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Frangos, Nicholas Efstratios

ON CONVERGENCE AND REGULARITY OF VECTOR-VALUED PROCESSES
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ON CONVERGENCE AND REGULARITY OF VECTOR-VALUED
PROCESSES Indexed BY DIRECTED SETS

Dissertation

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By
Nicholas E. Frangos, B.S., M.S.

* * * * *

The Ohio State University
1984

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I love you.
ACKNOWLEDGEMENTS

It is my pleasure to express my sincere thanks to my advisor Professor Louis Sucheston, for his guidance and encouragement. My thanks are also extended to Professors William Davis and Gerald Edgar for their interest in my dissertation.

My stay and studies here, in the United States, would have been impossible without the inexhaustible support from my family. Their warmth and encouragement reached me from overseas.

Last, but not least, I would like to thank Stella for sharing with me the joys and disappointments of my graduate studies.
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INTRODUCTION

In this dissertation we are concerned with three different topics. However all three topics are related in that they deal with convergence and regularity of vector-valued martingales and asymptotic martingales on directed sets. Below we give a brief summary of the results obtained. A more detailed account of the results can be found in the introductions of the three chapters.

In Chapter I we investigate the convergence of vector-valued pramarts and subpramarts indexed by directed sets, \((X_t, \mathcal{F}_t, t \in J)\). The notion of real-valued pramart and subpramart was introduced by Millet and Sucheston [45], generalizing that of martingale and amart. We show here that \(L_1\)-bounded, positive subpramarts taking values in a Banach lattice converge in norm if (and only if) the lattice has the Radon-Nikodym property (RNP). The same is true for Banach space-valued pramarts. We assume that the filtration \((\mathcal{F}_t)\) satisfies the Vitali covering condition \(V\). For \(J = \mathbb{N}\), a number of authors have investigated the same questions and have obtained partial or complete
results: Millet-Sucheston [44], Egghe [23, 24], Slaby [57, 58], Talagrand [64]. (See Section I-1. for more details.)

Chapter II is concerned with continuous parameter vector-valued processes \((X_t, \mathcal{F}_t, t \in \mathbb{R}^+)\). For the study of the paths of stochastic processes the upcrossing arguments are available in the real-valued case but not in the vector-valued case. Several authors, Doob [18], Dellacherie-Meyer [15], Edgar-Sucheston [21], investigated the regularity of the paths of real-valued processes by studying convergence of sequences \((X_{\tau_n})\) where \((\tau_n)\) is a monotone sequence of stopping times. Using the convergence results of Chapter I and the Kadec-Klee renorming theorem we obtain regularity properties for continuous parameter vector-valued, positive, submartingales and pramarts, provided that the vector space has RNP. The class of pramarts contains the class of martingales, quasimartingales and uniform amarts. Results in this direction were previously obtained by Pellaumail [53] for quasimartingales and Choi-Sucheston [13] for uniform amarts.

In Chapter III we introduce and investigate block martingales and submartingales \((X_t, \mathcal{F}_t, t \in J^m)\) (in particular, martingales and submartingales under the commutation assumption), and give some applications. The results of this chapter were obtained jointly with Professor Louis Sucheston. It is shown that under suitable conditions on the filtration \((\mathcal{F}_t, t \in J^m)\), \(L \log^{m-1} L\) bounded block martingales
essentially converge and $L \log^{m-1} L$ bounded, block submartingales essentially upper demiconverge. It is also shown that positive block martingales essentially lower demiconverge. The results are a consequence of a general theorem about Banach space-valued processes. Our approach yields applications of the same convergence theorem to differentiation of integrals in $\mathbb{R}^m$ and also to multiparameter Marcinkiewicz strong laws of large numbers, Edgar-Sucheston [22]. In the commutation case proofs are simplified by reducing demiconvergence to convergence.
CHAPTER I

CONVERGENCE OF VECTOR VALUED PRAMARTS AND SUBPRAMARTS

1-1. Introduction

In [45] Millet and Sucheston introduced the notion of pramart and subpramart indexed by directed sets, generalizing that of martingale and submartingale, and studied their properties. In particular convergence theorems were proved. In this chapter we obtain convergence theorems for analogous Banach-valued processes.

Let \((E, \mathcal{X}, \mathcal{F}, \tau)\) be a Banach lattice with the Radon-Nikodym property (RNP). Let \(\{X_t : t \in J\}\) be an \(E\)-valued, positive subpramart of class \((d)\), i.e., such that \(\lim \inf_{t \uparrow \tau} E\|X_t\| < \infty\).

In [24], for \(J = \mathbb{N}\), Egghe proved a subpramart convergence theorem under the additional assumption that there is a subsequence \(\{n_k\} \subset \mathbb{N}\) such that \((\int_A X_{n_k} \, d\mu)\) converges weakly for each \(A \in \mathcal{F} \cap \mathcal{F}_n\).
In [57], for \( J = \mathbb{N} \), Slaby proved a subpramart convergence theorem assuming that the Banach lattice \( E \) has an unconditional basis and no subspace of \( E \) is isomorphic to \( c_0 \). In [58] he drops the assumption about the unconditional basis.

In the present work, independently of [58], we prove the convergence result in the original, more general, setting of subpramarts adapted to a stochastic basis \((\mathcal{F}_t, t \in J)\) satisfying the Vitali condition.

Let now \((E, \mathcal{F}, \mathbb{P})\) be a Banach space with RNP. Let \((X_t, \mathcal{F}_t, t \in J)\) be an \( E \)-valued pramart of class \((d)\).

In [23], for \( J = \mathbb{N} \), Egghe proved a pramart convergence theorem under the assumption that the sequence \((X_n)\) has a uniformly integrable subsequence.

In [58], for \( J = \mathbb{N} \), Slaby proved a pramart convergence theorem assuming that the Banach space \( E \) is weakly sequentially complete.

The method used here for the convergence of subpramarts also applies to pramarts provided that \( E \) is a separable dual. Thus we could prove that if \((X_t)\) is a pramart of class \((d)\) taking values in a Banach space that is a separable dual then strong a.s. convergence obtains.

Subsequently, Talagrand [64], was able to prove the pramart convergence theorem for \( J = \mathbb{N} \), assuming only that the Banach space \( E \) has RNP.

Very recently, Ghoussoub and Maurey [29] proved that RNP is equivalent
to asymptotic norming property. Using their result we prove here that 
$E$-valued pramarts $(X_t, \mathcal{F}_t)$ of class (d) converge strongly a.s. if $E$
has RNP and $(\mathcal{F}_t)$ satisfies the Vitali condition $V$.

Section 2 gives basic definitions and recalls known results. In 
section 3 we develop the method for proving the subpramart convergence 
theorem. The main steps are two. First, using an extension of a lemma 
of Neveu, we prove that if $(X_t)$ is an $E^+$-valued subpramart of class 
(d) that converges to $X$ scalarly then $\|X_t\|$ converges to $\|X\|$. In 
the second step we identify the limit $X$, and using the Davis-
Ghoussoub-Lindenstrauss renorming theorem we obtain the convergence in 
norm. Some convergence results in order-continuous Banach lattices are 
also included. In section 4 we prove the pramart convergence theorem. 
A convergence result for reversed pramarts is also given, used in the 
following chapter to obtain convergence of positive vector-valued 
reversed submartingales.

I-2. Definitions and Basic Notions

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and $J$ a directed 
set filtering to the right with a countable cofinal subset. The relations 
are modulo sets of measure zero. The words almost surely may or 
may not be omitted. Let $(\mathcal{F}_t, t \in J)$ be an increasing family of 
sub-$\sigma$-fields of $\mathcal{F}$. A function $\tau : \Omega \to J$ is a simple stopping time
of \((\mathcal{F}_t)\) if it takes finitely many values and \(\{\tau = t\}\in\mathcal{F}_t\) for all \(t\in T\). Let \(T\) denote the set of all simple stopping times; under the natural order \(T\) is a set filtering to the right. For \(\sigma \in T\), write \(E^\sigma(\cdot)\) for \(E(\cdot|\mathcal{F}_\tau)\).

Let \((E,\mathbb{P})\) be a Banach lattice and \((X_t, t \in J)\) a family of \(E\)-valued, Bochner integrable random variables adapted to \((\mathcal{F}_t)\). The stochastic (in probability) limit is denoted by \(\text{s limit}\). The family \((X_t)\) is called a submartingale if \(\text{s lim} \left( X_{\sigma} - E^\sigma(X_{\tau}) \right)^+ = 0 \), \(\sigma \leq \tau; \sigma, \tau \in T\)\) i.e., for every \(\varepsilon > 0\) there exists \(\sigma_0 \in T\) such that \(\sigma_0 \leq \sigma \leq \tau\) implies

\[ P(\{ |X_{\sigma_0} - E^\sigma(X_{\tau})|^+ > \varepsilon \}) < \varepsilon. \]

Clearly for \(J = \mathbb{N}\) or \(\mathbb{R}^+\) every martingale is a submartingale.

Moreover, every positive adapted family \((Z_t, t \in J)\) with \(\text{s lim} \left( Z_{\tau} \right) = 0\) is obviously a submartingale.

Let \((E^\prime)^+\) denote the positive cone of the dual \(E^\prime\) of \(E\). If \((X_t)\) is a positive submartingale, then \((\|X_t\|)\) is a real-valued positive submartingale, i.e., satisfies \(\text{s lim} \left( \|X_{\sigma} - E^\sigma(X_{\tau})\| \right)^+ = 0 \), \(\sigma \leq \tau; \tau \in T\).
Indeed if $\sigma \leq \tau$, $\sigma, \tau \in \Gamma$, then for every $\varepsilon > 0$

$$\{\|X - E^\sigma(X)\| < \varepsilon\} \subseteq \{(\|X\| - E^\sigma(X))_+ < \varepsilon\} \subseteq \{(\|X\| - E^\sigma_+ X)_+ < \varepsilon\}. \quad (1)$$

Also $(\chi(X_t))$ is a real-valued submartingale for all $X \in (E')_+$. 

For an arbitrary Banach space $(E, \|\cdot\|)$, the family $(X_t, \tau \in \Gamma)$ is called a martingale if

$$\lim_{\tau \downarrow \sigma} (X_{\sigma} - E^\sigma(X)) = 0, \quad \text{i.e.,} \quad \sigma \leq \tau, \sigma, \tau \in \Gamma$$

for every $\varepsilon > 0$ there exists $\sigma \in \Gamma$ such that $\sigma < \sigma \leq \tau$ implies

$$P(\{\|X_{\sigma} - E^\tau(X)\| > \varepsilon\}) \leq \varepsilon. \quad (2)$$

Clearly the class of martingales is closed under linear combinations and contains the class of uniform amarts [5, 19].

If $(X_t)$ is a martingale then $(\|X_t\|)$ is a real-valued positive submartingale. Indeed, if $\sigma \leq \tau$, $\sigma, \tau \in \Gamma$, then for every $\varepsilon > 0$

$$\{\|X_{\sigma} - E^\tau(X)\| < \varepsilon\} \subseteq \{(\|X\| - E^\tau_+ X)_+ < \varepsilon\}. \quad (3)$$

Also $(\chi(X_t))$ is a real-valued martingale for all $X \in E'$. 

A filtration \((\mathcal{F}_t, t \in J)\) satisfies the **Vitali condition** \(V\) if for every adapted family of sets \((A_t)\) and for every \(\varepsilon > 0\), there exists a simple stopping time \(\tau \in T\) such that

\[
P(\limsup_{\tau \in T} A \setminus A_{\tau}) < \varepsilon.
\]

(The above formulation of the Vitali condition was given by Millet and Sucheston [46] page 344; see also Neveu [52] page 99 for an equivalent form not involving stopping times, first introduced by Krickeberg [36].)

It was observed in [52], page 100 that a filtration totally ordered by set inclusion satisfies \(V\). In particular this is the case if \(J = \mathbb{N}\).

We now state some known results for reference.

**Theorem I-2.1.** (Millet-Sucheston) [45] Let \((X_t, t \in J)\) be a real-valued, positive, integrable process. Then \((X_t)\) is a submartingale if and only if there exists a positive submartingale \((R, \mathcal{F}_\tau, \tau \in T)\) such that for every \(t\), \(R_{\tau} \leq X_{\tau}\) a.s. and if \(Z = X_t - R_t\) then

\[
s\lim_{\tau \in T, \tau > t} Z = 0. \quad R_t \text{ is given by: } \inf_{\tau \in T} E^t(X_{\tau}).
\]

To say that \((R, \mathcal{F}_\tau, \tau \in T)\) is a submartingale it is the same as saying that \((R_t, \mathcal{F}_t, t \in J)\) is a submartingale with the optional sampling property.
Theorem I-2.2. [45]. Let \( (X_t) \) be a real-valued submartingale which satisfies the assumption (d):

\[
(d): \quad \liminf_j E(X^+_t) + \liminf_j E(X^-_t) < \infty.
\]

Then the net \( (X_t, \tau \in T) \) converges stochastically to an integrable random variable.

Theorem I-2.3. ([45], see also [48] page 45.) Let \( E \) be a Banach space. Let \( f(\sigma, \tau) \) be an \( E \)-valued family of \( \mathcal{C} \sigma \) measurable random variables defined for \( \sigma, \tau \in T, \sigma \leq \tau \). Assume that for every \( t \in J \)

\[
1_{\{\sigma = t\}} f(\sigma, \tau) = 1_{\{\sigma = t\}} f(t, \tau) \quad \text{and}
\]

if \( A \in \mathcal{F} \) and \( \tau = \tau' \) on \( A \),

\[
1_A f(s, \tau) = 1_A f(s, \tau').
\]

If \( (f(\sigma, \tau)) \) converges stochastically to \( f_\infty \) and \( V \) holds then \( (f(\sigma, \tau)) \) converges almost surely to \( f_\infty \).
I-3. Convergence Theorems for Subpramarts

The following Proposition is an extension of Neveu's Lemma (see [52] page 109) to directed index sets. The main idea of the proof is the same.

**Proposition I-3.1.** Assume that $V$ holds. Let $(X^i_t, t \in J), i \in I)$ be a countable family of real-valued integrable submartingales that are subpramarts (e.g. submartingales with the optional sampling property). Assume also that \( \sup_J E[\sup_I X^i_t] < \infty \). Then each of the submartingales converges a.s. to an integrable limit $X^i (i \in I)$ and

\[
\lim \inf_J (\sup_I X^i_t) = \sup_I X^i \text{ a.s.}
\]

**Proof:** By Millet-Sucheston's theorem (Theorems I-2.2. and 2.3.), the a.s. limit $X^i = \lim_J X^i_t$ exists, for all $i \in I$. Let $X = \lim \inf_J (\sup_I X^i_t)$. By Fatou's lemma, which remains true for directed sets, $X$ is integrable. Since $\sup_I X^i_t \geq X^i_t, t \in J, X$ dominates each $X^i_t$, thus $X \geq \sup_I X^i_t$. In order to prove equality it is enough to show that $E(X) = E(\sup_I X^i_t)$. 

Let \((I,p \in \mathbb{N})\) be a sequence of finite subsets of \(I\) increasing to \(I\) as \(p \to +\infty\). The expectation \(\mathbb{E}(\sup_{I} X^i)\) then increases with \(p\) (\(p \in \mathbb{N}\)), and it also increases with \(t\) (\(t \in J\)) since 
\[(\sup_{I} X^i, t \in J)\] is a submartingale for every \(p \in \mathbb{N}\).

Notice that \((\sup_{I} X^i, t \in J)\) is also a submartingale since \(I\) is finite.

Since the upper bound \(S\) defined by 
\[S = \sup_{p \in \mathbb{N}} \mathbb{E}(\sup_{I} X^i) = \sup_{t \in J} \mathbb{E}(\sup_{I} X^i)\]
is finite, for every \(\varepsilon > 0\) there exists \(p \in \mathbb{N}\) and \(t \in J\) such that 
\[\mathbb{E}(\sup_{I} X^i) > S - \varepsilon\] if \(p > p\) and \(t > t\).

But \(X - \sup_{I} X^i = \lim \inf_{p} \sup_{I} X^i - \lim \sup_{I} X^i\)
\[= \lim \inf_{J} \sup_{I} X^i - \lim \sup_{J} \sup_{I} X^i\]
\[= \lim \inf_{J} \sup_{I} X^i - \lim \sup_{J} \sup_{I} X^i\]
\[= \lim \inf_{J} \sup_{I} X^i + \inf_{s \in J, t > s} \sup_{I} X^i\]
\[\leq \lim \inf_{J} (\sup_{I} X^i - \sup_{I} X^i)\].
Thus Fatou's lemma implies that

\[ E(X - \sup_I X^i) \leq \lim_{p} \inf_{J} E(\sup_I X^i - \sup_I x^i) \]

\[ < S - (S - \varepsilon) = \varepsilon \quad \text{for} \quad p > p_{\varepsilon} \]

Since \( \varepsilon \) is arbitrary, the proposition is proved. \( \square \)

If \( E \) is a separable Banach lattice, then there exists a countable set \( D \) in \( B(E') \cap (E')^+ \) such that for all \( x \) in \( E^+ \), \( \|x\| = \sup_D x(x) \), and \( \sup_D x(x) \leq \|x\| \) for any \( x \) in \( E \), where \( B(E') \) denotes the unit ball of \( E' \). The set \( D \) is called \textit{norming subset}

of \( (E')^+ \).

The case \( J = \mathbb{N} \) of the following lemma is due to L. Egghe [24]. Here we give a slightly different proof.

**Lemma 1-3.2.** Let \( E \) be a separable Banach lattice, and \( (X_t, t \in J) \) an \( E \)-valued positive subpramart. For each \( x \in D \) let \( R^X_t + Z^X_t = X(x_t) \) be the decomposition of the real positive subpramart \( (X_t) \) (Theorem 1-2.1.)

Then

\[ \lim \sup_{T} \sup_{D} Z^X_t = 0 \]
If \( V \) holds then \( \limsup_T \sup_{D \sigma} Z^X = 0 \) a.s.

**Proof:** Suppose that \( (u) \) fails. Then given any stopping time \( \sigma_0 \), there is \( \varepsilon > 0 \) and \( \sigma \geq \sigma_0 \) such that \( P(\sup_{D \sigma} Z^X > \varepsilon) > 2\varepsilon \). Now we can find a finite subset \( D' \) of \( D \), say \( D' = \{x_1, \ldots, x_k\} \), such that

\[
[A] \quad P(\sup_{D'} Z^X > \varepsilon) = P(\sup_{D'} (\sup_{\sigma} X(X) - R^X_\sigma) > \varepsilon) > \varepsilon.
\]

By [52], VI-1-1, p. 121, and using Theorem 1-2.1, there exists a sequence \( (\sigma^X_n) \) in \( T \) such that

\[
R^X = \inf_{\sigma \in \mathbb{N}} E^{\sigma}(X(X)) , \text{ a.s.}
\]

\[
= \lim_{n \to \infty} \inf_{k \leq n} E^{\sigma_k}(X(X)) , \text{ a.s.}
\]

Using a classical localization procedure (see [45], p. 93) we find a sequence \( (\tau^X_n) \) in \( T \) such that

\[
R^X = \lim_{n \to \infty} E^{\sigma}(X(X)) , \text{ a.s.}
\]
for each \( x \in (E')^+ \). Thus for each \( x \in \overline{D} \) there exists a stopping time \( \tau^x = \tau_0 \) such that \( P(\{E^\sigma(x(x)) - R^\sigma > \frac{\varepsilon}{2}\}) < \frac{\varepsilon}{2^k} \).

Therefore

\[ [B] \quad P(\{\sup_D (E^\sigma(x(x)) - R^\sigma) > \varepsilon\}) \leq \frac{\varepsilon}{2^k}. \]

But

\[ \{\sup_D (x(x) - R^\sigma) \geq \varepsilon\} \]

\[ = \{\sup_D (x(x) - E^\sigma(x(x))) \geq \varepsilon\} \cup \{\sup_D (E^\sigma(x(x)) - R^\sigma) \geq \varepsilon\}. \]

Because of \([A]\) and \([B]\) we have

\[ \frac{\varepsilon}{2} \leq P(\{\sup_D (x(x) - E^\sigma(x(x))) \geq \varepsilon\}). \]

If \( X_i(\sigma) = X_i(x) - E^\sigma(x(x)) \) \( i = 1, 2, \ldots, k \), we can then define the following sets
\[ A_1 = \{ X_1(\sigma) \geq \frac{\varepsilon}{2} \} \]
\[ A_2 = \{ X_1(\sigma) < \frac{\varepsilon}{2}, X_2(\sigma) \geq \frac{\varepsilon}{2} \} \]
\[ \vdots \]
\[ A_k = \{ X_1(\sigma) < \frac{\varepsilon}{2}, \ldots, X_{k-1}(\sigma) < \frac{\varepsilon}{2}, X_k(\sigma) \geq \frac{\varepsilon}{2} \} . \]

Then the \( A_i \)'s \( i = 1, 2, \ldots, k \) are disjoint, each belongs to \( \mathcal{F} \) and
\[ \bigcup_{i=1}^{k} A_i = \{ \sup_D (x(X) - E^\sigma(\chi_{X})) > \varepsilon \} = A . \]

Define
\[ \tau = \begin{cases} 
\tau & \text{on } A_i \\
\chi_i & \text{on } A^c 
\end{cases} \]

Then by the localization property, \( \tau \in \mathcal{T}, \sigma \leq \tau \) and
\[ A = \{ \sup_D (x(X) - E^\sigma(\chi_{X})) > \frac{\varepsilon}{2} \} . \]

Hence
\[
\frac{\varepsilon}{2} \leq P\left(\sup_{D} (x(X) - E^\sigma(x(X))) > \frac{\varepsilon}{2}\right)
\]

\[
= P\left(\sup_{D} x(X) - E^\sigma(x(X)) > \frac{\varepsilon}{2}\right)
\]

\[
\leq P\left(\sup_{D} x(X) - E^\sigma(x(X))^+ > \frac{\varepsilon}{2}\right)
\]

\[
= P\left(\sup_{D} x(X) - E^\sigma(x(X))^+ > \frac{\varepsilon}{2}\right)
\]

But then \((X_t)\) cannot be a submartingale, a contradiction. \(\square\)

Lemma I.3.3. Assume that \(V\) holds. Let \(E\) be a separable Banach lattice and \(D\) a countable norming subset of \((E^\prime)^+\). Let 
\((X_t, t \in J)\) be an \(E\)-valued positive submartingale which satisfies the assumption (d):

\[
(d) \quad \liminf_{J} E(\|X_t\|) < \infty.
\]

Suppose also that there exists an \(E\)-valued random variable \(X\) such that for all \(x \in D\),

\[
\lim_{J} x(X_t) = x(X) \text{ a.s.}
\]
Then

\[ \lim_{J} \| X \| = \| X \| \].

Proof: Let \( R^X_t + Z^X_t = \chi(X) \) be the decomposition of the real positive submart \((\chi(X))\), \( x \in D \). Then \( \{(R^X_t, t \in J), x \in D\} \) is a countable family of real valued positive integrable submartingales with the optional sampling property (Theorem 1-2.1.), such that

\[ \liminf_{J} E(\sup_{D} R^X_t) \leq \liminf_{J} E(\| X \|) < \infty \]. The last relation implies

\[ \sup_{J} E(\sup_{D} R^X_t) < \infty \] since \( (\sup_{D} R^X_t, t \in J) \) is a submartingale. We also have \( \lim_{J} R^X_t = \chi(X), x \in D \). Therefore Proposition 1-3.1. gives

(1) \[ \liminf_{J} (\sup_{D} R^X_t) = \sup_{D} \chi(X) = \| X \| \].

On the other hand

\[ 0 \leq \sup_{D} \chi(X) \leq \sup_{D} R^X_t + \sup_{D} Z^X_t \]

thus

\[ 0 \leq \| X \| - \sup_{D} R^X_t \leq \sup_{D} Z^X_t \].
By Lemma 1-3.2, we have

\[(2) \quad \lim_{J} (\|X_{t}\| - \sup_{D} R_{t}^{X}) = 0.\]

The real valued subpramart \((\|X_{t}\|, t \in J)\), being of class (d), converges. Using now (1) and (2) we have

\[\lim_{J} \|X_{t}\| = \|X\|.\]

**Lemma I-3.4.** Let \(E\) be a Banach lattice which is a separable dual

(i.e., \(E = F'\), for some Banach lattice \(F\)), and \((x_{t}, t \in J)\) an \(E^{+}\)-valued net, such that \(\lim \sup_{J} \|x_{t}\| < \infty\). Assume that \((x(x_{t}), t \in J)\) converges for all \(x\) in a countable dense subset of \(F^{+}\). Then there exists an element \(x\) in \(E\) such that

\[x(x_{t}) \rightarrow x(x) \quad \text{for all} \quad x \in F.\]

**Proof:** Easy. See e.g. [52] page 108.

Before stating the main theorem of this chapter we recall that a norm \(\| \cdot \|\) on a Banach space is said to have the **Kadec-Klee** property

(with respect to a countable and norming subset \(\Pi\) of the dual), if,

whenever \(x_{n} \rightarrow x\) for all \(x \in \Pi\) and \(\|x_{n}\| \rightarrow \|x\|\) then \(x_{n} \rightarrow x\)

strongly.
Davis, Ghoussoub and Lindenstrauss [14] proved the following fundamental renorming theorem for Banach lattices:

A Banach lattice, $E$, has an equivalent Kadec-Klee lattice norm if (and only if) it has an order continuous norm.

**Theorem 1-3.5.** Assume that $V$ holds. Let $E$ be a separable Banach lattice. The following are equivalent.

(i) $E$ has the Radon-Nikodym property.

(ii) Every $E$-valued positive submartingale $(X^t, t \in J)$ of class $(d)$, i.e., $\lim \inf_{J^t} E|X^t| < \infty$, converges a.s. in the norm topology to an $E$-valued integrable random variable.

**Proof:** $(i \Rightarrow ii)$. Since $E$ is separable with the Radon-Nikodym property by a theorem of Talagrand ([63], Theorem 1), $E$ is isometrically the dual of a Banach lattice $F$, i.e., $E = F'$.

Since $(X^t)$ is a real valued submartingale of class $(d)$, it converges a.s. and therefore $\lim \sup_{J^t} \|X^t(\omega)\| = \lim \sup_{J^t} \|X^t(\omega)\| < \infty$ for all $\omega$ in some set $\Omega$ with $P(\Omega) = 1$. Let now $D$ be a countable dense subset of $F^+$. For each $x \in D$, $(x(X^t))$ is a real valued submartingale of class $(d)$ and therefore $(x(X^t))$ converges for all $\omega$ in some set $\Omega$ with $P(\Omega) = 1$. Thus for $\omega \in \Omega = \bigcap_{x \in D} \Omega_x$, we have...
\[(a) \limsup_{t} \|X_t(\omega)\| < \infty.\]

\[(b) (X_t(\omega), t \in J) \text{ converges for all } \chi \in \mathcal{D}.\]

Hence by Lemma I-3.4, there exists an $E$-valued mapping $X$ on $\Omega_1$ such that for all $\omega \in \Omega_1$, $\lim_{t} \chi(X(\omega)) = \chi(X(\omega))$ for every $\chi \in \mathcal{F}$. Since $E$ is order continuous, by the renorming theorem, it admits an equivalent lattice norm $\|\cdot\|_1$ which is Kadec-Klee with respect to a countable norming subset $\eta$ of $\mathcal{F}^+$. Then by Lemma I-3.3, we have

$$\lim_{t} \|X_t\|_1 = \|X\|_1.$$ 

The Kadec-Klee property of the norm gives $\lim_{t} X_t = X$ a.s. in the norm topology.

Since $X$ is the strong limit of $(X_t, t \in J)$, it is measurable and integrable.

(ii + i) It was proved by Ghoussoub-Talagrand [28] that convergence of positive martingales in $E$ implies the Radon-Nikodym property; now, every martingale is a submartingale. \(\Box\)

Remark. The assumption that $E$ is separable is not a loss of generality if $J = \mathbb{N}$ since each $X_n$ being Bochner integrable is separable valued and so is the set $\{X_n, n \in \mathbb{N}\}$.
Corollary 1-3.6. Let $E$ be a Banach lattice with the Radon-Nikodym property. Let $(X_t, t \in J)$ be an $E$-valued, $L_1$-bounded positive submartingale. Then $(X_t, t \in J)$ converges in probability to an $E$-valued integrable random variable.

Proof: Since convergence in probability is given by a complete metric, the proof follows from Theorem 1-3.5. and Lemma III-4.1. □

We now obtain a convergence result in order-continuous Banach lattices.

Let $E$ be an order-continuous Banach lattice and $\gamma$ the countable norming subset of $(E')^+$ for the Kadec-Klee norm $\| \cdot \|_1$. An $E$-valued submartingale $(X_t, t \in J)$ is said to be of class $(od)$ if:

(i) $0 \leq X_t$

(ii) $\lim_{t \to \infty} \inf_J E_x X_t < \infty$

(iii) there exists an $E$-valued random variable $S$ such that $\chi(X_t \vee S) + \chi(S)$ a.s. for all $x \in \gamma$.

An example of a submartingale of class $(od)$ is the following:

$0 \leq X_t \leq E^t(S)$ for some integrable random variable $S$. The case $J = \mathbb{N}$ and $(X_n)_{n \in \mathbb{N}}$ a submartingale was considered by Davis-Ghoussoub-Lindenstrauss in [14].
Theorem 1-3.7. Assume that V holds. If E has an order continuous norm and \((X_t, t \in J)\) is a submart of class \((od)\), then \((X_t)\) converges in the norm topology to an E-valued integrable random variable.

Proof: The equality \(S \wedge X + S \vee X = S + X\) and condition (iii) imply \(\lim_j x(X_t \wedge S) = \lim_j x(X_t)\) a.s. for all \(x \in \eta\). Since order intervals in order-continuous Banach lattices are weakly compact, the family \((X_t \wedge S)\) has a weak limit point; let us say \(X\). But then \(\lim_j x(X_t \wedge S) = x(X)\) a.s. for all \(x \in \eta\), which is the same as \(\lim_j x(X_t) = x(X)\) for all \(x \in \eta\). Lemma 1-3.3. gives \(\lim_j \|X_t\| = \|X\|\) a.s. The Kadec-Klee property of the norm \(\|\cdot\|\) implies that \(\lim_j X_t = X\) a.s. in the norm topology of \(E\). Since \(X\) is the strong limit of \((X_t)\), \(X\) is measurable and integrable. □

Remark. A Banach lattice is order continuous if and only if every order bounded positive increasing sequence in \(E\) converges in the norm topology of \(E\). Therefore the necessity of order continuity follows from the fact that any positive increasing sequence in \(E\) is a deterministic submartingale.
I-4. Convergence Theorems for Pramarts

If $E$ is a separable Banach space, then there exists a countable set $D$ in $B(E')$ such that for all $x \in E$, $\|x\| = \sup_D x(x)$.

Lemma I-4.1. Let $E$ be a separable Banach space and $(X_t, t \in J)$ an $E$-valued pramart. For each $x \in D$, let $R^x_t + Z^x_t = |x(X_t)|$ be the decomposition of the real valued positive subpramart $|x(X_t)|$.

Then

\[ \limsup_{T \to \infty} \sup_{D \sigma} Z^x_T = 0. \]

If $V$ holds then $\limsup_{T \to \infty} Z^x_T = 0$.

Proof: The proof is identical to that of Lemma I-3.2.

We now observe that Lemma I-3.3. remains true for Banach space valued pramarts of class (d).

Before stating the convergence theorem for pramarts we recall that a separable Banach space $E$ has the Asymptotic norming property if there exists a separable Banach space $F$ such that $E$ is (isomorphic to) a subspace of $F'$ which verifies the following property:
if \( (x_n) \subseteq E \), \( x(x) = x(x') \) for all \( x \in F \), \( x' \in F' \), and

\[
\|x_n - x'\| \to 0.
\]

Ghoussoub and Maurey \cite{29} proved the following theorem:

A separable Banach space has the Radon-Nikodym property if and only if \( E \) has the Asymptotic norming property.

**Theorem 1-4.2.** Assume that \( V \) holds. Let \( E \) be a separable Banach space. The following are equivalent.

(i) \( E \) has the Radon-Nikodym property.

(ii) Every \( E \)-valued pramart \( (X_t, t \in J) \) of class \( (d) \) converges a.s. in the norm topology to an \( E \)-valued integrable random variable.

**Proof:** (i \( \Rightarrow \) ii) By Lemma 1-3.4. (see also the proof of Theorem 1-3.5.) there exists an \( F' \)-valued mapping on \( \Omega \) such that \( \lim_t X_t = X(X) \) for all \( x \in F \). Lemma 1-3.3. applies and we have \( \lim_t \|X_t\| = \|X\| \). The Asymptotic norming property of \( E \) now gives \( \lim_t \|X_t - X\| = 0 \).

(ii \( \Rightarrow \) i) It is well known for martingales \cite{16}, and every martingale is a pramart. \( \Box \)
Finally we prove the reversed pramart convergence theorem.

Write \(-J\) for \(J\) with the reversed ordering. Given a stochastic basis \((\mathcal{F}_t, t \in -J)\) the set \((-T, \leq)\) is filtering to the left. The family \((X_t, \mathcal{F}_t, t \in -J)\) of \(E\)-valued, Bochner integrable random variables is called a reversed pramart if

\[
\lim_{\sigma \leq \tau, \sigma, \tau \in -T} (X \circ E^0(X)) = 0
\]

Theorem 1-4.3. Let \(E\) be a Banach space and \((X_t, t \in -J)\) a reversed pramart. Then the net \((X_\tau, \tau \in -T)\) converges stochastically to an \(E\)-valued integrable random variable. If \(V\) holds then we have a.s. convergence.

Proof: Let \(\varepsilon > 0\). From the definition of reversed pramart there exists \(\tau_0 \in -T\) such that \(P(\{\|X_\sigma \circ E^0(X)\| < \varepsilon\}) < \varepsilon\) for \(\sigma \leq \tau \leq \tau_0, \sigma, \tau \in -T\). We observe now that the reversed martingale \((E^0(X), \sigma \leq \tau, \sigma \in -T)\) converges in \(L_1\) (see e.g. [50]). Thus \((E^0(X))\) converges in probability.
On the other hand

$$\|X - X\| \leq \|X - E^o(X)\| + \|\sigma E^o(X) - E^T(X)\| + \|E^T(X) - X\|.$$ 

Therefore the net $$(X, \sigma \in -T)$$ is Cauchy in probability and thus converges stochastically. Under $V$, there is a.s. convergence. (Theorem I-2.3.). □

It may be pointed out that this simple proof does not depend on the reduction to the real case.
II-1. Introduction

It is known that real-valued properly bounded processes 
\((X_t, t \in \mathbb{R}^+)\) that are martingales, submartingales, quasimartingales or 
amarts have almost surely left and right limits (see [51], [21]). The 
same is true if \((X_t)\) is a martingale, quasimartingale [53] or a 
uniform amart [13], taking values in a Banach space with the Radon-
Nikodym property (RNP). The proofs depend on convergence theorems for 
the analogous processes indexed by \(\mathbb{N}\) and \(-\mathbb{N}\), and thus there 
cannot be any regularity results for non-necessarily positive submartingales 
\((X_t)\) taking values in a Banach lattice \(E\) with RNP because 
\(L_1\)-bounded non-positive \(E\)-valued submartingales \((X_n, n \in \mathbb{N})\) need 
not converge (see [26]). However, Heinich [32] proved that there is 
convergence if \(X_n > 0\) , \(n \in \mathbb{N}\). Here the same is proved for 
reversed positive submartingales \((X_n, n \in -\mathbb{N})\) assuming only order-
continuity instead of RNP (Proposition II-4.3.). The stage is thus set for obtaining regularity theorems for vector-valued positive submartingales \( (X_t, t \in \mathbb{R}^+) \) (Theorem II-4.4.). We accomplish the translation of convergence results into regularity properties by using stopping time methods, passing to the real valued case, and returning to the vector-valued processes via the Kadec-Klee renorming theorem. The method, embodied in Theorems II-3.2. and 3.3. below, is quite general and can be applied to other processes; thus we prove regularity properties for continuous parameter vector-valued pramarts (Theorem II-4.7.). The class of pramarts includes the class of martingales, quasimartingales and uniform amarts. Exact definitions are given below.

II-2. Definitions and Basic Notions

Let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be a complete probability space. For each \( t \in \mathbb{R}^+ = [0, \infty) \), let \( \mathcal{F}_t \) be a sub-\( \sigma \)-field of \( \mathcal{F} \) which contains all the \( \mathbb{P} \)-null sets. The collection \( (\mathcal{F}_t, t \in \mathbb{R}^+) \) is assumed increasing and right continuous (i.e., \( \mathcal{F}_t = \bigcap_{s \leq t} \mathcal{F}_s \) for all \( t \in \mathbb{R}^+ \).

A function \( \tau : \Omega \rightarrow \mathbb{R}^+ \cup \{\infty\} \) is a stopping time for \( (\mathcal{F}_t) \) if \( \{\omega : \tau(\omega) \leq t\} \in \mathcal{F}_t \) for all \( t \in \mathbb{R}^+ \). An increasing sequence \( \tau_1 \leq \tau_2 \leq \ldots \) of stopping times is said to announce \( \tau \) if \( \lim_{n \to \infty} \tau_n = \tau \) and \( \tau_n < \tau \) (except on \( \{\tau = 0\} \)). Similarly a
decreasing sequence of stopping times is said to recall \( \tau \) if
\[
\lim_{n \to \infty} \tau^n = \tau \quad \text{and} \quad \tau > \tau^n \quad (\text{except on} \{\tau = \infty\})
\]. A predictable time is a stopping time that is announced by some sequence \( (\tau^n) \). A stopping time is called simple if it takes finitely many finite values.

Let \( T \) denote the set of all simple stopping times and \( T(S) \) the set of all simple stopping times with values in \( S \), where \( S \) is a countable dense subset of \( \mathbb{R}^+ \) containing 0.

Let \( (E, \| \cdot \|) \) be a Banach space. Let \( (X_t, t \in \mathbb{R}^+) \) be a family of \( E \)-valued, Bochner integrable random variables adapted to \( (\mathcal{F}_t) \).

The process \( (X_t) \) is called separable \( S \) if there exists a \( P \)-null set \( N \) such that for each \( \omega \) outside \( N \) the sample path \( x_t(\omega) \) is separable with respect to \( S \) in the strong topology of \( E \). The process \( (Y_t) \) is a modification of \( (X_t) \) if \( X_t = Y_t \) a.s. for all \( t \in \mathbb{R}^+ \). For unexplained terminology see [15].

We will often denote \( E(\cdot | \mathcal{G}_n) \) by \( E^n(\cdot) \), and \( E(\cdot | \mathcal{F}_\tau) \) by \( E^\tau(\cdot) \) for \( \tau \in T \).

II-3. Convergence

We start with the following Proposition which can be found in [21].
Proposition II-3.1. Let $\tau$ be a bounded stopping time (not necessarily simple) and let $\left( X_t, t \in \mathbb{R}^+ \right)$ be a real-valued process adapted to $(\mathcal{F}_t)$.

(a) There is a sequence $\tau_1 > \tau_2 > \ldots$ in $T(S)$ which recalls $\tau$ and such that, for almost all $\omega \in \Omega$, the two nets $\left( X_{\tau_n}(\omega) \right)_{n \to \infty}$ and $\left( X_{\tau_1}(\omega) \right)_{t \to t + \tau(\omega), t \in S}$ have the same cluster points in $[-\infty, \infty]$.

(b) Suppose that $\tau$ is predictable stopping time, announced by a sequence in $T(S)$. Then there is a sequence $(\tau_n)$ in $T(S)$ announcing $\tau$ and such that, for almost all $\omega \in \Omega$, the two nets $\left( X_{\tau_n}(\omega) \right)_{n \to \infty}$ and $\left( X_{\tau_1}(\omega) \right)_{t \to t + \tau(\omega), t \in S}$ have the same cluster points in $[-\infty, \infty]$.

Theorem II-3.2. Let $E$ be a separable Banach space. Let $\left( X_t, t \in \mathbb{R}^+ \right)$ be an $E$-valued process with $\mathbb{E} \left| X_t \right| < \infty$ for each $t \in S$.

(a) Let $\tau$ be a bounded stopping time. Suppose that for every sequence $(\tau_n)$ in $T(S)$ recalling $\tau$, $\lim_{-n} \left( X_{\tau_n} \right)$ exists strongly almost surely. Then $\lim_{t \to t + \tau(\omega), t \in S} X(\omega)$ exists strongly for almost all $\omega$ (the exceptional null set depends on $\tau$).
(b) Let $\tau$ be a bounded predictable stopping time which is announced by a sequence in $T(S)$. Suppose that, for every sequence $(\tau_n)$ in $T(S)$ announcing $\tau$, $\lim_{n \to \infty} X_{\tau_n}$ exists strongly almost surely. Then $\lim_{t + \tau(\omega), t \in S} X(\omega)$ exists strongly for almost all $\omega$ (the exceptional set depends on $\tau$).

Proof: (a) Let $X = \lim_{n \to \tau_n} X_{\tau_n}$. It is easy to see that $X$ does not depend on the choice of the sequence $(\tau_n)$ which recalls $\tau$.

Consequently $\lim_{n \to \infty} X_{\tau_n} = X$ a.s. for all $x \in \mathcal{E}$ and $\lim_{n \to \infty} X_{\tau_n} = X$ a.s. Now using Proposition II-3.1, we can choose sequences $(\tau_n)$ and $(\tau'_n)$ in $T(S)$ so that both $(\tau_n)$ and $(\tau'_n)$ recall $\tau$ and the two nets $(X_{\tau_n})_{n \to \infty}$ and $(X_{\tau'_n})_{n \to \infty}$, have the same cluster points in $[-\infty, \infty]$. But if a net has only one cluster point in $[-\infty, \infty]$, then it converges; so

$$\lim_{t + \tau(\omega), t \in S} X(\omega) = X(\omega) \text{ a.s. and}$$

$$\lim_{t + \tau(\omega), t \in S} ||X(\omega)|| = ||X(\omega)|| \text{ a.s.}$$
Since $E$ is a separable Banach space it admits an equivalent norm $\| \cdot \|_E^*$ which is Kadec-Klee with respect to a countable norming subset $D$ of $E'$ (whenever $x_n(x) \to x(x)$ for all $x \in D$ and $\| x_n \|_E^* \to \| x \|_E^*$ then $x_n \to x$ strongly) [6, p. 176]). Thus

(i) for all $x \in D$, $\lim_{t \to \tau(\omega), t \in S} x(t) = x(\tau(\omega))$ a.s.

(ii) $\lim_{t \to \tau(\omega), t \in S} \| x(t) \|_E^* = \| x(\tau(\omega)) \|_E^*$ a.s.

The Kadec-Klee property of the norm gives the strong convergence of $(X_t)_{t \to \tau}$ a.s. to $X$. 

(b) The proof of (b) is identical. □

**Theorem II-3.3.** Let $E$ be a Banach space, and $(X_t, t \in \mathbb{R}^+)$ an $E$-valued process.

(a) Suppose that, for every bounded stopping time $\tau$,

$$\lim_{t \to \tau(\omega), t \in S} X_t(\omega)$$

exists strongly for all $\omega \in \Omega$ with $P(\Omega_\tau) = 1$. Then for almost all $\omega$, the limit $X(\omega) =$

$$\lim_{s \to t, s \in S} X_s(\omega)$$

exists strongly for all $t \in \mathbb{R}^+$.

(b) Suppose that, for every bounded predictable stopping time $\tau$, announced by a sequence $(\tau_n)$ in $T(S)$,

$$\lim_{t \to \tau(\omega), t \in S} X_t(\omega)$$
exists strongly for all $\omega \in \Omega$ with $P(\omega) = 1$. Assume also that

$(X_t)$ has right limits. Then for almost all $\omega$, the limit

$$X = \lim_{t \to +} X(\omega)$$

exists strongly for all $t \in \mathbb{R}^+$. The proof (but not the statement) appears in [13] pp. 90 and 93. See also [18]. We sketch it here for completeness.

Proof: (a) Let $\varepsilon > 0$ and define the following stopping times by induction:

$$\tau_0 = 0$$

$$\tau_{a+1} = \inf\{t > \tau_a : [osc X(\omega) \text{ on } (\tau_a, t)] > \varepsilon\}$$

$$\tau = \sup_{\beta < \alpha} \{\tau_a : a < \beta\} \text{ for a limit countable ordinal } \beta.$$

If there exists a countable ordinal $\alpha$ such that $\tau_\alpha = \tau_{\alpha+1}$ on a subset of $\{\tau_\alpha < \infty\}$ with positive probability then, $(X_t)$ does not have right limits at $\tau_\alpha$ for large $\kappa$ on a non-null set, a contradiction. Otherwise $\tau_\alpha = \infty$ almost surely for every $\varepsilon > 0$ which implies that $(X_t)$ has right limits in the strong topology of $E$ almost surely.
(b) Let $\varepsilon > 0$ and define the following stopping times by induction:

$$
\tau_0 = 0
$$

$$
\tau_{n+1} = \inf \{ t > 0 : [\text{osc } X_n(\omega) \text{ on } (\tau_n, t)] > \varepsilon \}.
$$

Suppose that $(X_t)$ does not have left limits almost surely. Then there exists a sufficiently small $\varepsilon > 0$ such that $P(\tau < \infty) > 0$ where

$$
\tau(\omega) = \begin{cases} 
\lim_{n} \tau_n(\omega) & \text{if } \tau_n(\omega) < \infty \\
\tau_{n-1}(\omega) & \text{if } \tau_n(\omega) = \infty.
\end{cases}
$$

Since $(X_t)$ has right limits $\tau < \tau_n < \tau$ on $\{\tau < \infty\}$, so $\tau$ is a predictable stopping time. Thus $(X_t)$ does not have left limits at $\tau \wedge \kappa$ for large $\kappa$ on a set with positive probability, a contradiction. \qed

II-4. Applications

As a first application, we show that almost all the paths of a positive submartingale in a Banach lattice $E$ with the Radon-Nikodym property have left and right limits. We begin with a discrete parameter convergence result. The following Lemma can be found in [1].
Lemma II-4.1. Let $E$ be an order continuous Banach lattice. If $(x_n) \subseteq E$ is such that $0 \leq x_n \leq x$ and $(x_n)$ converges weakly to $0$ then $(x_n)$ converges strongly to $0$.

Remark. If $0 \leq x_n \leq y$ and $(x_n)$ converges weakly to $0$ and $(y_n)$ converges strongly to $y \in E$ then $(x_n)$ converges strongly to $0$.

Indeed $0 \leq x_n \wedge y \leq x$ implies that $(x_n \wedge y)$ converges weakly to $0$ and therefore strongly. The inequality $y \leq x_n \vee y \leq y \vee y$ implies that $(x_n \vee y)$ converges strongly to $y$. The equality $x_n \wedge y + x_n \vee y = x_n + y$ finishes the proof.

The next Proposition, a result of Brunel and Sucheston [8] is stated for easy reference.

Proposition II-4.2. Let $(X_n)$ be an $E$-valued sequence of random variables such that for almost all $\omega$, $(X_n(\omega))$ is weakly sequentially compact. If for every $\chi \in E'$, $(\chi(X_n(\omega)))$ converges to a finite limit for all $\omega$ in some set $\Omega$ with $P(\Omega) = 1$, then $(X_n)$ converges weakly a.s.

Proposition II-4.3. Let $E$ be an order continuous Banach lattice. Let $(X_n, n \in -\mathbb{N})$ be an $E$-valued, positive, integrable reversed submartingale. Then $(X_n, n \in -\mathbb{N})$ converges strongly a.s. to an $E$-valued integrable random variable.
Proof: By the definition of reversed submartingale [52]

\[ 0 \leq \frac{X_n}{n+1} \leq E_n^{n}(X_{-1}) , n \in \mathbb{N} . \]  

Thus

\[ 0 \leq E_n^{n}(X_{n+1}) - X_n \leq E_n^{n}(X_{-1}) , n \in \mathbb{N} . \]  

It is well known that the reversed martingale \( (E_n^{n}(X_{-1}), n \in \mathbb{N}) \) converges strongly a.s. and in \( L_1 \) to \( E_{-1}(X_{-1}) \), \( \mathcal{F} = \bigcap_{n \in \mathbb{N}} \mathcal{F} \) (see e.g. [50]).

If \( Z_n = E_n^{n}(X_{n+1}) - X_n \) then

\[ E^{-\infty}(X_{-1}) \leq Z_n \leq E^{-\infty}(X_{-1}) \leq E_n^{n}(X_{-1}) \leq E^{-\infty}(X_{-1}) \]  

thus

\[ \lim_{n \to \infty} Z_n = E_{-1}(X_{-1}) \] a.s. in the strong topology of \( E \).

On the other hand the equality

\[ Z_n \leq E^{\infty}(X_{-1}) + Z_n \wedge E^{\infty}(X_{-1}) = Z_n + E^{\infty}(X_{-1}) , n \in \mathbb{N} \]  

and the fact that for each \( x \in E' \) (\( x(Z_n), n \in \mathbb{N} \)) converges a.s. to 0, imply that for each \( x \in E' \) (\( x(Z_n \wedge E^{\infty}(X_{-1}))(\omega) \)) converges to 0 for all \( \omega \) in some set \( \Omega \) with \( P(\Omega) = 1 \). Since order intervals in order continuous Banach lattices are weakly compact, we deduce from Proposition II-4.2. that \( (Z_n \wedge E^{\infty}(X_{-1})), n \in \mathbb{N} \) converges to 0 weakly a.s. Thus \( (Z_n), n \in \mathbb{N} \) converges weakly to 0 a.s. and by the Remark strongly a.s. Since submartingales have the optional
sampling property we have that \((Z^n, n \in \mathbb{N})\) converges strongly to
\(0\) a.s. for every increasing sequence \((\tau_n, n \in \mathbb{N})\) of simple
stopping times. Then by Lemma III-4.1., the stochastic limit of the net
\(\left\{ E^\sigma(\tau^n) - X^n, \sigma \leq \tau \right\}\) is 0 as \(\tau \to \infty \tau \in T\). Thus \((X^n, n \in \mathbb{N})\)
is a reversed pramart (Section I-4.). The result now follows from
Theorem I-4.3., since the Vitali condition \(V\) is automatically satis-
fied in our case. \(\Box\)

Remark. The necessity of order continuity is trivial since any de-
creasing sequence is a deterministic reversed submartingale.

Theorem II-4.4. (a) Let \(E\) be an order continuous Banach lattice. If
\((X_t, t \in \mathbb{R}^+)\) is an \(E\)-valued separable, positive, integrable submartin-
gale, then almost all paths have right limits.

(b) Let \(E\) be a Banach lattice with the Radon-Nikodym property.
If \((X_t, t \in \mathbb{R}^+)\) is an \(E\)-valued, separable, positive, \(L^1\)-bounded sub-
martingale, then almost all paths have left limits.

Proof: (a) Let \(\tau\) be a bounded stopping time and \((\tau_n, n \in \mathbb{N})\) a
sequence in \(T(S)\) recalling \(\tau\). Then \((X^n, n \in \mathbb{N})\) is a reversed
\(\tau_n\)
submartingale. The assertion now follows from Proposition II-4.3.,
Theorem II-3.2.(a) and Theorem II-3.3.(a).

(b) Let $\tau$ be a bounded predictable stopping time and
$$(\tau_n, n \in \mathbb{N})$$
a sequence in $T(S)$ announcing $\tau$. Then $(X_n, n \in \mathbb{N})$
is an $L_1$-bounded positive submartingale which converges by Heinich's
theorem [32], thus Theorem II-3.2.(b) applies. Since $E$ has the Radon-
Nikodym property, it does not contain $c_0$ ([16], pp. 60 and 81) and
therefore $E$ is weakly sequentially complete ([39], p. 34), hence
order continuous. Because order continuity implies the existence of
right limits the result now follows from Theorem II-3.3.(b). □

The following result about continuous modifications is known for
the real valued case [51], and the proof (from Theorem II-4.4.) is the
same.

Theorem II-4.5. Let $E$ be an order continuous Banach lattice. Let
$$(X_t, t \in \mathbb{R}^+)$$
be an $E$-valued, separable, positive integrable submartingale. Then $(X_t)$
admits a right continuous modification if and only if
the function $E(X_t)$ of $t$ is right continuous in the norm topology of
$E$.

As a second application we show that pramarts taking values in a
Banach space have right limits whereas pramarts taking values in a
Banach space with the Radon-Nikodym property have both right and left
limits.

**Definition:** An adapted family \((X_t, t \in \mathbb{R}^+)\) of Bochner integrable random variables is an **ascending** [reversed or descending] **pramart** at a stopping time \(\tau\) if for each increasing sequence \((\tau_n, n \in \mathbb{N})\) in \(T\), converging to \(\tau\) one has

\[
s\lim_{n \to \infty} \|X_{\tau_n} - E^n(X_{\tau_{n+1}})\| = 0
\]

\((s\lim\text{ means limit in probability})\) [45].

\((X_t, t \in \mathbb{R}^+)\) is an ascending [reversed] pramart if it is an ascending [reversed] pramart at each stopping time.

\((X_t, t \in \mathbb{R}^+)\) is a **pramart** if it is both an ascending and reversed pramart.

If the stochastic limit above is replaced by the \(L^1\) limit then the process is called uniform amart [13].

It is clear that the class of pramarts contains the class of uniform amarts and hence also the class of quasimartingales [13].

We say that

(i) \((X_t, t \in \mathbb{R}^+)\) is of **class** (B) if, \(\sup_{\tau \in T} E\|X\| < \infty\)
(ii) \((X_t, t \in \mathbb{R}^+)\) is of class (AL) if, for all uniformly bounded increasing sequences \((\tau_n) \subset T\), we have \(\sup_n \mathbb{E} |X_{\tau_n}| < \infty\) [21]. Now we observe the following:

**Lemma 3.6.** Let \((X_t, t \in \mathbb{R}^+)\) be a pramart of class (AL) then 
\((X_{\tau_n}, n \in \mathbb{N})\) is a discrete parameter pramart of class (B), for every increasing sequence \((\tau_n)\) in \(T\) converging to a bounded stopping time \(\tau\).

**Proof:** Let \(\tau\) be a bounded stopping time and \((\tau_n)\) an increasing sequence in \(T\) converging to \(\tau\). Then for every increasing sequence \((\sigma_n, n \in \mathbb{N})\), \(\sigma_n \to \infty\) of simple stopping times for \((\mathcal{F}_t, n \in \mathbb{N})\), it is easily seen that \((\tau_n, n \in \mathbb{N})\) is an increasing sequence of simple stopping times for \((\mathcal{F}_t, t \in \mathbb{R}^+)\), converging to \(\tau\). Thus by the previous definition

\[
\lim_{n} \mathbb{E} \left[ X_{\tau_{\sigma_n}} - E(X_{\tau_{\sigma_n}} | \mathcal{F}_{\sigma_n + 1}) \bigg| \mathcal{F}_{\sigma_n} \right] = 0
\]

which is just the definition of the discrete parameter pramart 
\((X_{\tau_n}, n \in \mathbb{N})\).
Now since $\tau$ is bounded ($\tau, n \in \mathbb{N}$) is uniformly bounded, therefore $\sup_{n} \mathbb{E} |X_{\tau}| < \infty$ for all increasing sequences $\sigma_n, n \in \mathbb{N})$ for $(\sigma_{\tau}, n \in \mathbb{N})$. If $(\chi, n \in \mathbb{N})$ is not of class (B), that is, if $\sup_{\tau} \mathbb{E} |X_{\tau}| = \infty$ (the supremum is taken over all simple stopping times $\sigma_{\tau}$ for $(\sigma_{\tau})$), then by a Theorem of Krengel and Sucheston [35] (Theorem 2.4.) there exists an arbitrary (not necessarily simple) stopping time $\nu$ for $(\sigma_{\tau})$ such that $\mathbb{E} 1_{\{\nu < \infty\}} |X_{\nu}| = \infty$. Define $\sigma_n = \nu \wedge n$.

Then $(\sigma_n)$ is an increasing sequence of simple stopping times such that $\sup_{n} \mathbb{E} |X_{\tau}| = \infty$, a contradiction. 

**Theorem II-4.7.** (a) Let $E$ be an arbitrary Banach space. If $(X_t, t \in \mathbb{R}^+)$ is an $E$-valued, separable, integrable, reversed pramart, then almost all paths have right limits.

(b) Let $E$ be a Banach space with the Radon-Nikodym property. If $(X_t, t \in \mathbb{R}^+)$ is an $E$-valued, separable pramart of class (AL), then almost all paths are without oscillatory discontinuities.
Proof: (a) Let $\tau$ be a bounded stopping time and $(\tau_n, n \in \mathbb{N})$ a sequence in $\mathcal{T}(S)$ recalling $\tau$. Then $(X_n, n \in \mathbb{N})$ is a reversed $\tau_n$-martingale which converges by Theorem I-4.3. The assertion now follows from Theorem II-3.2.(a) and Theorem II-3.3.(a).

(b) Let $\tau$ be a bounded predictable stopping time and
$(\tau_n, n \in \mathbb{N})$ a sequence in $\mathcal{T}(S)$ announcing $\tau$. Then $(X_n, n \in \mathbb{N})$ is a martingale of class (B) (Lemma II-4.6.), and it has been proved in [44] Theorem 3.5.(b), that $\mathcal{E}$-valued martingales of class (B) converge strongly a.s. (see also Theorem I-4.2.). The result follows from Theorem II-3.2. and Theorem II-3.3. □

It may be pointed out that the proof of Theorem II-4.7.(b) does not depend on Theorem I-4.2. but on the rather simple Theorem 3.5. of [44].
CHAPTER III
CONVERGENCE AND DEMICONVERGENCE OF BLOCK
MARTINGALES AND SUBMARTINGALES

III-1. Introduction

Let $(\Omega, \mathcal{F}, P)$ be a probability space; $I$ a directed set filtering

to the right, $(\mathcal{F}_t, t \in I)$ an increasing filtration of sub-sigma-fields
of $\mathcal{F}$. Our main results concern processes $(X_t, \mathcal{F}_t), \ t \in I = J^m =
J_1 \times J_2 \times \ldots \times J_m$, where $J_i$ are directed sets filtering to the
right, and the order on $J^m$ is determined by $s = (s_1, \ldots, s_m) \leq t =
(t_1, \ldots, t_m)$ if $s_i \leq t_i$ for all $i \leq m$. In the first phases of the
multiparameter martingale theory begun by R. Cairoli [9], each $J_i$ was
\(\mathbb{N}\), and $P$ was the product of probability measures defined on the
coordinate filtrations $(\mathcal{F}_t)$. Later this assumption was relaxed to
conditional independence (F4 in Cairoli-Walsh [10]) or, equivalently,
commutation (cf. Meyer [40], p. 3). Here we typically assume that the
first filtration $(\mathcal{F}_t, t_1 \in J_1)$ has enough order for $L_1$-bounded
martingales to converge, or equivalently has a weak maximal inequality involving \( \limsup_{t} X_t \), and the other filtrations \( (\mathcal{F}_t, t \in J_i) \), \( 2 \leq i \leq m \) have a (stronger) weak maximal inequality involving \( \sup X_t \). For this it suffices that the first filtration has the covering condition \((C)\), introduced in [47], and the other filtrations satisfy a new regularity condition \((MR)\); we then call \((\mathcal{F}_t)\) regular. Conditions \((C)\) and \((MR)\) are stopping conditions involving multi-valued stopping times. We do not assume commutation, considering instead of martingales block martingales. This notion seems new if \( m > 3 \), but in two parameters a block martingale is exactly a 1-martingale in the sense of [48]. Under commutation, every martingale is necessarily a block martingale. The interest of block martingales and submartingales is that they arise in the natural context of laws of large numbers where \((F4)\) may fail; see the end of Section 4.

A typical result is that a block martingale bounded in \( L \log^{m-1} L \) converges essentially. Essential convergence is not very essential in this paper, because in all applications there is a countable cofinal subset, so that one obtains almost sure convergence. On the other hand, replacement of \( J_k = \mathbb{N} \) by more general index sets is important in applications to differentiation.

If there is convergence of a process indexed by \( \mathbb{N} \), then typically the analogous process indexed by \( J^m \) converges in probability without
additional integrability assumptions, but not essentially. This gives rise to the notion of \textit{demiconvergence}, first introduced in the two-parameter case for \( J = - \mathbb{N} \) in [22], and for \( J = \mathbb{N} \) in [49]. The case \( J = \mathbb{N} \) is more difficult, and the proofs given here are simpler than those in [49]. The stochastic limit is denoted by \( s \text{ lim} \); the letter \( e \) stands for \textit{essential}. We say that \( X \) \textit{demiconverges} if

\[
e \text{ lim sup } X_t = s \text{ lim } X_t \quad \text{(upper demiconvergence)} \quad \text{or} \quad e \text{ lim inf } X_t = s \text{ lim } X_t \quad \text{(lower demiconvergence)}.
\]

We show that \( L \log^{\alpha-1} L \) bounded block submartingales upper demiconverge, and positive block martingales lower demiconverge. The results are then applied to recover, in a somewhat stronger form involving "substantial sets", a theorem of Zygmund [66] about differentiation of integrals in \( m \) dimensions, along a net of rectangles with sides of not more than \( s < m \) different lengths. Zygmund's theorem is a generalization of the Jessen-Marcinkiewicz-Zygmund's theorem [33] in which \( s = m \); an intermediate result due also to [33] was recently derived from martingale theory by Shieh [56]. A demiconvergence version of Zygmund's theorem is also given.

Section 2 gives basic definitions. In Section 3 we prove maximal inequalities for positive submartingales under (C) and (MR). Under (C) these results are known for martingales [47], and the same article shows that (C) is sufficient for convergence of \( L_1 \)-bounded martingales. Recently Talagrand [65] proved that (C) is also
necessary, if there is cofinal countable subset. We believe that
\((\text{MR}_\alpha)\) may be necessary for our maximal inequalities, but this is here
not discussed. Section 3 also proves maximal inequalities for positive
block submartingales. Convergence and demiconvergence of block pro­
cesses is studied in Section 4. An extension to Banach lattices is
given in Section 5. Under the commutation assumption, we reduce in
Section 6 demiconvergence of block submartingales to convergence of
block martingales. The last, the seventh section contains applications
to differentiation of integrals.

III-2. Definitions and Basic Notions

Let \(J\) be a directed set filtering to the right. For a fixed
\(m \in \mathbb{N}\), define \(J^m = J \times J \times \cdots \times J\), \(J = J\); with the order
\(s = (s_1, \ldots, s_m) < t = (t_1, \ldots, t_m)\) if \(s_k < t_k\), \(k = 1, 2, \ldots, m\). The
set \(J^m\) is then filtering to the right. Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a complete
probability space, and let \((\mathcal{F}_t, t \in J^m)\) be an increasing net of
sub-\(\sigma\)-fields of \(\mathcal{F}\). For any integers \(k, \ell, 1 \leq k \leq \ell \leq m\), \(\mathcal{F}^{k, \ell}_t\)
is defined as the \(\sigma\)-field obtained by lumping together the \(\sigma\)-fields
on all the axes except for the \(k\)-th, \(k + 1\)-th, \(\ldots\), \(\ell\)-th ones. That
is,
If $k = \ell$ then $\mathcal{F}_t^{k-\ell}$ is denoted by $\mathcal{F}_t^k$. Obviously, if $k = 1$ and $\ell = m$, then $\mathcal{F}_t^{k-\ell} = \mathcal{F}_t^k$. We often denote $E(\cdot | \mathcal{F}_t)$ by $E^t(\cdot)$ and $E(\cdot | \mathcal{F}_t^{k-\ell})$ by $E^{k-\ell}_t(\cdot)$.

An integrable process $(X, \mathcal{F}, t \in \mathcal{J}^m)$ is a (sub)martingale whenever $E^s(X_t) = X_t (\geq)$ for $s \leq t$. An integrable process $(X, \mathcal{F}, t \in \mathcal{J}^m)$ is a block $k$-(sub)martingale for a fixed $k < m$ if

$$E^{1-k}_s(X_t) = X_{(s_1, \ldots, s_k, t, t+1, \ldots, t_m)} (\geq) \quad \text{for} \quad s \leq t.$$

An integrable process is a block (sub)martingale if it is a block $k$-(sub)martingale for all $k \leq m$. Thus a block (sub)martingale is necessarily a (sub)martingale.

Let $\mathfrak{F}$ denote the set of finite subsets of $\mathcal{J}$. An (incomplete) multivalued stopping time is a map $\tau$ from $\Omega$ (from a subset of $\Omega$ denoted $\mathcal{D}(\tau)$) to $\mathfrak{F}$ such that $R(\tau) = \bigcup \tau(\omega)$ is finite, and for every $t \in \mathcal{J}$,
\[(\tau = t) \overset{\text{def}}{=} \{ \omega \in \Omega : t \in \tau(\omega) \} \in \mathcal{F}_t \).

Denote by $M(\mathcal{IM})$ the set of (incomplete) multivalued stopping times. A simple stopping time is an element $\tau \in M$ such that $\tau(\omega)$ is a singleton for every $\omega$; the set of simple stopping times is denoted by $T$. For $\sigma, \tau \in \mathcal{IM}$, we say that $\sigma \preceq \tau$ if for every $s$ and $t$ such that $\{\sigma = s\} \cap \{\tau = t\} \neq \emptyset$, one has $s \leq t$. With this order, $\mathcal{IM}$ is a directed set filtering to the right.

The excess function of $\tau \in \mathcal{IM}$ is

\[ e = \sum_{\tau \in \mathcal{IM}} 1_{\{\tau = t\}} - 1_{D(\tau)} . \]

Let $\tau \in \mathcal{IM}$; for a positive stochastic process $(X_t, \mathcal{F}_t, t \in J)$, we set

\[ X(\tau) = \sup_{t \in \{\tau = t\}} X_t . \]

If $(A_t)$ is an adapted family of sets, i.e. $A \in \mathcal{F}_t$, we set

\[ A(\tau) = \bigcup_{t \in \{\tau = t\}} A_t . \]
Hence if \( X = 1 \), then \( X(\tau) = 1 \).

The letter \( e \) means "essential." Thus \( e \lim X \) is the essential limit of \( X \). In most applications, there is a countable cofinal subset and in this case the word "essential" can be replaced by "almost sure".

A filtration \((\mathcal{F}_t, t \in J)\) satisfies the covering condition \( C \), introduced in [47], if for every \( \varepsilon > 0 \) there exists a constant \( M > 0 \) such that for every adapted family of sets \((A_t, t \in J)\), there exists \( \tau \in IM \) with \( e < M \) and

\[
(1) \quad P[A(\tau)] > P[e \lim sup A_t] - \varepsilon.
\]

Let \( \alpha \) be a fixed number, \( 0 < \alpha \leq 1 \). A filtration \((\mathcal{F}_t, t \in J)\) satisfies the regularity condition \( \text{MR}_\alpha \) if for every \( \varepsilon > 0 \) there exists a constant \( M = M(\varepsilon, \alpha) > 0 \) such that for every adapted family of sets \((A_t, t \in J)\) there exists \( \tau \in IM \) with \( e < M \) and

\[
(2) \quad P[A(\tau) > \alpha (1 - \varepsilon)] P[e sup A_t].
\]

The set of \( \alpha, 0 < \alpha \leq 1 \) such that \((\mathcal{F}_t)\) satisfies \( \text{MR}_\alpha \) has a maximum, unless it is empty. Indeed, let \( \alpha \) be the supremum of all \( \alpha \).
such that \( \mathbf{MR} \) holds. Given \( \epsilon > 0 \), apply (2) with an \( \alpha \) and \( \alpha \) such that \( \alpha (1 - \epsilon) > \alpha (1 - \epsilon) \).

A stricter regularity condition \( \mathbf{R} \) was introduced in [46]: the stopping times \( \tau \) are required to be single-valued (that is, \( \epsilon = 0 \)). An example given in [42] (for a different purpose) shows that \( \mathbf{MR} \) is strictly weaker than \( \mathbf{R} \). It was observed in [49] that a filtration \( (\mathcal{F}_t) \) totally ordered by set-inclusion satisfies \( \mathbf{R} \).

The condition \( \mathbf{MR} \) implies condition \( \mathbf{C} \). Indeed, let \( \epsilon > 0 \) and \( (A_t) \) be an adapted family of sets. Choose \( s \in J \) such that

\[
P[e \sup_{t \geq s} A] - P[e \limsup_{t \geq s} A] \leq \epsilon.
\]

Let \( \tau \in IM \), \( \tau \geq s \), then

\[
P[e \limsup_{t \geq s} A_t \cap A(\tau)] \geq P[e \sup_{t \geq s} A_t \cap A(\tau)] - \epsilon = P[A(\tau)] - \epsilon.
\]

Applying now \( \mathbf{MR} \) to \( (B_t) \), \( B_t = A_t \) for \( t > s \) and \( B_t = \emptyset \) otherwise, one obtains an \( M = M(\epsilon, \alpha) \), and a \( \tau \in IM \) with \( e < M \), \( \tau \geq s \) and \( P[A(\tau)] \geq \alpha(1 - \epsilon) P[e \sup_{t \geq s} A_t] \). Since

\[
P[e \sup_{t \geq s} A_t] > P[e \limsup_{t \geq s} A_t],
\]

one has

\[
P[A(\tau)] \geq \alpha(1 - \epsilon) P[e \limsup_{t \geq s} A_t] > \alpha P[e \limsup_{t \geq s} A_t] - \epsilon.
\]

Hence

\[
P[e \limsup_{t \geq s} A_t \cap A(\tau)] \geq \alpha P[e \limsup_{t \geq s} A_t] - 2\epsilon
\]

which is equivalent with condition \( \mathbf{C} \) ([47], Theorem 1.1.).
A filtration \((\mathcal{F}_t, t \in \mathbb{J}^m)\) is called regular if \((\mathcal{F}_t^1, t \in \mathbb{J})\) satisfies condition \(C\), and for each \(k, 2 \leq k \leq m\), there exists \(\alpha_k, 0 < \alpha_k < 1\) such that for each fixed \(t_1 \in \mathbb{J}, \ldots, t_{k-1} \in \mathbb{J}\), \((\mathcal{F}_t^{k-1}, t \in \mathbb{J}_k)\) satisfies condition \(MR\).

If the probability space is of product type, i.e., \((\Omega, \mathcal{F}, \mathbb{P}) = \Pi (\Omega_k, \mathcal{F}_k, \mathbb{P}_k)\) and \(\mathcal{F} = \mathcal{F}_1 \otimes \cdots \otimes \mathcal{F}_m\), then the filtration \((\mathcal{F}_t^k, t \in \mathbb{J}^k)\) is regular if \((\mathcal{F}_t, t \in \mathbb{J})\) satisfies \(C\), and for each \(k > 2\), \((\mathcal{F}_t^k, t \in \mathbb{J}_k)\) satisfies \(MR\).

If \(\mathbb{J} = \mathbb{N}\), \(k = 1, 2, \ldots, m\), then necessarily the filtration \((\mathcal{F}_t^k)\) is regular.

### III-3. Maximal Inequalities

We now prove weak and strong maximal inequalities for positive submartingales.

**Theorem III-3.1.** (Maximal inequality under \(MR\)). Let \((\mathcal{F}_t, t \in \mathbb{J})\) be a filtration satisfying \(MR\). Fix \(v \in \mathbb{J}\). There is a constant \(c\) depending only on \((\mathcal{F}_t)\) such that for every positive submartingale \((X_t, \mathcal{F}_t)\) and every \(\lambda > 0\) one has, letting \(X^* = \mathbb{E} \sup_{t \leq v} X_t\),
If there is an upper bound \( \beta \) for the excess \( M(\epsilon, \alpha) \) as \( \epsilon \to 0 \), then

\[
P[X^* \geq \lambda] \leq \frac{\beta + 1}{\alpha \lambda} \int_{\{X^* \geq \lambda\}} X \, dP.
\]

**Proof:** As shown in the previous section, one can assume that \( \alpha \) is the largest number such that \( MR \) holds. Fix \( \epsilon > 0 \), \( \lambda > 0 \) and \( 0 < \delta < \lambda \). Let \( A_t = \{X > \lambda - \delta\} \) for \( t \leq \nu \); \( A_t = \emptyset \) otherwise.

There exist \( M = M(\epsilon, \alpha) \) and \( \tau \in IM \) such that \( \epsilon / \tau \leq M \) and

\[
P[A(\tau)] \geq \alpha(1 - \epsilon) P[e \sup_{\tau} A_t].
\]

Hence

\[
P[X^* > \lambda - \delta] \leq P[e \sup_{\tau} A_t]
\]

\[
\leq \frac{1}{\alpha (1 - \epsilon)} P[A(\tau)]
\]

\[
= \frac{1}{\alpha (1 - \epsilon)} P[U(\tau = t) \cap A_t]
\]

\[
\leq \frac{1}{\alpha (1 - \epsilon)} \sum_{t} P[(\tau = t) \cap A_t]
\]

\[
\leq \frac{1}{\alpha (1 - \epsilon)} \frac{1}{\lambda - \delta} \sum_{j} \int_{t \{\tau = t\} \cap A_t} X \, dP.
\]
Let $\delta > 0$. The relation (1) follows on letting $\epsilon = \frac{1}{2}$ and $c = 2[M(\frac{\epsilon}{2} , \alpha) + 1]$. If $M(c, \alpha) \leq \beta$ for all $\epsilon$, (2) follows on letting $\epsilon \to 0$. □

We recall some facts about Orlicz spaces (see also [34] and [52]).

Let $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing, left continuous function which is zero at the origin. Assume also that \( \lim_{t \to \infty} \phi(t) = \infty \). Let \( \phi(t) = \int_0^t \phi(s)ds \). For a random variable \( X \), let

\[
\|X\|_\phi = \inf\{a > 0 : E[\phi(\frac{|X|}{a})] \leq 1\}.
\]

We denote by \( L \) the Orlicz space of all those \( X \) for which \( \|X\|_\phi < \infty \). A process \( (X_t) \) is bounded in \( L \) if \( \sup_{t\phi} \|X_t\|_\phi < \infty \).

The function \( \phi(t) \) satisfies condition \( \Delta_2 \) (at infinity) if
\[ \limsup_{t \to \infty} \frac{\phi(2t)}{\phi(t)} < \infty. \text{ If } \Delta_2 \text{ is satisfied, then } E[\phi(|X|)] < \infty \text{ if and only if } \|X\| < \infty. \text{ In general } \|X\| \leq \max(1, E[\phi(|X|)]). \text{ Let } \phi_m(t) = t(\log^+ t)^m. \text{ The space } L_{\phi_m} \text{ is denoted } L_{\log^m L}. \text{ Since } \phi_m(t) \text{ satisfies the } \Delta_2 \text{ condition, a process } (X_t) \text{ is bounded in } L_{\log^m L} \text{ if and only if } \sup_t E[|X_t| (\log^+ |X_t|)^m] < \infty. \text{ We set } \phi_0(t) = t \text{ for all } t \geq 0 \text{ and consider } L_{\phi_0} = L_1 \text{ as an Orlicz space.}

\textbf{Lemma III-3.2.} \text{ Let } (\mathcal{F}_t, t \in J) \text{ be a filtration satisfying } MR. \text{ Fix } v \in J. \text{ Let } (X_t, \mathcal{F}_t, t \in J) \text{ be a positive submartingale. Then there is a constant } c \text{ depending on } (\mathcal{F}_t) \text{ such that for every } \eta \text{, and every } k \geq 0, \text{ one has}

\begin{align*}
(3) \quad &E[e^{\sup_{t \leq v} \phi_k(X_t)}] \\
&\leq \frac{e}{e-1} [\eta + |\log \eta| E[\phi_k(\frac{c}{\alpha} X_v)] + (k+1) E[\phi_{k+1}(\frac{c}{\alpha} X_v)]]
\end{align*}

\textbf{Proof:} Since the function } \phi_k(t) \text{ is convex and increasing, by Jensen's inequality } Y = \phi_k(X_t) \text{ is a positive submartingale with respect to } (\mathcal{F}_t). \text{ Let } Y_v^* = e^{\sup_{t \leq v} X_t}. \text{ From Theorem III-3.1, one has}
Hence applying Fubini and using the elementary inequality 
\( a \log^+ b \leq \log^+ a + \frac{b}{e} \), one obtains (see also [48], page 23)

\[
E[Y^*] = \int_0^\infty P(Y^* > \lambda) dP
\]

\[
\leq \eta + \int_0^\infty \frac{c}{\alpha \lambda} \int_{\{Y^* > \lambda\}} \phi_k(X) dP d\lambda
\]

\[
\leq \eta + \int \phi_k \left( \frac{c}{\alpha} X \right) \int_{\eta}^{Y^*} \frac{1}{\lambda} d\lambda dP \quad \text{(because } \frac{c}{\alpha} \geq 1)\]

\[
= \eta + \int \phi_k \left( \frac{c}{\alpha} X \right) [\log \frac{Y^*}{\eta} - \log \eta] dP
\]

\[
\leq \eta + \int [\phi_k \left( \frac{c}{\alpha} X \right) \log^+ \phi_k \left( \frac{c}{\alpha} X \right) + \frac{Y^*}{e} + |\log \eta| \phi_k \left( \frac{c}{\alpha} X \right)] dP
\]
(4) \[ \mathbb{E}[Y^*] \]

\[
\leq \frac{e}{e-1} \left[ \eta + |\log \eta| \mathbb{E}[\phi_k \left( \frac{C}{\alpha} X \right)] + \mathbb{E}[\phi_k \left( \frac{C}{\alpha} X \right) \log^+ \phi_k \left( \frac{C}{\alpha} X \right)] \right].
\]

Now applying

\[ t(\log^+ t)^k \log^+(t(\log^+ t)^k) \leq (k+1) t(\log^+ t)^{k+1} \]

one obtains

(5) \[ \phi_k \left( \frac{C}{\alpha} X \right) \log^+ \phi_k \left( \frac{C}{\alpha} X \right) \leq (k+1) \phi_{k+1} \left( \frac{C}{\alpha} X \right). \]

Inequality (3) now follows from (4) and (5). \[ \square \]

**Proposition 111-3.3.** Let \((\mathcal{F}_t, t \in \mathcal{J}^m)\) be a filtration such that for each \(1 \leq k \leq m\), there is \(\alpha_k, 0 < \alpha_k < 1\), such that \((\mathcal{F}_{t-k}^{1-k}, t \in \mathcal{J})\) satisfies MR for each \(t_1, t_2, \ldots, t_{k-1}\) fixed.

Fix \(v \in \mathcal{J}^m\). Let \((X_{t,v}, t \in \mathcal{J}^m)\) be a positive block submartingale. Then there exists a constant \(C_m\) (depending on the filtration), such that for every \(\delta > 0\) there exists a constant \(A(m,\delta) > 0\) with

(6) \[ \mathbb{E}[\exp \left( \mathbb{E} \left[ \phi_k \left( \frac{C}{\alpha} X \right) \right] \right) \]

\[ \leq \delta + A(m,\delta) \mathbb{E}[\phi_k \left( \frac{C}{\alpha} X \right) \mid \mathcal{F}_m v]. \]
$C_m$ may be chosen equal to \( \frac{c_1 \ldots c_m}{\alpha_1 \ldots \alpha_m} \) where \( c_1 = 2[M(\frac{1}{2}, \alpha_1) + 1] \),

\( i = 1, 2, \ldots, m \).

**Proof:** We use induction on \( m \) to prove (6). For \( m = 1 \) (6) holds with \( A(1, \delta) = \frac{e}{e-1} (|\log \delta| + 1) \): apply Lemma III-3.2. with \( k = 0 \) and \( \eta = \frac{\delta}{2} \). Suppose that (6) holds for \( m = n \). Let

\((X, \mathcal{F}, t \in \mathcal{J}^{n+1})\) be a positive block submartingale. For \( t = (t_1, \ldots, t_n, t_{n+1}) \in \mathcal{J}^{n+1} \), denote \((t_1, \ldots, t_n)\) by \( t' \). Let \( \mathcal{J}_{t'} = \mathcal{J}_{1}^{1-n} \). Since \( \mathcal{J}_{t'} = \mathcal{J}_{1}^{1-k} \) for all \( t \in \mathcal{J}^{n+1} \), it follows that for each fixed \( t_n+1 \in \mathcal{J}^{n+1} \), the process \((X, \mathcal{F}, t' \in \mathcal{J}^{n})\) is an \( n \)-parameter block martingale. Let

\( Y_{t'} = \sup_{t_n+1 \leq v} X_{t} \). Then \((Y_{t'}, \mathcal{J}_{t'}, t' \in \mathcal{J}^{n})\) is an \( n \)-parameter block submartingale. The filtration \((\mathcal{J}_{t'}, t' \in \mathcal{J}^{n})\) satisfies the assumptions of the theorem with \( m = n \) since \((\mathcal{J}_{t}, t \in \mathcal{J}^{n+1})\) does with \( m = n + 1 \). We have \( X^* = \sup_{v \geq t} X_t = e \sup_{t_{n+1} \leq v} \left( e \sup_{t_{n} \leq v} X_t \right) = \sup_{t_{n+1} \leq v} Y_{t'} = Y^* \). Applying the induction hypothesis to the block submartingale \((Y_{t'}, \mathcal{J}_{t'}, t' \in \mathcal{J}^{n})\),
one obtains

$$E[X^*] = E[Y^*] \leq \frac{\delta}{2} + A(n, \frac{\delta}{2}) \sup_{0 \leq k \leq n} E[\phi (C, Y, v)] .$$

The stochastic basis \((\mathcal{F} \left( v_1, \ldots, v_n, t \right), t_{n+1} \in \mathcal{J}_{n+1})\) satisfies condition MR. Thus Theorem III-3.1. can be applied to the one-parameter positive submartingale \((C \left( v_1, \ldots, v_n, t \right), t_{n+1} \in \mathcal{J}_{n+1})\).

From Lemma III-3.2. one has

$$E[\phi (C, Y, v)] = E[e \sup_{n+1 \leq v} \phi (C \left( v_1, \ldots, v_n, t \right), t_{n+1} \in \mathcal{J}_{n+1})]$$

$$\leq \frac{e}{e-1} \left[ \eta + |\log \eta| \right] E[\phi (C \left( v_1, \ldots, v_n, t \right), t_{n+1} \in \mathcal{J}_{n+1})] + (k+1) E[\phi (C \left( v_1, \ldots, v_n, t \right), t_{n+1} \in \mathcal{J}_{n+1})].$$

It follows that

$$\sup_{0 \leq k \leq n} E[\phi (C, Y, v)] \leq \frac{e}{e-1} \left[ \eta + (|\log \eta| + n+1) \right] \sup_{k \leq l \leq n+1} E[\phi (C, X, v)].$$

Choose now \(\eta_0\) so that \(A(n, \frac{\delta}{2}) \frac{e}{e-1} \eta_0 \leq \frac{\delta}{2}\) and apply (9) with \(\eta = \eta_0\) to (7), to obtain
\[
E[X^*_v] \leq \delta + A(n+1,\delta) \sup_{0 \leq k \leq n+1} E[\phi_1(C_{n+1}^*, X_v)]
\]

where \(A(n+1,\delta) = A(n, \frac{\delta}{2})(|\log \tau_0| + n + 1) \frac{e}{e-1}\). Thus \(6\) is true for all \(m\). \(\square\)

Lemma III-3.4. Let \((\mathcal{F}_t, t \in J)\) be a filtration satisfying \(C\). For every positive submartingale \((X_t, \mathcal{F}_t)\) and every \(\lambda > 0\) one has

\[
(11) \quad P[\text{e lim sup } X_t > \lambda] \leq \frac{1}{\lambda} \lim E[X_t].
\]

The lemma is proved in Section III-6., following Theorem III-6.2.

Proposition III-3.5. Let \((\mathcal{F}_t, t \in J^m)\), \(m > 1\), be a regular filtration. Let \((X_t, \mathcal{F}_t, t \in J^m)\) be a positive block submartingale. Then there exists a constant \(C_m^i\) (depending on the filtration), such that for every \(\delta > 0\) there is a constant \(A(m-1,\delta)\) with

\[
(12) \quad E[\text{e lim sup } X_t > \lambda] \leq [\delta + A(m-1,\delta)] \sup_{0 < k < m-1} \lim_{J^m} E[\phi_1(C^*, X_t)],
\]

\(C_m^i\) may be chosen equal to \(\frac{c_2 \cdots c_m}{\alpha_2 \cdots \alpha_m}\) where \(c_i = 2[M(\frac{1}{2}, \alpha_i) + 1]\), \(i = 2, \ldots, m\).
Proof: For fixed \( t_2, \ldots, t_m \in J \), the process \((X_{t_1,t_2,\ldots,t_m}, t_1 \in J)\) is a one parameter submartingale.

Hence \((e \sup_{t_1,t_2,\ldots,t_m} X_{t_1,t_2,\ldots,t_m}, t_1 \in J)\) is also a one parameter positive submartingale. The filtration \((\mathcal{F}_t, t \in J)\) satisfies \(\mathcal{C}\) since \((\mathcal{F}_t)\) is regular (Section III-2.). Therefore, by Lemma III-3.4.,

one has

\[
P\big[ e \lim_{t \to \infty} \sup_{t_1,t_2,\ldots,t_m} X_{t_1,t_2,\ldots,t_m} > \lambda \big] \leq \frac{1}{\lambda} \lim_{t \to \infty} E\big[ e \sup_{t_1,t_2,\ldots,t_m} (X_{t_1,t_2,\ldots,t_m}) \big].
\]

But \(e \lim_{t \to \infty} \sup_{t_1,t_2,\ldots,t_m} X_{t_1,t_2,\ldots,t_m} \leq e \lim_{t \to \infty} \sup_{t_1} (X_{t_1,t_2,\ldots,t_m})\), hence

\[
P\big( e \lim_{t \to \infty} \sup_{t_1} X_{t_1,t_2,\ldots,t_m} \big] \leq \frac{1}{\lambda} \lim_{t \to \infty} E\big[ e \sup_{t_1,t_2,\ldots,t_m} (X_{t_1,t_2,\ldots,t_m}) \big].
\]

For \( t \in J \) fixed, the process \((X_{t_1,t_2,\ldots,t_m}, t \in J^m)\) is a positive \((m-1)\)-parameter block submartingale. Indeed,

\[
E[X_{t_1,t_2,\ldots,t_m} | \mathcal{F}_{t_1,s_2,\ldots,s_m}] \geq X_{t_1,s_2,\ldots,s_m,t_{k+1},\ldots,t_m}.\]

Since \((\mathcal{F}_t)\) is regular, the filtration \((\mathcal{F}_t, t \in J^m)\) satisfies the assumptions of Proposition III-3.3. for \(m-1\). Hence
The proposition now follows from inequalities (13) and (14). □

III-4. Convergence and Demiconvergence

In this section we will apply maximal inequalities to obtain some convergence and demiconvergence results. We first recall the following basic lemma [52], page 96).

**Lemma III-4.1.** In order that the net \((x_t, t \in J)\) converge in a complete metric space on which it is defined, it suffices that \((x_n, n \in \mathbb{N})\) be convergent for all increasing sequences \((t_n, n \in \mathbb{N})\) in \(J\).

**Lemma III-4.2.** Let \((X_t, \mathfrak{F}_t, t \in J^m), m > 1\) be a martingale or positive submartingale bounded in \(L \log^{m-1} L\). Let \(X = \text{slim} X_t\). Then \(X + X\) in \(L \log^k L\) for all \(k \leq m-1\).

**Proof:** Since convergence in probability is determined by a complete metric, it follows that the (sub)martingale \((X_t)\) converges in
probability to a random variable $X$. Bounded in $L \log^{m-1} L$, $m > 1$, $(X_t)$ is necessarily uniformly integrable and therefore it converges to $X$ in $L_1$. Fatou's lemma holds also for directed index sets, hence $X \in L \log^{m-1} L$. Since $\int_A X dP = \int_A X dP (\leq)$ for all $s \leq t$, $A \in \mathcal{F}_s$, it follows that $\int_A X dP = \int_A X dP (\leq)$ for all $A \in \mathcal{F}_s$. Consequently $X_t = E^t(X) (\leq)$. Observe that $|X_t| \leq E^t(|X|)$ if $(X_t)$ is a martingale or a positive submartingale. By Jensen's inequality, one has

$$
\phi_k (|X_t - X|) \leq \frac{1}{2} \left[ \phi_k (2|X|) + \phi_k (2|X_t|) \right]
$$

$$
\leq \frac{1}{2} \left[ \phi_k (2|X|) + \phi_k (2E^t(|X|)) \right]
$$

$$
\leq \frac{1}{2} \left[ \phi_k (2|X|) + E^t(\phi_k (2|X|)) \right], \quad k > 0.
$$

Thus $(\phi_k (|X_t - X|), t \in J^m)$ is uniformly integrable for all $k \leq m-1$. But $\phi_k (|X_t - X|)$ converges to 0 in probability, since $|X_t - X|$ does, and therefore $\phi_k (|X - X|)$ converges to 0 in $L_1$ for all $k \leq m-1$, that is, $X + X$ in $L \log^k L$ for all $k \leq m-1$. 

The following is in a sense the main result of the chapter.

**Theorem III-4.3.** Let $(\mathcal{F}_t, t \in J^m), m > 1$, be a regular filtration. Let $E$ be a Banach space and $\pi : E \rightarrow \mathbb{R}^+$ a continuous map with
\[ \pi(0) = 0, \pi(x+y) \leq \pi(x) + \pi(y) \] for all \( x, y \in E \). Assume that there is a random variable \( X \) such that \( \pi(X - X) \) converges to 0 in \( L \log L \) for all \( k \leq m-1 \). Suppose that for each \( t_0 \in J_m \),

\[ (\pi(X - X), t_0, t \geq t_0) \] is a positive block submartingale. Then

\[ \lim_{t \to \infty} \pi(X - X) = 0. \]

**Proof:** Fix \( \varepsilon > 0, \lambda > 0 \). Choose \( \delta > 0 \) such that \( \delta < \frac{e \lambda}{6} \). Then choose \( \eta, 0 < \eta < \inf( \frac{e}{3}, \frac{e \lambda}{6} ) \), such that \( A(m-1, \delta) \eta < \delta \), where \( A(m-1, \delta) \) is the constant from Proposition III-3.5. Next choose \( X \), so that

\[ \sup_{0 < k < m-1} \lim_{t \to \infty} E[\phi(\pi(X - X))] \leq \eta. \]

Note that this implies \( E[\phi(\pi(X - X))] = C \pi(X - X) \leq \eta \), hence \( \pi(X - X) \leq \eta \) since \( C \geq 1 \). Then

\[ \mathbb{P}[\lim_{t \to \infty} \max_{0 \leq t \leq m} \pi(X - X) > \lambda] \leq \mathbb{P}[\lim_{t \to \infty} \max_{0 \leq t \leq m} (\pi(X - X) + \pi(X - X)) > \lambda] \leq \mathbb{P}[\lim_{t \to \infty} \max_{0 \leq t \leq m} \pi(X - X) > \frac{\lambda}{2}] + \mathbb{P}[\pi(X - X) > \frac{\lambda}{2}]. \]
\[
\frac{2}{\lambda} \sum_{k=0}^{m-1} \left[ \delta + A(m-1,\delta) \sup_{0 \leq t \leq m-1} \mathbb{E}[\phi(C^m_{X_t - X_t})] \right]
+ \frac{2}{\lambda} \pi(X - X)_t^1 \\
\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\]

Hence \( e \lim_{t \to t} \pi(X - X) = 0 \). \( \Box \)

**Theorem III-4.4.** Let \( (\mathcal{F}_t, t \in \mathbb{N}^m) \), \( m > 1 \), be a regular filtration.

(i) Let \( (X_t, \mathcal{F}_t, t \in \mathbb{N}^m) \) be a block submartingale bounded in \( L \log^{m-1} \) and let \( X = \lim_{t} X_t \). Then \( e \lim_{t} \sup_{t} X_t = X \) (upper demiconvergence).

(ii) Let \( (X_t, \mathcal{F}_t, t \in \mathbb{N}^m) \) be a positive (integrable) block martingale and let \( X = \lim_{t} X_t \). Then \( e \lim_{t} \inf_{t} X_t = X \) (lower demiconvergence).

(iii) Let \( (X_t, \mathcal{F}_t, t \in \mathbb{N}^m) \) be a block martingale bounded in \( L \log^{m-1} \). Let \( X = \lim_{t} X_t \). Then \( e \lim_{t} X_t = X \).

**Proof:** (i) Assume first that \( (X_t) \) is positive block submartingale.

Then by Lemma III-4.2., \( X_t + X \) in \( L \log^k \) for all \( k \leq m-1 \). Now
apply Theorem III-4.3. to the process \((X_t, \mathcal{F}_t)\) and \(\pi: \mathbb{R} \to \mathbb{R}^+, \pi(x) = x^+\) to obtain \(\ellim(X_t - X_t^+) = 0\). This shows that \\
\[\ellsup(X_t) \leq X_t\], and the equality follows from the general inequality \(X = s \lim X \leq \ellsup X_t\). The theorem is thus proved for positive block submartingales, and consequently for block submartingales which are bounded below by a constant. Let \((X_t)\) be a (not necessarily positive) block submartingale. Fix a real number \(a\), then the process \((X_t Va, \mathcal{F}_t)\) is also a block submartingale and thus \(\ellsup(X_t Va) = XVa\) by the first part of the proof. Since \(\ellsup(X_t V(-n)) = XV(-n)\) for all \(n > 0\) and since \(\ellsup X_t > \to\) by Fatou's lemma, it follows that \(\ellsup X_t = X_t\).

(ii) Let \((X_t, \mathcal{F}_t)\) be a positive integrable block martingale. Then \((X_t)\) converges in probability to a random variable \(X_t\). Let \(U_t = e^{-X_t}\); then \((U_t, \mathcal{F}_t)\) is a block submartingale bounded in \(L_{\infty}\), which converges in probability to \(U = e^{-X}\). By part (i), \(\ellsup U_t = U\). Fatou's lemma implies \(\ellinf X_t < \to\), thus \(\ellsup(-X_t) = (-X)\) or \(\ellinf X_t = X\).

(iii) Let \((X_t)\) be a block martingale bounded in \(L \log^{m-1} L\). By lemma III-4.2., \(X_t + X_t \in L \log^k L\) for all \(k \leq m-1\). Now apply Theorem III-4.3. to the process \((X_t, \mathcal{F}_t)\) and \(\pi: \mathbb{R} \to \mathbb{R}^+, \pi(x) = |x|\).
to obtain $\lim_{t \to \infty} |X_t - X| = 0$. □

Recall that if $J_k = \mathbb{N}$, $k = 1,2,\ldots,m$, then the filtration $(\mathcal{F}_t)$ is regular (section III-2). The case $m = 2$, $J_1 = J_2 = \mathbb{N}$ was obtained in [49], Theorem 2.1, and [48], Theorem 1.1.

Theorem III-4.5. Let $(\mathcal{F}_t, t \in J^m)$, $m > 1$, be a regular filtration. Let $(E, \mathcal{B})$ be a Banach space with the Radon-Nikodym property. Let $(X_t, \mathcal{F}_t)$ be an $E$-valued block martingale bounded in $L \log^{m-1}L(E)$, i.e., such that $\sup_t E[\|X_t\|(\log^+ \|X_t\|^{m-1})] < \infty$. Then $(X_t)$ converges essentially and in $L \log^{m-1}L(E)$ to a random variable $X$.

Proof: Since $(X_t)$ is uniformly integrable and $E$ has the Radon-Nikodym property, $(X_t)$ admits a representation $X_t = E^t(X)$ for an $E$-valued random variable $X$ and $E[\|X\|(\log^+ \|X\|^{m-1})] < \infty$. By Lemma III-4.2., $X_t \to X$ in $L \log^k L(E)$ for all $k \leq m-1$. Then Theorem III-4.3. applied to the process $(X_t, \mathcal{F}_t)$ and $\pi : E \to \mathbb{R}^+$, $\pi(x) = \|x\|$, gives $\lim_{t \to \infty} \|X_t - X\| = 0$. □

There is also a version of our results corresponding to the case where the directed set is filtering to the left. For the sake of simplicity we assume that $J_k = -\mathbb{N} = \{\ldots -3,-2,-1\}$ for all $k \leq m$. For any integers $k, \ell$, $1 \leq k \leq \ell \leq m$. 


An integrable process \((X_t, \mathcal{F}_t, t \in J^m)\) is a reversed block \(k\) (sub)martingale for a fixed \(k < m\), if \(E_s^{1-k}(X_t) = X(s_1, \ldots, s_k, t, t_{k+1}, \ldots, t_m) \geq \) for \(s \leq t\). An integrable process is a reversed block (sub)martingale if it is a reversed block \(k\) (sub)-martingale for all \(k < m\).

**Theorem III-4.6.** (i) Let \((X_t, \mathcal{F}_t, t \in J^m)\) be a reversed block submartingale such that \(X((-1,-1,\ldots,-1)}\) is \(L \log^{m-1} L\) integrable (hence \((X_t)\) is \(L \log^{m-1} L\) bounded). Let \(X = \lim s \sup X_t\). Then \(\lim \sup X_t = X\) a.s.

(ii) Let \((X_t, \mathcal{F}_t, t \in J^m)\) be a reversed positive (integrable) block martingale. Let \(X = \lim s \lim X_t\). Then \(\lim \inf X_t = X\) a.s.

(iii) Let \((X_t, \mathcal{F}_t, t \in J^m)\) be a reversed block martingale such that \(X((-1,-1,\ldots,-1)}\) is \(L \log^{m-1} L\) integrable. Let \(X = \lim e \lim X_t\). Then \(e \lim X_t = X\) a.s.
The proof is similar to the case filtering to the right, but simpler, because the process \((X - X, t \in J)\), where \(X = \lim_{k} X_{t}\), is now adapted and converges to 0 in \(L \log L\) for all \(k \leq m-1\).

An application to the multiparameter Marcinkiewicz theorem for \(p < 1\), similar to the one given for \(m = 2\) in [22], is possible. In this application the \(\sigma\)-fields are not of product type, and they do not satisfy the conditional independence assumption (F4) because the conditional expectations are with respect to fewer and fewer sums: one applies a multiparameter version of the classical Doob reversed martingale argument, extended in [22] to submartingales.

III-5. Banach Lattices

In this section we extend the demiconvergence results to random variables taking values in a separable Banach lattice \((E, \| \cdot \|)\). We at first consider the case \(E = L_{1}(\Omega, \mathcal{F}, P)\) where \((\Omega, \mathcal{F}, P)\) is a fixed probability space.

The following lemma is part of a more general theory developed in [7].

**Lemma III-5.1.** Let \((X_{t}, \mathcal{F}_{t}, t \in J)\) be an \(E\)-valued positive Bochner integrable martingale. Let \(Y_{t} : \Omega \times \Omega_{1} \rightarrow \mathbb{R}^{+}, Y_{t}(\omega, \omega_{1}) = X_{t}(\omega)(\omega_{1})\) \(P\)-a.s.
Then \((Y, \mathcal{F} \otimes \mathcal{F}, t \in J)\) is a positive martingale. Moreover, if \(X = e \liminf_{t} X, Y = e \liminf_{t} Y\), then \(P\)-a.s. \(X(\omega) \in E\) and \(X(\omega)(\cdot) = Y(\omega, \cdot)\).

**Proof:** Since \(X_t\) is strongly measurable, \(Y_t\) is measurable with respect to \(\mathcal{F} \otimes \mathcal{F}\). If \(s \leq t\) and \(A \in \mathcal{F}\), then \(\int A dP = \int A dP\). Thus for \(A \in \mathcal{F}, B \in \mathcal{F}\)

\[
\mu_s (A \times B) = \int_{A \times B} Y d(P \otimes P) = \int_{A} \int_{B} Y(\omega, \omega) dP(\omega) dP(\omega)
\]

\[
= \int_{B} \int_{A} X(\omega)(\omega) dP(\omega) dP(\omega) = \int_{A} \int_{B} X(\omega)(\omega) dP(\omega) dP(\omega)
\]

\[
= \int_{A \times B} Y d(P \otimes P) = \mu_s (A \times B).
\]

Therefore \(\mu_s, \mu_t\) are defined on the semi algebra \(\{A \times B : A \in \mathcal{F}, B \in \mathcal{F}\}\) of measurable rectangles, and by their definition are bounded and \(\sigma\)-additive. Thus \(\mu_s, \mu_t\) can be extended to measures on \(\mathcal{F} \otimes \mathcal{F}\). Therefore \((Y, \mathcal{F} \otimes \mathcal{F})\) is a martingale. \(\square\)

In Banach lattices with the Radon-Nikodym property, \(L_1\)-bounded positive submartingales indexed by \(\mathbb{N}\) converge a.s., as proved by Heinich [32] hence in probability. In the multiparameter case we have the following.
Theorem III-5.2. Let \((\mathcal{F}_t, t \in J^m), m > 1\) be a regular filtration. Let \((E, \mathcal{F}, \mathbb{P})\) be a Banach lattice with the Radon-Nikodym property.

(i) Let \((X_t, \mathcal{F}_t, t \in J^m)\) be an \(E\)-valued, \(L \log^{m-1} L\) bounded, positive block submartingale. Then the stochastic limit \(s \lim X = X\) exists and \(e \lim \inf (X_t - X)^+ \leq 0\).

(ii) Let \((X_t, \mathcal{F}_t, t \in J^m)\) be an \(E\)-valued, \(L_1\)-bounded, positive block martingale. Then the stochastic limit \(s \lim X = X\) exists and \(e \lim \inf X_t = X\).

Proof: (i) Let \((t_n)\) be an increasing sequence in \(J^m\). Then \((X_{t_n})\) is a positive submartingale, hence converges in probability. By Lemma III-4.1., the net \((X_{t_n})\) converges in probability, say to \(X\). By Lemma III-4.2. \(X_t - X\) in \(L \log^k L(E)\) for all \(k \leq m-1\). Then Theorem III-4.3. applied to the process \((X_t, \mathcal{F}_t, \mathbb{P})\), \(\pi : E \to \mathbb{R}_+^+, \pi(x) = \|x\|_1\) gives \(e \lim \inf (X_t - X)^+ \leq 0\).

(ii) Let \((X_t)\) be a positive \(L_1\)-bounded block martingale. As in (i) one shows that \(s \lim X = X\) exists. Let \(E\) have the Radon-Nikodym property then \(c_0\) is not contained in \(E\) ([16], pp. 60 and 81) and therefore \(E\) is weakly sequentially complete ([39], p. 34) hence order continuous. Thus \(E\) is order isometric to an ideal of an \(L_1^0(\Omega, \mathcal{F}, \mathbb{P})\) ([39], p. 25). Then the real-valued, positive block
martingale \( (Y, \mathcal{F} \otimes \mathcal{F}, t \in J^m) \) necessarily converges to \( Y \) in probability, where \( Y(\omega, \omega_1^t) = X(\omega)(\omega_1^t) \), and \( \lim_{t \to \infty} \inf_{t \in J^m} Y_t = Y \).

(Theorem III-4.4.) By Lemma III-5.1., \( \lim_{t \to \infty} \inf_{t \in J^m} X_t = X \). \( \square \)

The case \( m = 2, J = \mathbb{N} \), of Theorem 4.2. (i) was obtained in [49].

III-6. Relations between Convergence and Demiconvergence

We show here that in general convergence of martingales is equivalent with demiconvergence of submartingales. Under commutation, this extends to block martingales. It is also remarked that under commutation the proof of Theorem III-4.4. simplifies.

The following is joint version of Krickeberg and Riesz decomposition theorems in the directed index set case.

**Proposition III-6.1.** Let \( L_\phi \) be an arbitrary Orlicz space. Let 
\( (X, \mathcal{F}, t \in J) \) be a submartingale bounded in \( L_\phi \). Then \( X = Y_1^t - Y_2^t - S_t \) where \( (Y_1^t, \mathcal{F}_t) \), \( (Y_2^t, \mathcal{F}_t) \) are positive martingales bounded in \( L_\phi \) and \( (S_t, \mathcal{F}_t) \) is a positive supermartingale with \( E(S_t) < \infty \).

If \( (X_t) \) is a martingale then \( S_t = 0 \) for all \( t \).
Proof: Let \( s \in J \) be fixed but arbitrary. Set \( U_t = E^S(X_t) \), \( t \geq s \).

Then \( t < t' \) implies \( E^t(X_{t'}) > X_t \), consequently \( U_t = E^t(X_t) \). Thus \( (U_t, t \geq s) \) is increasing and therefore converges essentially to \( Y_s \). By Jensen's inequality

\[
E[\phi(|U_t|)] = E[\phi(|E^S(X_t)|)] \leq E[\phi(|X_t|)] .
\]

Since \( Y = \lim_{t \uparrow} U_t \) by Fatou's lemma \( E[\phi(|Y_t|)] \leq \lim_{t \uparrow} E[\phi(|X_t|)] \).

Since \( s \) is arbitrary, \( \sup_{s \in J^m} E[\phi(|Y_s|)] < \infty \). If \( s < s' \) then

\[
E^S(Y_s) = E^S(\lim_{t \uparrow} E^S(X_t)) = \lim_{t \uparrow} E^S(X_t) = Y_s , \text{ hence } (Y_t)_{t \geq s} \text{ is a martingale and } Y_t > X_t . \text{ We write } S = Y_t - X_t ; S_t \text{ is a positive supermartingale. For } t \geq s \ E^S(S_t) = E^S(Y_t) - E^S(X_t) = Y_t - E^S(X_t) , \text{ hence } \lim_{t \uparrow} E^S(S_t) = 0 . \text{ It follows that }
\]

\[
\lim_{t \uparrow} E[E^S(S_t)] = 0 , \text{ thus } E(S_t) = 0 . \text{ Define } Y^1_t = \begin{cases} \lim_{t \uparrow} E^S(Y^+), & t < s \\ \lim_{t \uparrow} E^S(Y^-), & t \geq s \end{cases} \text{ for both } (Y^+_t) \text{ and } (Y^-_t) \text{ are positive submartingales bounded in } L_\phi , \text{ the above argument shows that each } (Y^+_t)_{t \geq s} , i = 1, 2 \text{ is a positive martingale bounded in } L_\phi . \text{ Clearly } Y_t = Y^1_t - Y^2_t . \Box
Proposition IV-6.1. remains valid if $L_\Phi$ is replaced by $L_\Phi\cdot$

**Theorem III-6.2.** Let $(\mathcal{F}_t, t \in J)$ be a stochastic basis.

(i) Assume that all martingales bounded in $L_\Phi$ essentially converge. Then all submartingales bounded in $L_\Phi$ essentially demiconverge.

(ii) Assume that all positive submartingales bounded in $L_\Phi$ essentially upper demiconverge. Then all martingales bounded in $L_\Phi$ essentially converge.

**Proof:** (i) Let $(X_t, \mathcal{F}_t)$ be a submartingale bounded in $L_\Phi$. By Proposition III-6.1., $X_t = Y_t - S_t$ with $Y_t = Y_1^t - Y_2^t$ a martingale and $E(S_t) \to 0$. Let $X_t = s \lim X_t$, then $s \lim S_t = 0$ implies $s \lim Y_t = X_t$. The martingale $(Y_t, \mathcal{F}_t)$ is bounded in $L_\Phi$ and therefore essentially converges to $X_t$. Hence $e \limsup X_t \leq X_t$. Since $s \lim X_t \leq e \lim X_t$ always, $e \limsup X_t = X_t$.

(ii) Let $(Y_t, \mathcal{F}_t)$ be a positive martingale bounded in $L_\Phi$. Let $Y_t = s \lim Y_t$. Since $(Y_t)$ is also a positive submartingale

$e \limsup Y_t = Y_t$. Let $X_t = e^{-Y_t}$; then $(X_t)$ is an $L_\infty$-bounded positive submartingale. Hence $e \limsup e^{-Y_t} = e^{-Y_t}$ and

$e \liminf Y_t = Y_t$. Therefore $e \lim Y_t = Y_t$. If $(Y_t)$ is not
positive apply Proposition III-6.1. to represent $Y_t$ as a difference of two positive martingales bounded in $L^1$. □

Since condition $C$ is known to imply convergence of $L^1$-bounded martingales, ([47], Theorem 3.3.), we have the following:

**Corollary III-6.3.** Let $(\mathcal{F}_t, t \in J)$ be a stochastic basis satisfying condition $C$. Let $(X_t, \mathcal{F}_t)$ be an $L^1$-bounded submartingale and let $X$ be its limit in probability. Then $\lim_{t} \sup X_t = X$.

Let $(\mathcal{F}_t, t \in J)$ be any stochastic basis and $(X_t, \mathcal{F}_t)$ a positive submartingale. It was shown in [46], Theorem 1.4., that

$$P(\lim_{t} \sup X_t > \lambda) \leq \frac{1}{\lambda} \lim_{t} E[X_t].$$

The proof of Lemma III-3.4. now follows.

Observe that $L^1$-bounded submartingales need not converge not only under $C$, but even under the stronger Vitali condition $V$ ([37], [31]). We also note that since the covering conditions $V_p, 1 \leq p < \infty$, are necessary and sufficient for the convergence of $L^q$-bounded, martingales, $\frac{1}{p} + \frac{1}{q} = 1$, (Krickeberg [36] and A. Millet [41]), they are also necessary and sufficient for the essential upper demiconvergence of $L^q$-bounded submartingales. Analogous results hold for classes of Orlicz spaces ([38], [65]).
We now sketch an alternative proof of Theorem III-4.4, assuming commutation. Recall that

\[ \mathcal{F}^k_t = \mathcal{F}(t_1, \ldots, t_{k-1}, t_k, t_{k+1}, \ldots, t_m) \]

and \( E^k_t \) is the conditional expectation given \( \mathcal{F}^k_t \).

The commutation assumption is that the \( L_1 \)-operators \( E^k_t \) commute, [40] page 3. Observe that \( E^k_t \) commute if for every martingale \( X_t = E^t(X) \) and all \( k \leq m \).

\[ E^k_s(X_t) = X_{(t_1, \ldots, t_{k-1}, s, t_k, t_{k+1}, \ldots, t_m)} \]

Indeed for \( s \leq t \)

\[ X_{(s_1, \ldots, s_m)} = E^{\pi(1)}_s \cdots E^{\pi(m)}_s(X_t) \]

for any permutation \( \pi \) of \( (1, 2, \ldots, m) \). Since \( (X_t) \) converges to \( X_t \) in \( L_1 \), one obtains

\[ X_{(s_1, \ldots, s_m)} = E^{\pi(1)}_s \cdots E^{\pi(m)}_s(X_t) \]

This implies that the operators \( E^k_t \), \( k = 1, 2, \ldots, m \), commute for all
t. Conversely, if $\bigcap_{k=1}^{m} \mathcal{F}_t^k = \mathcal{F}_t$ for each $t$, and $E^k_t$ commute, then (1) holds. Indeed, then $E^1_t \ldots E^m_t(X) = E^\pi(1)_t \ldots E^\pi(m)_t(X)$ for any permutation $\pi$ of $(1,2,\ldots,m)$, hence $E^1_t \ldots E^m_t(X)$ is $\mathcal{F}_t^k$-measurable for all $k$, and therefore $\mathcal{F}_t$-measurable. Consequently $E^t(X) = E^1_t \ldots E^m_t(X) = E^\pi(1)_t \ldots E^\pi(m)_t(X)$. Let now $(X_t, \mathcal{F}_t)$ be a martingale. Then, replacing $X$ by $X_t$ and $t$ by $(t_1, \ldots, t_{k-1}, s, t_k, \ldots, t_m)$, one has

$$X_{(t_1, \ldots, t_{k-1}, s, t_k, \ldots, t_m)} = E[X_s | \mathcal{F}_s]_{(t_1, \ldots, t_{k-1}, s, t_k, \ldots, t_m)}$$

$$= E^1_t \ldots E^{k-1}_t E^k_s E^{k+1}_t \ldots E^m_t(X_s)$$

$$= E^k_s E^1_t \ldots E^{k-1}_t E^{k+1}_t \ldots E^m_t(X_s)$$

$$= E^k_s(X_s).$$

Relation (1) for $k = 1, 2, \ldots, \ell \leq m$ implies

$$E^1_s E^2_s \ldots E^\ell_s(X_s) = X_{(s_1, \ldots, s_\ell, t_{\ell+1}, \ldots, t_m)}.$$
Applying $E_{s}^{1-k}$ to both sides, one obtains that $(X_{t})$ is a block martingale. In particular, from the commutation assumption, which implies (1) for all $k$, it follows that every martingale is a block martingale; similarly every submartingale is a block submartingale.

Therefore demiconvergence of block submartingales reduces to convergence of block martingales. We may observe that the simple proof of convergence given in [62] assuming that each $J = \mathbb{N}$ extends to the case of Section 4. It suffices to apply the Lemma III-3.2, above, in the induction step in [62].

III-7. Applications

In the present section we give applications to differentiation of integrals in $\mathbb{R}^{m}$ $m \geq 1$. We first show that the regularity condition $R_{\alpha}$ is satisfied in the setting of differentiation in $m$-dimensional Euclidean space. Let $\mu$ denote the Lebesgue measure on $[0,1]^{m}$. Given a countable partition $t$ of $[0,1]^{m}$, by the diameter $d(t)$ of $t$ we mean the supremum of the diameters of the elements (atoms) of $t$.

Proposition III-7.1. Let $C$ be a collection of measurable subsets $C$ of $[0,1]^{m}$. Assume that $C$ is a family of substantial sets, i.e., there exists a constant $M$ such that every $C$ in $C$ is contained in an open ball $B$ with $\mu(B) \leq M \mu(C)$. Let $J$ be a non-empty family
of countable partitions (modulo sets of measure 0) of \([0,1]^m\) into elements of \(C\). \(J\) is ordered by refinement, i.e., if \(s,t \in J\), \(s \leq t\), then every element (atom) in \(s\) is a union of atoms in \(t\).

\(J\) is assumed filtering to the right. Then the filtration \((\mathcal{F}_t)\) of \(\sigma\)-fields generated by the partitions \(t\) satisfies the regularity condition \(R^\alpha\) with \(\alpha = M^{-1}3^{-m}\).

**Remark.** A simple example of such a family \(J\) is a family, ordered by refinement, of countable partitions of \([0,1]^m\) into parallelepipeds such that the ratio between the largest and shortest edges is bounded, say by \(a\). (If \(m = 2\), one can choose \(M = \frac{3a}{2}\).)

**Proof:** Let \((A_t)\) be an adapted family of sets and let \(A = \sup_t A_t\).

Since \(A = \bigcup_{t} C(t,i), C(t,i) \in t\), \(A\) is covered by the family \(\{C(t,i)\}\) except for a set of measure zero. Let \(\{B(t,i)\}\) be the corresponding family of open balls, i.e., \(C(t,i) \subseteq B(t,i)\) and

\[\mu(B(t,i)) \leq M \mu(C(t,i))\]. Let \(B = \bigcup_{t} B(t,i), B = \sup_t B_t\). Then \(A \subseteq B\). For every \(\epsilon > 0\), one can choose a finite collection of disjoint open balls \(B(t_1, \epsilon), \ldots, B(t_n, \epsilon)\), \(B(t_1, \epsilon), \ldots, B(t_n, \epsilon)\), \(B(t_1, \epsilon), \ldots, B(t_n, \epsilon)\), \(B(t_1, \epsilon), \ldots, B(t_n, \epsilon)\), \(B(t_1, \epsilon), \ldots, B(t_n, \epsilon)\).
such that

$$3^{-m}(1-\varepsilon)\mu(B) \leq \frac{1}{n} \sum_{1 \leq j \leq n} \frac{1}{l} \mu(B(t_j, \varepsilon_j)).$$

(For this result, due to J. Serrin and W. Rudin, see [55], page 164.)

Hence

$$3^{-m}(1-\varepsilon)\mu(A) \leq \frac{1}{n} \sum_{1 \leq j \leq n} \frac{1}{l} \mu(C(t_j, \varepsilon_j)).$$

Let $A = \bigcup_{j=1}^{n} C(t_j, \varepsilon_j)$, $j = 1, 2, \ldots, n$. Then the sets $A$ are disjoint and $A \in \mathfrak{F}$. Define

$$\tau = \begin{cases} 
  t_j & \text{on } A_j, \quad J = 1, 2, \ldots, n \\
  t_{n+1} & \text{on } (\bigcup_{j=1}^{n} A_j^c) 
\end{cases}.$$

Now $\tau$ is the desired single-valued stopping time $\tau$. \qed
Remark. Call sets \( B \subset \mathbb{R}^m \), \( V \)-balls if there is number \( a_m \) such that given \( \varepsilon > 0 \) and any collection \( \{ B_i \} \) of \( V \)-balls, there is a number \( M_\varepsilon \) and a finite subcollection \( B_1, \ldots, B_n \) such that
\[
P( \bigcup_{i=1}^n B_i ) > a_m (1-\varepsilon) P(\bigcup_i B_i) \quad \text{and} \quad \sum_{i=1}^n 1 - 1 \leq M_\varepsilon.
\]

Let \( C' \) be a collection of measurable sets, called \( V \)-substantial sets, such that there exists a constant \( M \) so that every \( C \in C' \) is contained in a \( V \)-ball \( B \) with \( \mu(B) \leq M \mu(C) \). Then the filtration \( (\mathcal{F}_t) \) generated by the collection \( C' \) satisfies the condition \( \alpha \) with \( \alpha = a M^{-1} \). The following theorem is still true if each \( C_1 \) is a collection of \( V \)-substantial sets.

**Theorem III-7.2.** Let \( 1 \leq s \leq m \) and consider positive numbers
\[
k_1, k_2, \ldots, k_s \quad \text{such that} \quad k_1 + \ldots + k_s = m.
\]
For each \( i \leq s \), let \( C_i \) be a collection of substantial subsets of \( [0,1]^i \). Let \( J_i \) be a family of countable partitions of \( [0,1]^i \) into elements of \( C_i \) such that for each \( \varepsilon > 0 \) there exists \( t_i \in J_i \) with \( d(t_i) < \varepsilon \). On \( [0,1]^m \) define
where $R = C_1 \times C_2 \times \ldots \times C_s$, $C_i \in C_1$, $x \in R$ and $f \in L_1([0,1]^m)$.

(i) If $f$ is positive, then

$$\liminf_{R} I(x) = f(x) \text{ a.s.}$$

as $R$ shrinks to $x$, i.e., $d(R) \to 0$.

(ii) If $f$ is $L_{\log}^{s-1}$ integrable, then

$$\lim_{R} I(x) = f(x) \text{ a.s.}$$

as $R$ shrinks to $x$.

Proof: For each $t \in J$ let $\mathcal{F}_t$ be the $\sigma$-field generated by the partition $t$. By the definition of conditional expectation, one has

$$E[f|\mathcal{F}_{t_1} \otimes \mathcal{F}_{t_2} \otimes \ldots \otimes \mathcal{F}_{t_s}](x) = \frac{\int_{R} f \, du}{\mu(R)} , \ x \in R .$$

For each $i$, the filtration $(\mathcal{F}_t, t \in J)$ satisfies condition $R^{t_i, i}_{i \alpha_i}$ with $a_i = M^{-1} 3_i$. Since $\lim_{i} d(t_i) = 0$, $\forall \mathcal{F}_t$ coincides with the
\( \sigma \)-field of all Lebesgue measurable sets on \([0,1]\). 

(i) If \( f \) is positive, then by Theorem III-4.4. (ii)

\[
\liminf E[f | \mathcal{F}_1 \otimes \cdots \otimes \mathcal{F}_k] = E[f | V_{\mathcal{F}_1} \otimes \cdots \otimes V_{\mathcal{F}_k}] = f \text{ a.s.} \]

(ii) If \( f \) is \( L \log^{s-1} L \) integrable, then by Theorem III-4.4. (iii) for \( s > 1 \) and by the martingale convergence theorem under \( C \) proved in [47], for \( s = 1 \)

\[
\lim E[f | \mathcal{F}_1 \otimes \cdots \otimes \mathcal{F}_k] = E[f | V_{\mathcal{F}_1} \otimes \cdots \otimes V_{\mathcal{F}_k}] = f \text{ a.s.} \] \( \square \)

A classical case (\( s = m = 2 \) and \( C \) intervals) of part (i) is due to Besicovich (see e.g. [30], p. 100.)

Notice that Theorem III-7.2. (ii) is still true with the same proof if \( f \) is a Banach-valued, strongly measurable, \( L \log^{s-1} L(E) \) integrable function; in the proof, apply Theorem III-4.5.

We now state Zygmund's theorem [66] on differentiation of integrals, a generalization of the theorem of Jessen-Marcinkiewicz-Zygmund ([33] or [30], p. 51).
Theorem III-7.3. (Zygmund). Let $1 < s < m$ and consider only intervals $R$ in $[0,1]^m$ whose sides have no more than $s$ different sizes. If $f$ is $L \log^{s-1} L$ integrable then $\lim_{R \to x} I_R(x) = f(x)$ a.s. as $R$ shrinks to $x$.

Proof: For each rectangle $R$ we have at most $s$ different sizes. Without loss of generality we can assume that the first $k_1$ coordinates are equal, then the next $k_2$ are equal, finally the last $k_s$ coordinates are equal, $k_1 + k_2 + \ldots + k_s = m$. Let $J_i$ denote the family of all partitions of $[0,1]^i$ into cubes. Then each $J_i$ is a collection of substantial sets. There are only finitely many possible orderings of coordinates to be considered. Therefore Theorem III-7.2. implies convergence. □
LIST OF REFERENCES


