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AN APPLICATION OF THE LAGUERRE TRANSFORM TO THE GI/G/1 QUEUE

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An Application of the Laguerre Transform
to the GI/G/1 Queue

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By

Joseph R. Litko, B.S., M.Sc.

* * * * *

The Ohio State University
1985

Reading Committee:
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Approved By
Walter C. Giffin, Adviser
Department of Industrial Engineering
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DEDICATION

To Carol, Suzy, and Joe
ACKNOWLEDGEMENTS

I wish to acknowledge the assistance of my research committee in reviewing and improving this document. I am particularly indebted to my adviser, Professor Walter Giffin, for his encouragement and assistance during this research and during my entire program at Ohio State University.

I thank my colleagues, Ed Perry, Tom Schuppe, Tony Sharon, and especially Robert Daley for being willing to listen to my ideas as this research developed.

Most of all, I thank my wife, Carol, and my children, Suzanne and Joseph, to whom this dissertation is dedicated. They have endured a lot over the past four years of graduate school.
Joseph Litko was born in Posen, Illinois in 1947. He received a B.Sc. in physics from Illinois Institute of Technology in June 1969. He entered Northwestern University on a graduate fellowship in September 1969, but left in June 1970 due to the military draft. After serving five years in the enlisted ranks, Captain Litko was commissioned in 1975. From May 1975 till February 1979, he was Officer-in-Charge of the Weather Communications Branch supporting the Air Force Global Weather Center. From March 1979 till May 1981 Captain Litko was the Director of Operations of the 1936 Communications Squadron at Lajes Field, Azores (Portugal). In June 1981 Captain Litko entered the Logistics Management curriculum at the Air Force Institute of Technology and received his M.Sc. degree in September 1982. Also in September 1982 he entered the Operations Research program in the Industrial Engineering Department at the Ohio State University. He has studied under Walter C. Giffin throughout his stay at Ohio State. Captain Litko was admitted to the Ph.D. candidacy in May 1984.
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Chapter I
THE GI/G/1 MODEL AND SOLUTION TECHNIQUES.

1.1 INTRODUCTION:
This chapter presents the GI/G/1 queueing model leading to Lindley's equation for the waiting time distribution at the arrival of the nth customer. Subsequent discussion covers various techniques which are used to "solve" this model. This begins with the Laguerre transform method as it currently exists and then considers the extensions to be made in this research.

The remainder of the techniques to be presented are divided into two broad categories, transformation methods and approximations. The transformation methods all bear some kinship to the Laguerre transform method because they take the original queueing problem and solve it in a different state space. The approximations category is included for a different reason. Namely, it will be shown in Chapter Five that knowledge of an approximate solution to the queueing problem is of great value in reducing the computational effort of the Laguerre transform method.
The queue GI/G/1 is described as follows. There is a single server and service times are drawn independently from some distribution $E(\cdot)$. The times between arriving customers are independent and follow a distribution $A(\cdot)$. This research also assumes a first come first served (FCFS) protocol. Consequently there is no need to distinguish between an arriving customer's waiting time and the unfinished work in the system at a customer's arrival. Define the following variables:

- $C_n$ = the $n$th customer arriving to the system
- $T_n$ = interarrival time between customer $C_{n-1}$ and $C_n$
- $X_n$ = service time for $C_n$
- $W_n$ = the waiting time (in queue) for $C_n$
- $W_{n+1}$ = unfinished work in the system immediately prior to $C_n$'s arrival

One can represent the queueing system as in Figure 1 and Figure 2. In both figures the bold arrows indicate the movement of a customer in joining the queue, occupying the server, and leaving the server. In Figure 1 the server is busy when $C_{n+1}$ arrives. Consequently $C_{n+1}$ waits a time $W_{n+1}$ until the server is free and then enters service. From Figure 1 we can see that $T_{n+1} + W_{n+1} = W_n + X_n$ if the server is busy when $C_{n+1}$ arrives. This equation applies
Figure 1: $c_{n+1}$ arrives to a busy server.

Figure 2: $c_{n+1}$ arrives and the server is idle, even if $w_n = 0$. The server being busy is equivalent to the condition $w_n + x_n \geq T_{n+1}$. 
In Figure 2 $w_{n+1} = 0$ since $w_n + x_n < T_{n+1}$. One can define a new random variable $Y_n$ as $Y_n = X_n - T_{n+1}$ so that

$$w_{n+1} = w_n + Y_n \quad \text{if} \quad w_n + Y_n \geq 0$$

$$= 0 \quad \text{if} \quad w_n + Y_n < 0$$

The $W_n$ (or the unfinished work) are a Markov process because we have restricted ourselves to examining the waiting times only at arrival epochs. That is, all we need to know to calculate $w_{n+1}$ is contained in $w_n$ and $Y_n$. Past history of the process is irrelevant. The distribution of $W_n$ is the waiting time of the $n$th arrival. Alternatively, it is the distribution of the unfinished work a server would see immediately prior to the $n$th arrival epoch.

Equation (1.2.1) leads naturally to Lindley's integral equation. Consider first the distribution of the random variable $Y_n = X_n - T_{n+1}$. Although the service and interarrival times are positive, there is no such restriction on $Y_n$. In fact, we know for any stable infinite population/capacity queue mean interarrival times must exceed mean service times. So clearly the distribution of $Y_n$ extends to the negative half line in most interesting cases. Define,

$$K(y) = \Pr[Y_n = X_n - T_{n+1} | Y_n = y]$$

$$= \int_R \Pr\{X_n \leq y + t | T_{n+1} = t\} a(t)dt$$
Both distributions \( B(\cdot) \) and \( A(\cdot) \) are defined only for positive arguments. The region of integration is therefore \([-y, \infty)\) for \( y < 0 \) and \([0, \infty)\) for \( y \geq 0 \). Define the distribution of waiting time for \( C_n \) as, \( W_n(x) = \Pr(W_n \leq x) \). For \( x \geq 0 \) equation (1.2.1) implies that

\[
W_{n+1} = \Pr\{W_n + Y_n \leq x\} = \int_0^\infty \Pr\{Y_n \leq x-z | W_n = z\} dW_n(z)
\]

(1.2.2) \[ W_{n+1}(x) = \int_0^\infty K(x-z) dW_n(z), \quad x \geq 0 \]

This last step follows since \( Y_n \) is independent of \( W_n \).

Equation (1.2.2) is Lindley's equation and it can be written in several other forms by making simple transformations of the variables. A form which is useful in succeeding sections is now given.

(1.2.3) \[ W_{n+1}(x) = \int_{-\infty}^x W_n(x-y) K(y) dy \quad x \geq 0 \]

\[ = 0 \quad x < 0 \]

Equation (1.2.3) is the starting point for the research. Although it is customary to indicate that \( W_n(x) = 0 \) for \( x < 0 \) there is an interpretation of the waiting time distribution for negative \( x \) as the distribution of server idle time. In later chapters this interpretation will be used to state a numerical procedure for obtaining interdeparture distributions for GI/G/1 queues.
If the queue is stable, the \( W(x) \) will approach some limiting distribution. That is,

\[
\lim_{n \to \infty} W_n(x) = W(x). 
\]

In this case we obtain an integral equation for the waiting time distribution such as,

\[
W(x) = \int_{-\infty}^{x} W(x-y)k(y)dy \quad x \geq 0
\]

\[
= 0 \quad x < 0
\]  

(1.2.4)

We will refer to the solution of equation (1.2.4) as the steady state solution of the GI/G/1 model.

The solution to equation (1.2.3) or any of its variants for each \( n \) will be referred to as the transient solution. It is clear that the subscript on the \( Y_n \) could have been maintained to allow the arrival and service distributions to change at each arrival epoch. This would complicate matters, but as long as the assumption of independence remains valid it would simply entail calculating a new pdf \( k(.) \) for each customer. In this case one could not make the simple transition to the steady state solution.

There is no general tractable solution to equations (1.2.3) or (1.2.4) for arbitrary choice of arrival and service distribution. The textbook technique for the steady state solution, spectral factorization [14], is usually illustrated for distributions which are handled by a simpler queueing model, e.g. M/G/1, M/M/1. Transient informa-
tion is obtained through simulation or by abandoning the GI/G/1 model in favor of a continuous diffusion process approximation. The Laguerre transform method retains the GI/G/1 model, is applicable to a wide though not limitless range of distributions, and handles both the steady state and transient cases.

1.3 LAGUERRE_TRANSFORM_SOLUTION_TO_GI/G/1.
This research is directed toward extending the applicability of a numerical solution to GI/G/1 first seen in Keilson, Kunn, and Sumita [9], and Sumita [25]. Schematically, the Laguerre transform technique converts the integral in equation (1.2.3) into a summation of discrete terms. These discrete terms are essentially the Laguerre function expansion coefficients of the arrival, service, and waiting time distributions. The solution thus obtained is easily converted back to the continuum by a Laguerre function expansion using the resultant coefficients from the discrete manipulations.

The literature relevant to the Laguerre transform solution to GI/G/1 begins with Keilson and Kunn [8]. They present the basic properties of the Laguerre functions under a convolution integral. Keilson and Kunn also show the convergence properties of Laguerre function expansions of probability distributions based on arguments similar to
those in Dym and McKean [4] for convergence of Fourier series. Further, Taylor series expansion methods are presented which provide a general purpose tool for obtaining Laguerre function expansions of probability distributions. Thus while no direct reference is made to GI/G/1 many of the required tools are provided.

In Keilson, Munn, and Sumita [9] a bilateral Laguerre transform is defined. This allows one to obtain expansions of functions on $(-\infty, \infty)$. The bilateral transform has a more obvious application to integrals as in equations (1.2.3) and (1.2.4). And in fact the basic iterative technique for solving GI/G/1 is given in this reference. It is interesting that since the current research was begun in ignorance of the bilateral transform technique an equivalent solution was developed based on the usual one-sided transform on $[0, \infty)$.

Perhaps the most comprehensive treatment of the bilateral transformation is given in Sumita [25]. He obtains useful expressions or algorithms for calculating the expansion coefficients for many common probability densities. Also restated is the basic technique for iterating equation (1.2.3) via the Laguerre transform. This iterative technique eventually yields the steady state solution to (1.2.4) if the arrival and service distributions do not change for each customer. The iterative method allows one
to observe the approach of the waiting time distribution to steady state.

Extensions of the Laguerre transform method to represent non-stationary situations are given in Susita [25] and in Keilson and Susita [10]. The queue which is approximated in both references is $M(\lambda(t))/G/1$. That is, a single server queue with general service distribution and time-dependent arrival rate. Although Lindley's equation applies only at arrival epochs, the quantity $\lambda(t)$ is approximated by $\lambda(t_k)$ where $t_k$ is the expected time of the $k$th arrival to the queue. This approximation preserves the flavor of the non-stationary input process while allowing the solution via the basic Laguerre transform technique.

Keilson and Susita [11] also apply the method to finding the total time in the system for a preempt/resume $G1/G/1$ queue with two customer classes. Each customer class has its own arrival and service distribution. Class I customers may preempt Class II customers, but when the system empties of all Class I customers, any preempted Class II customer resumes service where he left off. The solution is facilitated by defining an effective service time for the lower priority customers which takes into account their interruptions by the higher priority customers. Given this effective service time distribution a modified Lindley process similar to equation (1.2.1) is defined and the Laguerre transform technique is applied.
The research to be presented here will extend the Laguerre transform technique in several ways. The most important of these will now be previewed. First, in addition to an iterative method, a direct solution for the steady state waiting time distribution will be developed. At bottom this steady state solution is a restatement of equation (1.2.4) in terms of a system of linear equations in the expansion coefficients of the waiting time distribution. It might be argued that this steady state solution is a step backwards since it lacks information on the transient wait times. It will be shown, however, that the convergence of the iterative technique to steady state can be very slow. A direct solution is therefore intrinsically valuable and also valuable as an adjunct to the iterative method.

Second, this research shows the equivalence of a one-sided transform technique to the two-sided techniques in the current literature. In some ways the choice between the two is a matter of style. But the two sided technique calculates the distribution of "waiting times" on the negative x axis. These distributions correspond to server idle times and are later shown to be of direct use in obtaining stationary interval interdeparture distributions. The one-sided technique allows the same calculation as an option. One can calculate the interdeparture distributions
(via the server idle times) if desired or save computational effort by avoiding the calculation of the wait time distribution on the negative x axis.

Third, this research extends the Laguerre transform method to calculation of queue length probabilities under transient and steady state conditions.

Fourth, problems of dimensionality and scaling are addressed. These problems appear at the interface of the general Laguerre transform theory and the queueing problem. We essentially gauge or adjust the size of convolutions and systems of equations through the scaling of the queueing problem. One objective is to broaden the range of queues which might easily be solved on a microcomputer. Obviously, this also favorably impacts solution on a mainframe.

Having summarized the literature and the research to be presented here we characterize the Laguerre transform method as follows. The Laguerre transform method per se does not provide any analytical insights into the GI/G/1 queue or the Lindley process. But its virtue is in providing a numerical technique which, in principle, can be made arbitrarily accurate. It can be useful for studying GI/G/1 directly or as an aid in evaluating other approximations.
1.4 OTHER TRANSFORMATION METHODS

Perhaps the most basic imaginable transformation is to discretize the time axis. This approach is used by Neuts and Klimko [18], [19], [15] to study the transient and steady state behavior of the single server queue with finite capacity. Given a discrete time axis, $X_n$ is defined as the queue length at time $n$ and $Y_n$ is defined as the remaining service time at time $n$.

The assumptions of the discrete time model imply that the sequence $(X_n, Y_n), n > 0$ is a Markov chain with state space $\{0, 0\} \cup \{1, 2, \ldots, L_1\} \times \{1, 2, \ldots, L_2\}$ where $L_1$ is the maximum queue length and $L_2$ is the maximum service time. Neuts derives the transition probabilities of the Markov chain and discusses the practical limitations of the approach. The examples shown allowed queue lengths as high as 800, but in these cases the characterization of the arrival and service distributions must be comparatively coarse to maintain a reasonable limit on computational effort.

The direct output of the discrete time model is the queue length distribution at the discrete time points. Waiting times are inferred based on multiple convolutions of the discrete service distribution and using the residual service time distribution of the current customer. This is opposite the Laguerre transform technique where waiting
time distributions are the direct output and queue length distributions are calculated using multiple convolutions of the arrival distribution.

Ackroyd [2] uses the discrete Fourier transform to calculate transient and stationary queue length distributions for the M/G/1 queueing system. It was, in fact, this work by Ackroyd which stimulated the current research into GI/G/1. There is a strong resemblance between the two methods operationally and motivationally. Ackroyd cites the difficulties one encounters in numerically evaluating the Pollaczek-Khinchin transform formula [14]. Whereas in GI/G/1 there is the difficulty in applying spectral factorization methods.

Ackroyd's method is based on the following defining equation for M/G/1:

\[
q_{n+1} = \begin{cases} 
q_n - 1 + \xi_{n+1} & \text{for } q_n > 0 \\
\xi_{n+1} & \text{for } q_n = 0
\end{cases}
\]

(1.4.1)

where \( q_i \) = queue length immediately after the \( i \)th customer completes service.

\( \xi_i \) = number of customers that join the system while the \( i \)th customer is in service.

equation (1.4.1) summarizes the evolution of the M/G/1 system and leads to a recursion for the discrete probability density function for \( q_n \). Namely,
\( a_{n+1}(k) = [a_n(k+1) - a_n(0) \delta(k+1)] * e_{n+1}(k) + a_n(0) e_{n+1}(k) \)

where \( a_{n+1}(k) \) = probability that \( q_{n+1} = k \)

\( e_{n+1}(k) \) = probability that \( e_{n+1} = k \)

\( \delta(k+1) \) = discrete impulse function

\[
\begin{cases} 
1 & \text{for } k+1 = 0 \\
0 & \text{otherwise}
\end{cases}
\]

The distribution of \( e_{n+1}(k) \) is the Poisson transform of the service distribution, i.e.

\[
\int_0^\infty \frac{e^{xt}(xt)^k}{k!} b(t) dt.
\]

Equation (1.4.2) arises as follows. When \( q > 0 \) then according to (1.4.1) \( q_{n+1} \) is a sum of the random variables \( q_n - 1 \) and \( e_{n+1} \). The distribution of \( (q_n - 1) \) given that \( q_n > 0 \) is,

\[
\frac{a_n(k+1) - a_n(0) \delta(k+1)}{1 - a_n(0)}
\]

i.e. shift \( a_n(k) \) left and renormalize. Therefore the distribution of \( q_{n+1} \) is the convolution of (1.4.3) and \( e_{n+1}(k) \) weighted by the probability that \( q_n > 0 \) plus the distribution \( e_{n+1}(k) \) weighted by the probability that \( q_n = 0 \). Hence (1.4.2) follows.

Ackroyd cites five to six decimal place accuracy in queue length distribution computations over a range of traffic intensities \( \rho = .1 \) to \( .9 \). He is comparing his
result, obtained by iterating (1.4.2) to convergence, to the known steady state result. He says that the number of iterations to converge to the steady state varies from a few score at \( \rho = .1 \) to a few thousand at \( \rho = .9 \). We shall see in Chapter Five that these results are comparable to those obtained for GI/G/1 through the Laguerre transform methods.

Ott [21], [22] derives results, which given certain convergence requirements, will lead to approximations for the moments and the distribution of the stationary GI/G/1 waiting time distribution. These results are based on "infinite by infinite" sets of linear equations. Consequently, they bear at least a structural similarity to the direct steady state solution being presented in this research.

Ott constructs his equations from the Taylor series expansions of the arrival and service distributions and their Laplace Stieltjes transforms. One important result is an integral equation for the virtual waiting time distribution under stationary conditions. In discussing this equation Ott says that two suitable sets of functions \( (h_j(t))_{j=0}^\infty \) and \( (g_j(t))_{j=0}^\infty \) are needed for which

\[
A(t) = \sum_j a_j h_j(t)
\]

is convenient, and for which also

\[
\int_0^t (t-x)^k h_j(x) \, dx = \sum_i b_{i,k,j} g_i(t)
\]
is convenient. This would allow calculating the stationary virtual waiting time distribution.

In the above $\Lambda(t)$ is the arrival distribution and $(t - x)^k$ arises from the expansion of the service distribution. Ott's cited preliminary research considered the choice $h_k(x) = g_k(x) = x^k$ which he says is "probably not the optimal choice of systems." We suggest here that the Laguerre functions may in some ways be the optimal choice. Recognize that the integral (1.4.5) above would be nothing but a convolution of Laguerre functions in a different formulation of Ott's method.

Virtual wait time computation in steady state is a simple extension of the computation of actual wait times using results given in Cohen [3] for instance. That is,

$$V(t) = \begin{cases} 1 - \rho + \lambda \int_0^t W(t-u) \{1 - B(u)\} \, du & t > 0 \\ 0 & t \leq 0 \end{cases}$$

where $\rho =$ traffic intensity

$W(\cdot) =$ stationary actual or arriving customer wait time distribution

$E(\cdot) =$ service time distribution

$\lambda =$ arrival rate

The integral in (1.4.6) is a convolution and by the end of Chapter Four it will be clear that the steady state
Laguerre transform methods would make computation of (1.4.6) simple given the solution for \( W(\cdot) \). The direct or matrix solution for \( W(\cdot) \) will be presented in detail in Chapter Four.

1.2.2 APPROXIMATIONS FOR GI/G/1

The most familiar and oft-quoted approximation for the GI/G/1 waiting time is Kingman's heavy traffic approximation [12], [13]. In Kingman's nomenclature heavy traffic is the situation where the traffic intensity approaches but remains below \( \rho = 1 \). Some more modern authors treat heavy traffic as the situation \( \rho \geq 1 \). This discussion adheres to Kingman's definition.

Kingman's famous result is that in heavy traffic the waiting time distribution is approximately exponential with mean

\[
(1.5.1) \quad \frac{1}{2} \left( \text{arrival dist. variance} + \text{service dist. variance} \right) \text{mean interarrival time} - \text{mean service time}
\]

For this approximation to be valid Kingman says the denominator of (1.5.1) must be small compared to the square root of the numerator.

Fredericks [5] develops a class of approximations for the GI/G/1 waiting time distribution by inserting the functional form of Kingman's result into Lindley's equation (1.2.4). Essentially, Fredericks assumes that
\[ W(x) = 1 - \exp(-ax). \] Inserting this into Lindley's equation would give,

\[ (1.5.2) \quad 1 - C e^{-ax} = K(x) - C e^{-ax} \int_{-\infty}^{x} e^{ay} k(y) dy \]

If the equation \[ \int_{-\infty}^{\infty} e^{au} k(u) du = 1 \]
has a real non-zero root, \( a_e \), then a known result [5] is that

\[ 1 - W(x) \sim C e^{-a_e x} \]

i.e. the tail of the waiting time distribution is exponential.

Fredericks suggests that we choose, for instance, \( a = a_e \)
in (1.5.2) so that it will be valid as \( x \to \infty \). He suggests that the constant \( C \) be evaluated by requiring that (1.5.2) be valid at some other value of \( x \). This would imply that

\[ C(x) = \frac{(1 - K(x)) e^{ax}}{1 - \int_{-\infty}^{x} e^{ay} k(y) dy}. \]

\( C(x) \) can be evaluated, if only numerically, given a particular arrival and service distribution. This example is indicative of the methods Fredericks uses to develop his class of approximations. The general thrust is to find meaningful ways to evaluate the two constants \( C \) and \( a \), but retain the negative exponential functional form. Some of the results are quite good even at low (\( \rho = 0.2 - 0.4 \)) values of traffic intensity.
The relevance of the above to the current research is this. The Laguerre transform methods are numerical techniques, and hence are oblivious to the substantive results of queueing theory. But we can use these substantive results in matters of choice. The best example of this is in selecting the scale of the queueing problem. This idea will be developed further in Chapter Five.

With this motivation we return to the topic of approximations. Shanthikumar and Buzacott [23] is a thorough review of various approximations for the mean delay and queue length in GI/G/1. Their basic strategy is to compare several approximations over ranges of traffic intensity and squared coefficient of variation of the arrival and service distributions. Comparison is made against queues with generalized erlang arrival and service distributions whose solutions are known. The result is an evaluation of which approximation performs best over a particular range of queueing system parameters.

Kollerstrom [16] presents a second order heavy traffic approximation for GI/G/1. The main result is that,

\[ W(wy) \approx 1 - \exp(-y) + C(y-1) \cdot \exp(-y) \]

where \( W(y) \) is the waiting time distribution

\( C \) is a constant independent of \( y \)

\( w \) is a scaling constant
Kollerstrom suggests some numerical means of evaluating the constants $\bar{\nu}$ and $C$. The point of interest here is that as in Kingman's result the waiting time distribution takes on a simple functional form. In Kollerstrom's case we now have a second order approximation. He states that the constant $C$ is of the order $1/\bar{\nu}$. The constant $\bar{\nu}$ is of the order of the mean waiting time. After rearranging, one could group terms so that Kollerstrom's implied result for the pdf of the waiting time is an exponential and a "perturbing" second order erlang. Both of these distributions can have very compact Laguerre function expansions with the correct scaling of the problem.

Whitt [27], [28] presents two techniques for approximating a point process by a renewal process. In [28] he considers a $\sum_i G_i /G/1$ queue. As the symbols imply, the arrival process to the single server queue is the sum of independent renewal processes. In general, the individual arrival streams are not Poisson and therefore their sum is not a renewal process. Whitt gives two basic methods of approximating the superposed streams as a renewal process.

He uses the approximations to obtain some of the moments of the superposed arrival process. This is then used to obtain approximate solutions for the queue. It is significant that using numerical integration techniques one could easily obtain Laguerre function expansions to represent the
distribution in Whitt's approximate arrival process. Thus the techniques to be presented here could be applied to the case $\Sigma G_i/G/1$.

Furthermore, since this research will present a numerical method for obtaining the stationary interval interdeparture distribution, one might begin to deal with simple non-feedback networks of general queues in a detailed way. Since the departure process is not a renewal process it cannot be used as the input to a subsequent queue as though it were simply a renewal arrival process. The stationary interval interdeparture distribution obtained here would be the key ingredient in using the approximations in Whitt [27]. The prospect of systematically investigating networks is beyond the scope of the current research, however.

1.6 SUMMARY

The GI/G/1 model leads to Lindley's equation for the waiting time distribution of the $n$th arriving customer to the queue. A steady state version of Lindley's equation arises through the limit as the number of arriving customers tends to infinity. The problem is that the resulting equations are difficult or impossible to solve analytically.

The Laguerre function method is a numerical approach to solving these equations. The effort in the current research will be toward extending the Laguerre transform
method in ways outlined above. Furthermore, the research will show how established queueing results can significantly improve the performance of the method. This may make it practical to consider application to more complex queueing systems.
Chapter II
PROPERTIES OF THE LAGUERRE FUNCTIONS AND THE LAGUERRE TRANSFORM

2.1 INTRODUCTION
This chapter presents the background material needed to apply the Laguerre transform technique to the GI/G/1 queue. This includes the properties of the Laguerre functions themselves as well as the properties of the Laguerre transform as a linear integral operator. These properties can be found in Sumsita [25], Keilson [8], or Keilson, Nunn, and Sumsita [9], or in most cases proven easily from the Laguerre polynomial properties in Abramowitz and Stegun [1]. The shifting properties presented here are based on a one-sided transform and would thus appear different than bilateral results in [9] and [25]. The shifting and convolution results are crucial to the solution of Lindley's equation.

Convergence criteria are presented which gauge the quality of a Laguerre function expansion and there is a general discussion of the factors governing the rate of convergence of an expansion.
In addition, a bilateral Laguerre transform developed by Keilson, Nunn, and Sumita [9] is briefly discussed. And finally, a method of scaling functions is developed here. The idea behind scaling functions is this. Given a function \( g(x) = f(cx) \) we wish to obtain the expansion coefficients \( \{g_n\} \) directly in terms of the (known) expansion coefficients of \( f(x) \), \( \{f_n\} \). The rationale is that in some cases it is more efficient to obtain the \( \{g_n\} \) through scaling than through direct means.

### 2.2 Properties of the Laguerre Functions

The Laguerre functions are a complete set of functions on the interval \([0, \infty)\). Consequently, any continuous, square integrable function defined on \([0, \infty)\) will have a convergent Laguerre function expansion. The Laguerre functions \( l_n(x) \) are related to the Laguerre polynomials by \( l_n(x) = \exp(-x/2)L_n(x) \). The weighting function \( \exp(-x/2) \) ensures that \( |l_n(x)| \leq 1 \) on \([0, \infty)\). Some of the more useful properties of the Laguerre functions are presented below.

#### 2.2.1 Recurrence Relation

\[
(n+1)l_{n+1}(x) = (2n+1-x)l_n(x) - nl_{n-1}(x)
\]
2.2.2 Explicit definition:

\[ L_n(x) = e^{-x/2} \sum_{m=0}^{n} (-1)^m \frac{1}{m!} x^m \]

2.2.3 Rodrigues' formula:

\[ L_n(x) = \frac{e^{-x/2}}{n!} \frac{d^n}{dx^n} (x^n e^{-x}) \]

2.2.4 Generating function:

\[ \sum_{n=0}^{\infty} z^n L_n(x) = (1-z)^{-1} \exp \left[ -\frac{x}{2} \left( \frac{1+z}{1-z} \right) \right] \]

2.2.5 Orthogonality relation:

\[ \int_{0}^{\infty} e^{-x} L_n(x) L_m(x) dx = \delta_{nm} = \begin{cases} 1, & n=m \\ 0, & \text{o.w.} \end{cases} \]

2.2.6 Laplace transform:

\[ \phi(s) = \mathcal{L} (L_n(x)) = \int_{0}^{\infty} e^{-sx} L_n(x) dx = \frac{1}{(s+\frac{1}{2})} \left( \frac{S-\frac{1}{2}}{s+\frac{1}{2}} \right)^n \]

2.2.7 Integral of Laguerre function:

Let \[ I_n(t) = \int_{c}^{t} L_n(x) dx \]

Then \[ \mathcal{L} (I_n(t)) = \frac{1}{s} \left( \frac{S-\frac{1}{2}}{s+\frac{1}{2}} \right)^n \]

\[ = \frac{2}{s+\frac{1}{2}} \left( \frac{S-\frac{1}{2}}{s+\frac{1}{2}} \right)^{n-1} - \frac{2}{s+\frac{1}{2}} \left( \frac{S-\frac{1}{2}}{s+\frac{1}{2}} \right)^n - \frac{1}{s(s+\frac{1}{2})} \left( \frac{S-\frac{1}{2}}{s+\frac{1}{2}} \right)^{n-1} \]

Inverting the Laplace transform

\[ I_n(t) = 2[L_{n-1}(t) - L_n(t)] - I_{n-1}(t) \]
with \( I_0(t) = 2(1 - l_0(t)) \)

The recursion above for \( I_n(t) \) is computationally useful for integrating a function in terms of its Laguerre function expansion. Two explicit expressions, however, are

\[
I_n(t) = -2l_n(t) + 4\sum_{j=0}^{n-1} (-1)^n l_j(t) + 2(-1)^n
\]

or

\[
I_n(t) = -2\sum_{j=1}^{n} (-1)^{n-j} [l_j(t) - l_{j-1}(t)] + 2(-1)^n(1 - l_0(t))
\]

And as a particular case,

\[
l_0(t) = 2(-1)^n, \text{ since } \lim_{t \to \infty} l_0(t) = 0
\]

2.2.8 Convolutions:

\[
x \cdot l_n(x) \ast l_m(x) = \int_0^x l_n(x-y) l_m(y) \, dy
\]

Taking Laplace transforms,

\[
\mathcal{L}(l_n(x) \ast l_m(x)) = \mathcal{L}(l_n(x)) \cdot \mathcal{L}(l_m(x))
\]

\[
= \frac{1}{(s+1/2)} \left( \frac{s-1/2}{s+1/2} \right)^n \frac{1}{s+1/2} \left( \frac{s-1/2}{s+1/2} \right)^m
\]

\[
= \frac{1}{s+1/2} \left( \frac{s-1/2}{s+1/2} \right)^{m+n} \left( 1 - \frac{s-1/2}{s+1/2} \right)
\]

\[
= \mathcal{L}(l_{m+n}(x)) - \mathcal{L}(l_{m+n+1}(x))
\]

And by inverting,

\[
x \cdot l_n(x) \ast l_m(x) = l_{m+n}(x) - l_{m+n+1}(x)
\]

2.2.9 Shifting Left:

Given a Laguerre function \( l_n(x) \), we want an expression for that function shifted left by \( y \) units, i.e. \( l_n(x+y) \). The
generating function of $\ell_n(x+y)$ is,
\[
\sum_{n=0}^{\infty} z^n \ell_n(x+y) = (1-z)^{-1} \exp \left[ -\frac{x+y}{2} \left( \frac{1+2z}{1-z} \right) \right]
\]
\[
= (1-z)^{-1} \exp \left[ -\frac{x}{2} \left( \frac{1+2z}{1-z} \right) \right] \exp \left[ -\frac{y}{2} \left( \frac{1+2z}{1-z} \right) \right]
\]
\[
= \left\{ \sum_{n=0}^{\infty} z^n \ell_n(x) \right\} (1-z) \left\{ \sum_{k=0}^{\infty} z^k \ell_k(y) \right\}
\]

Equating the coefficients of $z$ on both sides of the expression,
\[
z^i: \quad \ell_i(x+y) = \sum_{n=0}^{i} \ell_n(x) \left\{ \ell_{i-n}(y) - \ell_{i-n-1}(y) \right\}, \quad i \geq 0
\]
with $\ell_{-1}(x)$ defined as zero.

It turns out that this simple shifting property and the convolution property are the keys to the numerical solution of the GI/G/1 queue. Sumita [25] proves a more general form of the shifting theorem when bilateral transforms are used. His shift theorem allows a shift to the right as well, i.e. $\ell_n(x-y)$. This will be presented in Section 2.6.

2.3 PROPERTIES OF THE LAGUERRE TRANSFORM
Next we present some properties of the Laguerre transform as an integral operator which transforms a continuous function into a discrete set of coefficients. The properties presented below follow directly from the properties of the Laguerre functions presented in the last section.
Given a function \( f(x) \) defined on \([0, \infty)\) its Laguerre function expansion is given by,

\[
f(x) = \sum_{n=0}^{\infty} f_n l_n(x)
\]

Subsequent sections will present the conditions under which such expansions converge.

The Laguerre coefficients, \( f_n \), are defined by

\[
f_n = \int_0^\infty f(x) l_n(x) \, dx
\]

It is also useful to define the Laguerre sharp coefficients as in Keilson [8]. Specifically, \( f_n^\# = f_{n+1} - f_n \). The Laguerre transform then is a mapping from the vector space containing \( f(x) \) onto the real numbers,

\[
T_e : f(x) \mapsto \{ f_n \}
\]

or \( T_e^\# : f(x) \mapsto \{ f_n^\# \} \)

2.3.1 Linearity

Given \( f(x) = \sum_{n=0}^{\infty} f_n l_n(x) \) and \( g(x) = \sum_{n=0}^{\infty} g_n l_n(x) \)

then,

\[
h(x) = a f(x) + b g(x) = \sum_{n=0}^{\infty} h_n l_n(x)
\]

with \( h_n = a f_n + b g_n \)
2.3.2 Inner product of two functions.

Given \( f(x) = \sum_{n=0}^{\infty} f_n \ell_n(x) \) and \( g(x) = \sum_{k=0}^{\infty} g_k \ell_k(x) \)

then,
\[
\int_{c}^{\infty} f(x)g(x)dx = \sum_{n=0}^{\infty} f_n \sum_{k=0}^{\infty} g_k \int_{c}^{\infty} \ell_n(x)\ell_k(x)dx = \sum_{n=0}^{\infty} f_n g_n
\]
since \( \int_{c}^{\infty} \ell_n(x)\ell_k(x)dx = \delta_{nk} \)

This will be written as, \( (f, g) = \sum_{n=0}^{\infty} f_n g_n \)

And as a particular case we have \( (f, \ell_n) = f_n \).

2.3.3 Integral of a function

\[
F(t) = \int_{t}^{\infty} f(x)dx = \int_{t}^{\infty} \left[ \sum_{n=0}^{\infty} f_n \ell_n(x) \right]dx
\]

and if \( \sum_{n=0}^{\infty} f_n < \infty \) the order of summation and integration may be interchanged so that,

\[
F(t) = \sum_{n=0}^{\infty} f_n \int_{c}^{\infty} \ell_n(x)dx
\]

By Property (2.2.7)

\[
F(t) = 2 \sum_{n=0}^{\infty} (-1)^n f_n + 2 \sum_{n=0}^{\infty} f_n \sum_{j=0}^{n} (-1)^{n-j} \left[ l_j(t) - l_{j-1}(t) \right]
\]

And by reversing the order of summation over \( n \) and \( j \) it can also be written as

\[
F(t) = 2 \sum_{n=0}^{\infty} (-1)^n f_n + \sum_{j=0}^{\infty} l_j(t) \sum_{n=j+1}^{\infty} (-1)^{n-j} (f_n - f_{n-1})
\]

If one recursively computes \( I_n(t) \) as in Property (2.2.7), then \( F(t) = \sum_{n=0}^{\infty} f_n I_n(t) \). As a particular case

\[
F(\infty) = \int_{c}^{\infty} f(x)dx = 2 \sum_{n=0}^{\infty} (-1)^n f_n
\]
In the case where \( \int f(x) \, dx = 0 \), then the first term in \( F(t) \) above, \( 2 \sum_{n=0}^{\infty} (-1)^n f_n \), is zero. Then \( F(t) \) also has a Laguerre function expansion whose coefficients are given by

\[
F_k = \langle F, l_k \rangle = 2 \sum_{j=k+1}^{\infty} \sum_{n=j+1}^{\infty} (-1)^n \binom{n-j}{k} (f_n - f_{n-1})
\]

\[
= 2 \sum_{n=k+1}^{\infty} (-1)^n f_n^\# f_{n+k+1}
\]

But in most cases of interest here \( f(x) \) is a probability density so that \( F(t) \) does not have a Laguerre function expansion. The survivor function of \( f(x) \) is of interest.

\[
F_s(t) = \int_0^t f(x) \, dx = \int_t^\infty f(x) \, dx - \int_0^t f(x) \, dx
\]

As long as \( \sum_{n=0}^{\infty} |f_n| < \infty \), then the coefficients of \( F_s(t) \) are

\[
\langle F_s, l_k \rangle = F_{sk} = 2 \sum_{n=0}^{\infty} (-1)^n f_n^\# f_{n+k+1}
\]

which follows readily from \( (F(\infty) - F(t), l_k) \).

2.3.4 Generating functions.

Given a function \( f_n(x) = \sum_{n=0}^{\infty} f_n \ell_n(x) \), define the generating functions

\[
T_f(z) = \sum_{n=0}^{\infty} f_n z^n, \quad 0 \leq |z| < 1
\]

and

\[
T_f^\#(z) = \sum_{n=0}^{\infty} f_n^\# z^n = (1-z) T_f(z), \quad 0 \leq |z| < 1
\]

It is also useful to note as in Keilson [8] that

\[
T_f^\#(z) = (1-z) \sum_{n=0}^{\infty} \int_0^\infty f(x) \ell_n(x) \, dx
\]
and if \( f(x) \) is integrable on \([0, \infty)\), then by Property (2.2.4)

\[
T_{f}^{\#}(z) = \int_{0}^{\infty} f(x) \exp \left[ -\frac{x}{z} \frac{(1+z)}{1-z} \right] dx
\]

and defining
\[
\phi_{f}(s) = \phi_{f}(\frac{1}{2} \frac{1+z}{1-z})
\]

Thus, if the Laplace transform of a function is known, one way to obtain its Laguerre expansion coefficients is by equating powers of \( z \) in

\[
T_{f}^{\#}(z) = \sum_{n=0}^{\infty} f_{n} z^{n} = \phi_{f}(\frac{1}{2} \frac{1+z}{1-z})
\]

### 2.2.5 Convolution of Two Functions

\( h(x) = f(x) \ast g(x) = \left\{ \sum_{n=0}^{\infty} f_{n} l_{n}(x) \right\} \ast \left\{ \sum_{m=0}^{\infty} g_{m} l_{m}(x) \right\} \)

\[
= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} f_{n} g_{m} (l_{n}(x) \ast l_{m}(x))
\]

and from Property (2.2.6)

\[
h(x) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} f_{n} g_{m} [l_{n+m}(x) - l_{n+m+1}(x)]
\]

And therefore,

\[
h_{k}^{c} = (h_{k}, l_{k}) = \sum_{n=0}^{k} f_{n} (g_{k-n} - g_{k-n-1}) = \sum_{n=0}^{k} f_{n} g_{n}^{\#}
\]
2.3.6 *Shifting a function left.*

\[ f(x+y) = \sum_{n=0}^{\infty} f_n \cdot \ell_n(x+y) = \sum_{n=0}^{\infty} f_n \cdot \sum_{j=0}^{n} \ell_j(x)[\ell_{n-j}(y) - \ell_{n-j-1}(y)] \]

by Property (2.2.9)

Reversing the order of summation,

\[ f(x+y) = \sum_{j=0}^{\infty} \ell_j(x) \cdot \sum_{n=j}^{\infty} f_n [\ell_{n-j}(y) - \ell_{n-j-1}(y)] \]

Thus any function of argument \( x+y \) can be split neatly into a product of functions in \( x \) and functions in \( y \).

2.3.7 *Inner product of shifted and unshifted functions.*

\[ h^s(x) = \int_{0}^{\infty} f(x+y)g(y)dy = \sum_{j=0}^{\infty} \ell_j(x) \sum_{n=j}^{\infty} f_n g_{n-j}^* \]

by Property (2.2.5)

And thus,

\[ h^s_k = (h^s, \ell_k) = \sum_{n=k}^{\infty} f_n g_{n-k}^* \]

This property and the convolution property turn out to be the keys to a numerical solution of Lindley's equation for the GI/G/1 queue. This will be presented in detail in Chapter Four. Shifting is in a sense the flip side of the convolution property. Of necessity we must terminate the expansion at some point. Assume that the expansion coefficients \( h_k = 0 \) for \( k > K \). Then, we have

\[ h^c_k = \sum_{k=0}^{K} f_n g_{k-n}^* \] (convolution)
and \( h_k^s = \sum_{n-k}^{K} f_n g_{n-k} \) (shift)

Thus the similarity of form between Properties (2.3.5) and (2.3.7) is clear. As usual the sum of the subscripts of \( f_i \) and \( g_j \) add up to the subscript of \( h_k^c \) in the convolution. In the shift the difference of the \( f_i \) and \( g_j \) subscripts give the \( h_k^s \) subscript.

2.3.9 Coefficients of \( xf(x) \) and \( x^2f(x) \)

From the recurrence relation for the Laguerre functions one can derive the coefficients of \( g(x) = xf(x) \) and \( h(x) = x^2f(x) \).

\[
\int_0^\infty g(x) l_n(x) dx = \sum_{k=0}^{\infty} f_k \int_0^\infty l_k(x) l_n(x) dx
\]

And using the recurrence relation (2.2.1) for \( x l_n(x) \),

\[
\int_0^\infty g(x) l_n(x) dx = \sum_{k=0}^{\infty} f_k \int_0^\infty l_k(x) \left[ (2n+1) l_n(x) - (n+1) l_{n+1}(x) - n l_{n-1}(x) \right] dx
\]

\[
= \sum_{k=0}^{\infty} f_k \left[ (2n+1) d_{k,n} - (n+1) d_{k,n+1} - n d_{k,n-1} \right]
\]

\[
= (2n+1) f_n - (n+1) f_{n+1} - n f_{n-1}
\]

\[
= -(n+1) f_{n+1}^\# + n f_n^\#
\]

And similarly,

\[
h_n = -(n+1) g_{n+1}^\# + n g_n^\#
\]
2.3.9 Moments of a Distribution

The moments of a distribution may be expressed easily in terms of the Laguerre coefficients. Define,

\[ \mu_i = \int_0^\infty x^i f(x) \, dx \]

Then the moments of \( f(x) \) are given by

\[ \mu_0^f = 2 \cdot \sum_{n=0}^\infty (-1)^n \frac{f_n}{n} = \sum_{n=0}^\infty (-1)^n \frac{f_n}{n} \quad (f = 0) \]

\[ \mu_i^f = 2 \cdot \sum_{n=0}^\infty (-1)^n n \frac{f_n}{n} \quad (i = 0) \]

\[ \mu_0^f = 4 \cdot \sum_{n=0}^\infty (-1)^n n \frac{f_n}{n} \]

\[ \mu_1^f = 16 \cdot \sum_{n=0}^\infty (-1)^n n^2 \frac{f_n}{n} \]

The results for the second moment will be demonstrated for illustration. The other results follow by similar algebra. As in Property (2.3.8), let \( g(x) = xf(x) \), \( h(x) = xg(x) \). Then,

\[ \mu_2^f = \mu_1^g = 8 \cdot \sum_{n=0}^\infty (-1)^n g_n + 2 \mu_0^g = 8 \cdot \sum_{n=0}^\infty (-1)^n [n^2 f_n - (n+1)^2 f_{n+1}] + 2 \mu_0^f \]

Put

\[ \sum_{n=0}^\infty (-1)^n n^2 f_n = - \sum_{n=0}^\infty (-1)^n (n+1)^2 f_{n+1} \]

and

\[ \sum_{n=0}^\infty (-1)^n (n+1) f_{n+1} = - \sum_{n=0}^\infty (-1)^n n f_n = -2 \mu_0^f \]
Therefore,
\[ \mu_{zf} = 16 \cdot \sum_{n=0}^{\infty} (-1)^n n^2 f_n^2 \]

And to show the other form,
\[ \mu_{zf} = 8 \cdot \sum_{n=0}^{\infty} (-1)^n \left[ n(f_n - f_{n-1}) - (n+1)(f_{n+1} - f_n) \right] + 2 \mu_{\mathrm{IF}} \]

Regrouping terms in the summation,
\[ \mu_{zf} = 8 \cdot \sum_{n=0}^{\infty} (-1)^n \left\{ 2n^2 f_n - (n+1)^2 f_{n+1} - (n-1)^2 f_{n-1} \right. \\
+ nf_n + (n+1)f_{n+1} - 2(n-1)f_{n-1} \\
- f_{n-1} \left. \right\} + 2 \mu_{\mathrm{IF}} \]

Therefore, since
\[ \sum_{n=0}^{\infty} (-1)^n n^2 f_n = - \sum_{n=0}^{\infty} (-1)^n (n+1)^2 f_{n+1} = - \sum_{n=0}^{\infty} (-1)^n (n-1)^2 f_{n-1} \]

and
\[ \sum_{n=0}^{\infty} (-1)^n n f_n = - \sum_{n=0}^{\infty} (-1)^n (n+1)f_{n+1} = - \sum_{n=0}^{\infty} (-1)^n (n-1)f_{n-1} \]

then
\[ \mu_{zf} = 32 \cdot \sum_{n=0}^{\infty} (-1)^n n^2 f_n + 4 \mu_{\mathrm{IF}} \]

2.9 Convergence of Laguerre Function Expansions.
There is no simple analysis available to treat the error introduced by truncating Laguerre function expansions. Sumita [25] presents useful bounds based on assumptions which essentially restrict their applicability to exponential type functions. Since this research is concerned with the GI/G/1 queue, these bounds are not generally useful here.
This section will introduce measures of convergence and fit which are applicable to the probability distributions to be used as the arrival and service distributions for the GI/G/1 queue. The error measures introduced in the next three sections are based on convergence in the mean, convergence of moments, and pointwise convergence.

2.4.1 Convergence in the mean
As was mentioned earlier, the Laguerre functions are a complete set of functions on \([0, \infty)\). This implies that a sequence of increasingly higher order approximations (expansions) of a continuous function \(f(x)\) will converge in the mean to \(f(x)\).

More formally, define \(\psi_n(x) = \sum_{i=0}^{n} f_i l_i(x)\) as an \(n\)th order expansion of \(f(x)\). We say that \(\psi_n(x)\) approximates \(f(x)\) in the mean whenever the \(f_i\) are chosen to minimize,

\[
\|f(x) - \psi_n(x)\|^2 = \int_0^\infty (f(x) - \psi_n(x))^2 \, dx
\]

This can be written as

\[
\|f(x) - \psi_n(x)\|^2 = \int_0^\infty f(x)^2 \, dx - \sum_{i=0}^{n} f_i^2 l_i^2 - \sum_{i=0}^{n} [f_i - f(x) l_i]^2
\]

And clearly the minimum is attained for \(f_i = (f, l_i)\).

The convergence in the mean implies that

\[
\lim_{n \to \infty} \|f(x) - \psi_n(x)\| = 0
\]

This leads directly to Plancherel's identity \([4]\).
\begin{equation}
\int_{0}^{\infty} f(x) dx = \sum_{i=1}^{\infty} f_i^2
\end{equation}

which applies to any square integrable function. As a practical matter then, one measure of the quality of a Laguerre function expansion is the quantity

\[ E_i = \| f(x) - \psi_n(x) \|^2 \]

Clearly for the neglected expansion terms we have,

\[ \sum_{i=n+1}^{\infty} f_i^2 = \| f(x) - \psi_n(x) \|^2 \]

So that in the worst case,

\[ |f_i| \leq \| f(x) - \psi_n(x) \|, \quad \text{for } i \geq n \]

Since any piecewise continuous function \( f(x) \) can be approximated in the mean by a continuous function \( g(x) \) [17], it follows that the Laguerre function expansion of a piecewise continuous function converges in the mean. For example, in the queueing context we might work with the uniform distribution. The foregoing remarks imply that the square sum criterion is still a good measure of the quality of the expansion in spite of the discontinuity of the uniform distribution.

2.4.2 **Convergence of moments.**

All of the methods to be applied to the GI/G/1 queue require truncating the expansions of probability densities after some reasonable number of terms. Consequently, two important questions are how fast the expansion coefficients decay and are they summable. Summability is of direct
interest when obtaining the moments of the distributions and allows one to use quantities such as,

\[ E_{20} = \int_c^\infty f(x)dx - \sum_{n=0}^N (-1)^n f_n^* I \int_c^\infty f(x)dx \]

\[ E_{21} = \int_c^\infty xf(x)dx - 4\sum_{n=0}^N (-1)^n n^2 f_n^* I \int_c^\infty f(x)dx \]

\[ E_{22} = \int_c^\infty x^2 f(x)dx - 16\sum_{n=0}^N (-1)^n n^2 f_n^* I \int_c^\infty x^2 f(x)dx \]

to measure the accuracy of an expansion.

Rather broad statements can also be made which relate the rate of decay of the Laguerre expansion coefficients to the characteristics of the functions being expanded. In general, expansions of smooth functions localized about the origin converge fairly rapidly. Smoothness in this usage can be roughly measured by the number of continuous derivatives of the function, although peakedness of the function is also important. Localization can be related to the magnitude of the moments of the distribution. All of these topics will now be considered.

Assume a function \( f(x) \) on \([0, \infty)\) which is integrable but has a discontinuity at \( x=t \) such that

\[ \lim_{h \to 0} \left[ f(t+h) - f(t-h) \right] \neq 0 \]

The expansion coefficients \( f_n \) are obtained from the integral

\[ f_n = \int_0^t \ell_n(x) f(x)dx + \int_t^\infty \ell_n(x) f(x)dx \]
and are well defined. Assume contrary to fact that the coefficients are absolutely summable so that \( \sum_{n=0}^{\infty} |f_n| < \infty \).

\[
f(t+h) - f(t-h) = \sum_{n=0}^{\infty} f_n [l_n(t+h) - l_n(t-h)]
\]

Taking limits,

\[
\lim_{h \to 0} [f(t+h) - f(t-h)] = \lim_{h \to 0} \sum_{n=0}^{\infty} f_n [l_n(t+h) - l_n(t-h)] \\
= \sum_{n=0}^{\infty} f_n \left\{ \lim_{h \to 0} [l_n(t+h) - l_n(t-h)] \right\}
\]

which is legitimate by the assumption of absolute summability. But \( \lim_{h \to 0} [l_n(t+h) - l_n(t-h)] = 0 \) and therefore the contradiction implies that the coefficients are not absolutely summable. This indicates that the expansion coefficients of a uniform distribution for example will be slow to decay to zero. It will be shown later that in a queueing context it is a combination of the arrival and service distributions whose rate of decay is important. The arrival distribution might be discontinuous for instance, but this might be compensated in a sense by the service distribution.

Keilson, Nunn [8] prove the following result which emphasizes the combined importance of smoothness and localization in determining the decay of the coefficients.
Theorem 2.1: Decay of the expansion coefficients

If (a.) the rth derivative of \( f(x) \), \( f^{(r)}(x) \), is continuous and bounded on \((0, \infty)\), \( r=0, 1, 2, \ldots, 2K \)

\[
(b.) \left[ \frac{1}{4} \frac{d}{dt} t \frac{d}{dt} \right]^r f(t) \in L_2(0, \infty), \quad r=0, 1, \ldots, K
\]

(\( L_2 \) defined as the class of square integrable functions)

then 

\[
f_n = O(n^{-K}), \quad n \to \infty
\]

Keilson and Nunn point out that (b) requires the existence of a 2Kth moment for \( f^2(x) \). That is,

\[
\int_0^\infty x^{2K} f^2(x) dx < \infty
\]

The theorem presents sufficient conditions that the expansion coefficients will ultimately decay rapidly. Condition (a) is based on smoothness and condition (b) on localization. The theorem can not say how large \( n \) must be before the rapid decay begins. If an analytic expression is available for the expansion coefficients one can easily bound the errors by directly determining suitably large \( n \).

The relation between the extent of the coefficients and the characteristics of the function being approximated can be quantified somewhat using concepts borrowed from Fourier transforms as in Sumita [25]. Define \( N_E \) as the extent of the Laguerre expansion coefficients with

\[
N_E = \sum_{n=0}^{\infty} (n+\frac{1}{2}) f_n^2 / \sum_{n=0}^{\infty} f_n^2.
\]
Let $T_{E}$ be the extent of the function being approximated with

$$T_{E} = \int_{c}^{\infty} x f(x) \, dx \int_{c}^{\infty} f(x) \, dx$$

Let $B_{E}$ be

$$B_{E} = \int_{c}^{\infty} \left[ \frac{df(x)}{dx} \right]^{2} \, dx \int_{c}^{\infty} f(x) \, dx$$

Then the following theorem applies [25].

**Theorem 2.2:** Extent of the coefficients.

Let (1.) $f(x) \in L_{2}$, and

(2.) $f_{n}$ have the property $n^{(3+\varepsilon)} f_{n} \to 0$, as $n \to \infty$.

Then,

(a.) $N_{E} = (1/4) T_{E} + B_{E}$

(b.) $N_{E} \geq (1/4) (T_{E} + 1/T_{E}) \geq 1/2$.

The second condition in Theorem 2.2, i.e. $n^{(3+\varepsilon)} f_{n} \to 0$, will be satisfied whenever the conditions of Theorem 2.1 are satisfied for $r \geq 3$. Theorem 2.2 is more useful for comparing two functions than as an absolute measure of the extent of a set of expansion coefficients.

An example will illustrate the point. Consider the values for three pdfs

$$f_{1}(x) = \exp(-.1x) \quad T_{E1} = 5 \quad B_{E1} = .05 \Rightarrow N_{E1} = 1.3$$

$$f_{2}(x) = .5\exp(-.5x) \quad T_{E2} = 1 \quad B_{E2} = .25 \Rightarrow N_{E2} = .5$$
\[ f_3(x) = 10 \exp(-10x) \quad T_{E3} = 0.05 \quad B_{E3} = 5 \Rightarrow N_{E3} = 5.0125 \]

\( N_{E2} \) achieves the minimum value which is logical since the Laguerre expansion of \((1/2)\exp(-x/2)\) has \(f_0 = 1, f_1 = 0^+\) for \(i > 0\). Also it is clear that in this case the penalty for peakedness \(N_{E3} = 5.0125\) is greater than the penalty for lack of localization \(N_{E1} = 1.3\). All three functions are equally smooth in that they have an infinite number of continuous derivatives.

2.4.3 Pointwise convergence

As a final indicator of how well a Laguerre function expansion fits the function in question one can add a direct approach—comparing values of the function with values of the expansion at a number of points. Two quantities are of interest, the maximum error and the average error over the sampled points.

The Laguerre expansion is an approximation

\[ S_N(x) = \sum_{n=0}^{N} f_n l_n(x) \]

The quantity of interest here is the error

\[ E(x) = f(x) - S_N(x) = \sum_{n=N+1}^{\infty} f_n l_n(x) \]

An empirical approach is to sample the expansion at a number of points, \(x_i\), \(i = 1, 2, \ldots, K\), within an interval where the function is appreciably non-zero. Two added measures of the quality of an expansion can then be introduced.
Observation of a large number of expansions shows that typically $E_{3m}=E(0)$, whenever the origin is in the set of sampled points. A heuristic explanation for this observation is that any term added to an expansion has its maximum at $x=0$, the maximum for all the Laguerre functions.

2.5 SCALING A FUNCTION

The objective is to obtain the expansion coefficients of an arbitrary function $g(x)=f(cx)$ in terms of the expansion coefficients $f_n$ of $f(x)$ and the scaling parameter $c$ via a general scaling procedure. As a practical matter it may be simpler in many cases to reapply whatever technique yielded the $f_n$ directly to the function $g(x)$ than to use the scaling technique developed here. But there will be other cases (e.g. if the coefficients were obtained through numerical integration) where scaling will be advantageous. The plan here is to review two theorems found in Sumita [25], and combine these to produce a scaling theorem. Then a few observations will extend the applicability of the theorem to a wider range of values of the scaling parameter. An algorithm for reducing the computation in obtaining the scaling matrix is given in Appendix A along with a few examples.
Theorem 2.3 [25]

Let \[ g(x) = \exp\left[-\frac{1}{2} (1-c)x\right] f(cx) \quad 0 < c < 1, \quad x > 0 \]
then \[ g_k = \sum_{n=k}^{\infty} f_n(k)c^k(1-c)^{n-k}, \quad k = 0, 1, 2, \ldots \]

Proof:
A somewhat different proof is given here than in the reference. From Abramowitz and Stegun [1],
\[ L_n(cx) = \sum_{j=0}^{n} L_j(x) \binom{n}{j} c^j (1-c)^{n-j} \]
Therefore,
\[ g(x) = \exp\left[-\frac{1}{2} (1-c)x\right] f(cx) = \exp\left[-\frac{1}{2} (1-c)x\right] \sum_{n=0}^{\infty} f_n L_n(cx) \]
\[ = \sum_{n=0}^{\infty} \exp\left[-\frac{1}{2} (1-c)x\right] f_n \sum_{j=0}^{n} L_j(x) \binom{n}{j} c^j (1-c)^{n-j} \]
\[ = \sum_{n=0}^{\infty} f_n \sum_{j=0}^{n} L_j(x) \binom{n}{j} c^j (1-c)^{n-j} \]
And \[ g_k = (g, l_k) = \sum_{n=0}^{\infty} f_n \sum_{j=0}^{n} \binom{n}{j} c^j (1-c)^{n-j} \]
Since \[ (l_j, l_k) = \delta_{jk} \]
\[ g_k = \sum_{n=k}^{\infty} f_n(k)c^k(1-c)^{n-k}, \quad k = 0, 1, 2, \ldots \]
It is also shown in the reference that the theorem applies to $c > 1$ if
\[ T_f(z) = \sum_{n=0}^{\infty} f_n z^n \]
has a radius of convergence greater than $(2c-1)$. This follows because
\[ \sum_{k=0}^{n} |\binom{n}{k} c^k (1-c)^{n-k}| \leq (2c-1)^n \]
which means that
\[ g_k = \sum_{n=0}^{\infty} f_n \sum_{j=0}^{n} \binom{n}{j} l_j c^j (1-c)^{n-j} \leq \sum_{n=0}^{\infty} f_n (2c-1)^n \]

**Theorem 2.4 [25]:** Exponential times a function.

Let
\[ g(x) = e^{-\theta x} f(x), \quad \theta > 0 \]

Then,
\[ G_m = \sum_{n=0}^{\infty} f_n p_{mn}(\theta) \]

where
\[ p_{mn}(\theta) = \int_0^{\infty} e^{-\theta x} l_m(x) l_n(x) dx. \]

**Proof:**
\[ g_m = (g, l_m) = \int f(x) e^{-\theta x} l_m(x) dx \]
\[ = \sum_{n=0}^{\infty} f_n \int l_n(x) l_m(x) e^{-\theta x} dx \]
\[ = \sum_{n=0}^{\infty} f_n p_{mn}(\theta) \quad \Box \]

A procedure is available in the reference to generate the $p_{mn}(\theta)$ matrix. But the algorithm given in the Appendix A does not require that the $p_{mn}(\theta)$ be generated. The
importance here is that the two theorems can be combined to give a fairly general scaling theorem.

**Theorem 2.5:** Scaling functions

Let

\[ g(x) = \exp\left[-\frac{1}{2} (1-c)x\right] f(cx) \]

and

\[ h(x) = \exp\left[\frac{1}{2} (1-c)x\right] g(x) \quad c>1 \]

with radius of convergence of \( T_f(u) \) greater than \( 2c-1 \).

Then

\[ h(x) = f(cx) \]

and

\[ h_m = \sum_{n=0}^{\infty} f_n \Gamma_{mn}(c) \]

with

\[ \Gamma_{mn}(c) = \sum_{k=0}^{n} \binom{n}{k} c^{n-k} (1-c)^k p_{m,n-k}(\theta) \]

\[ \theta = (c-1)/2 \]

**Proof:**

From Theorems 2.3 and 2.4, we have

\[ h_m = \sum_{k=0}^{\infty} g_k \Gamma_{mk}(\theta) \]

\[ G_k = \sum_{n=k}^{\infty} f_n \binom{n}{k} c^{k} (1-c)^{n-k} \]

Therefore,

\[ h_m = \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} f_n \binom{n}{k} c^{k} (1-c)^{n-k} \Gamma_{mk}(\theta) \]

Interchanging the order of summation,

\[ h_m = \sum_{n=0}^{\infty} f_n \sum_{k=0}^{n} \binom{n}{k} c^{n-k} (1-c)^k \Gamma_{m,n-k}(\theta). \]
The range of \( c \) over which the Theorem is valid should be considered. Theorem 2.4 was valid for \( \Theta > 0 \) which implies \( c > 1 \). But we can take a different view of the \( M_{mn}(c) \) matrix to extend the theorem to \( c < 1 \). The elements of \( M_{mn}(c) \) are really the values of the integrals

\[
M_{mn}(c) = \int_{0}^{\infty} l_m(x) l_n(cx) \, dx
\]

This can be seen since for

\[
g(x) = f(cx)
\]

\[
= \sum_{n=0}^{\infty} f_n l_n(cx)
\]

we have

\[
g_n = (g, l_n) = \sum_{n=0}^{\infty} f_n \int_{0}^{\infty} l_m(x) l_n(cx) \, dx
\]

\[
= \sum_{n=c}^{\infty} f_n M_{mn}(c)
\]

But if we consider the integral

\[
M_{mn}'(c) = \int_{0}^{\infty} l_m(x) l_n(x/c) \, dx, \quad c > 1
\]

and make the transformation \( x \rightarrow cx \), then

\[
M_{mn}'(c) = \int_{0}^{\infty} l_m(cx) l_n(x) \, dx, \quad c > 1
\]

\[
= c \int_{0}^{\infty} l_m(cx) l_n(cx) \, dx
\]

And therefore,

\[
M_{mn}(1/c) = c \cdot M_{mn}(c)
\]
So if one can obtain a scaling matrix for \( c > 1 \), \( 1/c \) times the transpose of that matrix allows one to scale with \( (1/c) < 1 \). Computing the scaling matrix via the expression in Theorem 2.5 is time consuming. Some of the intermediate steps can be avoided with the algorithm in Appendix A.

Two qualitative comments on the scaling matrix. First, as \( c \to 1 \) the scaling matrix becomes the identity matrix as expected. Second, the elements of the matrix become large quickly as \( c \) approaches 2. One should not view scaling via this matrix as a means to make radical expansions or contractions in scale.

### 2.6 Bilateral Laguerre Transform

The bilateral Laguerre transform enables one to expand functions defined on \( (-\infty, \infty) \). The same prerequisites to guarantee convergence apply in the bilateral case as in the one-sided transform. The bilateral transform was not applied directly in the queueing portion of the research because a one-sided transform procedure will lead to the same result with less computational effort in some cases. The reasons for this will be clear in Chapter Four.

There are some results from the bilateral transform which are used directly in obtaining expansion coefficients of probability distributions and indirectly in showing the equivalence of the one-sided and bilateral transforms relative to the queueing. These results will now be discussed.
A set of bilateral or extended Laguerre functions is defined such that,

\[ h_n(x) = \begin{cases} \ell_n(x)u(x), & n > 0 \\ -\ell_{n-1}(-x)u(-x), & n < 0 \end{cases} \]

where \( u(x) \) is the unit step function along the \( t \) axis.

If a function is symmetric about the origin then the definition above makes it trivial to obtain the expansion coefficients for the negative domain given those for the positive domain of the function. That is

\[ f_n = \int_{-\infty}^{\infty} f(x)h_n(x)\,dx \]

and for \( n < 0 \),

\[ f_n = -\int_{-\infty}^{\infty} f(-x)\ell_{-n-1}(-x)\,dx \]

Letting \( x \to -x \),

\[ f_n = -\int_{-\infty}^{\infty} f(-x)\ell_{-n-1}(x)\,dx \quad n < 0 \]

But since \( f(x) = f(-x) \), then

\[ f_n = -\int_{-\infty}^{\infty} f(x)\ell_{-n-1}(x)\,dx \quad n < 0 \]

which implies

\[ f_n = -f_{-n-1} \]

for all \( n \)

For the sharp coefficients

\[ f_{n*} = f_n - f_{n-1} \]

for all \( n \)

So that

\[ f_{n*} = -f_{n-1} + f_n \]
The above relations are used in two ways. First suppose that we have a function defined on \((-\infty, \infty)\) and we want the expansion coefficients of the portion on \((-\infty, 0)\). As in Figure 3 we can reflect the negative portion of \(f(x)\) and obtain the one-sided transform coefficients of \(f(-x)\). The symmetry rules will allow us to obtain the bilateral expansion coefficients immediately via equations (2.6.1) and (2.6.2). The only point where care is demanded is in noting that for the one-sided transform \(f_o^\# = f_o\) whereas for the bilateral transform \(f_o^\# = f_o - f_{-1}\).

\[ f_n^\# = -f_{-n}^\# \quad \text{for all } n \]
Second there is a bilateral shift theorem [25] which allows one to obtain the coefficients of a function that has been shifted left or right in terms of the coefficients of the unshifted function. This theorem is useful, for instance, in obtaining the expansion coefficients of a normal distribution centered away from the origin. Using the theorem one can first obtain the coefficients of a $N(0, \sigma^2)$ distribution which is a simpler task. The shift theorem can then be used to obtain the coefficients of the desired $N(\mu, \sigma^2)$ distribution. Proof of the theorem can be found in the references [9] and [25].

**Theorem 2.6: Bilateral transform shift theorem.**

Let \( g(x) = f(x-T), \quad T \text{ real} \)

Then \[
G_n = - \sum_{k=n+1}^{\infty} \sum_{m=-\infty}^{k} f_m \Delta h_m(T)
\]

where \[
\Delta h_m(T) = h_m(T) - h_{m-1}(T)
\]

**2.7 SUMMARY**

Several properties of the Laguerre functions and transform have been presented. The key properties in this research are those governing orthogonality, convolution, and shifting. These properties are central to the numerical solution of GI/G/1.
The number of terms needed to approximate a function by a Laguerre function expansion depends on the smoothness, localization, and peakedness of the function. Shrinking a function in toward the origin may make the function more localized, but beyond a certain point this actually makes the expansion converge more slowly.

Several useful and common sense measures of the fit of an expansion were presented. These were based on square integrability of the function, moments of the function, and pointwise convergence of the expansion.

It is possible to scale a function in terms of its expansion coefficients. One can obtain the coefficients of the scaled function directly in terms of the coefficients of the original function through a scaling matrix. This capability is useful when dealing with a function whose expansion coefficients are only obtainable through some tedious numerical technique.

The formalism of the bilateral Laguerre transform is an elegant extension of the usual Laguerre transform to \((-\infty, \infty)\). Its symmetry and shifting properties are especially useful in obtaining expansions for the normal distribution.
Chapter III

LAGUERRE EXPANSIONS OF PROBABILITY DISTRIBUTIONS.

3.1 INTRODUCTION

Having looked at some general properties of the Laguerre transform, Chapter Three will present Laguerre function expansions of some probability distributions. Many of the results for specific distributions can be found in other sources but are included here to give a fairly complete selection for reference.

Section 3.2 briefly discusses the principal techniques for obtaining Laguerre function expansions. Included here is a suggestion for a specific numerical integration procedure to obtain expansion coefficients when other methods fail. An error bound for the procedure is obtained. A recurrence relation is also developed which makes calculation of the expansion coefficients from a function's moments much simpler.

Section 3.3 will illustrate the techniques by application to a number of probability distributions. It will also apply the various measures of fit or convergence that were presented in Chapter Two to the distributions. Final-
ly, in Section 3.4 an approximation is suggested that can be used to calculate the moments of an (unknown) function when only a small number of its expansion coefficients are available.

3.2 TECHNIQUES FOR OBTAINING EXPANSIONS

There are basically three approaches to obtaining the Laguerre expansion coefficients of a function. These methods will be presented in roughly an order of increasing computational burden.

3.2.1 Analytic methods

If one can integrate

\[ I_n = \int_0^\infty f(x) L_n(x) \, dx \tag{3.2.1} \]

directly for all \( n \), or can construct a stable recurrence relation such as

\[ I_n = c_{n-1} I_{n-1} + c_{n-2} I_{n-2} + \ldots + c_{n-k} I_{n-k} \tag{3.2.2} \]

then the analytic method applies. This is clearly the technique of choice in terms of minimizing computational effort. However, \( f(x) \) must usually be of a simple form to accomplish this.

Recurrence relations typically result from integrating by parts or from employing the Laguerre function recursion,

\[ (n+1) L_{n+1}(x) = (2n+1-x)L_n(x) - nL_{n-1}(x) \tag{3.2.3} \]
Using (3.2.3) in (3.2.1) would yield,

\[(3.2.4) \quad (n+1)I_{n+1} = (2n+1)I_n - \int_0^\infty f(x)l_n(x)dx - nI_{n-1}\]

And of course the utility of this depends on the particular \( f(x) \). Some examples will be presented in Section 3.3.

### 3.2.2 Series expansion techniques

These techniques are presented in Keilson [8]. Series expansion techniques are based on expanding the Laplace transform of \( f(x) \), \( \phi_f(s) \), about the points \( s=1/2 \) or \( s=0 \).

Property (2.3.4) gave

\[(3.2.5) \quad \phi_f^\#(z) = \phi_f\left(\frac{1}{2}, \frac{1+\frac{z}{1-z}}{1-z}\right) = \sum_{n=0}^{\infty} f_{n}^\# z^n\]

and

\[
\phi_f\left(\frac{1}{2}, \frac{1+\frac{z}{1-z}}{1-z}\right) = \phi_f\left(\frac{1}{2}, \frac{z}{1-z}\right) = \int_0^\infty f(x) e^{-x/2} e^{-x/2/(1-z)} dx
\]

\[
= \sum_{k=0}^{\infty} \left( \int_0^\infty f(x) e^{-x/2} \frac{(x)^k}{k!} dx \right) \left( \frac{z}{1-z} \right)^k
\]

\[
= \sum_{k=0}^{\infty} h_k \left( \frac{z}{1-z} \right)^k = \sum_{k=0}^{\infty} h_k \sum_{m=0}^{\infty} (m-1)! \frac{z^m}{m!}
\]

Reversing the order of summation,

\[(3.2.6) \quad \phi_f\left(\frac{1}{2}, \frac{1+\frac{z}{1-z}}{1-z}\right) = \sum_{m=0}^{\infty} \frac{z^m}{m!} \sum_{k=1}^{m-1} h_k (k-1)
\]

Comparing (3.2.5) and (3.2.6) then,

\[(3.2.7) \quad f_{n}^\# = \sum_{k=1}^{n} h_k (k-1)
\]
The key is obviously performing the integral
\[ h_k = \int_0^\infty f(x) e^{-x^2} \frac{(-x)^k}{k!} \, dx \]

When the \( h_k \) in (3.2.9) cannot be obtained a second method, usually called the method of moments, is available. This is based on a Taylor series expansion of \( \phi_f(s) \) about \( s=0 \).

Let
\[ \phi_f(s) = \sum_{n=0}^\infty \alpha_n s^n \]

Then
\[ \alpha_n = \frac{(-1)^n}{n!} \mu_n \]

where
\[ \mu_n = \int_0^\infty f(x) x^n \, dx \]

If the moments of \( f(x) \) are available then the expansion coefficients of the Laplace transform, \( \phi_f(s) \), are easily available. As before,

\[ T_f^{\#}(\xi) = \phi_f \left( \frac{1}{2} + \frac{\xi}{1-\xi} \right) = \sum_{n=0}^\infty \alpha_n \left( \frac{1}{2} + \frac{\xi}{1-\xi} \right)^n \]

\[ \left( \frac{1}{2} + \frac{\xi}{1-\xi} \right)^n = \sum_{k=0}^\infty g_{nk} \xi^k \quad 0 \leq |\xi| < 1 \]

\[ \left( \frac{\xi}{1-\xi} \right)^r = \sum_{k=r}^\infty (k-1)^{r-1} \xi^k \quad r \geq 1 \]

and using the binomial expansion
\[ g_{nk} = \sum_{r=1}^{\min\{n,k\}} \binom{n}{r} \binom{k}{r-1} \xi^{n-r} (k-r)^r, \quad k \geq 1 \quad g_{no} = \left( \frac{1}{2} \right)^n, \quad k = 0. \]
Combining (3.2.11) and (3.2.12)

$$g_k \left( \frac{1}{2} + \frac{z}{1-z} \right) = \sum_{k=0}^{\infty} \left[ \sum_{j=0}^{\infty} \alpha_j g_{nk} \right] z^k$$

and therefore:

$$g_k = \sum_{n=0}^{\infty} \alpha_n g_{nk} \quad \text{with} \quad g_{nk} \text{ given by (3.2.14).}$$

Although it may not be obvious a simple recurrence relation can be established here which makes computation of the $g_{nk}$ simpler than straightforward use of equation (3.2.14). This recurrence is stated in the following theorem.

**Theorem 3.1:** **Coefficients for the Method of Moments.**

Given:

$$g_{nk} = \sum_{r=1}^{m} \left( \begin{array}{c} m \n k \end{array} \right) \left( \frac{\bar{z}}{2} \right)^{m-r} (k-r) \quad k \geq 1$$

then

$$(3.2.15) \quad g_{n+1,k} = \frac{k}{n} g_{nk} + \frac{1}{4} g_{n-1,k} \quad k \geq 1, \ n \geq 1$$

with $g_{0k} = 0$, $g_{1k} = 1$.

**Proof:**

$$\sum_{k=0}^{\infty} g_{n+1,k} z^k = \left( \frac{1}{2} + \frac{z}{1-z} \right)^{n+1} \quad 0 < |z| < 1$$

$$= \left( \frac{1}{2} + \frac{z}{1-z} \right)^2 \left( \frac{1}{2} + \frac{z}{1-z} \right)^{n-1}$$

$$= \left( \frac{1}{4} + \frac{z}{(1-z)^2} \right) \left( \frac{1}{2} + \frac{z}{1-z} \right)^{n-1}$$

$$= \frac{1}{4} \sum_{k=0}^{\infty} g_{n-1,k} z^k + \frac{z}{(1-z)^2} \left( \frac{1}{2} + \frac{z}{1-z} \right)^{n-1}$$

2nd term r.h.s. =

$$= \frac{1}{2} \left( \frac{z}{1-z} \right)^2 \left( \frac{1}{2} + \frac{z}{1-z} \right)^{n-1}$$

$$= \frac{1}{2} \sum_{j=0}^{n-1} \left( \begin{array}{c} n-1 \n j \end{array} \right) \left( \frac{z}{2} \right)^j \left( \frac{z}{1-z} \right)^{n-j+1}$$
Therefore

\[ \frac{1}{z} \sum_{j=0}^{n-1} \frac{n-j}{n} \left( \frac{j}{z} \right)^{-j} \left( \frac{z}{1-z} \right)^n j^{j+1} \]

let \( j \to n-j \)

\[ = \frac{1}{z} \cdot \frac{1}{n} \sum_{j=1}^{n} j\left( \frac{j}{z} \right)^{n-j} \left( \frac{z}{1-z} \right)^{(n-j)+1} \]

but

\[ \left( \frac{z}{1-z} \right)^{n-j+1} = \sum_{k=n-j+1}^{\infty} (l-1) \leq l \]

Therefore,

**2nd term r.h.s.**

\[ = \frac{1}{n} \sum_{j=1}^{n} \left( \frac{1}{j} \right)^{n-j} \sum_{l=n-j}^{\infty} \left( \frac{1}{l-1} \right) \leq \frac{1}{l} \]

\[ = \frac{1}{n} \sum_{j=1}^{n} \left( \frac{1}{j} \right)^{n-j} \sum_{k=j+1}^{\infty} \leq \frac{1}{k-1} \]

So we have:

\[ \sum_{k=0}^{\infty} g_{n+1,k} z^k = \frac{1}{4} \sum_{k=0}^{\infty} g_{n-1,k} z^k + \frac{1}{n} \sum_{j=1}^{n} \left( \frac{1}{j} \right)^{n-j} \sum_{k=j+1}^{\infty} \frac{1}{k-1} \leq l \]

Consider the coefficient of \( z^k \).

If \( k>n \), the upper limit of the 2nd sum r.h.s. is n

If \( n<k \), the upper limit of the 2nd sum r.h.s. is k

In either case \( l-1=k \) or \( l=k+1 \).

\[ z^k: \quad g_{n+1,k} = \frac{1}{4} g_{n-1,k} + \frac{1}{n} \sum_{j=1}^{\min(n,k)} \left( \frac{1}{j} \right)^{n-j} (k+1-j) (j-1) \]

\[ = \frac{1}{4} g_{n-1,k} + \frac{k}{n} \sum_{j=1}^{\min(n,k)} \left( \frac{1}{j} \right)^{n-j} (k-1) \]

Put by definition,

\[ g_{nk} = \sum_{j=1}^{\min(k,n)} \binom{n}{j} \left(\frac{1}{2}\right)^{n-j} (j-1)^{k-1} \]

Therefore,

\[ g_{n+1,k} = \frac{1}{4} g_{n-1,k} + \frac{k}{n} g_{n,k} \]

The application of the theorem makes the method of moments a very efficient way to obtain expansion coefficients. Instead of the computationally difficult expression (3.2.14), the simple recursion (3.2.15) is used. The length of the summation depends on whether and how fast the terms \( \alpha_n \) go to zero as \( n \) goes to infinity. The table below presents some values of the \( g_{nk} \) for illustrative purposes.

**Table 1**

**Values of the coefficients \( g_{nk} \).

\begin{center}

<table>
<thead>
<tr>
<th>( n )</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k )</td>
<td>( 5 )</td>
<td>2.73E+01</td>
<td>8.24E-01</td>
<td>6.07E-03</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>4.67E+03</td>
<td>3.18E+03</td>
<td>1.66E+02</td>
</tr>
<tr>
<td></td>
<td>15</td>
<td>1.63E+06</td>
<td>1.32E+06</td>
<td>4.27E+05</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>1.15E+07</td>
<td>1.46E+08</td>
<td>2.52E+08</td>
</tr>
<tr>
<td></td>
<td>25</td>
<td>5.79E+07</td>
<td>6.80E+09</td>
<td>5.32E+10</td>
</tr>
<tr>
<td></td>
<td>30</td>
<td>2.28E+08</td>
<td>1.72E+11</td>
<td>5.27E+12</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>7.50E+08</td>
<td>3.76E+13</td>
<td>1.03E+16</td>
</tr>
<tr>
<td></td>
<td>50</td>
<td>5.15E+09</td>
<td>1.95E+15</td>
<td>4.51E+18</td>
</tr>
<tr>
<td></td>
<td>60</td>
<td>2.02E+10</td>
<td>5.85E+16</td>
<td>7.14E+20</td>
</tr>
<tr>
<td></td>
<td>70</td>
<td>1.13E+11</td>
<td>1.05E+18</td>
<td>5.43E+22</td>
</tr>
<tr>
<td></td>
<td>80</td>
<td>3.73E+11</td>
<td>1.29E+19</td>
<td>2.38E+24</td>
</tr>
<tr>
<td>abs (alpha)</td>
<td>2.55E-06</td>
<td>1.06E-15</td>
<td>1.36E-26</td>
<td>1.83E-38</td>
</tr>
</tbody>
</table>
\end{center}
For any given coefficient $f_k$, one needs to examine the size of the product $q_{nk} x_n$, $n=1,2,3,...$ to determine a sufficient number of terms for a given accuracy. Later in the chapter the method of moments will be applied to the Weibull and normal distributions to illustrate its effectiveness. There are also cases such as the lognormal distribution which illustrate its limitations.

For the lognormal distribution,

$$f(x) = \frac{1}{\sqrt{2\pi} \sigma_Y x} \exp \left[ -\frac{1}{2} \left( \log x - \mu_Y \right)^2 \right]$$

where $\mu_Y, \sigma_Y^2$ are the mean and variance of a normal variate $Y$ such that $Y=\log X$.

Then

$$\mu_{xn} = \exp \left[ n\mu_Y + \frac{n^2}{2} \sigma_Y^2 \right]$$

and

$$x_n = \frac{(-1)^n}{n!} \mu_{xn}$$

In this case, the $x_n$ grow very rapidly and the method of moments is useless. Furthermore, literature search has failed to find published definite or indefinite integrals that would allow one to evaluate

$$\int_0^\infty f(x) l_{kn}(x) dx$$

for the lognormal distribution directly or indirectly. This suggests that for some distributions a numerical integration technique would be useful. This is the topic of the next section.
3.2.3 Numerical Integration Techniques

This section will present a particular numerical integration scheme which makes use of the recurrence relation of the Laguerre functions in computing expansion coefficients. The claim is that the technique is valuable because of its general applicability. An approximate error analysis will also be presented.

As was illustrated above the method of moments may be slow to converge, or will not converge, to the Laguerre expansion coefficients. We want the expansion coefficients,

$$f_n = \int_0^\infty f(x) L_n(x) \, dx, \quad n = 0, 1, 2, \ldots, N$$

(3.2-16)

If one attempts to compute each $f_n$ sequentially, i.e., $f_0$, then $f_1$, then $f_2$, ..., $f_n$, using a standard adaptive quadrature, it becomes increasingly time consuming to evaluate $L_n(x)$ for each higher order $n$. For instance, obtaining $f_{50}$ requires evaluating $\{L_{50}(x_i)\}$ where the $\{x_i\}$ are points dictated by the quadrature algorithm.

Evaluating each set of values $\{L_{n+1}(x_i)\}$ requires stepping through the recurrence relation $n-2$ times unless the values $\{L_n(x_i)\}$ and $\{L_{n-1}(x_i)\}$ are already known. The problem then is that with an adaptive quadrature a different set of points would be used in evaluating each succeeding coefficient. The actual points used are determined
solely by the error control procedure of the quadrature algorithm and the characteristics of the function to be integrated, i.e. \( g(x) = f(x) L_n(x) \).

The suggestion here is that one perform all of the integrations for the individual coefficients in parallel using a fixed point procedure. At each point \( x_i \) one evaluates each \( L_n(x_i) \), \( n=0,1,\ldots,N \), as well as \( f(x_i) \). The contribution of the product \( f(x_i)L_n(x_i) \) to each \( f_n \) is then accumulated and the algorithm moves to the next point \( x_j \). The \( x_i \) are chosen by global error control considerations rather than the local considerations of adaptive quadrature.

In this research 96 and 256 order Gauss-Legendre quadratures were used to perform the numerical integrations. Stroud and Secrest [24] is an excellent reference in this area. In fact the necessary weights and tables are given there for up to order 512. Using order \( K \) Gauss-Legendre quadrature we have,

\[
\int_a^b g(y) dy = \frac{(b-a)}{2} \sum_{i=1}^{K} w_i f(y_i) + R_K
\]

where \( y_i = \frac{(b-a)}{2} x_i + \frac{(b+a)}{2} \)

\( P_K(x) = K \)th order Legendre polynomial

\( R_K = \) remainder or error term

\( w_i = \) weight associated with \( P_K(x_i) \)

\( x_i = \) ith zero of \( P_k(x) \).
In order to integrate \( g(y) = f(y) \cdot \int g(y) \) on the interval \((a, b)\) we must simply evaluate \( g(y) \) at \( K \) points in that interval. The recommended procedure can be stated in the following algorithm.

1. Specify the highest order expansion coefficient desired, \( N \). Specify the order of the Gauss-Legendre quadrature, \( K \), and the interval of integration \((a, b)\).

2. Divide the interval \((a, b)\) into \( R \) subintervals if necessary to meet accuracy requirements.

3. Set the current subinterval index \( r = 1 \).

4. Evaluate \( \{ \ell_0(y_i) \}, \{ \ell_1(y_i) \}, \{ f(y_i) w_i \} \) for each \( y_i \) in the subinterval \( r \). Accumulate the sums \( \sum_i \ell_j(y_i) f(y_i) w_i \) into the \( f_o \) and \( f_i \) expansion coefficients, \( j = 0, 1 \).

5. Use the recursion (2.2.1) to evaluate each set \( \{ \ell_{n+1}(y_i) \} \) from \( \{ \ell_n(y_i) \} \) and \( \{ \ell_{n-1}(y_i) \} \). Accumulate the sum \( \sum_i \ell_{n+1}(y_i) f(y_i) w_i \) and add into the \( f_{n+1} \) expansion coefficient.

6. Let \( n = n + 1 \). If \( n > N \), go to step 7. If \( n \leq N \) go to step 5.

7. Let \( r = r + 1 \). If \( r > R \), quit. If \( r \leq R \), go to step 4.

The error analysis of the numerical integration is performed by considering the remainder term associated with the quadrature technique being used. In this case the analysis will dictate the maximum interval \((a, b)\) for a specified order \( K \) and desired error bound \( B_n^K \) on the \( n \)th order coefficient.
Stroud and Secrest [24] present a very general error analysis for Gaussian quadrature. The effort here is to give an error analysis that is specific to the Laguerre function integrations. The remainder term $R_K$ associated with $K$th order quadrature is given in Abramowitz and Stegun [1] as,

\begin{equation}
R_K = \frac{2(b-a)}{(2K+1)} \left( \frac{2K}{(2K+1)!} \int f^{(2K)}(\xi) \right)^{2K+1}
\end{equation}

where

\[ f^{(2K)} = 2K^{\text{th}} \text{ derivative of the integrand} \]

and other terms are as defined directly above.

The building blocks of the error analysis are the relations,

\begin{equation}
(-1)^m \frac{d^m}{dx^m} L_n(x) = \begin{cases} 
L_n^{(m)} & n \geq m \\
0 & n < m
\end{cases}
\end{equation}

\begin{equation}
|L_n^{(m)}(x)| \leq \frac{\Gamma(n+m+1)}{n! \Gamma(m+1)} e^{x/2} \quad m \geq 0
\end{equation}

where $L_n^{(m)}$ is a generalized Laguerre polynomial [1]. Since $L_n(x) = \exp(-x/2)L_n(x)$, we have

\[ \frac{d^m}{dx^m} [L_n(x) e^{-x/2}] = (-1)^m \sum_{i=0}^{\infty} e^{-x/2} L_i^{(m-i)} e^{-x/2} L_{n-i}^{(m-i)} \]

and using (3.2.20),
The \( D_{mn} \) are easily evaluated by noting that for

\[ n \geq m: \quad D_{mn} = \left( \frac{1}{2} \right)^{n-m} (\frac{3}{2})^n \]

\[ m < n: \quad D_{mn} = D_{m,n-1} + D_{m-1,n-1} \]

with \( D_{on} = 1 \) for all \( n \).

And therefore, for \( g_n(x) = f(x) \ell_n(x) \),

\[
\frac{d^j}{dx^j} g_n(x) = \frac{d^j}{dx^j} \left[ f(x) \ell_n(x) \right] \\
= \sum_{n=0}^{j} \left[ \frac{d^{j-m}}{dx^{j-m}} f(x) \right] \left[ \frac{d^m}{dx^m} \ell_n(x) \right] \\
\leq \sum_{m=0}^{j} \left[ \frac{d^{j-m}}{dx^{j-m}} f(x) \right] D_{mn} = B^j_n
\]

The feasibility of evaluating the \( B^j_n \) thus depends on evaluating the derivatives of \( f(x) \). If a bound on these derivatives can be obtained then we have the \( B^j_n \). When this cannot be done, an approximation can be made as follows. Let

\[ f(x) \approx \sum_{n=0}^{N} f_n \ell_n(x) \]

where the \( f_n \) are unknown except that \( |f_n| \leq 1 \). Then

\[
\left| \frac{d^{j-m}}{dx^{j-m}} f(x) \right| \leq \left| \sum_{k=0}^{N} \frac{d^{j-m}}{dx^{j-m}} \ell_k(x) \right| \\
\leq \sum_{k=0}^{N} D_{j-m,k}
\]
Therefore,

$$B_N^j \leq \sum_{m=0}^{j} \sum_{k=0}^{N} D_{j-m,k} D_{mn}$$

Furthermore, since the $P_{mn}$ are increasing in $n$, it suffices to evaluate,

$$(3.2.23) \quad B_N^j \leq \sum_{m=0}^{j} \sum_{k=0}^{N} D_{j-m,k} D_{kn}$$

The value $B_N^j$ will be used in the remainder term (3.2.18) to determine the order $K$ necessary to integrate over a given interval $(a,b)$ to a specified accuracy.

Before presenting some detailed evaluations of $B_N^j$ for various $N$, we present some crude results which put Gauss-Legendre quadrature in perspective. After replacing $f^{(2j)}$ by $B_N^{2j}$, the remainder term (3.2.18) is

$$R_j \leq \frac{[2(b-a)]^{2j+1} (j!)^4}{(2j+1) [(2j)!]^3} B_N^{2j}$$

For 96th order quadrature and an interval size $(b-a)=10$,

$$R_N^b \leq \frac{(20)^{193} (96!)^4}{(193) (192)^3} B_N^{193}$$

In order to attain a remainder bounded by $10^{-10}$ say, we need then $\log_{10} B_N^{192} < 217.69$. This generous limit is a result of the term $((2j)!)^3$ in the denominator of $R_j$. This serves to indicate, however, that one could integrate over a rather large interval with the 96th order quadrature. Consider
the function \( f(x) = 10e^{x} \) whose 193rd derivative is bounded by \( 10^{194} \). This function could be integrated on \((0, 20)\) with one set of 96th order weights and zeros.

To be more precise the operations in equation (3.2.23) were carried out for various \( N \), the order of the maximum desired coefficient. Results are reported below in Table 2.

### Table 2

**Bounds and Interval for Numerical Integration**

Remainder bounded by \( 1.0 \times 10^{-12} \), 96th order quadrature.

<table>
<thead>
<tr>
<th>Max Order</th>
<th>Bound on 192nd Derivative</th>
<th>Max Integration Interval</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>1.6 ( \times 10^{-8} )</td>
<td>1.3 ( \times 10^{2} )</td>
</tr>
<tr>
<td>60</td>
<td>4.9 ( \times 10^{1} )</td>
<td>1.2 ( \times 10^{2} )</td>
</tr>
<tr>
<td>70</td>
<td>1.5 ( \times 10^{11} )</td>
<td>1.1 ( \times 10^{2} )</td>
</tr>
<tr>
<td>80</td>
<td>4.5 ( \times 10^{20} )</td>
<td>9.4 ( \times 10^{1} )</td>
</tr>
<tr>
<td>90</td>
<td>1.3 ( \times 10^{30} )</td>
<td>8.4 ( \times 10^{1} )</td>
</tr>
<tr>
<td>100</td>
<td>3.8 ( \times 10^{39} )</td>
<td>7.5 ( \times 10^{1} )</td>
</tr>
</tbody>
</table>

The interpretation of Table 2 is this. Under the assumption that \( f(x) = \sum_{n=0}^{N} f_n \mathcal{L}_n(x) \) is a good approximation, then the bound on the derivative implies the limit on the integration interval. If the integration must be carried out over a larger interval, then one must either subdivide the interval or use a higher order quadrature to maintain the error bound.
3.3 EXPANSIONS FOR COMMON PROBABILITY DISTRIBUTIONS

This section obtains the expansion coefficients for the probability distributions which are used in the queueing research. It provides an illustration of the techniques which were described above and uses the measures of fit and convergence which were presented in the previous chapter.

3.3.1 NEGATIVE_EXPONENTIAL DISTRIBUTION

\[ f(x) = a \cdot \exp(-ax) \quad x \geq 0, \quad a > 0. \]

Using a direct approach

\[ f_n = \int_0^\infty a e^{-ax} \lambda_n(x) \, dx \]

so that

\[ f_n = a \phi_n(a) \]

\[ = \frac{a}{a+\frac{1}{2}} \left( \frac{a-\frac{1}{2}}{a+\frac{1}{2}} \right)^n \quad n = 0, 1, 2, ... \quad (3.3.1) \]

Therefore the \( f_n \) can be obtained from

\[ f_0 = \frac{a}{a+\frac{1}{2}} \]

\[ f_n = \left( \frac{a-\frac{1}{2}}{a+\frac{1}{2}} \right) f_{n-1} \quad n \geq 1 \]

The coefficients decay geometrically, alternating in sign when \( a < 1/2 \). This is the simplest set of coefficients that will be encountered. We note that for the special case \( a = 1/2 \) that \( f_i = \frac{1}{2^i} \) for \( i = 0 \) and \( f_1 = 0 \) otherwise.
3.3.2 Hyperexponential distribution

Let

\[ f(x) = b_1 f_1(x) + b_2 f_2(x) \]

with

\[ f_1(x) = a_1 e^{-a_1 x}, \quad f_2(x) = a_2 e^{-a_2 x} \]

Then by Property (2.3.1)

\[ f_n = b_1 f_{n1} + b_2 f_{n2} \]

(3.3.2)

\[ = \frac{a_1 b_1}{a_1 + \frac{1}{2}} \left( \frac{a_1 - \frac{1}{2}}{a_1 + \frac{1}{2}} \right)^n + \frac{a_2 b_2}{a_2 + \frac{1}{2}} \left( \frac{a_2 - \frac{1}{2}}{a_2 + \frac{1}{2}} \right)^n \]

3.3.3 Gamma distribution

\[ f(x) = \frac{\lambda^r x^{r-1} e^{-\lambda x}}{\Gamma(r)} ; \quad x > 0, \lambda > 0, r > 0. \]

There are two cases which should be distinguished—integer and non-integer values of \( r \). For positive integer values of \( r \) one can simply convolve the coefficients of an exponential distribution \( r \) times. That is, let

\[ g(x) = \lambda e^{-\lambda x} \]

Then

\[ f(x) = g(x) \ast g(x) \ast \ldots \ast g(x) \]

\[ = g(x)^{(r-1)\ast} \ast g(x) \]

where \( g(x)^{(r-1)\ast} \) is the \( (r-1) \) fold convolution of \( g(x) \)

so that

\[ f_n = \sum_{m=0}^{n} g_{n-m} \ast g_m \]
Of course one can also use the techniques that are described below for non-integer values of $r$.

Sumita [25] gives the following scheme for non-integer $r$ values. Let

$$g(x) = \frac{x^{r-1} e^{-(1/2\beta)x}}{T(r) (2\beta)^r} \; ; \; x > 0 , \; r, \beta > 0$$

$$\phi_g(s) = \frac{1}{(1+2\beta s)^r}$$

From Property (2.3.4)

$$T_g^\#(u) = \sum_{n=0}^{\infty} g_n u^n \quad 0 \leq |u| < 1$$

$$= \phi_g \left( \frac{1}{2} \frac{1+u}{1-u} \right)$$

$$= (1+\beta)^r (1-u)^r \left( 1 - \frac{1-\beta}{1+\beta} u \right)^{-r}$$

Then using the generalized binomial theorem to expand $(1-u)^r$ and $(1-\frac{1-\beta}{1+\beta} u)^{-r}$.

(3.3.3) $$g_n^\# = (1+\beta)^r \sum_{m=0}^{n} b_{n-m} c_m \; ; \; n \geq 0$$

with

$$b_n = \frac{n}{k=1} \left( 1 - \frac{1+r}{k} \right) ; \; n \geq 1 , \; b_0 = 1$$

$$c_n = \left( \frac{1-\beta}{1+\beta} \right)^n \sum_{k=1}^{n} \left( 1 - \frac{1-r}{k} \right) ; \; n \geq 1 , \; c_0 = 1$$

Note that when $r$ is a positive integer, $b_n = 0$ for $n > 1+r$. 
3.3.4 Method of Stages

The method of stages, see Kleinrock [14] for example, is essentially a method of modeling an arrival and/or service process as a combination of series and parallel exponential stages. The linearity of the Laguerre transform along with sections (3.3.1), (3.3.2), and (3.3.3) allows one to obtain the expansion coefficients of the distribution equivalent to the network of stages.

3.3.5 Normal Distribution

In the queuing context only distributions on \([0, \infty)\) are of interest. So in an exact sense the normal distribution will not arise. We can consider normal distributions which are truncated. Two methods of obtaining the expansion coefficients will be shown here—the method of moments and a unique analytical approach due to Samita [25].

Before beginning, a general strategy is outlined. We want the expansion coefficients of a truncated normal distribution with mean \(\mu\) and variance \(\sigma^2\). Instead of approaching the task directly we begin by obtaining the expansion for a normal distribution with mean \(\mu=0\). Given the coefficients for the \(N(0, \sigma^2)\) distribution on \([0, \infty)\), the corresponding coefficients on \((-\infty, 0)\) are given by

\[
f_{-n-1} = \frac{n}{n} f_n
\]
Once the coefficients are obtained for \((-\infty, \infty)\) the bilateral shifting theorem can be employed to position the distribution at \(\mu\).

The first method considered is the method of moments. Let

\[
I_n = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} x^n e^{-x^2/2\sigma^2} dx
\]

so that

\[
I_0 = \frac{1}{2}
\]

\[
I_{2k} = \frac{\sigma^{2k}}{2} \left[1 \cdot 3 \cdot 5 \ldots (2k-1)\right], \quad k = 0, 1, 2, \ldots
\]

\[
I_{2k+1} = \frac{2k k! \sigma^{2k+1}}{\sqrt{2\pi}}
\]

Then

\[
\alpha_{2k} = \frac{\sigma^{2k}}{2} \left[1 \cdot 3 \cdot 5 \ldots (2k-1)\right] = \frac{\sigma^{2}}{2k} \alpha_{2(k-1)}
\]

\[
\alpha_0 = \frac{1}{2}
\]

And

\[
\alpha_{2k+1} = -\frac{2k k! \sigma^{2k+1}}{\sqrt{2\pi} (2k+1)!} = \frac{\sigma^{2}}{(2k+1)} \alpha_{2k-1}
\]

\[
\alpha_1 = -\frac{\sigma}{\sqrt{2\pi}}
\]

For reasonable values of \(\sigma^2\) the \(\alpha_n\) go to zero rapidly and the method of moments may be applied efficiently. That is,

\[
\sum_{n=0}^{\infty} \alpha_n g_{nk}
\]

with the \(g_{nk}\) obtained as in Theorem 3.1.

Sumita [25] derives a recurrence relation for the coefficients of \(f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2}\), \(\sigma > 0\), \(x > 0\). Specifically,
for \( n \geq 3 \)

\[
f_n = \frac{1}{n} \left[ (3n-2-\delta^2) f_{n-1} - (3n-4-\delta^2) f_{n-2} + (n-2) f_{n-3} \right]
\]

with

\[
f_0 = \frac{e^{\sigma^2/2}}{\sqrt{2\pi}} \int_0^\infty e^{-y^2/2} dy
\]

\[
f_1 = (1 + \delta^2) f_0 - \frac{\sigma^2}{2\pi}
\]

\[
f_2 = (1 + 3/2 \delta^2 + \delta^4) f_0 - \frac{1}{4} (8 + 3 \delta^2) \frac{\sigma^2}{2\pi}
\]  

He points out that this forward recurrence relation is actually unstable. That is, any rounding error in \( f_0 \) will be amplified in the recursion and eventually error will overwhelm the coefficients being calculated. This actually occurs rather quickly. He therefore employs a technique due to J.C.P. Miller which allows one to use the backward recursion without knowledge of the starting values \( f_{n+1}, f_{n+2}, f_{n+3} \). This procedure is now summarized.

The backward recursion is

\[ (3.3.1) \quad f_n = \frac{1}{n+1} \left[ (3n+5-\delta^2) f_{n+1} - (3n+7+\delta^2) f_{n+2} + (n+3) f_{n+3} \right], \quad n \geq 0 \]

Define a matrix and a vector

\[
B_{n+1} = \frac{1}{n+1} \begin{bmatrix}
(3n+5-\delta^2), -(3n+7+\delta^2), (n+3)
\end{bmatrix}
\]

\[
f_{n+1} = \begin{bmatrix} f_{n+1}, f_{n+2}, f_{n+3} \end{bmatrix}^t
\]
\( B_{n+1} \) shifts \( f_{n+1} \) down by one to give \( f_n \). Define
\[ M_{n+1} = B_1 B_2 B_3 \ldots B_{n+1} \]
so that
\[ M_{n+1} \begin{bmatrix} f_n \\ f_{n+1} \end{bmatrix} = \begin{bmatrix} f_0 \\ f_1 \end{bmatrix} = \begin{bmatrix} f_c \\ f_1 \\ f_2 \end{bmatrix}^t \]

(3.3.2)

The matrix \( B_n \) is clearly of full rank for any \( n > 0 \), and therefore \( M_{n+1} \) is also of full rank. Choose any three independent vectors \( x_i \in \mathbb{R}^3 \) and define \( y_i = M_{n+1} x_i \), \( i = 1, 2, 3 \). The \( y_i \) must also be linearly independent and therefore span \( \mathbb{R}^3 \). Consequently, there exists \( \lambda = (\lambda_1, \lambda_2, \lambda_3) \) such that
\[ f_0 = \sum_{i=1}^{3} \lambda_i y_i = Y \lambda \]
with the matrix \( Y = (y_1, y_2, y_3) \). But it is also true that
\[ Y = M_{n+1} X \]
with \( X = (x_1, x_2, x_3) \)

Therefore \( f_0 = M_{n+1} X \lambda \). And since \( M_{n+1} \) is of full rank,
\[ f_{n+1} = M_{n+1}^{-1} f_0 = X \lambda = X Y^{-1} f_0 \]

A simple algorithm can be given then to generate the expansion coefficients for \( N(0, \sigma^2) \).

1. Pick \( x_1, x_2, x_3 \in \mathbb{R}^3 \) and linearly independent.
2. Generate \( y_1, y_2, y_3 \) by using the (stable) backward recursion.
3. Calculate \( X Y^{-1} \) and then \( f_{n+1} = X Y^{-1} f_0 \) using the directly calculable values shown above for \( f_0, f_1, f_2 \).
4. Use the backward recurrence relation (3.3.1) beginning with $f_{N+1}$ (as calculated in step 3) to obtain the expansion coefficients.

Sumita showed excellent results using this algorithm and points out that one can check the result by comparing $f_c$ as calculated directly with $f_0$ as calculated through the backward recursion.

This algorithm was implemented with 14 digit precision. Gauss-Jordan elimination with all pivots on the maximum element of the remaining submatrix was used to invert the Y matrix. Generally, the initial vectors were chosen as $\mathbf{x}_1 = \mathbf{e}_1$, $\mathbf{x}_2 = \mathbf{e}_2$, $\mathbf{x}_3 = \mathbf{e}_3$. It was determined that 14 digit precision was insufficient to implement the algorithm in the form stated above without perhaps some special starting vectors.

Consequently it was slightly modified as follows. A series of small steps to intermediate coefficients was substituted for the one large step to the final desired coefficient. For instance, six steps of ten coefficients are substituted for one step to the sixtieth coefficient. The output of each small step becomes the input vector to the next step. Of course after the final step one can still recalculate all of the coefficients from the highest order down to zero order to check the result. Table 3 compares the results of the algorithm with modification to results based on the method of moments for various values of $C^2$. 
Table 3

Normal distribution coefficients and measures of fit

<table>
<thead>
<tr>
<th>Method/Order</th>
<th>Sigma</th>
<th>%Error Squared</th>
<th>% Error 0th Moment</th>
<th>% Error 1st Moment</th>
<th>Max Term Summand</th>
</tr>
</thead>
<tbody>
<tr>
<td>Matrix/70</td>
<td>0.2304</td>
<td>1.80E-04</td>
<td>4.31E-04</td>
<td>3.17E-01</td>
<td>NA</td>
</tr>
<tr>
<td>Moments/70</td>
<td>0.2304</td>
<td>1.80E-04</td>
<td>4.29E-04</td>
<td>3.16E-01</td>
<td>9.52E+03</td>
</tr>
<tr>
<td>Matrix/60</td>
<td>1.0</td>
<td>1.14E-06</td>
<td>9.90E-04</td>
<td>3.00E-01</td>
<td>NA</td>
</tr>
<tr>
<td>Moments/60</td>
<td>1.0</td>
<td>1.94E-07</td>
<td>6.22E-04</td>
<td>1.91E-01</td>
<td>2.40E+07</td>
</tr>
<tr>
<td>Matrix/30</td>
<td>4.0</td>
<td>1.08E-06</td>
<td>4.10E-04</td>
<td>3.10E-02</td>
<td>NA</td>
</tr>
<tr>
<td>Moments/30</td>
<td>4.0</td>
<td>1.42E-07</td>
<td>1.10E-03</td>
<td>8.20E-02</td>
<td>7.53E+07</td>
</tr>
</tbody>
</table>

The entries under Max Term Summand are the largest values of $g_{nk}\alpha_n$ encountered in the method of moments. For higher values of $\sigma^2$ roundoff caused by large values of $g_{nk}\alpha_n$ effectively limits the number of coefficients which can be calculated. Also note that with only 30 coefficients the expansion for $\sigma^2=4$ is of comparable quality to the 60 and 70 term expansions for the other values of $\sigma^2$.

2.3.6 Uniform distribution

Because the uniform distribution is discontinuous at its end points the Laguerre expansions coefficients will decay slowly and will not be summable. Let

$$f(x) = \frac{1}{b-a} \quad a \leq x \leq b$$

Then the nth expansion coefficient is

$$f_n = \frac{1}{(b-a)} \left\{ \int_a^b l_n(x)dx - \int_0^a l_n(x)dx \right\}$$
Consequently,
\[ f_c = \frac{2}{(b-a)^2} \left\{ e^{-a^2/2} - e^{-b^2/2} \right\} \]

and using Property (2.2.7)
\[ f_n = \frac{2}{(b-a)^2} \left\{ l_{n-1}(b) - l_{n}(b) - l_{n-1}(a) + l_{n}(a) \right\} - f_{n-1}, \quad n \geq 1 \]

For the special case \( a=0 \), this simplifies to
\[ f_0 = \frac{2}{b} \left( 1 - l_0(b) \right) \]
\[ f_n = \frac{2}{b} \left\{ l_{n-1}(b) - l_{n}(b) \right\} - f_{n-1}, \quad n \geq 1 \]

3.3.7 Weibull distribution
\[ f(x) = \frac{\beta}{\delta} \left( \frac{x}{\delta} \right)^{\beta-1} \exp \left[ -\left( \frac{x}{\delta} \right)^\beta \right], \quad x \geq 0, \delta > 0, \beta > 0. \]

In the special case \( \beta=2 \), the expansion coefficients can be obtained from those of a \( N(0, \sigma^2) \) distribution by using Property (2.3.8) [25]. For the general case no simple form can be supplied. Two general methods will be discussed here: the method of moments and numerical integration.

The central moments of the Weibull distribution are given by,
\[ \mu_n = \delta^n \Gamma \left( 1 + \frac{n}{\beta} \right) \]

Whether one should use numerical integration or the method of moments depends on the values \( \delta \) and \( \beta \). If \( \beta \) is the result of say a regression, for instance \( \beta=2.5765 \), then one must obtain each of the values of \( \Gamma \left( 1 + \frac{n}{\beta} \right) \) directly whether by table lookup or some other means.
On the other hand if \( \beta \) is an integer, for example \( \beta = 3 \), then only two initial values of \( T(1+\eta/\beta) \) are needed—\( T(4/3) \) and \( T(5/3) \). The method of moments works easily in this case.

\[
\mu_n = \delta^n T(1+\eta/3) \\
\alpha_n = (-1)^n \delta^n T(1+\eta/3)
\]

and

\[
\alpha_{n+3} = -\frac{\delta^3 T(1+\eta/3)}{3(n+2)(n+1)} \alpha_n
\]

Large values of \( \delta \) will make it more difficult to obtain the expansion coefficients with either method. Table 4 compares the coefficients obtained through numerical integration and the method of moments for some values of \( \beta \) and \( \delta \).

<table>
<thead>
<tr>
<th>Method/#Coeff</th>
<th>Beta/Delta</th>
<th>%Error</th>
<th>%Err 0th</th>
<th>%Err 1st</th>
<th>Max Term</th>
<th>Max Term Summand</th>
</tr>
</thead>
<tbody>
<tr>
<td>NumInt/90</td>
<td>2/1</td>
<td>2.70E-08</td>
<td>4.14E-05</td>
<td>1.70E-02</td>
<td>NA</td>
<td>NA</td>
</tr>
<tr>
<td>NumInt/80</td>
<td>2/1</td>
<td>5.45E-09</td>
<td>1.99E-04</td>
<td>6.70E-02</td>
<td>5.61E-07</td>
<td>NA</td>
</tr>
<tr>
<td>NumInt/80</td>
<td>3/2</td>
<td>2.10E-07</td>
<td>1.48E-03</td>
<td>2.60E-01</td>
<td>NA</td>
<td>NA</td>
</tr>
<tr>
<td>NumInt/60</td>
<td>3/2</td>
<td>2.60E-04</td>
<td>3.04E-03</td>
<td>4.00E-01</td>
<td>2.87E+08</td>
<td>NA</td>
</tr>
</tbody>
</table>

Included are measures of fit developed earlier in the chapter. Figures 4 and 5 are plots of the two Weibull distri-
tutions and were obtained directly from the expansion coefficients.

\[ f(x) = 2xe^{-x^2} \]

mean = 0.866
2nd mom. = 1.00

**Figure 4:** Weibull distribution with $\beta = 2$ and $\delta = 1$. 
\[ f(x) = 0.375x^2 e^{-x^{3/8}} \]

mean = 1.786
2nd mom. = 3.611

Figure 5: Weibull distribution with \( \beta = 3 \) and \( \delta = 2 \).

3.3.6 Lognormal distributions

The moments of the lognormal distribution grow too rapidly to make use of the method of moments. Also lacking an analytic approach, numerical integration is an alternative. It is by far the most time-consuming method of obtaining the expansion coefficients. Figures 6 and 7 are plots of lognormal distributions based on expansion coefficients obtained by numerical integration. Each figure is accompanied by the measures of fit described earlier.
Figure 6: Lognormal distribution with mean 1.69.

\[ f(x) = \sqrt{\frac{2}{\pi}} \frac{1}{x} \exp[-2(\log x - .4)^2] \]

- \( E_{3m} = \text{max error} .0012 \text{ at } x = .001 \)
- \( E_{20} = \% \text{ error } \mu_0 = 2.2 \cdot 10^{-4} \)
- \( E_{21} = \% \text{ error } \mu_1 = .05 \)
- \( E_{22} = \% \text{ error } \mu_2 = 8.25 \)
- \( E_1 = \text{Abs error sq. sum} = 1.6 \cdot 10^{-6} \)

Figure 7: Lognormal distribution with mean 2.80.

\[ f(x) = \sqrt{\frac{2}{\pi}} \frac{1}{x} \exp[-2(\log x - 1)^2] \]

- \( E_{3m} = .0012 \text{ at } x = .05 \)
- \( E_{20} = \% \text{ error } \mu_0 = 4.6 \cdot 10^{-3} \)
- \( E_{21} = \% \text{ error } \mu_1 = .523 \)
- \( E_{22} = \% \text{ error } \mu_2 = 56.5 \)
- \( E_1 = \text{Abs error sq. sum} = 2.4 \cdot 10^{-7} \)
One might note that while the expansions are adequate in some ways (i.e. $E_1, E_2, E_3$) they are seriously lacking in terms of accuracy of the first and especially the second moments of the distribution. This might be remedied by obtaining many more coefficients but the coefficients are computationally dear when numerical integration is used. This leads naturally then to the topic of the next section which is how to extract some approximate information on the moments from an inadequate set of expansion coefficients.

3.4 MOMENTS OF DISTRIBUTIONS FROM SMALL COEFFICIENT SETS

If one begins with a given distribution then usually the moments are known or are directly calculable. However, in subsequent chapters sets of coefficients are manipulated to produce new sets of coefficients whose analytic counterpart is unknown. In this case the only knowledge of the moments of the distribution is in the coefficients.

In addition, it seems reasonable that one might trade off computational effort versus desired accuracy at least in the early stages of investigation while one is searching to find appropriate ranges for parameters. Specifically, a larger coefficient set will give a more accurate estimate of the distribution or its moments but the computational cost usually increases as $n^2$ or $n^3$ where $n$ is the number of output coefficients. So an approximation to the moments is also of interest for this reason.
If one were obtaining the moments of a function directly, then it would be possible to approximate these moments by integrating over a finite interval. That is,

\begin{align*}
(3.4.1) & \quad \mu_c = \int_c^b f(x)dx = \int_c^b x f(x)dx + \epsilon_c \\
(3.4.2) & \quad \mu_1 = \int_c^b x f(x)dx = \int_c^b x^2 f(x)dx + \epsilon_1 \\
(3.4.3) & \quad \mu_2 = \int_c^b x^2 f(x)dx = \int_c^b x^3 f(x)dx + \epsilon_2
\end{align*}

And we know that if the moments are finite that as \( b \to \infty \), \( \epsilon_i \to 0 \).

On the other hand if all that is available is a set of coefficients \( \{f_n\} \), then from Property (2.3.9)

\begin{align*}
(3.4.4) & \quad \mu_c \approx \sum_{n=0}^{N} (-1)^n f_n^+ \\
(3.4.5) & \quad \mu_1 \approx 4 \sum_{n=0}^{N} (-1)^n n f_n^+ \\
(3.4.6) & \quad \mu_2 \approx 16 \sum_{n=0}^{N} (-1)^n n^2 f_n^+
\end{align*}

Unless \( N \) is sufficiently large some or all of these approximations can be very poor as demonstrated above in Figures 6 and 7. If \( n^2 f_n^+ \) is not small compared to \( \mu_2 \), more terms are needed.

But if for some a priori reason one knows that \( f(x) = 0 \) for \( x > b \), then a more successful approximation of the moments comes from using (3.4.1), (3.4.2), and (3.4.3). That is,

\[ \mu_1 = \sum_{n=0}^{\infty} f_n \int_c^b x^1 l_n(x)dx. \]
This leads to

\[ \mu_c^b \approx \sum_{n=0}^{N} f_n I_n(b) \]  
(3.4.7)

\[ \mu_1^b \approx \sum_{n=0}^{N} nf_n \Delta I_n(b) \]  
(3.4.8)

\[ \mu_2^b \approx \sum_{n=0}^{N} ng_n^* \Delta I_n(b) \]  
(3.4.9)

where

\[ g_n^* = -(n+1)f_{n+1} + 2nf_n - (n-1)f_{n-1} \]
\[ I_n(b) = \int_c^b l_n(x)dx \]
\[ \Delta I_n(b) = I_n(b) - I_{n-1}(b) \]

Equations (3.4.7), (3.4.8), (3.4.9) can be put into several different forms but no significant simplification results.

One way to see the potential usefulness of this approximation is to compare the values of \( \mu_1^b \) versus \( \mu_1 \) and \( \mu_2^b \) versus \( \mu_2 \) as one increases the number of coefficients included in \( \{f_n\} \). This is illustrated for a Weibull and lognormal distribution in Figures 8 through 11, which show how the estimates of moments change as each coefficient is added to the expansion.

A final issue to consider is how sensitive are the values \( \mu_1^b \) and \( \mu_2^b \) to the choice of \( b \). How accurate must one be in assessing where \( f(x) = 0 \) for \( x > b \)?

\[ \frac{d}{db} \mu_1^b = \sum_{n=0}^{N} f_n (bl_n(b)) \approx bf(b) \]

and similarly

\[ \frac{d}{db} \mu_2^b \approx b^2f(b). \]
This would indicate that as long as \( f(x) = 0 \) for \( x > b \) that the
Figure 10: First moment for Weibull as coefficient set is increased.

Figure 11: Second moment for Weibull as coefficient set is increased.

approximation will be relatively insensitive. In fact as
$b \to \infty$ the approximation will reduce to the usual result.

To illustrate this relative insensitivity Figures 12 and 13 plot the approximate moments as a function of $b$ for the same lognormal and Weibull distributions.

<table>
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<tr>
<th>Upper Limit</th>
<th>5</th>
<th>7</th>
<th>9</th>
<th>11</th>
<th>13</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st Moment</td>
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<tr>
<td>Scale</td>
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<tr>
<td>2nd Moment</td>
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<td>Scale</td>
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</tbody>
</table>

Figure 12: Varying the upper limit of integration - lognormal

Thus, over a wide range of values, the approximation is insensitive to the choice of the upper limit, $b$. 
Figure 13: Varying the upper limit of integration - Weibull

3.5 SUMMARY

Many common probability densities are easily expanded in terms of Laguerre functions. While closed form expressions are only available in simple cases, some powerful numerical techniques are available.

The method of moments approach to obtaining expansion coefficients can be efficiently implemented via a recurrence relation. This recursion reduces the effort involved by an order of magnitude over a straightforward approach to the method of moments.

Numerical integration, while tedious, can be used with predictable results. Error bounds are established for this
technique and one can use the stable recurrence relation for the Laguerre functions to good advantage in the integrations.
Chapter IV

BASIC LAGUERRE TRANSFORM TECHNIQUES FOR GI/G/1

4.1 INTRODUCTION

Conceptually this chapter splits into two parts--transient and steady state techniques for numerically solving GI/G/1. Previous work has emphasized the iteration of Lindley's equation (1.2.3) to arrive at the steady state solution for the waiting time distribution. But it may be that only a steady state result is desired or that one would like to study perturbations of the steady state. In these cases it may be desirable to have a technique which yields the steady state solution directly. This is especially so if the steady state method offers advantages in computational effort or robustness.

All of the steady state methods follow from the solution of Lindley's integral equation (1.2.4) for the waiting time distribution. The solution given here is based on approximating the kernel of that equation by a Laguerre function expansion. The shift and convolution properties of the Laguerre functions lead to a set of simultaneous linear equations for the expansion coefficients of the waiting
time distribution. These may be solved by any standard technique. Solution of these equations through matrix inverse techniques provides a structure for the error analysis. Steady state queue length and interdeparture distributions can be obtained from the steady state waiting time solution using Laguerre function techniques developed in previous chapters.

Since the techniques for the transient waiting time have been demonstrated elsewhere [9], [10], [25], the emphasis in this chapter is on extending them and on extracting more information for a given effort. This is accomplished by showing the equivalence of a technique based on a one-sided transform versus a bilateral Laguerre transform. In addition to transient waiting time distributions, the Laguerre transform is also applied to queue lengths and interdeparture distributions. The application to interdeparture distributions paves the way for possible application to tandem queues.

9.2 TRANSIENT WAITING TIME DISTRIBUTIONS VIA ONE-SIDED TRANSFORM

In Chapter One the GI/G/1 model was reviewed ending with Lindley's equation which is repeated here for convenience.

\[ W_{n+1}(x) = \int_{-\infty}^{\infty} W_n(x-y)k(y)dy \quad x \geq 0 \]

\[ = 0 \quad x < 0 \]
The obstacle to presenting a solution based on the one-sided transform is the function $k(y)$ which is defined on $(-\infty, \infty)$. Fortunately this research was begun in ignorance of the bilateral techniques so that the obstacle was overcome.

The problem posed by $k(y)$ is easily remedied by first splitting equation (4.2.1) into two parts.

\[
(4.2.2) \quad W_{n+1}(x) = \int_{-\infty}^{x} W_n(x-y)k(y)dy + \int_{0}^{x} W_n(x-y)k(y)dy
\]

Then in the first integral in (4.2.2) make the transformation $y \rightarrow -y$ so that

\[
\int_{-\infty}^{x} W_n(x-y)k(y)dy = \int_{0}^{\infty} W_n(x+y)\bar{k}(y)dy
\]

where

\[
\bar{k}(y) = k(-y), \quad \text{for } y \geq 0
\]

We have then,

\[
(4.2.3) \quad W_{n+1}(x) = \int_{0}^{\infty} W_n(x+y)\bar{k}(y)dy + \int_{0}^{x} W_n(x-y)k(y)dy, \quad x > 0
\]

Taking the derivative of (4.2.3) with respect to $x$ on the open interval $(0, \infty)$ yields for $x > 0$

\[
(4.2.4) \quad W_{n+1}(x) = \int_{0}^{\infty} W_n(x+y)\bar{k}(y)dy + \int_{0}^{x} W_n(x-y)k(y)dy + W_n(x)k(x)
\]

where

\[
(4.2.5) \quad W_n(0) = 1 - \int_{0}^{\infty} W_n(x)dx
\]
All of the terms in equation (4.2.4) are easily computed via techniques described in Chapter Two. The first term follows from Property (2.3.7). Namely,

\[ \int_0^\infty \omega_n(x+y)k(y)dy = \sum_{j=0}^\infty l_j(x) \sum_{i=j}^\infty \omega_{in} k_{i-j} \]

The second term follows from the convolution property (2.3.5). Specifically,

\[ \int_0^\infty \omega_n(x-y)k(y)dy = \sum_{j=0}^\infty l_j(x) \sum_{i=0}^j \omega_{in} k_{j-i} \]

The third term follows from Property (2.3.3) as

\[ W_n(0) = 1 - 2 \sum_{k=0}^\infty (-1)^k \omega_{kn} \]

And therefore

\[ W_n(0)k(x) = W_n(0) \sum_{j=0}^\infty k_j l_j(x) \]

Combining (4.2.6) through (4.2.9) we obtain

\[ \omega_{n+1}(x) = \sum_{j=0}^\infty \omega_{n} l_j(x) \sum_{i=0}^j \omega_{in} k_{i-j} + \sum_{i=0}^j \omega_{in} k_{j-i} + W_n(0) k_j \]

Therefore to obtain the mth expansion coefficient of \( \omega_{n+1}(x) \) we use \( \omega_{m,n+1} = (\omega_{n+1}(x), l_m(x)) \)

\[ \omega_{m,n+1} = \sum_{i=0}^m \omega_{in} k_{m-i} + \sum_{i=m}^\infty \omega_{in} k_{i-m} + W_n(0) k_m \]

Equation (4.2.11) can also be stated in terms of the sharp coefficients with a little algebra as

\[ \omega_{in,m+1}^\# = \sum_{i=m}^\infty \omega_{in} k_{i-m}^\# + \sum_{i=0}^m \omega_{in} k_{i-m}^\# + W_n(0) k_m^\# \]
If one begins with the queue empty and idle then \( w_1(0) = 1 \) and \( w_{m_1} = 0 \) for all \( m > 0 \). That is the first customer waits zero time units with probability one. The transient solution is composed of the iterations of (4.2.11), using (4.2.8) to obtain \( w_n(0) \) at each iteration. It is also possible to start with other initial conditions by simply supplying the Laguerre expansion coefficients of \( w_1(x) \) and the value \( w_1(0) \).

What remains is to supply the formulas which yield \( \{ z_j^\#: \} \) and \( \{ k_j^\# \} \). In Chapter One we had

\[
K(y) = \Pr \{ Y = X - T < y \}
\]

where \( Y \) is the service time distributed as \( B(x) = \Pr(X < x) \) and \( T \) is the interarrival time distributed as \( A(t) = \Pr(T < t) \). If \( A(\cdot) \) and \( B(\cdot) \) are defined on \([0, \infty)\) then

\[
(4.2.13) \quad K(y) = \int_0^\infty \Pr \{ x < y + z \mid T = z \} a(z) \, dz, \quad y > 0
\]

For the case \( y < 0 \),

\[
(4.2.14) \quad K(y) = \Pr \{ X - T \leq y \} = 1 - \Pr \{ T - X \leq -y \} = 1 - \int_0^\infty \Pr \{ T - y + z \leq X = z \} b(z) \, dz
\]

\[
= 1 - \int_0^\infty A(-y + z) b(z) \, dz, \quad y \leq 0
\]
Taking derivatives with respect to $y$ in (4.2.13) and (4.2.14)

(4.2.15) \[ k(y) = \int_{-\infty}^{y} b(u+y) a(u) \, du \quad y > 0 \]

(4.2.16) \[ k(y) = \int_{-\infty}^{y} a(-u+y) b(u) \, du \quad y \leq 0 \]

Letting $y \rightarrow -y$ in (4.2.16) then

(4.2.17) \[ k(-y) = \overline{k}(y) = \int_{-\infty}^{y} a(u+y) b(u) \, du \quad y > 0 \]

The equations (4.2.15) and (4.2.17) yield immediately the expansion coefficients $\{ \overline{k}_j \}$ and $\{ k_j \}$ using Property (2.3.7).

(4.2.18) \[ k_j = \sum_{i=-\infty}^{\infty} b_i a_{i-j} \]

(4.2.19) \[ \overline{k}_j = \sum_{i=-\infty}^{\infty} a_i b_{i-j} \]

Before comparing this to the bilateral transform solution, we examine the computational effort involved in iterating (4.2.11). The expansions of $k(y)$ and $\overline{k}(y)$ are truncated at order $N$ say. This determines the number of operations in each iteration of (4.2.11). Then (4.2.11) becomes

\[ \omega_{m,n+1} = \sum_{i=m}^{N} \omega_{i,m} \overline{k}_{i-m}^{\#} + \sum_{i=0}^{m} \omega_{i,m} k_{m-i}^{\#} + \omega_n(x) k_m \]

Each $\omega_{m,n+1}$ requires $N+2$ additions and multiplications giving a total of $(N+1)(N+2)$ additions and multiplications.
4.3 BILATERAL_TRANSFORM_SOLUTION_FOR_WAITING_TIME.

The bilateral Laguerre transform leads to the following equation involving the Laguerre expansion coefficients [9].

\[ C_m = W_n(0)k_m + \sum_{i=0}^{\infty} w_{i,n} k_{m-i} \quad -\infty < m < \infty \]

where \[ w_{m,n} = \begin{cases} 0 & m < 0 \\ \sum_{j=-\infty}^{0} w_{j,n} & m = 0 \\ w_{m,n} & m \geq 1 \end{cases} \]

The coefficients \[ w_{m,n} \] are non-zero for negative values of \( m \). For negative \( m \) values these coefficients lead to the expansion coefficients for the waiting time distribution on the negative half line. That is,

\[ w_{m,n+1} = \sum_{j=-\infty}^{m} w_{j,n+1} \]

And for \( m < 0 \) the \( w_{m,n+1} \) are the expansion coefficients on the negative \( x \) axis. Similarly for values of \( m \geq 0 \) the \( w_{m,n+1} \) yield the expansion coefficients for the function on the positive \( x \) axis.

The waiting time distribution for negative \( x \) is usually summarized in the value \( W_n(0) \) since a negative waiting time or unfinished work does not make physical sense. But another way of looking at the situation is useful. The waiting time distribution on the negative \( x \) axis is really the reflection of the server idle time distribution at arrival epochs.
The bilateral transform technique therefore yields the waiting time distribution and the distribution of server idle time. But there is some price paid in computational effort. Assume that the expansion coefficients for the $k$ are non-zero on $-N \leq \xi \leq N$ as was done with the one-sided technique. Then the effort per iteration in calculating $w_{m,N+1}$ is

\[
\begin{align*}
N+2 & \text{ multiplications} & N \geq 0 \\
\text{and additions} & \text{ and additions} & \text{ and additions}
\end{align*}
\]

Thus the total number of multiplications and additions in an iteration is

\[
(N+1)(N+2) + \sum_{m=-1}^{-N} (N-m) = (N+1)(N+2) + \frac{N(N-1)}{2}
\]

Looking at the order $N^2$ terms, the bilateral transform goes as $(3/2)N^2$ while the one-sided transform goes as $N^2$. So if one can give up the distribution of server idle time, the effort per iteration can be reduced by about $1/3$.

It turns out that the distribution of server idle time at arrival epochs is needed to obtain the interdeparture distribution. This will be presented later in the chapter. The server idle time distribution is also available from the one-sided transform technique. But in this case the effort per iteration becomes equal to that of the bilateral transform technique.
Before proceeding we demonstrate the exact equivalence of the two techniques. Two subsidiary relations are needed.

(1.) In equation (4.3.1) we had
\[ \omega^\text{#}_0 = \sum_{j=-\infty}^{0} \omega^\text{#}_j \]
\[ = \omega^H_0 \]

(2.) Consider the bilateral transform of \( k(y) \), \(-\infty < y < \infty\). The expansion coefficients of the left half of \( k(y) \) are the \( k_j \) for \( j<0 \). If this left half of \( k(y) \) were reflected about the origin, then the symmetry rules (see Section 2.6) of the bilateral transform would give the expansion coefficients of the reflected portion of \( k(y) \) which corresponds to the function \( \tilde{k}(y) \). Specifically,
\[ \tilde{k}_j = -k_{-j-1}, \quad j \geq 0 \]
This implies that \( \tilde{k}_0 = -k_{-1} \) and \( \tilde{k}^\#_j = k_{-j}^\# \). Equation (4.3.1) can be written as
\[ \omega^\text{H#}_{m,n+1} = W_n(0)k^\#_m + \sum_{i=0}^{m-1} \omega^\#_{i} k^\#_{m-i} + \sum_{i=m+1}^{\infty} \omega^\#_{i} k^\#_{m-i} + \omega^\#_{mn} k^\#_c \]
But in the last term
\[ \omega^\#_{mn}(k^\#_c - k^\#_{-1}) = \omega^\#_{mn} k^\#_0 + \omega^\#_{mn} k^\#_c \]
And for \( j>0 \), \( k^\#_j = \tilde{k}^\#_j \). Therefore,
\[ \omega^\text{H#}_{m,n+1} = W_n(0)k^\#_m + \sum_{i=c}^{m} \omega^\#_{i} k^\#_{m-i} + \sum_{i=m}^{\infty} \omega^\#_{i} k^\#_{i-m} \]
which is equivalent to (4.2.12).
To summarize, the one-sided technique developed here allows one to decouple the problem into two parts. Separate solutions are developed for the waiting time distribution on the positive and negative x axis. If only the positive portion is needed, then computational effort can be saved.

4.4 STEADY STATE WAITING TIME DISTRIBUTION.

Starting from equation (1.2.3) we consider the limit as the number of arriving customers to the queue grows. If the queue is stable, then a limiting distribution for $W_{n+1}(x)$ will exist. Formally, the conditions for the existence of a steady state waiting time distribution are as follows [12]. Let $U_n = X_n - T_n$ be i.i.d. so that $U_n = U$. Then if $0 < E|U| < \infty$, the queue is stable if and only if $E(U) < 0$. For the distributions considered here it is sufficient that the mean service rate exceed the mean arrival rate.

In the limit as $n \to \infty$ we may write,

\[ W(x) = \int_{-\infty}^{x} W(x-y)k(y)dy \]

for the steady state waiting time distribution. Using the same transformations as in the transient case we arrive at an expression for the steady state pdf.

\[ (4.4.3) \quad W(x) = \int_{0}^{\infty} W(x+y)\bar{k}(y)dy + \int_{0}^{x} W(x-y)k(y)dy + W(0)k(x) \]

for $x > 0$
Equation (4.4.3) is the steady state counterpart of equation (4.2.4). Equation (4.2.4) led to the iteration schema for the expansion coefficients of $w_{n+1}(x)$. Consequently the steady state version of the solution for the expansion coefficients is simply equation (4.2.11) sans the iteration subscript.

So we have,

$$
\omega_m = \sum_{i=0}^{m} \omega_i k_{m-i} + \sum_{i=m}^{\infty} \omega_i k_{i-m} + W(0) k_m, m = 0, 1, \ldots
$$

for the steady state expansion coefficients. If one truncates the system of equations (4.4.4) then the steady state solution is given by a finite set of linear equations. But one might ask what theoretical grounds support this common sense approach. The principal question is whether

$$
\omega^{(N)}(x) = \sum_{n=0}^{N} \omega_n \ell_n(x)
$$

converges to the steady state $w(x)$ as $N \to \infty$.

This convergence can be demonstrated in two general ways, via Fredholm theory and via projection methods for Wiener-Hopf integral equations. The important results will now be summarized.

We begin with an equation equivalent to (4.4.3), namely

$$
\omega(x) = \int_{\mathbb{C}} k(x-y) \omega(y) \, dy + k(x) W(0), \ x > 0
$$

Equation (4.4.5) is recognizable as a Fredholm equation of the second kind. The kernel of this equation is the function $k(x-y)$. If the kernel were degenerate we could write,
Degeneracy implies that the kernel could be represented by a finite number of terms in the product form. In this case a theorem [17] guarantees that (4.4.5) would reduce to a system of \( p \) linear equations in \( p \) unknowns. The functions \( k(x,y) \) normally encountered will not be degenerate however.

But since the bilateral Laguerre functions are a complete set of functions on \((-\infty, \infty)\), \( k(x,y) \) can be approximated arbitrarily closely by a finite expansion. Consequently \( k(x,y) \) can be approximated by a degenerate kernel and it can be shown (see Lichnerowicz [17] for instance) that the approximation converges to the steady state solution as the number of expansion terms is increased.

Gohberg and Feldman [6] detail projection methods for solving Wiener-Hopf integral equations. In fact their presentation is carried out using essentially the Laguerre functions as a basis for the projection. Equation (4.4.5) can be written in operator form as

\[
A u_f(x) = \hat{W}(0) k(x)
\]

where \( f(x) = W(0) k(x) \)

\[
A = I - \int_0^\infty k(x-y) dy
\]

The following theorem is given in the reference [6] and has been adapted to the notation used here.
Theorem 4.1: Projection Solution

Let \( k(.) \in L_{2}(-\infty, \infty) \), and let the operator \( A \) be defined as above. If \( \lim \inf_{n \to \infty} \lambda_{n}^{-1} < \infty \), where \( \lambda_{n} = \| (A \ell_{k}, \ell_{j}) \|_{j, k=0}^{n-1} \), then

\[
\sum_{k=0}^{n-1} (A \ell_{k}, \ell_{j}) \omega_{k} = (\ell_{j}, \ell_{j}) \quad j = 0, 1, 2, \ldots, (n-1)
\]

beginning with some \( n \) has the unique solution \( \{ \omega_{n}^{j} \}_{j=0}^{n} \) and as \( n \to \infty \) the functions

\[
\omega^{n}(x) = \sum_{j=0}^{n-1} \omega_{j}^{n} \ell_{j}(x)
\]

converge in the norm of \( L_{2} \) to a solution of (4.4.5) whatever the function \( f(t) \in L_{2} \) may be. \( \square \)

Two final comments are necessary. First, \( \lambda_{n} \) is the norm of the matrix whose \( kj \) element will be shown to be

\[
(A \ell_{k}, \ell_{j}) = \delta_{jk} + k^{j}_{-k}
\]

As long as this matrix is not the null matrix the norm will be non-zero and \( \lambda_{n}^{-1} \) will be bounded.

Second, one can examine the system (4.4.6) and show it identical to the system (4.4.4). First derive the \( m \)th element of (4.4.6).

\[
(4.4.7) \quad (\ell_{m}, \ell_{m}) = W(c) \int_{c}^{\infty} k(x) \ell_{m}(x) dx = W(c) k_{m}
\]

And using the bilateral shift theorem the \( i \)th expansion coefficient of \( k(x-y) \) is
\[ k_i = \sum_{j=i+1}^{\infty} \sum_{m=-\infty}^{\infty} k_{j-m} \Delta h_m(y) \]
\[ \Delta h_m(y) = h_m(y) - h_{m-1}(y) \]

In this case \( y > 0 \) and therefore
\[ h_j(y) = 0 \quad j < 0 \]
\[ h_j(y) = l_j(y) \quad j \geq 0 \]

This gives
\[ k_i = \sum_{j=i+1}^{\infty} \sum_{m=0}^{\infty} [k_{j-m} - k_{j-m-1}] l_m(y) \quad \text{and} \quad k(y) = \sum_{i=0}^{\infty} k_i l_i(y) \]

Consequently,
\[ \int_{0}^{\infty} k(x-y) l_k(y) dy = - \sum_{i=0}^{\infty} l_i(x) \sum_{j=i+1}^{\infty} [k_{j-m} - k_{j-m-1}] \delta_{km} \]
\[ = - \sum_{i=0}^{\infty} l_i(x) \sum_{j=i+1}^{\infty} [k_{j-k} - k_{j-k-1}] \]

But,
\[ \sum_{j=i+1}^{\infty} [k_{j-k} - k_{j-k-1}] = - k_{i-k} \]

Using this,
\[ \int_{0}^{\infty} k(x-y) l_k(y) dy = \sum_{i=0}^{\infty} l_i(x) k_{i-k} \quad \text{and} \quad (A l_k, l_m) = \delta_{km} - k_{m-k} \]

Inserting (4.4.8) and (4.4.7) into (4.4.6) we have
\[ \sum_{k=0}^{n-1} \omega_k (\delta_{km} - k_{m-k}) = W(0) k_m \quad 0 \leq m \leq n-1 \]

The summation can be split into three parts,
\[ \omega_m = \sum_{k=0}^{m-1} \omega_k k_{m-k} + \sum_{k=m+1}^{n-1} \omega_k k_{m-k} + k_{m-k} \omega_m + \omega(0) k_m \]
But since,
\[ k_{m-k}^* = \overline{k}_{k-m}^* \quad m \neq k \]
\[ k_{n}^* = k_n - k_{-1} \]

then converting to the one-sided transform gives
\[ k_0 \rightarrow k_0^* \quad -k_{-1} \rightarrow \overline{k}_0 = \overline{k}_0^* \]

And at last
\[ (4.4.10) \quad \omega_m = \sum_{k=0}^{m} \omega_k k_{m-k}^* + \sum_{k=m}^{n-1} \omega_k \overline{k}_{k-m}^* + W(0)k_m, \quad 0 < m \leq n - 1 \]

which is the same as (4.4.4), restricted to a finite number of terms.

Finally we can get an idea of the errors involved in truncating equation (4.4.4) to obtain (4.4.10). The matrix associated with equations (4.4.10) is shown below in Figure 14. The last row of the matrix represents a normalization of the integral of \( w(x) \), i.e.,
\[ I = \int_{0}^{\infty} w(x) \, dx = W(0) + 2 \sum_{n=0}^{\infty} (-1)^n \omega_n \]

If the highest expansion order is \( N \), then there are \( N+2 \) equations for \( W(0) \) and the \( N+1 \) coefficients \( \omega_j \).

For the \( j \)th row, \( 0 \leq j \leq N \), the neglected terms on the r.h.s. are
\[ \sum_{i=N+1}^{\infty} k_{i-j} \omega_i \leq \sum_{i=N+1}^{\infty} |\omega_i||\overline{k_{i-j}}| \equiv \varepsilon_j \]
Write (4.4.10) in matrix form as \( Kw = \begin{pmatrix} \frac{\alpha}{\beta} \\ \frac{\beta}{\alpha} \end{pmatrix} \). The solution by matrix inverse techniques gives the solution vector as the last column of \( K^{-1} \). One can control errors by controlling the magnitude of the neglected \( w_i \). This will be considered in Chapter Five.

\[
\begin{bmatrix}
  k_0 + k_0^{-1} & k_1 & k_2 & \ldots & k_N & k_0 & 0 \\
  k_1 & k_0 + k_0^{-1} & k_1 & \ldots & k_N & k_1 & 0 \\
  k_2 & k_1 & k_0 + k_0^{-1} & \ldots & k_{N-2} & k_2 & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
  k_N & k_{N-1} & k_{N-2} & \ldots & k_0 + k_0^{-1} & k_N & 0 \\
  2 & -2 & 2 & \ldots & 2(-1)^N & 1 & 1
\end{bmatrix}
\]

Figure 14: Steady state coefficient matrix.

4.5 INTERDEPARTURE DISTRIBUTIONS--STEADY STATE AND TRANSIENT.

The preliminary development here is taken from Cohen [3]. Let,

\[ W_n = \text{actual waiting time of the } n\text{th customer} \]
\[ I_n = \text{service time of the } n\text{th customer} \]
\[ B_n = \text{departure time of the } n\text{th customer} \]
\[ U_n = \text{arrival time of the } n\text{th customer} \]
\[ Q_{U_n} = \text{the # of customers in the system just prior to } U_n \]
The Laguerre transform method is ideally suited to obtaining the distribution of \( D_{n+1} \). When \( (W_n + X_n - T_{n+1}) \) is negative then \( D_{n+1} \) is simply a convolution of a service time \( (X_{n+1}) \) and \( (W_n + X_n - T_{n+1}) \). The \( X_n \) are the i.i.d. service times with pdf \( b(\cdot) \). And \( (W_n + X_n - T_{n+1}) < 0 \) is simply distributed as \( W_{n+1}(x) \) for \( x < 0 \). This \( W_{n+1}(x) \) will be given by Lindley's equation on the negative \( x \) axis, i.e.

\[
W_{n+1}(x) = \int_{-\infty}^{x} W_n(x-y) k(y) dy, \quad x < 0
\]

Taking the derivative with respect to \( x \) on \( (0, -\infty) \)

\[
\omega_{n+1}(x) = \int_{-\infty}^{x} \omega_n(x-y) k(y) dy + W_n(0) k(x), \quad x < 0
\]

Letting \(-y \rightarrow y, \quad \infty\)

\[
\omega_{n+1}(x) = \int_{-x}^{\infty} \omega_n(x+y) \overline{k(y)} dy + W_n(0) k(x), \quad x < 0
\]
where as before \( \bar{k}(y) = k(-y) \) is the reflection of the left half of \( k(y) \). Let \( y \rightarrow y-x \) so that

\[
\omega_{n+1}(x) = \int_{-\infty}^{\infty} \omega_n(y) \bar{k}(y-x) \, dy + \mathcal{W}_n(0)k(x) \quad x < 0
\]

Finally let \( x \rightarrow -x \) and define \( \bar{w}_{n+1}(x) = w_{n+1}(-x) \), \( x > 0 \) as the reflection of \( w_{n+1}(x) \) for \( x < 0 \). Therefore

\[
(4.5.2) \quad \bar{w}_{n+1}(x) = \int_{0}^{\infty} \omega_n(y) \bar{k}(x+y) \, dy + \mathcal{W}_n(0)k(x) \quad x > 0
\]

A graphical look at equation (4.5.2) is shown in Figure 5. The lines labeled \( a, b, c \) are the integration paths which yield the values of \( \bar{w}_{n+1}(x) \) at the points \( x_1, x_2, x_3 \). The result is to allow writing (4.5.1) as the convolution of two positive variables,

\[
(4.5.3) \quad D_{n+1} = X_{n+1} + \max [0, \bar{W}_{n+1}]
\]

where \( \bar{w}_{n+1} \) has pdf \( \bar{w}_{n+1}(x) \) with probability mass \( (1-W_{n+1}(0)) \) at the origin.

All of the machinery developed in the last section for the waiting time distribution extends to equations (4.5.2) and (4.5.3). Consequently for \( \bar{w}_{n+1}(x) \) we have in the transient period,

\[
(4.5.4) \quad \bar{w}_{j,n+1} = \sum_{i=j}^{\infty} k_i \omega_{i-j,n}^\# + \mathcal{W}_n(0)\bar{k}_j \quad j = 0, 1, 2, ...
\]

and in steady state,

\[
(4.5.5) \quad \bar{w}_j = \sum_{i=j}^{\infty} k_i \omega_{i-j}^\# + W(0)\bar{k}_j \quad j = 0, 1, 2, ...
\]
We note that the equations for $\bar{w}_j$ or $\bar{w}_{j,n+1}$ are in terms of the coefficients of the waiting time distribution. So equations (4.5.4) and (4.5.5) require solving for the waiting time distribution after which the expansion for $\bar{w}_{n+1}(x)$ or $\bar{w}(x)$ is available.

To obtain the interdeparture distribution define

$$
\bar{d}_{n+1}(x)dx = \Pr[x \in D_{n+1} \leq x+dx]
$$

the pdf of $D_{n+1}$.

Then from (4.5.3)

\[\text{Figure 15: Waiting time for } x<0\]
\[ d_{n+1}^{(x)} = \int_0^x b(x-y) \overline{w}_{n+1}(y) \, dy + [1 - \overline{w}_{n+1}(0)] b(x), \quad x \geq 0 \]

or in steady state
\[ d(x) = \int_0^x b(x-y) \overline{w}(y) \, dy + [1 - \overline{w}(0)] b(x), \quad x \geq 0 \]

And for the expansion coefficients
\[ d_{j,n+1} = \sum_{i=0}^{j} \overline{w}_{j-i} b_{j-i} + [1 - \overline{w}_{n+1}(0)] b_{j}, \quad j = 0, 1, 2, \ldots \]

and in steady state
\[ d_{j} = \sum_{i=0}^{j} \overline{w}_{j-i} b_{j-i} + [1 - \overline{w}(0)] b_{j}, \quad j = 0, 1, 2, \ldots \]

### 4.6 Queue Lengths—Steady State and Transient

As in the interdeparture distribution, the preliminary development here follows Cohen [3]. All random variables are defined as in the beginning of Section 4.5.

Let \( R_0 = 0, \quad W_1 = 0, \quad U_1 = 0. \) That is, the first customer arrives to an empty queue at time zero and enters service. Then the event that the \((n+j)\)th arriving customer finds \( j \) customers in the system upon arrival is,
\[
\{ R_{n-1} < U_{n+j} \leq R_n \} \quad n=1, 2, \ldots ; \quad j = 0, 1, 2, \ldots
\]

It is also clear that
\[ \Pr\{ R_{n-1} < U_{n+j} \leq R_n | W_1 = 0 \} = \Pr\{ R_{n-1} < U_{n+j} | W_1 = 0 \} - \Pr\{ R_n < U_{n+j} | W_1 = 0 \} \]

for \( j = 1, 2, \ldots \)
(4.6.2) \( \Pr \{ R_n < U_{n+j} \mid W_i = c \} = \Pr \{ W_n + X_n < U_n + T_{n+1} + \ldots + T_{n+j} \} \)

\[ = \Pr \{ W_n + X_n - T_{n+1} < T_{n+2} + T_{n+3} + \ldots + T_{n+j} \} \]

And therefore,

(4.6.3) \( \Pr \{ R_n < U_{n+j} \mid W_i = 0 \} = \Pr \{ \overline{W_{n+1}} < T_{n+2} + T_{n+3} + \ldots + T_{n+j} \} \)

And consequently,

(4.6.4) \( \Pr \{ R_n < U_{n+j} \mid W_i = 0 \} = \int_0^\infty \left( \int_{A^{-1}(u)} 1 \right) d_u \Pr \{ \overline{W_{n+1}} < u \mid W_i = 0 \} \)

Equation (4.6.2) is based on this idea. For the event \( \{ B_n < u_{n+j} \} \) to occur fewer than \( j \) arrivals must occur between the time the \( n \)th customer arrives and departs. The total time \( C_n \) is in the system is, of course, \( W_n + X_n \). We simplify later computation by noting, as in equation (4.6.3) that a convolution of the arrival distribution can be traded for an additional computation of a waiting time distribution, i.e. \( \overline{W_{n+1}} \).

Then using (4.6.4) in (4.6.1) we get, for \( n > 1 \),

\( \Pr \{ Q_{U_{n+j}} = j \mid W_i = 0 \} = \text{Probability of } j \text{ in the system at the } (n+j)\text{th arrival epoch} \)

(4.6.5) \( \Pr \{ Q_{U_{n+j}} = j \mid W_i = 0 \} = \int_0^\infty \left( \int_{A^{-1}(u)} 1 \right) d_u \Pr \{ \overline{W_{n+1}} < u \mid W_i = 0 \} \)

\[ - \int_0^\infty \left( \int_{A^{-1}(u)} 1 \right) d_u \Pr \{ W_{n+1} < u \mid W_i = 0 \} \]
For the case \( n = 1, j = 0, 1, 2, \ldots \) we must evaluate \( \Pr(B_0 < U_{i+j} < R_1 \mid W_1 = 0) \). For \( j = 0 \), Cohen gives \( \Pr(Q_{U_1} = 0 \mid W_1 = 0) = 0 \), but this is an artifact of the system definition and of no particular interest. Therefore for \( n = 1 \),

\[
(4.6.6) \quad \Pr\{Q_{U_{i+j}} \mid W_1 = 0\} = \begin{cases} 0 & j = 0 \\ \int_0^\infty \left[ -B(u) \right] dA^j(u) & j = 1, 2, \ldots 
\end{cases}
\]

For example, for \( C_4 \) to find three customers in the system, three arrivals must have occurred while \( C_1 \) is still in service (including \( C_{i-1} \)).

In the steady state,

\[
\rho_j = \lim_{n \to \infty} \Pr\{Q_{U_{n+j}} = j \mid W_1 = 0\}
\]

\[
(4.6.7) = \begin{cases} W(0) & j = 0 \\ \int_0^\infty \left[ A^j(u) - A^{j-1}(u) \right] dA^j(u) & j = 1, 2, \ldots 
\end{cases}
\]

The Laguerre transform method makes calculation of (4.6.5), (4.6.6), and (4.6.7) a mechanical process. For instance, assume we want to know

\[
\Pr\{Q_{U_{n+j}} = j \mid W_1 = 0\} \text{ for } 0 \leq j \leq 5; \ n = 1, 2, \ldots
\]

Then we must compute,

\[
1 - A^j(u) = \int_u^\infty A^j(x) dx \text{ for } 0 \leq j \leq 5
\]

Define

\[
g^j(u) = 1 - A^j(u)
\]
then
\[ g_n^j = -2 \sum_{m=0}^{\infty} \alpha_{n+m+1} \quad \text{by Property (2.3.3)} \]

The \( \alpha_k \) are obtained as usual from the convolution property. Next we need
\[
d_k \Pr \{ W_n \leq u \mid W_1 = 0 \} = \begin{cases} 
\omega_n(u) du, & u > 0 \\
\omega_n(0) \delta(u) du, & u = 0
\end{cases}
\]

Therefore,
\[
\left[ \prod_{\mathcal{G}} \Lambda_1^j(u) \right] d_k \Pr \{ W_n \leq u \mid W_1 = 0 \} = \sum_{k=0}^{\infty} \alpha_k \omega_n \left( \sum_{i=0}^{\infty} g_i^j \omega_i \right) d_k \omega_n \left( \sum_{i=0}^{\infty} g_i^j \omega_i \right) du + \omega_n(0)
\]
\[
= \sum_{k=0}^{\infty} g_k^j \omega_k + \omega_n(0)
\]

We make explicit the assumption that the arrival distribution is continuous with no point mass at the origin so that \( \Lambda_1^j(0) = 0 \). Then we have,
\[
(4.6.8) \quad P_{n+j} = \Pr \{ Q_{n+j} = j \mid W_1 = 0 \} = \sum_{k=0}^{\infty} \left[ g_k^j \omega_k - g_k^{j-1} \omega_k \right] - \Delta W_{n+1}(0)
\]
where \( \Delta W_{n+1}(0) = W_{n+1}(0) - W_n(0) \)

for \( n=2,3,4,\ldots \) and \( j=1,2,\ldots \)

Of course for \( j=0 \),
\[
P_{n0} = W_n(0)
\]

While for \( n=1 \) we need,
\[
1 - B(u) = \int_u^\infty b(x) dx
\]
Define \( b^s(x) = 1 - B(x) \)

then by property (2.3.3) \( b^s_n = -2 \sum_{m=0}^{\infty} (-1)^m b^{sN}_{n+m+1} \)

So that for \( n=1 \)

\[ (4.6.9) \quad P_{i+j, j} = \begin{cases} 0 & j = 0 \\ \sum_{k=0}^{\infty} b^s_k a^{-k} & j > 0 \end{cases} \]

Using the same concepts in equation (4.6.7), the steady state solution is,

\[ (4.6.10) \quad P_j = \begin{cases} W(0) & j = 0 \\ \sum_{k=0}^{\infty} (g^{j-1}_k - g^j_k) \omega_k & j > 0 \end{cases} \]

If we consider the matrix of numbers \( p_{kj} \) defined by equations (4.6.8) for \( k=1, 2, \ldots \) and \( j=0, 1, 2, \ldots \), then we know that

\[ p_{kj} = 0 \quad j > k \]

\[ \sum_{j=0}^{k-1} p_{kj} = 1 \cdot \]

4.7 SUMMARY

Two methods are available for obtaining steady state waiting time distributions for GI/G/1. One can iterate through each arrival epoch or solve directly with a system of linear equations. The iterative method also gives the waiting time distributions during the transient period.
Besides waiting time distributions the Laguerre transform method can provide steady state and transient interdeparture distributions and queue length distributions. The interdeparture and queue length distributions require only a slight amount of effort beyond that expended in obtaining the waiting time distributions.
Chapter V
APPLYING THE METHODS

5.1 INTRODUCTION

The equations in Chapter Four give little appreciation for the practical considerations involved in applying the Laguerre transform method. This chapter will highlight these considerations as well as develop new ideas which might be the subject of future research.

The objectives of this chapter will be previewed now so that they might be more clearly seen in the examples which follow.

1. Provide a convincing demonstration of the method over a reasonably large range of queuing system parameters.

2. Show that the scaling of a problem has a crucial impact on computational effort. It is thus a prime practical consideration.

3. Show that the steady state solution via matrix inverse does not compete with the iterative method, but is in fact complementary to it.
4. Suggest methods in which heuristic approximations can be used to improve the performance of the Laguerre transform method.

5.2 THE SCALING PROBLEM

5.2.1 Introduction

The general thrust of this section can be illustrated by a trivial $M/M/1$ example. Suppose that the arrivals to a queue follow the pdf $a(t) = \lambda \exp(-\lambda t)$, and the service times follow the pdf $b(t) = \mu \exp(-\mu t)$. Then the known result is that the waiting times go as

$$\omega(t) = (1-\rho)\delta(t) + \lambda(1-\rho)e^{\mu(1-\rho)t}$$

where $\rho = \lambda/\mu$

Given previous results for the expansion of exponential distributions we know that the expansion coefficients decay as,

$$a(t) : \left(\frac{\lambda - \frac{1}{2}}{\lambda + \frac{1}{2}}\right)^n$$

$$b(t) : \left(\frac{\mu - \frac{1}{2}}{\mu + \frac{1}{2}}\right)^n$$

$$\omega(t) : \left(\frac{\mu(1-\rho) - \frac{1}{2}}{\mu(1-\rho) + \frac{1}{2}}\right)^n$$

$n = 0, 1, 2, \ldots$
As far as the steady state solution via the matrix technique given in Chapter Four is concerned, the key is to have the coefficients for \( w(t) \) decay rapidly. Clearly this would allow one to solve the problem with a small matrix. The obvious strategy is to scale the problem to obtain \( \mu(1-\rho)=1/2 \). Then the expansion of \( w(t) \) would have coefficients \( w_0=\rho/2 \) and \( w_i=0 \) for \( i>0 \). One could invert a 2x2 matrix and obtain a virtually exact solution.

But this strategy imposes constraints on the scaling of the arrival and service distributions. For instance, in heavy traffic we might have \( \rho=0.9 \). This implies \( \lambda=5 \) and \( \mu=4.5 \) in order to obtain \( \mu(1-\rho)=1/2 \).

The arrival and service coefficients would then decay as \( (.8)^n \) and \( (9/11)^n \) respectively—a rather slow decay. On the other hand, one might have used \( \lambda=0.477 \) and \( \mu=0.53 \) which also yields \( \rho=0.9 \). The arrival and service distributions would converge rapidly. However, the coefficients of \( w(t) \) would decay as \( (.447/.553)^n \) which would require solution via a relatively large matrix to attain reasonable accuracy.

For the exponential distributions involved it is a trivial matter to generate any number of coefficients right down to the smallest representable number of the computer one is using. So clearly one scales to accommodate a compact expansion for \( w(t) \). The problem of obtaining the
expansion coefficients for other distributions is far less trivial especially when numerical techniques must be used.

These arguments also have implications for the transient technique of iterating Lindley's equation at arrival epochs. If the final or "converged" solution for \( w(t) \) requires a large coefficient set then we must retain many terms at each iteration. But return to the \( \text{M}/\text{M}/1 \) example. If the final solution for \( w(t) \) goes as \( \exp(-t/2) \), it should logically be possible to discard insignificant terms at each iteration as the expansion converges to its final values. The remainder of this section is devoted toward extending this reasoning toward the more difficult \( \text{GI}/\text{G}/1 \) case.

5.2.2 Scaling the distributions

Chapter Three introduced an expression for the extent of a coefficient set. Namely,

\[
N_E = \frac{1}{4} T_E + B_E \geq \frac{1}{2}
\]

\[
= \frac{1}{4} \frac{\int_{0}^{\infty} x f^2(x) \, dx}{\int_{0}^{\infty} f^2(x) \, dx} + \frac{\int_{0}^{\infty} x \left[ \frac{d}{dx} f(x) \right]^2 \, dx}{\int_{0}^{\infty} f^2(x) \, dx}
\]

The idea of minimizing \( N_E \) by scaling the distributions is suggested, in another context, in Suhita [25]. But minimizing \( N_E \) does not ensure that the queueing problem is any
easier to solve. Since three distributions are involved in queueing, it is unlikely that all three would be minimized by the same scale factor. Consequently we now obtain analytic expressions for \( N_E \) for distributions of interest later in the chapter. These expressions will give a measure of how far from the minimum various scaling decisions place the problem.

In general we want to make a transformation \( x \rightarrow cx \) on \( f(x) \) so that \( N_E \) is a function of \( c \), the scale factor. That is,

\[
N_E(c) = \frac{1}{4} \, T_E(c) + B_E(c)
\]

Considering now the individual integrals in \( N_E(c) \) define

\[
\begin{align*}
I_1(c) &= \int_0^\infty x f^2(cx) \, dx = \frac{1}{c^2} \, I_1(1) \\
I_2(c) &= \int_0^\infty x \left[ \frac{d}{dx} f(cx) \right]^2 \, dx = I_2(1) \\
I_3(c) &= \int_0^\infty f^2(cx) \, dx = \frac{1}{c} \, I_3(1)
\end{align*}
\]

Therefore,

\[
N_E(c) = \frac{1}{4} \, \frac{1}{c^2} \, \frac{I_1(1)}{I_3(1)} + \frac{1}{c} \, \frac{I_2(1)}{I_3(1)}
\]

\[
= \frac{1}{4} \cdot \frac{1}{c} \, T_E(1) + c \, B_E(1)
\]

Now the principal distributions to be used in the examples that follow are the lognormal and Weibull distribu-
tions. We can obtain analytic expressions for \( N_E(c) \) for these distributions. Restating here the functional form of these distributions,

Weibull: \[ f(x) = \frac{\beta}{\delta} \left( \frac{x}{\delta} \right)^{\beta-1} \exp\left[-\left(\frac{x}{\delta}\right)^\beta\right] \]

\( \beta = \) shape parameter.

\( \delta = \) scale parameter.

Lognormal: \[ f(x) = \frac{1}{\sqrt{2\pi \sigma^2}} \cdot \exp\left[-\frac{1}{2\sigma^2} (\log x - \mu)^2\right] \]

\( \mu = \) mean of underlying normal.

\( \sigma^2 = \) variance of underlying normal.

After a considerable amount of manipulation we can obtain,

Weibull: \[ N_E(c) = \frac{1}{\pi^{2/3} \Gamma(2-1/\beta)} \left\{\frac{1}{4c} + c\left[\frac{\beta(\beta-1)}{\pi(2-2/\beta)}\right]\right\} \]

\( \beta > 1, \ \delta = 1 \)

Lognormal: \[ N_E(c) = \exp(\mu - \sigma^2/4) \left\{\frac{1}{4c} + \frac{c}{2\sigma^2} \exp(\sigma^2 - 2\mu)\right\} \]

We examine some specific examples now. First consider a lognormal distribution with a squared coefficient of variation, \( c^2 = 2 \). For the lognormal we have,
mean: \( \mu_{x_1} = \exp(\mu_y + \sigma_y^2/2) \)

2nd moment: \( \mu_{x_2} = \exp(2\mu_y + 2\sigma_y^2) \)

so that

\[
\sigma_y^2 = \frac{\mu_{x_2} - \mu_{x_1}^2}{\mu_{x_1}^2} = \exp\sigma_y^2 - 1
\]

\[
\Rightarrow \exp\sigma_y^2 = 3
\]

Therefore, \( \sigma_y^2 = \log_e 3 = 1.0986. \)

In this example take \( \sigma_y = 1.1 \), and initially choose \( \mu_y = 0. \)

By differentiating the expression for \( N_E(c) \) with respect to \( c \) we find the minimum extent expansion coefficients for this particular lognormal.

\[
\frac{dN_E(c)}{dc} = 0
\]

\[
\Rightarrow c = \frac{\sigma_y}{\sqrt{2} \exp(\sigma_y^2/2)} = .4278774
\]

Therefore, the scaled distribution with \( \sigma_y^2 = 1.1 \) and \( \mu_y = 0 \) with the minimum extent expansion coefficients is given by

\[
cF(cx) = \frac{1}{\sqrt{2\pi\sigma_y^4}} x \exp\left[-\frac{1}{2\sigma_y^4} \log_e \left(4278774x \right)\right]
\]

\[
= \frac{1}{\sqrt{2\pi\sigma_y^4}} x \exp\left[-\frac{1}{2\sigma_y^4} (\log_e x - .8489)^2\right]
\]

In other words, the scaling is equivalent to changing \( \mu_y \) from 0 to .8489. This results in a lognormal distribution with mean of \( \mu_{x_1} = 4.0508 \). The squared coefficient of variation remains unchanged at about two.
For the Weibull distribution consider a case with $\beta = 3$ and $\delta = 1$. Then,

$$\frac{dN_E(c)}{dc} = 0$$

$$\Rightarrow \quad c \approx 0.2425$$

Therefore the scaled Weibull distribution with $\beta = 3$ and $\delta = 1$ with the minimum extent expansion coefficients is given by,

$$c^f(cx) = 3x^2(0.2425)^3 \exp[-(0.2425x)^3]$$

This is equivalent to the Weibull distribution with $\beta = 3$ and $\delta = 1/0.2425 = 4.1244$. For this distribution we have: mean = 3.6827, and squared coefficient of variation = .13224.

A significant change of the scale factors from those given above for the lognormal and Weibull will result in an expansion which requires more terms to produce a given accuracy. Figure 16 below illustrates this dependence of $N_E(c)$ on $c$ for the Weibull and lognormal distributions discussed above.

The effect of scaling upon accuracy of the expansions will be seen directly in later sections of this chapter in the context of the queueing problems.

The curves in Figure 16 are a very quantitative illustration of the tradeoffs between smoothness (peakedness) and localization. They are also a useful input in deciding
how to scale a given queueing problem. The other needed tool is of course a rough predictor of the steady state waiting time distribution. This is the topic of the next section.
5.2.3 Scaling the overall queuing problem

There is no shortage of approximations for the steady state waiting time distribution. It is not the purpose of this research to choose among them. The chosen approximation will, after all, serve only as a rough guide in scaling and will not necessarily affect the accuracy of the result. In fact, in the examples shown later, the simple $M/M/1$ result for waiting time was used as the guide to scaling. This will now be illustrated for the specific queues to be considered here.

The remainder of this chapter applies the theory presented in Chapter Four to six $GI/M/1$ queues. GI/M/1 queues were selected because several known results are available and because one can choose a wide variety of interarrival distributions. The coefficient of variation of the arrival distribution can be ranged over values larger and smaller than one. To further illustrate the power of the Laguerre transform method, the Weibull and lognormal densities will be used so that only numerical techniques are available for obtaining the expansions. Thus proper scaling becomes essential because of the effort in obtaining the expansion coefficients.

Assume an arrival process with rate $\lambda$ and a server with rate $\mu$. Further, assume that the steady state waiting time distribution is dominated by an exponential term with
parameter $\mu - \lambda$ (the $M/M/1$ case). Under these assumptions the waiting time expansion coefficients decay as

$$
\left( \frac{\mu - \lambda - 1/2}{\lambda - \lambda + 1/2} \right)^n
$$

If we are solving for the steady state coefficients using an $(N+1) \times (N+1)$ matrix and would like a final coefficient, $w_{M*}$ of about $10^{-9}$ using no more than a $31 \times 31$ matrix, then we need

$$
\left( \frac{\mu - \lambda - 1/2}{\mu - \lambda + 1/2} \right)^{30} \approx 10^{-9}
$$

This roughly implies $1/6 \leq \mu - \lambda \leq 3/2$. Seriously violating this constraint will require solving a larger matrix to fulfill the constraint on the magnitude of the final coefficient.

For a given traffic intensity, $\rho = \frac{\lambda}{\mu}$, the scaling problem is fully defined. That is,

$$
\mu - \lambda \in \left[ \frac{1}{6}, \frac{3}{2} \right]
$$

At $\rho = .9$:

$$
\{ (\mu, \lambda) : \mu \in \left[ \frac{5}{3}, 15 \right] ; \lambda \in \left[ 1.5, 13.5 \right] ; \frac{\lambda}{\mu} = .9 \}
$$

At $\rho = .1$:

$$
\{ (\mu, \lambda) : \mu \in \left[ 185, \frac{5}{3} \right] ; \lambda \in \left[ 0.185, 1/6 \right] ; \frac{\lambda}{\mu} = .1 \}
$$

to obtain $\omega_{30} \approx 10^{-9}$

These guidelines are easily restated in terms of the means of the distributions involved. That is,

At $\rho = .9$:

$$
\{ (\text{mean}_5, \text{mean}_a) : \text{mean}_5 \in \left[ 0.07, 6 \right] ; \text{mean}_a \in \left[ 0.07, 6 \right] ; \frac{\text{mean}_5}{\text{mean}_a} = .9 \}
$$
There is a one-to-one correspondence between points in the arrival and service mean intervals at a given value of \( \rho \). And hopefully for the distributions involved there is some point in those intervals where both distributions are easily expanded. If not, one must be prepared to increase the dimension of the problem.

Two final comments apply. First, the discussion above does not "solve" the scaling problem but it does structure it. Second, one is not tied to any particular approximation for the waiting time distribution. For instance, when the arrival distribution's coefficient of variation is greater than one, the mean of the waiting time distribution increases. This effect is captured, for example, by Kingman's [12] heavy traffic approximation. Various approximations can be used to modify one's estimate for the intervals on \( \mu \) and \( \lambda \) or the mean arrival and service interval which will give a rapidly convergent waiting time distribution.

5.3 APPLICATIONS

5.3.1 Introduction
This section illustrates the Laguerre transform methods presented in Chapter Four and the heuristic scaling principles developed earlier in this chapter. It begins with a general discussion of some computational issues important
to the computer implementation. Known results for the GI/M/1 queue are then presented. And finally a large set of examples follows. The examples may seem somewhat repetitive, but they do serve to illustrate various aspects of the Laguerre transform methods. Further, they will also highlight the complementary uses of the matrix solution and the iterative solution to Lindley's equation.

5.3.2 Computational issues

Some of the conditions under which these results were obtained will now be discussed. A minor goal of the research was to develop these methods for a microcomputer. Thus, all of the computations were done on an Apple Macintosh using Apple's version of Pascal. This version of Pascal provides for 15-16 digit double precision accuracy.

Some computations were duplicated on the Ohio State University's IBM 3081 computer. This will give the reader some basis of comparison for later comments on program run times. Specifically, when performing numerical integrations, it was found that Amdahl execution times were on average .0013 of those of the Macintosh.

The matrix inversions to obtain steady state waiting time distributions were done using Gauss-Jordan elimination with all pivots on the maximum element of the remaining submatrix. Solutions can be obtained much faster via triangularization and backward elimination. But the use of an
inverse technique provided excellent accuracy and allowed an easy application of the bordering method.

That is, after an initial inverse of order $m \times n$ was obtained, a new row and column was added and bordering was used to find the new inverse. The new row and column consisted of expansion coefficients assumed negligible in the initial matrix. Thus by repeated borderings one obtains a sequence of solution vectors so that one may see directly the effect of neglecting expansion terms.

The transient solution via iterating Lindley's equation was implemented adaptively in the following sense. At each iteration (arrival epoch) the final five waiting time expansion coefficients were examined. If all five coefficients were below a certain threshold (normally $10^{-14}$) in absolute value, four were discarded. Conversely, if the last coefficient exceeded a certain value (normally $10^{-12}$), provision was made to include a new coefficient at the next iteration.

This adaptive technique can not be justified analytically. But it does capitalize on the concept of scaling to produce a compact final solution set. Since each iteration requires order $n^2$ effort, some cases resulted in as much as a 16 to 1 reduction in computation over non-adaptive implementation. Furthermore, with the matrix technique to cross check the final result there is a solid means to empirical-
ly justify the adaptive method. The results must speak for themselves.

Finally, convergence of the transient solution to steady state was detected as follows. The absolute value of the change in each of the first 21 coefficients was summed at each iteration, i.e.

\[ \text{sum} = \sum_{i=0}^{20} |\omega_{i,n+1} - \omega_{i,n}| \]

When this sum dropped below an arbitrary threshold (normally $10^{-3}$) the iterations were stopped. The solution at this point could be compared to the steady state (matrix) solution and if desired the iterations could be restarted with a smaller threshold.

5.3.3 Known results for GI/H/1

The results quoted here can be found in Kleinrock [14]. For the G/H/1 queue the pdf of the steady state waiting time is given by

\[ \omega(y) = (1-\gamma)\delta(y) + \mu \gamma (1-\gamma) e^{-\mu(1-\gamma)y} \] (5.3.1)

The value $\gamma$ is the (unique) root in the range $0 < \gamma < 1$ of the equation

\[ \gamma = \mathcal{R}(\alpha(x)) \bigg|_{x=\mu(1-\gamma)} \] (5.3.2)

In words: $\gamma$ equals the Laplace transform of the arrival pdf evaluated at $s=\mu-\mu\gamma$ where $\mu$ is the rate of the service pdf.
In the cases to be employed as examples (lognormal and weibull) the Laplace transform is not analytically available. The technique to be used is to take \( W(0) \) as determined by the Laguerre transform method and equate \( \chi = 1 - W(0) \), which follows from equation (5.3.1). We then numerically evaluate (integrate) as in
\[
x = \int_c^\infty e^{-\mu(1-\chi)y} a(y)dy
\]
and check whether \( x = \chi \). This numerical check allows using interarrival distributions which challenge the Laguerre transform techniques even though an analytic expression for the Laplace transform is unavailable.

Steady state queue length results are given by

\[
(5.3.3) \quad p(j)=\text{probability of } j \text{ customers in the system}
\]
\[
= (1-\chi)\chi^j \quad \text{for } j=0,1,2,\ldots
\]

Thus, in the examples, when reference is made to checking the fit of the waiting time distribution, we mean a comparison to equation (5.3.1). Checking the value of \( W(0) \), the steady state probability of zero wait, refers to equation (5.3.2). Comparison of the steady state queue length distribution to known values is a reference to equation (5.3.3).
5.3.4 Results

The first division of results is according to the arrival distributions—lognormal in one case weibull in the other. The squared coefficient of variation of the arrival distribution was fixed at some value and other parameters changed to produce a range of queueing problems. Specifically, for the lognormal $c_a^2=2$ and for the weibull $c_a^2=1.32$.

Within the group of examples for each distribution we consider three values of traffic intensity, $\rho = .1, .5, .9$. For each value of $\rho$ we consider steady state and transient waiting time, queue length, and interdeparture distributions. Transient results are not verified at specific arrival epochs but are checked in the limit as they approach the (known) steady state result.

Since the queueing problem becomes most interesting at $\rho = .9$ (high traffic intensity) more detailed comparisons are carried out there. It is also of interest to compare the effort required to iterate to the steady state solution to the effort involved in the matrix solution. Finally, all of the graphs shown are based on the expansion coefficients not direct plots of functions.
5.3.4.1 Lognormal arrivals, traffic intensity=.1

At a traffic intensity of $\rho=.1$ the interarrival pdf mean is relatively large, in this case 5.9899. The specific pdf used was,

$$f(x) = \frac{1}{\sqrt{2\pi}x} \exp\left[-\frac{1}{2}(\log x - 1.24)^2\right]$$

To obtain an accurate expansion for $f(x)$ it was necessary to numerically integrate on the interval $(0,200)$. The function $f(x)$ and some measures of fit are shown in Figure 17.

**Figure 17:** Lognormal arrival distribution—traffic intensity=.1
Figure 18 shows the rapid convergence of the transient to the steady state waiting time distribution. Convergence occurred at iteration 10. Overall accuracy is evaluated by comparing

\[ x(a) \mid s = \mu e^{-s} \in [0.07038190, 0.07038194] \]

with

\[ \gamma = 1 - W(0) = 0.070381934 \]

The agreement is thus excellent. The measures of fit in Figure 18 compare the known waiting time distribution with the steady state solution. Maximum error refers to the largest deviation from the known function in a set of at least sixty sample points in the domain shown in the figure. Average absolute error refers to the average value of the absolute deviation over the same sampled points. These same terms are used throughout this section.

Table 5 below shows the progression of the queue length distribution at various arrival epochs and in steady state. The steady state values can be compared with the known values in the last column. Note that at arrival epoch four the values shown should sum to one since there must be zero, one, two, or three in the system. This is not necessarily true at other arrival epochs. At the fourth arrival epoch the sum of the p(i) is 1.000000. In all future cases where queue length distributions are displayed the first displayed result will be chosen so that it should sum to one.
Figure 18: Waiting time distributions—lognormal arrivals, traffic intensity = .1

Table 5

Queue distribution—lognormal arrivals, traffic intensity = .1

<table>
<thead>
<tr>
<th></th>
<th>4th Arr Epoch</th>
<th>10th Arr Epoch</th>
<th>Steady State</th>
<th>Calculated Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>p(0)</td>
<td>.929845</td>
<td>.929618</td>
<td>.929618</td>
<td>.929618</td>
</tr>
<tr>
<td>p(1)</td>
<td>.065391</td>
<td>.065425</td>
<td>.065425</td>
<td>.065428</td>
</tr>
<tr>
<td>p(2)</td>
<td>.004502</td>
<td>.004601</td>
<td>.004601</td>
<td>.004605</td>
</tr>
<tr>
<td>p(3)</td>
<td>.000262</td>
<td>.000320</td>
<td>.000320</td>
<td>.000324</td>
</tr>
</tbody>
</table>

Figure 19 shows the rapid convergence of the interdeparture distributions to steady state values.
Only a rough check can be provided here using the mean value. The mean of the interdeparture distribution calculated from its expansion coefficients is 5.97512. The mean of the interarrival distribution as calculated from its expansion coefficients is 5.97516. So we have the logical result that the mean departure interval and arrival interval are equal.
5.3.4.2 Lognormal arrivals, traffic intensity=0.5

At $\rho = 0.5$ an interarrival distribution with a mean of about 4.1 was used. This is very near the lognormal whose expansion coefficients have minimum extent. Figure 20 shows this arrival distribution and some measure of fit. Note that using the same number of coefficients (110) as in the next lognormal example, the fit of this lognormal is better since it is much nearer the minimum extent lognormal.

Figure 20: Lognormal arrival distribution, traffic intensity=0.5

true mean: 4.09596
% error in mean: 0.074
max error: 0.0014 at $x=0.001$
ave. abs. error: 0.000026
Overall accuracy of the steady state waiting time is checked by comparing
\[ s = \mu (1 - \gamma) \]
with
\[ 1 - W(0) = \gamma = 0.549945042 \]

Convergence of the transient to the steady state solution came at arrival epoch 134. Figure 21 shows the waiting time distribution at several arrival epochs and includes some measures of fit for the distribution at arrival epoch 134 to the known solution.

\[
\begin{align*}
\text{max error:} & \quad 0.00000002 \text{ at } x=0 \\
\text{ave. abs. error:} & \quad 0.00000004 \\
\text{true mean:} & \quad 2.502528 \\
\% \text{ error in mean:} & \quad 0.0002
\end{align*}
\]

\[ W_n(x) \]

Figure 21: Waiting time distributions—lognormal arrivals, traffic intensity = .5

Transient and steady state queue lengths are shown in Table 6 along with known values.
Table 6

Queue distributions—lognormal arrivals, traffic intensity = .5.

<table>
<thead>
<tr>
<th></th>
<th>5th Arr Epoch</th>
<th>15th Arr Epoch</th>
<th>50th Arr Epoch</th>
<th>Steady State Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>p(0)</td>
<td>.505566</td>
<td>.456517</td>
<td>.450113</td>
<td>.450055</td>
</tr>
<tr>
<td>p(1)</td>
<td>.279509</td>
<td>.251211</td>
<td>.247529</td>
<td>.247499</td>
</tr>
<tr>
<td>p(2)</td>
<td>.141840</td>
<td>.137604</td>
<td>.136120</td>
<td>.136108</td>
</tr>
<tr>
<td>p(3)</td>
<td>.058492</td>
<td>.074703</td>
<td>.074849</td>
<td>.074855</td>
</tr>
<tr>
<td>p(4)</td>
<td>.014593</td>
<td>.039935</td>
<td>.041152</td>
<td>.041160</td>
</tr>
<tr>
<td>p(5)</td>
<td>N/A</td>
<td>.020837</td>
<td>.022620</td>
<td>.022633</td>
</tr>
<tr>
<td>p(6)</td>
<td>N/A</td>
<td>.010488</td>
<td>.012429</td>
<td>.012444</td>
</tr>
<tr>
<td>p(7)</td>
<td>N/A</td>
<td>.005017</td>
<td>.006825</td>
<td>.006841</td>
</tr>
<tr>
<td>p(8)</td>
<td>N/A</td>
<td>.002238</td>
<td>.003743</td>
<td>.003759</td>
</tr>
</tbody>
</table>

The values of the queue length probabilities at arrival epoch five sum to one as should be the case. Note that, in general, the accuracy of the probabilities will ultimately decrease for larger values of queue lengths. Recall that these probabilities are calculated using repeated convolutions of the interarrival distribution. These convolutions are essentially performed in terms of the original expansion coefficients. This eventually becomes an inadequate representation of the convolved distribution even if the dimension of the discrete coefficient convolution is allowed to grow at each convolution. This is particularly true when the original distribution has a large mean. In this case p(8) depends on a distribution with mean of about 32.8.
The convergence of the interdeparture distributions, $d_n(x)$, to steady state is shown in Figure 22. Again, mean value of the interdeparture and interarrival distributions agrees.

5.3.4.3 Lognormal arrivals, traffic intensity=.9

At traffic intensity $\rho=.9$ the interarrival distribution has a relatively small mean, in this case .67. Figure 23
shows this distribution and some measure of fit. At \( \rho = 0.9 \) we expect that the queueing system will require many more arrival epochs to converge to the steady state waiting time distribution. And, in fact the transient solution technique converges \((10^{-6})\) in 4556 arrival epochs. The transient technique gives a steady state value of \( W(0) = 0.07226270 \). This can be compared to the value from the matrix technique, \( W(0) = 0.07226265 \).

\[
\begin{array}{c}
\text{Figure 23: Lognormal arrival distribution, traffic intensity = 0.9.}
\end{array}
\]

Figures 24 and 25 show the waiting time distribution at various arrival epochs and in steady state. There is less than a ten percent change in \( W(0) \) after the 200th arrival.
epoch, and a one percent change after the 70.0th. This suggests that having the alternative method for the steady state solution will save several thousand iterations and much effort. The 4556 iterations on the Macintosh took 31 hours. Each iteration was typically manipulating a string of about 40 coefficients. Poor choice of scale factor could have increased the number of coefficients significantly and made the run length even longer. By contrast, the matrix method took about 35 minutes to solve a 40x40 set of equations. We return to this topic when considering the weibull distribution and in the conclusions.
The quality of the steady state solution can be evaluated by comparing

$$\mathcal{L}(\alpha, \gamma) s= \mu(1-\gamma)$$

with

$$\gamma = 1 - W(0) = 0.927737346$$

The measures of fit shown in Figure 25 are for the steady state solution.

Table 7 shows the convergence of the transient to the steady state queue length distribution. The last column contains the calculated values of the steady state queue length distribution for comparison. Even at $p(15)$ we have
Table 7

Queue distributions—lognormal arrivals, traffic intensity = .9

<table>
<thead>
<tr>
<th></th>
<th>6th Arr Epoch</th>
<th>10th Arr Epoch</th>
<th>50th Arr Epoch</th>
<th>Steady State Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>p(0)</td>
<td>.252736</td>
<td>.197800</td>
<td>.109427</td>
<td>.072263</td>
</tr>
<tr>
<td>p(1)</td>
<td>.245393</td>
<td>.188712</td>
<td>.101976</td>
<td>.067046</td>
</tr>
<tr>
<td>p(2)</td>
<td>.216246</td>
<td>.172424</td>
<td>.094609</td>
<td>.062205</td>
</tr>
<tr>
<td>p(3)</td>
<td>.162485</td>
<td>.148524</td>
<td>.087305</td>
<td>.057703</td>
</tr>
<tr>
<td>p(4)</td>
<td>.092799</td>
<td>.118209</td>
<td>.080100</td>
<td>.053539</td>
</tr>
<tr>
<td>p(5)</td>
<td>.030342</td>
<td>.084560</td>
<td>.073007</td>
<td>.049669</td>
</tr>
<tr>
<td>p(6) N/A</td>
<td>.052239</td>
<td>.066068</td>
<td>.046080</td>
<td>.046075</td>
</tr>
<tr>
<td>p(7) N/A</td>
<td>.026156</td>
<td>.059322</td>
<td>.042751</td>
<td>.042745</td>
</tr>
</tbody>
</table>

The result $p(15) = 0.023464$ which compares to the calculated value $0.023458$. In the case at hand, the mean of the arrival distribution is .67. The 15th convolution produces some distribution whose mean is still only about 10. There is little difficulty in maintaining the accuracy of the convolution in this case. The original expansion (see Figure 23) was not particularly accurate—the final coefficient was of the order $10^{-4}$. The peakedness of the distribution would require many more than the 110 coefficients used to produce a very accurate expansion. But the key point is that the available coefficients are accurate. The mechanics of the discrete convolution allow us to convolve to a more accurate expansion, i.e. one whose tail coefficients are smaller because the distribution they represent is less peaked.
It is thus quite understandable that the convolved coefficients in the $\rho = .9$ case will remain an accurate representation of the convolved distribution. The selection of the mean interarrival interval is a consequence of the scaling guidelines. And the end result is that in cases where more queuing occurs, the method allows accurate calculation of the distribution at higher queue lengths.

Figure 26 shows the convergence of the transient interdeparture distributions, $d_\eta(x)$, to the steady state. The mean interdeparture and interarrival intervals again agree. Qualitatively one can also observe that the depart-
ture distribution's form roughly follows the service dis-
tribution (exponential). This can be compared to the case
for $\rho=.1$ where the departure distribution is somewhere
between the exponential service and lognormal arrival dis-
tributions. This is expected since at $\rho=.9$ the server is
nearly always busy. We would thus expect nearly exponen-
tially distributed departures. A detailed examination of
the coefficients would reveal that the distribution is not
truly exponential.

5.3.4.8 Weibull arrivals, traffic intensity=.1

The weibull distribution squared coefficient of variation
is so low ($c^2_\lambda=.132$) compared to the lognormal used in the
last section ($c^2_\lambda=2$) that we expect to see a reduction in
queueing and more rapid convergence for any given traffic
intensity. And, in fact, for $\rho=.1$ there is virtually no
transient problem to solve. Steady state values and curves
will be displayed instead.

Figure 27 shows both the interdeparture and interarrival
distributions. The interdeparture distribution is clearly
non-zero at the origin reflecting a small probability of
waiting for the server. Using the steady state calcula-
tions yields

$$\lambda(\alpha(y)|s=\mu(1-\gamma)) \in [0.0415769, 0.0415770]$$
mean arrival interval: 6.000
mean departure interval: 5.994

Figure 27: Interarrival and interdeparture distributions—
weibull arrivals, traffic intensity=.1

and

\[ 1 - W(0) = \gamma = .00415770 \]

which is excellent agreement.

Figure 28 shows the steady state waiting time distribution and measures of fit to the known solution. Table 8 compares the queue length distribution to known values.
Figure 28: Waiting time distribution—Weibull arrivals, traffic intensity = .1

Table 8

<table>
<thead>
<tr>
<th>Steady State</th>
<th>Calculated Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>p(0)</td>
<td>.9958423</td>
</tr>
<tr>
<td>p(1)</td>
<td>.0041404</td>
</tr>
<tr>
<td>p(2)</td>
<td>.0000172</td>
</tr>
<tr>
<td>p(3)</td>
<td>7.2E-08</td>
</tr>
</tbody>
</table>
5.3.4.5 Weibull arrivals, traffic intensity=.5

Figure 29 shows the waiting time distribution at iterations two, five, and for steady state. Again we can check the value of \( W(0) \) with

\[
\mathcal{L}(\alpha(t)) \in [0.262963200, 0.262963205]
\]

and

\[
1 - W(0) = \gamma = 0.262963201
\]

Figure 29: Waiting time distributions—Weibull arrivals, traffic intensity=.5

Table 9 displays the convergence of the queue lengths to steady state values and compares these to known values.
### Table 9

<table>
<thead>
<tr>
<th>Queue distributions -- weibull arrivals, traffic intensity = 0.5</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>6th Arr Epoch</strong></td>
</tr>
<tr>
<td>---------------------</td>
</tr>
<tr>
<td>p(0)</td>
</tr>
<tr>
<td>p(1)</td>
</tr>
<tr>
<td>p(2)</td>
</tr>
<tr>
<td>p(3)</td>
</tr>
<tr>
<td>p(4)</td>
</tr>
<tr>
<td>p(5)</td>
</tr>
</tbody>
</table>

**Figure 30:** Interdeparture and interarrival distributions -- weibull arrivals, traffic intensity = 0.5

Finally, Figure 30 illustrates the interarrival and interdeparture distributions. This is a case where the shape of the interdeparture distribution is a mix of the weibull arrival and the exponential service distributions.
5.3.4.6 Weibull arrivals, traffic intensity=.9

As was the case with the lognormal example, the transient solution technique takes many (2320) iterations to converge to steady state values that are much more easily calculated with the matrix technique. In this case the run time was about eight hours. This is disproportionately shorter than the lognormal case at \( \rho = .9 \) because the iterations were done with about 25 coefficients versus the 40 in the lognormal case. Recall that the transient technique as implemented adaptively selects the number of coefficients needed, although this selection is a direct result of the scaling of the problem.

Figure 31 shows the convergence of the waiting time distributions to steady state.

Note that the distributions at arrival epoch 200 and in steady state are barely distinguishable. This suggests that the last 2000 odd iterations might easily be replaced by a type of merge to the steady state solution as calculated by the matrix inverse technique. This will be discussed further in the conclusions.

The steady state result is again checked with

\[
\lambda(\alpha(\cdot)) \mid_{s=\mu(1-\xi)} \in [8282705, 8282707]
\]

with

\[
1-W(c) = \xi = .8282704
\]
One final comment on the waiting time distribution. Table 10 is a comparison of every fifth coefficient of the waiting time distribution as calculated from the matrix technique with those from the transient or iterative technique. This lends evidence that both techniques are highly stable and compatible.

Table 11 shows the convergence of queue length distributions to steady state values and compares these to known results. As was the case with lognormal arrivals at $\rho = .9$, "compressing" the arrival distribution (small mean) allows one to maintain reasonable accuracy at higher queue lengths.
Table 10

Waiting time expansion coefficients.

<table>
<thead>
<tr>
<th>Coeff Order</th>
<th>Matrix Method</th>
<th>Iterative Method (Arr. Epoch 2320)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>.228778722</td>
<td>.228778729</td>
</tr>
<tr>
<td>5</td>
<td>-.004109075</td>
<td>-.004109073</td>
</tr>
<tr>
<td>10</td>
<td>.00073803</td>
<td>.00073802</td>
</tr>
<tr>
<td>15</td>
<td>-.000001326</td>
<td>-.000001325</td>
</tr>
<tr>
<td>20</td>
<td>.000000024</td>
<td>.000000024</td>
</tr>
<tr>
<td>25</td>
<td>-.000000001</td>
<td>.000000000</td>
</tr>
<tr>
<td>W(0)</td>
<td>.171729503</td>
<td>.171729509</td>
</tr>
</tbody>
</table>

Table 11

Queue distributions—Weibull arrivals, traffic intensity=.9.

<table>
<thead>
<tr>
<th></th>
<th>6th Arr Epoch</th>
<th>50th Arr Epoch</th>
<th>200th Arr Epoch</th>
<th>Steady State</th>
<th>Calculated Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>p(0)</td>
<td>.3851027</td>
<td>.206157</td>
<td>.176010</td>
<td>.171729</td>
<td>.171729</td>
</tr>
<tr>
<td>p(1)</td>
<td>.294412</td>
<td>.170042</td>
<td>.145750</td>
<td>.142265</td>
<td>.142239</td>
</tr>
<tr>
<td>p(2)</td>
<td>.188428</td>
<td>.139084</td>
<td>.120622</td>
<td>.117825</td>
<td>.117812</td>
</tr>
<tr>
<td>p(3)</td>
<td>.093979</td>
<td>.112647</td>
<td>.099752</td>
<td>.097618</td>
<td>.097580</td>
</tr>
<tr>
<td>p(4)</td>
<td>.032343</td>
<td>.090227</td>
<td>.082437</td>
<td>.080862</td>
<td>.080823</td>
</tr>
<tr>
<td>p(5)</td>
<td>.005735</td>
<td>.071359</td>
<td>.068056</td>
<td>.066952</td>
<td>.066943</td>
</tr>
<tr>
<td>p(6)</td>
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<td>.056144</td>
<td>.055469</td>
<td>.055447</td>
</tr>
<tr>
<td>p(7)</td>
<td>N/A</td>
<td>.042777</td>
<td>.046249</td>
<td>.045966</td>
<td>.045925</td>
</tr>
</tbody>
</table>

Finally, Figure 32 shows the progression of interdeparture distributions, \( d_\eta(x) \), to the steady state distribution. As in the lognormal case the general appearance of the curve becomes that of the exponential service distribution while the mean is that of the interarrival distribution.
5.4 SUMMARY
The examples shown indicate that the Laguerre transform methods can withstand detailed scrutiny of their accuracy. They provide access numerically to detailed system behavior not otherwise available.

Since computational effort must remain an important consideration, particularly in a microcomputer implementation, one must incorporate knowledge of queueing results into the numerical methods. Scaling as described earlier is a way of doing this. In a sense we assume an approximate answer and use this to select our scale factor.

Figure 32: Interdeparture distributions--weibull arrivals, traffic intensity=.9
If transient information is desired then one must use the iterative solution of Lindley's equation. At high traffic intensities it becomes very costly to iterate to very near the steady state solution. In the examples shown it was only at $\rho=.1$ that the iterative method obtained the steady state values more efficiently than the matrix method.

The iterative method very quickly came within ten percent of the final values and used about ninety percent of the iterations to cover the remaining distance. A more practical approach would then combine the two methods. This would increase the range of traffic intensities over which the method is viable. The previous inability to accommodate higher traffic intensities has been noted in the literature [9] as a difficulty of the method.
Chapter VI

CONCLUSIONS

6.1 COEFFICIENT GENERATION

Several distributions were presented in Chapter Three whose coefficients are easily generated, e.g. exponential, gamma, normal, uniform. Numerical techniques based on moments of the distribution or on numerical integration can deal with most other common distributions. Consider Chapter Three's extensive application to the weibull and lognormal distributions.

The method of moments is generally faster than numerical integration but works well only when higher order moments grow slowly. When this is not the case roundoff error accumulates and the technique becomes inaccurate, especially in a microcomputer application limited to 14-16 digit accuracy.

Numerical integration, on the other hand, sums a small number of terms of magnitude less than one and roundoff is not a significant problem. A procedure based on Gauss-Legendre quadrature and the stable Laguerre function recursion has worked well throughout this research. The practi-
call limit seems to be in dealing with extremely long-tailed functions, i.e. functions that are not negligible beyond say $x=300$.

Although $x=300$ is an arbitrary limit, the problem surely arises at some point because the Laguerre function recursion must start off with $J_0(x) = \exp(-x/2)$. Eventually this approaches the smallest representable number of the computer. Furthermore, the recursion may not perform well at these extremes. This is an area which requires further testing.

6.2 COMPUTING MOMENTS

Even when an expansion is an accurate representation of a function it does not follow that the moments of the function can be accurately calculated from the given expansion. This is true even when theory assures us that the moments can be calculated given a sufficiently long sequence of expansion coefficients. The appropriate summations simply converge more slowly to the true values of the moments--more expansion terms are needed.

When faced with a limited number of expansion coefficients we might form an estimate of the moments based on integrating the Laguerre functions over the region where the function "appears" to be non-zero. This is clearly an approximation, but one that works well in some cases as
shown in Chapter Three. Perhaps its best use is in preliminary work when one is adjusting system parameters to attain a particular value. After the survey work, one might invest the computational effort to obtain convergence of the moments in the standard fashion.

6.3 SCALING

Several ideas travel under the heading of scaling. First, it is possible to change the scale of a function's expansion using a scaling matrix developed here (see Appendix A). This essentially gives the expansion coefficients of the scaled function directly in terms of the coefficients of the original function. The purpose of this is to avoid redoing whatever computation was needed to get the original expansion coefficients. Clearly its utility depends on the relative accuracy and computational effort of scaling versus the other method. It might be useful in scaling the output of a lengthy queueing computation. But it would not be employed to scale an exponential distribution's coefficients since it would be simpler (and more accurate) to generate the new expansion directly.

Second, we have the idea of the minimum extent coefficient set. This refers to a function scaled so that its expansion coefficients converge as rapidly as possible. In the context of the queueing problem attaining the minimum
for one function involved (say the arrival distribution) may produce horrendous results for the others.

So finally we have the idea of scaling the overall queueing problem. This is not a precise operation. To be precise we would need to know the solution to the queue. Instead, this scaling seeks to reduce the dimension of the problem by assuming an approximate solution and scaling so that this solution will be compactly represented. This interacts necessarily with the second notion of scaling which evaluates the impact of the scale choice on the arrival and service distributions. Through these techniques one can have some control over computational effort and accuracy.

6.4 TRANSIENT AND STEADY-STATE TECHNIQUES

Current literature has emphasized the iteration of Lindley's equation via the two-sided Laguerre transform. The departure in this research is threefold. First, a one-sided transform method is developed. Second, a direct solution for the steady state waiting time distribution is developed. Third, methods are developed to give transient and steady state interdeparture and queue length distributions.

Using a one-sided transform reduces computational effort by one third, by eliminating calculation of the waiting
(idle) time distribution on the negative x axis. But more important than this is pointing out the use of the idle time distribution in calculating interdeparture distributions. Thus if interdeparture distributions are needed, either the one-sided transform or two-sided transform will require the same effort. If they are not needed the one-sided method saves computation. The availability of the interdeparture distribution provides a general numerical method for studying tandem queues.

The direct matrix solution for the steady state waiting time distribution is particularly advantageous at high traffic intensity. At high traffic intensity the queue converges slowly to its steady state values. This must be reflected in the iterative method which moves from one arrival epoch to the next. The direct solution short circuits this problem.

Furthermore, it appears that information from the two methods may be usefully combined. The transient solution's rate of convergence is initially rapid. With the steady state solution in hand via the matrix method it is easier to decide when to stop iterating through the arrival epochs.

Queue length distributions, both transient and stationary, can be calculated based on the waiting time distributions and the interarrival distribution. Maintaining the
accuracy of the queue length calculations depends most
directly on maintaining the accuracy of repeated convolu-
tions of the arrival distribution.

6.5 THE LAGUERRE TRANSFORM METHOD

The Laguerre transform method is a versatile and accurate
tool for working with the general single server queue.
Future research promises to extend it to tandem queues and
even small networks. Network implementation could be based
on some approximations for sums of renewal processes by
Whitt [28]. These approximations appear to lend themselves
to numerical calculation of the necessary Laguerre expan-
sion coefficients.

The computational effort involved in some cases suggests
the Laguerre transform technique be used in conjunction
with other methods and approximations. The methods devel-
oped here could supply the fine detail and precision when
needed.

Finally, once one has paid the "overhead" costs so to
speak, the Laguerre transform method becomes a very useful
tool for working on stochastic processes other than queues.
It is very suitable for microcomputer implementations while
still maintaining high accuracy.
Appendix A
THE SCALING MATRIX

A.1 INTRODUCTION

The most concise statement of the algorithm for generating the scaling matrix is via the computer program used to do this. The relevant subroutine (labelled procedure scale) is contained in the program listing at the end of this appendix. We will now consider two examples to highlight some of the practical problems of scaling.

A.2 EXAMPLE ONE—SCALING DOWN

Example one is composed of two scaling operations. Begin with an exponential distribution $f(x) = .4\exp(-.4x)$. In the first scaling operation a scale factor of $1.21$ is applied. This will produce an exponential distribution $f(x) = .484\exp(-.484x)$. In the second operation a scaling factor of $1.1$ is applied twice and will result in $f_1(x) = .44\exp(-.44x)$ and then $f_2(x) = .484\exp(-.484x)$. The end result of both operations should be the same in other words. Table 12 compares the end result of these two scaling operations to the true values of the coefficients.
<table>
<thead>
<tr>
<th>N</th>
<th>original coefficients .4exp(-.4x)</th>
<th>known coefficients .484exp(-.484x)</th>
<th>S(1.1x)S(1.1) output coefficients</th>
<th>S(1.21) output coefficients</th>
<th>max summation term</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>4.44444444444e-1</td>
<td>4.918699187e-1</td>
<td>4.918699187e-1</td>
<td>4.918699187e-1</td>
<td>4.0221e-1</td>
</tr>
<tr>
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<td>-7.997884857e-3</td>
<td>-7.997884857e-3</td>
<td>-7.997884857e-3</td>
<td>4.3883e-2</td>
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<tr>
<td>2</td>
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<td>1.300469082e-4</td>
<td>1.300469082e-4</td>
<td>1.300469082e-4</td>
<td>8.3781e-3</td>
</tr>
<tr>
<td>3</td>
<td>-6.096316111e-4</td>
<td>-2.114583874e-6</td>
<td>-2.114583874e-6</td>
<td>-2.114583874e-6</td>
<td>1.3648e-3</td>
</tr>
<tr>
<td>4</td>
<td>6.774035123e-5</td>
<td>3.438347762e-8</td>
<td>3.438347762e-8</td>
<td>3.438347762e-8</td>
<td>2.6097e-4</td>
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<td>-5.590809370e-10</td>
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<td>9.097469434e-12</td>
<td>9.097469434e-12</td>
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<td>-1.478170235e-13</td>
<td>-1.478170235e-13</td>
<td>1.7111e-6</td>
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<tr>
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<td>2.403528837e-15</td>
<td>2.403528837e-15</td>
<td>2.403528837e-15</td>
<td>3.2762e-7</td>
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<tr>
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<td>-3.908176970e-17</td>
<td>-3.908176970e-17</td>
<td>-3.908176970e-17</td>
<td>6.3860e-8</td>
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<td>6.354759301e-19</td>
<td>6.354759301e-19</td>
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<td>-1.033294195e-20</td>
<td>-1.033294195e-20</td>
<td>2.4273e-9</td>
</tr>
<tr>
<td>12</td>
<td>1.573647183e-12</td>
<td>1.680153636e-22</td>
<td>1.680153636e-22</td>
<td>1.680153636e-22</td>
<td>4.6501e-10</td>
</tr>
<tr>
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<td>4.442205468e-26</td>
<td>4.442205468e-26</td>
<td>1.7908e-11</td>
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<tr>
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<td>1.174487532e-29</td>
<td>1.174487532e-29</td>
<td>6.9728e-13</td>
</tr>
<tr>
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<td>3.105261503e-33</td>
<td>3.105261503e-33</td>
<td>3.105261503e-33</td>
<td>2.7652e-14</td>
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<td>3.655672596e-20</td>
<td>8.210095034e-37</td>
<td>8.210095034e-37</td>
<td>8.210095034e-37</td>
<td>1.1004e-15</td>
</tr>
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<td>21</td>
<td>-4.061858499e-21</td>
<td>-1.334974057e-38</td>
<td>-1.334974057e-38</td>
<td>-1.334974057e-38</td>
<td>2.183e-16</td>
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<td>2.170689534e-30</td>
<td>2.170689534e-30</td>
<td>2.170689534e-30</td>
<td>4.3922e-17</td>
</tr>
<tr>
<td>24</td>
<td>5.571822727e-24</td>
<td>5.739148794e-44</td>
<td>5.739148794e-44</td>
<td>5.739148794e-44</td>
<td>1.7577e-18</td>
</tr>
</tbody>
</table>
Both scaling operations produce about the same result over the range where the scaling is accurate—to about the 12th coefficient. The last column in Table 12 shows the maximum term in the summation

$$f_k^s = \sum_{j=0}^{T} S_{1,21}(k,j) f_j^u$$

- $f_k^s$ = scaled coefficient
- $f_j^u$ = unscaled coefficient

For instance, for the twelfth coefficient there is some $j$ for which $S_{1,21}(12,j) f_j^u = 4.6501 \times 10^{-10}$. Note that this number is larger than the coefficient we wish to calculate by a factor of $10^{12}$. Keeping track of this maximum term is one mandatory measure in order to recognize the accuracy limit of a scaling operation. For the 25th coefficient the maximum term is $10^{27}$ times the desired coefficient. Obviously 16 digit precision as used here will not suffice. Overall it is not disappointing to produce accurate coefficients down to $10^{-20}$, however.

A second accuracy consideration is where to terminate the summation shown above. Just to provide some feel for the subject Table 13 shows the "corners" of the scaling matrix. This particular matrix was generated as a square 47x47 matrix. The area of interest is the lower right hand corner of the matrix.
Table 13

Corners of the scaling matrix

<table>
<thead>
<tr>
<th>Upper left hand corner of S matrix</th>
<th>Lower left hand corner of S matrix</th>
<th>Lower right hand corner of S matrix</th>
</tr>
</thead>
<tbody>
<tr>
<td>9.049774e-1</td>
<td>1.007395e-44</td>
<td>-9.990778e+0</td>
</tr>
<tr>
<td>-8.599333e-2</td>
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<td>-5.403762e+2</td>
</tr>
<tr>
<td>8.171311e-3</td>
<td>9.885030e-40</td>
<td>1.636325e+3</td>
</tr>
<tr>
<td>7.764595e-4</td>
<td>1.407286e-37</td>
<td>-2.765382e+3</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1.636325e+3</td>
</tr>
<tr>
<td></td>
<td></td>
<td>-5.403762e+2</td>
</tr>
<tr>
<td></td>
<td></td>
<td>-1.126867e+2</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1.968010e+4</td>
</tr>
</tbody>
</table>

These terms are of order $10^{-2}$. If the product of these terms and the unscaled coefficients is small compared to the size of the unscaled coefficients then the size of the matrix is sufficient. If not, a rectangular matrix can be generated. Table 14 shows the lower right hand corner of a 47X80 scaling matrix with the same scale factor as above—1.21.

A rough way of putting the problem is whether one is scaling toward or away from the minimum extent coefficient...
Table 14

**Extending the columns of the scaling matrix**

<table>
<thead>
<tr>
<th>lower right hand corner of S matrix</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2.014637e-10</td>
</tr>
<tr>
<td>1.823377e-8</td>
</tr>
<tr>
<td>-1.640394e-6</td>
</tr>
<tr>
<td>1.463303e-4</td>
</tr>
</tbody>
</table>

set for the function. The example above was scaling the distribution $f(x)=-.4\exp(-.4x)$ toward the minimum for an exponential distribution, i.e. $f(x)=-.5\exp(-.5x)$. We should then expect to come away with a smaller set of coefficients than was begun with.

A.3 **EXAMPLE TWO—SCALING UP**

Example two is just the flip side of example one. We begin with the distribution $f(x)=.484\exp(-.484x)$ and generate a small set of 15 expansion coefficients. We apply a single scale factor $1/1.21 \approx .826$ to this distribution which should yield the expansion coefficients of $f(x)=.4\exp(-.4x)$. Then a scale factor of $1/1.1 \approx .909$ is applied twice to the original coefficients. The results of the two scaling operations should be identical and are shown in Table 15 below.

In this example the original 15th order expansion is converted up to a 25th order expansion of the scaled distribution. There is very little inaccuracy in the resultant coefficients whether one applies the factor .909 twice or the factor .826 once.
<table>
<thead>
<tr>
<th>N</th>
<th>original coefficients</th>
<th>known coefficients [0.484 exp(-0.484x)]</th>
<th>S([0.909] X S[0.909]) output coefficients</th>
<th>S([0.826]) output coefficients</th>
<th>max summation term</th>
</tr>
</thead>
<tbody>
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<td>4.444444444e-1</td>
<td>4.444444444e-1</td>
<td>4.444444444e-1</td>
<td>5.3861e-1</td>
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<td>-4.938271605e-2</td>
<td>-4.938271605e-2</td>
<td>5.1180e-2</td>
</tr>
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<td>5.486968450e-3</td>
<td>4.6233e-3</td>
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</tr>
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<td>1.032469917e-8</td>
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<td>-1.147188796e-9</td>
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<tr>
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<td>1.274654218e-10</td>
<td>1.274654218e-10</td>
<td>5.9348e-13</td>
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<td>-1.748496870e-13</td>
<td>7.5414e-14</td>
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<tr>
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<td>1.942774300e-14</td>
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</tr>
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<td>2.390486790e-16</td>
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<td>2.961094802e-18</td>
<td>2.961094802e-18</td>
<td>2.961094802e-18</td>
<td>1.0735e-20</td>
</tr>
<tr>
<td>19</td>
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<td>-3.290105336e-19</td>
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<tr>
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<td>3.655672596e-20</td>
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<td>-4.061858438e-21</td>
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</tr>
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<td>0.000000000e+0</td>
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<td>4.513176040e-22</td>
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<td>-6.190913539e-25</td>
<td>-6.190913539e-25</td>
<td>1.6796e-25</td>
</tr>
</tbody>
</table>
As before the last column shows the maximum term in the summation which produces the output coefficient. In this case the maximum term is of the same order of magnitude as the output coefficient and roundoff error never becomes a serious consideration. Example two is a case where scaling is away from the minimum extent coefficient set.

A-9 PROGRAM LISTING

```plaintext
program Smatrix;

var
  S : array[1..50, 1..90] of double;
    (the scaling matrix)
  V : array[1..51] of double;
    (vector used to generate next row of scaling matrix)
  c, b : double;
    (c is the scale factor)
  nrow, ncol, inmax, outmax : integer;
    (the maximum number of rows and columns to be generated)
    (the max order input and output coefficients)
  incoeff, outcoeff : array[0..100] of double;
    (the unscaled and scaled coefficients)
  maxterm : array[0..100] of double;
    (tracks roundoff error in matrix mult.)

procedure scale;
  (calculates the scaling matrix)

  var
    row, col, krow, kcol, j, i : integer;
      (row, col index the current row and column being generated)
      (krow, kcol control the sign of operations within a row and column)
      (j, i are indices for loops)

  begin

    (initialize variables)
    S[1, 1] := 2 / (c + 1);
    b := (c - 1) / (c + 1);
    V[1] := b;
    krow := -1;
```
program listing continued

(do first row)
   for j := 2 to ncol do
      begin
         S[1, j] := -V[1] * S[1, j - 1];
      end;

(update the vector V)
   V[1] := b * V[1];

(do the remainder of the S matrix)
   for row := 2 to nrow do
      begin
         krow := -krow;  (set row sign)
         kcol := krow;  (set initial column sign)

         (do the diagonal element)
         S[row, row] := S[row - 1, row - 1] + 2 * b * S[row - 1, row];

         (do the below diagonal elements within the row)
         for col := 1 to row - 1 do
            begin
               kcol := -kcol;  (set the column sign)
               S[row, col] := kcol * S[col, row];  (equate to transpose element)
            end;

         (do the above diagonal elements within the row)
         for col := row + 1 to ncol do
            begin
               i := col - row - 1;
               for j := 1 to row do
                  begin
                  end;
            end;

         (update the V vector)
         V[row + 1] := V[row] + b;  (calculate the new element of V)
         for l := row downto 2 do
            begin
               V[l] := V[l - 1] + b = V[l];  (update the old elements)
            end;
         V[1] := b * V[1];
      end;  (go to the next row)
   end;  (end of procedure scale)
procedure cocalc;
    var
        i, j : integer;
        temp : double;
    begin
        for i := 0 to outmax do
            begin
                outcoeff[i] := 0;
                maxterm[i] := 0;
                for j := 0 to inmax do
                    begin
                        temp := incodff[j] * S[i + 1, j + 1];
                        outcoeff[i] := outcoeff[i] + temp;
                        if abs(temp) > maxterm[i] then
                            begin
                                maxterm[i] := abs(temp);
                            end;
                    end;
                outcoeff[i] := c * outcoeff[i];
            end;
    end;  (procedure cocalc)
BIBLIOGRAPHY


