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VIBRATION AND BUCKLING OF LAMINATED COMPOSITE PLATES WITH
ARBITRARY BOUNDARY CONDITIONS

The Ohio State University

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VIBRATION AND BUCKLING OF LAMINATED COMPOSITE PLATES WITH ARBITRARY BOUNDARY CONDITIONS

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

by

Behnam Baharlou, B.S., M.S.

* * * * *

The Ohio State University

1985

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1985
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CHAPTER I
INTRODUCTION

Laminated composite plates are widely used at the present because of their many significant advantages. These plates are made up of layers (or lamina or plies) where, in each layer, parallel fibers of high strength material (e.g., glass, boron, carbon) are embedded in a homogeneous matrix material (e.g., epoxy). If the lamina are symmetrically arranged relative to its midplane, the plate may be characterized by the same uncoupled equations of bending and stretching as those of homogeneous, anisotropic plate theory. Indeed, in the special case of cross-ply symmetric laminates (i.e., layers arranged with fiber directions all either parallel or perpendicular), the equations reduce to the more simple ones of homogeneous, orthotropic plate theory. However, in the general case of unsymmetric laminates, bending and stretching are coupled, and the problems of vibration and buckling are governed by a set of eighth order coupled differential equations.

A vast range of theoretical and experimental research on the determination of the behavior of these structural elements has been in progress over the past two decades. A number of
references are listed in survey papers by Leissa\textsuperscript{1-4}, Bert\textsuperscript{5-9} and Bert and Francis\textsuperscript{10} dealing with vibration and buckling of laminated composite plates. The theory of thin laminated plates was well developed by 1970. Due to complexity of the differential equations, and especially the existence of coupling between bending and stretching for unsymmetrical laminates, exact analytic solutions can be found only for a few simple cases, whereas the theory identifies a total of sixteen different possible combinations of boundary conditions for each edge of the plate. The approximate methods are the only available tools to study the great number of problems that do not have exact solutions. Fourier series, Galerkin, and Ritz methods are some of the examples of these approximate approaches that have been used in vibration and buckling analysis of laminated plates. However, most of the approximate methods are not capable of giving a general solution for all the cases that might arise. Finite element methods, which can handle all of these cases, usually need a large number of terms for convergence, which requires long computing time. Accuracy of the results obtained by any of these approximate methods remain in doubt, unless they are examined against the results obtained from alternative methods, especially for those cases that experimental data are not available.

The major objective of this study is to develop an accurate and efficient theoretical method to find frequencies
and buckling loads for all the possible combinations of the geometric boundary conditions that exist for a rectangular laminated composite plate. Within the limitation of the classical theory of thin laminated plates, the formulation is done for a general case such that the developed computer program is even capable of handling the problems of vibration, uniaxial buckling, and shear buckling of unsymmetrically laminated plates. The Ritz method is used with simple polynomials as displacement functions. These ordinary polynomials are modified such that they satisfy the geometric boundary conditions, as is required by Ritz method. This is done by a technique which is explained in Appendix B.

Chapter II reviews briefly a large number of references that have dealt with the problems of vibration and buckling of orthotropic, anisotropic, and general laminated composite plates. The review is limited to publications that are within the theory of thin laminated composite plates and, as is pointed out in this chapter, the number of numerical results obtained for the vibration frequencies or buckling loads of rectangular plates is far less than one would expect.

Chapter III explains how the energy formulation based on the theory of thin laminated composite plates, which is briefly explained in Appendix A, is used to solve the problems of vibration or buckling of rectangular laminated plates for any kind of edge conditions that might exist. In this chapter it is described how the simple polynomials are
taken as displacement functions in a way that geometrical boundary conditions are satisfied. The effectiveness of the developed numerical method is demonstrated in Chapter IV. These numerical results are obtained and compared with the solutions previously presented for vibration frequencies and buckling loads in the references reviewed in Chapter II. Special attention is given to convergence of solutions, while the numerical results obtained by the program are checked against available closed form exact solutions. Chapter V provides numerical results for a large number of previously unsolved problems. Vibration frequencies as well as buckling loads for plates having various edge conditions are obtained. Orthotropic, anisotropic, and unsymmetrically laminated composite plates are studied. New results are given for general unsymmetric laminated plates, as well as anti-symmetric cross-ply and angle-ply plates.

The next two chapters explain how the same approach is developed to find the natural frequencies of shallow circular cylindrical shells of rectangular planform. Chapter VI summarizes the formulation to derive the potential energy of deformation in terms of displacements for thin laminated composite shells. Then the energy expression is simplified for shallow circular cylindrical shells. Having the energy expression, the Ritz method with polynomials as displacement functions is used to develop the analytical approach to obtain the vibration frequencies. Numerical results obtained by this
approach are presented in Chapter VII.
CHAPTER II
HISTORICAL COMMENTS

The number of papers relevant to the vibration and buckling of orthotropic, anisotropic, and general laminated composite plates has increased largely over the past two decades. In a monograph by Leissa\textsuperscript{1}, published in 1969, only twenty one references were identified which had dealt with vibration of rectangular orthotropic plates and no results had been obtained for generally anisotropic or unsymmetrically laminated plates. More recent surveys by Bert\textsuperscript{5-9}, Bert and Francis\textsuperscript{10}, and Leissa\textsuperscript{2,3} list more than one hundred references dealing with these problems. However, the number of references dealing particularly with vibration and buckling problems using the classical linear theory (that is, not considering large amplitude displacements or postbuckling) of laminated composite plates is significantly less.

An early work by Reissner and Stavsky\textsuperscript{11} was published in 1961. This paper, which was based on Stavsky's dissertation\textsuperscript{12}, studied coupling between bending and stretching in composite plates. Sarkisyan and Movsisyan\textsuperscript{13} proposed a perturbation method for the buckling analysis of simply supported anisotropic plates.
In 1969, exact solutions for vibration and buckling of anti-symmetric and angle-ply plates having simply supported boundary conditions were presented by Whitney and Leissa\textsuperscript{14} based upon the former's dissertation\textsuperscript{15}. Whitney\textsuperscript{16}, in the same year, presented an approximate solution for the shear buckling of rectangular anti-symmetric cross-ply laminated plates having simply supported boundary conditions.

Chamis\textsuperscript{17} used the Galerkin method to analyze buckling of simply supported orthotropic plates. The axes of orthotropy were not parallel to the plate edges; therefore, the anisotropic theory was used. Numerical results were obtained for a number of cases and compared with experimental results.

In 1969, Ashton and Waddoups\textsuperscript{18} applied the Ritz method for biaxial and shear buckling of rectangular orthotropic plates and compared the theoretical results with experimental results obtained by Ashton and Love\textsuperscript{19,20}. Besides these two papers, Ashton presented a number of relevant publications by himself\textsuperscript{21-24} and with Halpin and Petit\textsuperscript{25}, and Anderson\textsuperscript{26} during the same year. Bert and Mayberry\textsuperscript{27} analyzed the free vibrations of laminated composite plates with clamped edges.

The theory of laminated composite plates was brought together by Ashton and Whitney\textsuperscript{28} in an excellent book published in 1970. This reference organizes the progress that had been made about the subject up to then. In addition to explaining the theory completely, different approaches like Galerkin method, Ritz method, and Fourier series approxima-
tion, which had been utilized up to that time to solve the problems of vibration and buckling of laminated composite plates, are discussed in this book. A great number of numerical results both theoretical and experimental are brought together, that help the reader to understand the subject.

After 1969, the rate of publications about the laminated composite plates increased. For example, a survey paper by Leissa\textsuperscript{3} identifies 17 references published between 1973 and 1976, which had dealt with rectangular orthotropic plates only. Whitney and Leissa\textsuperscript{29} in 1970, published a paper dealing with laminated plates having a type of simply supported boundary conditions for which exact solutions are not available. Using a Fourier series method, they obtained a solution for cross-ply and angle-ply unsymmetrically plates. The numerical results were given for the fundamental frequencies and critical buckling loads. Whitney also presented additional results\textsuperscript{30-34} for a number of symmetric and unsymmetric cases. Kicher and Mandel\textsuperscript{35,36} obtained experimental results for buckling of laminated plates. The experimental results are compared with those obtained from theory. Two types of boundary conditions are studied, all edges simply supported, and two opposite sides simply supported and others free. More experimental results on buckling of laminated plates were presented by Viswanathan et al\textsuperscript{37,38}.

Jones\textsuperscript{39} presented the exact solution for uniaxial buckl-
ing of simply supported anti-symmetric cross-ply laminated plates. He later on in 1975 published a book about mechanics of composite materials. In this outstanding text, he summarizes the theory of laminated composite plates in a well organized chapter. Discussing the theory and its complications, he reviews the progress that had been made up to that date.

Lin and King showed how the exact solution could be obtained for anti-symmetric cross-ply and angle-ply laminated plates having two opposite sides simply supported, and arbitrary edge conditions on the two other sides. In the same paper they used an asymptotic method to find vibration frequencies for a set of laminated plates with clamped edges. Numerical results are given for some cases and comparison is made between asymptotic method solutions and those of Ritz and Fourier series.

Minich and Chamis used a finite element method and obtained natural frequencies and nodal patterns for cantilevered plates. Housner and Stein made parametric studies, and gave extensive numerical results for simply supported, clamped and rotational elastic edge constrained plates, loaded axially and in shear. The Southwell plot static method, was used by Chailleux et al to obtain experimental results for critical buckling loads of laminated plates. Numerical results were given for plates having two opposite sides simply supported and the other two simply supported or
developed a formula to find vibration frequencies of anisotropic plates having any boundary conditions from the frequencies of the corresponding isotropic plate. He then used this result to develop an optimal design procedure to maximize the fundamental frequencies of plates. In 1978, Dickinson published a paper about buckling and vibration of rectangular isotropic and orthotropic plates. He demonstrated how a simple approach, originally developed by Warburton, using characteristic beam functions in Rayleigh's method could be applied to specially orthotropic plates. He showed how with minor modification to Warburton's frequency parameter expressions, the natural frequencies of orthotropic plates may be obtained by using Warburton's original table, and effects of uniform inplane forces can be also considered. To find buckling loads the same approach would be taken with setting the frequency parameters equal to zero. Numerical results are given using this simple method and compared with those obtained for vibration frequencies of orthotropic plates using Ritz method by Dickinson and Bassily and Dickinson. The results are given for the cases of a cantilever plate and a plate with clamped edges along two opposite sides and free along the other two. The vibration frequencies of a square clamped plate under hydrostatic in-plane loads, are calculated and are compared with the results from a series solution given by Dickinson himself. Also an example is made by considering the buckling
of clamped orthotropic plate. The numerical result is compared with the one given by Wittrick. In 1979 Crawley presented experimental results for natural frequencies of graphite/epoxy cantilever plates and curved panels. These experimental results were compared with numerical results calculated by finite element analysis. In the same year, Crawley and Dugundji presented a method based on a partial Ritz (Kantorovich) analysis for estimating the natural frequencies of cantilever composite plates. The non-dimensionalized frequency parameters were compared with results from the finite element method. Sakata in a two-part article described a generalized reduction method to derive an approximate formula for estimating the natural frequency of orthotropic plates by using the natural frequency of isotropic plates. Bucco et al. used the finite strip method combined with the deflection contour method to obtain the fundamental frequency of orthotropic plates of arbitrary shape. The numerical results for fundamental frequency parameters of square specially orthotropic plates for a set of values of material constants are obtained. These results are compared with values obtained by using the expression given earlier by Maurizi and Laura. Kuttler and Siglillito presented upper and lower bounds for the frequencies of clamped orthotropic plates. The numerical results found by their method was compared with results presented previously by Sundara et al., Dickinson.
and Marangoni et al.\textsuperscript{60}.
CHAPTER III

ANALYSIS OF VIBRATION AND BUCKLING OF COMPOSITE LAMINATES

The major objective of this chapter is to develop a theoretical approach, using a modified Ritz method with simple polynomials as displacement functions, to analyze the problems of vibration and buckling of rectangular laminated composite plates. The classical lamination theory, which is explained briefly in Appendix A, is used to derive the energy expression needed. Basic assumptions and notations which are introduced in the appendix are used throughout this chapter. The approach is capable of finding vibration frequencies or buckling loads for rectangular plates with arbitrary boundary conditions.

3.1 Energy Formulation

The potential energy of deformation for an elastic body is well known.

$$V_1 = \frac{1}{2} \int \left( \sigma_x \varepsilon_x + \sigma_y \varepsilon_y + C_z \varepsilon_z + \sigma_{xy} \varepsilon_{xy} + \sigma_{xz} \varepsilon_{xz} + C_{yz} \varepsilon_{yz} \right) dV$$

Neglecting transverse normal and shear strains, based on assumptions for thin plates, equation (3.1) simplifies and the expression for the strain energy of deformation for the rectangular plates shown in figure (A-1) can be written as
\[ V_1 = \frac{1}{2} \int_{-\frac{b_1}{2}}^{\frac{b_1}{2}} \int_{-\frac{b_2}{2}}^{\frac{b_2}{2}} \left( \sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \sigma_{xy} \varepsilon_{xy} \right) dx dy dz \]  

(3.2)

For a laminated plate, the volume integral of (3.2) must be separated into piecewise integration through the thickness of the plate, and the stress-strain relationship for each layer is given by relation (A.3). Then equation (3.2) will be

\[
V_1 = \frac{1}{2} \int_{-\frac{a_1}{2}}^{\frac{a_1}{2}} \int_{-\frac{a_2}{2}}^{\frac{a_2}{2}} \left[ \sum_{k=1}^{n} \left( \begin{array}{c}
\sigma_{11}^k \varepsilon_x + \sigma_{12}^k \varepsilon_y + \sigma_{16}^k \varepsilon_{xy} \\
\sigma_{12}^k \varepsilon_x + \sigma_{22}^k \varepsilon_y + \sigma_{26}^k \varepsilon_{xy} \\
\sigma_{66}^k \varepsilon_{xy}
\end{array} \right) \right] dx dy dz
\]

(3.3)

where \( n \) is the total number of layers.

Now the expression for the strain energy of the plate due to bending (3.3) can be written in terms of displacements of the midsurface of the plate by using the equations (A.7).

\[
V_1 = \frac{1}{2} \int_{-\frac{b_1}{2}}^{\frac{b_1}{2}} \int_{-\frac{b_2}{2}}^{\frac{b_2}{2}} \left[ \sum_{k=1}^{n} \left( \begin{array}{c}
\bar{\sigma}_{11}^k u_x^2 + 2\bar{\sigma}_{12}^k u_x v_y + \bar{\sigma}_{22}^k v_y^2 + \\
\bar{\sigma}_{12}^k (2u_x u_y + 2u_x v_x) + \bar{\sigma}_{26}^k (2v_y u_x + 2v_y v_x) + \bar{\sigma}_{66}^k \\
(2u_x v_y + v_x v_y)^2
\end{array} \right) \right] dx dy dz - \sum_{k=1}^{n} \left[ 2\bar{\sigma}_{11}^k u_x w_{xx} + 2\bar{\sigma}_{12}^k (v_y v_x) + \bar{\sigma}_{66}^k \right]
\]

(3.3)
\[ u_{,x}w_{,xx} + v_{,x}w_{,xx} + 2u_{,x}w_{,xy} + 26^k_{26} (u_{,y}w_{,yy} + v_{,x}w_{,yy} \]
\[ + 2v_{,y}w_{,xy}) + 46^k_{66} (u_{,y}w_{,xy} + v_{,x}w_{,xy}) \right] dz + \sum_{k=1}^{n} \int_{h_k}^{h_{k+1}} 11^k_w_{,xx} \]
\[ + 22^k_{12} w_{,xx}w_{,yy} + 22^k_{22} w_{,yy} + 42^k_{16} w_{,xx}w_{,xy} + 42^k_{26} w_{,yy}w_{,xy} \]
\[ + 42^k_{66} w_{,xy}^2 \right] z^2 dz \right\} dx dy \quad (3.4) \]

Since the displacements \( u, v, \) and \( w \) are not functions of \( z \) and \( \delta_{ij} \) are constant within \( k \)th layer, the equation (3.4) can be written in the following form.

\[
V_1 = 1/2 \left\{ \int_{\text{Area}} \left[ A_{11}u_{,x}^2 + 2A_{12}u_{,x}v_{,y} + A_{22}v_{,y}^2 + 2A_{16}(u_{,x} \]
\[ u_{,y} + u_{,x}v_{,x} + 2A_{26}(v_{,y}u_{,y} + v_{,y}v_{,x}) + A_{66}(u_{,y} + \]
\[ v_{,x})^2 - 2B_{11}u_{,x}w_{,xx} - 2B_{12}(v_{,y}w_{,xx} + u_{,x}w_{,yy}) - 2B_{22} \]
\[ v_{,y}w_{,yy} - 2B_{16}(u_{,y}w_{,xx} + v_{,x}w_{,xx} + 2u_{,x}w_{,xy}) - 2B_{26} \right\} \]

* The factor "2" in two places is the only difference between equation (3.5) and energy expression in reference 28.
\[
\begin{align*}
&u'y'_{w,y} + v'_{x,w,y} + 2v'y'_{w,xy} - 4d_{66}(u'y'_{w,xy} + v'_{x,w,xy}) \\
&+ D_{11}w'_{xx} + 2D_{12}w'_{xx}w'_{yy} + D_{22}w'_{yy} + 4D_{16}w'_{xx}w'_{xy} + \\
&4D_{26}w'_{yy}w'_{xy} + 4D_{66}w'_2 \right\} d\text{Area} \\
&\text{(3.5)}
\end{align*}
\]

where \( u, v, \) and \( w \) are mid-plane displacements in the \( x, y, \) and \( z \) directions respectively. The \( A_{ij} \) are extensional stiffnesses, the \( B_{ij} \) are coupling stiffnesses, and \( D_{ij} \) are bending stiffnesses. These stiffnesses are given by equations (A.10).

The potential energy of inplane loads due to transverse deflection is \(^{28}\).

\[
V_2 = 1/2 \iint_{\text{Area}} \left( N_x w'_{x,x} + N_y w'_{y,y} + 2N_{xy} w'_{x,y} \right) d\text{Area} \quad \text{(3.6)}
\]

where \( N_x, N_y, \) and \( N_{xy} \) are the total inplane stress resultants.

The kinetic energy, neglecting rotary inertia and tangential inertia, is

\[
T = 1/2 \iint_{\text{Area}} w'_{tt}^2 d\text{Area} \quad \text{(3.7)}
\]

where \( \rho \) is mass per unit area and \( w' \) denotes partial differentiation with respect to time.
3.2 Ritz method

For small amplitude, free vibration, the displacements of the midsurface can be expressed as

\[
\begin{align*}
    u(x,y,t) &= U(x,y) \sin \omega t \\
    v(x,y,t) &= V(x,y) \sin \omega t \\
    w(x,y,t) &= W(x,y) \sin \omega t
\end{align*}
\] (3.8)

The maximum kinetic energy occurs at zero displacement (maximum velocity) and is equal to

\[
T_{\text{max}} = \frac{1}{2} \rho \omega^2 \int \int w^2 \, dA
\] (3.9)

The maximum potential energy due to bending and stretching (\(V_{\text{1max}}\)) is given in equation (3.5) with \(U, V, W\) replacing \(u, v, w\) and, similarly, the maximum potential energy due to the inplane loads is found from equation (3.6). The maximum of the total potential energy is then

\[
V_{\text{max}} = V_{\text{1max}} + V_{\text{2max}}
\] (3.10)

The displacement function in terms of nondimensional coordinates \(\xi\) and \(\eta\) are taken to be simple, algebraic polynomials. These are chosen because they: (1) are capable of dealing with arbitrary boundary conditions, (2) form mathematically complete sets of functions, thereby permitting
convergence to the exact solution as sufficient terms are taken and (3) are relatively simple to manage in algebraic calculations.

\[ u(\xi, \eta) = \sum_{i=0}^{J} \sum_{j=0}^{J} R_{ij} \xi^i \eta^j \]

\[ v(\xi, \eta) = \sum_{k=0}^{K} \sum_{l=0}^{L} S_{kl} \xi^k \eta^l \]  \hspace{1cm} (3.11)

\[ w(\xi, \eta) = \sum_{m=0}^{M} \sum_{n=0}^{N} p_{mn} \xi^m \eta^n \]

where \( \xi = 2x/a \) and \( \eta = 2y/b \)  \hspace{1cm} (3.12)

Since the Ritz method requires that the displacement functions at least satisfy the geometric boundary conditions, the assumed displacement functions of equations (3.11) are subjected to certain constraints. These constraint equations could be different for each problem. The general method of application of the constraints will be discussed later in this chapter. However for a plate with free edges, the equations (3.11) could be used directly because there is no constraint on the displacements.

For the free vibration problems, the Ritz method requires minimization of the functional \( T_{\text{max}} - V_{\text{max}} \) which is accomplished by setting
Equations (3.13) are simultaneous, linear, characteristic equations. The number of unknowns equals the number of equations. These equations can be written in the form

\[ (K - \lambda^2 M)g = 0 \]  \hspace{1cm} (3.14)

where \( K \) and \( M \) are stiffness and mass matrices resulting from equations (3.5) and (3.9), respectively, and \( \lambda \) is a frequency parameter.

For the case of buckling, the frequency is zero and the energy functional is in fact, \( v_{\text{max}} \), and a similar analysis yields to an equation in the same form of equation (3.14). But in this case the stiffnesses and load matrices are resulting from equations (3.5) and (3.6), respectively, and \( \lambda \) is a buckling parameter.

Equation (3.14) represents an eigenvalue problem. For a nontrival solution, the determinant of the coefficient matrix is set equal to zero. The roots of the determinant are the eigenvalues. Substituting each eigenvalue back into the equations generating the determinant yields the
corresponding eigenvector, and substituting the eigenvector into the displacement functions will give the mode shape for each eigenvalue.

3.3 Boundary conditions

As we mentioned earlier, boundary conditions have to be satisfied by the displacement functions. The Ritz method requires that at least the geometric boundary conditions be satisfied. The three kinds of commonly used boundary conditions for an isotropic plate are free, simply supported, and clamped. For laminated plates the situation is more complicated. However, the geometric boundary condition equations are considered as constraint equations, and are given below.

1) \( x = a/2 \);

\[
\begin{align*}
U &= 0; \\
\sum_{i=0}^{I} R_{ij} &= 0 \\
V &= 0; \\
\sum_{k=0}^{K} S_{kl} &= 0 \\
W &= 0; \\
\sum_{m=0}^{M} P_{mn} &= 0 \\
\frac{\partial W}{\partial x} &= 0; \\
\sum_{m=0}^{M} mP_{mn} &= 0
\end{align*}
\]
2) \( x = -a/2 \);

\[ U = 0; \quad \sum_{i=0}^{I} (-1)^i R_{ij} = 0 \]

\[ V = 0; \quad \sum_{k=0}^{K} (-1)^k S_{kl} = 0 \]

\[ W = 0; \quad \sum_{m=0}^{M} (-1)^m P_{mn} = 0 \]

\[ \frac{\delta W}{\delta x} = 0; \quad \sum_{m=0}^{M} (-1)^{m-1} P_{mn} = 0 \]

3) \( y = b/2 \);

\[ U = 0; \quad \sum_{j=0}^{J} R_{ij} = 0 \]

\[ V = 0; \quad \sum_{l=0}^{L} S_{kl} = 0 \]

\[ W = 0; \quad \sum_{n=0}^{N} P_{mn} = 0 \]

\[ \frac{\delta W}{\delta y} = 0; \quad \sum_{n=0}^{N} n P_{mn} = 0 \]

4) \( y = -b/2 \);

\[ U = 0; \quad \sum_{j=0}^{J} (-1)^j R_{ij} = 0 \]

\[ V = 0; \quad \sum_{l=0}^{L} (-1)^l S_{kl} = 0 \]

\[ W = 0; \quad \sum_{n=0}^{N} (-1)^n P_{mn} = 0 \]

\[ \frac{\delta W}{\delta y} = 0; \quad \sum_{n=0}^{N} (-1)^{n-1} n P_{mn} = 0 \]
For each type of boundary condition the required combination of constraint equations are chosen.

Defining vector \( g \) in the following form;

\[
\{g\}^T = \{R_{00} \ R_{01} \ldots \ R_{IJ} \ S_{00} \ldots \ldots \ldots \ S_{KL} \ P_{00} \ldots \ldots \ldots P_{MN}\}^T
\]  
(3.16)

The required constraint equations may be written as;

\[
\{\bar{g}\} = [G]\{g\} \tag{3.17}
\]

where \( \{\bar{g}\} \) is now a reduced column matrix, and \([G]\) is called the constraint matrix.

Following the method explained in Appendix B, the equations (3.14) can be changed to

\[
(K' - \lambda^2 M')g = 0 \tag{3.18}
\]

where

\[
K' = G^T K G \\
M' = G^T K G \tag{3.19}
\]

Thus the geometric boundary conditions which are required by the Ritz method, are satisfied.
CHAPTER IV

NUMERICAL RESULTS,
COMPARISON WITH OTHERS

The major objective in this chapter is to demonstrate the effectiveness of the analytical method presented in Chapter 3. Numerical results are compared with ones available in the works of others. The advantages and limitations of the method are discussed for both vibration and buckling. The more simple problems are discussed first. There are many papers written dealing with uncoupled orthotropic and anisotropic plates\textsuperscript{1-10} (by uncoupled we mean that the solution depends only on transverse displacement). As the degree of difficulty of the problems increases, the number of available publications dealing with the subject decreases. However some references mentioned in Chapter 2, especially the ones that have presented the exact solutions for certain special problems, provide enough information about the subject to enable us to establish the effectiveness of the present method.

An effort will also be made to study all the parameters involved in the problem. The importance of plate aspect ratio and material properties are
demonstrated, as well as the capability of the computer program based on the analysis of Chapter 3 to handle various combinations of boundary conditions. The accuracy of the program results are checked against reliable references and available data.

4.1 Specially orthotropic laminates

A specially orthotropic laminate is made of either a single layer of an orthotropic material which has its principal material directions aligned with the plate edges, or multiple orthotropic layers that are arranged symmetrically about the laminate middle surface, again with the fibers of each layer parallel to an edge. In both cases the stiffnesses $A_{16}$, $A_{26}$, $B_{ij}$, $D_{16}$, and $D_{26}$ are equal to zero. In other words, neither bending-twisting coupling nor bending-extension coupling exist. Equation (3.1) reduces to the following:

$$
V_1 = \frac{1}{2} \left\{ \iint \left\{ \frac{A_{11}u_x^2}{\text{Area}} + \frac{2A_{12}u_xv_y}{\text{Area}} + \frac{A_{22}v_y^2}{\text{Area}} + \frac{A_{66}(u_y + v_x)^2}{\text{Area}} \right\} \text{dArea} + \frac{1}{2} \iint \left\{ \frac{D_{11}w_{xx}^2}{\text{Area}} + \frac{2D_{12}w_{xx}w_{yy}}{\text{Area}} + \frac{D_{22}w_{yy}^2}{\text{Area}} + \frac{4D_{66}w_{xy}^2}{\text{Area}} \right\} \text{dArea} \right\}
$$

(4.1)

Note that the inplane displacements $u_x$ and $v_x$ are not coupled with normal displacement $w$. Hence the first term on the
right hand side of the equation (4.1) can be taken as an arbitrary constant with respect to the variation of $w$. Then the equation (4.1) simplifies to

$$V_1 = 1/2 \iint \left\{ D_{11} w_{xx}^2 + 2D_{12} w_{xx} w_{yy} + D_{22} w_{yy}^2 + \right. $$

$$\left. \frac{\text{Area}}{4D_{66} w_{xy}^2} \right\} \text{dArea} + \text{constant term} \quad (4.2)$$

For the problem of free vibration without inplane forces the differential equation of transverse motion can be written as

$$D_{11} w_{xxxx} + 2(D_{12} + D_{66}) w_{xxyy} + D_{22} w_{yyyy} + \rho w_{tt} = 0 \quad (4.3)$$

This relatively simple differential equation has an exact solution for the cases where at least two opposite sides are simply supported. This exact solution, and also numerical results using approximate methods which are available for this problem, are used to study the present analytical method.

For the plate having all its boundaries simply supported the exact solution for vibration frequencies is as follows:
\[ \omega^2 = \frac{n^4}{\rho} \left[ D_{11} \left( \frac{m}{a} \right)^4 + 2(D_{12} + 2D_{66}) \left( \frac{m}{a} \right)^2 \left( \frac{n}{b} \right)^2 + D_{22} \left( \frac{n}{b} \right)^4 \right] \]  

(4.4)

For a square plate with

\[ \frac{D_{11}}{D_{22}} = 10 \quad \text{and} \quad \frac{D_{12} + 2D_{66}}{D_{22}} = 1 \]

the five lowest frequencies are presented in Table 4.1.

This Table also shows the convergence study which is carried out using various upper limits M and N of the summations in the last of equations (3.11). Using 25 (5x5) degrees of freedom, the exact solution is obtained up to 5 significant figures. To have the same accuracy for the second mode, 64 (8x8) terms are required. The same number of degrees of freedom gives accurate results for the third mode up to 4 significant figures. Note that the third mode converges very slowly while the results are much better for the fourth and fifth modes.

Tables 4.2 and 4.3 show the comparison between the results of the present method and those given by Dickinson for the two cases of a cantilever plate, and a plate clamped along two parallel edges and free along the other edges. The results are calculated using 64 (8x8) degrees of freedom. Frequency parameters are given in terms of \( \left( \frac{\rho \nu^2 a^2 b^2}{\pi^4 H} \right)^{1/2} \) with
<table>
<thead>
<tr>
<th>MxN</th>
<th>1st mode</th>
<th>2nd mode</th>
<th>3rd mode</th>
<th>4th mode</th>
<th>5th mode</th>
</tr>
</thead>
<tbody>
<tr>
<td>4x4</td>
<td>3.60547</td>
<td>5.84390</td>
<td>13.0525</td>
<td>14.4592</td>
<td>14.4763</td>
</tr>
<tr>
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</tr>
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<td>5x6</td>
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<td>5.83102</td>
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<td>14.4704</td>
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</tr>
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<td>5.83102</td>
<td>10.5779</td>
<td>13.0003</td>
<td>14.4223</td>
</tr>
<tr>
<td>7x5</td>
<td>3.60555</td>
<td>5.84337</td>
<td>10.5779</td>
<td>13.0003</td>
<td>14.4282</td>
</tr>
<tr>
<td>5x7</td>
<td>3.60555</td>
<td>5.83102</td>
<td>10.4414</td>
<td>13.0524</td>
<td>14.4704</td>
</tr>
<tr>
<td>7x6</td>
<td>3.60555</td>
<td>5.83102</td>
<td>10.5779</td>
<td>13.0003</td>
<td>14.4223</td>
</tr>
<tr>
<td>6x7</td>
<td>3.60555</td>
<td>5.83102</td>
<td>10.4414</td>
<td>13.0003</td>
<td>14.4223</td>
</tr>
<tr>
<td>7x7</td>
<td>3.60555</td>
<td>5.83102</td>
<td>10.4414</td>
<td>13.0003</td>
<td>14.4223</td>
</tr>
<tr>
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<td>5.83097</td>
<td>10.4414</td>
<td>13.0000</td>
<td>14.4222</td>
</tr>
<tr>
<td>exact</td>
<td>3.60555</td>
<td>5.83095</td>
<td>10.4403</td>
<td>13.0000</td>
<td>14.4220</td>
</tr>
</tbody>
</table>

Table 4.1 Convergence studies for a square, simply supported, specially orthotropic plate.
<table>
<thead>
<tr>
<th>Mode* m</th>
<th>Ritz\textsuperscript{48}</th>
<th>Rayleigh\textsuperscript{47}</th>
<th>Present method</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 0</td>
<td>1.383</td>
<td>1.384</td>
<td>1.38304</td>
</tr>
<tr>
<td>1 1</td>
<td>1.649</td>
<td>1.666</td>
<td>1.65407</td>
</tr>
<tr>
<td>1 2</td>
<td>3.219</td>
<td>3.278</td>
<td>3.21538</td>
</tr>
<tr>
<td>1 3</td>
<td>6.985</td>
<td>7.072</td>
<td>6.98706</td>
</tr>
<tr>
<td>2 0</td>
<td>8.668</td>
<td>8.668</td>
<td>8.66625</td>
</tr>
<tr>
<td>2 1</td>
<td>8.999</td>
<td>9.008</td>
<td>9.00737</td>
</tr>
<tr>
<td>2 2</td>
<td>10.215</td>
<td>10.320</td>
<td>10.20170</td>
</tr>
</tbody>
</table>

Table 4.2 Frequency parameter $\sqrt{\frac{h\omega^2ab^2}{\rho H}}$ for a square cantilevered plate.

* m is the number of nodal lines parallel to, and including, the clamped edge and n the number parallel to the free edges.
Table 4.3 Frequency parameter $\sqrt{\frac{\mu \omega^2 a^2 b^2}{n^4}}$ for a square orthotropic plate clamped along edges $x=a/2$ and $x=-a/2$ and free along the other sides.

<table>
<thead>
<tr>
<th>Mode*</th>
<th>Ritz$^{48}$</th>
<th>Rayleigh$^{47}$</th>
<th>Present method</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 0</td>
<td>8.802</td>
<td>8.808</td>
<td>8.80109</td>
</tr>
<tr>
<td>2 1</td>
<td>8.930</td>
<td>8.939</td>
<td>8.92916</td>
</tr>
<tr>
<td>---</td>
<td>-----</td>
<td>-----</td>
<td>12.05715</td>
</tr>
<tr>
<td>3 0</td>
<td>24.26</td>
<td>24.27</td>
<td>24.26094</td>
</tr>
<tr>
<td>3 1</td>
<td>24.44</td>
<td>24.45</td>
<td>24.43356</td>
</tr>
<tr>
<td>3 2</td>
<td>25.12</td>
<td>25.15</td>
<td>25.15511</td>
</tr>
</tbody>
</table>

* $m$ is the number of nodal lines parallel to and including, the clamped edges and $n$ the number parallel to the free edges.
\[
\frac{D_{11}}{H} = 15.0825, \quad \frac{D_{22}}{H} = 1.03692, \quad \frac{D_{66}}{H} = 0.37557 \text{ where } \\
H = D_{12} + 2D_{66}. \text{ Here it should be mentioned that the parameter } H = D_{12} + 2D_{66} \text{ is introduced due to the fact that in the differential equation of motion (4.3) } D_{12} + 2D_{66} \text{ appears as one factor.}
\]

For the buckling problem the last term on the right hand side of equation (4.3) will change by the term due to inplane loads \((N_{x}w_{x}^{\prime}, N_{y}w_{y}^{\prime}, N_{xy}w_{xy}^{\prime})\). The exact solutions presented by Jones and also calculated buckling loads for different aspect ratios are presented in Table 4.4 for the chosen material properties \\
\[
\frac{D_{11}}{D_{22}} = 10 \text{ and } \frac{D_{16} + 2D_{66}}{D_{22}} = 1.
\]

When the aspect ratio \((a/b)\) is smaller than 2.6, the first buckling load appears when we have half waves in \(x\) and \(y\) directions \((m = 1, n = 1)\). The results are accurate up to 5 digits using only 25 \((5x5)\) terms. But for the aspect ratios greater than 2.6 shown in Table 4.4 the number of half waves in the direction of loading is two and the calculated results using same number of terms is accurate only up to 2 digits. When the number of terms used is increased to 64 \((8x8)\) terms the accuracy for the chosen aspect ratios \((2.7, 2.9, \text{ and } 3.1)\) is improved to 5 digits.
<table>
<thead>
<tr>
<th>Aspect ratio</th>
<th>a/b</th>
<th>Exact value</th>
<th>5x5</th>
<th>8x8</th>
</tr>
</thead>
<tbody>
<tr>
<td>m = 1</td>
<td>1</td>
<td>13.000</td>
<td>13.000</td>
<td>---</td>
</tr>
<tr>
<td></td>
<td>1.5</td>
<td>8.6944</td>
<td>8.6944</td>
<td>---</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>8.5000</td>
<td>8.5000</td>
<td>---</td>
</tr>
<tr>
<td></td>
<td>2.5</td>
<td>9.8500</td>
<td>9.8500</td>
<td>---</td>
</tr>
<tr>
<td>m = 2</td>
<td>2.7</td>
<td>9.3095</td>
<td>9.3469</td>
<td>9.3095</td>
</tr>
<tr>
<td></td>
<td>2.9</td>
<td>8.8587</td>
<td>8.9505</td>
<td>8.8587</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>8.5944</td>
<td>8.7237</td>
<td>---</td>
</tr>
<tr>
<td></td>
<td>3.1</td>
<td>8.5648</td>
<td>8.5918</td>
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</tr>
<tr>
<td></td>
<td>3.3</td>
<td>8.3956</td>
<td>8.4186</td>
<td>---</td>
</tr>
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<td></td>
<td>3.5</td>
<td>8.3278</td>
<td>8.3473</td>
<td>---</td>
</tr>
<tr>
<td></td>
<td>3.7</td>
<td>8.3443</td>
<td>8.3609</td>
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<td></td>
<td>3.9</td>
<td>8.4323</td>
<td>8.4462</td>
<td>---</td>
</tr>
</tbody>
</table>

Table 4.4 Buckling load parameters for rectangular specially orthotropic plates under uniform compression, \( N_x \). The plates are simply supported along all edges.
4.2 **Mid-Plane Symmetric Laminated Plates**

Mid-plane symmetric laminated plates do not have any coupling between bending and extension ($B_{ij}$ are zero). This will simplify the problem to that of homogeneous anisotropic plates. The potential energy reduces to the following

$$
V_1 = \frac{1}{2} \int \left\{ D_{11} w_{xx}^2 + 2D_{12} w_{xx} w_{yy} + D_{22} w_{yy}^2 + 4D_{16} w_{xx} w_{xy} + 4D_{26} w_{yy} w_{xy} + 4D_{66} w_{xy}^2 \right\} \text{dArea}
$$

(4.5)

The twist coupling stiffnesses $D_{16}$ and $D_{26}$ are the difference between this problem and specially orthotropic laminates which were discussed in previous section. The existence of these terms in the differential equation of transverse motion makes an exact solution impossible. However, numerical results for a few cases are available.

Table 4.5 shows the calculated results for the buckling loads for a rectangular symmetric angle-ply plate under uniform uniaxial compression. The plate is simply supported on all edges. The plates have typical properties of boron/epoxy composite material, i.e., $E_{22}/E_{11} = 0.1$, $G_{12}/E_{11} = 0.03$, and $\nu_{12} = 0.3$. The aspect ratio is 1.13 and the non-dimensional buckling load presented is $N_x a^2/E_{11} h^3$. 
Figure 4.1 shows the results presented by Ashton and Whitney for the same case which is calculated by 49(7x7) terms using Ritz method with double sine series as displacement functions. Note that 36(6x6) terms are used for the polynomial approach.

Crawley presented experimental results for graphite/epoxy cantilever plates. He also used a finite element method to find the five lowest natural frequencies of the same plates. The experimental and numerical results for two aspect ratios of 1 and 2 for 8-ply graphite/epoxy plates with orientation \(0_2/\pm 30^\circ\) (see Appendix A for definition) are tabulated in Table 4.6. This table also shows the results obtained by the present method using 36(6x6) terms. Material properties used in calculations are:

\[
\begin{align*}
E_1 &= 18.5 \times 10^6 \text{ psi} \\
E_2 &= 1.6 \times 10^6 \text{ psi} \\
\gamma_{12} &= 0.25 \\
G_{12} &= 0.65 \times 10^6 \text{ psi} \\
\text{density} &= 0.055 \text{ lb/in}^3 \\
\text{nominal ply thickness} &= 0.0052 \text{ in}
\end{align*}
\]
<table>
<thead>
<tr>
<th>Lamination angle (Degrees)</th>
<th>Buckling loads $N_\alpha a^2 / E_{11} h^3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
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</tr>
<tr>
<td>10.0</td>
<td>1.17598</td>
</tr>
<tr>
<td>20.0</td>
<td>1.24645</td>
</tr>
<tr>
<td>30.0</td>
<td>1.35592</td>
</tr>
<tr>
<td>40.0</td>
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<td>1.45262</td>
</tr>
<tr>
<td>60.0</td>
<td>1.27029</td>
</tr>
</tbody>
</table>

Table 4.15 Buckling loads $N_\alpha a^2 / E_{11} h^3$ for rectangular symmetric angle-ply plates.

**FIG. 4.1** Buckling loads for rectangular symmetric angle-ply plates under uniform compression. (After Ashton and Whitney, Ref. 28)
<table>
<thead>
<tr>
<th>Mode</th>
<th>Experimental results</th>
<th>Finite element method</th>
<th>Present method</th>
</tr>
</thead>
<tbody>
<tr>
<td>1B</td>
<td>234.2</td>
<td>261.9</td>
<td>262.13</td>
</tr>
<tr>
<td>1T</td>
<td>362</td>
<td>363.5</td>
<td>363.67</td>
</tr>
<tr>
<td>1C</td>
<td>728.3</td>
<td>761.8</td>
<td>771.37</td>
</tr>
<tr>
<td>2B</td>
<td>1449</td>
<td>1662</td>
<td>1642.1</td>
</tr>
<tr>
<td>2C</td>
<td>1503</td>
<td>1709</td>
<td>1652.7</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Mode</th>
<th>Experimental results</th>
<th>Finite element method</th>
<th>Present method</th>
</tr>
</thead>
<tbody>
<tr>
<td>1B</td>
<td>58.3</td>
<td>65.37</td>
<td>65.39</td>
</tr>
<tr>
<td>1T</td>
<td>148</td>
<td>137.5</td>
<td>137.89</td>
</tr>
<tr>
<td>2B</td>
<td>362.7</td>
<td>408.3</td>
<td>408.70</td>
</tr>
<tr>
<td>2T</td>
<td>508</td>
<td>525.6</td>
<td>527.15</td>
</tr>
<tr>
<td>1C</td>
<td>546</td>
<td>588.3</td>
<td>589.40</td>
</tr>
</tbody>
</table>

Table 4.6 Five lowest frequencies (Hz) of cantilever 8-ply graphite/epoxy plates.
4.3 **Unsymmetrical Laminated Plates**

General laminated plates cannot be handled by anisotropic plate theory because extensional and bending stiffnesses couple the displacements $u$, $v$, and $w$. All the terms in equation (3.5) can then exist. The resulting differential equations of motion for thin plates have an order of eight. Four boundary conditions are required for each edge (see Appendix A) and the number of combinations they might have is very large. However, a few of these cases are studied in the literature. In fact, closed form exact solutions are available for frequencies and also buckling loads for antisymmetric cross-ply and antisymmetric angle-ply rectangular plates having certain special boundary conditions. These results are compared with the present work's in this section.

References 14 and 39 present the exact solutions for frequencies and also buckling loads for some antisymmetric angle-ply and cross-ply laminates.

An antisymmetric angle-ply laminate is made of an even number of layers of orthotropic laminae with each laminae alternately oriented at $+\theta$ and $-\theta$ to the principal axis of the plate. This particular arrangement of the layers will lead to

$$A_{16} = A_{26} = 0, \quad D_{16} = D_{26} = 0$$

Also all the bending-extension coupling stiffnesses $(B_{ij})$ vanish except $B_{16}$ and $B_{26}$. Reference 14 presents exact
solutions for vibration frequencies and buckling loads of antisymmetric angle-ply laminated plates with all edges simply supported. The solution for natural frequencies of a plate with the simply supported boundary conditions S3 (See Appendix A for definition) is given in closed form in reference 14. The numerical results for a graphite/epoxy composite with $E_1/E_2 = 40$, $G_{12}/E_2 = 0.5$, and $\nu_{12} = 0.25$ are given for different values of lamination angles in Table 4.7. This Table also presents the numerical results from the present method using totals of 54 terms and 75 terms for a 2-ply antisymmetric laminate. Column three gives the fundamental vibration frequencies using $36(6x6)$ terms for $w$ and $9(3x3)$ terms for $u$ and $9(3x3)$ terms for $v$. The results are accurate up to 2 digits. Column four presents the results for the same case using 75 degrees of freedom, $25(5x5)$ for $w$ $25(5x5)$ for $u$, and $25(5x5)$ for $v$. The results in this case are accurate up to five significant figures. The importance of coupling is also shown here. Although the number of transverse displacement terms ($w$) is reduced from 36 to 25, increasing the number of inplane displacement terms has improved the results. In the Table 4.8 the number of displacement functions are kept constant while the coupling is decreased by increasing the number of plies from two to four to six. For this case the lamination angle is taken to be 30 degrees.
<table>
<thead>
<tr>
<th>Lamination Angle</th>
<th>Exact solution</th>
<th>Number of terms used</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>6x6 for w</td>
</tr>
<tr>
<td>0</td>
<td>18.8052</td>
<td>18.8052</td>
</tr>
<tr>
<td>30</td>
<td>14.2035</td>
<td>14.4935</td>
</tr>
<tr>
<td>45</td>
<td>14.6376</td>
<td>14.9618</td>
</tr>
</tbody>
</table>

Table 4.7 Fundamental vibration frequencies for square antisymmetric angle-ply laminated plates. Frequency parameter is given in terms of $wa^2\sqrt[n]{\rho/Eh^3}$.

<table>
<thead>
<tr>
<th>Number of plies</th>
<th>2</th>
<th>4</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exact solution</td>
<td>14.2035</td>
<td>22.1753</td>
<td>23.3548</td>
</tr>
<tr>
<td>Present method</td>
<td>14.4935</td>
<td>22.2649</td>
<td>23.3954</td>
</tr>
</tbody>
</table>

Table 4.8 Fundamental vibration frequency $wa^2\sqrt[n]{\rho/Eh^3}$ for square anti-symmetric angle-ply plates, made of 2, 4, and 6 plies.
Lin and King\textsuperscript{41} determined the exact solution for natural frequencies of antisymmetric angle-ply laminates for cases where a pair of opposite edges are simply supported (S3) by using a procedure which is the same as the one employed by Forsberg\textsuperscript{62} in a study of cylindrical shell vibrations. Numerical results for a square plate consisting of graphite/epoxy lamina \((\frac{E_1}{E_2} = 40, \frac{G_{12}}{E_2} = 0.5, \nu_{12} = 0.25)\) with edges \(y = b/2\) and \(y = -b/2\) simply supported S3, and edges \(x = a/2\) and \(x = -a/2\) supported according to three conditions of clamping (Cl, C2, and C3) are presented in Table 4.9. The same Table also shows calculated fundamental vibration frequencies vs. lamination angle, using the present computer program, based on the Ritz method with polynomials as displacement functions. The number of terms used, are 36(6x6) for \(w\), 25(5x5) for \(u\), and 25(5x5) for \(v\). Values in parentheses which are given for lamination angles of 45, 75, and 85 degrees are calculated using 36(6x6) terms for \(w\), 35(5x7) terms for \(u\), and 25(5x5) terms for \(v\). Apparently 2 more terms in the \(y\) direction (for these frequencies) are needed for the tangential displacement \(u\) to obtain convergence to a reasonable solution. By reasonable solution we mean that, since for the Cl boundary condition we have greater fixity (all the components of displacement are fixed) than the C3 condition, we expect a lower fundamental frequency for C3 than for Cl conditions. The fact that the principal of ordering of frequencies is not violated here
<table>
<thead>
<tr>
<th>$\theta$</th>
<th>S3C3</th>
<th></th>
<th>S3C2</th>
<th></th>
<th>S3C1</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>degrees</td>
<td>Exact value</td>
<td>Present method</td>
<td>Exact value</td>
<td>Present method</td>
<td>Exact value</td>
<td>Present method</td>
</tr>
<tr>
<td>5</td>
<td>36.248</td>
<td>36.2478</td>
<td>36.682</td>
<td>36.6828</td>
<td>36.683</td>
<td>36.6828</td>
</tr>
<tr>
<td>15</td>
<td>---</td>
<td>26.6136</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>35</td>
<td>---</td>
<td>21.1309</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>55</td>
<td>17.661</td>
<td>17.6769</td>
<td>17.617</td>
<td>17.6491</td>
<td>17.692</td>
<td>17.6941</td>
</tr>
<tr>
<td>85</td>
<td>18.776</td>
<td>18.7789</td>
<td>17.978</td>
<td>18.4242</td>
<td>18.777</td>
<td>18.7772</td>
</tr>
</tbody>
</table>

Table 4.9 Fundamental vibration frequencies $\omega b^2 \sqrt{\rho / E_2 h^3}$ of square laminated angle-ply plates with $y = -b/2$, $b/2$ simply supported and $x = -a/2$ and $x = a/2$ clamped.
can be proven by studying the algorithm used in the computer program. This is explained in the following demonstration. Consider two cases of C3 and C1 boundary conditions at edges $y = -b/2$ and $y = b/2$. For the case of C1, using 2x2 terms for $u$, originally we take $u$ as

$$u = R_{00} + R_{10} \xi + R_{01} \eta + R_{11} \xi \eta + R_{02} \eta^2 + R_{12} \xi^2 +$$

$$R_{03} \eta^3 + R_{13} \xi \eta^3$$

Then we have two conditions on $u$.

$$u(\xi,-1) = 0 \text{ and } u(\xi,1) = 0$$

which implies

$$R_{00} + R_{01} + R_{02} + R_{03} = 0$$

$$R_{00} - R_{01} + R_{02} - R_{03} = 0$$

$$R_{10} + R_{11} + R_{12} + R_{13} = 0$$

$$R_{10} - R_{11} - R_{12} - R_{13} = 0$$

Then

$$u = R_{00}(1 - \eta^2) + R_{10}(1 - \eta^2) + R_{01}(\eta - \eta^3) +$$

$$R_{11}(\eta - \eta^3)$$

(4.7)

For the case of C3, there are no conditions on $u$. Therefore, using 2x2 terms for $u$, we have;
Comparing equations (4.7) and (4.8) indicates that the displacement functions are not the same for these two cases. A similar situation exists when we use 25(5x5) terms for displacement functions. Depending upon the effectiveness of each term, the accuracy of the solution could change.

Ref. 41 also presents numerical results for fundamental frequencies of square antisymmetric angle-ply plates clamped on all edges by using an asymptotic method developed by Bolotin\textsuperscript{63, 64}. The numerical results for graphite/epoxy laminated plates which are presented in that paper and the results calculated by Whitney\textsuperscript{34} using Fourier series analysis are compared with the results of the present method in Table 4.10. Note that the Ritz method is used to develop the present approach and the frequencies found are upper bounds, while the results calculated by asymptotic analysis are uncertain to be upper bound or lower bound solutions. Unlike Whitney's results the calculated frequencies for the Cl boundary condition are found to be more than C3, which is expected due to greater fixity of Cl boundaries. It should be mentioned that 25(5x5) terms for w and the same number of terms for u and v were used.
Another case presented in reference 41 was that of antisymmetric, cross-ply laminated plates. These laminates are composed of an even number of orthotropic layers with principal material directions alternating at 0 and 90 degrees to the plate edges. These laminates have the following stiffnesses:

\[
\begin{align*}
A_{11}, & \quad A_{12}, A_{22} = A_{11}, \quad \text{and} \quad A_{66}, \\
B_{11}, & \quad \text{and} \quad B_{22} = -B_{11}, \\
D_{11}, & \quad D_{12}, D_{22} = D_{11}, \quad \text{and} \quad D_{66}.
\end{align*}
\]

All other stiffnesses vanish. The exact solution for the free vibration frequencies for antisymmetric cross-ply laminates having simply supported S2 boundary conditions was found by Leissa and Whitney\textsuperscript{14}. The solution will be discussed later in this chapter.

Table 4.11 shows the calculated natural frequencies of a graphite/epoxy, antisymmetric cross-ply laminate (\(\frac{E_1}{E_2} = 40\), \(\frac{G_{12}}{E_2} = 0.5\), \(\nu_{12} = 0.25\)) for different aspect ratios. The laminate is clamped C1 on all edges and the results of the present method using 25(5x5) terms for \(w\), 25(5x5) terms for \(u\), and 25(5x5) terms for \(v\) are compared with asymptotic solutions\textsuperscript{41} and Fourier series solutions\textsuperscript{34}. When the same number of terms are chosen to estimate the natural frequency of the plate with the same material properties, but with simply supported S2 boundary conditions, 4 digit accuracy is observed compared to the exact solution presented by Leissa and Whitney\textsuperscript{14}. The numerical results are tabulated.
<table>
<thead>
<tr>
<th>θ</th>
<th>Asymptotic solution</th>
<th>Whitney's solution</th>
<th>Present method</th>
<th>Asymptotic solution</th>
<th>Whitney's solution</th>
<th>Present method</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>36.880</td>
<td>37.582</td>
<td>37.2833</td>
<td>36.401</td>
<td>39.458</td>
<td>36.8363</td>
</tr>
<tr>
<td>15</td>
<td>27.378</td>
<td>28.052</td>
<td>27.8344</td>
<td>27.156</td>
<td>31.337</td>
<td>27.5622</td>
</tr>
<tr>
<td>35</td>
<td>23.189</td>
<td>23.799</td>
<td>23.6312</td>
<td>23.111</td>
<td>25.316</td>
<td>23.6025</td>
</tr>
<tr>
<td>45</td>
<td>22.838</td>
<td>23.488</td>
<td>23.3089</td>
<td>22.835</td>
<td>25.316</td>
<td>23.3473</td>
</tr>
</tbody>
</table>

Table 4.10 Fundamental frequencies, $\omega b^2 \sqrt{\frac{v}{E_2 h^3}}$, of angle-ply square clamped plates. ($E_1/E_2 = 40$, $G_{12}/E_2 = 0.5$, $\nu_{12} = 0.25$)
Table 4.11 Fundamental vibration frequencies of antisymmetric cross-ply laminated plates vs. aspect ratio. Comparison of asymptotic, Fourier, and Ritz solutions for a two-layer cross-ply plate clamped Cl on all sides.

<table>
<thead>
<tr>
<th>Aspect ratio</th>
<th>$\omega b^2 \sqrt{\rho / E_2 h}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>a/b</td>
<td>Asymptotic solution</td>
</tr>
<tr>
<td>1</td>
<td>23.638</td>
</tr>
<tr>
<td>2</td>
<td>17.111</td>
</tr>
<tr>
<td>3</td>
<td>16.640</td>
</tr>
<tr>
<td>4</td>
<td>16.543</td>
</tr>
<tr>
<td>5</td>
<td>16.509</td>
</tr>
</tbody>
</table>
in Table 4.12. Table 4.12 also presents the exact solution for the buckling parameter for the plate under uniform uniaxial compression. Using same number of terms the accuracy of the program is less (3 digits). Note that for both cases boundary conditions are simply supported S2 on all edges.

The exact solution for buckling of antisymmetric angle-ply laminates is given by Whitney and Leissa\textsuperscript{14} for the case of a plate simply supported S3 on all edges. The results for 2-ply and 4-ply laminates composed of graphite/epoxy ($\frac{E_1}{E_2} = 40, \frac{G_{12}}{E_2} = 0.5, \nu_{12} = 0.25$) for different values of lamination angles are presented in Table 4.13. For the numerical calculations 86 terms are used; 36(6x6) for \( w \), 25(5x5) for \( u \) and 25(5x5) for \( v \). Note that when the number of layers is increased from 2 to 4 the coupling decreases and the accuracy of the numerical solution increases. Changing the lamination angle has the same influence on the results.
Table 4.12 Frequency parameter and buckling load parameter for a square antisymmetric cross-ply plate of graphite/epoxy with S2 conditions on all sides.

<table>
<thead>
<tr>
<th>((\omega b^2 \sqrt{\rho/D_{22}})/\pi^2)</th>
<th>Present method</th>
<th>Exact solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>((N_x b^2/\pi^2 D_{22}))</td>
<td>0.86492</td>
<td>0.8648</td>
</tr>
<tr>
<td>0.74807</td>
<td>0.7479</td>
<td></td>
</tr>
</tbody>
</table>

Table 4.13 Buckling loads for square antisymmetric angle-ply laminated plates under uniform uniaxial compression.

<table>
<thead>
<tr>
<th>Angle degree</th>
<th>Buckling parameter (N_x b^2/\pi^2 D_{22})</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>2-ply</td>
</tr>
<tr>
<td></td>
<td>Exact solution</td>
</tr>
<tr>
<td>0</td>
<td>35.831</td>
</tr>
<tr>
<td>15</td>
<td>21.734</td>
</tr>
<tr>
<td>30</td>
<td>20.440</td>
</tr>
<tr>
<td>45</td>
<td>21.6865</td>
</tr>
<tr>
<td>60</td>
<td>--</td>
</tr>
<tr>
<td>90</td>
<td>--</td>
</tr>
</tbody>
</table>
In the present chapter, the computer program which was utilized in chapter IV is employed to find numerical results for a number of previously unsolved problems. Due to the capability of the program for handling the problems of free vibration, vibration under inplane loads, uniaxial buckling, biaxial buckling, and uniform shear buckling for all possible combinations of geometric boundary conditions of rectangular plates, the total number of the cases which can be studied by the present method is far more than could be dealt within this chapter. However, a variety of practical cases are studied in this chapter to serve the purpose of understanding better the behavior of rectangular laminated composite plates.

In order to organize the problems, an outline similar to chapter IV is followed. In other words, orthotropic plates are studied first. Then the numerical results for a number of anti-symmetric laminates are presented. Finally, a generally unsymmetric laminated plate is analyzed.

5.1 Specially Orthotropic Laminates

A specially orthotropic laminate, as is defined in chapter IV, is made of symmetrically (with respect to the middle
plane) arranged layers of orthotropic material with the axes of material orthotropy parallel to the edges of the plate. The differential equation of transverse motion with constant inplane forces is as follows:

\[
D_{11}w_{,xxxx} + 2(D_{12} + 2D_{66})w_{,xxyy} + D_{22}w_{,yyyy} + \rho w_{,tt} = N_x w_{,xx} + N_y w_{,yy} + 2N_{xy} w_{,xy} \tag{5.1}
\]

when there is no inplane shear involved (\(N_{xy} = 0\)), equation (5.1) has an exact solution of the form:

\[
w = \bar{w}\sin \frac{mnx}{a} \sin \frac{nny}{b} \tag{5.2}
\]

This solution also satisfies the simply supported boundary conditions. Table 4.1 showed the natural frequencies of the first 5 modes for a square simply supported specially orthotropic plate with the properties:

\[
\frac{D_{11}}{D_{22}} = 10, \quad \frac{D_{12} + 2D_{66}}{D_{22}} = 1
\]

To study the effect of boundary conditions for a plate with the same properties, two cases of a completely free and completely clamped edges are chosen.

Table 5.1 presents the first five natural frequency parameters \((\omega a^2 \sqrt{\rho/D_{22}})/\pi^2\) for plates with completely free boundary
conditions. The number of terms used are $36(6 \times 6)$ and the results are given for the aspect ratios $(a/b)$ of 0.5, 1 and 2. Numerical results for plates clamped on all edges are tabulated in Table 5.2. Considering the convergence studies carried out in Table 4.1 for the simply supported plate, the results given in Tables 5.1 and 5.2 are expected to have good accuracy. The first natural frequency for a simply supported plate is accurate up to 6 significant figures when $36(6 \times 6)$ terms are used, while the third mode frequency parameter is accurate up to 2 digits. However, except for the fact that the numerical results for the first mode are expected to be more accurate than for the higher modes, and that all frequencies are upper bounds on the exact values, no further conclusion can be made.

As is expected, the natural frequency parameters increase when the number of constraints on the edges of the plate increase. For a free plate there are no constraints on its edges and the natural frequency parameter for the first nontrivial vibration mode (i.e., other than rigid body motion) is equal to $1.183$ when $a/b = 1$, while for a simply supported plate it is $3.606$ (more than twice). For the square clamped plate the frequency parameter of the first mode increases to $7.719$. In this case there are two constraints per edge and the frequency parameter is more than twice the one for simply supported plate. The effect of boundary condition is observed to be more for the square
<table>
<thead>
<tr>
<th>( \frac{a}{b} )</th>
<th>( \frac{a}{b} = 0.5 )</th>
<th>( \frac{a}{b} = 1 )</th>
<th>( \frac{a}{b} = 2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5645</td>
<td>1.183</td>
<td>2.383</td>
<td></td>
</tr>
<tr>
<td>0.5762</td>
<td>2.239</td>
<td>7.083</td>
<td></td>
</tr>
<tr>
<td>1.341</td>
<td>3.302</td>
<td>8.693</td>
<td></td>
</tr>
<tr>
<td>1.591</td>
<td>6.313</td>
<td>9.190</td>
<td></td>
</tr>
<tr>
<td>2.464</td>
<td>7.138</td>
<td>10.30</td>
<td></td>
</tr>
</tbody>
</table>

Table 5.1 Natural frequency parameter \((\omega a^2/n^2)\sqrt{\rho/D_{22}}\) for rectangular specially orthotropic plates with free edges.

<table>
<thead>
<tr>
<th>( \frac{a}{b} )</th>
<th>( \frac{a}{b} = 0.5 )</th>
<th>( \frac{a}{b} = 1 )</th>
<th>( \frac{a}{b} = 2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.243</td>
<td>7.719</td>
<td>12.08</td>
<td></td>
</tr>
<tr>
<td>7.529</td>
<td>10.10</td>
<td>22.78</td>
<td></td>
</tr>
<tr>
<td>8.185</td>
<td>15.06</td>
<td>26.87</td>
<td></td>
</tr>
<tr>
<td>9.654</td>
<td>20.18</td>
<td>34.47</td>
<td></td>
</tr>
<tr>
<td>12.14</td>
<td>21.74</td>
<td>41.06</td>
<td></td>
</tr>
</tbody>
</table>

Table 5.2 Natural frequency parameter \((\omega a^2/n^2)\sqrt{\rho/D_{22}}\) for rectangular specially orthotropic plates with clamped edges.
plate with material properties used in this study \( \frac{D_{11}}{D_{22}} = 10, \frac{D_{12} + 2D_{66}}{D_{22}} = 1 \) than the isotropic plate where the plate has the same stiffnesses in all directions (i.e., \( \frac{D_{11}}{D_{22}} = 1 \)). To verify this point the frequency parameters for isotropic plates having completely free, simply supported, and clamped edges are calculated by the present method and tabulated in Table 5.3. The frequencies are given in terms of \( \left( \frac{wa^2\sqrt{\rho/D}}{n^2} \right) \) where \( D = D_{11} = D_{22} \). As the results show, the ratio of fundamental frequency of the clamped isotropic plate over the fundamental frequency of the simply supported one is less than 2 (1.82); and comparing the simply supported plate with the free plate, the frequency ratio is 1.46.

The orthotropic plates which are studied here are stiffer in the \( x \) direction \( \left( \frac{D_{11}}{D_{22}} = 10 \right) \). The characteristic mode shapes corresponding to fundamental frequencies of these plates for the square simply supported plates, where the exact solutions are available, are discussed in Ref. 28. Similar behavior is expected for plates with free and also clamped boundary conditions. For this reason double frequencies (i.e., degenerate modes) seen in Table 5.3 for isotropic plates, which correspond to mode shapes identical to each other but with node lines rotated 90 degrees with respect to each other, are not seen for specially orthotropic plates. Such cases have different frequencies when the material properties are not the same in both directions.
Tables 5.1 and 5.2, show the fundamental natural frequencies for three aspect ratios. It is observed that when the aspect ratio \( \frac{a}{b} \) is equal to 0.5 the frequency parameter for the clamped plate \( \omega_c \) is more than twelve times the one for the free plate \( \omega_f \).

\[
\frac{\omega_c}{\omega_f} = 12.83 \quad \text{when } \frac{a}{b} = 0.5
\]

(5.3)

The same ratios for the other aspect ratios are as follows:

\[
\frac{\omega_c}{\omega_f} = 6.52 \quad \text{when } \frac{a}{b} = 1
\]

(5.4)

\[
\frac{\omega_c}{\omega_f} = 5.07 \quad \text{when } \frac{a}{b} = 2
\]

These results are expected because when \( \frac{a}{b} \) is less than 1, the edges at \( x = \pm \frac{a}{2} \) are longer. These are the edges that, when the constraints applied to them, primarily reinforce the stiffer direction of material orthotropy.

For isotropic plates the ratio \( \frac{\omega_c}{\omega_f} \) is the same for aspect ratios of 0.5 and 2 because they are identical case rotated by 90 degrees.

<table>
<thead>
<tr>
<th>FFFF</th>
<th>SSSS</th>
<th>CCCC</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.365</td>
<td>2.000</td>
<td>3.647</td>
</tr>
<tr>
<td>1.999</td>
<td>5.000</td>
<td>7.402</td>
</tr>
<tr>
<td>2.486</td>
<td>5.000</td>
<td>7.437</td>
</tr>
<tr>
<td>3.576</td>
<td>8.000</td>
<td>10.96</td>
</tr>
<tr>
<td>3.576</td>
<td>10.01</td>
<td>13.28</td>
</tr>
</tbody>
</table>

Table 5.3 Natural frequency parameter \( \frac{w a^2 \sqrt{\rho / D}}{h^2} \) of square isotropic plates with free, simply supported, and clamped edges. \((\nu=0.3)\)
To study buckling, three types of inplane loading are considered.

1. Uniaxial loading
2. Hydrostatic loading
3. Shear loading

The same material properties \( \frac{D_{11}}{D_{22}} = 10, \frac{D_{12} + 2D_{66}}{D_{22}} = 1 \) are chosen as used earlier in this section, and numerical results are summarized in Table 5.4. The buckling parameter \( \frac{N_x b^2}{\pi^2 D_{22}} \) for four important types of boundary combinations is given in column 1 for uniaxial compression. In each case, the loading is applied to the clamped or simply supported edges. The exact solution for the plate with all the boundaries simply supported is available and the accuracy of the results found by the present method for this case is discussed in Chapter 4. For the hydrostatic loading when the loading in the x direction is equal to the loading in the y direction, the buckling parameter \( \frac{N_x b^2}{\pi^2 D_{22}} = \frac{N_y b^2}{\pi^2 D_{22}} \) is given for two kinds of boundaries namely all simply supported and all clamped edges. The CFCF and SFSF cases were not considered for this problem because of the impracticality of loading the free edges. The same types of edge conditions are studied for the pure shear loading. The plate aspect ratio \( a/b \) is chosen to be one for all the numerical results given in Table 5.4 and the number of terms used are 25(5x5).

Since the plate is 10 times stiffer in the x direction, it is expected that for the uniaxial loading cases the
buckling parameter of the CFCF plate is larger than the one of a SSSS plate. As Table 5.4 shows, for the CFCF the buckling parameter is 39.79, while for the SSSS case it is 13.00. The effects of clamping at the edges \( x = \pm a/2 \) is more pronounced if we compare the two cases of CFCF and SFSF plates. The buckling parameter for the CFCF plate is about 4 times the one for SFSF plate. ( For larger aspect ratios one would expect this effect be less. )

The numerical results for hydrostatic and shear loading parameters are given for plates with completely simply supported and completely clamped edges and as expected the clamped plates have larger buckling parameters than simply supported ones. Another interesting point about the results gathered in Table 5.4 is that the buckling parameter for a simply supported square plate under hydrostatic loading is one half of the one under uniaxial loading. However this conclusion cannot be generalized for all the aspect ratios and different kinds of material properties.

The vibration frequency parameters for the first five modes of square simply supported and clamped specially orthotropic plates are tabulated in Table 5.5. For each boundary condition two types of inplane loading are applied. First, uniaxial loading and second, biaxial loading. In both cases the inplane loads are one half of the critical loads required to buckle the plates. The critical buckling loads are already calculated and presented in Table 5.4.
<table>
<thead>
<tr>
<th>Edge condition</th>
<th>Uniaxial loading</th>
<th>Hydrostatic loading</th>
<th>Shear loading</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\frac{N_{xb}^2}{\pi^2 D_{22}}$</td>
<td>$\frac{N_{yb}^2}{\pi^2 D_{22}}$</td>
<td>$\frac{N_{xyb}^2}{\pi^2 D_{22}}$</td>
</tr>
<tr>
<td>CCCC</td>
<td>46.25</td>
<td>16.75</td>
<td>47.03</td>
</tr>
<tr>
<td>SSSS</td>
<td>13.00</td>
<td>6.500</td>
<td>28.08</td>
</tr>
<tr>
<td>CFCF</td>
<td>39.79</td>
<td>----</td>
<td>----</td>
</tr>
<tr>
<td>SFSP</td>
<td>9.846</td>
<td>----</td>
<td>----</td>
</tr>
</tbody>
</table>

Table 5.4 Buckling parameters for square specially orthotropic plates having various boundary conditions.

<table>
<thead>
<tr>
<th>Simply supported plate</th>
<th>Clamped plate</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>Uniaxially loaded</td>
<td>Biaxially loaded</td>
</tr>
<tr>
<td>2.54951</td>
<td>2.54950</td>
</tr>
<tr>
<td>5.25784</td>
<td>4.22846</td>
</tr>
<tr>
<td>10.2664</td>
<td>8.89952</td>
</tr>
<tr>
<td>12.0140</td>
<td>12.4138</td>
</tr>
</tbody>
</table>

Table 5.5 Vibration frequency parameters for specially orthotropic plates with simply supported and clamped edges under uniaxial and biaxial loading.
For the plate with simply supported boundary conditions an exact solution is possible. The uniaxial inplane loading parameter is taken to be 6.5 which is one half of the buckling load found by both exact solution and present method. Similarly for the biaxial loading the inplane loadings in both directions \( \left( \frac{N_x b^2}{\pi^2 D_{22}} \right) \) and \( \left( \frac{N_y b^2}{\pi^2 D_{22}} \right) \) are equal to 3.25, which is one half of hydrostatic buckling load for the same plate. As the exact solution verifies, the vibration frequency parameter of the first mode is exactly the same for both cases. This is an exception for some modes with equal number of half waves in \( x \) and \( y \) directions, but the other modes such as the second and third do not have the same vibration frequencies.

The inplane compressive loadings, as expected, decrease the vibration frequencies and, depending on characteristic shapes for different modes, the effects of inplane loading vary. For the plate with clamped boundary conditions, the vibration frequency parameters when the plate is loaded biaxially are observed to be less than when the plate is loaded uniaxially. For example, the natural frequency parameter for the first vibration mode of a clamped plate with no inplane loading is 7.719, with uniaxial loading it is 6.192, and with biaxial loading it is 5.511. The simply supported plate has the same vibration frequency for both cases. This observation is in harmony with the results found for buckling load parameters. The buckling load
parameters given in Table 5.4 show that the hydrostatic loading for a completely clamped plate is less than one half of the uniaxial buckling load, while for a simply supported plate it is exactly one half. In other words, biaxial loading, compared to uniaxial loading, is more effective for a square clamped plate than a simply supported one having the same material properties \( \frac{D_{11}}{D_{22}} = 10, \frac{D_{12} + 2D_{66}}{D_{22}} = 1 \). For isotropic plates, the hydrostatic buckling load for a simply supported square plate is exactly one half of the uniaxial buckling load, which is similar to the orthotropic plate, and when the plate has clamped edges the ratio is about the same (0.53). Note that the ratio for the orthotropic plate is 0.36.

5.2 Buckling of generally orthotropic plates

As indicated in chapter 4, mid-plane symmetric laminated plates when the axes of material orthotropy are not parallel to the plate edges, are equivalent mathematically to homogeneous anisotropic plates. For these plates membrane-bending coupling does not exist \( B_{ij} = 0 \) but the appearance of \( D_{16} \) and \( D_{26} \) terms in the governing equations increases the complexity of the analysis. The equilibrium equation for uniaxial buckling is

\[
D_{11} w_{,xxxx} + 4D_{16} w_{,xxyy} + 2(D_{12} + 2D_{66}) w_{,xxyy} + 4D_{26} w_{,xyyy} + D_{22} w_{,yyyy} = N_x w_{,xx}
\]

(5.5)
Table 5.6 presents the uniaxial buckling loads for plates having all the boundaries simply supported (SSSS), and plates with clamped boundaries along the edges where the load is applied and simply supported along the other two sides (CSCS). The plates are made of graphite/epoxy layers with properties:

- \( E_{11} = 19.0 \times 10^6 \) psi
- \( E_{22} = 1.5 \times 10^6 \) psi
- \( \nu_{12} = 0.30 \)
- \( G_{12} = 0.80 \times 10^6 \) psi

The laminate orientation is \((0/45/0/0/-45/0)_{2s}\). The plate thickness \( t = 0.1232 \) in. for all the cases, and plates have the length \( a = 9.625 \) in. The numerical results are presented for plates having different width. In addition to the results calculated by the present method, Table 5.6 presents the experimental results and also approximate solutions by using different methods provided by Sandorff.65

As the numerical results show, when the width of the plate is decreased the difference between experimental and analytical results increases. For example, when the width of the plate is 3.003 in. \((b/h = 24.4)\) the experimental results are in good agreement with numerical results calculated by the present method, and theoretical solution is 11% higher than the experimental result. When the width of the plate is 2.003 \((b/h = 16.3)\) the percentage of difference between theoretical and experimental results increases to
<table>
<thead>
<tr>
<th>Plate width</th>
<th>Experimental results</th>
<th>Present method</th>
<th>Solutions provided by Sandroff</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>SSSS</td>
<td>SSSS</td>
<td>SSSS</td>
</tr>
<tr>
<td>3.003</td>
<td>4530</td>
<td>4326</td>
<td>4414</td>
</tr>
<tr>
<td></td>
<td>4290</td>
<td>5070</td>
<td></td>
</tr>
<tr>
<td>2.503</td>
<td>5910</td>
<td>6330</td>
<td>6462</td>
</tr>
<tr>
<td></td>
<td>5740</td>
<td>7004</td>
<td></td>
</tr>
<tr>
<td>2.003</td>
<td>8140</td>
<td>9711</td>
<td>9920</td>
</tr>
<tr>
<td></td>
<td>8360</td>
<td>10462</td>
<td></td>
</tr>
<tr>
<td>1.503</td>
<td>(9580)</td>
<td>17380</td>
<td>17920</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>18156</td>
<td>17930</td>
</tr>
</tbody>
</table>

Table 5.6 Buckling load calculations and experimental values (lb/in) for graphite/epoxy laminate.

1 Boundary conditions are CSCS.
2 The results are provided by using computer programs based on various approximate methods.
3 Reject: panel "oilcanned" at 7000 lb/in, indicating edge supports too loose.
This is expected because the present method is based on the theory of thin plates which is good when the width to thickness ratio \( \frac{b}{h} \) is more than 20. Results presented in Table 5.6 also show that when the aspect ratio \( \frac{a}{b} \) increases, the difference between solutions for CSCS and SSSS cases decreases. When the aspect ratio \( \frac{a}{b} \) is 6.4 (when \( b = 1.503 \)) the solution for CSCS is only 4% more than the solution for the plate with all edges simply supported.

5.3 **Antisymmetric cross-ply laminates**

An antisymmetric cross-ply laminate is made of an even number of orthotropic layers. The principal material direction of these layers are alternating at 0 and 90 to the plate edges. This class of laminates have the exten- tional stiffnesses \( A_{11}, A_{12}, A_{22} = A_{11} \) and \( A_{66}, \) bending-extension coupling stiffnesses \( B_{11} \) and \( B_{22} = -B_{11} \), and bending stiffnesses \( D_{11}, D_{12}, D_{22} = D_{11}, \) and \( D_{66}. \) Due to the existence of \( B_{11} \) and \( B_{22} \) the differential equations of motion are coupled. However, since for this laminate

\[
A_{16} = A_{26} = B_{12} = B_{16} = B_{26} = B_{66} = D_{16} = D_{26} = 0
\]

(5.6)

the equations of equilibrium given in Appendix A simplifies to:
\[ A_{11}u_{xx} + A_{66}u_{yy} + (A_{12} + A_{66})v_{xy} - B_{12}w_{xxx} = 0 \]

\[ (A_{12} + A_{66})u_{xy} + A_{66}v_{xx} + A_{11}v_{yy} + B_{11}w_{yyy} = 0 \]

\[ D_{11}(w_{xxx} + w_{yyy}) + 2(D_{12} + 2D_{66})w_{xxyy} - B_{11}(u_{xxx} - v_{yyy}) + \rho w_{tt} = N_{x}w_{xx} + 2N_{xy}w_{xy} + N_{y}w_{yy} \]

(5.7)

The exact solution for plate with simply supported boundary conditions S2 at all edges is relatively simple and was discussed in Chapter 4. For the free vibration frequencies the equations of motion are satisfied by

\[ u = U \cos \frac{m \pi x}{a} \sin \frac{n \pi y}{b} \sin \omega t \]

\[ v = V \sin \frac{m \pi x}{a} \cos \frac{n \pi y}{b} \sin \omega t \]

\[ w = W \sin \frac{m \pi x}{a} \sin \frac{n \pi y}{b} \sin \omega t \]

(5.8)

The vibration frequencies for a number of plates having various combinations of boundary conditions are presented in Table 5.7. The laminate is taken to be made of 2 layers of graphite/epoxy with material properties

\[ \frac{E_{11}}{E_{22}} = 40, \quad \frac{G_{12}}{E_{22}} = 0.5, \quad \nu_{12} = 0.25 \]
The two layers have the same thickness and the plate's non-dimensionalized stiffnesses are found to be

\[
\begin{bmatrix}
A_{ij} & B_{ij} & D_{ij}
\end{bmatrix} =
\begin{bmatrix}
0.513302028 & 0.006259780 & 0 \\
0.006259780 & 0.513302028 & 0 \\
0 & 0 & 0.001041667
\end{bmatrix}
\]

\[
\begin{bmatrix}
A_{ij} & B_{ij} & D_{ij}
\end{bmatrix} =
\begin{bmatrix}
-0.122065723 & 0 & 0 \\
0 & 0.122065723 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
A_{ij} & B_{ij} & D_{ij}
\end{bmatrix} =
\begin{bmatrix}
0.042775169 & 0.000521648 & 0 \\
0.000521648 & 0.042775169 & 0 \\
0 & 0 & 0.001041667
\end{bmatrix}
\]

(5.9)

The frequency parameter \( \omega a^2 \sqrt{\rho/E_2 h^3} \) is found by using 25(5x5) terms for each of the displacements \( u, v, \) and \( w \). The accuracy of the results when the same number of terms were used for the solution for a S2S2S2S2 plate in Chapter 4 was up to 4 significant figures.

Table 5.8 presents values of the buckling parameter \( \frac{N_x b^2}{E_2 h^3} \) for square antisymmetric cross-ply plates (2-ply plate made of the same material used for the numerical results presented in Table 5.7 for free vibration frequencies). These values are calculated for plates having S2S2S2S2,
C1C1C1Cl, C1F4C1F4, and S1F4S1F4 boundary conditions. The values of buckling parameter \( \frac{N_b^2}{E_2 h^3} = \frac{N_y b^2}{E_2 h^3} \) for plates under hydrostatic loading is presented for a S2S2S2S2 and a plate with C1C1C1Cl boundary conditions. For the simply supported plate (S2S2S2S2) the exact solution is possible for two cases and is presented also. The hydrostatic buckling load is one half of the uniaxial buckling load, confirming that they have identical mode shapes (one half wave in each direction). The shear buckling load parameter \( \frac{N_{xy} b^2}{E_2 h^3} \) is also presented in table 5.8 for two kinds of boundary conditions namely a plate with all boundaries simply supported (S2S2S2S2) and a plate with clamped edges (C1C1C1Cl). As the results show, the shear buckling load for the plate with all boundaries clamped (C1) is almost twice the one for the plate with all boundaries simply supported.

The vibration frequency parameters of the same plate with S2S2S2S2 and C1C1C1Cl edge conditions are calculated when the plates are under uniaxial and biaxial loading. The numerical results are tabulated in Table 5.9. For uniaxially loaded plates the inplane load is taken to be one half of required load for buckling. For the case when the plates are under biaxial loading, \( N_x \) and \( N_y \) are taken to be one half of critical hydrostatic buckling load. The vibration frequency parameter is found to be nearly the same for both cases of uniaxial inplane loading and hydrostatic
<table>
<thead>
<tr>
<th>Boundary conditions</th>
<th>Frequency parameter $\omega a^2\sqrt{\rho / E_2h^3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>F4F4F4F4</td>
<td>4.826</td>
</tr>
<tr>
<td>F2F2F2F2</td>
<td>4.838</td>
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<tr>
<td>F1F1F1F1</td>
<td>4.842</td>
</tr>
<tr>
<td>S2F2F2F2</td>
<td>2.402</td>
</tr>
<tr>
<td>C2F2F2F2</td>
<td>2.786</td>
</tr>
<tr>
<td>S2S2S2S2</td>
<td>11.17</td>
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<td>C2S2C2S2</td>
<td>18.72</td>
</tr>
<tr>
<td>C2C2C2C2</td>
<td>24.01</td>
</tr>
<tr>
<td>C1C1C1C1</td>
<td>24.03</td>
</tr>
</tbody>
</table>

Table 5.7 Fundamental vibration frequency parameters for square 2-ply antisymmetric cross-ply laminated plates having different boundary conditions.
<table>
<thead>
<tr>
<th>Plate boundary condition</th>
<th>Uniaxial buckling parameter</th>
<th>Hydrostatic buckling parameter</th>
<th>Shear buckling parameter</th>
</tr>
</thead>
<tbody>
<tr>
<td>S2S2S2S2</td>
<td>12.63</td>
<td>6.316</td>
<td>31.43</td>
</tr>
<tr>
<td></td>
<td>(12.63)*</td>
<td>(6.315)*</td>
<td></td>
</tr>
<tr>
<td>C1C1C1C1</td>
<td>41.13</td>
<td>24.32</td>
<td>60.41</td>
</tr>
<tr>
<td>C1F4C1F4</td>
<td>21.71</td>
<td>--</td>
<td>--</td>
</tr>
<tr>
<td>S1F4S1F4</td>
<td>11.60</td>
<td>--</td>
<td>--</td>
</tr>
</tbody>
</table>

Table 5.8 Buckling loads calculated for square anti-symmetric cross-ply plates having various combinations of boundary conditions.

* Exact solutions
in-plane loading when the plate is simply supported \( S2 \) along all its edges. Even for the plate with clamped edges \( (C1C1C1C1) \) the results found for both cases are very close (up to 2 digits).

<table>
<thead>
<tr>
<th>Plate boundary condition</th>
<th>Uniaxial loading ( N_x = N_x^{c}/2 )</th>
<th>Biaxial loading ( N_x = N_y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S2S2S2S2 )</td>
<td>7.8955</td>
<td>7.8957</td>
</tr>
<tr>
<td>( C1C1C1C1 )</td>
<td>17.785</td>
<td>17.008</td>
</tr>
</tbody>
</table>

Table 5.9 Vibration frequency parameter \( \omega a^2 \sqrt{\rho/E_h^3} \) for square cross-ply plates under inplane loads.
5.4 **Vibration frequencies of a generally unsymmetric laminate**

The differential equations of motion for a generally unsymmetric laminated plate as are given in Appendix A (equations A.12) are very complicated. The existence of all extensional stiffnesses ($A_{ij}$), coupling stiffnesses ($B_{ij}$), and bending stiffnesses ($D_{ij}$) in these equations makes it impossible to find an exact solution for them. The expression for the potential energy of deformation of a generally unsymmetric laminated plate is given by equation (3.5). None of the references given in Chapter 2 has dealt with this problem and there are no numerical results available for a generally laminated plate. However one of the greatest advantages of composite plates is that they can be tailored to efficiently meet the design requirements of strength, stiffness, and other parameters all in various directions. A generally unsymmetric laminated plate is as important as a cross-ply or an angle-ply plate, and the fact that it is more difficult to study the behavior of the generally laminated plates does not mean that they are less important.

Consider a laminate composed of 2 layers of orthotropic material. One layer has the principal-axes of material direction parallel to the plate edges, and the principal axes of material direction of the other one makes an angle of $\theta$ with the plate edges. Both layers are made of the same
material and have the same thickness. When the angle $\theta$ is zero, the plate is a specially orthotropic, and when the angle $\theta$ is $90^\circ$, the plate is an antisymmetric cross-ply laminated plate. For these two special cases the exact solution is possible when the plate is simply supported (S2) on all its edges. Figures 5.1, 5.2, and 5.3 show the variation of plate stiffnesses $A_{ij}$, $B_{ij}$, and $D_{ij}$ vs. the angle $\theta$. The material properties are chosen to be the ones for graphite/epoxy ($E_1/E_2 = 40$, $G_{12}/E_2 = 0.5$, and $\nu_{12} = 0.25$) and the stiffnesses are nondimensionalized. Figure 5.4 shows the first five natural frequencies $\omega a^2 \sqrt{\rho/E_2 h^3}$ vs. the angle $\theta$, for a square simply supported (S2) plate. Table 4.14 presents the numerical results for selected angles up to six significant figures. These results are calculated by using 25(5x5) terms for $u$, 25(5x5) terms for $v$, and 25(5x5) terms for $w$.

As Figure 5.4 shows, the natural frequency of the first vibration mode is a minimum when the angle is $90^\circ$. This is the case where the plate is an antisymmetric cross-ply plate. All the elements of coupling stiffnesses matrix ($B_{ij}$) are zero but $B_{11} = -B_{22}$ and their absolute value is maximum (See Figure 5.2). As Figures 5.1 and 5.3 show, at this angle $A_{22}$ and $D_{22}$ have their maximum value too, while $A_{11}$ and $D_{11}$ are minimum. Note that $A_{12}$ and $D_{12}$ are small compared to $A_{11} = A_{22}$ and $D_{11} = D_{22}$, but they are not zero. When the angle $\theta$ is equal to zero
and the plate is specially orthotropic, the coupling stiffnesses vanish. In this case the natural frequency of the first mode is at its maximum value. Again it should be mentioned that, except $D_{16}$ and $D_{26}$, the other bending stiffnesses are small compared to $D_{11}$, but they are not 0.

Figure 5.4 and Table 5.10 show that the frequency of the first mode does not continuously decrease when the angle $\theta$ increases. In fact, when the angle is between $20^\circ$ and $30^\circ$ the curve has a relative minimum, and when the angle is between $30^\circ$ and $40^\circ$ it has a relative maximum point. The behavior of higher modes is different. For example, the frequency of the second mode is smaller when the angle $\theta$ is $0^\circ$ and it is minimum when the angle $\theta$ is between $20^\circ$ and $30^\circ$.

The curves for the third and fourth modes are drawn in Figure 5.4 as if they cross in the vicinity of $\theta = 47^\circ$. Without extensive and more precise calculations, one cannot be sure whether they cross, or whether they approach each other and veer away from each other, as was demonstrated by Leissa for other eigenvalue problems having approximate solutions.
Fig. 5.1 Variation of extensional stiffnesses $A_{ij} = A_{1j}/E_1h$ vs. the angle $\theta$. 
Fig. 5.2 Variation of coupling stiffnesses $\bar{B}_{ij} = B_{ij}/E_1 h^2$ vs. the angle $\theta$. 
Fig. 5.3 Variation of bending stiffnesses $\bar{D}_{ij} = D_{ij}/E_1 h^3$ vs. the angle $\theta$. 
Fig 5.4 Variation of frequency parameter $wa^2/\sqrt{\rho/E_2h^3}$ vs. the angle $\theta$. 

74
<table>
<thead>
<tr>
<th>θ (Degree)</th>
<th>1st mode</th>
<th>2nd mode</th>
<th>3rd mode</th>
<th>4th mode</th>
<th>5th mode</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>18.8052</td>
<td>23.1880</td>
<td>34.4991</td>
<td>73.0539</td>
<td>75.5257</td>
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<tr>
<td>15</td>
<td>17.1072</td>
<td>21.2404</td>
<td>35.5607</td>
<td>57.8635</td>
<td>66.6790</td>
</tr>
<tr>
<td>30</td>
<td>15.5252</td>
<td>20.6098</td>
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<td>60.3255</td>
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<tr>
<td>33</td>
<td>15.7142</td>
<td>20.9448</td>
<td>40.4697</td>
<td>49.9392</td>
<td>60.9971</td>
</tr>
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<td>36</td>
<td>15.9596</td>
<td>21.4865</td>
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<td>49.6397</td>
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<td>37</td>
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<tr>
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<td>85.4485</td>
</tr>
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</table>

Table 5.10 Natural frequencies $\omega a^2\sqrt{\rho/E_2h^3}$ for generally unsymmetric laminated plates with simply supported boundary conditions.
CHAPTER VI
ANALYSIS OF VIBRATION AND BUCKLING OF LAMINATED SHELLS

In the previous chapters an analytical method based on the Ritz method with polynomials as displacement functions was established to solve the problems of vibration and buckling of rectangular laminated plates. The purpose of this chapter is to demonstrate how this approach may be generalized to find vibration frequencies and buckling loads of thin laminated shallow circular cylindrical shells of rectangular planform.

6.1 Energy formulation

The potential energy of deformation in terms of shell coordinates $\kappa, \beta$, and $z$ is $^{67}$.

$$ U = \frac{1}{2} \int_V \left( C_{\kappa \kappa} e_{\kappa} + C_{\beta \beta} e_{\beta} + C_Z e_Z + C_{\kappa A} e_{\kappa A} + C_{kZ} e_{kZ} + C_{\beta \beta} e_{\beta} \right) dv $$

where $z$ is the coordinate normal to the shell surface and parameters $\kappa$, and $\beta$ represent orthogonal curvilinear coordinates of the midsurface. The volume element $dV$ in terms of shell coordinates is:
\[ dV = (1 + z/R_\alpha)(1 + z/R_\beta)ABd\xi d\eta dz \] (6.2)

In equation (6.2) \( A \) and \( B \) are surface metric coefficients and \( R_\alpha \) and \( R_\beta \) are radii of curvature.

Assume normals to the undeformed middle surface remain straight and normal to the deformed middle surface and they do not extend (Kirchhoff hypothesis for thin shells). Then

\[ e_{\alpha z} = e_{\beta z} = e_z = 0 \] (6.3)

Applying equations (6.3) reduces the equation (6.1) to the following form:

\[ U = 1/2 \int_{V} \left( \sigma_{\alpha \alpha} e_{\alpha} + \sigma_{\beta \beta} e_{\beta} + \sigma_{\alpha \beta} e_{\alpha \beta} \right) dV \] (6.4)

Assuming a state of plane stress in the kth lamina of the shell the stress-strain relations are:

\[
\begin{bmatrix}
\sigma_{11} & \sigma_{12} & \sigma_{16} \\
\sigma_{12} & \sigma_{22} & \sigma_{26} \\
\sigma_{16} & \sigma_{26} & \sigma_{66}
\end{bmatrix}_k
\begin{bmatrix}
e_{11} \\
e_{22} \\
e_{66}
\end{bmatrix}_k
= \begin{bmatrix}
\sigma_{11} & \sigma_{12} & \sigma_{16} \\
\sigma_{12} & \sigma_{22} & \sigma_{26} \\
\sigma_{16} & \sigma_{26} & \sigma_{66}
\end{bmatrix}_k
\begin{bmatrix}
e_{11} \\
e_{22} \\
e_{66}
\end{bmatrix}_k
\] (6.5)

Equation (6.5) is introduced in Appendix A and is the same as equation (A.3). Substituting equation (6.5) into
equation (6.4) and separating the volume integral into piecewise integration through the thickness,

\[ U = \frac{1}{2} \sum_{k=0}^{n} \left[ \left( \sigma_{11}^k \varepsilon_\alpha + \sigma_{12}^k \varepsilon_\beta + \sigma_{16}^k \varepsilon_{\alpha\beta} \right) \varepsilon_\alpha + \left( \sigma_{12}^k \varepsilon_\alpha + \sigma_{22}^k \varepsilon_\beta + \sigma_{26}^k \varepsilon_{\alpha\beta} \right) \varepsilon_\beta \right] (1 + z/R_\alpha)(1 + z/R_\beta) AB dz d\alpha d\beta \quad (6.6) \]

The total strains in equation (6.6) are written in terms of the middle surface strains and changes in curvature as follows:

\[ \varepsilon_\alpha = \frac{1}{(1 + z/R_\alpha)} \left( \varepsilon_\alpha + z K_\alpha \right) \]

\[ \varepsilon_\beta = \frac{1}{(1 + z/R_\beta)} \left( \varepsilon_\beta + z K_\beta \right) \quad (6.7) \]

\[ \varepsilon_{\alpha\beta} = \frac{1}{(1 + z/R_\alpha)(1 + z/R_\beta)} \left[ (1 - z^2/R_\alpha R_\beta) \varepsilon_{\alpha\beta} + z(1 + z/2R_\alpha + z/2R_\beta) K_{\alpha\beta} \right] \]

\( \varepsilon_\alpha, \varepsilon_\beta, \) and \( \varepsilon_{\alpha\beta} \) are middle surface strains, \( K_\alpha, K_\beta \) are the middle surface changes in curvature and \( K_{\alpha\beta} \) is middle surface twist. Relations (6.7) are used by Novozhilov, Byrne, Flugge, Goldenveiser, and Lur'ye for isotropic thin shells. Since \( z/R_i \) is less than unity for a thin shell it could be
expanded into a geometric series,

\[
\frac{1}{1 + z/R_i} = \sum_{n=0}^{\infty} (-z/R_i)^n
\]  

Substituting the expression for the total strain in terms of middle surface strains and curvature given in equation (6.7) and replacing \((1 + z/R_i)^{-1}\) by its series expansion given by equation (6.8), then neglecting terms raised to power of \(z\) greater than 2 and carrying out the integration over the thickness, one obtains:

\[
U = \frac{1}{2} \int \left\{ A_{11} \varepsilon_{\alpha}^2 + B_{11} (2 \kappa_{\alpha} \varepsilon_{\beta} - 2 \varepsilon_{\beta}^2/R_{\alpha}) + D_{11} (3 \varepsilon_{\alpha}^2/R_{\alpha}^2 + \kappa_{\alpha}^2 - 4 \kappa_{\alpha} \varepsilon_{\alpha}/R_{\alpha}) + 2A_{12} \varepsilon_{\alpha} \varepsilon_{\beta} + 2B_{12} \left( (\varepsilon_{\alpha} \kappa_{\beta} + \varepsilon_{\beta} \kappa_{\alpha}) - (1/R_{\alpha} + 1/R_{\beta}) \varepsilon_{\alpha} \varepsilon_{\beta} \right) + 2D_{12} \left[ \varepsilon_{\alpha} \varepsilon_{\beta} (1/R_{\alpha}^2 + 1/R_{\beta}^2 + 1/R_{\alpha} R_{\beta}) + \kappa_{\alpha} \kappa_{\beta} + (1/R_{\alpha} + 1/R_{\beta}) (\varepsilon_{\alpha} \kappa_{\beta} + \varepsilon_{\beta} \kappa_{\alpha}) + 2A_{16} \varepsilon_{\alpha} \varepsilon_{\alpha \beta} + 2B_{16} \left[ \varepsilon_{\alpha \beta} (\kappa_{\alpha} - \varepsilon_{\alpha}/R_{\alpha}) - \varepsilon_{\alpha} (1/R_{\alpha} + 1/R_{\beta}) \varepsilon_{\alpha \beta} + K_{\alpha \beta} \right] + 2D_{16} \left[ \varepsilon_{\alpha} (1/R_{\alpha}^2 + 1/R_{\beta}^2) \varepsilon_{\alpha \beta} - K_{\alpha \beta}/2R_{\alpha} - K_{\alpha \beta}/2R_{\beta} + \varepsilon_{\alpha \beta} (\varepsilon_{\alpha}/R_{\alpha}^2 - K_{\alpha}/R_{\alpha}) + (K_{\alpha} - \varepsilon_{\alpha}/R_{\alpha}) \left[ (1/R_{\alpha} + 1/R_{\beta}) \varepsilon_{\alpha \beta} + K_{\alpha \beta} \right] \right] \right\} \right. 
\]
\[ + A_{22} \varepsilon_\beta^2 + B_{22} (2K_\beta \varepsilon_\beta - 2\varepsilon_\alpha^2/R_\alpha) + D_{22} (3\varepsilon_\beta^2/R_\beta^2 + K_\beta^2 - 4K_\beta \varepsilon_\beta/R_\alpha) + 2A_{26} \varepsilon_\beta \varepsilon_\alpha \varepsilon_\beta + 2B_{26} \left[ \varepsilon_\alpha \varepsilon_\beta (K_\alpha - \varepsilon_\beta/R_\beta) - \varepsilon_\beta \left[ (1/R_\alpha + 1/R_\beta) \varepsilon_\alpha + K_\alpha \right] \right] + 2D_{26} \left[ \varepsilon_\alpha^2 \left[ (1/R_\alpha^2 + 1/R_\beta^2) \varepsilon_\alpha + K_\alpha \right] - K_\alpha \varepsilon_\alpha/2R_\alpha - K_\alpha \varepsilon_\alpha^2/2R_\alpha \right] + \varepsilon_\alpha \varepsilon_\beta \left[ (1/R_\alpha^2 - K_\alpha^2/R_\beta^2) + (K_\alpha - \varepsilon_\beta/R_\beta) \right] \left[ (1/R_\alpha + 1/R_\beta) \varepsilon_\alpha + K_\alpha \right] (2B_{26} \varepsilon_\alpha \varepsilon_\beta) \right] + A_{66} \varepsilon_\alpha^2 - \left[ (1/R_\alpha + 1/R_\beta) \varepsilon_\alpha + K_\alpha \right] (2B_{66} \varepsilon_\alpha \varepsilon_\beta) \right] \right] + K_\alpha \varepsilon_\alpha/2R - K_\alpha \varepsilon_\alpha^2/2R_\alpha \right] \right) \right] \right] \] (6.9)

Equation (6.9) represents the total strain energy of deformation for a thin laminated composite shell where \( A_{ij}, B_{ij}, \) and \( D_{ij} \) are:

\[ A_{ij} = \sum_{k=0}^{N} (\tilde{Q}_{ij})_k(z_k - z_{k-1}) \]

\[ B_{ij} = 1/2 \sum_{k=1}^{N} (\tilde{Q}_{ij})_k(z_k^2 - z_{k-1}^2) \] (6.10)

\[ D_{ij} = 1/3 \sum_{k=1}^{N} (\tilde{Q}_{ij})_k(z_k^3 - z_{k-1}^3) \]
The mid-surface strains in terms of displacements $u$, $v$, and $w$ may be written in the following form:

\[
\begin{align*}
\varepsilon_u &= \frac{1}{A} \frac{\partial u}{\partial \nu} + \frac{v}{AB} \frac{\partial A}{\partial \beta} + \frac{w}{R_{\kappa}} \\
\varepsilon_v &= \frac{u}{AB} \frac{\partial B}{\partial \nu} + \frac{1}{B} \frac{\partial v}{\partial \zeta} + \frac{w}{R_{\zeta}} \\
\varepsilon_{\nu \beta} &= \frac{A}{B} \frac{\partial}{\partial \beta} \left( \frac{u}{A} \right) + \frac{B}{A} \frac{\partial}{\partial \kappa} \left( \frac{v}{B} \right)
\end{align*}
\]

(6.11)

Changes in middle surface curvatures, $\kappa_\alpha$ and $\kappa_\beta$, and the middle surface twist, $\kappa_{\nu \beta}$, are:

\[
\begin{align*}
\kappa_\alpha &= \frac{1}{A} \frac{\partial}{\partial \nu} \left( \frac{u}{R_\alpha} - \frac{1}{A} \frac{\partial W}{\partial \nu} \right) + \frac{1}{AB} \left( \frac{v}{R_\beta} - \frac{1}{B} \frac{\partial W}{\partial \beta} \frac{\partial A}{\partial \beta} \right) \\
\kappa_\beta &= \frac{1}{AB} \left( \frac{u}{R_\alpha} - \frac{1}{A} \frac{\partial W}{\partial \nu} \right) \frac{\partial B}{\partial \nu} + \frac{1}{B} \frac{\partial}{\partial \alpha} \left( \frac{v}{R_\beta} - \frac{1}{B} \frac{\partial W}{\partial \beta} \right) \\
\kappa_{\nu \beta} &= \frac{A}{B} \frac{\partial}{\partial \beta} \left[ \frac{1}{A} \left( \frac{u}{R_\alpha} - \frac{1}{A} \frac{\partial W}{\partial \nu} \right) \right] + \frac{B}{A} \frac{\partial}{\partial \alpha} \left[ \frac{1}{B} \left( \frac{v}{R_\beta} - \frac{1}{B} \frac{\partial W}{\partial \beta} \right) \right] + \\
&\quad \frac{1}{R_\alpha} \left( \frac{1}{B} \frac{\partial u}{\partial \beta} - \frac{v}{AB} \frac{\partial A}{\partial \beta} \right) + \frac{1}{R_\beta} \left( \frac{1}{A} \frac{\partial v}{\partial \alpha} - \frac{u}{AB} \frac{\partial A}{\partial \alpha} \right)
\end{align*}
\]

(6.12)

Equations (6.12), change in curvature-displacement equations, are used by Novozhilov, Byrne, Goldenveizer, and Lur'ye and equations (6.11) are almost universally accepted by all the theories developed for thin shells.
6.2 Shallow shells

A shallow shell is defined as a "shell whose smallest radius of curvature at every point is large compared with the greatest length measured along the middle surface of the shell". Based on assumptions used in developing shallow shell theory, equation (6.9) greatly simplifies. Using rectangular cartesian coordinates x and y instead of curvilinear coordinates \( \alpha \) and \( \beta \), equation (6.4) reduces to:

\[
U = \frac{1}{2} \int \left( \frac{\partial}{\partial x} \varepsilon_x + \frac{\partial}{\partial y} \varepsilon_y + \frac{\partial}{\partial xy} \varepsilon_{xy} \right) \sqrt{\varepsilon_x \varepsilon_y} \, dxdydz
\]

For constant curvatures \( R_x', R_y', \) and \( R_{xy}' \), metrics \( \alpha \) and \( \beta \) become constant, and in equations (6.7), all the terms involving \( z/R_i \) are neglected. Then:

\[
\varepsilon_x = \xi_x + z \xi_x, \quad \varepsilon_y = \xi_y + z \xi_y, \quad \varepsilon_{xy} = \xi_{xy} + z \xi_{xy}
\]

(6.14)

where equations (6.11) reduce to:

\[
\begin{align*}
\xi_x &= u_x + w/R_x \\
\xi_y &= v_y + w/R_y \\
\xi_{xy} &= u_x + v_y - 2w/R_{xy}
\end{align*}
\]

(6.15)

and curvature changes will simplify to:
Equation (6.9) may be written in terms of displacements as follows:

\[
\begin{align*}
\kappa_x &= -w_{xx} \quad \kappa_y = -w_{yy} \quad \kappa_{xy} = -2w_{xy} \\
\end{align*}
\tag{6.16}
\]

\[
\begin{align*}
U &= \frac{1}{2} \left( \begin{bmatrix}
A_{11} (u_x + w_{Rx})^2 + 2A_{12} (u_x + w_{Rx}) (v_y + w_{Ry}) + \\
- \frac{1}{2} \end{bmatrix} \\
&- \frac{1}{2} + \frac{1}{2} \right)
\end{align*}
\]

\[
\begin{align*}
&+ A_{22} (v_y + w_{Ry})^2 + 2A_{16} (u_x + w_{Rx}) (v_{,x} + u_{,y} - 2w_{Rxy}) + \\
&+ A_{66} (v_{,x} + u_{,y} - 2w_{Rxy})^2 - 2B_{11} (u_x + w_{Rx}) w_{,xx} - \\
&- 2B_{12} \left[ (v_y + w_{Ry}) w_{,xx} + (v_{,x} + w_{Rx}) w_{,yy} \right] - 2B_{22} (v_y + w_{Ry}) w_{,yy} - 2B_{16} \left[ (u_{,y} + v_{,x} - 2w_{Rxy}) w_{,xx} + 2(u_{,x} + w_{Rx}) w_{,xy} \right] - 2B_{26} \left[ (u_{,y} + v_{,x} - 2w_{Rxy}) w_{,yy} + 2(v_{,y} + w_{Ry}) w_{,xy} \right] - 4B_{66} (u_{,y} + v_{,x} - 2w_{Rxy}) w_{,xy} + \\
&+ D_{11} w_{,xx} + 2D_{12} w_{,xx} w_{,yy} + D_{22} w_{,yy} + 4D_{16} w_{,xx} w_{,xy} + \\
&+ 4D_{26} w_{,yy} w_{,xy} + 4D_{66} w_{,xy} \right) \text{d}x \text{d}y 
\tag{6.17}
\end{align*}
\]
Note that the integration is taken over the projected planform with the dimensions axb on the x-y basis. Equation (6.17) is derived based on shallow shell theory assumptions and is much more simple than the expression given in equation (6.9) for strain energy of deformation which is based on generalizing Novozhilov theory of thin isotropic shells to laminated shells. However, equation (6.17) may also be derived directly from equation (6.9) by neglecting proper terms that are ignorable for shallow shells (terms involving z/R₁).

6.3 **Shallow circular cylindrical shells of rectangular planform**

A thin circular cylindrical shallow shell of rectangular planform is shown in Figure 6.1. The thickness of the shell is denoted by h, and a and b represent length and width of the projected area on xy plane respectively. The three orthogonal components of displacement of the middle surface of the shell, are u and v (tangential to the middle surface parallel to xz-plane and yz-plane, respectively) and w (normal to the middle surface). Substituting 1/Rₓ = 0, 1/Rᵧ = 1/R, and 1/Rₓᵧ = 0 in equation (6.17), the potential energy of deformation simplifies to:
Figure 6.1  Circular cylindrical shell of rectangular planform
\[ U = \frac{1}{2} \left( \sum_{i=1}^{2} \left[ A_{11} u_{,xx}^2 + 2A_{12} \left( u_{,x}(w/R) + u_{,x}v_{,y} \right) + A_{22} \left( v_{,y}^2 + (w/R)^2 + 2v_{,y}(w/R) \right) + 2A_{16} (u_{,x} u_{,y} + u_{,x} v_{,x}) + 2A_{26} \left( (w/R) v_{,x} + (w/R) u_{,y} + v_{,y} u_{,y} + v_{,y} v_{,x} \right) + A_{66} (u_{,x} + v_{,x})^2 \right] - 2B_{11} u_{,x} w_{,xx} - 2B_{12} \left( (w/R) w_{,xx} + v_{,y} w_{,xx} + u_{,x} w_{,yy} \right) - 2B_{16} (u_{,y} w_{,xx} + v_{,x} w_{,xx} + 2u_{,x} w_{,xy}) - 2B_{26} \left( 2(w/R) w_{,xy} + u_{,y} w_{,yy} + v_{,x} w_{,xy} + 2v_{,y} w_{,xy} \right) - 4B_{66} (u_{,y} w_{,xy} + v_{,x} w_{,xy}) + D_{11} w_{,xx}^2 + 2D_{12} w_{,xx} w_{,yy} + D_{22} w_{,yy}^2 + 4D_{66} w_{,xx} w_{,yy} + 4D_{26} w_{,yy} w_{,xy} + 4D_{66} w_{,xy}^2 \right) dx dy \]

(6.18)

Equation (6.18) is the general expression for strain energy of shallow thin, circular cylindrical laminated shells.

Although this expression may be simplified furthermore for mid-surface symmetrical laminated shells when \( B_{ij} = 0 \), all the components of displacement will remain in the expression.
(unlike the plates). Therefore, even for homogeneous isotropic shells the components of displacement ($u$, $v$, and $w$) remain coupled and four boundary conditions per edge exist (similar to unsymmetric laminated plates). The reason for the coupling of the components of displacement is the existence of curvature ($1/R$) in the expression of the strain energy of deformation. When $1/R = 0$, equation (6.17) reduces to the expression for potential energy of deformation for a flat plate which was discussed in Chapter 3.

The kinetic energy expression, neglecting rotary inertia and tangential displacement terms will have the following form:

$$T = \frac{1}{2} \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{a}{2}}^{\frac{a}{2}} \rho \left( w_{,t}^2 \right) dx dy$$  \hspace{1cm} (6.19)

where $\rho$ is mass per unit area and $w_{,t}$ denotes partial differentiation with respect to time.

The expression for the potential energy of inplane loads may be written in the following form:

$$V_2 = \frac{1}{2} \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{a}{2}}^{\frac{a}{2}} \left( N_x w_{,x}^2 + N_y w_{,y}^2 + 2N_{xy} w_{,x} w_{,y} \right) dx dy$$  \hspace{1cm} (6.20)

Expressions (6.19) and (6.20) are exactly the same as the energy expressions which were used for the plates and
in fact, the only difference between the analysis of the shells and plates is the formulation of strain energy of deformation. To find the natural frequencies and buckling loads, the same steps which were discussed in Chapter 3 for the plates are applied for the shells.
CHAPTER VII
NUMERICAL RESULTS FOR SHELLS

In this chapter the analytical method developed in the previous chapter, is used to calculate the natural frequencies and buckling loads of laminated circular cylindrical shells of rectangular planform. As it was explained in Chapter 6, the introduction of thickness and curvatures in the formulations for laminated shells adds additional complications to the problem. The number of publications dealing with the subject are limited. Therefore, the exclusive comparison which was made between the numerical results calculated by the present method and those of the other approaches for the laminated plates, can not be done here. The objective here is to demonstrate the effects of curvature on the vibration frequencies and buckling loads of laminates. The analysis in this chapter is by no means complete and the results are given for a selected number of important cases.

7.1 Symmetrically laminated shells

Due to the existence of curvature, the governing equations for shells are far more complicated than those of the plates. Even for isotropic shells which are mathematically the simplest cases of symmetrically laminated shells, all
the components of displacement remain in the energy expression. The Ritz method with algebraic polynomials as displacement functions has been applied successfully by Leissa, Lee, and Wang\textsuperscript{68} for a special case of vibration of cantilevered shell. Since basically they have used the same approach as the one developed here, the excellent results presented by them shows how accurate the results of the approach would be.

Crawley\textsuperscript{52} presents numerical results for a more complicated case i.e., symmetrically laminated graphite/epoxy shells. He presents experimental results for natural frequencies of cantilevered shells and compares them with numerical results calculated by a finite element analysis. Table 5.1 presents five lowest natural frequencies of an 8-ply laminate with orientation (0°/±30)\textsubscript{s}. This Table also shows the numerical results calculated by the present method using 36(6x6) terms for each of the displacements \(u, v,\) and \(w\). Material properties used in calculations are:

\[
E_1 = 18.5 \times 10^6 \text{ psi.} \\
E_2 = 1.60 \times 10^6 \text{ psi.} \\
\nu_{12} = 0.25 \\
G_{12} = 0.65 \times 10^6 \text{ psi.} \\
\text{density} = 0.055 \text{ lb/in}^3
\]

The shell has the length \(a = 6''\) and width \(b = 3''\) and the thickness of the ply \(t = 0.0052\) in. The radius of the
curvature $R = 5$ in., and the frequencies are given in Hz. As Table 7.1 shows, the results calculated by finite element analysis and the present approach are generally very close and their maximum difference is at the frequency of the third vibration mode where a difference of 3% between the results exists. Since both methods provide upper bound frequencies the smaller result for each case is more accurate.

<table>
<thead>
<tr>
<th>Experimental results$^{52}$</th>
<th>Finite element method$^{52}$</th>
<th>Present method</th>
</tr>
</thead>
<tbody>
<tr>
<td>161.</td>
<td>165.7</td>
<td>167.7</td>
</tr>
<tr>
<td>254.1</td>
<td>289.6</td>
<td>286.2</td>
</tr>
<tr>
<td>555.6</td>
<td>597.1</td>
<td>617.4</td>
</tr>
<tr>
<td>670.</td>
<td>718.5</td>
<td>721.7</td>
</tr>
<tr>
<td>794.</td>
<td>833.3</td>
<td>848.4</td>
</tr>
</tbody>
</table>

Table 7.1 Natural frequencies (Hz) of 6"x3" 8-ply graphite/epoxy cylindrical shells.
To study the effect of curvature on symmetrically laminated shells, the numerical results are calculated for uniaxial buckling loads of specially orthotropic laminates with material properties:

$$\frac{D_{11}}{D_{22}} = 10, \quad \frac{D_{16} + 2D_{66}}{D_{22}} = 1$$

Note that the shell is stiffer in x-direction where the curvature is zero. The numerical results for uniaxial buckling parameter $N_x b^2/\pi^2 D_{22}$ are summarized in Table 7.2. These results are calculated by using 25(5x5) terms for each of the displacements $u$, $v$, and $w$. The aspect ratio $(a/b)$ is taken to be 1 and the width to thickness ratio $b/h = 100$ which corresponds to a shell well within the accepted limits of thin shell theory. The buckling loads are obtained for typical shallowness parameters $(b/R)$ of 0, 0.2, and 0.5. The upper limit $b/R = 0.5$ is within the limitation of shallow shell theory, and $b/R = 0$ corresponds to flat plate. The boundary conditions are chosen to be C2C2C2, S2S2S2S2, C2F2C2F2, and S2F2S2F2 which are the same important cases that were used to study the specially orthotropic laminated plates in Chapter 5. As expected, by increasing the shallowness ratio $(b/R)$, the buckling load parameter increases. Depending on the edge conditions, the rate of this change in buckling parameter varies. For example, the buckling load of the S2S2S2S2 case increases from 13.00 for the flat plate to 52.41 for the shell with $b/R = 0.5$. The buckling load
parameter for the C2F2C2F2 case is 39.79 where b/R = 0, and is 50.12 where b/R = 0.5. In fact, the buckling load parameter of the S2S2S2S2 shell is a little more than the buckling load parameter of the C2F2C2F2 when the shallowness ratio b/R = 0.5. For the flat plate the buckling load of the S2S2S2S2 plate is one third of the C2F2C2F2.

<table>
<thead>
<tr>
<th>Boundary condition</th>
<th>Uniaxial buckling load $N_x b^2/\mu^2 D_{22}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>b/R = 0</td>
</tr>
<tr>
<td>C2C2C2C2</td>
<td>46.29</td>
</tr>
<tr>
<td>S2S2S2S2</td>
<td>13.00</td>
</tr>
<tr>
<td>C2F2C2F2</td>
<td>39.79</td>
</tr>
<tr>
<td>S2F2S2F2</td>
<td>9.846</td>
</tr>
</tbody>
</table>

Table 7.2 Effect of curvature on the uniaxial buckling parameter of square orthotropic shells having various boundary conditions.
7.2 Antisymmetric cross-ply laminates

This class of laminates and their properties were discussed in previous chapters. The effectiveness of the bending-stretching stiffnesses \( B_{11} \) and \( B_{22} = -B_{11} \) on the vibration frequencies and buckling loads of laminated plates was studied for various combinations of boundary conditions. In this section the effect of curvature is studied on vibration frequencies and uniaxial buckling loads of cross-ply laminates. The boundary conditions are taken to be simply supported (S2) on all the edges, where the exact solution is possible for the special case of zero curvature (flat plate). The laminates are made of two layers with material properties:

\[
E_1/E_2 = 40, \quad G_{12}/E_2 = 0.5, \quad \nu_{12} = 0.25
\]

These are the same typical properties which were used in Chapters 4 and 5 where the behavior of cross-ply plates were studied. Table 7.3 shows how the fundamental frequencies and uniaxial buckling loads of these laminates change when the shallowness ratio \( (b/R) \) varies. The aspect ratio \( a/b = 1 \), and the width to thickness ratio \( b/h = 100 \) for all the cases. The \( b/h \) ratio chosen is well within the accepted limits of thin theory. The numerical results are calculated for shallowness ratios \( b/R \) of smaller and equal to 0.5, which are within the limitations of shallow shell theory.
The cross-ply laminates are not symmetric with respect to the middle surface and physically two sets of problems exist when one is studying the effect of curvature. First, the direction of the fibers of the outer layer of the shell is in the x direction (the generating axis of the shell). Second, the direction of the fibers of the inner layer is in the x direction. This effect is studied by calculating the results for both cases. As the results show, the difference between the two cases is not significant for the uniaxial buckling load. For vibration frequencies the results are observed to be different for the cases when the shallowness ratio is 0.2 and 0.3.
<table>
<thead>
<tr>
<th>Curvature direction</th>
<th>$N_x b^2/E_2 h^3$</th>
<th>$\omega a^2/\rho/E_2 h^3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>b/R</td>
<td>0.5 0.4 0.3 0.2 0.1 0.0 0.1 0.2 0.3 0.4 0.5</td>
<td></td>
</tr>
<tr>
<td></td>
<td>85.76 73.67 56.28 32.06 17.51 12.63 17.49 32.17 56.62 73.64 85.74</td>
<td></td>
</tr>
</tbody>
</table>

Table 7.3 Buckling load parameters and fundamental natural frequencies of cross-ply laminates with simply supported boundary conditions having various curvatures, $b/h = 100$, $a/b = 1$
CHAPTER VIII
CONCLUDING REMARKS

The Ritz method was used in this study to determine the vibration frequencies and buckling loads of rectangular laminated composite plates with arbitrary edge conditions. The energy expression used was based on the linear theory of thin laminated plates, which is valid for cases where the plate's width is at least 20 times its thickness. Ordinary algebraic polynomials were chosen as displacement functions because they were capable of dealing with arbitrary boundary conditions, and they are also mathematically complete set of functions, thereby permitting convergence of the solutions to the exact solution by using sufficient terms. To satisfy the arbitrary boundary conditions, constraint matrices were used to modify the coefficient matrix of the linear characteristic equations derived from minimization of the energy functionals \( T_{\text{max}} - V_{\text{max}} \).

The accuracy of the approach was demonstrated by comparing the numerical results calculated by this method with those of the exact solutions which were available for some cases. Furthermore, a number of reliable approximate solutions provided by others for various combinations of boundary conditions were observed to be in good agreement with results calculated by the present method.
The developed method was used to study the behavior of orthotropic, anisotropic, and generally laminated plates. A number of previously unsolved problems were studied. The natural frequencies and critical buckling loads were calculated for plates with various important combinations of boundary conditions. The antisymmetric cross-ply plate was used to study the effect of edge conditions for cases where bending and stretching coupling exists and the natural frequencies, vibration frequencies with in-plane loads were found for a number of cases. In addition, problems of uniaxial buckling, biaxial buckling, and shear buckling of cross-ply plates were studied.

By developing the energy expression for shallow shells, the same approach was generalized to handle the problems of vibration and buckling of shallow shells of rectangular planforms. The numerical results were calculated for circular cylindrical shells with different material properties. The effects of width to curvature ratio (shallowness ratio) were demonstrated for specially orthotropic and cross-ply laminates.

Due to the priority of finding the natural frequencies and critical buckling loads, and the great number of unsolved problems and variety of variables involved, no attempt was made to analyze the mode shapes. However, the approach is capable of finding this information if it is desired. The only additional step will be to substitute
the eigenvector corresponding to each eigenvalue back into the original displacement expressions.

One of the advantages of taking algebraic polynomials as displacement functions, is that the solution may be generalized relatively easy to handle the variable thickness, varying inplane loads (for example, gravity type inplane load), and variable curvature. This can be seen in Chapters 3 and 6, where the exact integration can be done easily even if these complicating effects are added.

Since the Ritz method provides upper bound solutions for the problems, the desired degree of accuracy may be reached for any particular problem by adding enough terms to displacement functions as long as the result is converging. Since the tangential inertia is basically ignored in these problems, considerable computational time can be saved (if required) by reducing the size of the characteristic determinant. The technique to achieve this, is explained in reference 68.

Another way to save computational time, is taking the advantage of any possible symmetry that may exist for any individual case. For example, a cantilevered cross-ply plate has one plane of symmetry (if the plate is clamped on the edge $x = -a/2$, the $xz$ plane is the plane of symmetry). For solving this case, since the modes are either symmetric or antisymmetric with respect to this plane, even powers of $y$ in displacement functions may be taken separately for
finding corresponding eigenvalues for symmetric modes and odd powers of $y$ may be used to find corresponding eigenvalues for antisymmetric modes. These two uncoupled determinants have one half of the size of the original characteristic determinant and solving them requires much less computational time than solving the original one.

The computer program developed based on the formulation derived in this dissertation may be used to verify any alternative solution which is within the limitations of the theory of thin laminated plates and shallow shells. The capability of the approach to provide reasonable solutions for all the possible combinations of geometric boundary conditions for any type of laminates will provide valuable information in optimum design of these structural elements. The experimental results for any problem may be used to verify the limitations of the theory for any individual case if numerical results are provided based on that theory, and the approach developed here is capable of providing such information for the linear theory of composite plates.
REFERENCES


64. Bolotin, V.V., "The edge effect in the oscillations of elastic shells", Prikladnava Matematika i Mekhanika, 24, (1960) 831-842.


APPENDIX A

CLASSICAL THIN LAMINATED PLATE THEORY
This appendix presents a summary of basic assumptions and fundamental equations of classical theory of thin laminated rectangular plates. The complete theory may be found in references 28 and 40.

A.1 Assumptions

A typical coordinate system is shown in Figure A.1. The x and y axes are parallel to the plate edges and 0 is its center. The x-y plane is the mid-plane of the plate and normal to the plate is z axis (outward in the figure). The displacements in x, y, and z directions are denoted by \( \bar{u} \), \( \bar{v} \), and \( \bar{w} \), respectively.

The theory is developed using the following assumptions

1. The plate is constructed of an arbitrary number of orthotropic layers. Each layer obeys Hooke's law.
2. The thickness of the plate (h) is small compared to its width (b) and its length (a).

![Plate coordinates](image)

*Fig. A.1 Plate coordinates*
3. The displacements \( \bar{u}, \bar{v}, \) and \( \bar{w} \) are small compared to the thickness \( h \).

4. In-plane strains \( \varepsilon_x, \varepsilon_y, \gamma_{xy} \) are small compared to unity.

5. Transverse shear strains \( \gamma_{xz}, \gamma_{yz} \), and also normal strain \( \varepsilon_z \) may be neglected.

6. Tangential displacements \( \bar{u}, \) and \( \bar{v} \) are linear functions of the coordinate \( z \) (i.e., the Kirchhoff hypothesis).

7. The plate has constant thickness.

8. Rotary inertia terms are negligible.

9. There are no body forces.

10. Transverse shear stresses \( \tau_{xz} \) and \( \tau_{yz} \) vanish on plate surfaces \( z = \pm h/2 \).

A.2 Stress-strain relationship for each lamina

For a lamina of an orthotropic material under plane stress condition, the stress strain relationships in the principal material directions are as follows:

\[
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_{12}
\end{bmatrix} =
\begin{bmatrix}
Q_{11} & Q_{12} & 0 \\
Q_{12} & Q_{22} & 0 \\
0 & Q & Q_{16}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\gamma_{12}
\end{bmatrix}
\]

(A-1)

where \( Q_{ij} \) (reduced stiffnesses) may be written in terms of engineering constants in the following form:
\[ Q_{11} = E_1/(1 - \nu_{12}\nu_{21}) \]

\[ Q_{12} = \nu_{12}E_2/(1 - \nu_{12}\nu_{21}) = \nu_{21}E_1/(1 - \nu_{12}\nu_{21}) \quad (A.2) \]

\[ Q_{22} = E_2/(1 - \nu_{12}\nu_{21}) \]

\[ Q_{66} = G_{12} \]

If the principal material directions make an angle of \( \theta \) with the plate coordinates (see Fig. A.2), the stress-strain relationship in \( x \) and \( y \) directions may be written in the following form:

\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\sigma_{xy}
\end{bmatrix} =
\begin{bmatrix}
\bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\
\bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\
\bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{bmatrix} \quad (A.3)
\]

where \( \bar{Q}_{ij} \) (transformed reduced stiffnesses) may be written in terms of \( Q_{ij} \) (reduced stiffnesses) as follows:

![Fig. A.2 Lamination Angle](image-url)
\[ \bar{\sigma}_{11} = Q_{11} \cos^4 \sigma + 2(Q_{12} + 2Q_{66}) \sin^2 \epsilon \cos^2 \epsilon + Q_{22} \sin^4 \epsilon \]

\[ \bar{\sigma}_{12} = (Q_{11} + Q_{22} - 4Q_{66}) \sin^2 \epsilon \cos^2 \epsilon + Q_{12}(\sin^4 \epsilon + \cos^4 \epsilon) \]

\[ \bar{\sigma}_{22} = Q_{11} \sin^4 \epsilon + 2(Q_{12} + 2Q_{66}) \sin^2 \epsilon \cos^2 \epsilon + Q_{22} \cos^4 \epsilon \]

\[ \bar{\sigma}_{16} = (Q_{11} - Q_{12} - 2Q_{66}) \sin^3 \epsilon \cos \epsilon + (Q_{12} - Q_{22} + 2Q_{66}) \sin^3 \epsilon \cos \epsilon \]

\[ \bar{\sigma}_{26} = (Q_{11} - Q_{12} - 2Q_{66}) \sin^3 \epsilon \cos \epsilon + (Q_{12} - Q_{22} + 2Q_{66}) \sin^3 \epsilon \cos \epsilon \]

\[ \bar{\sigma}_{66} = (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66}) \sin^2 \epsilon \cos^2 \epsilon + Q_{66}(\sin^4 \epsilon + \cos^4 \epsilon) \]

(A.4)

Equation (A.3) represents the stress strain relationship for each lamina. For the kth layer of a multilayered laminate
the stress strain relationship in short notation could be written as:

\[ \{ \sigma \}_k = [\bar{Q}]_k \{ \varepsilon \}_k \]  \hspace{1cm} (A.5)

**A.3 Strain-displacement relations**

Using the Kirchhoff-Love hypothesis the displacements \( \bar{u}, \bar{v}, \) and \( \bar{w} \) of a laminate may be written in terms of middle plane displacements \( u, v, \) and \( w \) as follows:

\[
\begin{align*}
\bar{w} &= w \\
\bar{u} &= u - z(w_x) \\
\bar{v} &= v - z(w_y)
\end{align*}
\]  \hspace{1cm} (A.6)

Then the strains in terms of displacements of the mid-plane are:

\[
\begin{pmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{pmatrix} =
\begin{pmatrix}
u_x \\
v_y \\
u_y + v_x
\end{pmatrix} - z
\begin{pmatrix}
w_{xx} \\
w_{yy} \\
2w_{xy}
\end{pmatrix}
\]  \hspace{1cm} (A.7)

**A.4 Force and moment integrals**

The resultant inplane forces and bending moments are obtained by integrating the stresses in each layer of lamina
through the laminate thickness. Integrating equations (A.3) yields the following relations for the plate:

\[
\begin{bmatrix}
N_x \\
N_y \\
N_{xy} \\
M_x \\
M_y \\
M_{xy}
\end{bmatrix}
= \int \begin{bmatrix}
\frac{1}{2}
\\
\frac{1}{2}
\\
\frac{1}{2}
\\

\end{bmatrix}
\begin{bmatrix}
\sigma_{xx} \\
\sigma_{yy} \\
\sigma_{xy}
\end{bmatrix}
dz = \begin{bmatrix}
A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\
A_{12} & A_{22} & A_{16} & B_{12} & B_{22} & B_{26} \\
A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\
B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\
B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\
B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66}
\end{bmatrix}
\begin{bmatrix}
u_{x} \\
v_{y} \\
u_{x} + v_{y}
\end{bmatrix}
\begin{bmatrix}
x \\
y
2w_{xy}
\end{bmatrix}
\]

(A.8)

where

\[
(A_{ij}, B_{ij}, D_{ij}) = \int_{h/2}^{h/2} \sigma_{ij} \begin{bmatrix}
1, z, z^2
\end{bmatrix} dz
\]

(A.9)

Equations (A.9) for a N-layered laminate will simplify to:

\[
A_{ij} = \sum_{k=0}^{N} (\sigma_{ij})_k(z_k - z_{k-1})
\]

\[
B_{ij} = \frac{1}{2} \sum_{k=1}^{N} (\sigma_{ij})_k(z_k^2 - z_{k-1}^2)
\]

\[
D_{ij} = \frac{1}{3} \sum_{k=1}^{N} (\sigma_{ij})_k(z_k^3 - z_{k-1}^3)
\]

(A.10)
A.5 Vibration of laminated plates

The differential equations of motion in terms of the force and bending moment resultants, assuming infinitesimal deformation and neglecting rotary inertia are \(^{28}\):

\[
\begin{align*}
N_{xx'} + N_{xy,y} &= 0 \\
N_{xy,x} + N_{yy,y} &= 0 \\
M_{xx''} + 2M_{xy,xy} + M_{yy,y} + N_{w,xx} + 2N_{xy,w,xy} + \\
N_{y,w,yy} + q &= \rho w_{tt}
\end{align*}
\]  \(\text{(A.11)}\)

where a comma denotes differentiation of the principal symbol with respect to the subscript. Note that inertias associated with inplane displacement components, \(u\) and \(v\), are neglected. In equations (A.11) \(C\) represents transverse loading and for the problem of free vibration is zero. When the inplane loads are zero, equations (A.11) may be written in terms of displacements by using relations (A.8) and the differential equations for the free vibration of laminated plates may be written as follows:
\[ A_{11} u_{xx} + 2A_{16} u_{xy} + A_{66} u_{yy} + A_{16} v_{xx} + \]

\[ (A_{12} + A_{66}) v_{xy} + A_{26} v_{yy} - B_{11} w_{xxx} - \]

\[ 3B_{16} w_{xxy} - (3_{12} + 2B_{66}) w_{xyy} - B_{26} w_{yyy} = 0 \]

\[ A_{16} u_{xx} + (A_{12} + A_{66}) u_{xy} + A_{26} u_{yy} + A_{66} v_{xx} \]

\[ 2A_{26} v_{xy} + A_{22} v_{yy} - B_{16} w_{xxx} - (B_{12} + 2B_{66}) w_{xxy} \]

\[ - 3B_{26} w_{xyy} - B_{22} w_{yyy} = 0 \] (A.12)

\[ D_{11} w_{xxx} + 4D_{16} w_{xxy} + 2(D_{12} + 2D_{66}) w_{xyy} + \]

\[ 4D_{26} w_{xyy} + D_{22} w_{yyy} - B_{11} u_{xxx} - 3B_{16} u_{xxy} - \]

\[ (3_{12} + 2B_{66}) u_{xyy} - B_{26} u_{yyy} - B_{16} v_{xxx} - \]

\[ (3_{12} + 2B_{66}) v_{xxy} - 3B_{26} v_{xyy} - B_{22} v_{yyy} = \rho_{w}^{tt} \]
A.6 Stability of laminated plates

The governing differential equations for the buckling of laminated plates may be written as:

\[
\begin{align*}
N_{x,x}^b + N_{xy,y}^b &= 0 \\
N_{xy,x}^b + N_{y,y}^b &= 0 \quad \text{(A-13)} \\
M_{x,xx}^b + 2M_{xy,xy}^b + M_{yy,yy}^b + N_{w,xx}^i &= 0 \\
&\quad + 2N_{xy,w,xy}^i + N_{y,w,yy}^i = 0
\end{align*}
\]

where subscript "i" denotes stress resultants at the prebuckled state and subscript "b" denotes the resultant bending moment and force due to the buckling.
A.7 Boundary conditions

Since the differential equations for laminated plates are coupled and have order of eight, they require four boundary conditions to be stated per each edge. These boundary conditions are classified in three major sets: free (F), simply supported (S), and clamped (C), depending on transverse displacement conditions. In other words, these three conditions are chosen similar to those of isotropic plates. An edge is free if there is no restriction on $w$ along the edge. The edge is simply supported if $w = M_n = 0$ and it is clamped if $w = \gamma w/\partial n = 0$ ($n$ denotes normal to the edge).

Furthermore each of free, simply supported, or clamped boundary conditions will have four cases according to degree of inplane restraints. The notation used for each of these cases is summarized in Table A.1.

The notation introduced here is similar to those of Whitney's. It should be mentioned that these are not the only possibilities of the boundary conditions. Another type of boundary condition, almost never considered, would have $\partial w/\partial n = 0$ and $\gamma M_{ns}/\partial s + Q_n = 0$. Elastic supports would be other possible, more general, condition.
Table A.1 Notation used for boundary conditions along an edge. \( n \) and \( s \) denote normal and tangential to the plate edge. For buckling the conditions are in terms of variation from pre-buckled state.
Stacking sequence terminology is used to describe the orientation of the layers of a laminate. For laminates with equal thickness layers a listing of the layers and their orientation is used, for example, \((0/60/35)\). This notation refers to a laminate made of three layers of equal thickness. The principal material direction of the first layer (the layer at the top) makes an angle of \(0^\circ\) with the \(x\)-axis of the laminate. The principal material direction of the second layer makes an angle of \(60^\circ\) with the \(x\) coordinate and the third layer's angle of orientation is \(35^\circ\). Some simplifications in writing of the stacking sequence may be possible. For example, instead of writing \((0/17/22/22/22/15)\), one may write \((0/17/22_3/15)\), and for \((20/45/-45)\) one may write \((20/\pm45)\). For symmetric laminate the simplest representation of, for example, \((90/35/35/90)\) is \((90/35)_s\). A stacking sequence like \((90/35/90/35/35/90/35/90)\) may be written \((90/35)_2s\). Having the stacking sequence and the material properties of the layers one may find the stiffnesses easily by using the relations described in this Appendix.
APPENDIX B

ENFORCING THE GEOMETRIC BOUNDARY CONDITIONS
The displacement functions may be written as:

\[ u = \sum_{i=1}^{M} u_i \varphi_i(x, y) \]

\[ v = \sum_{i=1}^{N} v_i \Theta_i(x, y) \quad \text{(B.1)} \]

\[ w = \sum_{i=1}^{P} w_i \psi_i(x, y) \]

The Rayleigh-Ritz method requires that the functions \( \varphi_i \), \( \Theta_i \), and \( \psi_i \) are chosen such that they satisfy at least the geometric boundary conditions. However, any one of these functions may be written in terms of a linear combination of a complete set of functions as follows:

\[ \varphi_i = \sum_{m} a_m \bar{\varphi}_m \]

\[ \Theta_i = \sum_{n} b_n \bar{\Theta}_n \quad \text{(B.2)} \]

\[ \psi_i = \sum_{p} c_p \bar{\psi}_p \]
Then the equations (B.1) can be written;

\[ u = \sum_{i=1}^{N} u_i \left( \sum_{m} a_m \phi_m \right) \]
\[ v = \sum_{i=1}^{M} v_i \left( \sum_{n} b_n \theta_n \right) \]  
\[ w = \sum_{i=1}^{P} w_i \left( \sum_{p} c_p \psi_p \right) \]  

(B.3)

or

\[ u = \sum_{m} u'_m \phi'_m \]
\[ v = \sum_{n} v'_n \theta'_n \]  
\[ w = \sum_{p} w'_p \psi'_p \]  

(B.4)

where \( u'_m \), \( v'_n \), and \( w'_p \) are constrained and boundary conditions are satisfied. Depending on boundary conditions for each case, the constraint equations are known; therefore, it is straightforward to find the elements of the matrices of transformation relating \( u_i \) to \( u'_m \), \( v_i \) to \( v'_n \), and \( w_i \) to \( w'_p \).
Defining vector $g$ as

\[ g_i = u_i \quad 1 \leq i \leq M \]
\[ g_i = v_i \quad M \leq i \leq M + N \quad (B.5) \]
\[ g_i = w_i \quad M + N \leq i \leq M + N + P \]

Similarly vector $\bar{g}$ can be introduced in the following form:

\[ \bar{g}_i = u_i' \quad 1 \leq i \leq m \]
\[ \bar{g}_i = v_i' \quad m \leq i \leq m + n \quad (B.6) \]
\[ \bar{g}_i = w_i' \quad n + m \leq i \leq m + n + p \]

The relation between $\bar{g}$ and $g$ may be found from constraint equations, i.e.,

\[
\{g_i\} = [G]\{\bar{g}_i\}
\]

and

\[
\{g_i\}^T = \{\bar{g}_i\}^T [G]^T \quad (B.7)
\]
The energy functionals in the matrix form are written as:

\[ \bar{E} = \frac{1}{2} \{ g_1 \}^T \{ E \} \{ g_1 \} \] (B.8)

Then using equations (B.7), the functionals may be written in the following form:

\[ \bar{E} = \frac{1}{2} \{ \bar{g}_i \}^T \{ G \}^T \{ E \} \{ G \} \{ \bar{g}_i \} \] (B.9)

The advantage of equation (B.9) is that the matrix \( E \) may be evaluated from unconstrained displacement functions. Then by using the proper constraint equations for any given boundaries, the matrices \( G \) and \( G^T \) can be found.

This method has been applied successfully by Kalro\textsuperscript{70} for isotropic rectangular plates.
APPENDIX C

PROGRAM LISTING
IMPLICIT REAL*(A-H, O-Z)
INTEGER U1, U2, U3, U4, U5, U6, U7, U8
INTEGER V1, V2, V3, V4, V5, V6, V7, V8
INTEGER W1, W2, W3, W4, W5, W6, W7, W8
DIMENSION CC(100, 100), CT(100, 100), T(151, 151)
DIMENSION V(151, 151), CD(151, 151), CON(151, 151)
DIMENSION CONT(151, 151), A(3, 3), B(3, 3), D(3, 3)
DIMENSION WK(8000)
DIMENSION ROMGA(151), BETA(151), IS(151), WKI(151)
COMPLEX*16 ALFA(151), CMN(151, 151), OMEGA(151)
COMPLEX*16 OMEGA
COEF=1.0D0
PICRC=3.1415926535890D0

THIS PROGRAM IS DESIGNED TO SOLVE THE PROBLEM OF
VIBRATION AND BUCKLING OF LAMINATED COMPOSITE PLATES
AND CIRCULAR CYLINDRICAL SHELLS WITH ARBITRARY
BOUNDARY CONDITIONS. THE MAIN PROGRAM REQUIRES TWO
SUBROUTINES "EDGE" AND "MATDATA". THE DATA INPUT
FORMAT IS BRIEFLY EXPLAINED IN SUBROUTINE MATDATA.
HOWEVER THE MAIN PROGRAM HAS 4 READING STATEMENTS
AS FOLLOWS:
FIRST READING LINE HAS THE FORMAT 611. THE NUMBER
OF DEGREES OF FREEDOM OF EACH OF THE DISPLACEMENTS
ARE GIVEN HERE. FIRST DIGIT IS THE NUMBER OF TERMS
CHosen FOR TRANSVERSE DISPLACEMENT "W" IN THE X
DIRECTION AND THE SECOND ONE IS THE NUMBER OF TERMS
IN Y DIRECTION. 3RD AND 4TH CORRESPOND TO DEGREES OF
FREEDOM OF THE DISPLACEMENT COMPONENT U IN X AND Y
DIRECTIONS AND THE LAST TWO ARE THE NUMBER OF TERMS
FOR V. FOR EXAMPLE WHEN THE USER DECIDES TO HAVE
25(5x5) TERMS FOR W AND 12(3x4) TERMS FOR U AND
12(2x6) TERMS FOR V, THE FIRST INPUT LINE WILL BE
553426
THE NEXT THREE INPUT LINES ARE USED TO ENFORCE
BOUNDARY CONDITIONS. THE FORMAT FOR THESE THREE IS
811. FIRST ONE READS THE CONSTRAINTS ON TRANSVERSE
DISPLACEMENT W. FIRST COLUMN IS 1 IF THE DISPLACE-
MENT AT EDGE X=-a/2 IS CONSTRAINED OTHERWISE IT IS
0. THE SECOND COLUMN IS 1 IF THE SLOPE AT THE SAME
EDGE IS ZERO. THE NEXT THREE PAIRS OF COLUMNS DEAL
WITH THE THE OTHER THREE EDGES IN CLOCKWISE DIRECTION.
THE NEXT TWO LINES FOLLOW THE SAME ORDER FOR EDGE
CONDITIONS ON U AND V. FOR EXAMPLE IF THE USER
DECIDES TO SOLVE A C1S2F4C3 PROBLEM THE THREE LINES
INPUT WILL BE:
11100011
10100000
10000010
THE REST OF THE DATA SHOULD BE PROVIDED BY SUBROUTINE
MATDATA
CALL MATDATA(NX, NY, NXY, NVIB, ASR, BOR, HOB, A, B, D)
READ 803, IPRED, IQRED, MRED, NRED, KRED, LRED
803 FORMAT(61I1)
READ 999, W1, W2, W3, W4, W5, W6, W7, W8
READ 999, U1, U2, U3, U4, U5, U6, U7, U8
READ 999, V1, V2, V3, V4, V5, V6, V7, V8
999 FORMAT(8I1)
JAU = U3 + U4 + U7 + U8
JBU = U1 + U2 + U5 + U6
JAW = W3 + W4 + W7 + W8
JBW = W1 + W2 + W5 + W6
JAV = V3 + V4 + V7 + V8
JBV = V1 + V2 + V5 + V6
M1 = MRED + JBU
N1 = NRED + JAU
IGENU = MRED * NRED
K1 = KRED + JBV
L1 = LRED + JAV
IGENV = LRED * KRED
IP1 = IPRED + JBW
IQ1 = IQRED + JAW
IGENW = IPRED * IQRED
M1N1 = M1 * N1
K1L1 = K1 * L1
IQ1P1 = IQ1 * IP1
MTOT = M1N1 + K1L1 + IQ1P1
ILEFT = IGENU + IGENV + IGENW
IF (IGENU .EQ. 0) GO TO 60
CALL EDGE(U1, U2, U3, U4, U5, U6, U7, U8, M1, N1, CC, CT)
DO 501 I = 1, MIN1
DO 501 J = 1, IGENU
CON(I, J) = CC(I, J)
501 CONTINUE
IF (IGENV .EQ. 0) GO TO 61
CALL EDGE(V1, V2, V3, V4, V5, V6, V7, V8, K1, L1, CC, CT)
DO 502 I = 1, K1L1
DO 502 J = 1, IGENV
CON(I + M1N1, J + IGENU) = CC(I, J)
502 CONTINUE
IF (IGENW .EQ. 0) GO TO 62
CALL EDGE(W1, W2, W3, W4, W5, W6, W7, W8, IP1, IQ1P1, CC, CT)
DO 503 I = 1, IQ1P1
DO 503 J = 1, IGENW
IIP = I + M1N1 + K1L1
JJP = J + IGENU + IGENW
CON(IIP, JJP) = CC(I, J)
503 CONTINUE
IF (JJP .EQ. 0) GO TO 62
CALL EDGE(W1, W2, W3, W4, W5, W6, W7, W8, IP1, IQ1P1, CC, CT)
DO 503 I = 1, IQ1P1
DO 503 J = 1, IGENW
IIP = I + M1N1 + K1L1
JJP = J + IGENU + IGENW
CON(IIP, JJP) = CC(I, J)
503 CONTINUE
62 CONTINUE
IF((MPI/2)*2 .EQ. MPI) MMM=1
IF(MMM .EQ. 1 .OR. NNN .EQ. 1) GO TO 111
IF(MPI*NPJ .LT. 1) GO TO 111
R=A16*(I*N+J*M)/(MPI*NPJ)
111 CONTINUE
IF(MMM .EQ. 1 .OR. NNN .EQ. 1) GO TO 112
IF(MPI .LE. 1) GO TO 201
R=A11*I*M/((MPI-1)*(NPJ+1))
201 CONTINUE
IF(NPJ .LE. 1) GO TO 202
R=A66*J*N/((MPI+1)*(NPJ-1))+R
202 CONTINUE
112 R=4.0D0*R
V(IR,JR)=R
2 CONTINUE
DO 3 KC=1,K1
K=KC-1
DO 3 LC=1,L1
L=LC-1
R=0.0D0
JR=JR+1
MMM=0.0D0
NNN=0.0D0
T(IR,JR)=0.0D0
KPI=K+I
LPJ=L+J
IF((KPI/2)*2 .EQ. KPI) MMM=1
IF((LPJ/2)*2 .EQ. LPJ) NNN=1
IF(KPI*LPJ .LT. 1) GO TO 113
R=(A12*I*L+A66*j*K)/(KPI*LPJ)
211 CONTINUE
113 CONTINUE
IF(MMM .EQ. 1 .OR. NNN .EQ. 1) GO TO 114
IF(KPI .LE. 1) GO TO 212
R=R+A16*I*K/((KPI-1)*(LPJ+1))
212 CONTINUE
IF(LPJ .LE. 1) GO TO 213
R=R+A26*J*L/((KPI+1)*(LPJ-1))
213 CONTINUE
114 R=4R
3 CONTINUE
C
DO 4 IPC=1,IP1
IP=IPC-1
DO 4 IQC=1,IQ1
IQ=IQC-1
JR=JR+1
MMM=0
NNN=0
R=0.0D0
T(IR,JR)=0.0D0
IPPI=IP+I
IQPJ=IQ+J
IF((IPPI/2) * 2 .EQ. IPPI) MMM=1
IF((IQPJ/2) * 2 .EQ. IQPJ) NNN=1
IF(MMM .EQ. 1 .AND. NNN .EQ. 1) GO TO 115
IF(MMM .NE. 1 .AND. NNN .NE. 1) GO TO 115
IF(MMM .EQ. 0) GO TO 116
IF(IPPI .LE. 1) GO TO 205
R=B16*J*IP*(IP-1)/(IPPI-1)*IQPJ)
205 CONTINUE
IF(IQPJ .LE. 2) GO TO 206
R=R+B26*J*IQ*(IQ-1)/(IPPI+1)*(IQPJ-2))
206 CONTINUE
IF(IPPI .LE. 1) GO TO 207
R=0.0D0*B16*IP*IQ/((IPPI-1)*IQPJ)+R
207 CONTINUE
IF(IQPJ .EQ. 0) GO TO 115
R=R-A26*TET*J/((IPPI+1)*IQPJ)
GO TO 115
116 CONTINUE
IF(IPPI .LE. 2) GO TO 208
R=B11*IP*(IP-1)/((IPPI-2)*IQPJ+1))
208 CONTINUE
IF(IQPJ .LE. 1) GO TO 209
R=R+B12*I*IQ*(IQ-1)/(IPPI*(IQPJ-1))
R=2.0D0*B66*J*IP*IQ/(IPPI*(IQPJ-1))+R
209 CONTINUE
IF(IPPI .EQ. 0) GO TO 115
R=R-A12*TET*I/((IPPI)*(IQPJ+1))
115 CONTINUE
R=-4.0D0*IR
V(IR,JR)=R
4 CONTINUE
1 CONTINUE

C

IF(K1L1.LE.0) GO TO 11
DO 11 IC=1,K1
I=IC-1
DO 11 JC=1,L1
J=JC-1
IR=IR+1
JR=0
DO 12 MC=1,M1
M=MC-1
DO 12 NC=1,N1
N=NC-1
JR=JR+1
R=0.0D0
MMM=0
NNN=0
T(IR, JR)=0.0D0
MPI=M+I
NPJ=N+J
IF((MPI/2)*2 .EQ. MPI) MMM=1
IF((NPJ/2)*2 .EQ. NPJ) NNN=1
IF(MMM .EQ. 1 .OR. NNN .EQ. 1) GO TO 117
IF(MPI*NPJ .LT. 1) GO TO 214
R=(A12*J*M+A66*I*N)/(MPI*NPJ)
214 CONTINUE
117 IF(MMM .EQ. 0 .OR. NNN .EQ. 0) GO TO 118
IF(MPI .LT. 1) GO TO 215
R=A16*I*M/((MPI-1)*(NPJ+1))
215 CONTINUE
IF(NPJ .LE. 1) GO TO 216
R=R+A26*J*N/((MPI+1)*(NPJ-1))
216 CONTINUE
118 R=4.0D0*R
V(IR, JR)=R
12 CONTINUE
C
DO 13 KC=1,K1
K=KC-1
DO 13 LC=1,L1
L=LC-1
JR=JR+1
R=0.0D0
T(IR, JR)=0.0D0
MMM=0
NNN=0
LPJ=L+J
KPI=K+I
IF((KPI/2)*2 .EQ. KPI) MMM=1
IF((LPJ/2)*2 .EQ. LPJ) NNN=1
IF(MMM .EQ. 1 .OR. NNN .EQ. 1) GO TO 119
IF(KPI*LPJ .LT. 1) GO TO 217
R=A26*(I*L+J*K)/(KPI*LPJ)
217 CONTINUE
119 CONTINUE
IF(MMM .EQ. 0 .OR. NNN .EQ. 0) GO TO 121
IF(LPJ .LE. 1) GO TO 218
R=A22*J*L/((KPI+1)*(LPJ-1))
218 CONTINUE
IF(KPI .LE. 1) GO TO 219
R=R+A66*I*K/((KPI-1)*(LPJ+1))
219 CONTINUE
121 R=4.0D0*R
V(IR, JR)=R
13 CONTINUE
DO 14 IPC=1, IP1
IP=IPC-1
DO 14 I QC=1, IQ1
IQ=IQC-1
JR=JR+1
MMM=0
NNN=0
R=0.0D0
T(IR, JR)=0.0D0
IPPI=IP+I
IQPJ=IQ+J
IF((IPPI/2)*2 .EQ. IPPI) MMM=1
IF((IQPJ/2)*2 .EQ. IQPJ) NNN=1
IF(MMM .EQ. 0 .OR. NNN .EQ. 1) GO TO 122
IF(IQPJ .LE. 2) GO TO 221
R=B12*J*IP*(IP-1)+2.0D0*B66*I*IP*IQ
R=R/((IPPI-1)*IQPJ)
221 CONTINUE
R=R+B22*J*IQ*(IQ-1)/((IPPI+1)*(IQPJ-2))
IF(IQPJ .EQ. 0) GO TO 122
R=R-A22*TET*J/((IPPI+1)*IQPJ)
122 CONTINUE
IF(MMM .EQ. 1 .OR. NNN .EQ. 0) GO TO 123
IF(IPPI .LE. 2) GO TO 222
R=B16*I*IP*(IP-1)/((IPPI-2)*(IQPJ+1))
222 CONTINUE
IF(IPPI .EQ. 0 .OR. IQPJ .LE. 1) GO TO 223
R=R+(B26*I*IQ*(IQ-1)+2.*B26*J*IP*IQ)/(IPPI*(IQPJ-1))
223 CONTINUE
IF(IPPI .EQ. 0) GO TO 123
R=-A26*TET*I/(IPPI*(IQPJ+1))
123 R=-4.0D0*R
V(IR, JR)=R
14 CONTINUE
11 CONTINUE
C
DO 21 IC=1, IP1
I=IC-1
DO 21 JC=1, IQ1
J=JC-1
JR=0
IR=IR+1
IF(MIN1 .LE. 0) GO TO 22
DO 22 MC=1, M1
M=MC-1
DO 22 NC=1, N1
N=NC-1
JR=JR+1
MMM=0
NNN=0
R=0.0D0
T(IR, JR)=0.0D0
MPI = M + I
NPJ = N + J

IF((MPI/2)*2 .EQ. MPI) MMM = 1
IF((NPJ/2)*2 .EQ. NPJ) NNN = 1
IF(MMM .EQ. 1 .OR. NNN .EQ. 1) GO TO 124
IF(MPI .LE. 2) GO TO 237

R = B11*I*(I-1)*M/((MPI-2)*(NPJ+1))

237 CONTINUE
IF(MPI .LE. 2) GO TO 239
R = R + (B12*J*(J-1)*M + 2.*B66*I*J*N)/(MPI*(NPJ-1))

239 CONTINUE
IF(MPI .EQ. 0 .OR. NPJ .LE. 1) GO TO 124
R = R - A12*TET*M/(MPI*(NPJ+1))

124 CONTINUE
IF(MMM .NE. 1 .OR. NNN .EQ. 1) GO TO 125
IF(MPI .LE. 1 .OR. NPJ .EQ. 0) GO TO 226
R = B16*I*(I-1)*N + 2.*B16*I*J*M/((MPI-1)*NPJ)

226 CONTINUE
IF(LPJ .LE. 2) GO TO 227
R = R + B26*J*(J-1)*L/((KPI+1)*(NPJ-2))

227 CONTINUE
IF(LPJ .EQ. 0) GO TO 126
R = R - A26*TET*L/((KPI+1)*LPJ)

125 CONTINUE

133
IF(MMM .EQ. 1 .OR. NNN .NE. 1) GO TO 127
IF(KPI .LE. 2) GO TO 228
R=B16*I*(I-1)*K/((KPI-2)*(LPJ+1))

228 CONTINUE
IF(LPJ .LE. 1) GO TO 229
IF(KPI .LE. 0) GO TO 229
R=R+(B26*J*(J-1)*K+2*B26*I*J*L)/(KPI*(LPJ-1))

229 CONTINUE
IF(KPI .EQ. 0) GO TO 127
R=-A26*TET*K/(KPI*(L+J+1))+R

127 CONTINUE
R=-4.0D0*R
V(IR,JR)=R

23 CONTINUE
DO 24 IPC=1,IP1
IP=IPC-1
DO24 IQC =1,IQ1
IQ=IQC-1
JR=JR+1
MMM=0
NNN=0
T(IR,JR)=0.0D0
R=0.0D0
TNX=0.0D0
TNY=0.0D0
TNXY=0.0D0
TVIB=0.0D0
IPPI=IP+1
IOPJ=IQ+J
IF((IPPI/2)*2 .EQ. IPPI) MMM=1
IF((IOPJ/2)*2 .EQ. IOPJ) NNN=1
IF(MMM .NE. 1 .OR. NNN .NE. 1) GO TO 128
IF(NVIB .NE. 1) GO TO 413
TVIB=1.0D0/(4.0D0*(IPPI+1)*(IOPJ+1))

413 CONTINUE
IF(NX .NE. 1) GO TO 402
TNX=1.0D0*I/IP/((IPPI-1)*(IOPJ+1))

402 CONTINUE
IF(NY .NE. 1) GO TO 403
TNY=1.0D0*J/IP/((IPPI+1)*(IOPJ-1))
TNY=ASR2*TNY

403 CONTINUE
IF(IPPI .LE. 1 .OR. IOPJ .LE. 1) GO TO 231
R=D12*(I*(I-1)*IQ*(IQ-1)+J*(J-1)*IP*(IP-1))
R=(R+4.*D66*I*J*IP*IQ)/((IPPI-1)*(IOPJ-1))

231 CONTINUE
IF(IPPI .LE. 3) GO TO 232
R=D11*I*(I-1)*IP*(IP-1)/((IPPI-3)*(IOPJ+1))+R

232 CONTINUE
IF(IOPJ .LE. 3) GO TO 232
R=D11*I*(I-1)*IP*(IP-1)/((IPPI-3)*IOPJ+1))+R

232 CONTINUE
IF(IQPJ .LE. 3) GO TO 233
R=R+D22*J*(J-1)*IQ*(IQ-1)/((IPPI+1)*(IQPJ-3))

233 CONTINUE
R=R+A22*TET*TET/((IPPI+1)*(IQPJ+1))
IF(IPPI .LE. 1) GO TO 243
R=R-B12*TET*IP*(IP-1)/((IPPI-1)*(IQPJ+1))
R=R-B12*TET*I*(I-1)/((IPPI-1)*(IQPJ+1))
R=R-2*B26*TET*IP*(IP-1)/((IPPI-1)*(IQPJ+1))

243 CONTINUE
IF(IQPJ .LE. 1) GO TO 244
R=R-B22*TET*J*(J-1)/((IPPI+1)*(IQPJ-1))
R=R-B22*TET*IQ*(IQ-1)/((IP+I+1)*(IQPJ-1))

244 CONTINUE
128 CONTINUE
IF(MMM .EQ.1 .OR. NNN .EQ. 1) GO TO 129
IF(NXY .NE. 1) GO TO 405
IF(IPPI .LE. 0 .OR. IQPJ .LE. 0) GO TO 405
TNXY=ASR3*(J*IP+I*IQ)/(IPPI*IQPJ)

405 CONTINUE
IF(IPPI .LE. 2 .OR. IQPJ .EQ. 0) GO TO 234
R=D16*(I*(I-1)*IP*IQ+I*J*IP*(IP-1))/((IPPI-2)*IQPJ)

234 CONTINUE
IF(IPPI .EQ.0 .OR. IQPJ .LE. 2) GO TO 235
R=R+D26*(J*(J-1)*IP*IQ+I*J*IQ*(IQ-1))/(IPPI*(IQPJ-2))

235 CONTINUE
R=2.*D0*R
129 CONTINUE
R=4.0D0*R
V(IR,JR)=R
T(IR,JR)=TNX+TNY+TNXY+TVIB

24 CONTINUE
21 CONTINUE

C ENFORCING CONSTRAINTS

CALL VMULFF(CONT,T,ILEFT,MTOT,MTOT,151,151,CD,151,IER)
CALL VMULFF(CD,CON,ILEFT,MTOT,ILEFT,151,151,T,151,IER)
CALL VMULFF(CONT,V,ILEFT,MTOT,MTOT,151,151,CD,151,IER)
CALL VMULFF(CD,CON,ILEFT,MTOT,MTOT,151,151,V,151,IER)

C SOLVING EIGENVALUE PROBLEM

CALL EIGZF(V,150,T,150,ILEFT,2,ALFA,BETA,CMN,150,WK,IER)
DO 902 I=1,ILEFT

IF(BETA(I) .EQ. 0) BETA(I)= 1.D-50
OMGA(I)=ALFA(I)/BETA(I)
OMEGA=OMGA(I)
IF(NVIB .EQ. 0) OMEGA=CDSQRT(OMEGA)
ROMGA(I)=OMEGA

902 CONTINUE
CALL VSRTPD(ROMGA,ILEFT,IS)
PRINT 995, (ROMGA(I), I=1, ILEPT)
995 FORMAT (8E12.5)
STOP
END
SUBROUTINE EDGE(L1, L2, L3, L4, L5, L6, L7, L8, MAX, NAX, CC, CT)

THIS SUBROUTINE COSTRUCTS THE CONSTRAINTS MATRIX

IMPLICIT REAL*8(A-H, O-Z)
DIMENSION CC(100,100), CM(40,100), AA(40,100), BB(40,40)
DIMENSION BBI(40,40), C(40,100), CT(100,100), T(100,100)
DIMENSION WKRA(40)
JA=L3+L4+L7+L8
JB=L1+L2+L5+L6
IM=MAX-JB
IN=NAX-JA
IGEN=IM*IN
MMAX=MAX*NAX
NA=NAX-JA
JGAN=MAX*JA
IGAN=MMAX-JGAN
BC1=L5
BC2=L6
BC3=L1
BC4=L2
BC5=L7
BC6=L8
BC7=L3
BC8=L4
DO 320 I=1, IGAN
DO 320 J=1, IGAN
CC(I,J)=0.0D0
IF(I .EQ. J) CC(I, J) = 1.0D0
320 CONTINUE
IF(JA .EQ. 0) GO TO 350
I=0
IF(BC5 .NE. 1) GO TO 440
DO 440 IK=I, MAX
K=IK-1
I=I+1
J=0
DO 440 KN=1, NAX
N=KN-1
DO 440 KM=1, MAX
M=KM-1
J=J+1
CM(I,J)=0.0D0
IF(M .EQ. K) CM(I, J) = 1.0D0
440 CONTINUE
IF(BC6 .NE. 1) GO TO 450
DO 450 IK=1, MAX
K=IK-1
I=I+1
J=0
DO 450 KN = 1, NAX
N = KN - 1
DO 450 KM = 1, MAX
M = KM - 1
J = J + 1
CM(I, J) = N
450 CONTINUE
IF (BC7 .NE. 1) GO TO 460
DO 460 KN = 1, NAX
N = KN - 1
DO 460 KM = 1, MAX
M = KM - 1
J = J + 1
CM(I, J) = 0.0 DO
IF (M .EQ. K) CM(I, J) = ((-1)**N)
460 CONTINUE
IF (BC8 .NE. 1) GO TO 470
DO 470 IK = 1, MAX
K = IK - 1
I = I + 1
J = 0
DO 470 KN = 1, NAX
N = KN - 1
DO 470 KM = 1, MAX
M = KM - 1
J = J + 1
CM(I, J) = 0.0 DO
IF (M .EQ. K) CM(I, J) = ((-1)**N) * N
470 CONTINUE
DO 300 I = 1, JGAN
DO 300 J = 1, IGAN
300 AA(I, J) = -CM(I, J)
DO 310 I = 1, JGAN
DO 310 J = 1, IGAN
JJ = IGAN + J
310 BB(I, J) = CM(I, JJ)
CALL LINVIF(BB, JGAN, 40, BBI, 0, WKREA, IER)
CALL VMULFP(BBI, AA, JGAN, JGAN, IGAN, 40, 40, C, 40, IER)
DO 321 I = 1, JGAN
II = IGAN + I
DO 321 J = 1, IGAN
321 CC(II, J) = C(I, J)
DO 330 J = 1, IGAN
I = 0
DO 330 II = 1, MAX
DO 330 IP = 1, NAX
I = I + 1
IQ = II + (IP - 1) * MAX
330 CT(I, J) = CC(IQ, J)
I=1-I
N=KN-1
X
N=MAX
Do 420 KN=1',MAX
J=0
I=1+I
Do 420 KI=1',MAX
If (RCS. NE. I) Go to 420
Continue
If (RNS. RQ. II) CM(I)',F) = 1,00
J=1+I
CM(I)',F) = 0,00
N=KN-1
X
N=MAX
Do 420 KN=1',MAX
J=0
I=1+I
Do 420 KI=1',MAX
If (RCS. NE. I) Go to 420
Continue
If (RNS. RQ. II) CM(I)',F) = 1,00
J=1+I
CM(I)',F) = 0,00
N=KN-1
X
N=MAX
Do 420 KN=1',MAX
J=0
I=1+I
Do 420 KI=1',MAX
If (RCS. NE. I) Go to 700
Continue
350 CONTINUE
340 CC(I)',F) = (I)',F)
J=I+I
Do 30 JF=1',MAX
J=0
I=1+I
Do 30 JF=1',MAX
J=0
Do 30 JI=1',MAX
NA=MAX-7A
IF(BC4 .NE. 1) GO TO 430
DO 430 IL=1, NA
L=IL-1
I=I+1
J=0
DO 430 KM=1, MAX
M=KM-1
DO 430 KN=1, NAX
N=KN-1
J=J+1
CM(I,J)=0.0D0
IF(KN .EQ. IL) CM(I,J)=((-1)**M)*M
430 CONTINUE
JJEN=JB*NA
CALL VMULPF(CM, CC, JJEN, MMAX, IGAN, 40, 100, C, 40, IER)
DO 600 I=1, JJEN
DO 600 J=1, IGEN
600 AA(I,J)=-C(I,J)
DO 610 I=1, JJEN
DO 610 J=1, IGEN
610 BB(I,J)=C(I,J)
CALL LINVIF(BB, JJEN, 40, BBI, 0, WKAREA, IER)
DO 620 I=1, IGEN
DO 620 J=1, IGEN
CT(I,J)=0.0D0
IF(I .EQ. J) CT(I,J)=1.0D0
620 CONTINUE
CALL VMULFF(BBI, AA, JJEN, JJEN, IGEN, 40, 40, C, 40, IER)
DO 621 I=1, JJEN
II=I+IGEN
DO 621 J=1, IGEN
621 CT(II,J)=C(I,J)
CALL VMULFF(CC, CT, MMAX, IGAN, IGEN, 100, 100, TI, 100, IER)
DO 650 I=1, MMAX
DO 650 J=1, IGEN
CC(I,J)=TI(I,J)
650 CT(J,I)=TI(I,J)
700 CONTINUE
DO 660 I=1, MMAX
DO 660 J=1, IGEN
660 CT(J,I)=CC(I,J)
RETURN
END
SUBROUTINE MATDATA(NX,NY,NXY,NVIB,ASR,BOR,HOB,A,B,D)
IMPLICIT REAL*8(A-H,O-Z)
DIMENSION A(3,3),B(3,3),D(3,3)

THIS SUBROUTINE IS NEEDED TO PROVIDE THE DATA
REQUIRED BY THE MAIN PROGRAM. USER HAS THE
CHOICE OF ORGANIZING THE INPUT PARAMETERS IN THIS
SUBROUTINE. THE PARAMETERS NEEDED TO BE DEFINED
ARE AS FOLLOWS:

NX:
IF THE CASE OF UNIAXIAL BUCKLING IS TO BE SOLVED
NX SHOULD BE GIVEN 1 (NX=1)
NOTE THAT THE LOADING FOR THIS CASE IS IN THE X
DIRECTION.

NY:
IF THE PROBLEM OF UNIAXIAL BUCKLING IS BEING
SOLVED, NY=1 (UNIAXIAL LOADING IN Y-DIRECTION)

NXY:
NXY=1, IF THE PLATE IS UNDER SHEAR LOADING

IF NATURAL FREQUENCIES ARE TO BE FOUND THEN NX, NY,
AND NXY ALL SHOULD BE TAKEN AS ZERO.
FOR THE VIBRATION PROBLEM NVIB=1.

FOR BUCKLING NVIB=0

ASR:
ASPECT RATIO

BOR:
WIDTH TO RADIUS RATIO (REQUIRED FOR SHELLS)

HOB:
THICKNESS TO WIDTH RATIO (REQUIRED FOR SHELLS

A, B, AND D MATRICES ARE DEFINED TO SUPPLY THE
STIFFNESSES OF THE LAMINATE.

A(1,1)=A11 A(1,2)=A12 A(1,3)=A16
A(2,2)=A22 A(2,3)=A26
A(3,3)=A66

B(1,1)=B11 B(1,2)=B12 B(1,3)=B16
B(2,2)=B22 B(2,3)=B26
B(3,3)=B66

D(1,1)=D11 D(1,2)=D12 D(1,3)=D16
D(2,2)=D22 D(2,3)=D26
D(3,3)=D66

NOTE THAT THE REST OF THE ELEMENTS OF THESE MATRICES
ARE NOT NEEDED.