sufficient to serve as the central engines of AGNs and quasars because of the constraint of the total stellar energy being negative for stability. Inclusion of the salient relativistic contributions to the energy (Reynolds 1979) yields positive terms with the changeover to positive energies occurring at around $10^6$ solar masses. Moreover, rotation does not appear to be enough to stave off this effect for long. Fahlman and Anand (1971) considered rapidly rotating supermassive stars in the post-Newtonian approximation to general relativity and observed that uniform rotation permits stable masses up to only about $3 \times 10^6$ solar masses. Thus the quasar question seems to remain outside the SMS domain but the fundamental problem of whether a supermassive star can form remains unaddressed in any rigorous sense. Finally, the work of von Hoerner and Saslaw
(1976) must also be addressed. They considered $10^4$ to $10^8$ solar mass protostellar clouds that were taken to be nonrelativistic, bounded spheres. The ratio of thermalized kinetic energy to overall kinetic energy during the approximately free fall collapse was assumed to be constant and the effects of shocks were ignored. The net result was that structures of $3 \times 10^6$ solar masses and below halt their collapse due to the dominance of nuclear energy generation while more massive configurations than about $4 \times 10^7$ solar masses collapse indefinitely and achieve the Schwarzschild radius before nuclear ignition can halt the collapse, thus apparently resulting in black holes. While that investigation considered the proper range of masses it was not a truly hydrodynamic computation since the methods employed were purely analytic with a number of simplifying assumptions as well. Essentially it was a one point integration over the entire cloud. Moreover, as the detailed lower mass calculations had previously shown, shocks contribute extremely significant dynamical as well as energy effects to collapsing protostellar clouds.

It seems clear that the only manner by which to begin to conclusively answer the question of whether a supermassive star can form is to apply more realistic initial conditions in solving the equations of classical hydrodynamics and the post-Newtonian equations arising from general relativity (when necessary). It is precisely this approach that this dissertation will adopt. Chapter 2 presents many of the physical considerations that may be expected to be of prime importance to the dynamics of collapsing supermassive clouds and addresses the question of which effects we shall incorporate in our
numerical treatment. Chapter 3 presents the appropriate Newtonian formulation of the problem along with the numerical hydrodynamics and thermodynamics required to achieve a solution in the protostellar context at hand. Chapter 4 gives the necessary post-Newtonian theory followed by its related numerical treatment. Finally, Chapter 5 presents the detailed numerical models and results along with analytic extensions to the calculations, an associated discussion, and suggestions for future investigations — most of which are currently under active consideration.
Prior to the numerical development and solution of the equations of classical and post-Newtonian hydrodynamics, it is advisable to consider some of the primary physical effects that are likely to prove significant in promoting (or retarding!) the development of a supermassive protostar. The intent of this chapter is to do precisely this in the light of the previous model calculations for conventional low mass protostars. In particular, it has become widely accepted in modern astronomy and astrophysics that the origin of the solar system is best addressed in the context of the Nebular Hypothesis in which the solar system is viewed as having formed from a large, rotating cloud of interstellar gas contaminated with dust grains. Although problems remain with this scenario, it appears likely that the general process was along these lines and it has become equally popular to consider star formation as taking place in an analogous fashion. In other words, if we may temporarily neglect magnetic effects, we define a protostellar cloud as being a region of interstellar gas (and possibly dust depending upon the region and the epoch under consideration) that has somehow exceeded the Jeans mass criterion and thus become gravitationally unstable to collapse. Here we are not particularly concerned with the details of the process leading up to this point. Rather we merely note
that current potential mechanisms for producing such density enhancements in the interstellar medium may include spiral density waves and supernovae shock fronts. In any event, if one considers a constant density, constant pressure, static gas and applies a small perturbation to the density and to the velocity then recourse to the continuity and momentum equations of classical hydrodynamics plus Poisson's equation relating the gravitational potential to the density distribution leads to a wave equation for the perturbations. Solution of this equation in the context of plane waves leads ultimately to the Jeans length $\lambda_J$ with

$$\lambda_J = \left[ \frac{\pi kT}{\rho \mu H} \right]^{1/2}$$

(2-1)

and $\rho$ and $T$ give the unperturbed density and temperature, $\mu$ is the mean molecular weight, $H$ is the mass of the hydrogen atom, and the other symbols have their standard meanings (Lang 1980). If, as is more interesting for the protostellar situation, the analysis is repeated in spherical rather than Cartesian geometry then one obtains an analogous Jeans length but with a slightly different numerical coefficient,

$$\lambda_J = 0.89 \left[ \frac{kT}{\rho \mu H} \right]^{1/2}$$

(2-2)

The critical importance of this length is that lengths longer than this limit are gravitationally unstable. We may therefore translate the length obtained above into a Jeans mass defining a mass limit above which the region becomes gravitationally unstable. All our protostellar models will therefore exceed the Jeans criterion to ensure an initially global collapse phase.
SECTION 2.1 TURBULENCE AND FRAGMENTATION

One of the major problems in modern star formation theory is the lack of a detailed, semirigorous understanding of the role of fragmentation in the formation process. Indeed, from a qualitative standpoint, it would appear that formation of conventional mass stars should be extremely difficult in the case of no magnetic fields present due to simple Jeans mass considerations. In such a situation it would seem likely that substantial density enhancements corresponding to at least $10^4 M_\odot$ should arise given plausible initial conditions for the cloud. Obviously to form conventional-mass stars with masses at least $10^2$ times smaller than this global value would therefore require some sort of fragmentation process to occur. Early reviews of this fragmentation problem have been presented by Layzer (1964) and Hunter (1967). More recently, Low and Lynden-Bell (1976) and Silk (1977) have addressed the question of opacity-limited fragmentation appropriate to the protostellar problem, Tohline (1980) has considered the question of fragmentation of primordial protostellar clouds, and Silk (1982) has examined the likelihood of fragmentation taking place during the early dynamical collapse phase of a protostellar cloud. A random, probabilistic theory of fragmentation has also been developed by Larson (1973). Tohline has concluded that the inclusion of slight, but nonnegligible, pressure effects act to inhibit fragmentation to stellar-sized regions at least during the early essentially free fall dynamical timescale of the configuration. Silk (1982) however examined the pressure-free problem both for a cloud initially at rest and even for a uniformly expanding cloud and found that fragmentation is rather more likely. He noted that
incorporation of a uniform pressure should not greatly alter his results
in the case of a cloud significantly beyond the Jeans criterion. In
this work, however, the role of detailed opacity effects, cloud cooling
mechanisms, density gradients, and magnetic fields was not directly
examined. Thus, at the early collapse phases we have significant
disagreements as to the likelihood of fragmentation. Numerical studies
of the later phases however seem to indicate that fragmentation becomes
extremely likely in the presence of pressure effects and/or rotational
effects. Indeed, it has been noted that the efficiency of fragmentation
during the early phases increases if the collapse is nonspherical as
should be the case if even a modest rotation is present in the initial
cloud. The basic physical idea therefore leading to the formation of
solar-sized fragments is that during the initial collapse phase the
conventional hydrodynamic models have indicated essentially an isothermal
collapse. However the density obviously increases. Now, since the
Jeans length is proportional to \((T/p)^{1/2}\) it is thus apparent that during
this early dynamical phase the Jeans length steadily decreases. This
in turn should enable successively smaller zones to become gravitation­
ally unstable and to fragment out of the original cloud. Indeed, if
such fragmentation occurs, perhaps successively within individual
fragments as well, before significant compressional heating effects allow
the temperature and opacity to rise then conventional sized protostellar
masses might result. Numerical examples of this process are given by
in the conventional mass protostar context is also presented by Reddish
(1978) in which random fragmentation (by mass and volume), hierarchichal
fragmentation as just outlined, shock-induced fragmentation, and molecule formation-induced fragmentation are all considered.

Obviously, if one is in the business of trying to produce a supermassive protostar, fragmentation is the last thing that would be desirable to take place. However, as we shall ascertain later in this chapter, quite straightforward entropy considerations constrain the formation of a supermassive protostar from realistic initial conditions to take place in a dissipative fashion. Namely, shock fronts, turbulence effects, fragmentation and collisions would all qualitatively act in this fashion. Indeed, it has long been recognized that shock fronts are a natural result in the lower mass protostellar calculations and as we shall see they also arise in the supermassive protostellar models as computed here. It also has been recognized in the supermassive star literature that one potential way to form such an object would be via stellar collisions and coalescence. However, in this work we are primarily concerned with a nebular origin for SMSs in a spherical collapse context. Thus the detailed role of stellar collisions and turbulence plus turbulence-induced fragmentation with subsequent coalescence is beyond this dissertation. One can, however, crudely examine the potential role of turbulence-induced fragmentation and recoalescence without recourse to detailed hydrodynamic treatments. Indeed, Nakano (1966) considered such coalescence as did Taff and Savedoff (1973), who incorporated total and partial coalescence both with and without equipartition of the kinetic energy. Their work is in the context of conventional mass objects but of obvious use here since both treatments found rather rapid evolution to an equilibrium
situation. In this case the number of configurations within unit range about mass \( m \) is given by

\[
N(m) = C m^{-\kappa}
\]  

(2-3)

where \( C \) is a constant and \( \kappa \) is a constant depending rather critically upon the details of the model's fundamental assumptions. If total coalescence is assumed then \( \kappa \) falls between 1.3 and 1.8 depending on the presence or lack of kinetic energy equipartition and the details of the fragments' velocities and cross sections. If partial coalescence is adopted then the critical factor is whether the excess mass is in the form of one or two pieces. In the former case the appropriate value is about 1.6 while in the latter case \( \kappa \) tends to infinity! In other words, fragmentation dominates and the mass ultimately becomes totally subdivided into countless small regions. It is thus apparent that the true role of coalescence in physically realistic contracting protostellar clouds is poorly understood given the range of potential values for \( \kappa \).

It is apparent however that the role of fragmentation in the supermassive star context should act as a critical limiting factor given the form of equation (2-3). Indeed, it would appear highly unlikely that such a size fragment would ever result.

Given the fact that one of the dissipative processes available to a cloud is turbulence and that fragmentation appears to be a major obstacle to the formation of a supermassive star it thus seems crucial to at least attempt to assess the potential role of turbulence-induced fragmentation in supermassive clouds. One could envision turbulence
leading to fragments which could grow to supermassive size only via further accretion of mass or through coalescence with other fragments. Thus, it is appropriate to consider the likelihood and severity of the turbulent fragmentation problem. Consider a turbulence moving with velocity $v$ of length $L$. If $\rho$ is the density and $\mu$ is the coefficient of dynamic viscosity then the Reynolds number is given by

$$Re = \frac{Lvo}{\mu}. \tag{2-4}$$

If $Re > 1$ the motion of the turbulence will not be damped via viscous gas effects. Substituting the form of $\mu$ (Lang 1980, p. 226) appropriate for a non-ionized gas it may be shown that

$$Re \approx 2 \times 10^4 \frac{pLvT^{-1/2}}{\mu} > 1 \tag{2-5}$$

where the inequality corresponds to a nondamped situation.

If we are concerned with gravitationally unstable fragments then the appropriate turbulence length is the Jeans length $\lambda_J$ (or larger).

Namely,

$$L = f\lambda_J = f \cdot 6 \times 10^7 \frac{L}{T^{1/2} \mu^{-1/2} \rho^{-1/2}} \tag{2-6}$$

where the parameter $f > 1$. In other words, the condition for undamped turbulence becomes upon substitution of (2-6) into (2-5):

$$\rho^{1/2}v_f > 8.3 \times 10^{-13} \tag{2-7}$$

after some trivial arithmetic and rearrangement. Now, as $f$ is strictly larger than one, it promotes the inequality and hence the instability.
We may thus cast the inequality into logarithm format as

\[ \frac{1}{2} \log(p) + \log(vf) > -12.1 \]  

(2-8)

for undamped turbulence. Figure 1 shows a plot of this relationship with the straight line corresponding to the case of marginal damping due to viscous gas effects as predicted by equation (2-8) through this simple Reynolds number argument. Note that the area above the line corresponds to an undamped situation while the area below the line implies a situation where the turbulence is damped. Thus, a Jeans-unstable turbule (size \( \lambda_j \)) would have to be moving with a turbulent velocity of about \( 2.5 \times 10^{-3} \text{ cm/sec} \) or slower to be damped appreciably by viscosity if \( \rho \approx 10^{-19} \text{ g/cm}^3 \). For higher densities it is apparent that the constraint becomes even more severe. In the case of \( \rho \approx 10^{-12} \text{ g/cm}^3 \) the value corresponds to a turbulent velocity under \( 8 \times 10^{-7} \text{ cm/sec} \). Thus, in accordance with the more intricate calculations cited earlier, our idealized treatment here seems to indicate that as a protostellar collapse proceeds eventually fragmentation seemingly becomes inevitable.

At higher temperatures the gas eventually will become fully ionized and the coefficient of dynamic viscosity changes, thus making the Reynolds number for a fully ionized gas

\[ \text{Re} = 5 \times 10^{14} \frac{\rho Z^4 L v \ln(\Lambda)}{T^{5/2} A_{1/2}^{1/2}} \]  

(2-9)

in the notation of Lang (1980), where \( \Lambda = 1.3 \times 10^{4} T^{3/2}/N_e^{1/2} \) is the ratio of the gas Debye radius to the impact parameter. For a cloud
Figure 1
Undamped and Viscous Damped Zones for Nonionized Protostellar Cloud In Terms of Log(vf) Versus Log(Density)
predominately consisting of hydrogen one finds

\[
\text{Re} = 5 \times 10^{14} \frac{\rho_l v}{T^{5/2}} \ln \left[ 1.3 \times 10^4 \frac{T^{3/2}}{N_e^{1/2}} \right] \tag{2-10}
\]

where \(N_e\) is the electron number density and other quantities are as before. But \(N_e = N_i\) and \(\rho = N_e (m_e + m_i) = N_e (m_e + m_i)\). Thus we have approximately \(N_e = 6 \times 10^2\rho\). So the corresponding Reynolds number is

\[
\text{Re} = 5 \times 10^{14} \frac{\rho_l v}{T^{5/2}} \ln \left[ 1.7 \times 10^{-8} \left( \frac{T^3}{\rho} \right)^{1/2} \right] > 1 \tag{2-11}
\]

for a non-damped fully ionized case. Again applying the Jeans criterion one obtains the transcendental relationship

\[
4.2 \times 10^{-22} \frac{\rho^{1/2} v_f}{T^2} \ln \left[ 1.7 \times 10^{-8} \left( \frac{T^3}{\rho} \right)^{1/2} \right] > 1 . \tag{2-12}
\]

If we adopt a temperature, say \(10^{4}\)K, then we may tabulate \(\log(v_f)\) versus \(\log(\rho)\) in direct analogy to Figure 1 except now approximately including the effects of ionization. The resulting values are given in Table 1 below.

**Table 1**

<table>
<thead>
<tr>
<th>(\log(\rho))</th>
<th>(\log(v_f))</th>
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<tbody>
<tr>
<td>-16</td>
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<td>-15</td>
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<td>-9</td>
<td>-10.9</td>
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<td>-8</td>
<td>-11.3</td>
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</tbody>
</table>
Even without precisely considering the equation of state it is readily apparent that ionization worsens the situation as far as damping out turbulent eddys based on a Reynolds number treatment. Once again, turbulent fragmentation appears inevitable.

If we accept the development of fragments, a natural followup question significant to the supermassive protostar arises. Namely, what is the likelihood of fragments escaping versus coalescing through collisions? Conceivably coalescence would accelerate (at least to some degree) the formation timescale for a supermassive star. Such a question may be approximately addressed by considering the relative timescales required for a fragment to reach the surface $t_e$ versus the characteristic dynamical time of the entire cloud. Obviously if the dynamical time is substantially smaller than the escape time then collisions are more probable.

We consider the escape time first. Such a problem has been considered by many authors. For example, von Weizsacker (1947) applied the concept in an attempt to dump the excess angular momentum inhibiting protostellar formation. Let $L$ denote the mixing length (the path length a turbule moves before the intervening gas retards and "reabsorbs" it), $R$ be the cloud radius, $v$ be the turbule velocity. Then, the number of steps to escape is given by

$$ N = \left( \frac{R}{L} \right)^2 \quad (2-13) $$

and the time elapsed per step is simply $L/v$. Thus the escape time is

$$ t_e = \frac{L}{v} \left( \frac{R}{L} \right)^2 \quad (2-14) $$

Conversely, the dynamical (free fall) time is roughly

\[ t_{ff} = \frac{2101}{\rho^{1/2}} \text{ sec} \]  

so upon taking the ratio \( t_e / t_{ff} \) one obtains

\[ \frac{t_e}{t_{ff}} = \frac{R^2}{2101v} \frac{R}{L} < 1 \]  

where the inequality corresponds to escape with multiple collisions unlikely. Casting this expression into logarithmic form we therefore find for escape from the cloud the condition

\[ \log(v) > \frac{1}{6} \log(\rho) + \frac{1}{3} \log(M/M_\odot) + \log(R/L) + 7.57 . \]  

For any physically realistic choice of \((\rho, M/M_\odot, \text{ and } R/L)\) it appears that the inequality will not be satisfied and fragments may indeed collide and coalesce. Equivalently put, the escape time is somewhat longer than the cloud global dynamical time. However, at this point recall the more rigorous investigations cited earlier in the sense that the mass spectrum does not remain a function of time for a prolonged period of time, i.e. to build up successively more massive protostellar fragments through mutual collisions becomes increasingly more unlikely as the mass desired becomes larger. If one incorporates the effect of the spectrum of fragments into the treatment above, then only marginal gains apparently arise with respect to the timescale ratio.

Thus, in conclusion, it would appear that the problem of fragmentation is of a highly severe nature for formation of a supermassive protostar. Its rigorous treatment however is outside the scope of a
spherically symmetric hydrodynamic computation and we will therefore not consider it again in this work. Rather, we tacitly assume that one or more mechanisms act to inhibit the fragmentation thereby limiting its role in the dynamical evolution in the supermassive clouds under study here. Certainly, observations tend to indicate that conventional-mass star formation does not take place in regions of considerable turbulence but rather in the more "quiescent" zones. Equivalently we assume that turbulence is not initially present in our clouds nor does it develop to a significant point at later times.

SECTION 2.2 DUST DYNAMICS AND RADIATION PRESSURE

The presence of a dust component as well as a gas component in the initial protostellar cloud can have tremendous effects on both the energy transport properties of the cloud as well as the dynamical properties of the cloud during the later phases of its evolution. In particular, a number of workers have considered the role of dust in the formation of conventional-mass stars. In an investigation of the basic physical factors tending to produce an upper limit to the masses of conventional size protostars, Larson and Starrfield (1971) noted that the formation timescales are such that a central "stellar" object will tend to form well before the outermost material has had a chance to complete its infall — a strongly nonhomologous collapse. As the luminosity of this central object grows along with its density, the role of radiation pressure obviously will grow as well. In particular, if the luminosity reaches a high enough value then it is apparent that if dust grains are mixed in with the infalling gas they will react to
the radiation pressure, arrest their infall, and set up a dust cocoon at a radius from the central object where the radiation pressure effect just balances the pressure due to the dynamically infalling material. Indeed, it is apparent that the dust flow could be reversed into an outflow if conditions are sufficiently extreme. The dynamical role of the dust thus affects the cloud gas dynamics through any frictional coupling between the gas and the dust. Unfortunately for the aspiring supermassive protostar, at the densities typical in the protostellar environment it would appear that the frictional drag produced by the dust on the gas is effective enough to readily transmit the drag into the gas field. Kahn (1974) treated the role of dust in an even more detailed fashion noting that the grains would absorb the starlight from the central object and reemit it in the infrared. The problem is thus worsened since the dust cocoon built up can also be rather opaque to the reradiated energy providing an additional drag effect upon the gas. Moreover, Kahn noted that as the central protostar grows, the luminosity tends to grow more rapidly than the mass thereby exacerbating the situation. A shell of gas and dust is built up at approximately the grain melting distance from the central protostar and ultimately the inflow is reversed into an outflow potentially breaking up into fragments. The corresponding upper limit to protostars computed by Kahn for current estimated cloud conditions turns out to be around 40 M. However, his treatment is analytical assuming that the flow is stationary with a constant mass flux throughout the cloud (i.e. a time independent formalism suffices), that the effects arising from gas pressure are negligible, that the central object obeys the mass luminosity relationship (MLR), and that
a one-component treatment of the dust and gas together is a close enough approximation. Yorke and Krugel (1977) relaxed most of the assumptions in their hydrodynamical calculations of 50 and 150 solar mass clouds. In particular, they performed a time dependent treatment thereby incorporating the effects of shocks and treated the dust and gas in a two component hydrodynamical formulation capable of following different flow velocities in the dust and gas fields. Qualitatively their results tend to agree with Kahn in the sense that the mass inflow is ultimately arrested. Unfortunately the manner in which they treat the central core object is rather idealized in the sense that it is implicitly in hydrostatic equilibrium and assumed to have reached the main sequence when it has accreted a total of fifteen solar masses. The detailed role of core oscillations on the protostellar formation timescale is thus completely ignored.

We can very roughly judge the approximate role of dust in inhibiting the formation of supermassive protostars by considering the ratio of radiation pressure to dynamical pressure in the style of Larson and Starrfield (1971). Thus

$$\frac{P_{\text{rad}}}{P_{\text{dynam}}} = \frac{L/4 \pi r^2 c}{\rho v^2}$$

(2-18)

where $L$ is the luminosity of the central object and all other symbols have their standard meanings. Tacitly we are including only the radiation pressure effects of the direct starlight and ignoring the role of reradiated light from the dust grains themselves. If one does adopt
the MLR for the central object, a power law wavelength dependence of the
grain opacity, the ideal gas law, and a central blackbody source then
the point at which the ratio exceeds unity is under thirty solar masses
and it achieves a value of 18+ for a sixty solar mass cloud. In other
words, for the current interstellar medium, dust acts as a very strict
constraining mechanism prohibiting the buildup of stars with much more
than the conventional mass range let alone a supermassive object!
Since the thrust of this dissertation is to begin to assess whether
there appears to be any chance for the formation of supermassive objects
it is apparent that we should consider only dust free protostellar
clouds. While certainly inappropriate for the solar environment, this
approximation is physically realistic for the earlier epochs such as
pre-Population II objects. Moreover, it is currently applicable in
those cases that the initial cloud temperature is sufficiently high to
melt all the grains prior to the dynamical collapse. Ice mantles would
melt above grain temperatures of around 150-200°K while the precise
grain melting temperature obviously depends fairly sensitively upon the
grain composition. Melting values of 1200 - 3500°K would seem most
likely however.

SECTION 2.3 RADIATIVE HEATING

A trivially obvious, but nevertheless significant, effect that will
tend to inhibit the development of a supermassive protostar particularly
through mass accretion is radiative heating. This process can be
subdivided into two rather different mechanisms. The first is the
inevitable pressure increase in the gas that will accompany any increase
in the gas temperature due to energy transported by radiative transfer from the central object. Clearly this can be treated only by a reasonably accurate approximation to the radiative transfer problem along with sufficient knowledge of the opacity properties of the medium and by the proper protostellar equation of state for the gas. We will fully incorporate this effect into our solutions of the hydrodynamical energy equation at both the Newtonian and post-Newtonian levels. The radiative transfer term in the energy equation will be treated in the diffusion approximation and the full tabular range of opacity values from low to high temperatures will be incorporated. The details of this procedure along with the relevant equation of state will appear in the next chapter.

The other radiative heating mechanism arises in the case of an ambient dust field in the original protostellar cloud. As noted by Larson and Starrfield (1971), if dust is present then it will tend to absorb the protostellar object's light, convert it into infrared radiation (particularly in the innermost zones of the cloud) and then reradiate it to the outer regions. This infrared component thus will also act to heat up the cloud. Indeed, depending upon the dust opacity and density it is possible to envision a protostellar core shielded from the outer regions of the cloud in which only this dust mechanism provides a significant heating and thus a significant pressure effect. However, we have already assumed in the previous section that our supermassive clouds are dust free. Thus, the effect of this "diluted" radiation field is negligible for our models and will not be incorporated into the solution of the energy equation.
As noted previously, one characteristic behavior of the protostellar collapse problem for the conventional range of masses is the strongly nonhomologous nature of the collapse. In particular, this has been observed by essentially every conventional hydrodynamic calculation regardless of the numerical details of the specific algorithm employed. Indeed, both nonrotating spherical clouds and rotating spheroidal clouds appear to undergo this process. It was Larson (1969) who first strongly emphasized this aspect of the low mass protostellar models. The basic situation is that the central regions actually collapse rapidly enough to form a central kernel, or as Larson refers to it, an embryo. At the same time the outer zones of the cloud are still in dynamical infall while the central temperature rises dramatically. Equivalently put, the core timescale is much shorter than the cloud's global timescale. Thus, even low mass stars apparently originate through mass accretion onto this kernel. It is therefore apparent that the critical factor determining the ultimate mass of the resultant stellar object is the timescale required to form this core and sufficiently evolve it to high enough temperatures and pressures so that further mass infall is arrested. This problem has been considered in the high, but still conventional, mass range up to ten solar masses by Larson (1972). He noted that for masses above about three solar masses, the evolution is directly to the main sequence well before the radiation can significantly be transported to the outer regions. Moreover, the object obeys the radiative pre-main sequence trajectory on the HR Diagram and does not undergo a Hayashi phase as do stars below 1.5 solar masses.
This is of obvious significance to the potential formation of a supermassive protostar. Once the central kernel's temperature becomes high enough to ionize the surrounding gas in the outer regions of the cloud, a drastic pressure and temperature jump will occur in these regions. Indeed, based on the earlier low mass computations, Larson and Starrfield (1971) estimate this jump to be above two orders of magnitude for the temperature with an even steeper jump for the pressure when this ionized hydrogen (HII region) becomes dominant. Thus, the infall of the outer layers is halted and reversed and only the inner layers of the cloud can continue to accrete onto the core due to their much greater density. Unfortunately, unless artificially contrived, this amount of material added to the core object is invariably only a small fraction of its ultimate total mass. Moreover, we note that the potential severity of this HII region formation problem is heightened by the assumptions we have previously been forced to adopt for our supermassive cloud. In particular we have assumed that no dust is present at any point in the cloud, to avoid its dynamical properties during the later phases. This in turn implies that the ionizing radiation from the central object will more readily be able to reach the very outermost layers rather than being converted into infrared to any degree by intervening dust.

The problem therefore boils down to one of timescales. Namely what is the pre-main sequence time required for contraction (core formation) compared to the timescale for accretion (the global hydrodynamical timescale)? For a conventional massive star the Larson models would imply a value of around $10^4$ to $10^5$ (at most!) years for contraction. The accretion timescale however is on the order of $10^6$ years. Moreover, as
more massive protostars are considered, the contraction timescale steadily shortens. It would therefore appear that even if all the various fragmentation mechanisms do not operate and even if the dynamical effects of dust are not present to halt the infall (e.g. pre-Population II chemical mix or an initially high temperature cloud), the HII region formation process and accompanying pressure enhancement will act to suppress further accretion onto the core after the initially strongly nonhomologous collapse likely for a supermassive cloud. Indeed, Larson and Starrfield (1971) estimate this effect by balancing radiative ionizations and recombinations in a conventional Stromgren sphere fashion to determine the size of the resulting HII region and the maximum size and mass attainable for the central protostar. They find a value of about twenty-five solar masses. However the situation becomes a little less stringent if the original cloud was at a higher temperature than the 20°K value assumed by them and/or the initial density of the cloud is significantly above the Jeans criterion (the cloud is compressed to an extra degree). In these cases it would be possible to achieve the observed upper conventional mass limit of around sixty solar masses. Moreover, if turbulent energy is present in the cloud to enhance the kinetic energy then the mass limit also increases. Of course in this case we would be faced again with the problem of fragmentation arising to limit the ultimate core mass anyway.

It would thus appear that the potential for formation of a supermassive star is bleak at best. However, if one is prepared to adopt very extreme initial conditions as might indeed be appropriate for the early universe, the potential for an SMS is not absolutely ruled out. Indeed,
based on the Jeans criterion we note that as the initial temperature is taken to be larger, the zone unstable to collapse increases as does the Jeans mass. Furthermore, for mixtures depleted in heavy elements one might anticipate such higher initial temperatures since the most effective cooling mechanisms operate through these heavier components. Happily, as the initial temperature is taken to be larger the qualitative trend is to increase the mass of the resultant protostar. It is thus apparent that if one is willing to also adjust the initial density of the cloud material then it should be possible to build a star with a mass into the SMS domain. Indeed, if the cloud is initially at, or above, the temperature required for hydrogen ionization (nominally $10^4 \text{K}$) then a sudden pressure jump as previously outlined will not occur.

We may estimate the mass of the eventual star in a very idealized fashion by use of the free fall time of the cloud. The free fall time is given by

$$t_{ff} = \frac{\sqrt{3\pi}}{32G\rho}$$

(2-19)

and physically corresponds to the time required for collapse of a sphere to infinite density if pressure effects are negligible (Hunter 1962 and Boss 1979). As previously noted, the lifetime of an equilibrium supermassive star is about $10^6$ years and is independent of the mass of the star. If we crudely equate the lifetime to the free fall time of the initial cloud and assume all the mass goes into forming the star, then an initial cloud density of about $4 \times 10^{-21} \text{ g/cm}^3$ or greater is implied. Conversely, we may cast this result in terms of the initial cloud mass and radius to find that
\[ \log(R) \leq 17.7 + \frac{1}{3} \log(M/M_\odot) \]  

(2-20)

to meet the free fall timescale criterion (again assuming a fixed $10^6$ years lifetime). This corresponds to the region under curve A in Figure 2 of the next page.

An additional constraint is provided by the Jeans mass criterion. Of course the free fall condition is based upon the assumption of a pressure free collapse and thus is internally inconsistent with the isothermal Jeans treatment. Hence one cannot analytically combine them in a self-consistent fashion. However, it is possible to compare how stringent the two criteria are in the context of a supermassive cloud. Recalling the (spherical) Jeans length of equation (2-2) and replacing the density in terms of the cloud mass and radius one finds the critical radius for collapse to be

\[ \log(R) = 17.7 + \log(M/M_\odot) - \log(T/\mu) \]  

(2-21)

where $\mu$ is the mean molecular weight. Figure 2 exhibits for $\mu = 2.4$ the critical radii versus mass for temperatures of $10^6 K$, $100^6 K$, and $1000^6 K$ (curves B, C, and D respectively). As expected, the radius constraint strengthens as temperature increases so as to permit the gravitational energy to dominate the thermal energy. But, we also note that the more stringent constraint in the supermassive domain is provided by the free fall timescale which would require the initial clouds to be substantially compressed compared to the fundamental Jeans analysis. Indeed, in the case of the conventional mass protostellar models if one computes the ratio of thermal to gravitational energy of
Figure 2

Free Fall Formation Timescale Constraints and Jeans Mass Criterion Constraints Arising for the Idealized Formation of a Supermassive Star
the initial hypothesized clouds, the values typically are on the order of 0.5 thus indicating that the beginning departure from gravitational/thermal balance is fairly close to the Jeans constraint. From our rough analysis here we note that this will not be the case for the supermassive models which must all be gravitationally dominated so as to avoid the timescale dilemma.

However, we must also note that the free fall timescale is an overly stringent one. Although the dynamical collapse phase and subsequent evolution of stars onto the main sequence has been noted to proceed faster for more massive objects, at least within the conventional mass range, it is apparent that ultimately during the collapse pressure will provide a nonnegligible role in the subsequent dynamics. Indeed, based on the nonhomologous nature of the calculated collapses for traditional mass clouds, one might qualitatively envision formation of a central stellar kernel that possesses a substantially longer main sequence lifetime than the SMS value adopted above. Growth would then occur via accretion onto this kernel with a concurrent shortening of the nuclear burning lifetime as more mass was added. Furthermore, it would appear likely that upon exceeding the range of $100 M_\odot$ pulsational instabilities would greatly complicate the structure, tending to act to disrupt the configuration. The relative efficiencies of infall, expansive disruption, and presumably also a rather stiff stellar wind tending to impede further infall (if any at this point) might greatly extend the available time. However it must also be recognized that if a strong stellar wind is able to form, it probably would overwhelm the infall. Thus, it would appear that although marginal gains in the permitted formation time may
arise via the nonhomologous scenario and inclusion of pressure effects, they are probably not enough to give rise to a supermassive star.

Another mechanism, nevertheless, does appear to have this potential. If the range of low mass models are examined in detail, particularly those of Larson, it becomes apparent that the hydrodynamical algorithms chosen tend to limit the effect of core bounces and oscillations. Indeed, a feature of practically every protostellar hydrodynamic model is an initial bounce/oscillatory phase that ultimately dies out. This was noted by Larson (1968) in his original Ph.D. thesis whose numerical methods provided the basis for his subsequent studies. The central kernel formed after the early dynamic collapse phase and invariably underwent oscillations. More recent workers such as Boss (1979, 1980) have also observed an impressively long oscillatory phase in the low mass models. Moreover, if one examines Larson's thesis the explicit comment is made, "... the motion is quite complicated, since many modes are excited ... since these pulsations are not particularly of interest anyway for the present purposes, we have favored numerical methods which tend to suppress such motions ..." Certainly this is reasonable in the low mass situation since the accretion phase required onto the kernel is not all that extended. However, in the case of a supermassive star, and potentially even for extremely high mass conventional stars, substantial accretion would be required in the face of very tight, additional evolutionary timescales. If such an oscillatory phase were to arise in the core region of a supermassive cloud and if it extended long enough to stave off nuclear ignition until much more
mass was added to this pulsating kernel, then the case for formation of a supermassive star would be considerably enhanced. Unfortunately, the core oscillatory frequencies and damping behavior depend rather critically upon the mass and it is not possible to calculate this behavior in detail, using the conventional hydrodynamic methods. To do so would require extremely fine-zoned models, thereby tremendously increasing the computational effort. However, one can apply analytic approximations to the pulsations and damping behavior to estimate their roles in the SMS context. In particular, when we consider the actual supermassive models in the last chapter, it will be noted that core oscillations substantially more powerful than those in the conventional mass cases can arise due to the gravitational dominance and resulting overshoot beyond hydrostatic equilibrium. Thus, we have a potentially vital mechanism that may alleviate, at least in part, the otherwise stiff timescale constraint governing supermassive star formation.

SECTION 2.5 CORE ENERGY CONSTRAINTS

For the moment, let us turn our attention to the structure of the probable core object developed after the initial dynamic collapse phase. Neglecting the question of oscillations as raised in the previous section we are faced with a structure essentially in hydrostatic equilibrium if the low mass models are indicative at all of the likely behavior. Then we may apply the virial theorem

\[ E_{\text{grav}} + 2E_{\text{therm}} = 0 \]  \hspace{1cm} (2-22)

where \( E_{\text{grav}} \) is the core gravitational energy and \( E_{\text{therm}} \) is the internal
thermal energy of the ionized core gas. However, the total initial energy (where by initial here we mean the energy per particle after thermalization of the kinetic energy associated with infall) associated with the cloud $E_{\text{tot}}$ is given by

$$E_{\text{tot}} = E_{\text{grav}} + E_{\text{therm}} + E_{\text{diss}} + E_{\text{rad}}$$  \hspace{1cm} (2-23)

where $E_{\text{diss}}$ and $E_{\text{rad}}$ represent the energies of dissociation plus ionization and the total escaping energy due to radiative transfer respectively. Now, as has been previously noted, $E_{\text{grav}}$ will tend to dominate (particularly for a supermassive configuration) over $E_{\text{tot}}$ so upon zeroing the left side of (2-23), temporarily neglecting radiative transport, and combining (2-23) with (2-22) one obtains

$$E_{\text{grav}} = -2E_{\text{diss}}$$  \hspace{1cm} (2-24)

where we observe that $E_{\text{diss}} \approx 10$ eV. But when the kernel attains the main sequence then it will also obey the virial theorem embodied in equation (2-22). Moreover, at that point $E_{\text{therm}} \approx 1000$ eV. Thus we have a discrepancy of approximately one hundred times between equations (2-22) and (2-24) in the predicted gravitational energy associated with the ultimate main sequence kernel versus its initial quasi-hydrostatic state. Furthermore, it may be noted that $E_{\text{grav}} \approx 1/R$ so this implies a one hundred fold discrepancy in radius! Physically, the matter amounts to asking whether the initial core object will have a size comparable to the main sequence size of the same mass star or whether it will be about one hundred times larger. Obviously, as noted by Shu (1977) who studied this situation for conventional mass protostars, the resolution of the
question lies with an understanding of how effective radiative transfer is in transporting energy away from the structure. We observe that the low mass models indicate an initially isothermal phase with most of the energy going into a conventional free fall. Parenthetically, we observe that the supermassive models to be outlined in the last chapter will show a similar tendency. Thus, the only radiative transfer mechanism capable of resolving this dilemma is the bounding shock that has been computed to set up around conventional low mass protostars and certainly would be anticipated around an SMS kernel. Indeed, based on the values quoted previously, it would appear that only if $E_{\text{rad}}$ is some one hundred times larger than $E_{\text{diss}}$ will the object initially attain a size comparable to a main sequence size. The numerical details of answering this question obviously lie in the radiative transfer algorithm chosen and in the technique to be employed in treating protostellar shock fronts. Due to the finite difference nature of conventional hydrodynamic schemes it is apparent that the shock can only be treated in a rather approximate fashion and thus the resultant core size will be somewhat suspect. Here we will choose widely applied techniques to treat both the radiative transfer and shock front situations. Namely, we shall apply the diffusion approximation to the former and we shall use an artificial viscosity (shock smearing) approach for the latter. Both techniques have been extensively applied in low mass protostellar calculations so we would anticipate less of a probability of systematic differences with the low mass models due to subtle numeric effects. The appropriateness of these approximations will be considered in the next chapter as they are fully presented in a numerical fashion. It is obvious however that
this core size problem is intimately connected to the question of oscillatory damping timescales and their effectiveness in extending the formation timescale allowed for a supermassive protostar. From an energy standpoint, the question simply becomes how much of the energy is available for the kernel to use in its oscillations in contrast to the amount lost through radiative transfer in the shock front. Apparently, the less efficient is the radiative transfer, the better the likelihood of a higher mass star.

SECTION 2.6 ENTROPY CONSTRAINTS

As has been noted previously, many workers have considered the question of the equilibrium properties of a hydrogen burning supermassive star. One such property of course is the entropy associated with such a structure which will consist of two terms

$$ s_{\text{SMS}} = s_{\text{rad}} + s_{\text{gas}} $$

(2-25)

where the first is the contribution due to the radiation (also acting as a pressure support mechanism) and the second is the contribution to the gas (taken to be hydrogen). If one considers the gas as essentially a small perturbation to the structure then it can be shown (Shapiro and Teukolsky 1983), that the entropy is given by

$$ s_{\text{SMS}} = \frac{4aHnT^3}{3\rho} + k[\ln(T^3/\rho^2) - 21.0] $$

(2-26)

where $H$ is the mass of hydrogen, $a$ is the radiation density constant, and $k$ is the Boltzmann constant. All other notation is as before. Now, the entropy associated with the hydrogen plasma precursor to the supermassive
star is merely given by the second term in equation (2-26) with the appropriately different values of temperature and density. Thus it is apparent that the entropy associated with the final supermassive equilibrium state is substantially larger than that associated with the plasma cloud predecessor. This is simply as would be expected from the second law of thermodynamics. However, the magnitude of the difference becomes apparent when one writes the SMS entropy in terms of the total mass of the star as

$$s_{\text{SMS}} = 0.942k(M/M_\odot)^{1/2}$$

(2-27)

which corresponds to Shapiro and Teukolsky's equation (17.2.8). This is strictly appropriate for a fully ionized hydrogen star supported by radiation pressure with gas pressure being negligible i.e. ann = 3 polytrope. If we adopt $T = 10^4$K and $\rho = 10^{-8}$ g/cm$^3$ for the plasma cloud ultimately giving rise to a $10^6 M_\odot$ SMS then one finds (cgs)

$$s_{\text{SMS}} = 1.3 \times 10^{-13}$$

$$s_{\text{pla}} = 6.0 \times 10^{-15}$$

(2-28)

and it is apparent that to produce an equilibrium SMS, the formation scenario must be a dissipative one. Potential processes that would appear likely to account for this difference include turbulence, fragmentation with subsequent collisions between fragments, and obviously shock fronts. Each of these has been noted in the previous sections and to treat the supermassive star formation problem in a spherically symmetric, hydrodynamic fashion we have already ruled out turbulence.
and fragmentation processes. Thus we are left with the shock front mechanism for this present work. It is interesting to note that throughout the low mass models cited previously, the supersonic infall velocities have led to shocks. Certainly, one would expect, qualitatively at least, even more powerful shocks to arise for the SMS protostar. However, as we have just seen, a rather more fundamental reason underlying this behavior for conventional high mass stars as well as potential supermassive stars is to be found in the underlying thermodynamics.
CHAPTER 3 - THE NEWTONIAN NUMERICAL FORMULATION

Since the pioneering efforts of Larson (1968,1969,1972) in computing the evolution of protostellar clouds through application of the classical equations of hydrodynamics to the conventional mass range, it has become apparent that the best approach to the protostellar problem is largely a numeric one rather than a purely analytic one. Such models have clearly shown many of the earlier analytic approximations employed are not justifiable in the protostellar context. Thus, one cannot assume homologous collapse, or an incompressible fluid, or even attempt to linearize the fundamental equations of motion due to the invariable supersonic nature of the infall. Moreover, at least for the most general treatment, not even a purely radial flow can be adopted. Indeed, except during the early phases when the various viscosity mechanisms can largely be ignored and the collapse is essentially isothermal anyway, one is forced to consider a solution of the complete system of equations arising from classical hydrodynamics. The problem is thus a formidable one both from a human as well as from a computer's standpoint. Nevertheless, considerable progress has been made in computing such models to varying levels of sophistication as cited earlier. Particularly noteworthy extensions to the original treatments include Black and Bodenheimer (1975) and Boss (1979,1984) in terms of considering rotation in
the case of low mass protostars and Yorke and Krugel (1977) for their two fluid formalism treating dust and gas for conventional high mass protostars. Indeed, given the variety of specific numerical algorithms employed and the agreement of results, Stahler, Shu, and Taam (1980) have introduced a hybrid zoned approach to permit further time extension of the models through solution of sets of ordinary, rather than partial, differential equations. Their method depends upon zoning the low mass models in terms of the various physical zones naturally arising from the hydrodynamic calculations. While the approach is potentially quite powerful, it is obvious that in the case of a supermassive cloud whose detailed behavior has not yet been studied we may not employ such an algorithm here. Rather the fundamental equations, a set of coupled partial differential equations, must be solved and it is that topic that is the subject of this chapter.

In particular, examination of the variety of solution methods previously employed reveals that none of these treatments are completely satisfactory for the supermassive case at hand either due to the various approximations employed or due to the use of ad hoc initial conditions. As a result, a considerable amount of effort has gone into personally developing a FORTRAN computer code capable of treating a supermassive cloud at both the Newtonian and post-Newtonian general relativistic levels. Advantages of the code include excellent mass conservation properties, excellent momentum conservation capabilities, high computational efficiency (comparable to the best values quoted in the open protostellar literature), and good expansion capabilities. The latter
point has been emphasized particularly since a number of future extensions to these calculations including the effects of rotation and dust have been contemplated since the beginning of code development.

Although the derivation of the conventional equations describing a hydrodynamically evolving system has been widely presented in the literature, it would appear appropriate to consider some of the fundamental steps here. This is particularly true since the ultimate form of the equations to be applied in the supermassive context is somewhat different, except for the continuity equation, from the typical form in which they are generally quoted (Landau and Lifshitz 1982). Hence, in this chapter we will analytically develop the classical equations of hydrodynamics in a form particularly appropriate to the protostellar problem and then consider the numerical details necessary to achieve a solution to these equations.

The traditional starting point is to consider some physical configuration composed of N point particles with no internal degrees of freedom. In general, this system will evolve in some time-dependent fashion from its initial conditions as uniquely specified by the 3N canonical coordinates \( q_i \) \( (i = 1, 2, \ldots, 3N) \) and the 3N canonical momenta \( p_i \) \( (i = 1, 2, \ldots, 3N) \). The solution trajectory is followed through solving Hamilton's equations of classical physics where the Hamiltonian \( H \) of the system implicitly depends upon all the various \( q_i \) and \( p_i \). Thus any desired macroscopic parameter of interest describing the configuration would then be known. However, it is apparent, for any realistic situation consisting of a large number of particles, the approach would
be impractical. Moreover, the solution would demand knowledge of the initial conditions which are never truly known in any physically interesting situation let alone for such a complex system as a protostellar cloud. Thus such a direct approach must be abandoned.

Instead, one introduces a $3N + 3N = 6N$ dimensional phase space to study the system's time evolution. Such a "r-space" entails the coordinates $q^1, \ldots, q^{3N}$ and $p^1, \ldots, p^{3N}$ so it is evident that the system's state is completely specified by the particular location of the corresponding point in r-space and that some unique path therein depicts the system's time evolution. Upon envisioning a collection of systems in different states and otherwise identical to the true system and letting this ensemble's density in r-space be denoted by $D = D(q, p, t)$ then in the limit of a large number of such systems, the number of points in the infinitesimal volume element $dq dp$ at time $t$ centered on $(q, p)$ is $n = D(q, p, t) dq dp$. Moreover, if $\vec{v}$ and $V$ represent the $6N$ dimensional velocity and gradient operator in r-space and $V$ is an arbitrary volume bounded by surface $S$ in r-space with unit vector outward normal $\hat{n}$ then

$$\int_S \hat{n} \cdot \vec{v} D dS = - \frac{3}{\partial t} \int_V D dV \tag{3-1}$$

which is merely the conservation statement that the net flux out through $S$ corresponds to the time rate of decrease of points within $V$ assuming that the total number of systems is constant. If one applies Gauss' divergence theorem, exchanges the order of differentiation and integration, and recognizes that the volume $V$ is completely arbitrary so the integrand must vanish identically since the integral does then the result is a continuity equation in r-space:
\[
\frac{\partial \mathbf{D}}{\partial t} + \mathbf{v} \cdot (\nabla \mathbf{D}) = 0 \quad .
\] (3-2)

However, expanding the gradient operator and applying Hamilton's equations transforms equation (3-2) ultimately into

\[
\frac{\partial \mathbf{D}}{\partial t} + \sum_{i=1}^{3} \hat{q}_{i} \frac{\partial \mathbf{D}}{\partial q_{i}} + \sum_{i=1}^{3} \hat{p}_{i} \frac{\partial \mathbf{D}}{\partial p_{i}} = R 
\] (3-3)

where \( R \) is the creation or destruction rate (if the total number of systems is not constant). This inhomogeneous form of equation (3-2) is the Boltzmann equation which has been widely discussed. It forms the basis for obtaining the classical equations of hydrodynamics through applying an integral moment analysis.

SECTION 3.1 DERIVATION OF THE FUNDAMENTAL EQUATIONS OF CLASSICAL HYDRODYNAMICS

If we let \( f \) denote the number density in the Boltzmann equation and represent the net force acting on the particles as \( \mathbf{F} \) then we may more compactly express (3-3) as

\[
\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \frac{\mathbf{F}}{m} \cdot \nabla f = R 
\] (3-4)

In the special case with \( R = 0 \) we have the collisionless Boltzmann equation, or Vlasov equation of plasma physics. Now, apply the vector identity for \( \nabla \cdot (\mathbf{v} f) \), let \( \mathbf{u} \) denote the fluid velocity and \( n \) be the particle number density, set \( \mathbf{w} = \mathbf{v} - \mathbf{u} \), and carefully distinguish between the Stokes derivative \( d/dt = 3/t + \mathbf{u} \cdot \nabla \) and \( 3/t \). Then it is possible to obtain after considerable mathematical manipulations the expression...
Here, we have adopted \( R = 0 \) and defined
\[
\bar{a} = \frac{q}{m} \left[ \bar{E} + (1/c)\bar{w} \bar{x} \bar{B} + (1/c)\bar{u} \bar{x} \bar{B} \right] + \frac{F_{\text{grav}}}{m} - \frac{d\bar{u}}{dt} = \bar{w} \cdot \bar{v} \quad (3-6)
\]
where one in general has allowed for electric, magnetic, and gravitational effects all of which will be spatially as well as time dependent.

We are now in a position to apply the conventional moment analysis to derive the fundamental equations of classical hydrodynamics. The first moment is
\[
\int \frac{\partial f}{\partial t} d\bar{w} + \int \nabla \cdot [\bar{w} (\bar{w} + \bar{u})] d\bar{w} + \int \nabla \cdot (\bar{a} f) d\bar{w} = 0 \quad (3-7)
\]
But, interchanging operational order on the first term, this results in
\[
\int \frac{\partial f}{\partial t} d\bar{w} = \frac{\partial}{\partial t} \int f(\bar{r}, \bar{w}, t) d\bar{w} = \frac{\partial}{\partial t} \int f(\bar{r}, \bar{v}, t) d\bar{v} = \frac{\partial n(\bar{r}, t)}{\partial t} \quad (3-8)
\]
by the definition of the particle number density. Considering the second term one has
\[
\int \nabla \cdot [\bar{w} (\bar{w} + \bar{u})] d\bar{w} = \nabla \cdot [\int \bar{v} \bar{f} d\bar{v} - \bar{u} \bar{f} d\bar{v}] + \nabla \cdot \bar{u} \bar{f} d\bar{w}
\]
\[
= \nabla \cdot \left[ n(\bar{r}, t) \bar{u}(\bar{r}, t) - \bar{u}(\bar{r}, t) n(\bar{r}, t) \right]
\]
\[
+ \nabla \cdot \bar{u} \bar{f} d\bar{w}
\]
\[
= \bar{v} \cdot \bar{u} \bar{f} d\bar{w}
\]
\[
= 0 + \bar{v} \cdot \bar{u} \bar{f} d\bar{w}
\]
\[
= \bar{v} \cdot (\bar{u} \bar{n}) \quad (3-9)
\]
And the third term is
\[
\int \nabla \cdot (\bar{a} f) d\bar{w} = 0 \quad (3-10)
\]
by the divergence theorem since \( f \) vanishes (assumed rapidly enough) as \( w \to \infty \). Thus, collecting the results (3-8,9,10) we find that (3-7) becomes

\[
\frac{\partial n}{\partial t} + \nabla \cdot (n \mathbf{u}) = 0
\]  

(3-11)

requiring conservation of the total number of particles. Multiplying through by the particle mass or charge yields the related continuity expressions

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 
\]  

(3-11a)

\[
\frac{\partial q}{\partial t} + \nabla \cdot (q \mathbf{u}) = 0 
\]  

(3-11b)

Thus, the first moment yields the continuity equation (3-11) in its various forms. Moreover, we note that this expression is not solvable since the particle velocity \( \mathbf{u}(\mathbf{r},t) \), which is unknown, is required to obtain \( n(\mathbf{r},t) \) given \( n(\mathbf{r},0) \).

In an attempt to remedy this problem, we take the second moment while multiplying through by the particle mass as well. Thus

\[
\int \rho \frac{\partial f}{\partial t} d\mathbf{w} + \int \rho \nabla \cdot [(\tilde{\mathbf{w}}+\mathbf{u})f] d\mathbf{w} + \int \rho \nabla \cdot (\tilde{\mathbf{w}} f) d\mathbf{w} = 0 
\]  

(3-12)

Now the first term is simply zero in direct analogy to the analysis that led to a zero in the first term on the right hand side of equation (3-9). The second term becomes

\[
\int \rho \nabla \cdot [(\tilde{\mathbf{w}}+\mathbf{u})f] d\mathbf{w} = m \sum_{j=1}^{3} \frac{\partial}{\partial x_j} \int \rho \mathbf{w}_j f d\mathbf{w} 
\]  

(3-13)
again upon recognition of the zero nature of the first term. Finally, 
the third term is given by

$$\int m w_i \nabla \cdot (\tilde{a} \tilde{f}) d\tilde{w} = m \int w_i \nabla \cdot (\tilde{w} \tilde{a} \tilde{f}) d\tilde{w} - m \int (\tilde{a} \tilde{f}) \cdot \nabla w_i d\tilde{w}$$

(3-14)

where we immediately note that by the divergence theorem the first term 
on the right side is zero. Indeed, upon substituting for the forces 
term the proper combination of electric, magnetic, and gravitational 
forces, defining the pressure tensor (a symmetric tensor of rank two) 
so that

$$\int m w_i \nabla \cdot [(\tilde{w} + \tilde{u}) \tilde{f}] d\tilde{w} = \sum_{j=1}^{3} \frac{3}{\partial x_j} p_{ij}$$

(3-15)

where

$$p_{ij} = m w_i w_j f(\tilde{r}, \tilde{w}, t) d\tilde{w}$$

(3-16)

and performing a considerable amount of mathematical manipulations one 
ultimately obtains

$$\rho [(\partial \tilde{u} / \partial t) + (\tilde{u} \cdot \nabla) \tilde{u}] = \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial t} \frac{\vec{E} + \vec{b} \times \vec{B}}{m c} - \rho \nabla \phi_{grav} - \nabla \times \vec{\mathbf{F}}$$

(3-17)

where in principle each species has its corresponding equation (3-17). 
Note however that collisions between different species were not included 
in the previous analysis. Thus, if such effects are significant, then 
equation (3-17) would need to be modified. An approximation for the 
inclusion of collisions between species i and j = 1, ..., n would be to 
add

$$- \sum_{j=1}^{n} \rho_{ij} (\tilde{u}_i - \tilde{u}_j) v_{ij}$$

(3-18)
to the right side of (3-17). Here $v_{ij}$ denotes the mean collision frequency between species $i$ and $j$ for momentum transfer from all other particles types.

It is readily apparent that solution of (3-17), the momentum transfer equation, is nontrivial for arbitrary $\mu$, $\vec{v}$, and $\vec{F}_{\text{grav}}$. Also complicating the situation is $\vec{F}$. Great simplification arises in the case that electric and magnetic field effects are negligible ($\vec{E}=\vec{B}=0$) and the pressure is isotropic ($\nabla \cdot \vec{P} = \nabla P$). Then (3-17) in the absence of interspecies collisions becomes

$$p \left( \frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla) \vec{u} \right) = - \rho \nabla \phi - \nabla P \quad (3-19)$$

where the gravitational potential $\phi$ is given by Poisson's equation

$$\nabla^2 \phi = 4\pi G \rho \quad (3-20)$$

Thus, the second moment yields the momentum (transfer) equation (3-17) or (3-19).

However, just as the continuity equation alone was incomplete, so is the combination of the continuity plus momentum equations incomplete. Namely, $n(r,t)$ and $\vec{u}(r,t)$ may not be found from $n(r,0)$ and $\vec{u}(r,0)$ since the pressure tensor $\vec{P}$ term is present. Thus one must take the third moment of the Boltzmann equation, namely,

$$\int \omega_i \omega_j \frac{\partial f}{\partial \omega_j} d\omega + \int \omega_i \omega_j \nabla \cdot [\nabla (\vec{u} \cdot \vec{u})] d\omega + \int \omega_i \omega_j \nabla \cdot (\vec{a} \vec{f}) d\omega \quad (3-21)$$

Simplifying the terms in the third moment proceeds directly in analogy
with the previous two cases but does involve a substantial amount of manipulation. As these manipulations are not particularly relevant for our immediate purposes and have been treated widely throughout the hydrodynamics literature, we omit them here, merely noting that after multiple applications of vector identities including the divergence theorem and upon defining the heat flow tensor, a third rank completely symmetric tensor, as

\[ Q_{ijk} = m/\nu_i \nu_j \nu_k \int d\bar{\nu} \]

the third moment of the Boltzmann equation becomes

\[
\frac{3 \partial u_j}{\partial t} + \nabla \cdot (\bar{u} p_{ij}) + \sum_{k=1}^{3} \left( \frac{3}{\partial x_k} Q_{ijk} + p_{ik} \frac{3u_j}{\partial x_k} + p_{jk} \frac{3u_i}{\partial x_k} \right) \\
= \sum_{k, \ell=1}^{3} \frac{9}{mc} B_\ell \left( c_{\ell j} p_{ik} + c_{\ell k} p_{ij} \right) 
\]

which is transparently a heat flow equation. We note thus that the continuity, momentum, and heat equations are yet incomplete, since this new quantity \( Q_{ijk} \) has arisen in the third moment. In general, as progressively higher moments are taken, the system of equations does not close itself without some auxiliary constraint or assumption. Rather a hierarchy of moment equations is generated.

Thus, instead of pursuing the moment analysis further, progress is made upon examining the form of the heat flow equation more closely to recast it into an even more transparent form. Now, we let \( i = j \) in equation (3-23), apply one half the trace, and exploit the fact that
\( p_{ik} \) is symmetric in \( i,k \) and \( \varepsilon_{kli} \) is antisymmetric with respect to any pair of indices thereby creating a sum over the elements of an anti-symmetric array so as to zero the last term in (3-23). Interchange operations where necessary, recognizing that the random kinetic energy per unit volume is given by

\[
\frac{3kT}{2} = \int \frac{f(k_\omega^2) \omega^2}{f \omega} \, dw
\]  

(3-24)

and defining the heat flux vector \( \vec{H} \) as the flux of kinetic energy flowing per unit area per unit time relative to a system moving with the fluid velocity \( \vec{u} \),

\[
\vec{H} = \int (k_\omega^2) \vec{u} f(\vec{r}, \vec{w}, t) \, dw
\]  

(3-25)

equation (3-23) ultimately takes on the more readily understandable form of

\[
\frac{3}{\partial t} \left( n \left( \frac{3}{2} kT + \frac{1}{2} \mu \vec{u}^2 \right) \right) + \sum_i \frac{3}{\partial x_i} \left( n u_i \left( \frac{3}{2} kT + \frac{1}{2} \mu \vec{u}^2 \right) \right) 
\]

\[
= - \sum_{i,k=1}^{3} \frac{3}{\partial x_k} (p_{ik} u_i) + n \sum_i q E_i u_i 
\]

\[- \sum_i n u_i \nabla \Phi \_{\text{grav}} - \sum_{k=1}^{3} \frac{\partial H_k}{\partial x_k} .
\]  

(3-26)

In this form we see that the third moment is an energy transfer equation. In particular, the volume integral of the left hand side gives the rate of change of energy in the material while the right hand side's volume integral yields the rate of work done by pressure and viscous stresses, the electric and gravitational fields, plus heat added through the outer
boundary.

If we assume an isotropic distribution function then \( p_{ij} = p \delta_{ij} \). Moreover, suppose \( E = 0 \) and if \( L \) denotes the time rate of change of the energy per unit volume due to radiative transfer and \( E \) is the specific internal energy then (3-26) becomes

\[
\frac{3}{3t} (\rho E) + \nabla \cdot (\rho E \mathbf{u}) + \rho \mathbf{v} \cdot \mathbf{u} = L
\]

(3-27)

where \( \rho \) is the density as before.

SECTION 3.2 SPECIALIZATION TO NONROTATING CONFIGURATIONS

Collecting the continuity, momentum, and energy equations developed in the previous section we have

\[
\frac{3}{3t} \rho + \nabla \cdot (\rho \mathbf{u}) = 0
\]

(3-28a)

\[
\rho \left( \frac{3}{3t} \mathbf{u} + (\mathbf{u} \cdot \mathbf{v}) \mathbf{u} \right) = -\rho \nabla \Phi - \nabla P
\]

(3-28b)

\[
\frac{3}{3t} (\rho E) + \nabla \cdot (\rho E \mathbf{u}) + \rho \mathbf{v} \cdot \mathbf{u} = L
\]

(3-28c)

In the absence of rotation, the natural choice of coordinates is spherical polar since only radial and time derivatives survive. Upon recalling the fundamental definitions of \( \nabla \), \( \nabla \mathbf{A} \), and \((\mathbf{u} \cdot \mathbf{v}) \mathbf{u}\) and focusing our attention on (3-28b) we see that

\[
\frac{3}{3t} (\rho \mathbf{u}) = \rho \frac{3}{3t} \mathbf{u} + \mathbf{u} \frac{3}{3t} \rho = \rho \frac{3}{3t} \mathbf{u} + \mathbf{u} (\nabla \cdot (\rho \mathbf{u}))
\]

(3-29)

where the last term arises from (3-28a). Substituting this result into
the momentum equation, letting \( \overline{u} + \overline{v} \), and retaining the \( \overline{r} \) term in \((\overline{u} \cdot \overline{v})\overline{u}\) alone as is appropriate for a purely radial flow yields for the momentum transport equation

\[
\frac{\partial (\rho v r)}{\partial t} + \nabla \cdot (\rho v \overline{v} \overline{r}) = -\rho \frac{\partial \phi}{\partial r} - \frac{\partial P}{\partial r} \quad (3-30)
\]

Comparison with the form developed in the previous section reminds us that the \( \phi \) here is the gravitational potential. This assumed that all other effects such as electric and magnetic fields plus external potentials (including centrifugal) are negligible.

Thus, for a nonrotating, nonmagnetic supermassive protostar with purely radial motions the equations to be solved at the classical Newtonian level are (Eulerian form)

\[
\frac{\partial \rho}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \rho v \right) = 0 \quad (3-31a)
\]

\[
\frac{\partial (\rho v)}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \rho v^2 \right) = -(\rho \frac{\partial \phi}{\partial r} + \frac{\partial P}{\partial r}) \quad (3-31b)
\]

\[
\frac{\partial (\rho E)}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \rho Ev \right) = -\frac{P}{r^2} \frac{\partial (r^2 v)}{\partial r} + L \quad (3-31c)
\]

where we also have the auxiliary relations

\[
\frac{\partial^2 \phi}{\partial r^2} = 4\pi G \rho \quad (3-32a)
\]

\[
E = E(X,Y,Z,T,\rho) \quad (3-32b)
\]

\[
P = P(X,Y,Z,T,\rho) \quad (3-32c)
\]
with \( X, Y, Z \) being the fractions by mass of hydrogen, helium, and all other components respectively.

It might be asked retrospectively at this point why choose this Eulerian representation rather than the more conventional Lagrangian representation so widely applied in conventional stellar interiors calculations? The answer lies in two key parts. The first is that the Eulerian viewpoint is rather more directly envisioned; thus when we select a particular hydrodynamic solution scheme, its physical aspects will be rather more transparent. The second point is the more significant one; many of the conventional low mass protostellar calculations ranging from those of Larson (1968) to Boss (1979) have been most successful numerically in employing some variation of the fundamental Eulerian approach rather than the Lagrangian one. Indeed, when we turn to a selection of the hydrodynamic algorithm later in this chapter, it will become apparent that the method to be chosen directly complements this approach. Moreover, the fundamental hydrodynamic equations have been deliberately cast into conservation form which will permit much better numerical conservation properties of mass and momentum than often arise through competing formulations. Although mass and momentum (in the absence of nonconservative "driving terms" on the right hand side of the momentum transfer equation) should be strictly conserved, it is more often the case that the particular numerical algorithms employed tend to rather poorly conserve these quantities. By adopting the conservation form of the underlying equations it will be possible to avoid many of these unphysical numerical problems.
SECTION 3.3 THE DIFFUSION APPROXIMATION

It has been recognized for some time now (Larson 1969) and Hayashi 1966) that the initial phases of conventional low mass protostellar collapse occur in a nearly isothermal fashion. Naturally, compressional heating takes place; however, the heat is efficiently radiated away in the infrared region of the spectrum via the dust grains which are characterized by the same temperature as the protostellar gas due to collisional transfer of kinetic energy. Until densities of approximately $10^{-12}$ g/cm$^3$ are achieved, the material is optically thin with the protostellar core being the first region to hold the energy released via the gravitational compression. Thus, until this time, it would be sufficient, in principle, to merely solve the continuity and momentum equations. Once the nonisothermal regime is entered, however, the energy equation must be solved to follow the changing temperature. In particular, the energy equation's formal solution is greatly complicated by the presence of $L$, the time rate of change of energy per unit volume due to radiative transfer effects. Some simplification does arise however in the practical solution of the radiative transfer problem if the protostellar gas, dust grains, and photons may be treated as having the same temperature at a given location within the configuration. This is because the structure's free fall time, the characteristic time connected with the dynamical collapse to significantly change the structure, is much longer than the time required for the photon field to achieve equilibrium with the matter field (Boss 1979). However, in the case of conventional high mass protostars one cannot so glibly adopt
this approximation. Indeed, as we have already seen, due to the photon field's radiation pressure (among other effects) the dust will tend to act as a separate dynamical entity ultimately setting an upper limit to the protostellar mass achievable out of a dusty gas cloud. Thus, one should incorporate a two-component fluid formalism with the necessary frictional coupling (drag) treatment between the dust and the gas in the style of Yorke and Krugel (1977). However as we have already decided that our supermassive clouds will be dust free so as to promote the chances for obtaining a supermassive star, we need not worry about this complication and thus can adopt an equilibrium temperature at every point within the configuration that is assumed to characterize both the photon field and the matter field.

Assuming Local Thermodynamic Equilibrium (LTE) everywhere and at all times, we now look for a tractable manner in which to treat the radiative transfer problem. Previous workers have generally used the diffusion approximation in connection with conventional protostellar collapse calculations (Larson 1969, Black and Bodenheimer 1975, Tscharnuter 1975, and Boss 1979). Others have applied the Eddington approximation, the ratio of the third moment of the radiation field to the first moment is one third (Unno and Kondo 1976 and Kondo 1978). Both the diffusion approximation and the Eddington approximation are correct for the optically thick case in which an isotropic radiation field ensues. However, the Eddington approximation is somewhat more general as the diffusion approximation may be derived from the Eddington approximation as the limiting case of infinite optical depth.
Then the mean intensity equals the Planck function integrated over frequency. It is through solution of a second equation for the mean intensity that the Eddington approximation better treats a nonisotropic radiation field than does the diffusion approximation. We note that while the Eddington approximation is undoubtedly a superior technique, the diffusion approximation is a more conventional and straightforward approach. As extensive post-dissertation investigations are planned, it seems most appropriate to use the diffusion approximation so that low mass literature models may more directly serve as checks on the computer code evolved and so that the initial computational burden remains tolerable. At a later stage, the Eddington treatment may be incorporated as a natural extension. Indeed, when multidimensional calculations are performed for rotating supermassive protostellar clouds, the Eddington method should, and will, be applied.

At first glance however it seems that the diffusion approximation would be a rather poor choice for a protostellar calculation. The fundamental assumption in the diffusion approximation (Mihalas 1978) is that the photon mean free path, $1/\kappa_\rho$, is substantially shorter than the characteristic distance $d_T$, over which an appreciable temperature change occurs. In the case of a stellar interior, a photon undergoes many absorptions and emissions as it eventually achieves the surface so the diffusion approximation is appropriate due to the large optical depth. In the case of the central core of a typical low mass protostar, the material is again optically thick so that the diffusion approximation is legitimate. However, what about the situation in the outer regions
of a protostar which are optically thin? Larson (1969) has noted that the outer zone's temperature should increase as extensive collapse takes place due to the heat emitted from the central protostellar core. The diffusion approximation fails to do this for the low mass models calculated and effectively acts as the imposition of a constant temperature outer boundary condition on the solution. Thus, we immediately note that the results from the diffusion approximation are not good in the outermost regions of the structure. The reason why it is fairly useful for the central and intermediate regions stems from the T(\(r\)) profile produced in the solution. The diffusion approximation yields (Boss 1979)

\[
L = \nabla \cdot \frac{4}{3 \kappa_R \rho} \nabla (\sigma T^4) \tag{3-33}
\]

where \(L\) is the time rate of change of energy per unit volume due to radiative transfer effects, \(\kappa_R\) is the Rosseland mean opacity, and \(\sigma\) is the Stefan-Boltzmann constant \((5.66956 \times 10^{-5} \text{ erg cm}^{-2} \text{ deg}^{-4} \text{ s}^{-1},\) Allen 1976\). When \(\kappa_\rho\) is small then \(1/\kappa_\rho\) is large so that \(L\) in (3-33) is large. Indeed, it becomes so large that it becomes the main term, compared to the convective transport, in the energy equation (3-31c). Hence, if a temperature gradient were to arise physically, or through numerically propagated errors, then the radiative transfer factor tends to eliminate such a gradient. As a result, the temperature tends to remain isothermal in the optically thin regions which is precisely the expected situation before the highly advanced stages of the collapse set in. Thus, since the diffusion approximation yields valid T(\(r\)) in the
optically thin as well as optically thick zones through a prolonged
collapse range, it serves as a decent initial approximation to the
correct run of temperature with position within the protostellar cloud.
In other words, to some extent, the use of the diffusion approximation
in protostellar calculations hinges on an ends justifying the means type
of logic. We do note, however, that the presence of the Rosseland mean
opacity within the diffusion approximation (3-33) is logically consistent
with the latter's formulation. Mihalas (1978) notes that the approxima-
tions required in the diffusion approximation are identical to the
approximations demanded by the Rosseland mean opacity, namely, the
radiation moments tend to the limits \( K_v + \frac{1}{3} J_v \) and \( J_v + B_v \).

SECTION 3.4 THE OPACITY

As noted in the previous section, simultaneous use of the diffusion
approximation and the Rosseland mean opacity is a self-consistent
approach toward solving the energy equation. Of course, all information
is lost regarding deviations of the frequency dependence of the photon
field from a blackbody dependence. However, given the complex discontinu-
ous nature of the low temperature opacity sources plus the remaining
uncertainties about precise numerical values, it seems a reasonable first
approach to use the Rosseland mean. At a later point the computationally
more demanding frequency dependent opacities along with the Eddington
approximation to the radiative transfer problem may be incorporated.

A prime reason for the sizeable problem with low temperature,
protostellar opacities is the presence of dust grains in the material.
Not only is the composition of these grains uncertain, the question of
whether equilibrium or nonequilibrium conditions obtain has yet to be completely resolved. This has serious consequences for more than just the opacity, Section 3.5 for example. Here we assume LTE so that the same temperature holds for the matter and radiation components. In spite of the inherent difficulties, a number of workers have contributed low temperature opacities to the literature. For temperatures up to 3000°K, Gaustad (1963) treated grain and certain molecular opacities. Auman (1966) and Auman and Bodenheimer (1967) considered H₂O opacities between 1680 and 4200°K. Numerous molecular opacities were incorporated by Tsuji (1966,1971) for 1680 to 5000°K. Kellman and Gaustad (1969) considered H₂O, SiO₂, and graphite effects for 500°K or colder. Christy (1966) and Keeley (1970) developed approximation formulae to the classic Cox and Stewart (1965) stellar interiors opacity tables. Cox and Stewart (1970a,b) extended the opacities to 1500°K but neglected molecular effects. Iron grain opacities with T = 500 to 1500°K were treated by Cameron and Pine (1973) in their study of the solar nebula. Alexander (1975) computed low temperature (T = 700 to 10,000°K) Rosseland mean opacity tables for five chemical compositions and included several effects. Finally, DeCampli (1979) (see also Boss 1979 for a description) has evolved a sophisticated opacity routine which yields the Rosseland mean opacity, \( \kappa_R \), given the temperature, density, and composition. Included are Fe, FeO, FeS, H₂O, SiO₂, and graphite grains with certain grain formation, size, and equilibrium assumptions. Many of the previously cited opacity studies are incorporated in his algorithm.
Past protostellar investigations thus have had to contend with the opacity problem to varying degrees of sophistication but always with an implicit concern for the accuracy of the values employed. The same situation exists here. We will employ a dual mode approach. For temperatures below 700°K we will make the appropriate interpolations and extrapolations amongst the DeCampli values as previously cited. We note here that there is something of a logical inconsistency in doing so. Namely, we are incorporating the effects of dust into the computations at the opacity level but have tacitly assumed the cloud to be dust free as far as the dynamics of the system are concerned. Of course, in the case of Population II the heavy element contribution is so small that this is in all likelihood a good approximation. In the case of Population I it will become apparent in the last chapter that the probability of forming a supermassive star is rather small due to other physical effects so there again it would seem to be a fair approximation in terms of the eventual results. Indeed, we note that this level of sophistication in the opacity treatment has not generally been the case. A number of the very significant previous low mass protostellar models such as Larson (1969) and Black and Bodenheimer (1975) have used a constant value, \( \kappa_R = 0.15 \text{ cm}^2/\text{g} \), in this range. The result is reasonably consistent with the relevant low temperature opacity sources: \( \text{H}_2\text{O} \) ice, \( \text{SiO}_2 \), and graphite (Kellman and Gaustad 1969) and Fe grains (Cameron and Pine 1973). Thus to afford more direct comparison with the low mass models in the literature as test cases against the computer code ultimately developed here, some of the early testing was performed with the constant
approximation. However, for all the supermassive models quoted herein the DeCampli values have been applied.

Boss (1979) has noted the effects of using this constant value versus applying the DeCampli opacity routine in low mass protostellar models. He considered a Black and Bodenheimer (1975) test case of a two solar mass cloud with \( \rho = 1.43 \times 10^{-16} \text{ g/cm}^3 \) (initially uniform), \( T = 105.2^\circ\text{K} \), \( X = 0.68 \), \( Y = 0.29 \), and \( Z = 0.03 \). Initially, the true opacity is higher than 0.15 cm\(^2\)/g and the model starts to heat up before the constant treatment model begins to heat. At approximately 150°K the H\(_2\)O ice melts which causes the opacity to drop below the constant approximation and the temperature rise moderates with values below those found via the constant opacity. By 500°K the effects of graphite, silicate, and iron grains cause the true opacity to again exceed the constant approximation and the model eventually bounces when the central density and temperature have reached \( 8 \times 10^{-9} \text{ g/cm}^3 \) and 1100°K. The bounce in the case of the constant opacity model was found to take place at a central density eight times smaller. Thus the qualitative behavior of the models agree rather well although there is a lack of precise numerical agreement of when certain phases occur.

For values above 700°K up to 10,000°K we will apply the Alexander (1975) tables with a weighted interpolation for the intermediate case to link with the low temperature approach. The principal advantages here are accessibility, application of the internally consistent results of a single investigator, and the relatively wide range of temperatures and densities over which table interpolation can take place. One disadvantage is that we are formally limited to the five chemical mixes.
treated in his work although various potential interpolation approaches between mixtures might be applied. However, the Alexander tables include a number of opacity effects. Both bound-free and free-free absorption by H and H^- are included according to the Gingerich (1964) method. H_2 free-free absorption is treated (Somerville 1964 and Carbon, Gingerich, and Latham 1969) as is Mg I and Si I bound-free absorption (Gingerich 1964) and free-free absorption by He (Kurucz 1970). Also considered is Rayleigh scattering from H_2, H, and He (Dalgarno 1962, Dalgarno 1962, and Kurucz 1970 respectively). The final quasi continuous opacity source included is electron scattering (Kurucz 1970). Also incorporated are several molecular bound-bound opacity sources taken as straight-mean averages over 100 cm^{-1} intervals. These include H_2O in the red and infrared (Auman 1967 and Burch and Gryvnak 1966), the CN red system (Johnson, Marenin, and Price 1972), the fundamental and first three overtone CO bands (Beebe 1969), and the visual and infrared opacity arising from six TiO bands based on Collins (1974) and Collins and Fay (1975). Additional opacity sources were treated in a more approximate manner. These are Tsuji's (1966) treatment of the OH and H_2O rotation bands, and Main and Bauer (1967) and Fay's (1973) studies of band-averaged opacities due to C_2H, NH_2, CH_2, C_3, SiO_2, HCN, HCO, C_2, MgH, CH, OH, and NH. Atomic line blanketing is included via Mutschlecner and Keller (1970,1972) by semi-empirically altering their solar results. Finally, silicate grain absorption and scattering effects were treated via the conventional Mie theory for dust grains. Alexander (1975) notes several points of caution in his paper. Principally, these are the adoption of LTE even in the case
of very low pressure gases, the neglect of certain opacity contributors due to the lack of sufficient fundamental data, and the use of straight-mean opacities for bound-bound molecular contributions.

Finally, for the high temperature range above $\log(T) = 3.80$ we adopt the conventional tables typically applied for stellar interiors models; namely we use the Cox and Stewart opacity tables. The relevant articles here are Cox and Stewart (1970a,b) which represent upgrades of the earlier Cox and Stewart opacities. However, further upgrades have taken place since then. We choose these earlier values because they permit more direct numerical contact with a broader range of potential literature test cases and additional computational factors, particularly the specific internal energy expression and the equation of state, are expected to yield more significant numerical errors for the extremely high temperatures ($T \approx 10^5 \text{K}$ and above) than the opacities should. However, see also Cox and Stewart (1964) for further specifications on techniques and interpolation algorithms in terms of density, temperature, and chemical mixtures. To make contact with the Alexander (1975) values previously cited, the appropriate Cox and Stewart values were interpolated between adjacent mixtures to match the Alexander values.

Thus, throughout the range of temperatures and densities the tables are adjusted numerically for the fundamental five Alexander (1975) mixes. Although a variety of interpolation techniques were examined, the one ultimately selected was a linear interpolation in $\log(c_R)$, $\log(T)$, and $\log(\rho)$ for values within the tables. For the few instances of values needed just outside table ranges an analogous extrapolation was applied.
SECTION 3.5 THE SPECIFIC INTERNAL ENERGY

The time rate of change of the energy per unit volume due to radiative transfer effects, $L$, is not the only complicating factor required in the solution of the energy equation (3-31c). Also present is $E$, the total specific internal energy. Physically, the compressional energy arising from the gravitational potential energy released during the protostellar collapse is partly lost to internal energy. In general, $E$ depends in a nontrivial manner upon the thermodynamic properties: temperature, density, and chemical composition. Fortunately, however, these dependences have already been listed by Larson (1968) and Black and Bodenheimer (1975). These are the analytic expressions we will apply in the solution of the energy equation.

First, we note that the metals abundance will be small. Indeed, two extreme situations probably arise. At high temperatures the metals should become ionized while at low temperatures the protostellar dust grains may serve to tie up most of the metals. Since $Z$ is small however we will neglect this point and represent the translational energy contribution due to the metals by

$$E_M = 1.247 \times 10^8 \frac{Z}{A_M} T$$  \hspace{1cm} (3-34)$$

where $A_M (= 16.78$, Cameron 1969) is an average atomic number based on abundance studies. Helium will also provide a translation contribution, this time given by

$$E_{\text{He}} = 3.118 \times 10^7 Y T$$  \hspace{1cm} (3-35)$$
where Y is the helium mass fraction. Finally, hydrogen, hydrogen plus ions, and electrons yield the component

$$E_{H, H^+, e^-} = 1.247 \times 10^8 X y (1 + x) T$$

(3-36)

Here X is the mass fraction of hydrogen, y is the fraction of total hydrogen dissociated, and x is the fraction of hydrogen which has been dissociated and ionized. Note that we neglect ionization of all elements heavier than hydrogen here which is consistent with the treatment of the equation of state in the next section.

The only factors then left to consider are the specific internal energy contributions due to hydrogen ionization and dissociation and the contribution due to the degrees of freedom available to molecular hydrogen. The hydrogen dissociation term is

$$E_{H_2d} = 2.144 \times 10^{12} X y$$

(3-37)

while the ionization term is

$$E_{H_1} = 1.303 \times 10^{13} X y x$$

(3-38)

with the same notation as before. In particular, x and y may be determined from the pressure law components, Section 3.6. Namely, the fraction of total hydrogen dissociated is

$$y = \left[ p_H / (p_H + 2 p_{H_2}) \right]$$

(3-39a)

while the fraction of dissociated hydrogen which is ionized is expressed as
\[ x = \frac{p_{H^+}}{p_{H^+} + p_H} \]  \hspace{1cm} (3-39b)

The molecular hydrogen's contribution to E should include all the translational, rotational, and vibrational degrees of freedom. For low temperatures, T below 600°K, the translational and two rotational degrees of freedom prove to be the significant energies (Woolley et al. 1948 and Larson 1968). As the temperature rises above this boundary, the vibrational degrees of freedom must be considered as well. The precise contributions depend upon the ratio of ortho- to para-hydrogen which itself is determined by the protostellar temperature. Recalling now that the Schrodinger equation admits antisymmetric and symmetric solutions, we note that all integer spin particles have symmetric wavefunctions while all half-integer spin particles have antisymmetric wavefunctions. Indeed it is the antisymmetric wavefunction's vanishing when two such particles are in identical states that is the foundation of the Pauli Exclusion Principle. In the case of random formation of hydrogen molecules, three-fourths of them have parallel spins of the two protons and one-fourth have antiparallel spins; in other words, \( S = 1 \) (triplet) and \( S = 0 \) (singlet) respectively. Thus, three-fourths are ortho-hydrogen and one-fourth are para-hydrogen. The reason why the energy contribution depends upon the ratio is that the two states provide different partition function factors since they have odd and even rotational quantum numbers and it is the partition function which determines \( E_{H_2} \). The para- form dominates at very low temperatures. Also significant however are the more complex physical processes which determine the equilibrium ratio (DeCampli et al. 1978) such as proton
interchange reactions. These in turn are set by the flux of the ionizing radiation so the task of predicting the ortho/para ratio is not simple. Osterbrock (1962) found that in the conditions expected in the protostellar problem the para-form dominated even up to \( T = 200^\circ K \) due to the cool dust grains (nonequilibrium conditions) acting as sites for hydrogen molecular formation. At higher temperatures he found a ratio of \( 3/1 \) in favor of the ortho-form. DeCampli (1978) has noted that the molecular hydrogen contribution to the specific internal energy at low temperatures (equilibrium conditions) differs significantly from the result expected from Osterbrock's values. Here, we follow the Larson (1968) and Boss (1979) approach and simply apply Woolley et al. (1948) for empirically determined forms for \( E_{H_2} \) in the two temperature ranges. Other workers, Black and Bodenheimer (1975) for example, attempt to take the ratio factor into account. Their 1975 paper exhibits an explicit dependence on the ortho/para hydrogen ratio in their \( E_{H_2} \) expression. As the ratio itself is still somewhat uncertain, the basic empirical approach seems adequate. For low temperatures

\[
E_{H_2} = 1.031 \times 10^8 T \left[ X(1 - y) \right] \quad (T < 600^\circ K) \quad (3-40)
\]

where the second term, the one in brackets, corrects for the fact that the protostellar material is not pure hydrogen. For high temperatures the expression becomes

\[
E_{H_2} = \left[ 8.30 \times 10^3 T^2 + 9.32 \times 10^7 T + 2.99 \times 10^9 \right] \left[ X(1 - y) \right] \quad (T \geq 600^\circ K) \quad (3-41)
\]
Hence, the total specific internal energy in units of ergs/gram is given via the sum of these contributions or

\[ E(\text{ergs/g}) = E_{H_2} + E_{H_2^d} + E_{H_1} + E_{H^+,e^-} + E_{He} + E_M \]  \hspace{1cm} (3-42)

SECTION 3.6 THE EQUATION OF STATE

One of the auxiliary relations required for the solution of the hydrodynamic equations developed previously is the detailed dependence of pressure upon density, temperature, and chemical composition, in other words an equation of state. At the outset we note that earlier workers have approached this problem in varying degrees of sophistication. This is particularly the case for spherically symmetric studies. For instance Gerola and Glassgold (1976) incorporate salient molecular abundances. On the other hand, it is apparent that any numerical equation of state subroutine developed will be accessed a tremendous number of times during a collapse sequence. This problem will become even more critical when post-dissertation nonspherical studies are conducted necessitating a multidimensional treatment. Thus, is behooves us to include only as much sophistication as is demanded for a realistic treatment of the problem.

It has been known for some time now (Field et al. 1966) that the hydrogen in the cool, highly dense, interstellar clouds leading to conventional star formation is largely molecular. It seems likely that a similar situation arises in the case of supermassive star formation if possible under current conditions. Recall for instance the scenario due to Reddish (1978) in which pressure instabilities lead to protogalactic
cloud fragmentation and supermassive star formation at a rate controlled by the $H_2$ formation rate which was the original cause of the instabilities due to nonuniform $H_2$ formation onto grain hosts. As contraction proceeds however, dissociation and ionization will set in just as they apparently do in the low mass case. However, for typical protostellar conditions it is fortunate that the $H_2$ is completely dissociated before any appreciable ionization occurs (Larson 1968). Of course, this stems from the 4.98 eV dissociation energy contrasted to the 13.6 eV ionization energy. For example, Boss (1979) notes that at a density of $10^{-11}$ g/cm$^3$ complete dissociation has been achieved by 2750°K while appreciable ionization does not set in before 4500°K. The situation is even more favorable for the lower densities expected to be initially present. For $10^{-19}$ g/cm$^3$, the dissociation is essentially complete by 1500°K and it is at 3500°K that ionization effects start to become significant. Hence, it is apparent that the dissociation and the ionization of hydrogen may be treated separately.

But additional simplifications arise for the early phases of the collapse, $T < 3000°K$, since helium ionization and $H^-$ formation is insignificant (Allen 1976). Moreover, for low temperatures the metals should be tied up in dust grains and $Z$ is small anyway even for stellar Populations I and II. Thus, initially, ionization of metals is taken to be negligible.

Thus, the interstellar gas is assumed to be an ideal gas with mass fractions $X$, $Y$, and $Z$ of hydrogen, helium, and heavier elements (all) respectively. Furthermore, to make the situation more tractable
we also assume local thermodynamic equilibrium (LTE) which is consistent with previous sections. The ionization is then governed by the standard Saha equation. We then note that the total gas pressure $P$ is given by the sum of the constituent pressures:

$$P = P_H + P_{H^+} + P_{H_2} + P_{He} + P_M + P_e$$  \hspace{1cm} (3-43)

where $M$ stands for metals, $e$ for electrons, and the other symbols are self-explanatory.

Addressing the contributing terms now,

$$P_H^2 = P_{H_2}K_{H_2}^2(T)$$  \hspace{1cm} (3-44)

where the dissociation constant's functional dependence upon the temperature, $T$, is given via

$$K_{H_2}^2(T) = 3.49 \times 10^8 T \exp(-52490/T) \text{ dynes/cm}^2$$  \hspace{1cm} (3-45)

according to Larson (1968) from partition function calculations by Tsuji (1966) and Tatum (1966). But we also have

$$P_H + 2P_{H_2} = X \rho kT/m_p$$  \hspace{1cm} (3-46)

where $m_p$ is the proton mass. Thus,

$$P_{H_2} = \frac{1}{2}[X\rho kT/m_p - P_H]$$  \hspace{1cm} (3-47)

and

$$P_H^2 = P_{H_2}K_{H_2} = \frac{1}{2}[X\rho kT/m_p - P_H]K_{H_2}$$  \hspace{1cm} (3-48a')
which admits the solution

\[ P_H = - \frac{K_{H_2}}{4} \pm \frac{1}{2} \left[ \left( \frac{K_{H_2}^2}{4} \right) + \left( 2XpTK_{H_2}/m_p \right) \right]^{1/2} \]  \hspace{1cm} (3-48a)

Noting that \( K_{H_2} \) is intrinsically positive, for a physically real \( P_H \) (\( \geq 0 \)) we choose the + sign on the second term. Hence, \( P_H \) is given via equation (3-48a) and

\[ P_{H_2} = \frac{P_H}{K_{H_2}} \]  \hspace{1cm} (3-48b)

Next, we apply the Saha equation for \( P_{H^+} \) and have

\[ P_{H^+} = \frac{P_H}{P_e} K_H(T) \]  \hspace{1cm} (3-49)

where

\[ \log_{10}(K_H(T)) = -13.598 \frac{5040}{T} + 2.5 \log_{10}(T) - 0.4772 \]  \hspace{1cm} (3-50)

But since the electrons are assumed to arise only via hydrogen ionization (all other species' ionizations are ignored), \( P_e = P_{H^+} \). Also, we assume total dissociation prior to ionization. Thus,

\[ P_H + P_{H^+} = XpTK/m_p \]  \hspace{1cm} (3-51)

It is interesting to note that we have an overabundance of equations now since \( P_H \) was obtained earlier. However, the spirit of the various approximations outlined earlier imply application of \( P_H(K_{H_2}) \) for cases of low temperatures.
The $P_{\mathrm{H}}$, $P_{\mathrm{H}^+}$, $P_e$ solution may be achieved as follows. First we note that $P_e = P_{\mathrm{H}^+}$. Then, applying (3-49) and (3-51) we find the quadratic expression

$$P_{\mathrm{H}^+}^2 + K_H P_{\mathrm{H}^+} - K_H \frac{\rho kT}{m_p} = 0$$ \hspace{1cm} (3-52)

with the corresponding solutions

$$P_{\mathrm{H}^+} = -\frac{K_H}{2} \pm \frac{1}{2} \left[ K_H^2 + (4K_H \rho kT/m_p) \right]^{1/2}$$ \hspace{1cm} (3-53)

Again for a physically real result we choose the + sign on the radical. Then $P_{\mathrm{H}}$ is found from $P_{\mathrm{H}^+}$ and $K_H$ and as we have already noted $P_e = P_{\mathrm{H}^+}$.

It now remains to specify the choice between use of $K_H$ vs $K_{H_2}$ in the ultimate solution. In practice then we must choose a break point between considering dissociation versus considering ionization. We adopt a break at $K_{H_2} = 1000 \times \rho kT/m_p$. For $K_{H_2}$ equal to or below this value we consider only dissociation while for values exceeding this boundary we consider only ionization. This is the same semi-arbitrary break point applied by Boss (1979) in his studies of rotating low mass protostars. To verify this value we note that by factoring,

$$P_{\mathrm{H}} = \frac{K_{H_2}}{4} \left[ (1 + (8\rho kT/K_{H_2} m_p))^{1/2} - 1 \right]$$ \hspace{1cm} (3-54)

And for $K_{H_2}/(\rho kT/m_p)$ large then expanding the radical gives

$$P_{\mathrm{H}} \approx \frac{\rho kT}{m_p} \left[ 1 - (2\rho kT/K_{H_2} m_p) \right] = (P_{\mathrm{H}} + 2P_{H_2})(1 - \frac{2}{1000})$$ \hspace{1cm} (3-55)
Hence, for the above choice of 1000, the hydrogen is 99.8% dissociated thereby verifying the break point selection.

Finally, we turn to consideration of implementation and the actual calculations. It is apparent that five situations may arise including machine overflows and underflows. The first is the straightforward case of no dissociation and no ionization. Then $K_{H_2} = 0$ and $P_H = P_{H^+} = P_e = 0$ while $P_{H_2}$ is given via (3-47) and

$$P_{He} = \frac{Y_{pkT}}{4m_p}, \quad P_M = \frac{Z_{pkT}}{A_{m_p}}$$

The second case is the situation of partial dissociation and no ionization in which event $K_{H_2} \leq 1000 \frac{Y_{pkT}}{m_p}$ with $P_{H^+} = P_e = 0$, $P_{He}$ and $P_M$ are given via (3-56), and $P_H$ is given by (3-48a) with (3-48b) yielding $P_{H_2}$. The third possibility is the numerically induced limit of complete dissociation but vanishingly small ionization. Then $K_H$ is set to zero, $P_{H_2} = P_{H^+} = P_e = 0$ with $P_{He}$ and $P_M$ given by (3-56) and $P_H$ given via (3-51). The fourth possibility is complete dissociation and partial ionization with the ionization constant numerically below the machine imposed upper limit. Then $P_{H_2} = 0$, $P_{H^+}$ is obtained from (3-53), $P_H$ immediately follows from (3-49) since $P_e = P_{H^+}$, and $P_{He}$ and $P_M$ are found from (3-56). The fifth, and final, situation occurs when the ionization constant attains or exceeds the machine limit. Hence we have complete dissociation and complete ionization and the trivial limit of $P_H = P_{H_2} = 0, P_e = P_{H^+}$. $P_{H^+}$ comes from (3-51), and (3-56) again yields $P_{He}$ and $P_M$. The parallelism between limiting case 1 and 5 is evident since only a numerical (mass) constant differs.
Practically, however, it would be best to set particular boundaries for no dissociation and complete ionization rather than applying the generally unrealistic machine-imposed limits. In the case of dissociation we select a cutoff corresponding to 99.99% association. Then if one applies the previously found forms for the partial pressures $P_{H}$ and $P_{H_{2}}$, incorporates the analytic form for the dissociation constant and goes through a fair amount of algebraic manipulation ultimately a temperature constraint falls out corresponding to no (0.01%) dissociation. Similarly one can examine the practical boundary between $K_{H_{2}}$ and $K_{H}$. Application of the previously quoted $K_{H_{2}}$ break and substitution of the explicit form for $K_{H_{2}}$ again provides a temperature constraint after some algebra corresponding to when case 2, as previously cited, applies. Finally, one can treat the situation for complete ionization. In this case we again adopt 99.99% ionization as being "complete" and substitute the analytic forms for $P_{H}$ and $P_{H^{+}}$ from the proceeding pages. The situation is considerably more complex than arises for the dissociation cutoff derivation. In particular, the result is an inequality in $\log(K_{H})$ with the temperature bound up on the right hand side. However, the functional dependence of $K_{H}$ in terms of the temperature is both inversely proportional within an exponent and also directly proportional. Thus the net result for the cutoff for ionization ultimately may be cast into a transcendental constraint on the temperature which, although much more complex than the constraint for no dissociation, is directly analogous to that limit formally in terms of imposing a temperature/density "boundary".
SECTION 3.7 ENERGY GENERATION

Most of the previous sections in this chapter have been concerned with factors that arise to complicate the practical solution of the energy equation of classical hydrodynamics. One such additional factor is the onset of nuclear burning, hence energy generation, in the core of a protostar during the very late stages of its evolution just prior to its settling on the main sequence. Indeed, we have not formally incorporated it into the explicit form of the energy equation as quoted previously since its significance arises only so late during the SMS computations. It is the purpose of this section to remedy this matter.

We note that basically three nuclear burning scenarios are of interest here. First, it has been observed at a few positions within the low mass calculations that deuterium burning can make marginal alterations to the structure and timescales. However, as most such calculations have ignored its effect we shall do so as well merely noting that salient thermonuclear rates are provided by Harris et al. (1983). The other two fusion schemes are the conventional Proton-Proton Chain (PP) and Carbon-Nitrogen-Oxygen Cycle (CNO) which power conventional low and high mass stars on the main sequence. In particular, as it would appear likely that the nonhomologous collapse should lead to a "low" mass protostellar core with possible additional growth by accretion, one must incorporate both reaction rates into the calculations.

For the analytic approximations we adopt the expressions quoted by Clayton (1968) tacitly noting the underlying assumptions therein. The salient power laws therefore are:
The total nuclear burning energy contribution is thus simply the sum of these two terms and what must be added into the energy equation (3-31c) is

$$E_{\text{tot}} = \rho c_{\text{tot}} = \rho c_{\text{PP}} + \rho c_{\text{CNO}}$$  \hspace{1cm} (3-59)

Finally, we note that all symbolic notations are as in previous sections with $X_{\text{CN}}/X = 0.02$ for stellar Population I mixtures.

SECTION 3.8 THE MOVEABLE RADIAL GRID

As previously outlined in Section 3.2 we have chosen to adopt an Eulerian formalism rather than a Lagrangian treatment based on the practical experiences of previous investigators who have treated low mass conventional protostellar hydrodynamical models. However, it turns out that to optimize the numerical properties of the models, particularly conservation of mass and momentum, it is best not to employ a purely Eulerian approach. Indeed, what has proven best in the numerical methods applied earlier is a radial grid that is constrained to follow the flow in a globally optimized manner (to be defined and outlined here) rather than either a fixed Eulerian grid or a Lagrangian grid attached to specific mass elements. Thus for our supermassive cloud models we shall utilize such an optimized grid and this in turn means that our true
algorithm is best viewed as a hybrid Eulerian method. Indeed, in Section 3.10 we shall note that introduction of such a moveable grid introduces additional terms in the hydrodynamic equations as previously formulated. The attendant complications however are more than outweighed by the numerical accuracy benefits. In following this approach we are basically extending the methods of Larson (1969), Black and Bodenheimer (1975), and Boss (1979) into the supermassive regime.

A valid initial question is what is meant by a globally optimized moveable radial grid? By this we mean that the grid spacings and grid concurrent motions upon which the partial differential equations are cast into finite difference form and solved yield numerically induced inaccuracies that approach the smallest possible values subject to the numerical truncation and roundoff errors inherent in any computer calculation. To achieve this one must construct a grid that is non-uniformly spaced with the finest spacings arising at the center of the grid and the coarsest spacings at the outer grid boundary. Indeed, if we let $\Delta r_k$ denote the radial spacing between the centers of grid cell $k$ and grid cell $k-1$ (cell numbers increase outward!) then such an optimization may be approximately met by imposing the Larson condition

$$\Delta r_k = C \Delta r_{k-1} , \quad k > 2$$

(3-60)

where $C$ is a numerical constant to be determined in such a way so as to meet the optimum criterion and is constrained to be greater than unity. It is possible to show (Boss 1979) that the finite difference replacements for the conventional hydrodynamic partial differential equations are most accurate when the constant $C$ approaches unity. This
corresponds to a uniformly spaced grid. In such a situation the formal errors in the numerical values for derivatives would minimize. However, one also finds empirically that as the infall progresses (at least for the low mass models) and as the grid adjusts its infall so as to follow the flow the grid spacings become so small that the errors actually decrease in spite of the increasing values for the constant C from one timestep to the next. This is in the case of a fixed number of grid points throughout the model's time evolution.

To obtain the value for C we note that solution of the gravitational Poisson equation to be outlined in the next section requires the constraint that

\[ \Delta r_2 = \Delta r_1 \]  

(3-61)

so if we can determine C then all the other grid spacings will follow from the recurrence relation (3-60) upon selecting an initial distribution of grid spacings, say uniformly spaced, for the initial model before the first timestep. Now, let the number of radial grid points be represented by N and furthermore let us introduce the scaled radial grid spacing variable \( x \) where \( x = r/R \) and R is the total (fixed in space and time) radius of the protostellar cloud. Then we may impose the trivial condition that the total sum of the scaled grid spacings must equal unity. Thus,

\[ 1 = \Delta x_1 + \Delta x_2 + \Delta x_3 + \ldots + \Delta x_{N-1} + \Delta x_N \]  

(3-62)

or upon imposing conditions (3-61) for the second grid point and (3-60)
for the rest of the grid points we have

\[ 1 = \Delta x_1 + \Delta x_1 + C\Delta x_1 + C^2\Delta x_1 + C^3\Delta x_1 + \ldots + C^{N-2}\Delta x_1 \quad (3-63) \]

The powers of the constant \( C \) arise from recursive application of condition (3-60) so as to write each grid spacing ultimately in terms of the first grid spacing. Note also that we do not count the central point, where conditions will indeed be computed, as one of the grid points for purposes of this treatment. Upon factoring out the common \( \Delta x_1 \) it is apparent that we are faced with a conventional geometric series along with an additional additive term of unity as well. Thus one can rewrite equation (3-63) as

\[ (1 + \left[ (1 - C^{-N})/(1 - C) \right]) \cdot \Delta x_1 = 1 \tag{3-64} \]

after application of the expression for a geometric series. At this point the conventional approach is to define a new variable, say \( K \), such that \( K = C - 1 \). Then equation (3-64) becomes the polynomial constraint

\[ (1 + K)^{N-1} - BK - 1 = 0 \tag{3-65} \]

where the coefficient \( B \) is given by

\[ B = (1 - \Delta x_1)/\Delta x_1 \quad (3-66) \]

Thus, it is apparent that to set the computational grid at each timestep so as to optimally follow the inflow one must solve the polynomial equation (3-65) which is of degree \( N-1 \) where \( N \) is the total number of radial grid cells. In some of the computations as many as 200 grid cells were
applied. To solve a polynomial of degree 199 is a nontrivial objective. However, generally it was found that grids of 50 points were sufficient so the computational problem was reduced substantially. Indeed, it was found that a conventional Newton-Raphson (Burden et al. 1981), usually worked well. However, it should be noted that throughout the final hydrodynamic code the results were obtained to a double precision level so as to minimize truncation and roundoff. To set the grid so as to achieve this accuracy in the densities, flow velocities, and temperatures required double double precision (FORTRAN Q precision) at times. This was particularly true during the later stages of the calculations when the grid became highly nonuniform. However, note that the definition of K and the condition that C be greater than unity set a corresponding constraint on K. Namely K must lie between 0 and 1. Surprisingly, however, it was noted that no particular efficiency gains arose upon using the value of K from the previous timestep as the initial "guess" for K at the current timestep compared to using a fixed value centered in the K range as the initial "guess".

Finally, we observe that during the moderate to late stages of the calculations the flow motions became rather complex involving outward bounces. It became apparent that the optimum numerical approach in such situations was to abandon the previously moving grid and instead allow it to remain fixed in space during such an outward bounce. The condition imposed was to zero the grid velocities if the gas flow velocity in the first grid cell was positive (outflow). Upon its return to negative values, the optimized grid motion was activated again as above.
SECTION 3.9 SOLUTION OF THE POISSON EQUATION

The fundamental driving force for protostellar formation from the initial interstellar clouds of course is Newtonian gravity. Its effect arises in the momentum (transfer) equation, equation (3-31b), in the first term on the right hand side: the gravitational potential energy gradient. To compute this requires knowledge of the run of potential throughout the structure which is in turn specified by solution of the conventional Poisson equation (3-32a). Now the literature on solving such equations is legion so we do not consider all the possible solution algorithms here. Rather we note that the particular algorithm imposed must satisfy the constraints of high accuracy, reasonable computational efficiency, (hopefully) previously demonstrated success in the low mass protostellar domain, and capability for extension to nonspherical structures as would arise from rotational (and magnetic) effects as are envisioned in post-dissertation investigations.

Such an algorithm indeed exists and has been applied by Boss (1979) in his investigations of low mass rotating protostellar clouds. The method is to apply expansions in spherical harmonic functions for the gravitational potential $\phi$ as well as for the density $\rho$ to thereby effect a complete separation of variables. Numerical integrations are then performed and ultimately the problem can be cast into a finite difference formalism which can be highly efficiently solved due to the tridiagonal nature of the ultimate matrix of algebraic equations. The complete set of details is given in Boss' dissertation and we therefore do not take the time to repeat his full treatment. Rather, we cite only the most
significant aspects. Furthermore, we note that for the purposes of this dissertation, the models are spherically symmetric; thus the actual computations reduced to that special case greatly increasing efficiency. However, the "machinery" is potentially present for direct incorporation of rotation.

We begin by writing the potential and density in terms of the spherical harmonic expansions:

\[
\phi(r, \theta, \phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \phi_{\ell m}(r) Y_{\ell m}(\theta, \phi) \quad (3-67)
\]

\[
\rho(r, \theta, \phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \rho_{\ell m}(r) Y_{\ell m}(\theta, \phi) \quad (3-68)
\]

where the subscripted potential and density coefficients within the summations are complex numbers. The method obviously is to substitute these expressions into the three dimensional Poisson equation, achieve the separation of variables, multiply by the complex conjugate of the spherical harmonic functions \( Y_{\ell m} \) and perform the \( \theta \) and \( \phi \) angular integrations. Then, upon application of the orthogonal nature of the expansion one obtains a total differential equation for the potential coefficients in terms of the density coefficients. Namely, one must solve

\[
\frac{d^2 \phi_{\ell m}(r)}{dr^2} + \frac{2}{r} \frac{d \phi_{\ell m}(r)}{dr} - \frac{\ell(\ell+1)}{r^2} \phi_{\ell m}(r) = 4\pi \rho_{\ell m}(r) \quad (3-69)
\]
In the protostellar case one can invoke symmetry about a rotational equator so further simplifications arise with regard to the density coefficients especially. In particular, upon imposing the boundary condition that the potential vanishes at infinite distances beyond the structure and upon invoking the expressions for the density coefficients,

\[ \rho_{\ell m}(r) = 2 \int_0^{\pi/2} \int_0^\infty \rho(r,0,\phi) Y_{\ell m}^*(0,\phi) \sin(\theta) \, d\theta \, d\phi \]  

(3-70a)

\[ \rho_{\ell m}(r) = 0 \]  

(3-70b)

where (3-70a) applies for even $\ell+m$ and (3-70b) holds for odd $\ell+m$, we note that the differential equation (3-69) is cast into the appropriate finite difference form as specialized to a nonuniform grid. Proof of the need for this replacement form is to be found in Appendix B of Boss (1979) so we do not justify it here again. Instead we merely note that (3-69) eventually becomes

\[ a_i \phi_{\ell m}(r_{i-1}) - b_i \phi_{\ell m}(r_i) + c_i \phi_{\ell m}(r_{i+1}) = 4\pi G \rho_{\ell m}(r_i) \]  

(3-71)

where for the general nonuniform spatial grid the approximation is above first order accuracy while for uniform grids it is second order accurate and where the coefficients $a_i$, $b_i$, and $c_i$ are given by

\[ a_i = \frac{1}{\Delta r_i} \left( 2 \frac{r_i - \Delta r_{i+1}}{\Delta r_i + \Delta r_{i+1}} \right) \]  

(3-72a)

\[ b_i = \frac{\ell(\ell+1)}{r_i^2} + 2 \frac{r_i + \Delta r_i - \Delta r_{i+1}}{\Delta r_i \Delta r_{i+1}} \]  

(3-72b)
in terms of the radial grid spacings as outlined in the previously treated section.

The full power of this approach is best recognized now that it is apparent that the finite difference form (3-71) is tridiagonal. Namely, it involves initially unknown $\phi_{lm}$ at adjacent grid locations and no farther away! Now, it is well known that tridiagonal matrix systems of equations such as (3-71) obey solution algorithms much speedier than say a conventional Gaussian elimination given the size of the grid. Thus, we have an efficient method directly expandable to nonspherically symmetric situations that has proven itself on low mass models. Here we will momentarily skip the details of the tridiagonal algorithm needed to solve system (3-71,3-72) because as it will turn out, we will be able to cast the energy equation into a tridiagonal form as well given the spherical symmetry and it is more logical to consider the solution details at that point, Section 3.12. However, we do note that solving (3-71) does require knowledge of the density coefficients $\rho_{lm}$ on the right hand side. These in turn are given by (3-70a,b) so we must also achieve integrations over the mass distribution. The details of these integrations are covered by Boss (1979) but the basic approach is to apply a Filon quadrature scheme in the general case. In achieving the density coefficients for the SMS models we have adopted this same algorithm which considerably simplifies for spherical symmetry but again for later rotational investigations the full "machinery" is available.
Also, in imposing a tridiagonal solution algorithm we are constrained to equal grid spacings for $r_1$ and $r_2$ as shall be demonstrated later.

Finally we observe that upon solving the system we still do not know the central gravitational potential energy. To obtain this, one may integrate over the appropriate volumes. Namely, by assumption the first grid cell begins at radial position $\Delta r_1/2$ since the radial positions $r_i$ ($i = 1, 2, 3, \ldots, N$) correspond to the cell centers rather than cell boundaries. The central mass interior to first grid point is given by the volume integral out to $r_1$ weighted by the density. One simply treats the potential at the center differing from that at the first grid location in a fashion due to the appropriately weighted density sphere. For non spherically symmetric cases the approximation becomes

$$\phi(0) = \phi(r_1) - \frac{4\pi G}{6} \left[ \frac{\rho(0) + \rho(\Delta r_1)}{8} \right] \Delta r_1^2$$  \hspace{1cm} (3-73)

where the coefficients in the second factor arise from taking a volume weighted average. In our spherically symmetric case this expression becomes

$$\phi(0) = \phi(r_1) - \frac{4\pi G}{6} \left[ \frac{\rho(0) + \rho(r_1)}{2} \right] \Delta r_1^2$$  \hspace{1cm} (3-74)

where we have a straight mean.
SECTION 3.10 EXPLICIT EULERIAN HYDRODYNAMICS VIA THE DONOR CELL METHOD

We are now in a position to select the particular hydrodynamic algorithm to be applied in solving the classical equations. In doing so we note that the fundamental factors governing our selection are basically those we imposed in solving the Poisson equation. Namely, we desire good computational speed, high numerical accuracy, evidence of successful past application to low mass protostellar models if possible, and the potential for relatively straightforward extensions to the contemplated future SMS research efforts, particularly rotation. The fundamental equations however are partial, rather than total, differential equations. Thus the extensive body of numerical algorithms ranging from Runge-Kutta to multistep and extrapolation algorithms developed for ordinary equations are useless for our purposes. However, given the wide range of partial differential equations that have arisen in the engineering and scientific literature, a variety of solution algorithms have been developed. These may be conveniently subdivided into the finite element and the finite difference methods (Burden et al. 1981). The advantages of the finite element methods largely lie in their ability to readily handle boundary conditions imposed over rather irregularly shaped boundaries. Thus they have been widely applied in engineering contexts and are not particularly of further interest for our purposes. The finite difference methods, on the other hand, approximate the partial derivatives by difference equations over finite ranges and it is this approach that has seen considerable application in the protostellar context. We therefore select a finite difference approach.
Given this general choice we now must choose between the two basic types of finite difference algorithms. These are the implicit and the explicit methods. In this respect, a superb summary of the available hydrodynamic algorithms (particularly for engineering applications) is provided by Roache (1982). We note immediately that the implicit methods are potentially quite powerful because of their capabilities to take large timesteps. In general the implicit methods apply advance values in the spatial derivatives which in turn develop a simultaneous set of difference equations for all the grid densities, velocities, and temperatures that must be solved together so as to advance the model to the new time. In general such a computation requires a multidimensional Newton-Raphson approach as was applied by Reynolds (1979) in his treatment of mass accretion onto ad hoc existing SMSs. However, if the computational effort required per timestep is examined it is apparent that such an iterative approach is relatively slow. Moreover, if one is considering a physically realistic problem then a practical limitation does arise to the allowable implicit timestep in that one cannot take steps long enough to allow the fundamental variables to change by more than a small fraction compared to their current orders of magnitude. Indeed, if longer steps are allowed then spurious results can arise due to the algorithm not recognizing the onset of significant physical alterations such as shocks during the interim time period. Thus, if an implicit method is selected then considerable computations are demanded in spite of the fact that the timesteps are relatively far apart. The explicit algorithms apply the variable values at the current timestep
In order to determine the values at the new time. In other words, if the time difference term appears on the left hand side then all the terms on the right hand side involve known or previously calculated quantities at the current time. Explicit approaches therefore do not require the computational expense inherent in a multidimensional Newton-Raphson treatment since the variables are obtained separately. However, they have a major disadvantage in terms of a limited timestep that may be applied to ensure numerical solution stability. This timestep constraint is known as the Courant-Friedrichs-Lewy condition or CFL timestep and is given via

\[ \Delta t \leq \frac{\Delta r}{(v_a + |v|)} \]  

(3-75)

Roache (1982) explicitly proves this condition via a formal stability analysis. However, we can understand its relevance from a physical standpoint since the right hand side of (3-75) represents the time required for a physical "message" to be transported across a cell of width \( \Delta r \) due to sonic effects as well as a hydrodynamical flow. Here \( v_a \) is the fluid's local sound speed given by \( \sqrt{\gamma P/\rho} \) and \( v \) is the fluid flow velocity. In other words, physical information would be lost if larger steps were taken since modifications to the hydrodynamical and thermodynamic parameters could take place on a faster timescale than the algorithm could follow. Moreover, this represents a necessary but not necessarily sufficient condition for solution stability. In general the timesteps required are somewhat smaller particularly in the case of a
multidimensional, nonspherically symmetric generalized flow. In the case of a nonuniformly spaced grid as we have already adopted then one would have to compute CFL timesteps throughout the grid volume and select a timestep sufficiently shorter than the shortest CFL step to ensure solution stability. Moreover, the development of strong shocks has been noted to further reduce the CFL step size condition in the protostellar context (Black and Bodenheimer 1975). This is definitely in agreement with the experiences arising from the SMS models since we ultimately selected such an explicit algorithm. Thus, both types of methods have disadvantages. The implicit methods are impractical for the earlier phases of a collapsing SMS cloud due to the computational effort required. This is because a central density kernel fairly rapidly develops which constrains the timesteps to be at least comparable to the CFL step size and at each timestep an iterative Newton-Raphson would have to be applied. On the other hand, the explicit methods are impractical for the late phases of a protostar because then the structure evolves on a thermal timescale rather than on a hydrodynamical timescale. At that point although each timestep demands relatively little computational time the total number of timesteps required to substantially evolve the model is enormous. However, we do observe that since a rigorous SMS model has yet to be computed by other workers it would be more logical to apply an explicit algorithm capable at least of correctly handling the physics until swamped by the step constraint. Moreover, we note that a variety of low mass protostellar investigations ranging from Larson to Boss have been most successful through use of explicit approaches. Finally, as one
future extension of this work is the incorporation of rotational effects, it is almost essential to follow an explicit method since no successful fully implicit multidimensional protostellar code apparently has been developed.

Having chosen an explicit method we must now select the particular algorithm to be employed. Again, reference to Roache (1982) proved invaluable. The specific method ultimately chosen is often referred to as the second upwind differencing method. Equivalently, and rather more descriptively, it has also been called the donor cell method. The basic idea behind the algorithm is that one cell donates mass or momentum to the next cell based on the algebraic sign of the fluid flow velocity at the interface between adjacent grid cells. Specifically, we have defined the cells to be centered on the radial positions \( r_i \), \( i = 1, 2, \ldots, N-1, N \). Thus, the density, velocity, and temperature values are found at those locations. To determine the values at the interface we simply take the average between the cells. Namely, if we desire the fluid velocity at the interface between cell \( k \) and cell \( k+1 \) we take it to be

\[ v^n = \frac{1}{2}(v^n_k + v^n_{k+1}) \]  

(3-76)

where the subscripts refer to the spatial grid cell location and the superscripts denote the current time level.

While a number of other algorithms may be formally more accurate it is ironic that in many practical applications such solution accuracy judgements are of little use. Moreover, the donor cell approach is readily physically envisioned, a factor of prime importance during tests.
Furthermore, the algorithm possesses a first order finite difference approach that tends to slightly diffuse the variables but aids in the numerical stability properties of the scheme. This stability can be additionally enhanced through selection of a sufficiently small timestep. Also, when the equations are initially written in conservation form as we have chosen in the first sections of this chapter then a conservative hydrodynamic scheme like the donor cell algorithm provides excellent numerical conservation properties of mass and momentum (ignoring for now the nonconservative "driving" terms on the right hand side of the momentum equation). Further enhancement of these accuracy properties can be achieved by selection of finer grid spacings and constraining the grids to follow the flow as already outlined. In addition, this algorithm will always yield positive densities since the mass flowing out of any given cell is taken as being proportional to that cell's density. Also, fluid disturbances are numerically transported in the direction of the fluid flow (downstream) since the flow direction determines the donation direction. This is as one would desire on an intuitive physical basis. These last two criteria are not always met by the hydrodynamic schemes available in the literature. For example, in Yorke and Krugel (1977) two continuity equations were used for the gas component. One tended to gain mass and the other tended to lose mass. Their results were rather arbitrarily combined in such a way so as to conserve mass! Finally, we also have a low mass test case against which the SMS donor cell code can be numerically tested. Boss (1979) chose this algorithm for his low mass rotating protostellar models.
Prior to developing the finite difference replacements for the continuity and momentum equations however, we introduce the modifications that arise due to the moving grid of Section 3.8. In particular, consider the continuity equation (3-31a). Suppose for now that the grid motion is globally uniform and focus our attention on a radial cell bounded by surfaces $S_A$ and $S_B$ with cell thickness $dr$. Such a cell is illustrated in Figure 3. Now, if the grid motion is uniform then the surface velocities are equal and the velocity of the flow relative to the moving grid is simply given by $\vec{v}_{rel} = \vec{v} - \vec{v}_g$ where $\vec{v}_g$ is the grid velocity. Thus, at this level the continuity equation would become

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \rho (v - v_g) \right) = 0$$

(3-77)

with notation as before. However, in general the grid flow will not be uniform but will tend to follow the local flow pattern subject to the global optimization constraint previously cited. Thus, the volume of any particular grid cell will be subject to change from one time level to the next time level. We must therefore account for this volume effect as well. Again adopting superscripts as denoting the time level we have the initial and final volumes respectively given via

$$V_{oln} = 4\pi r^2 dr$$

$$V_{oln+1} = 4\pi r^2 [dr + (v_{gB} - v_{gA}) \ dt]$$

(3-78)

where the subscripts A and B refer to surfaces A and B. Now, it is simplest to envision the required modifications in the limit of zero
Figure 3

Radially Moving Grid Cell in the Context of the Donor Cell Algorithm
fluid flow velocity. Then, the time rate of change of density is given by

\[
\frac{dp}{dt} = \frac{N_{\text{cell}}}{4\pi r^2} \left( \frac{1}{dr} + \frac{1}{v_{\text{BB}} - v_{\text{GA}}} \right) \frac{1}{dt} \tag{3-79}
\]

\[
= \rho_{\text{cell}} \frac{(v_{\text{BB}} - v_{\text{GA}})(dt/dr)}{1 + (v_{\text{BB}} - v_{\text{GA}})(dr/dt)} \frac{1}{dt}
\]

\[
= \rho (v_{\text{BB}} - v_{\text{GA}})/dr \quad (dt + 0) \tag{3-80}
\]

and the corresponding modification to the equation that takes into account the effect of a variable grid motion with attendant cell volume changes becomes

\[
\frac{3p}{3t} + \frac{1}{r^2} \frac{3}{3r} [r^2 \rho (v - v_g)] + \frac{\rho}{r^2} \frac{3}{3r} (r^2 v_g) = 0 \quad \tag{3-81}
\]

This is the form of the continuity equation that we will replace with the related finite difference forms arising from the donor cell method.

Furthermore, we are in a position to note now that the treatment of grid motion as performed above for the continuity equation was general in that one could have selected any particular quantity to be "advected" with the same results for contributions due to measuring a relative flow and due to a changing cell volume. In particular, suppose the variable under consideration is the momentum \( \rho v \) rather than just the density. Then the changes arising in the momentum equation (3-31b) will be analogous to those that have appeared in (3-81). We thus write the momentum equation as
\[
\frac{3}{2}(p v) + \frac{1}{r^2} \frac{3}{3x} \left[ r^2 p v (v - v_g) \right] = - \frac{p v}{r^2} \frac{3}{3x} \left( r^2 v_g \right) - \rho \frac{3}{3x} \frac{\partial \phi}{\partial x} - \frac{3}{r^2} \frac{3}{3x} \left( r^2 qr \right) \quad (3-82)
\]

where the last term on the right hand side that appeared "magically" is the momentum transfer contribution due to artificial viscosity. Artificial viscosity will be discussed in the next section.

We can now apply the donor cell method to obtain the finite difference replacements. As the details are to be found in Roache (1982) and Boss (1979) we do not repeat them here. Instead, we exhibit the ultimate form for the \( i \)-th grid cell density arising from such a treatment of the continuity equation. It is given by

\[
\rho_{i+1} = \rho_i \frac{r_i}{r_{i+1}} \left( \frac{r_{i+1}^n - r_i^n}{\Delta r_i} \right) - \frac{\Delta t}{r_{i+1}^n} \left( \frac{r_{i+1}^{n+1} \rho_{i+1}^{n+1} - r_i^{n+1} \rho_{L,L}^{n+1}}{\Delta r_i^{n+1} + \Delta r_{i+1}^{n+1}} \right) \quad (3-83)
\]

in which the superscripts \( n \) and \( n+1 \) refer to the current and next times respectively. The leading term arises from the contribution due to the moving grid. Note that if no grid motion takes place then no alteration occurs due to the first term at least in the cell density. The second term arises because of the fluid flow relative to the moving grid -- the "advective" term. The auxiliary quantities appearing there are defined as follows:

\[
V_R = \frac{1}{2} (v_i - v_{g_i} + v_{i+1} - v_{g_{i+1}}) = \text{outer surface relative } v
\]

\[
V_L = \frac{1}{2} (v_i - v_{g_i} + v_{i-1} - v_{g_{i-1}}) = \text{inner surface relative } v
\]

\[
(3-84)
\]
and the radial grid quantities are given by

\[ r_{i+\frac{1}{2}} = \frac{1}{2}(r_i + r_{i+1}) \]  
(3-87)

\[ \Delta r_k = r_k - r_{k-1} \]  
(3-88)

with \( k = i \) or \( k = i+1 \). Note especially that the densities and velocities as applied in (3-83) are defined at the old time level \( n \) whereas the grid quantities in the advective term are all defined at the new time level \( n+1 \), as the grid has been moved to the new position. This is in keeping with the explicit nature of the donor cell algorithm. Also, when the central density \( i = 0 \) and the boundary density \( i = N \) (for \( N \) total grid points) are to be obtained then it is apparent that some complications arise. Namely, we impose the outer boundary condition that the velocity is zero so material may flow out of the outermost cell (even into it from the next cell inside if \( v_{N-1} \) is positive) but no material can flow into the grid through the outermost boundary of cell \( N \). For the central density we note that the velocity of the central point is zero so contributions arising to alter the density develop only through the sign of the flow speed of cell 1 and due to central cell volume changes via the moving grid. Numeric application of these boundary conditions plus...
solution of (3-83) for the intervening cells permits the complete span of grid cell densities at the new time level \( n+1 \) to be determined.

Now we turn to the donor cell replacement for the momentum equation. In this case we know the values for the densities at time \( t^{n+1} \) but have the nonconservative driving terms to replace on the right hand side in terms of differences as well as the conventional advective and grid motion terms. Once again the spirit will generally be to define the physical quantities, except for the time difference term, in terms of the old time level \( n \) whereas the grid spacings and positions are mostly in terms of the new time level \( n+1 \). Thus, the finite difference form becomes (Boss 1979)

\[
\frac{\rho_{i+1} v_{i+1}^{n+1} - \rho_{i} v_{i}^{n+1}}{\Delta t} + \frac{1}{r_{i}^{n+1/2}} \left( \frac{r_{i+1}^{n+1/2} X_{R}^{n} - r_{i-1}^{n+1/2} X_{L}^{n}}{L_{i}^{n+1} + \Delta r_{i+1}^{n+1}} \right) = - \frac{\rho_{i} v_{i}^{n}}{\Delta t} \left[ 1 - \frac{r_{i}^{2}}{r_{i}^{n+1/2}} \left( \frac{r_{i+1}^{n} - r_{i-1}^{n}}{r_{i+1}^{n+1/2} - r_{i-1}^{n+1/2}} \right) \right] - \rho_{i} \left[ \frac{S_{i+1}^{n+1/2} - S_{i-1}^{n+1/2}}{S_{i}^{n+1} - S_{i}^{n}} \right] - \frac{\rho_{i}^{n} \phi_{i+1}^{n+1} - \rho_{i}^{n} \phi_{i-1}^{n+1}}{\Delta t} \left( \frac{r_{i+1}^{2} Q_{i+1}^{n+1/2} - r_{i-1}^{2} Q_{i-1}^{n+1/2}}{L_{i}^{n+1} + \Delta r_{i+1}^{n+1}} \right)
\]

where the first term represents a forward time difference, the second one represents the advection of the momentum \( \mathbf{X} = \rho \mathbf{v} \) with respect to the moving grid, the first term on the right side corrects for cell volume changes, the second approximates the potential gradient, the third treats the pressure gradient, and the fourth is due to the artificial viscosity.
Generally the notation directly matches that applied for the continuity equation. The relative velocities \( V_R \) and \( V_L \) are given via (3-84). The advection parameters \( X_R \) and \( X_L \) are defined in exact analogy to the conditions in (3-85) and (3-86) except now we have the product of density times velocity rather than merely density. However, the subscript choices are precisely the same as for the continuity equation treatment. The grid spacing parameters and grid location radii are defined as in (3-87) and (3-88). Finally, we observe that some new auxiliary quantities appear within the finite difference expressions approximating the potential, pressure, and artificial viscosity gradients. We shall define the artificial viscosity in the next section and so will not continue to pursue it here. However, the \( S_p \), \( S_q \), and \( S_M \) parameters arise through approximating the derivatives on a nonuniformly spaced grid. These may be shown to be given by (Boss 1979)

\[
S_p = \frac{\Delta r_i}{\Delta r_i + \Delta r_i} \frac{1}{\Delta r_i + \Delta r_i} \quad (3-90)
\]

\[
S_M = \frac{\Delta r_i + \Delta r_i}{\Delta r_i} \frac{1}{\Delta r_i + \Delta r_i} \quad (3-91)
\]

\[
S_o = S_p - S_M \quad (3-92)
\]

Further, we note that the relevant boundary conditions are \( v_0 = 0 \) and \( v_N = 0 \). Thus, solution of (3-89) provides the full grid range of flow velocities since the grid densities at this new time were previously obtained and the grid velocities are given once one sets the first grid point via \( \Delta r^{n+1} = \Delta r^n + v_i^n \Delta t \).
SECTION 3.11 ARTIFICIAL VISCOSITY

As has already been cited in Section 2.5, shocks are widely observed to occur numerically in the low mass conventional protostellar models. Moreover, application of straightforward entropy constraints indicated that one should expect them to also arise in the SMS protostellar context. Thus, we must be prepared to handle them within the hydrodynamical computer code. Initially one might envision application of the conventional shock fitting conditions, the Rankine-Hugoniot relations. However, in the protostellar case this proves impractical since it is impossible to know a priori just where the shock fronts will occur. Furthermore, they may not be assumed to remain spatially fixed. Thus, we turn to the conventional shock modeling approach originally due to von Neumann and Richtmyer, the artificial viscosity technique.

This method, discussed in Roache (1982), essentially empirically incorporates an additional term within the momentum equation (as we have already seen) and within the energy equation. The problem arising due to the shock(s) of course is the inability to represent such a sharp discontinuity within the context of a finite difference formalism. Thus, the numerical effect of the new term in the momentum equation is to spread out the shock over several grid points while still obeying the conservation constraints across the shock front. Thus, the values of physical variables immediately around the front are not highly accurate but hopefully otherwise the results are good and one is able to treat such a front no matter of where it chooses to develop within the computational grid.
While this shock smearing technique has been widely applied throughout the literature from Larson (1968) to Boss (1984), it is still somewhat disturbing to incorporate such a term into the basic hydrodynamical equations on a purely empirical basis. However, we note in passing that in addition to the original article, von Neumann and Richtmyer (1950), outlining the fundamental approach, a particularly telling article is Roache (1972). In it the topic of artificial viscosity is examined in considerable depth. Especially, it is noted that the simple existence of truncation errors in the finite difference equations arising in the numerical approaches can inadvertently create an effective "artificially viscous behavior". Moreover, although other authors had noted this effect, Roache (1972) demonstrated that several of those numerical hydrodynamic algorithms often described as being artificial viscosity free: Lax-Wendroff, Leith, Moretti, and MacCormack algorithms, actually do have an implicit artificial viscosity. Furthermore as we have already observed in Section 3.10 the donor cell method exhibits a diffusive tendency in the mass and momentum which actually promotes solution stability. Thus, with the implicit presence of such artificial viscosity already within our algorithm the incorporation of an explicit artificial viscosity is at least acceptable.

From a physical analysis standpoint, the artificial viscosity enters into the momentum equation as an effective pressure term Q which for a general structure possessing nonradial motions will be a tensor. However in our case we are confined to radial flow so a single component will suffice -- $Q_{rr}$. The name viscosity is rather more accurate than pressure
since the expression actually entails flow velocity differences. Also from a formal standpoint we observe that the conventional shock jump conditions themselves do not describe the physical characteristics within the shock but rather do so far upstream and downstream; thus the artificial viscosity approach can smear the shock over multiple grid cells while attempting to conserve mass, momentum, and energy on a global basis. Thus, when the shock is smeared out the numerical trend is to convert the energy of the pressure oscillations behind the front into heat as a real viscosity would. Hence, the effect of artificial viscosity must be incorporated within the energy equation as well.

The explicit form for $Q^{rr}$ is given by

$$Q^{rr} = -\rho \ C_q (\Delta r)^2 \left| \frac{3v}{\Delta r} \right| \frac{\partial v}{\partial x}$$

(3-93)

where the gas density is $\rho$, $v$ is the fluid velocity, and the parameter $C_q$ is an adjustable empirical parameter nominally close to unity. In particular, the choice of $C_q$ is the factor that determines the total number of grid cells over which a shock front will be smeared during the calculations. The explicit form for $Q$ thus reinforces its behavior as a viscosity effect. In practice, for our calculations we shall select a value of one for $C_q$.

We are now in a position to display the finite difference form for the artificial viscosity term in equation (3-82). In direct analogy to the treatment of the advective term's finite difference replacement, we apply the approximation.
where again the superscripts refer to the time level while the subscripts denote the spatial grid cell. Moreover, we define the $Q$ values via

\[ Q_i^{rr} = -\min\left[0, \frac{k_c}{Q_i^{pp}} (\rho_i + \rho_{i-1})|v_i - v_{i-1}|(v_i - v_{i-1})\right] \]  

(3-95a)

\[ Q_{i+1}^{rr} = -\min\left[0, \frac{k_c}{Q_{i+1}^{pp}} (\rho_{i+1} + \rho_i)|v_{i+1} - v_i|(v_{i+1} - v_i)\right] \]  

(3-95b)

Thus, we may observe that in the case of a compression front arising during a radial infall both velocities are negative but $v_i < v_{i-1}$ so the last factor in (3-95a) will be negative while all other factors are intrinsically positive. Hence, the artificial viscosity will be activated upon selecting the minimum value within brackets. On the other hand, for a rarefaction case then the inequality will be reversed and the value of zero will be selected thus eliminating any explicit artificial viscous term. An analogous situation obviously obtains for (3-95b) and $Q_{i+1}^{rr}$.

Note also that the leading negative sign in both equations makes the artificial viscosity an intrinsically positive number and our choice of the finite difference replacement (3-94) coupled with definitions (3-95) guarantees global momentum conservation through the donor cell formalism.

Finally, we observe that due to the presence of an explicit artificial viscosity term in the momentum equation, (3-94) should be added to the viscosity free form to give equation (3-89). For internal consistency one must also incorporate an artificial viscosity term into the energy
equation. In particular, the required term to be added on the right hand side of equation (3-31c) is given by

\[- Q^{rr} \frac{3v}{3r}\]  \hspace{1cm} (3-96)

as noted by Boss (1979) and Schulz (1964). Indeed, one can show that this admits global energy conservation upon integrating over the protostellar structure assuming that \( Q = 0 \) at the surface as is indeed the case in the actual models. The physical cause of (3-96) is to take into account the heat produced by the damping of the aft pressure oscillations behind the shock. When we apply this term in the next section's consideration of practical solutions of the energy equation, the derivative will be finite differenced in direct analogy to the forms previously used in the continuity and momentum equations.

SECTION 3.12 TRIDIAGONAL IMPLICIT SOLUTION OF THE ENERGY EQUATION

At the classical level it only remains to consider the approach to be taken in solving the energy equation. Without question this was the most difficult area from a practical implementation standpoint. But in Section 3.3 we have already opted for use of the diffusion approximation embodied in equation (3-33) which does considerably simplify the situation. However, we note that the specific internal energy \( E \), the pressure \( P \), and the temperature \( T \) all enter into the energy equation in a nontrivial manner. Thus, we must select a fundamental variable to be solved for from amongst these three. There are advantages in choosing either \( E \) or \( T \) but if \( T \) is selected then one can develop a highly efficient tridiagonal algorithm for solving the energy equation. This
is the route we shall follow. However we note at the outset that certain approximations will be required, to be specified shortly, that are best at low to moderate temperatures. Moreover, the method is limited to application for spherically symmetric structures and will have to be abandoned for post-dissertation rotational studies. As our donor cell algorithm will ultimately be constrained by short timesteps during the final stages of model calculations these objections are not of overriding concern at this point.

If we generalize the energy equation (3-33) to include the effects of a moving grid, artificial viscosity, and the diffusion approximation for the radiative transfer term then it becomes

\[
\frac{3(\rho E)}{\partial t} = - \frac{1}{r^2} \frac{\partial}{\partial r} \left[ r^2 \rho E (v - v_g) \right] - \frac{\rho E}{r^2} \frac{\partial}{\partial r} (r^2 v) + \rho \nu_{\text{Nuc}} \\
- \frac{P}{r^2} \frac{\partial}{\partial r} (r^2 v) + \frac{1}{r^2} \frac{\partial}{\partial r} \left( \frac{16 r^2 \sigma T^3}{3 \kappa_R} \frac{\partial T}{\partial r} \right) \\
- Q^{rr} \frac{\partial v}{\partial r}
\] (3-97)

where $E$ is the specific internal energy, $\rho$ is the density, $v$ is the gas flow velocity, $v_g$ is the (local) grid velocity, $P$ is the pressure, $T$ is the temperature, $Q^{rr}$ is the artificial viscosity, and $\kappa_R$ is the Rosseland mean opacity. We have also included the effects of nuclear fusion which only arise at the very end phases of certain models.

Now it has been recognized since Larson (1968) that a practical solution of the energy equation in the protostellar context is only
achieved in an implicit approach. Application of an explicit algorithm further constrains the allowable timestep to be several orders of magnitude smaller than even the CFL timestep! Thus, we shall select an implicit approach. Furthermore, the physical origin of the terms arising in the energy equation are readily identifiable. The first term on the right is due to the energy advection with respect to the moving computational grid. The second is due to the nonuniform grid velocity and the attendant alterations in cell volumes. The third represents the nuclear burning effects. The fourth expresses the heat gains/losses due to compressional heating/cooling effects. The fifth term corresponds to the diffusion approximation for the radiative transfer contribution. Finally, the sixth term yields the energy contribution due to artificial viscosity which arises only when strong shock fronts are set up.

Having selected \( T \) as the fundamental variable one might immediately proceed to Section 3.5 and Section 3.6 to substitute the explicit forms for the specific internal energy and the equation of state so as to cast everything in terms of the density, velocity, and temperature of which the first two are known at the new time upon solution of the continuity and momentum equations as per the previous sections. However, such a method is impractically cumbersome and Boss (1979) has shown that at low and moderate temperatures one can make the following approximations for the specific internal energy and pressure as far as the formal solution of (3-97) is concerned:

\[
E_{i}^{n+1} = C_{i}^{n+1} \quad (3-98a)
\]
where the pseudo specific heats are given by the values at the old time level

\[ C_E = \frac{E^n}{T^n_1} \]  \hspace{1cm} (3-99a)

\[ C_P = \frac{P^n}{T^n_1} \]  \hspace{1cm} (3-99b)

Note that we are not saying here that the formal expressions for the specific internal energy and the equation of state are not being used. Indeed, \( P = P_{\text{gas}} + P_{\text{rad}} \) with \( P_{\text{gas}} \) as per Section 3.6. Also, for computing the specific internal energy \( E \) the method of Section 3.5 is applied. Rather what (3-98) and (3-99) represent is a manner of casting (3-97) into a form expressed solely in terms of the grid temperatures (all simultaneously unknown) at the new time with all other quantities being known values. Upon solution of the energy equation by the method to be outlined next, the logical followup is to update the grid energies and pressures through the computed temperatures and the actual equations of state. Thus, these expressions are solely applied for reducing the energy equation to a readily solvable form.

We can however gain some indication of the error introduced into the calculations during the very late stages by examining the effect on the pressure since at this point for very high temperatures the radiation...
pressure contribution greatly exceeds the gas pressure contribution.

Then

\[(P^{n+1}/P^n)_{\text{real}} = (T^{n+1}/T^n)^4 \quad (3-100a)\]

\[(P^{n+1}/P^n)_{\text{apprx}} = (T^{n+1}/T^n) \quad (3-100b)\]

where the second equation corresponds to the linearized "specific heat" treatment. Now if we further let

\[T^{n+1} = T^n + \delta T^n = T^n(1 + \delta) \quad (3-101)\]

then we can determine the error produced in the compressional heating/cooling term due to the approximation as a function of how rapidly the temperature changes between the current time and the new time. In this respect \(\delta\) represents the percent change in the temperature during the timestep. We tabulate the percent errors arising in Table 2 below.

<table>
<thead>
<tr>
<th>(\delta)</th>
<th>((P^{n+1}/P^n)_{\text{real}})</th>
<th>((P^{n+1}/P^n)_{\text{apprx}})</th>
<th>% Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0001</td>
<td>1.0004</td>
<td>1.0001</td>
<td>0.03</td>
</tr>
<tr>
<td>0.001</td>
<td>1.0040</td>
<td>1.0010</td>
<td>0.30</td>
</tr>
<tr>
<td>0.005</td>
<td>1.0202</td>
<td>1.0050</td>
<td>1.49</td>
</tr>
<tr>
<td>0.01</td>
<td>1.0406</td>
<td>1.0100</td>
<td>2.94</td>
</tr>
<tr>
<td>0.02</td>
<td>1.0824</td>
<td>1.0200</td>
<td>5.77</td>
</tr>
<tr>
<td>0.03</td>
<td>1.1255</td>
<td>1.0300</td>
<td>8.49</td>
</tr>
<tr>
<td>0.04</td>
<td>1.1699</td>
<td>1.0400</td>
<td>11.10</td>
</tr>
<tr>
<td>0.05</td>
<td>1.2155</td>
<td>1.0500</td>
<td>13.62</td>
</tr>
</tbody>
</table>
We thus note that even in the case of a five percent change in the temperature over the timestep, the error produced in the pressure ratio is nominally about ten percent. However, by these later stages the approximations inherent in the adopted equation of state and formula for the specific internal energy will be at these levels anyway. Of course the true solution of the energy equation is nonlinear so it is not clear precisely how these errors propagate through the solution but at least the above analysis provides an estimate.

Now we turn to the finite difference replacement for equation (3-97) upon applying these approximations. We shall treat each term in order for clarity. Our replacement for the time derivative term on the left side is simply

\[
\frac{(\rho C_P T)^{n+1}_i - (\rho E)^n_i}{\Delta t}
\]  \hspace{1cm} (3-102)

again where subscripts denote grid positions and superscripts imply the time level. The leading (advection) term on the right is replaced in precisely the same manner as the advection term in the momentum equation. Namely, we use

\[
\frac{1}{\frac{r_{n+1}^2}{r_1}} \left( \frac{r_{i+\frac{1}{2}} X_{R} V_R n+1 - r_{i-\frac{1}{2}} X_{L} V_L n+1}{\Delta t} \right) - \frac{r_{n+1}^2}{r_1} \left( \frac{\Delta r_{n+1}^1 + \Delta r_{n+1}^{i+1}}{2} \right)
\]  \hspace{1cm} (3-103)

where the only difference with the earlier formalism lies in the fact the current time level values are used for the \( X_{R,L} \) where \( X = \rho C_P T \) and the choice of subscripts is precisely as in equations (3-85) and (3-86).
The second term on the right arises from the changing cell size due to grid motion and is differenced via

\[
\frac{(\rho E)^n_i}{\Delta t} \left( 1 - \frac{r_i^n}{r_i^{n+1}} \frac{(r_i^{n+1} - r_i^{n-1})}{(r_i^{n+1} - r_i^{n-1})} \right)
\]

which is the energy equation analogue of the first term on the right hand side of the finite difference momentum equation (3-89). The third term due to nuclear energy generation is trivial since it merely takes on the subscript values \(i\). The fourth term due to compressional heating is treated as

\[
-c_p T_i^{n+1} \left[ (A_1 v_{i+1} - (A_1 - A_2) v_i - A_2 v_{i-1}) \right]
\]

\[/(\Delta r_i + \Delta r_{i+1}) + \frac{2v_i^{n+1}}{r_i} \] \[j^{n+1}
\]

where the auxiliary quantities \(A_1\) and \(A_2\) are given by

\[
A_1 = \frac{\Delta r_i}{\Delta r_{i+1}}
\]

\[
A_2 = A_1^{-1}
\]

The two term nature of (3-105) arises from expanding the product inside the original partial derivative in (3-97). Next comes the radiative diffusion term which is the most complicated since it entails treating a derivative inside a derivative. The formal analysis is done by Boss (1979) so we do not repeat it here but rather quote the resultant form for this fifth term when finite differenced:
where we define the auxiliary quantities as follows:

\[ t_p = \frac{1}{\Delta r_{i+1}} \left[ \frac{1}{(\Delta r_i + \Delta r_{i+1})} \right] \]  
(3-108a)

\[ t_M = \frac{1}{\Delta r_i} \left[ \frac{1}{(\Delta r_i + \Delta r_{i+1})} \right] \]  
(3-108b)

and

\[ W_j = \frac{3 \tau_j^2}{\Delta r_i^2} \kappa_{R_j}^\alpha j \]  
(3-109)

with \( j = i \) or \( j = i+1 \) or \( j = i-1 \) and all quantities within (3-109) are defined at the old time level \( n \) except for the radial positions. In general the radial positions and spacings are as defined at the new time level \( n+1 \) throughout (3-107 to 109). Finally we have the sixth term which arises when the artificial viscosity shock smearing algorithm is activated to treat a strong compressional shock. This term is approximated similarly to the approach in (3-105) via a finite difference representation keyed to a nonuniformly spaced grid. Thus we have

\[ - \frac{1}{2} \left( q_i^{n+1} q_{i+1}^{n} \right) \left[ (A_1 v_i^{i+1} - (A_1 - A_2) v_i - A_2 v_{i-1}) \right] \frac{1}{(\Delta r_i + \Delta r_{i+1})} \]  
(3-110)
where $A_1$ and $A_2$ are defined by equations (3-106a,b).

To proceed we observe that if the expressions (3-102) through (3-110) are collected then it can be recognized that all quantities are known except for the full range of grid temperatures at time level $n+1$. Thus, we have a simultaneous set of equations to solve for $T_0$, $T_1$, $T_2$, $\ldots$, $T_{N-1}$, $T_N$. This is the classic case of any implicit solution approach. However, to obtain the solution we must impose the appropriate boundary conditions. In particular, we choose an isothermal outer boundary condition and retain $T_N$ at a fixed value through the model timesteps. At the center, to close the system of equations we will apply a spatial average to the run of temperatures. Thus dropping the time level notation now we have

$$T_0 = 2T_1 - T_2$$

(3-111)

which merely states that the first grid cell temperature is assumed to be the average of the central value and the second grid cell temperature. Fortunately we have chosen our moveable grid in such a manner so as to yield the finest zoning at the center thus partially justifying this treatment.

Formally, we may cast the matrix system of equations arising from the finite difference treatment into the form

$$a_i T_{i-1} - b_i T_i + c_i T_{i+1} = d_i$$

(3-112)

$$i = 1, 2, \ldots, N-1$$

where the $a_i$, $b_i$, $c_i$, $d_i$ are known in terms of the appropriate sums of
the finite difference factors as displayed previously and the T's are to be found. Now, in the spirit of the tridiagonal nature of the matrix let us try the solution form

\[
T_i = \omega_i T_{i+1} + g_i \quad (3-113)
\]

\[
T_{i-1} = \omega_{i-1} T_i + g_{i-1} \quad . \quad (3-114)
\]

Substituting these expressions into equation (3-112) so as to eliminate \(T_{i-1}\) yields the condition

\[
T_i = \frac{c_i}{b_i - a_i \omega_{i-1}} T_{i+1} + \frac{d_i + a_i g_{i-1}}{b_i - a_i \omega_{i-1}} . \quad (3-115)
\]

Direct comparison of (3-113) and (3-115) immediately gives

\[
\omega_i = \frac{c_i}{b_i - a_i \omega_{i-1}} \quad (3-116a)
\]

\[
g_i = \frac{d_i + a_i g_{i-1}}{b_i - a_i \omega_{i-1}} \quad (3-116b)
\]

and we see that we can recursively build up the vectors \(\omega\) and \(g\) if their first elements \(\omega_0\) and \(g_0\) are known. But if we let \(i = 0\) in (3-113) then

\[
T_0 = \omega_0 T_1 + g_0 = 2T_1 - T_2 \quad \text{where the second equality comes from condition (3-111). Thus, } \omega_0 = 2 \quad \text{and we can immediately build our } \omega \text{ vector from equation (3-116a). However we also note that } g_0 = -T_2 \text{ which is an unknown quantity. To eliminate } T_2 \text{ one must apply the outer boundary}
condition. This points out the global nature of the energy equation/radiative transfer problem. Our outer boundary condition is that $T_N = T = \text{constant}$.

Suppose we define the notational replacement

$$g_i = \frac{d_i + a_i g_{i-1}}{b_i - a_i \omega_{i-1}} = \xi_i + n_i g_{i-1} \quad . \quad (3-117)$$

Then we can back substitute within (3-117) for $g_i$ in terms of the earlier vector components $g_j$ ($j<i$) and ultimately cast everything in terms of the $\xi$ values and $n$ values. Upon doing so it may be found that the value for $g_i$ is given by

$$g_i = \xi_i + n_i \xi_{i-1} + n_i n_{i-1} \xi_{i-2} + n_i n_{i-1} n_{i-1-2} \xi_{i-3} + \ldots + n_i n_{i-1} n_{i-2} \xi_i - n_i n_{i-1} n_{i-2} n_i T_2 \quad . \quad (3-118)$$

Similarly we may recursively build up an expression for the $i$-th grid cell temperature in terms of the $\omega$ and $g$ vectors through use of equation (3-113). The result is

$$T_i = \omega_i \omega_{i+1} \omega_{i+2} \omega_{i+3} \omega_{i+4} \ldots \omega_{N-2} \omega_{N-1} \text{T}$$

$$+ \omega_i \omega_{i+1} \omega_{i+2} \omega_{i+3} \omega_{i+4} \ldots \omega_{N-2} g_{N-1}$$

$$+ \omega_i \omega_{i+1} \omega_{i+2} \omega_{i+3} \omega_{i+4} \ldots \omega_{N-3} g_{N-2} + \ldots$$

$$+ \omega_i g_{i+1} + \omega_i g_{i+1} + g_i \quad . \quad (3-119)$$
It is now apparent that we can close our g vector system since we have an analytic expression for the i-th component in terms of totally known quantities $a_j$, $b_j$, $d_j$, and $T_2$ which in turn can be expressed itself through (3-119) in terms of the known $\omega$ components and the g components. The latter however are again given by (3-118). Thus, combination of (3-118) and (3-119) ultimately permits one to solve for $T_2$ in terms of sums and products of the $\xi$ and $\eta$ values along with the $\omega$ values with the only other factor appearing in the expression being the fixed outer boundary temperature $T$. Thus all these quantities are known in terms of the matrix coefficients $a$ through $d$ so we can determine $T_2$. Once $T_2$ is known then $g_0$ is known and we may recursively build the g vector through equation (3-116b). As the algebra and the explicit expression for $T_2$ are relatively messy we do not reproduce them here. We do note however that considerable care must be exercised in building up the sums involved in the expression for $T_2$ on any computer with inherent truncation error. Indeed, in the hydrodynamical program developed, the sum contributions are initially sorted in order of magnitude and accumulated from the smallest to the largest values so as to minimize introducing numerical errors. Generally, the terms contributing to the sums vary considerably due to the large variation of physical conditions throughout the model's computational grid during a protostellar calculation.

While the above algorithm formally solves the two point boundary value problem embodied in the energy equation and also serves with slight modifications to solve the Poisson equation of Section 3.9 (compare equation (3-71) for the gravitational potential coefficients
with equation (3-112) for the temperatures), at times problems can arise. This is particularly the case with strong shock fronts that arise during the later phases of evolution and when strong bounces occur. One very serious problem can arise if the shock front sets up during the middle range of the calculations at a spatial location close to the first two grid cells. Then the extrapolation of (3-111) for the central temperature can lead to a spuriously small central temperature because $T_2$ is much larger than $T_1$ and the run of interior temperatures is simply not adequately represented by a linear approximation. In such a case a rather better central temperature approximation is to use the ratio of central densities (new time versus old time) to proportionally extrapolate the central value and close the system of equations. However this alters the $\omega$ and $g$ vectors since $\omega_0$ and $g_0$ take on new values if such an alternate approach is applied. Nevertheless, in direct analogy to the treatment applied previously, one can recursively build up the modified vectors consistently with this approximation. Another even more serious problem can arise if a large enough timestep is taken so as to permit any cell within the computational grid to become completely drained of energy (too efficient advective cooling). Then the previously outlined tridiagonal approach becomes unphysical and yields negative temperatures! The solution to this dilemma is generally to take smaller size timesteps.

Typically during the calculations it was found that the tridiagonal algorithm worked well. However, to address the above problems as well as to provide a computational check on the accuracy of the solution, two
alternative solution methods were also applied at times. The first was a conventional Gaussian elimination with partial pivoting so as to check the accuracy of the recursive tridiagonal method if desired. The second was a Broyden (quasi Newton) algorithm appropriate for massive protostellar envelopes in which a constant luminosity versus spatial location can be assumed. In other words, the contributions due to shocks cannot be significant within the region being treated in this manner. Basically, the Broyden algorithm consists of computing the system's function vector and system's Jacobian matrix by using the old temperatures (time level n) as an initial guess for the temperatures at the new time level n+1. The matrix is inverted and a test is performed on the solution convergence based on the maximum percent difference between the old temperatures guess and the new values. A tolerance level is selected by the user and the temperature values are iterated until convergence is achieved to within the desired tolerance or until some maximum number of iterations are performed in which case a warning is output along with the best temperature values found.

Thus, during a given protostellar model calculation as many as four different recovery methods were available if the primary tridiagonal approach failed for some reason. The first was to try the Gaussian elimination. The second was to use the modified central temperature treatment and reapply the tridiagonal algorithm. The third was to apply the Broyden algorithm if shocks were noted to be absent or not making appreciable contributions to the luminosity. The fourth was to cut the timestep by a factor of ten recursively until the conventional method
succeeded. Normally, the tridiagonal algorithm achieved a solution on the first try. When it did not, the ultimate resolution of the problem generally lay in cutting the timestep size. When this failed usually the problem was due to a shock adjacent to the first two grid cells with the central linear extrapolation yielding spuriously low central temperature values and thus propagating throughout the interior to middle part of the grid (global solution!). Switching to the alternative central temperature extrapolation temporarily until the grid could flow inward sufficiently to circumvent this situation proved the key approach.

SECTION 3.13 SUMMARY OF CLASSICAL PROGRAM LOGIC

With the temperature solution now treated, we have considered all the salient aspects of the supermassive protostellar problem at a classical hydrodynamic level. We therefore pause a moment to summarize the practical implementation of the previous sections of this chapter within the hydrodynamic computer code developed to treat the SMS models.

Initially the code sets the conditions for a new model sequence or for continuing a previously calculated sequence. By this we mean that the variable values are initialized including densities, temperatures, velocities, compositions, grid velocities, etc. Next, the Poisson equation is solved via the tridiagonal method to obtain the run of gravitational potential energy throughout the computational grid. Then, the program examines the grid spacings, velocities, and local sound speeds so as to compute the maximum allowed timestep. This would be the CFL step of equation (3-75) except that a user-definable multiplicative factor is present thereby allowing the user to further cur the stepsize
based on past experience with the protostellar models so as to ensure solution stability. Recall that the CFL stepsize is a necessary but not necessarily sufficient condition for solution stability. Then the optimized radial grid is moved so as to follow the current flow as closely as possible. In particular, the position of the first grid point is adjusted to follow the local flow there and equation (3-66) is solved thereby providing the parameter C which permits the rest of the grid positions to be calculated according to condition (3-60). This in turn permits the run of grid velocities to be computed. Next, the continuity equation is solved via the finite difference formalism of equation (3-83) for the run of grid densities. Then we are in a position to explicitly obtain the gas flow velocities at this new time via equation (3-89) since the densities required therein have been obtained. Next, the energy equation is solved via the implicit tridiagonal method of the previous section for the run of temperatures throughout the model. If the primary method fails then the backup methods are attempted until success is achieved, perchance at the expense of cutting the stepsize somewhat and thereby necessitating the recomputation of the gas densities and velocities over the smaller step. At this point solution accuracy for the temperatures can also be monitored by recourse to the backup methods as well if the primary method succeeded. Given the temperatures we are now in a position to update the pressures and specific internal energies via the equation of state and via the analytic approximation for \( E \) from Sections 3.5 and 3.6. At this point we have all the thermodynamic and hydrodynamic parameters of interest computed at the new time. But as a check on the solution accuracy afforded by the donor cell approach
the program computes the level of global mass conservation through summing up all the cell contributions. Finally, the program prints the table of variable values at the current time if desired by the user including a notation as to which solution algorithm was applied to the energy equation with success (if multiple iterations were required), the current time, stepsize, the stepsize's adjustment factor compared to the CFL step, and a variety of parameters indicating program status such as the level of conservation attained at the current step. Then one is ready to solve the Poisson equation again and run through the sequence if another timestep is required for additional model evolution.

SECTION 3.14 TEST CASES VISITED

A truly critical aspect of computational hydrodynamics throughout the sciences as well as in engineering applications is to initially ascertain whether a given computer code can successfully reproduce existing literature results usually obtained via alternative algorithms. While this does not in itself ensure that the method has converged to the true solution of the governing partial differential equations since one is invariably solving a finite difference replacement to the PDEs, still it indicates internal consistency amongst the variety of FDE methods. It is thus the purpose of this section to outline some of the tests that have been applied to verify the accuracy of the donor cell code.

One particularly valuable test case is that of a pressureless collapse in which the pressure terms arising in the momentum equation are artificially zeroed and only the continuity and momentum equations are
solved. The energy equation is unnecessary since the pressure does not appear in the momentum equation. The prime value of this test lies in the fact that the pressureless collapse is solvable in terms of analytic functions. In particular, Hunter (1962) has demonstrated that the solution for a collapsing pressureless sphere of finite radius $R$ and initially in a stationary state is given via

\begin{align}
\rho &= \rho_0 \sec^6(\bar{\theta}) \\
v &= -2r \frac{d\bar{\theta}}{dt} \tan(\bar{\theta}) \\
R &= R_o \cos^2(\bar{\theta})
\end{align}

if the physical time $t$ is related to the parameter $\bar{\theta}$ by

$$\bar{\theta} + \frac{1}{8} \sin(2\bar{\theta}) = t \sqrt{8\pi G \rho_0 / 3}$$

and $\rho_0$ and $R_o$ represent the initial cloud density (uniform) and the initial cloud radius. Thus, a pressureless collapse is characterized by a constant density throughout the cloud at any given epoch during the infall. Also, the cloud radius shrinks. However, in our hybrid Eulerian donor cell algorithm we have adopted a fixed outer cloud boundary which is inconsistent with condition (3-122). Thus, to apply the pressureless case as a test one must adjust the treatment of the outer boundary density and velocity so as to be consistent with condition (3-122). In his low mass donor cell development Boss (1979) specifies the alterations required (his equation (5-3)) and so, for the purposes of this test only, we followed suit. In particular, a pressureless collapse for $10^{-15} \text{g/cm}^3$
as the initial density was followed for over half the free fall time as defined in equation (2-19). For the same timestep size as will be applied to the protostellar models, $0.1\Delta t_{\text{CFL}}$, the numerical results were found to agree throughout the model grid to within better than 2.4% of Hunter's analytic predictions. Boss (1979) found a similar situation with a coarser grid when he obtained agreement better than 5%. This test is very sensitive to the ability of the donor cell method to accurately transport mass. If any region within the grid has mass numerically transported too quickly or too slowly then the corresponding numerical free fall timescale in that region will be artificially too fast or too slow respectively compared to the true value. Thus, the numerical error in the density would progressively worsen as the model's evolution was continued.

Another useful test case is provided by the results of Bodenheimer and Sweigart (1968) who treated isothermal collapse of protostellar clouds. In particular, one model was a 10 solar mass atomic hydrogen cloud possessing an initial density of $10^{-15}\text{g/cm}^3$, an initial temperature of 100$^\circ\text{K}$, a zero initial velocity distribution, and a constant volume boundary condition. Since Boss (1979) applied his low mass protostellar code to this test case we do so also. Figures 4 and 5 provide the numerical results obtained after one free fall time had elapsed. The solid curves correspond to the results of Bodenheimer and Sweigart (1968). The dots correspond to the results of Boss (1979). The x's correspond to the results obtained via the SMS donor cell code (classical level only). We observe that both donor cell treatments while largely mutually consistent tend to systematically differ from the earlier results. This
Figure 4

Log(Density) Versus Radius for a Ten Solar Mass Protostellar Test Cloud
Figure 5

Log(-v) Versus Radius for a Ten Solar Mass Protostellar Test Cloud
may be understood when it is recognized that Bodenheimer and Sweigart's
treatment was a Lagrangian approach. Formation of a central density
plateau simply reflects the initial lack of pressure gradients (at least
until the rarefaction wave set up by the outer boundary condition hits
the plateau) and the analogy to a pressureless collapse. A Lagrangian
code is intrinsically better at handling sharp edges and boundaries due
to its mass element following nature. Thus, the donor cell algorithms
tend to deviate somewhat from the Lagrangian's result of a fairly sharp
boundary to the central density plateau due to its inherently Eulerian
nature. However, the agreement with Boss' results in the density profile
is gratifying. Moreover, we observe that the velocity profiles also
agree rather well between the donor cell approaches although showing some
systematic differences with the Lagrangian method. Indeed, the agreement
of the SMS donor cell computed velocity in the outer regions of the
protostellar cloud to Bodenheimer and Sweigart's velocities is actually
marginally better than Boss' agreement with the earlier values.

Finally, we shall mention one more amongst the variety of test
cases applied to the SMS code. This is the collapse (nonisothermal) of
a one solar mass cloud. Such a configuration was treated in a semi-
analytic fashion using the polytropic approximation by Hayashi and
Nakano (1965); Bodenheimer (1968), Winkler (1978), and Boss (1979) have
considered the structure from a numerical hydrodynamical standpoint. It
therefore affords an excellent test case for the hybrid Eulerian code
we have outlined here. The SMS code was applied with the same number of
grid points and the same stepsize, $0.5\Delta_{CFL}$, as used by Boss (1979).
Upon comparing the evolution of central temperature as a function of the central density, the behavior was noted to agree excellently with Figure 5-10 of Boss (1979). In particular, curve d of that figure does a superb job of representing not only Boss' results but those provided by the SMS code as well. Moreover, we observe that while at higher densities curve d is systematically above the curves representing the other workers previously cited, this occurs because of the neglect of the formation of H− ions and the neglect of ionization of elements heavier than H, particularly He. Thus the computed pressure is lower than the actual value and the models tend to go to higher temperatures so as to achieve the proper pressure values. Nevertheless, the qualitative trends are the same as those found by the previous workers. Moreover, Winkler's treatment represents a highly sophisticated thermodynamic treatment beyond the accuracy levels of the other workers. It is quite pleasant to note that amongst all the curves in that figure, except for Boss' curve which coincides with ours, Winkler's curve is in the closest agreement with our curve. At the final phase of the computations the donor cell algorithm with the tridiagonal implicit energy solution gives a central temperature of nominally $9 \times 10^{4}$°K whereas Winkler finds $6 \times 10^{4}$°K. However, the protostellar core radii computed by the donor cell algorithm and by Winkler's approach both agree at three solar radii. Hence, although the pressure law introduces numerical errors at high temperatures, at lower temperatures it agrees well with the more intricate Winkler (1978) formalism and moreover the physical dimensions of the ultimate structure are in agreement as well.
In Chapter 1 we encountered an intriguing duality within the supermassive star context. While the structural properties are set at the classical Newtonian level, the ultimate stability properties of the configuration are determined at the post-Newtonian level. Hence it is not enough to treat the collapsing supermassive protostellar cloud via the approach of the previous chapter. One must also at least approximately take into account General Relativity. Appenzeller (1972a,b) has done so in the SMS context in the style of a conventional multidimensional Newton Raphson approach as has Reynolds (1979). These formulations parallel the standard stellar interiors techniques and as such do not directly tie in with the hybrid Eulerian hydrodynamic formalism we have adopted for the classical level treatment. It therefore remains to adopt a post-Newtonian formalism that parallels as closely as possible the classical formalism.

Indeed, a variety of treatments have been developed for considering hydrodynamics at the post-Newtonian approximation level. Overviews are provided by Weinberg (1972) and Misner, Thorne, and Wheeler (1973). Furthermore, practical computational properties at this level are given in Centrella, LeBlanc, and Bowers (1985). A particularly attractive development is that of Chandrasekhar (1965). Of course, all the analyses
ultimately reduce to computing the same properties but as in the classical treatments, subtle differences exist in their practical numerical implementation. It is fortunate that in the SMS protostellar case, the dynamics are strongly affected only near the onset of the relativistic instability so only such a first order treatment is required. From an implementation standpoint it will prove most useful to adopt the development of Misner, Thorne, and Wheeler (1973). Upon doing so we shall see in the next sections that it is possible to ultimately cast such a treatment into a formulation that directly extends that of the previous chapter. In other words, all the power of the donor cell algorithm may ultimately be brought to bear upon the problem at the relativistic level as well!

SECTION 4.1 THE FUNDAMENTAL EQUATIONS OF POST-NEWTONIAN HYDRODYNAMICS

We therefore begin with the generalized Parameterized Post-Newtonian (PPN) Equations quoted by Misner, Thorne, and Wheeler (1973) page 1088:

\begin{align}
\rho \frac{\partial t}{\partial t} + (\rho v_j)^j = 0 + O(\rho, \epsilon) \\
\rho \frac{\partial n}{\partial t} + t_{jk} v_j, k = 0 + O(\rho, \epsilon) \\
\rho \frac{\partial v_j}{\partial t} - \rho u_j + [t_{jk}(1 + 3\gamma u)]_k - t_{jk,k}(\gamma v^2 + \Pi) \\
- \frac{t_{jk} t_{kl,e}}{\rho^*} + \frac{d}{dt} [(2\gamma + 2)U v_j - \frac{1}{2}(7\Delta_1 + \Delta_2) v_j +]
\end{align}
Here $\frac{d}{dt}$ is the time derivative with the matter, namely

$$
\frac{d}{dt} = \frac{\partial}{\partial t} + v_k \frac{\partial}{\partial x_k}
$$

while the commas imply partial derivatives with respect to the stated components. Specifically, these equations are generalized in the sense that they arise from an analysis at the post-Newtonian level for the range of competing relativistic theories. Equation (4-1) is the post-Newtonian analogue of the classical continuity equation in terms of the relativistic variable $\rho^*$. Equation (4-2) is the post-Newtonian form of the classical energy conservation equation where $\Pi$ is the internal energy density per unit baryon mass density. Finally equation (4-3) is the relativistic analogue of the classical momentum equation. It is
trivially apparent that most of the complexity arising at the post-Newtonian level is expressed in the form of this relationship. However, if we specialize to General Relativity then the parameters take on the values

\[
\begin{align*}
\gamma &= \beta_1 = \beta_2 = \beta_3 = \beta_4 = \Delta_1 = \Delta_2 = 1 \\
\zeta &= \eta = \alpha_1 = \alpha_2 = \alpha_3 = 0
\end{align*}
\]

and considerable simplification obviously arises. Furthermore, the $t_{jk}$ are the stress components taken relative to the matter rest frame. For simplification we ignore shears in the stress and take

\[
t_{ij} = P \delta_{ij}
\]

where $P$ is the isotropic pressure. Also, $v^i_j$ is the coordinate velocity of the matter frame relative to the PPN coordinates with the latter given by $\bar{w}$ relative to some universal rest frame. With the $\alpha$ values zeroed for GR then only $v$ survives. Finally we observe that the notation applied here is precisely as in Misner et al. (1973) so the units used are "geometrized units".

Turning now to each of the post-Newtonian expressions we see that the baryon conservation equation (4-1) is simply

\[
\frac{\partial \rho^*}{\partial t} + \nabla \cdot (\rho^* \bar{v}) = 0
\]

where the relativistic density is given via the form

\[
\rho^* \equiv \rho_o (1 + \frac{1}{2}v^2 + 3\gamma U) = \rho_o (1 + \frac{1}{2}v^2 + 3U)
\]
with \( \rho_0 \) of course being the rest mass density and through our choice of units the speed of light, \( c \), is unity.

The energy conservation equation, with energy flow through the matter neglected consistent with the PPN treatment, becomes

\[
\rho_0 \frac{d\Pi}{dt} + \rho_0 \delta_{jk} \frac{\partial}{\partial x_k} v_j = 0 \quad (4-10)
\]

where the internal energy density per unit baryon-mass density is simply given by

\[
\Pi = \frac{\rho - \rho_0}{\rho_0} \quad (4-11)
\]

Or in more compact notation we have the energy expression

\[
\rho_0 \frac{\partial \Pi}{\partial t} + \rho_0 v \cdot \nabla \Pi + \nabla \cdot \Pi = 0 \quad (4-12)
\]

Turning to the post-Newtonian momentum equation and applying the parameters of equation (4-5) and equation (4-6), as well as adopting the isotropic pressure form embodied in equation (4-7), permits us to rewrite (4-3) as

\[
\rho \frac{d\v}{dt} - \rho \frac{\partial U}{\partial x_j} + \frac{\partial}{\partial x_k} [P \delta_{jk} (1 + 3U)]
\]

\[
- (\delta_{jk} \frac{\partial P}{\partial x_k}) (\frac{1}{2} v^2 + \Pi) - \frac{P \delta_{jk}}{\rho} \delta_{k\ell} \frac{\partial P}{\partial x_{\ell}}
\]

\[
4\rho \frac{d}{dt} (U v_j - v_j) - v_j \rho \frac{\partial v}{\partial t} + v_k \delta_{kj} \frac{\partial P}{\partial t} +
\]
Or, equivalently upon simplifying (4-13) as well as placing it into the more compact notation we obtain for the post-Newtonian Euler equation the expression

\[ \frac{d}{dt} \rho * U - \frac{\rho}{\rho} U + \frac{3}{2} \rho U^2 + 3 \rho (\nabla U) \mathbf{F} - (\nabla \mathbf{F}) (\nabla U^2 + 2) - \frac{\rho}{\rho} \mathbf{F}^2 \]

\[ + 4 \rho \frac{3}{dt} (\mathbf{U} - \mathbf{V}) \]

\[ + 4 \rho \sum_{i=1}^{3} \frac{3}{4} \rho \frac{\partial \mathbf{F}}{\partial \mathbf{x}_i} \mathbf{V}_i - 2 \rho \mathbf{F} \cdot \mathbf{V} - \rho (\nabla U^2 + 3 \rho \mathbf{F}) \mathbf{V} \]

\[ = 0 \]  

(4-14)

with the notation as previously defined.

SECTION 4.2 SPECIALIZATION TO SUPERMASSIVE STRUCTURES

Equations (4-8), (4-12), and (4-14) represent the system to be solved at the post-Newtonian level for a supermassive structure. However, we may cast them into an even more transparent form appropriate to the SMS problem upon recalling that in our case spherical symmetry is always present and only radial and time derivatives survive. Thus, for example the post-Newtonian momentum equation becomes formally
Thus, upon rearranging and simplifying the above expression as well as casting the remaining post-Newtonian equations into the similar notation we have the following system to solve for our SMS models:

\[
\begin{align*}
\rho \frac{d\mathbf{v}}{dt} - \rho \frac{\partial \mathbf{U}}{\partial t} + \frac{\partial P}{\partial t} + 3 \frac{\partial}{\partial \mathbf{r}}(P\mathbf{U}) - (\mathbf{v}^2 + \Pi) \frac{\partial \mathbf{P}}{\partial t} \\
- \frac{P}{\rho^*} \frac{\partial P}{\partial \mathbf{r}} + 4\rho \frac{\partial}{\partial t}(\mathbf{Uv} - \mathbf{V}_r) - \nu \frac{\partial \mathbf{U}}{\partial t} + \nu \frac{\partial \mathbf{P}}{\partial t} \\
+ \frac{1}{2} \rho \frac{\partial}{\partial t}(\mathbf{V}_r - \mathbf{V}_r) + 4\rho \mathbf{v} \frac{\partial \mathbf{V}_r}{\partial \mathbf{r}} - 2\rho \frac{\partial \mathbf{w}}{\partial \mathbf{r}} \\
- \rho [v^2 + (3P/\rho^*)] \frac{\partial \mathbf{w}}{\partial \mathbf{r}} = 0
\end{align*}
\]

(4-15)

Thus, upon rearranging and simplifying the above expression as well as casting the remaining post-Newtonian equations into the similar notation we have the following system to solve for our SMS models:

\[
\begin{align*}
\frac{\partial \rho^*}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial \mathbf{r}}(r^2 \rho^* \mathbf{v}) = 0 \\
(4-16)
\end{align*}
\]

\[
\rho_o \frac{\partial \Pi}{\partial t} + \rho_o \mathbf{v} \frac{\partial \Pi}{\partial \mathbf{r}} + \frac{P}{r^2} \frac{\partial}{\partial \mathbf{r}}(r^2 \mathbf{v}) = 0 \\
(4-17)
\]

\[
\begin{align*}
\rho \frac{\partial \mathbf{v}}{\partial t} + \rho \frac{\partial}{\partial \mathbf{r}}(4\mathbf{Uv} - 4\mathbf{V}_r) - \nu \frac{\partial \mathbf{U}}{\partial t} + \nu \frac{\partial \mathbf{P}}{\partial t} + \rho \frac{\partial}{\partial \mathbf{r}}[h(V_r - V_r)] \\
+ \rho \mathbf{v} \frac{\partial \mathbf{v}}{\partial \mathbf{r}} - \rho \frac{\partial \mathbf{U}}{\partial \mathbf{r}} + \frac{\partial P}{\partial \mathbf{r}} + 3 \frac{\partial}{\partial \mathbf{r}}(P\mathbf{U}) - (\mathbf{v}^2 + \Pi) \frac{\partial \mathbf{P}}{\partial \mathbf{r}} \\
- \frac{P}{\rho^*} \frac{\partial P}{\partial \mathbf{r}} + 4\rho \mathbf{v} \frac{\partial}{\partial \mathbf{r}}(\mathbf{Uv} - \mathbf{V}_r) + 4\rho \mathbf{v} \frac{\partial \mathbf{V}_r}{\partial \mathbf{r}} - 2\rho \frac{\partial \mathbf{w}}{\partial \mathbf{r}} \\
- [v^2 + (3P/\rho^*)] \frac{\partial \mathbf{w}}{\partial \mathbf{r}} = 0
\end{align*}
\]

(4-18)

In particular, note the advantage of equation (4-16) over Chandrasekhar's (1965) equation in terms of his alternative variable \( \sigma \). The advantage lies in the fact that the above form is a conservation form which, when
used with a conservative numerical hydrodynamical solution scheme, allows one to ensure $\rho^*\text{ conservation}$. Chandrasekhar (1965) also does include the general form of (4-16) in terms of $\rho^*$ but chooses to develop the post-Newtonian Euler equation in terms of $\sigma$.

We may now also specify the auxiliary parameters remaining within the post-Newtonian level equations. First of all we observe that tied up within equation (4-18) are spatial and time derivatives of the variables $U$, $V$, $W$, and $\psi$. Indeed, in their absence and in the limit of $c^2$ large then (4-18) reduces to the classical equation of Newtonian hydrodynamics of the previous chapter as one should expect. These auxiliary quantities are spatially and time dependent gravitational potentials. In particular, $U(x,t)$ is the ordinary Newtonian potential $\phi$ except with a change of the leading algebraic sign. Namely,

\begin{equation}
U(x,t) = \int \frac{\rho_0(x',t)}{|x - x'|} \, d^3x' \tag{4-19}
\end{equation}

\begin{equation}
V_r(x,t) = \int \frac{\rho_0(x',t) \, v(x',t)}{|x - x'|} \, d^3x' \tag{4-20}
\end{equation}

\begin{equation}
W_r(x,t) = \int \frac{\rho_0(x',t) [(x-x') \cdot v(x',t)] (x - x'_r)}{|x - x'|^3} \, d^3x' \tag{4-21}
\end{equation}

\begin{equation}
\psi(x,t) = \int \frac{\rho_0(x',t)}{|x - x'|} \, (v^2 + U + \frac{1}{2} \Pi + \frac{3}{2} \frac{P}{\rho_0}) \, d^3x' \tag{4-22}
\end{equation}

where it is apparent that the post-Newtonian gravitational potentials are simply functions weighted by the appropriate velocity or energy. Recall that we are working here in geometrized units so that conversion
to "physical" units is achieved by multiplying through by the appropriate "unity" factor(s) as specified by Misner, Thorne, and Wheeler (1973) page 36. Moreover, an additional simplification arises between $V_r$ and $W_r$ for our spherically symmetric case. Comparison between (4-20) and (4-21) reveals that for our case

$$W_r = V_r = \int \frac{\rho \rho \nu}{|x - r|^2} \nu 4\pi r^2 \, dr'$$  \hspace{1cm} (4-23)

SECTION 4.3 RELATIVISTIC APPLICATION OF THE DONOR CELL ALGORITHM

We may now cast the fundamental post-Newtonian equations into the donor cell framework. In particular, recognizing that the spirit of the donor cell treatment is a numerically conservative one in mass and momentum, we require an initial form of the equations in conservation mode. As previously noted, the relativistic continuity equation (4-16) is already in that form so upon introducing the numerically useful artifact of a moving grid (local velocity given by $v_g$) we have

$$\frac{\partial \rho^*}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial \rho} \left( r^2 \rho^* (v - v_g) \right) = -\rho^* \frac{1}{r^2} \frac{\partial}{\partial \rho} \left( r^2 v_g \right)$$  \hspace{1cm} (4-24)

which is the relativistic analogue of the classical level continuity equation (3-81). One may therefore apply precisely the same finite difference formalism with the only alteration being that the density is the relativistic density rather than the conventional one. The result is a finite difference relationship of exactly the same form as (3-83) but for $\rho^*$ rather than $\rho$ where
\[ \rho^* = \rho^o (1 + \frac{1}{2}v^2 + 3U) \]  

(4-25)

Thus we note immediately that solution of the continuity equation at the post-Newtonian level can proceed prior to considering the relativistic momentum equation. However, one cannot disentangle the classical density \( \rho^o \) from the resultant \( \rho^* \) yet because of the presence of the velocity within the form of \( \rho^* \). To do this we must examine the momentum equation.

Upon turning to the momentum equation we note first of all that it is not in the desired conservation form. However, upon strategically grouping the terms in (4-18) we may rewrite it as

\[
(1 + 4U) \left( \rho^* \frac{\partial v}{\partial t} + \rho^* v \frac{\partial v}{\partial x} \right) + v \frac{\partial \rho}{\partial t} 
+ \rho^* \left( 3v \frac{\partial U}{\partial t} - \frac{7}{2} \frac{\partial v}{\partial x} - \frac{1}{2} \frac{\partial v}{\partial t} \right) 
+ \rho^* (3v^2 - 1) \frac{\partial U}{\partial x} - 2 \rho^* \frac{\partial v}{\partial x} 
+ [1 + 3U - \frac{1}{2}v^2 - \Pi - (P/\rho^*)] \frac{\partial \rho}{\partial x} = 0 .
\]

(4-26)

We note that this is still not in conservation form of course but now apply the identity

\[
\rho^* \frac{\partial v}{\partial t} = \frac{\partial (\rho^* v)}{\partial t} - v \frac{\partial \rho^*}{\partial t} = \frac{\partial (\rho^* v)}{\partial t} + v \cdot (\rho^* \nabla v)
\]

(4-27)

where the second equality stems from equation (4-8). Also apply the fact
that

\[ \rho \mathbf{v} \frac{\partial \mathbf{v}}{\partial r} = \rho \mathbf{v} \cdot \nabla \mathbf{v} = \nabla \cdot (\rho \mathbf{v} \mathbf{v}) - \mathbf{v} \cdot \nabla \cdot (\rho \mathbf{v}) \]  (4-28)

Combination of (4-27) and (4-28) permits us to substitute into the second factor of the first term of equation (4-26) and formally thus cast it into conservation form. If we also correct for a numerically nonuniformly moving computational grid in the same fashion as before then the relativistic Euler equation becomes

\[
\frac{3(\rho \mathbf{v})}{\partial t} = -\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \rho \mathbf{v} \mathbf{v} \mathbf{v} \right) - \rho \mathbf{v} \frac{\partial}{\partial r} \left( \frac{3}{2} \mathbf{v} \mathbf{v} \right)
- \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 Q \mathbf{v} \mathbf{v} \mathbf{v}) - \frac{\mathbf{v}}{(1 + 4U)} \frac{\partial P}{\partial r}
- \frac{\rho}{(1 + 4U)} \left( 3 \mathbf{v} \frac{\partial U}{\partial t} - 4 \mathbf{v} \frac{\partial U}{\partial r} \right)
- \frac{\rho}{(1 + 4U)} \left( (3U^2 - 1) \frac{\partial U}{\partial r} - 2 \mathbf{v} \frac{\partial U}{\partial r} \right)
- \frac{1}{(1 + 4U)} \left[ 1 + 3U - 4U^2 - \pi - (P/\rho) \right] \frac{\partial P}{\partial r}
= 0 \]  (4-29)

Note that this directly parallels the classical level momentum equation (3-82) except for the extra post-Newtonian terms such as the spatial and time derivatives of the post-Newtonian gravitational potentials. Now that the equation is in conservation form, we observe that some provision
must be made for evaluating the time derivatives of the quantities \( U, V_r, \) and \( P \) in (4-29). Formally at least, it would appear that the solution could not proceed since these factors in turn depend upon \( \rho_0 \) and \( v \) which themselves are unknown at this point. Our solution for \( \rho^* \) does not yet allow determination of the rest mass densities because \( v \) is present within \( \rho^* \) and \( v \) must in turn be obtained from the momentum equation. Thus, it would seem we are at an impasse. However, the spirit of the post-Newtonian formulations previously cited at the beginning of this chapter is to use the values obtained at the current new time level from the classical Newtonian treatment as approximations to the correct values. Thus the solution algorithm of the previous chapter has indeed provided us with estimates of the correct \( \rho_0 \) and \( v \). These in turn are substituted into equations (4-19, 20, and 22) to approximate the post-Newtonian potentials at this new time. Thus, a simple difference approach between these values and the values from the previous time yields the required approximation to the time derivatives of the potentials in (4-29). Moreover, we have a pressure estimate as well from the classical treatment so it can be handled in exactly the same fashion. Thus, solution of the relativistic momentum equation can proceed.

We now turn briefly to the donor cell finite difference replacement for equation (4-29). From a practical standpoint, we can compute the potential values via the same algorithm of Section 3.9 except by weighting the integrands with the corresponding weight functions embodied in (4-19 through 4-22). To approximate the left side we apply the same forward time difference as in (3-89). The advective term (leading term
on the right side of (4-29) is differenced with precisely the same expression as the second term in (3-89). The only formal difference is that $X \rightarrow \rho v^*$ instead of $\rho_o v$. The rules for choosing the appropriate grid cell subscripts on the $X$ values are the same as conditions (3-85) and (3-86). Next comes the term correcting for the effect of the grid motion being nonuniform, volumetric effect. Once more the treatment matches the classical approach with the replacement expression matching the first term on the right hand side of equation (3-89) again with the relativistic density involved rather than simply the rest density. The third term on the right side of (4-29) arises from incorporation of artificial viscosity at the post-Newtonian level. Its finite difference replacement exactly equals the last term on the right side of equation (3-89). Furthermore, the rules for computing the $Q$ values are again embodied in (3-95a and b).

However, the remaining terms in equation (4-29) involve, or arise from, the PPN analysis and not the classical treatment. Fortunately, however, their replacements are comparatively straightforward. All time derivatives on the right side of (4-29) are replaced with forward time differences using the Newtonian values computed for the new time as approximations to the correct values as per the discussion on the preceding page. The spatial derivatives on the right side may be handled in a fashion directly paralleling the classical approach. Recall that the classical potential and pressure derivatives were replaced with a specialized finite difference formalism of high accuracy for a nonuniform grid spacing. As a result the auxiliary quantities $S_p$, $S_H$, and $S_o$ arose.
Now, in the context of the relativistic momentum equation, we again apply those finite difference forms for the $U$, $\psi$, and $P$ spatial derivatives arising in the last two terms of the left side of (4-29). Specifically, let $Y = U$, $\psi$, or $P$. Then,

$$\frac{3Y}{2r} + S^p_{i+1}Y^{n+1} - S^o_{i}Y^{n+1} - S^p_{i-1}Y^{n+1} = 0 \quad (4-30)$$

where the superscripts refer to the time level and the subscripts on the $Y$ values represent the grid cell numbers. This is the approximation at the $i$-th cell for the partial derivative. Note that in keeping with the explicit nature of the donor cell algorithm the $Y$ values are defined in terms of their values at the previous time while the weighting quantities $S^p$, $S^o$, and $S^p_N$ are defined at the new time level. These in turn are given by equations (3-90, 91, and 92) as in the conventional Newtonian case. All nonconstant leading factors multiplying these derivatives in (4-29) are taken to be defined at the old time level $n$. Again, this is in keeping with an explicit hydrodynamical algorithm. Thus, the only unknowns in the momentum equation are the velocities which are solved for in terms of the values at the previous time level. As at the classical level, we impose the boundary conditions $v_o = 0$ and $v_N = 0$ (fixed outer boundary for a constant volume boundary condition).

Finally, it remains to consider the third of the equations of post-Newtonian hydrodynamics, the energy conservation relation (4-17). But we note that this relationship is the same as energy conservation at the Newtonian level so it suffices to solve again the classical energy equation via the treatment of Section 3.12. The only difference is that we
now apply the values of density and velocity as computed at the post-
Newtonian level through the previous finite difference solutions of
equations (4-24) and (4-29).

A final, albeit technical, choice is to select a criterion for
implementation of the post-Newtonian treatment versus treating the
collapse in a purely Newtonian manner. Such a criterion is very
conveniently afforded by the conventional Schwarzschild radius, $R_{\text{Sch}}$, where

$$R_{\text{Sch}} = \frac{\sqrt{2GM}}{c^2} \quad . \quad (4-31)$$

Previous workers such as Appenzeller and Reynolds have noted that at a
ratio of $R/R_{\text{Sch}} \approx 10^3$ the effects of General Relativity begin to become
significant. However, to provide a numerical safety margin so that a
sudden activation of the post-Newtonian treatment does not induce non-
physical, spurious results and so that we do not neglect it for too long
a period, we impose the condition that at

$$R/R_{\text{Sch}} \leq 10^5 \quad (4-32)$$

the post-Newtonian numerical analysis is activated by the computer code.
Indeed, a variety of tests indicated that this safety margin was more
than ample in providing agreement between collapse calculated with the
GR included versus without the GR included. Moreover, it was observed
that inclusion of the post-Newtonian analysis invariably led to slightly
enhanced velocities even at such very distended structural phases due to
the effectively larger gravitational "force" exerted through inclusion
of the overall "mass-energy" effect. This is precisely as one would
expect on an intuitive basis.

SECTION 4.4 SUMMARY OF GLOBAL PROGRAM LOGIC

Having solved the hydrodynamic problem now at both the classical Newtonian level as well as the relativistic post-Newtonian level, it is appropriate to summarize the overall approach taken by the supermassive protostellar FORTRAN code as ultimately developed and implemented. In this vein, Figure 6 provides a pseudo-flow chart exhibiting this global program logic.
SET CONDITIONS FOR NEW/CONTINUED MODEL SEQUENCE

- COMPUTE CURRENT GRID POTENTIALS
- DETERMINE MAXIMUM ALLOWED TIMESTEP
- MOVE OPTIMIZED RADIAL GRID
- OBTAIN $\rho_{\text{Newt}}(t + \Delta t)$
- OBTAIN $v_{\text{Newt}}(t + \Delta t)$
  - ISOTHERMAL MODEL?
    - Yes
    - OBTAIN $T_{\text{Newt}}(t + \Delta t)$
      - CONVERGENCE/ACCURACY TEST(S)
        - Pass
          - UPDATE $P$, $E$
        - Fail
          - ITERATE OR CUT STEP
    - No
      - CONVERGENCE/ACCURACY TEST(S)
        - Pass
          - UPDATE $P$, $E$
        - Fail
          - ITERATE OR CUT STEP
  - No
    - CHECK CONSERVATION BEHAVIOR
      - PASS
        - UPDATE $P$, $E$
      - Fail
        - ITERATE OR CUT STEP

DONE? → STOP

PRINT TABLE?

- No
  - NEED POST NEWT GR?
    - Yes
      - COMPUTE ESTIMATED PN POTENTIALS
        - NOW COMPUTE $(\rho^*, v)_{\text{PN}} + \rho$
          - OBTAIN $T$ VIA $(\rho, v)_{\text{PN}}$ AS BEFORE
    - No
      - PRINT TABLE?
        - Yes
          - UPDATE $P$, $E$
        - No
          - PRINT TABLE?
            - Yes
              - UPDATE $P$, $E$
            - No
              - PRINT TABLE?

Figure 6
Flowchart of Global Program Logic at the Newtonian and Post-Newtonian Levels
CHAPTER 5 - NUMERICAL SOLUTIONS AND RESULTS

We turn now to a consideration of some of the numerical results obtained with the FORTRAN protostellar hydrodynamic computer code via the methods of the previous two chapters. Given the large range of potential initial conditions one is best guided by the analysis of Chapter 2, especially Figure 2 to select initial values that meet the Jeans mass criterion. This effect was trivially observed numerically in models deliberately selected with too low an initial density. Moreover, this condition must be exceeded enough so that the condition imposed by curve A in Figure 2 is also approximately met. Recall that line A corresponds to the criterion that the free fall timescale not exceed the main sequence lifetime of the supermassive star, nominally $10^6$ years. Of course, that criterion is set assuming a pressureless collapse so that it should be overly stringent compared to a collapse that is actually slowed down by internal pressure effects. On the other hand, it is apparent intuitively that if one is willing to select extreme enough initial conditions, particularly an abnormally large value for the initial density of the cloud, then eventually an SMS would have to result due to the sheer dominance of gravitational potential energy compared to the cloud thermal energy tending to support against
collapse. While the purpose of this work is not merely an exercise in numerology, nevertheless a wide range of supermassive cloud models were computed through varying the initial density, temperature, and chemical composition. In particular, density values between $10^{-24}$ g/cm$^3$ and $10^{-12}$ g/cm$^3$ were tried. Temperature values between 10°K and 200°K were applied. Finally, the sensitivity toward the chemical composition was not observed to be particularly strong but the two standard chemical mixes of $(X,Y,Z) = (0.700, 0.299, 0.001)$ and $(0.600, 0.380, 0.020)$ were generally used. For the overall mass of the clouds a value of $10^6 M_\odot$ was normally selected although a few models of $10^5 M_\odot$ and $10^7 M_\odot$ were computed. In every case the boundary conditions imposed upon the numeric solutions were those of a fixed outer boundary (constant total volume), through which no material was allowed to flow (constant total mass), and a fixed outer temperature condition. Also, the clouds were taken to be initially of uniform density and of uniform temperature. Finally, to ensure solution stability, the computational timesteps applied were consistently $0.1\Delta t_{CFL}$ or smaller.

In particular, given the wide range of models computed, three specific ones serve well as representative cases of the hydrodynamical and thermodynamical behavior of the collapsing supermassive clouds studied here. We define these to be Models A, B, and C. Their corresponding initial conditions are presented in Tables 3, 4, and 5. It was observed throughout the range of supermassive models that the most critical initial parameter tending to dominate the subsequent evolution of the cloud was the ratio of thermal energy to gravitational energy.
Indeed, from an intuitive standpoint one would expect this to be the case. This ratio is given by the expression

\[
\frac{E_{\text{thermal}}}{E_{\text{grav}}} = \frac{\frac{5}{2} \frac{kT}{\mu H}}{\frac{GM}{R}} = \frac{5}{2} \frac{k}{\mu H} \frac{RT}{M} = 1.5 \times 10^{15} \frac{RT}{M}
\]

(5-1)

where \( H \) is the mass of hydrogen, \( M \) is the total mass, \( R \) is the total radius, \( \mu \) is the mean molecular weight, and \( T \) is the temperature. We define this ratio to be \( \alpha \). Nominal values for a conventional mass protostellar cloud are on the order of 0.5 (Larson 1968 and Boss 1979); however we note that for our models the denominator will generally be some \( 10^6 \) times bigger. In other words, we observe our first fundamental difference between low mass and supermassive protostellar cloud dynamics in that the supermassive clouds are all initially strongly gravitationally dominated compared to their thermal effects. Reference to Tables 3, 4, and 5 confirms this for our reference cases.

Model A was deliberately chosen to depict the behavior of an initially low temperature supermassive configuration. Such a structure was not expected to lead to a supermassive star due to the strongly nonhomologous collapse that was anticipated to occur in analogy with the low mass literature models. Pressure effects would not be able to stave off the collapse for long, ultimately leading to two different structural timescales: a core timescale and an envelope timescale with the former being much shorter than the latter. Indeed, this was observed numerically to be the case. After collapsing from rest, the supermassive cloud rapidly built up a supersonic infall. Recall that in the presence of
such low densities and pressures the sonic velocity will itself be rather small, on the order of $10^4$ to $10^5$ cm/second depending upon the specific pressure and density values. Thus, it proved relatively easy for the excess gravitational potential to cause the velocities to achieve this condition. This is also common in the low mass literature models.

Observed as well was the tendency to set up two density profiles; one corresponded to the envelope and one corresponded to the interior. Such behavior has also been observed in the low mass models and is well understandable from a mathematical as well as from an intuitive basis.

Consider the continuity equation of classical hydrodynamics, (3-31a).

Now, because the core regions are so much smaller than the envelope zone the code is constrained to take very small timesteps compared to those potentially available if only the envelope was treated. In turn, this implies that from a practical computational basis essentially only the core density profile is observed to change with time while the envelope density profile, after some initial alterations, hardly evolves at all. Thus, within the envelope we can effectively set the density derivative to zero and just consider the second term in (3-31a). In other words,

$$\frac{3}{3r} (r^2 \rho v) = 0 \quad (5-2)$$

to an excellent approximation within the envelope. If so then we see trivially that

$$r^2 \rho v = \text{constant} = \kappa' \quad (5-3)$$

But since the envelope infall velocity is constant spatially (at least
to within an order of magnitude or so), this implies an envelope density profile given by

$$\rho \propto \frac{1}{r^2}$$  \hspace{1cm} (5-4)

which is precisely the behavior of our supermassive cloud model as well as the conventional low mass models. From a more physical standpoint, it would appear that if the structure is initially nearly in hydrostatic equilibrium, certainly a better approximation for the lower mass clouds, then there is an early subsonic phase during which time the various regions of the cloud can sonically "communicate" with one another due to the fact that the sound travel time across the cloud is much shorter than the timescale required for extensive gravitational collapse. This in turn would tend to lead to smoothing out of the initial discrepancies between the cloud self-gravity and pressure gradient. Indeed, Shu (1977) has demonstrated that an asymptotic similarity solution to the collapse shows such a behavior apparently arising from such an approach toward detailed dynamical "balance" within the structure. On the other hand, in the core regions of the cloud in which one cannot make the time independent approximation for the density, the pressure will tend to resist the gravitational collapse. Thus, one should expect a somewhat shallower density profile as a function of radius. Indeed, Shu (1977) has demonstrated again through a formal asymptotic similarity solution that the profile should be proportional to $r^{-3/2}$. This behavior was also observed in our supermassive models, at least approximately. We do note, however, that the density profiles
tended to be somewhat steeper than the "nominal" slopes apparently due to the enhanced gravitational dominance over the pressure in the supermassive case compared to the classical mass range.

The general trend in Model A thus was for a steadily increasing central density, temperature, and pressure until the core regions overshoot the equilibrium values and the velocities immediately adjacent to the core become positive thereby implying an outflow. In other words, the core is observed to bounce. Such behavior has also been observed for the conventional mass models, Boss (1979) for example. In the case of Model A this bounce took place at a central density of around $7 \times 10^{-10}$ g/cm$^3$ and a central temperature above 400°K. The nonhomologous nature of the collapse is thus readily evident since envelope densities were at least $10^7$ times smaller. Thereafter, the central density and temperature dipped temporarily only to start growing again. In particular, Figure 7 depicts the density profile throughout the cloud after 0.9 pressureless free fall timescale had elapsed, approximately $5.9 \times 10^{12}$ seconds. At this point the central core had attained a density of $1.7 \times 10^{-7}$ g/cm$^3$ and a temperature of over 8400°K. The core mass was just over four solar masses and both the density and temperature were growing rapidly. It was apparent that a conventional mass star would be the ultimate result although it was impossible to follow the system's evolution to that point due to the excessively small timesteps required by the explicit hydrodynamical scheme. Nevertheless, the qualitative behavior tied in well with the lower mass models. In Figure 8 we show the velocity profile throughout the structure again at $0.9 t_{ff}$. The
Table 3

Initial Conditions for Model A

Total Mass = \(10^6 \, M_\odot\)

Total Radius = \(1.6808 \times 10^{19} \, \text{cm}\) (fixed)

Initial Density = \(10^{-19} \, \text{g/cm}^3\) (uniform)

Initial Temperature = \(10^6 \, \text{K}\) (uniform)

\(X = 0.700\)

\(Y = 0.299\)

\(Z = 0.001\)

Initial Ratio of Thermal to Gravitational Energy = \(\alpha = 0.00013\)

Initial Timestep Size = 0.1 \(\Delta t_{\text{CFL}}\)
Figure 7

Log(Density) Versus Log(Radius) for Model A at $0.9t_{ff}$
Figure 8

$\text{Sgn}(v) \log(\text{Abs}(v))$ Versus $\log(\text{Radius})$ for Model A at $0.9t_{\text{ff}}$
Figure 9

Log(Temperature) Versus Log(Radius) for Model A at 0.9 \( t_{ff} \)
zero outer boundary condition is readily noted as is the supersonic nature of the infall. Note also the order of magnitude constancy of the outer envelope infall velocity as invoked in the density profile treatment earlier. Finally, in Figure 9 we exhibit the run of temperature at this same time throughout the supermassive cloud. Observe particularly the steep inner temperature profile as well as the outer isothermal nature of the cloud at the original assumed temperature of 10°K.

Model B, initial conditions specified in Table 4, was also a $10^6 M_\odot$ cloud but with an initial temperature of 100°K. Thus, although the gravitational energy dominated the thermal energy again, it did so by approximately an order of magnitude less. The only other difference with Model A initially was in the choice of a more nearly Population I chemical mixture rather than the very low "metals" abundance adopted for Model A. Initially the behavior was similar to Model A in the development of a strongly supersonic infall. As before, bounces were observed as well. Both the central density and temperature grew steeply after a given bounce damped out. In Figure 10 we exhibit the run of density throughout the cloud after 193,729 years had elapsed from the initially uniform cloud. This corresponds to 0.92 $t_{ff}$. By this time the central density had achieved $3.26 \times 10^{-8}$ g/cm$^3$. The central temperature was $2.08 \times 10^3$°K. Figure 11 provides the run of velocity with radius. Again, a strongly supersonic infall is evident. Finally, Figure 12 depicts the variation of temperature with radial location. Obviously the outer regions are still isothermal with a strong temperature gradient near the core boundary. Thereafter, conditions changed rapidly. The central density and temperature quickly grew with time. We next consider the
Table 4

Initial Conditions for Model B

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<td>Total Radius</td>
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</tr>
<tr>
<td>Initial Density</td>
<td>$10^{-19} , \text{g/cm}^3$      (uniform)</td>
</tr>
<tr>
<td>Initial Temperature</td>
<td>$100^\circ \text{K}$             (uniform)</td>
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<td>Initial Timestep Size</td>
<td>$0.1 , \Delta t_{\text{CFL}}$</td>
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</table>
Figure 10

Log(Density) Versus Log(Radius) for Model B
Figure 11

\[ \text{Sgn}(v) \log(\text{Abs}(v)) \text{ Versus Log(Radius) for Model B} \]
Figure 12

Log(Temperature) Versus Log(Radius) for Model B
Log(Density) Versus Log(Radius) for Model B at Second Time

Figure 13
Figure 14

Sgn(v) log(Abs(v)) Versus Log(Radius) for Model B at Second Time
Figure 15

Log(Temperature) Versus Log(Radius) for Model B at Second Time
Figure 16
Log(Density) Versus Log(Radius) for Model B at Final Time
**Figure 17**

Sgn(v)Log(Abs(v)) Versus Log(Radius) for Model B at Final Time
Figure 18

Log(Temperature) Versus Log(Radius) for Model B at Final Time
physical situation for Model B after just less than a decade had elapsed. Figure 13 displays the density profile at that epoch. Note that the density in the zones immediately around the core ($r < 10^{14}$ cm) has definitely increased as has the slope of the density profile in these regions. At this time the central density had reached $1.37 \times 10^{-5}$ g/cm$^3$. Observe also the effect of the moving computational grid in that the data points are at slightly smaller radii values than they were at the earlier plotted epoch. The grid is moving in to follow the flow. Figure 14 depicts the variation of velocity. We see the characteristic supersonic infall clearly. Finally, Figure 15 shows the temperature curve. The innermost zones have jumped by almost an order of magnitude in temperature during this short interval due to the compressional heating effect. The outer regions remain isothermal however. As previously cited at this point the model was evolving rapidly although the timesteps were constrained to be very small, at least compared to those allowed earlier. We depict in Figures 16 through 18 the structure as last computed when the timesteps grew so small as to make further evolution of the model impossible. This time corresponds to less than a year having elapsed since the previous figures. Figure 16 shows the run of density. We note that it is essentially the same as in Figure 13, not having had time to change substantially. Figure 17 gives the variation of velocity with radius. Here we see a striking difference since the velocities in the innermost regions are positive corresponding to a bounce taking place after the structure overshot the equilibrium conditions in the center. Indeed, at this point, the central temperature
was computed to be $1.56 \times 10^6 \, \text{K}$. Moreover, one would expect with the sudden velocity discontinuity at about $10^{14} \, \text{cm}$ evident in Figure 17 that a shock front should have formed. Indeed, the temperature profile for this moment, plotted in Figure 18, displays a near temperature discontinuity at this same spatial location although smeared out somewhat by the artificial viscosity method we have been forced to adopt. The shock luminosity is obviously huge and this is in keeping with the conventional low mass protostellar treatments. Yorke and Krugel (1977) for example found that shocks could at times dominate the luminosity over the core. The core mass at this point was about $3.7 \, M_\odot$ with the central density having dipped to $1.28 \times 10^{-5} \, \text{g/cm}^3$ due to the bounce outflow at this time. While it is evident that these oscillations may continue for some time, the high temperature and the characteristic bounding shock front around the core that has been seen by the conventional low to moderate mass literature models would seem to indicate that the core is well on its way to becoming a conventional mass star. Indeed, given the mass, radius, and luminosity we can calculate the Kelvin timescale for the core since

$$t_K = 2 \times 10^7 \frac{M^2}{LR}$$

(5-5)

So we adopt a blackbody approximation for the core alone (ignoring the bounding shock front) and use $R = 5 \times 10^{12} \, \text{cm}$, $M = 7.36 \times 10^{33} \, \text{g}$, and $T = 10^6 \, \text{K}$. Then $L = 4\pi R^2 \sigma T^4 = 4.6 \times 10^{12} \, L_\odot$ for a Kelvin time of substantially less than a year. Of course this approach is very crude since the outward bounce will cause a lower average temperature as the
structure readjusts itself around the proper equilibrium value. Still, it would appear that what should result here is a moderate mass conventional star rather than a SMS. Once the star has settled on the main sequence some growth via accretion probably could take place but development of a stiff stellar wind would surely inhibit much further growth let alone growth into the SMS mass range!

It would appear then that a distended interstellar cloud exceeding the Jeans mass criterion but still within the density range considered by workers within the lower mass protostellar range is doomed to yield a conventional mass star regardless of the cloud's overall mass due to the nonhomologous nature of the collapse. As a result it is natural to begin investigation of equal mass clouds that begin with substantially higher densities. In particular, several such models were examined including unphysically high initial densities at which point it appeared that the intuitive concept of an SMS ultimately having to result if the density is arbitrarily increased at the start is indeed correct. However, the question of whether such extreme initial conditions \( \rho > 10^{-12} \text{ g/cm}^3 \) correspond to physically significant initial values is open to debate. Moreover, it is apparent that one could begin with an ad hoc initial density distribution within the cloud that also could potentially yield an SMS if the geometrical effect of most of the mass originally being in the outermost layers could be avoided so as to nullify the nonhomologous collapse constraint. While interesting these approaches would appear more along the lines of numerical "games" than anything else. However, we shall consider a last specific model with an increased initial density as our Model C.
In this case we adopt an initial density of $10^{-16} \text{ g/cm}^3$ which is obviously high but still within the extreme range of serious low to moderate mass protostellar literature calculations. Once again a temperature of $100^\circ \text{K}$ and a nearly Population II mixture was chosen. The initial properties of Model C are to be found in Table 5. As with the other supermassive models, this one is strongly gravitationally dominated compared to its initial thermal content. With the order of magnitude reduction in $\alpha$ compared to Model B, the attendant smaller physical radius, and the higher initial density yielding a shorter free fall time-scale, Model C collapsed quite rapidly. Figure 19 displays the density profile after some 6614 years from the starting uniform density distribution. This corresponds to $0.99 t_{\text{ff}}$. Particularly note the exceptionally steep density profile in the outermost envelope regions compared to the density profiles exhibited for the earlier models at comparable time-scales. At this time the central density was $1.51 \times 10^{-8} \text{ g/cm}^3$ and the central temperature was approximately $2270^\circ \text{K}$. Figure 20 reveals the variation of velocity; again a supersonic flow is apparent, nearly constant in the outer regions. Finally, temperature is plotted in Figure 21 and it is apparent that the outermost regions can no longer remain isothermal due to the smaller physical size of the cloud. Radiative transfer effects are apparently significant. The next three figures catch Model C at a time less than one year after the previous plots. Figures 22, 23, and 24 exhibit the density, velocity, and temperature at this new epoch when the central density had climbed to $1.93 \times 10^{-6} \text{ g/cm}^3$ with a central temperature of $3.25 \times 10^4^\circ \text{K}$. Particularly observe that
In the velocity plot we see the characteristic indication of the beginning of an outward bounce due to an equilibrium overshoot. The velocity of the first grid cell is positive producing a velocity discontinuity in the curve. Otherwise, the infall is still highly supersonic. Moreover, contrasting Figures 21 and 24 it is obvious that the temperatures in the innermost regions have jumped appreciably. It is further apparent that the outermost cell temperature would have already increased as well due to the energy transfer through the cloud if we were not imposing the constant outer boundary temperature condition. We next step ahead just over three decades in time to see the bounce in a prominent role. Figure 25 displays the density profile at 6647 years after the model sequence starting point. This is just over ten years short of a full free fall time having elapsed. Note that the bounce has dropped the central density enormously now. The current value is only $8.38 \times 10^{-13}$ g/cm$^3$ and the central temperature has fallen back to almost 105°K! As the supersonically infalling gas collides with the outward rebounding material, a shock front is being set up. Moreover, the core mass at the previous timestep was 4.56 M$_\odot$ but due to the outward bounce it has fallen to only $1.98 \times 10^{-6}$ M$_\odot$. Figure 26 reveals the velocity profile and the velocity discontinuity is now obvious between the inner rebounding zones and the outer material still falling inward. Finally, in Figure 27 we see the temperature profile at this time and it is apparent that a substantial shock front is building at the location of the velocity discontinuity. Moreover, the cooling of the innermost zones of the protostellar cloud are also evident.
Table 5

Initial Conditions for Model C

Total Mass = $10^6 M_\odot$

Total Radius = $1.6808 \times 10^{18} \text{ cm}$ (fixed)

Initial Density = $10^{-16} \text{ g/cm}^3$ (uniform)

Initial Temperature = $100^\circ \text{K}$ (uniform)

$X = 0.700$

$Y = 0.299$

$Z = 0.001$

Initial Ratio of Thermal to Gravitational Energy $\equiv \alpha = 0.00013$

Initial Timestep Size = $0.1 \Delta t_{\text{CFL}}$
Figure 19

Log(Density) Versus Log(Radius) for Model C
Figure 20

Sgn(v)Log(Abs(v)) Versus Log(Radius) for Model C
Figure 21

Log(Temperature) Versus Log(Radius) for Model C
Figure 22

Log(Density) Versus Log(Radius) for Model C at Second Time
Figure 23

$\text{Sgn}(v) \log(\text{Abs}(v))$ Versus $\log(\text{Radius})$ for Model C at Second Time
Figure 24

Log(Temperature) Versus Log(Radius) for Model C at Second Time
Figure 25

Log(Density) Versus Log(Radius) for Model C at Final Time
Figure 26

Sgn(v)Log(Abs(v)) Versus Log(Radius) for Model C at Final Time
Figure 27

Log(Temperature) Versus Log(Radius) for Model C at Final Time
Unfortunately at this point we are unable to evolve Model C substantially further due again to severe timestep constraints. However, we are faced with an interesting possibility in the supermassive star context. Namely, will the subsequent oscillations last? It is apparent that the bounce is quite powerful, possessing a substantial amount of kinetic energy. Obviously also we are seeing a fundamental difference in the dynamical behavior of supermassive clouds depending upon their initial density and concurrent ratio of thermal to gravitational energy. For the very strongly gravitationally dominated structures an overly powerful bounce develops. Moreover, given the fact that the oscillating core only accounts for a small fraction of the total mass of the cloud, it is apparent that the oscillations will eventually damp. However, the damping timescale in turn determines the time available for additional mass to accrete onto the core as well as for kinetic energy to be built up in the infalling envelope gas. Thus, it would be nice to be able to continue the hydrodynamic calculations. However, such an approach is clearly impractical. Moreover, it is apparent that such a coarsely zoned hydrodynamic technique is inadequate for resolving such a complex dynamical behavior.

We can however begin to gain some idea of the damping timescale of the core for Model C and for the other high initial density models which systematically exhibited a similar oscillatory behavior. This may be achieved by examining the physical mechanisms that will lead to the core being damped and carrying out the salient analytic approximations to estimate how efficient these mechanisms are. It would appear that two
primary mechanisms exist. The first of these is the dynamical pressure exerted on the outermost core regions due to the infalling envelope material. The second is due to viscous damping effects within the core material itself. We turn first therefore to an estimation of the role of dynamical pressure.

Certainly a major effect tending to damp the core oscillations should be the loss of pulsational kinetic energy due to the work the outer regions of the core must perform against the dynamic pressure of the infalling envelope material. Here we compute the approximate damping timescale due to this effect. Consider a core oscillating with a single period $P$ and steadily accreting mass at a rate

$$\frac{dM_c}{dt} = 4\pi R_c^2 \rho_I v_I$$  \hspace{1cm} (5-6)

where $M_c$ and $R_c$ are the core mass and core radius while $\rho_I$ and $v_I$ are the density and speed of infall of the incoming gas. Now, during a core contraction, the energy gained is simply given by

$$W_+ = P_+ \Delta V = P_+ 4\pi R_c^2 \Delta R$$  \hspace{1cm} (5-7)

where the pressure is trivially the force per unit surface area or equivalently the time rate of change of the momentum per unit area. Thus,

$$P_+ \Delta p_+ = \frac{\Delta P}{\Delta t} 4\pi R_c^2 = \frac{(v_I - v_c) \Delta M}{4\pi R_c^2 (P/2)}$$  \hspace{1cm} (5-8)

where $v_c$ is the core surface velocity. But

$$\Delta M = 4\pi R_c^2 (v_I - v_c) \rho_I (P/2)$$  \hspace{1cm} (5-9)
so that the work gain during contraction becomes

$$W_+ = 4\pi R_c^2 (v_I - v_c)^2 \rho_I \Delta R \quad . \quad (5-10)$$

We similarly may calculate the energy lost during the core expansion phase as

$$W_- = P_- \Delta V = P_- 4\pi R_c^2 \Delta R$$

$$= 4\pi R_c^2 (v_I + v_c)^2 \rho_I \Delta R \quad . \quad (5-11)$$

Hence, during an oscillation the amount of energy expended as work is given by

$$\Delta E = W_+ - W_-$$

$$= -16\pi R_c^2 \rho_I v_I v_c \Delta R \quad . \quad (5-12)$$

The time rate of change of core kinetic energy due to this effect is therefore

$$\frac{\Delta E}{\Delta t} = -16\pi R_c^2 \rho_I v_I v_c \frac{\Delta R}{P} = -4v_c (\Delta R/P) \frac{dM_c}{dt} = -2v_c^2 (dM_c/dt) \quad (5-13)$$

since $v_c = 2\Delta R/P$. It is apparent that the result is only approximate and the core parameters will evolve over long periods. However, as we are interested only in the approximate damping timescale to an order of magnitude, the ratio

$$t_D = \frac{KE_c}{(-\Delta E/\Delta t)} \quad (5-14)$$
at least should suffice to determine if the core oscillations are rapidly suppressed by the infalling gas. Here, $KE_c$ is the core pulsational kinetic energy given by

$$KE_c = \frac{1}{2} \int \rho v^2 dV = 2\pi \int_0^{R_c} \rho v^2 r^2 dr$$  \hspace{1cm} (5-15)$$

where we assume that the core is rebounding around an essentially hydrostatic configuration. Since we are interested in a timescale only, for a range of supermassive cloud models, we consider typical core density and pulsational velocity profiles approximated by the empirical expressions

$$\rho = \rho_o [1 - (r/R_c)^\ell]^m$$  \hspace{1cm} (5-16)$$

$$v = v_c (r/R_c)^n$$  \hspace{1cm} (5-17)$$

where $\rho_o$ is the central density and we constrain the parameters to obey $\ell > 0$, $m = 0, 1$, and $n \geq 0$. Then upon substituting expressions (5-16) and (5-17) into equation (5-15) we find for the core pulsational kinetic energy upon performing the integration

$$KE_c = 2\pi \rho_o v_c^2 R_c^3 N$$  \hspace{1cm} (5-18)$$

where

$$N = 1/(2n + 3) \hspace{1cm} \text{if } m = 0 \hspace{1cm} (5-19)$$

$$N = \ell/[(2n+3)(2n+3+\ell)] \hspace{1cm} \text{if } m = 1 \hspace{1cm} (5-20)$$

Moreover, upon forming the ratio in (5-14) we ultimately obtain upon
application of (5-18) through (5-20) the form

$$t_D = \frac{N R_c \rho_o}{4 v_I \rho_I} \quad (5-21)$$

Applying the definitions (5-19, 20) we see that the parameter N typically ranges between about 1/30 and 1/3. Thus, we may build a table of damping timescales for the widest possible range of ratios $R_c/v_I$ and $\rho_o/\rho_I$. This is done in Table 6 below.

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</tbody>
</table>

Here, the values are presented as logarithms of the ratio damping time in seconds. Note for comparative purposes that the pressureless free fall timescale for a $10^{-16}$ g/cm³ structure is approximately $2 \times 10^{11}$ sec.

As the envelope is still in dynamical infall at the last computed stage of Model C this also represents the timescale for global structural changes due to the envelope. We therefore observe that a prolonged
oscillatory phase is indeed potentially possible for a supermassive cloud based on this highly idealized analysis of the role of the dynamical pressure due to infalling material. However, it could require rather extreme values for the ratio of core size to infall velocity as well as for the ratio of central core density to the density of the infalling inner envelope gas material. Certainly, crossing the shock front bounding the protostellar core will induce an abrupt drop in the infall velocity by at least an order of magnitude prior to impact on the core’s surface layers. At the same time we already know that the central core density will (on average) greatly exceed the inner envelope density. If we select $R_c = 10^{14}$ cm (note that $10^{13}$ cm is more typical for the low mass literature protostellar models) and infall velocity of $10^6$ cm/sec then to meet the quoted free fall timescale would require a density ratio of at least $10^4$. Certainly, such a ratio would not be unreasonable. On the other hand, without a more sophisticated analysis the above results must be considered as only very approximate. Nevertheless, it would appear that core oscillations represent a potentially viable mechanism for staving off direct evolution onto the main sequence during which time substantial mass may be added to the core. Certainly the effect would appear to be a significant one for conventional high mass stars!

We noted previously that a second damping effect could arise to rob the core of its pulsational kinetic energy. This is gas viscosity within the core itself. It must be noted that the classical equations of hydrodynamics solved by the computer code do not contain viscous terms as is certainly appropriate during the very early phases of the dynamical
infall. However, at the later phases it is not so clear that viscosity will continue to be negligible. Thus, there is something of a logical inconsistency in now applying the viscosity as a damping mechanism considering that the oscillations were built up without viscosity as is standard in protostellar calculations. Again, for our purposes we desire only an order of magnitude estimate of the role of viscosity. Such is provided by recalling the form of the coefficient of dynamic viscosity $\mu$ as given by Lang (1980) for hydrogen atoms:

$$\mu = 5.7 \times 10^{-5} T^{1/2}$$

(5-22)

in units of g/cm-sec. The coefficient of kinematic viscosity in turn is given by the ratio

$$\nu = \mu/\rho$$

(5-23)

and we assume that the viscous damping time is very roughly given by

$$t_{\text{damp}} \propto \frac{R_c^2}{\nu}$$

(5-24)

If we adopt the core values of $10^{-6}$ g/cm$^3$, $10^4$ K, and $R_c = 10^{14}$ cm then condition (5-24) would predict a damping timescale of approximately $2 \times 10^{24}$ seconds! It would appear therefore that the dominant damping effect will indeed be the pressure due to the dynamically infalling envelope material.

Thus, in summary the case for formation of a supermassive star based on the models specifically cited here as representative of the overall group of models computed would appear to be somewhat brighter
than has typically been considered in the astrophysical literature. In particular, if one is willing to adjust the initial density value high enough then obviously an SMS will result irrespectively of the nonhomologous infall characterizing protostellar collapse. On the other hand, if one adopts nonuniform density distributions (not treated in these models) then again it should be possible to defeat the nonhomologous constraint if most of the mass is not initially present in the outermost cloud regions. Distended protostellar clouds with conventional initial uniform densities in the range of $10^{-19}$ g/cm$^3$ however appear to consistently yield low to moderate mass stars due to the nonhomologous infall. Further accretion onto this "kernel" should be halted by a stellar wind for the moderately high mass objects. Moreover, one encounters the problem of the central star's main sequence lifetime dramatically shortening as additional mass is added onto it. Thus, even if the infall was not completely halted for some reason, it would still appear impractical to build up a SMS through an accretion process. Of course such a timescale analysis ignores the role of pulsational instabilities in high mass stars. Finally, what is most exciting is that the models appear to indicate that if high, but nevertheless conventional, initial density values are used the models will undergo a strong bounce and oscillatory phase. The dynamics of the bounce are complex and not well studied by an explicit hydrodynamic computer code due to the fact that the configuration is then evolving on a thermal rather than on a hydrodynamical timescale. On the other hand, it is also not well studied by many of the conventional Lagrangian hydrodynamical computer algorithms.
which tend to suppress such oscillatory phases. Furthermore, any conventional hydrodynamic approach with a finite difference formalism tends to utilize a relatively coarse grid which is itself ill-suited to following oscillatory behavior in any serious detail. An additional complication lies in the use of artificial viscosity. While necessary from the standpoint of handling the inevitable shocks that arise, it has a definite effect upon the bounce amplitudes observed in the SMS models. A similar fact has been noted by Boss (1979) in his investigation of low mass models which also underwent oscillatory phases. Nevertheless, upon performing a very simplified dynamical pressure damping analysis through energy considerations, it would appear that this oscillatory phase is protracted in the case of a supermassive cloud with an initial density greater than $10^{-16}$ g/cm$^3$. Certainly this would appear to be significant for building up conventional high mass protostars. Whether or not it can suffice to extend the central object's stellar formation timescale enough to permit the remaining infalling material to build up a supermassive star however would appear to be marginally likely at best. Nevertheless, a definitive answer can only be obtained outside the context of the conventional protostellar hydrodynamical algorithms.

SECTION 5.1 SUGGESTIONS FOR FUTURE WORK

The supermassive protostellar problem is a rich one. As a result, a number of potential extensions to this work are under active consideration. Chief among these are development of a practical numerical asymptotic formalism for the hydrodynamical equations so as to permit
their numerical solution over time frames spanning such thermal phases as the oscillatory intervals encountered here. Indeed, such an analysis was attempted but proved to be unwieldy to implement on the computer due to the presence of the energy equation. Shu and his co-workers have noted a similar problem at the low mass level, having presented only an asymptotic formalism appropriate to the early isothermal stages of a low mass protostar. Nevertheless, if such a formulation could be extended to the nonisothermal situation then not only the low mass protostars could be more accurately studied over these later phases, but in addition the ultimate question of the oscillatory damping timescale could be answered within the supermassive context. The value of such a formulation is obvious throughout the protostellar literature.

From a more immediate standpoint, it would be significant to examine the role of oscillations in permitting the development of conventional high mass protostars. Such a scenario was not considered by Larson and Starrfield's previously cited paper on the factors inhibiting high mass star formation but as we have seen here is potentially very significant in spite of the particular chemical mixture of the cloud. Thus, presumably both high mass Population I and II objects could arise in this manner.

It would also be of interest to consider the ability of nonuniform initial density distributions to circumvent the nonhomologous collapse scenario. Such investigations should be significant both at the conventional high mass range as well as at the supermassive star mass range in attempting to avoid the fundamental formation timescale problem arising from the geometrical fact that most of the cloud's mass is
initially very distant from the cloud center. This in turn permits the nonhomologous collapse to build up a comparatively low mass central stellar kernel. Highly concentrated initial density distributions would avoid this problem somewhat although their physical reality would be open to question if the profile was too extreme.

From a thermodynamic standpoint it would be interesting to consider the role of partial ionization of elements other than hydrogen in the supermassive context. Recall that our treatment ignored formation of $\text{H}^{-}$ ions as well as ionization of elements heavier than hydrogen. One could incorporate the salient partial ionization tables such as those of Fontaine et al. (1977) into the computer code in an attempt to examine their significance upon the hydrodynamics. Furthermore, in this context one could upgrade the energy equation solution algorithm from use of the diffusion approximation to use of the Eddington approximation.

Finally, a number of additional factors could be incorporated into the hydrodynamic computer code to assess their effects upon collapsing supermassive protostellar clouds. Certainly at some point the role of dust should be investigated in a two-component fluid formalism including time dependency rather than the steady state treatment of Kahn as cited earlier. The donor cell algorithm should be well suited for such an investigation. Furthermore, one could consider rotating supermassive protostellar clouds in an attempt to examine the potential formation of objects like Ozernoy’s magnetoids if in addition magnetic field effects were incorporated. Lastly, the role of fragmentation and coalescence could be treated albeit outside the spirit of a conventional hydrodynamic
approach as we have pursued here. Indeed, throughout this investigation we have tacitly assumed that turbulence and fragmentation do not occur. The presence of either or both of these phenomena would certainly pose a major obstacle to the formation of a supermassive star through a conventional Larson-type protostellar collapse.
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