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THE FRACTURE MECHANICS OF A SLIT CRACK WITH CRACK-TIP DUAL ZONES

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THE FRACTURE MECHANICS OF A SLIT CRACK
WITH CRACK-TIP DUAL ZONES

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By

Arvind K. Nagar, M.M.E, P.E., B.S.M.E.

* * * * *

The Ohio State University
1984

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Dedicated to my father
ACKNOWLEDGMENTS

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1.1. Background Information

Fractographic investigations of many serious accidents in modern structures such as welded ships, storage tanks, pressure vessels and large rotors[1]* revealed that pre-existing cracks in most cases were responsible for these failures. These fractures were truly brittle and occurred at applied stress levels below one-third of the normal yield strength. It has consequently been established that many materials that are ductile under normal operating conditions may become brittle when subjected to low temperatures, heat treatment, alloying, high strain rate or hydrostatic tensile stresses[2]. Griffith[3] postulated a brittle crack extension theory based on an energy approach, requiring that the surface energy to form additional crack surfaces is equal to the released elastic strain energy. A continuum mechanics based analytic approach was proposed by Irwin[4] who suggested examining the crack-tip stress fields for three distinct modes of deformations according to whether the resulting displacements tend to open the crack, an opening mode (mode-I); an inplane sliding of crack faces (mode-II); or an antiplane shear deformation with sliding of crack faces normal to the plane (mode-III). Such an analysis for a linearly elastic body shows that inverse square root singularities of the stresses occur.

* Numbers in brackets indicate references.
at the tip of a sharp crack. A brittle fracture occurs at a critical value of the crack-tip stress gradients. A measure of such stress gradients is the stress intensity factor which depends upon the applied stresses, the length of the crack and the geometry of the body.

Although the concept of a brittle material model is found to be very useful, most engineering materials tend to exhibit yield strength and plastically deformed regions are formed at a crack-tip. A strain-hardened material of appreciable thickness develops a state of plane strain with a triaxial stress field at the crack-tip. As a result the effective yield strength is substantially increased and the extent of inelastic deformation is limited. It has been experimentally established[5] that the state of plane strain prevails when the deformation zone is smaller than the thickness. Characteristically, the materials with a high yield strength exhibit such deformation patterns of small scale yielding. Various techniques are available to determine the extent of crack-tip plasticity[6]. Irwin[1] proposed an estimation of the spread of plasticity by assuming finite stresses in the yielded regions and a force balance concept. Dugdale[7] obtained a more rigorous solution for the plastic zone size assuming narrow band shaped yielded regions, negligible strain-hardening of the material and by removing stress singularities at the plastic zone tips. The effect of plasticity is to increase the displacements and reduce the material stiffness at the crack-tip. Thus under small scale yielding the displacement of the crack-tip (CTOD) is non-zero.

This widely applied area in fracture mechanics where the crack-tip is free of plasticity effects or where the spread of crack-tip
yielded regions is limited to a small scale is called linear elastic fracture mechanics (LEFM). In LEFM, the stress intensity factor is calculated using an effective crack length, the original crack plus the attached plastic zones at each end of the crack-tip. Also the stress intensity factor, the crack-tip opening displacement and Griffith's elastic strain energy release rate (crack extension force) are equivalent fracture parameters.

1.2. Purpose and Scope of Research

A series of recent microscopic investigations of the crack-tip region in several ductile materials under tensile deformation reveal that there exists a small zone between the crack-tip and the plastic zone where no dislocations are present[8,9,10]. Since the dislocation pile-ups represent plastically deformed zones, these zones are no longer directly ahead of the crack-tip. Recent crack-tip fields analysis[11, 12] for the presence of a dislocation free zone are confined to anti-plane shear deformations (mode-III) and are based on dislocation theories. A solution based on a conventional continuum mechanics approach for a practically more important case of the tensile mode of deformation, and which realistically simulates the conditions of experimental observations is lacking.

A purpose of this dissertation is to formulate a theoretical model for the presence of a dislocation free zone in a plastically deformed region ahead of the crack tip in tension and to study the effects of these dual zones on the crack-tip fields. A thru central crack is assumed to be present in a two dimensional infinite medium which is
subjected to uniformly applied remote stresses resulting in a tensile mode of deformation perpendicular to the crack plane. The material is assumed to be relatively thin and therefore in a state of plane stress. As the applied stresses increase, a dislocation free zone and a plastic region surrounds each crack-tip. The physical model of the problem assumes the remaining material to be linear, homogeneous and isotropically elastic. The dislocation free zone directly ahead of the crack-tip is a region where the stresses are higher than the material yield strength, and thus capable of moving a dislocation. Following previous approaches by Thomson[13] and Weertman[14] the stresses and strains in this zone are assumed to have a constant slope equal to the elastic modulus of the material. For this reason, in the continuum approach this zone is referred to as an elastic zone. The transverse displacement (perpendicular to the crack plane) distributions over the elastic zone are assumed to be uniform and equal to the crack-tip opening displacement since no dislocations are present there. A similar assumption was made in studies for anti-plane shear modes of deformations[11,12].

The plastically deformed region is assumed to be a perfectly plastic continuum capable of arbitrary transverse displacements. The displacements parallel to the crack plane in this region are assumed to be negligible. The stresses in the plastic zone are assumed to be bounded and uniformly distributed with a value equal to the material yield strength. Such plastic zones with no strain-hardening and thru the thickness relaxation[15] are cross (x) shaped, representing planes of maximum shear stresses. Along the crack plane, they have narrow
elongated band shapes. Thus their lateral dimensions (perpendicular to the crack plane) are small compared to their length.

Applying the principle of superposition, the problem is treated in two parts: a mixed boundary value problem with boundary conditions imposed only along the crack plane and a problem for an uncracked infinite plane medium subjected to uniform remote tensile stresses. The mixed boundary value problem is formulated using Papkowich-Neuber harmonic function. Due to the symmetry of the two dimensional problem and non-existence of shear stresses along the crack plane, the four part mixed boundary conditions are represented by a quadruple set of integral equations. These equations are solved using a modified finite Hilbert transform technique. The analytic expressions for stresses and displacements are obtained in terms of elliptic integrals without directly solving for the unknown function of the integral equations. Based on these results an inverse square root singularity of stresses exists at the crack-tip. As the plastic zone size increases, a relaxation of the normal stresses in the elastic zone occurs as the later zone is allowed to reduce in size. The relations for the crack-tip stress intensity factor, CTOD and the plastic zone length are reduced to simple expressions involving Heuman's Lambda Functions. The plastic zone length depends primarily on the applied remote stresses while the crack-tip stress intensity factor is predominantly a function of the elastic zone length. The CTOD, however, varies with both the elastic zone size as well as the applied stresses.
As a limiting case of the generalized solution, an important problem of three coplanar cracks with specified displacements between the cracks is solved. For another special case, a solution of the triple crack problem is obtained which is in agreement with known results[16,17]. Finally by letting the length of the elastic zone go to a vanishing limit, the solution of the crack problem with Dugdale plastic zones is obtained. These results agree with those reported by Goodier and Field[18] and Rice[19] using complex stress functions. Similarly, for the case of small scale yielding, the plastic zone size reduces to the well known relation in terms of the applied stress material yield strength and the crack size obtained by Dugdale[7].

The exact forms of the results obtained here show a qualitative correlation with some of the results reported for antiplane shear deformation cases[11,12].
CHAPTER II
LITERATURE REVIEW ON PHYSICAL MODELS
OF THE CRACK-TIP

Several concepts which have led to physical models of crack problems in fracture mechanics are note-worthy. Barenblatt[20] introduced a notion of cohesive forces (molecular attractions) at the crack edges which neutralizes the singularity effects of applied forces. Irwin[4] argued that for all practical purposes, small scale crack-tip plasticity effects may be studied by extending the crack by segments equal to the plastic zone lengths. Dugdale[7] applied these concepts to determine the extent of crack-tip plasticity under tensile deformation. These concepts have opened new areas of knowledge whose development covers a large portion of the field of fracture mechanics. These and some modern developments[11-14] in crack-tip modeling, comprise the subject matter for this chapter.

2.1. Barenblatt Model of Cohesive Forces

Barenblatt[20] considers a perfectly elastic material with an internal crack. Under a tensile mode of deformation, the crack opens and the two faces at the ends come together with zero elastic displacements and zero slope as shown by the solid line in Figure 1. He argued that large cohesive forces of molecular attractions, of the order of the theoretical cohesive strength of the material, could exist at the
Figure 1. An opening mode of a crack based on Barenblatt model of cohesive forces[20].
end faces. A representation of the cohesive force distribution acting over the faces of the crack edges is shown in the figure. A presence of such forces is responsible for the smooth (zero slope) closure of the two faces by working against the applied forces. The cohesive forces and the applied forces induce stress singularities of opposite characters. Thus the net result is that the stresses remain nonsingular (finite) at the crack tips, $|x| = a$. The usual assumption of traction-free crack faces in linear elasticity is valid everywhere along the crack except in the cohesive end zones. The Barenblatt hypothesis limits the distribution of cohesive forces to a very small zone which extends inward from the tip. Such a distribution and the edge length over which these forces act is independent of the applied loading or crack size, but depends upon a given material and its operating environment. A solution of the Barenblatt model is discussed by Goodier[20] who suggested its treatment as a three-part mixed boundary value problem with all boundary conditions written on the crack plane.

2.2. Dugdale Model of Crack-Tip Plasticity

A representation of Dugdale's model of crack-tip plasticity for a thin central crack in a plane infinite medium for an opening mode of deformation is given in Figure 2. In a non-hardening material of small thickness, the plastic zones developed at the crack-tip have narrow band shapes[25]. Such deformed regions may be replaced by additional cuts at the crack edges. The action of the plastic continuum may be replaced by application of forces along the faces of the new cuts. For a perfectly plastic material, these forces may be taken to be equal to the material
Figure 2. Dugdale's model of crack-tip plasticity. $T$ is an applied remote stress and $s$ is the length of the narrow plastic zone[7].
Figure 3. A comparison of Dugdale's analytic and experimental results[7].
yield strength. Since the length of the plastic zone is large compared to the interatomic spacings[21], the Dugdale problem may be treated by the methods of linear elasticity. Assuming that the stresses must be bounded in the yielded regions and by removing the linear elastic singularity from the tip of the hypothetical crack, Dugdale derived a relation for the plastic zone size in terms of the applied remote stresses. His analytic results using his own notations are represented by a solid curve in Figure 3. To verify his analytic predictions, Dugdale conducted experiments on an 18 gage, cold roll steel with central and single edge slits[7]. His experimental data points (Figure 3) show an excellent agreement with the theoretical results. In terms of the Barenblatt concept described earlier, Dugdale assumed the restraining stresses in confined narrow regions of yielding to be constant as shown by a horizontal line in Figure 4. In general for a strain-hardening material, the restraining stresses vary approximately parabolically with the displacements in these regions. This relation is schematically represented by the nonlinear curve in Figure 4 and shows that the Dugdale approximation was reasonable. Goodier and Field[18] completed the solution of the Dugdale problem by obtaining an analytic form for the displacements and deriving an expression for the crack-tip opening displacement. Similar results were reported by Rice[19] and Burdekin and Stone[23]. A parallel study yielding analogous results, for an antiplane shear mode of deformation based on the theory of dislocations, was conducted by Bilby, Cottrell and Swinden[24].

The Dugdale model has been applied to various other physical and loading configurations. Rosenfield, et.al[25] modified the Dugdale
Figure 4. The restraining stress as a function of displacement in the plastic zone [22].
model for strain-hardening materials by approximating the restraining stresses in yielded zones by piecewise uniform distributions. Harrop [26], and Theocaris and Gdouts[27] parabolically distributed restraining stresses. Kanninen[28] obtained a solution of the Dugdale problem for the case of linearly varying applied remote stresses. Keer[29] extended the Dugdale model to a three dimensional problem with a penny shaped crack. Theocaris applied the yield strip model to an infinite medium containing two cracks[30] and to nonmetallic materials including polycarbonate[31]. A further extension of the Dugdale scheme to a linear array of cracks is reported by Bilby and Smith[32].

Other notable studies in related areas include: elevated temperature crack growth by Reidel[33], fast fracture and crack arrest by Achenbach[34] and fatigue crack closure by Budiansky and Hutchinson[35].

2.3. Crack-Tip Elastic Zone Models

Recently, Weertman[14] proposed a unified view of fracture for a Griffith crack and an Irwin-Orowan crack using a crack-tip stress singularity approach rather than the surface energy. To establish proof of such a view, he assumes an elastic zone at the crack-tip and a true stress-strain behavior for the material in this zone as shown in Figure 5. It is noted that the stresses are elastic up to a yield stress, plastic up to the order of the theoretical cohesive strength of the material and is elastic again to infinite stress and strain. Since the displacements in the plastic zone at the crack-tip are elastic at smaller distances and the crack-tip is a free surface from which the dislocations can be removed by image forces, it is reasonable to assume under such a
Figure 5. A schematic stress-strain relation of the material in the crack-tip elastic zone[14].
case that the stress-strain behavior is elastic again as shown by the upper linear curve in Figure 5.

A similar physical model of the crack-tip with an inner elastic core region between a plastic continuum and an atomically sharp crack-tip is proposed by Thomson[13]. The stress distributions ahead of the crack assumed in this model are shown in Figure 6. For reference, the stress distributions based on Barenblatt and Orowan postulates are also included. The stresses vary with the inverse of the square root of the radial distance and become infinite at the crack-tip. Thomson proposes that it is possible for a ductile material to undergo brittle fracture if the crack-tip is prevented from blunting and the dislocation content at the tip remains limited.

2.4. Crack-Tip Dislocation-Free Zone Models

In a very recent series of experiments on ductile materials using an electron microscope, it has been observed that the region immediately ahead of the crack-tip under tensile deformation is often free of dislocations[8,9,10]. The dislocations are emitted from the crack-tip, pass through the dislocation-free zone and pile up in the plastic zone. The dislocations are line defects. When a crystal is loaded beyond its yield point, the dislocation sources begin to operate and give rise to slip on planes of maximum shear stresses. The sources of dislocations feed the slip plane with dislocation loops. If the applied shear stress on slip planes is greater than the friction stress, the dislocations are set in motion. When slip occurs, the applied shear stress relaxes to the value of friction stress and is accommodated by resulting plastic strains.
Figure 6. Stress distributions ahead of the crack-tip for an elastic region model[13].
Thus the dislocation pile-ups may be used to account for the plastic deformation.

Two dislocation-free zone models\([11,12]\) for anti-plane deformation have been proposed. A schematic representation of the physical model with a dislocation free zone and a plastic zone ahead of the crack-tip and corresponding distribution of dislocations reported by Chang and Ohr is shown in Figure 7. The displacements in the dislocation free zone are assumed to be constant and the stresses higher than the friction stress (material yield strength) are allowed to prevail. The stresses in the plastic zone are assumed to be constant including its two end points. The dislocation distribution function is obtained by applying a condition of equilibrium. During the equilibrium state, the net force on a dislocation due to the applied shear stress and the friction stress must be zero. This condition leads to a set of singular integral equations from which the dislocation distributions may be derived.

A similar dislocation free zone model for antiplane shear deformation is reported by Majumder and Burns\([12]\). A distribution of dislocations over the plastic zone derived from this model is shown in Figure 8.

Both models described above are based upon the theory of dislocations and the concepts of Bilby, Cottrell and Swinden (BCS) models\([24]\). It is noted that for a BCS model, the dislocation distribution function approaches infinity at the inner end of the plastic zone which corresponds to the crack-tip as shown by the dashed curve in Figure 8.
Figure 7. A schematic view of dislocation distributions over the crack and the plastic zones[11].
Figure 8. Distribution of dislocations in the plastic zone [12].
CHAPTER III
METHODS OF SOLUTION

An interest in analytic solutions of crack problems began with Inglis's solution of an elastic plate with an elliptic hole[36]. An elliptic hole, when flattened by letting its minor axis reduce to a vanishing limit, may be assumed to be representative of a crack. Since then, various methods of solving the crack problems in elastic bodies have been introduced. An illustration of the development of various approaches to crack problems is presented in Figure 9. It is interesting to note that the analytic function theory plays an important role in the methods of crack analysis. It is beyond the scope of this chapter to discuss all the approaches listed in Figure 9. Only the methods commonly applied to the mathematical formulation and solution of the physical models described in Chapter II are reviewed.

3.1. Stress Function Method

A powerful method to solve two-dimensional mixed boundary value problems based on complex stress functions was introduced by Muskhelishvili[37]. The stress function concept stems from the stress representation of governing equations in linear elasticity. The Muskhelishvili approach has the capability of handling plane problems in finite or infinite regions by using conformal mapping techniques. Conformal mapping involves transformation of points or curves in one...
Figure 9. Development of analytic methods for crack problems.
complex plane to another, preserving the magnitude and the sense of the angle between the curves. An application of conformal mapping to crack problems is a mapping from an ellipse in a complex plane to a unit circle in another. Any simply connected region may be mapped into a unit circle. The main difficulty lies in finding a proper mapping function. However, once the appropriate function is found, the solution is very simple. A special case of the complex stress function method by incorporating specific boundary conditions was introduced by Westergaard[38]. The literature witnesses a large number of problems solved by this technique[18,19,23,25].

The stress function method is well suited to solve the Dugdale problem of crack-tip plasticity. Dugdale's results[7] for the stresses in the remaining ligament zone and the resulting plastic zone relation was derived based on this method. Goodier and Field[18], Rice[19], and Burdekin and Stone's[23] extensions of the solutions of the Dugdale model were all based upon this approach. A major advantage of this method is its simplicity in determining the displacements. But it is restricted to two dimensional plane problems.

3.2. Displacement Vector Approach

Another major approach to solve problems of linear elasticity is to solve for the displacements from Navier's representation of governing equations of equilibrium in terms of displacements. A direct solution for the displacements from these equations is quite difficult. A widely used method is based on potential functions. The method of potentials is derived from a theorem proposed by Helmholtz[39]. According to this
Theorem, any analytic vector may be expressed as a combination of a gradient of a scalar function and a curl of a vector function. Thus a displacement vector such as the one used in Navier equations can be expressed by four potential functions, which, if harmonic, lead to displacements satisfying Navier's equation. One for the scalar function and the other three for three components of a vector function. Since displacement vector has only three components, it is now well agreed that at least one potential function may be selected arbitrarily[39]. There are a number of potential functions available in the literature. The two potential functions with widest applicability are: a Galerkin vector and a Papkovich-Neuber set of functions. In searching for a more general solutions, Galerkin introduced displacement functions that are bi-harmonic. Papkovich and Neuber proposed four harmonic functions independent of Galerkin. However it has been shown that these functions may be obtained from the Galerkin vector. Although quite a large number of problems have been solved using potential functions[39], their main application to crack problems is found in the mathematical formulation of the physical model. The mathematical model of the problem of this dissertation is based upon Papkovich-Neuber harmonic functions.

3.3. Method of Integral Transforms

Recently, the analytic solutions of crack problems have been obtained using the integral transforms. For most two dimensional problems, the Muskhelishvili stress functions and Papkovich-Neuber potential functions may be represented by integral transforms such as Fourier, Hankel, Mellin, Stieltjes or Hilbert. Thus such transforms may be introduced either during the problem formulation phase or at the solution
stage of equations representing the boundary conditions of the problem. Their introduction during the formulation phase leads to a set of integral equations representing the stresses and the displacements specified at various boundaries. Therefore, in order to obtain an analytic solution of a crack problem with this method, such integral equations must be solved.

A solution of dual integral equations involving trigonometric kernels was first sketched by Tranter[40] who attempted to solve them by transforming their kernels into Bessel series or Bessel kernels. Noble's applications of Erdelyi-Kober operators to dual integral equations led to Fredholm integral equations of the second kind[41]. Similarly Cooke's method led to and required solutions of Fredholm integral equations of the first kind[42]. Other contributors to the solutions of these equations are: Titchmarsh, Busbridge and Copson[43]. It is clear from these discussions that most of these solutions required numerical techniques to get the final solution.

Sneddon[44] treated the Griffith crack problem as a mixed boundary value problem with an internal pressure acting on the crack. Using Papkovich-Neuber harmonic functions in terms of Fourier cosine transforms, he derived a set of dual integral equations representing the specified stresses over the crack region and vanishing displacements elsewhere along the crack plane of the symmetric problem. A solution of these equations was obtained by introducing an unknown function in terms of the Bessel function of the zero th order. Subsequently the methods of integral transforms have been applied to several crack problems including two equal colinear Griffith cracks, an infinite row
of parallel and colinear cracks, radial cracks originating from an internal circular hole and star cracks under pressure[44]. The Hankel integral transform was applied to formulate a problem of a penny shaped surface crack in a three dimensional elastic medium and the final solution was obtained by introducing an unknown function involving a Fourier sine transform.

In some cases, especially for finite mediums, the methods of integral transforms lead to Fredholm integral equations of the first or the second kind and require numerical treatments[43].

3.4. Dislocation Theory Approach

A powerful method to solve problems of antiplane shear deformation (Mode III) in elastic-perfectly plastic materials is based on the continuum theory of dislocations introduced by Bilby, Cottrell and Swinden [24]. The method of dislocations is referred to as an elastic analysis of line defects which are responsible for constant displacement discontinuity (Burger's vector of a dislocation) including analysis of the generation and motion of fields of such defects.

The method is based on the assumption that plastic deformations aheads of a crack may be represented by a continuous array of dislocations. Such an assumption is reasonable for perfectly plastic cases. In reality, the dislocations are distributed in a discrete manner but such a distribution makes the mathematical solution quite difficult. The formulation based on continous pile-up of dislocations at their equilibrium states leads to singular integral equations[11,12]. At equilibrium, the stress on a dislocation due to an external applied load is equal to the friction stress of a dislocation. The resulting
singular integral equations may be conveniently solved for simple geometric by the methods of Mushelishvili[45]. In addition to the BCS model (an analog of Dugdale model for mode-III), the solutions of several problems including the more advanced dislocation free zone models of Chang and Ohr[11] and Majumdar and Burns[12] are based upon the theory of dislocations approach.
CHAPTER IV
FINITE HILBERT TRANSFORM

It was pointed out in the previous chapter that the use of Papkovich-Neuber harmonic functions in the form of integral transforms in solving mixed boundary value problems often leads to integral equations. A solution of such equations based on a modified finite Hilbert transform technique was proposed by Srivastava and Lowengrub [46]. These authors developed this method by modifying a theorem on finite Hilbert transforms originally proposed by Tricomi[47] to solve an aerofoil equation in aero-elasticity.

4.1. Tricomi's Theorem

The finite Hilbert transform is defined by

$$F_x[\phi(y)] = \frac{1}{\pi} \int_{-1}^{1} \frac{\phi(y)}{y-x} \, dy$$

(4.1.1)

where $F_x$ denotes the finite Hilbert transform of some function $\phi(y)$. According to Tricomi's theorem[47], a solution of

$$F_x[\phi(y)] = f(x)$$

(4.1.2)

is given by

$$\phi(x) = -\frac{1}{(1-x^2)^{1/2}} \cdot F_x[(1-y^2)^{1/2}f(y)] + \frac{c}{(1-x^2)^{1/2}}$$

(4.1.3)
Using the convolution theorem with
\[ \phi_1(x) = (1-x^2)^{1/2} \quad (4.1.4) \]

and
\[ \phi_2(x) = \phi(x) \quad (4.1.5) \]

the solution of \( \phi(x) \) may be expressed in the following form:
\[ \phi(x) = -\frac{1}{\pi} \int_{-1}^{1} \frac{1}{(1-y^2)^{1/2}} \frac{f(y)dy}{y-x} + \frac{c}{(1-x^2)^{1/2}} \quad (4.1.6) \]

where
\[ c = \frac{1}{\pi} \int_{-1}^{1} \phi(y) dy \quad (4.1.7) \]
is a constant.

4.2. Srivastava and Lowengrub Modification

If
\[ F_y[h(x)] = \frac{1}{\pi} \int_{a}^{b} \frac{h(x)}{x-y} \, dx = p(y) \quad (4.2.1) \]

where \( p \) is sufficiently smooth and integrable function of \( y \) in the interval \([a,b]\) then the solution of \( h(x) \) is given by the inverse finite Hilbert transform, i.e.,
\[ F_x^{-1}[p(y)] = h(x) = -\frac{1}{\pi} (\frac{x-a}{b-x})^{1/2} \int_{a}^{b} (\frac{b-y}{y-a})^{1/2} \frac{p(y)dy}{y-x} + \frac{c}{[(x-a)(b-x)^{1/2}]^2} \quad (4.2.2) \]
where \( c \) is a constant to be determined from the boundary conditions of the problem\[45\]. For a particular case when \( x = t^2 \), the above theorem becomes:

\[
\text{If } \int_a^b \frac{2t \cdot h(t^2)}{t^2-y^2} \, dt = p(y), \quad y \in (a,b) \quad (4.2.3)
\]

then

\[
h(t^2) = - \frac{1}{\pi} \left( \frac{t^2-a^2}{b^2-t^2} \right)^{1/2} \int_a^b \left( \frac{b^2-y^2}{y^2-a^2} \right)^{1/2} \cdot \frac{2y \cdot p(y) \, dy}{y^2-t^2} \\
+ \frac{c_1}{[(t^2-a^2)(b^2-t^2)]^{1/2}}. \quad (4.2.4)
\]

To simplify integrations over certain regions, it has become common practice to express the solution for \( h(t^2) \) in an alternate form given by

\[
h(t^2) = - \frac{2}{\pi} \left( \frac{t^2-a^2}{b^2-y^2} \right)^{1/2} \int_a^b \left( \frac{y^2-a^2}{b^2-y^2} \right)^{1/2} \cdot \frac{y \cdot p(y) \, dy}{y^2-t^2} \\
+ \frac{c_2}{[(t^2-a^2)(b^2-t^2)]^{1/2}}. \quad (4.2.5)
\]

To derive (4.2.5) the following identity\[46\] has been used:

\[
\left[ \frac{(t^2-a^2)(b^2-y^2)}{(b^2-t^2)(y^2-a^2)} \right]^{1/2} \left[ 1 + \frac{y^2-t^2}{t^2-a^2} \right] \\
= \left[ \frac{(b^2-t^2)(y^2-a^2)}{(t^2-a^2)(b^2-y^2)} \right]^{1/2} \left[ 1 + \frac{y^2-t^2}{b^2-t^2} \right] \quad (4.2.6)
\]
It is clear from the technique presented above that to apply this method, it is necessary to reduce the integral equations to a finite Hilbert transform form such as (4.2.1) or (4.2.3). This requires suitable representation of the unknown function of the integral equations. Such a representation must be tailored so as to satisfy at least one of the integral equations automatically and when applied to others to lead to the desired form (4.2.1). In general, the greater the number of integral equations in a set, the more difficult it is to find the proper unknown function representation.

The wide applicability of the method is demonstrated by a large number of solutions of crack problems reported in the current literature[48-51]. Lowengrub and Srivastava[48] extended its use to cracked bodies of finite geometry. Konishi[49] solved a set of triple integral equations for the problem of a dual crack system in anisotropic medium. Dhaliwal and Singh[50] applied the modified finite Hilbert transform to solve the problem of an infinite elastic strip with three coplanar Griffith cracks. Dhawan and Dhaliwal[51] obtained stress intensity factors and the energy required to open a triple crack system in an infinite transversely isotropic medium.

The method of finite Hilbert transforms is relatively simple and efficient in solving dual and triple integral equations since the task of determining a representation of the unknown function is considerably reduced. In some cases, especially those involving weight functions or finite bodies, this technique, like other integral transform methods, leads to Fredholm integral equations which require iterative procedures or numerical treatments for their solutions.
The solutions of problems leading to quadruple integral equations, available in the literature, are rare. Dhawan and Dhaliwal[51] obtained a solution for a quadruple set with two homogeneous equations. Similarly Nakai, Tanaka and Yamashita[59] solved a quadruple set of integral equations with two homogeneous equations representing a crack closure model for small cracks. In this dissertation, the finite Hilbert transform technique is applied to solve quadruple integral equations with only one homogeneous equation to obtain a completely analytic solution.
CHAPTER V
PROBLEM FORMULATION

In this chapter a mathematical model of the physical problem is formulated using the Papkovich-Neuber form of solution for displacement[39]. The underlying assumptions of the formulation are discussed. An incorporation of such assumptions simplifies the formulation substantially and a mixed boundary value problem is formulated. A representation of the mixed boundary value conditions with this approach leads to a quadruple set of integral equations with sine and cosine kernels and an unknown weight function. Three equations of the quadruple set are non-homogeneous.

5.1. Problem Statement and Assumptions

Consider a homogeneous, elastic perfectly-plastic two-dimensional infinite medium with a central crack of length 2a under mode-I deformation due to tensile stresses \( \sigma_\infty \) applied at infinity as shown in Figure 10. The origin of the cartesian coordinate system is located at the center of the crack. The x-axis is directed along the crack plane and the y-axis is parallel to the applied remote stresses and thus, both axes represent planes of symmetry. Due to the resulting high stresses at the crack-tip, a dislocation-free zone (DFZ), extending from a to b, is assumed to be present. Similarly the plastic zones,
Figure 10. A crack-tip elastic zone model of Dugdale plastic yielding under tensile mode of deformation.
occupying the region from b to c along the x-axis are assumed to form ahead of the dislocation-free zone. The region past the plastic zone (x > c) is called the remaining ligament.

Since the dislocation-free zone was observed in thin ductile materials, it is assumed that the material is relatively thin. The centrally located through crack is straight and is assumed to lie along the x-axis. The dislocation-free zone ahead of the crack-tip is a very small region of high stresses, larger than the friction stress (yield strength). These stresses are sufficient to cause a dislocation to move and form an equilibrium zone where the stresses are equal to the material yield strength[24]. A dislocation is a representation of strain, therefore the condition of no dislocation implies that the displacements are either constant or zero. Since the crack-tip deforms under applied loading and the plastic continuum ahead of the DFZ is capable of transverse displacements, the displacement in the DFZ must be non-zero. From the continuity of displacements, it follows that the displacements in this region are uniform. A similar assumption is made in anti-plane shear models[11,12] based on continuum theory of dislocations. Regarding the constitutive relations for the material in the DFZ, it is assumed that the material is capable of resisting stresses beyond the yield strength. It may be noted that the true stress-strain curve has this characteristic. Such an assumption is necessary in order to obtain an analytic solution. However, assumptions similar to this have been made in previous studies. For example, Thomson's elastic case model[13] and Weertman's hot butter knife approach[14]. Weertman discusses the possibility of such a stress-strain curve from
micro-structural considerations. It is shown that the stresses and strains beyond the cohesive strength may be approximated by a linear curve with a slope equal to the elastic modulus of the material.

The plastic continuum zone is a perfectly plastic material and the yield stresses are uniformly distributed there. For strain-hardened materials, it is shown that the restraining stresses in the plastic zones vary parabolically with the displacements as shown in Figure 4. Such restraining stresses differ slightly from the constant value assumption for perfectly plastic materials. Furthermore, the plastic continuum is capable of undergoing transverse displacements. However, the deformations parallel to the crack-plane are neglected. Under the assumed state of plane stress and negligible strain-hardening, through the thickness relaxation occurs and the plastic zones are cross (x) shaped representing planes of maximum shear stress. Thus the formed plastic zones are narrow bands and their dimensions along the y-axis are small[21,25]. Such narrow bands are observed for many practical materials including cold rolled steel, high strength steel, silicon steel, stainless steel[25] and polycarbonate[31].

With an application of the principle of superposition and the observations of symmetry, the above stated problem may be treated as two individual problems: a mixed boundary value problem for the whole space with stresses applied only on the crack-plane (problem A), and an uncracked whole space subjected to uniform remote stresses (problem B), shown in Figure 11. Due to the symmetry about the crack plane \( y = 0 \), Problem A is further reduced to one for the half-space \( y > 0 \) on whose boundary, the shear stresses must vanish (Figure 12). All of the
Figure 11. An infinite uncracked plane medium with applied remote tensile stresses.
Figure 12. A mixed boundary value problem in a half space representing a crack under internal pressure, a crack-tip elastic zone and attached Dugdale plastic yield strips subjected to adjusted tensile stresses (problem - A).
mixed boundary conditions for the two problems are as follows:

**Problem A (reduced)**

\[
\begin{align*}
\sigma_{yy}(x,0) &= -\sigma_\infty \quad 0 < |x| < a \\ u_y(x,0^+) &= u_0 \quad a < |x| < b \\ \sigma_{yy}(c,0) &= \sigma_{yp} - \sigma_\infty \quad b < |x| < c \\ u_y(x0) &= 0 \quad |x| > c \\ \sigma_{xy}(x,0) &= 0 \quad \text{for all } x
\end{align*}
\]

**Problem B**

\[
\begin{align*}
\sigma_{yy}(x,z=\infty) &= \sigma_\infty \quad \text{for all } x \\ \sigma_{xy}(x,z=\infty) &= 0 \quad \text{for all } x
\end{align*}
\]

In equations (5.1.1) through (5.1.7), \(u_0\) is the value of the normal transverse displacement \(u_y\) in the elastic zone and equals one-half of the value of the crack-tip opening displacement. \(\sigma_\infty\) is the applied remote tensile stress and \(\sigma_{yp}\) the material yield strength. \(\sigma_{yy}\) and \(\sigma_{xy}\) represent the normal and shear stresses respectively in the whole space.

**5.2. Papkovich-Neuber Formulation and Derivation of Integral Equations**

Since the stresses in the plastic continuum are held at yield stress and the dislocation-free zone is assumed to be an elastic region with linearly related stresses and strains, an analytic solution may
obtained using the linear theory of elasticity (Chapter III). In the absence of body forces and inertia terms, Navier's equations of equilibrium in displacements are

\[(\lambda + G)u_{j,j} + G u_{i,jj} = 0 \quad (5.2.1)\]

where \(\lambda\) is Lame's constant, \(G\) is the shear modulus and standard convention for tensor summation and differentiation apply. A solution for the displacement vector field using Helmholtz's theorem[39] may be written in terms of the gradient of a scalar function \(\phi\) and the curl of a vector field \(\psi\) as follows:

\[u_i = \phi_i + e_{ijk} \psi_k,j \quad (5.2.2)\]

with

\[\nabla^2 \phi = \nabla^2 \psi = \text{constant} \quad (5.2.3)\]

and \(e_{ijk}\) is the permutation symbol.

Papkovich[39] transformed equation (5.2.2) in terms of a Galerkin vector \(F_i\) so that the displacements become

\[2G u_i = 2(1-v)F_{i,jj} - F_{j,jj} \quad (5.2.4)\]

Let

\[F_{i,jj} = \frac{1}{2(1-v)} \phi_i \quad (5.2.5)\]

\[F_{j,j} = \phi_0 \quad (5.2.6)\]

then a substitution of these equations in (5.2.4) yields
\[ 2G u_i = \phi_i - \phi_{0,i} \quad \text{(5.2.7)} \]

The vector field \( \phi_i \) and the scalar function \( \phi_0 \) in equations (5.2.5) and (5.2.6) are related by the following equation:

\[ \nabla^2 \phi_0 = \frac{1}{2(1-v)} \phi_{i,i} \quad \text{(5.2.8)} \]

A general solution of \( \phi_0 \) in this equation is given by [55]

\[ \phi_0 = \frac{1}{4(1-v)} x_j \phi_j + \phi_0 \quad \text{(5.2.9)} \]

where \( \phi_0 \) is an arbitrary scalar function.

Substituting

\[ \phi_i = \frac{\phi_j}{4(1-v)} \quad \text{(5.2.10)} \]

in equations (5.2.9) and (5.2.10), the displacements in equation (5.2.7) become

\[ 2G u_i = -\text{grad} F + 4(1-v)\phi_i \quad \text{(5.2.11)} \]

where the scalar function \( F \) is given by

\[ F = \phi_0 + x_i \phi_i \quad \text{(5.2.12)} \]

For a two dimensional problem, equation (5.2.11) becomes

\[ 2G u_x = -\frac{\partial F}{\partial x} + 4(1-v)\phi_1 \quad \text{(5.2.13)} \]

\[ 2G u_y = -\frac{\partial F}{\partial y} + 4(1-v)\phi_2 \quad \text{(5.2.14)} \]

with

\[ \nabla^2 \phi_i = 0 \quad i = 0,1,2 \quad \text{(5.2.15)} \]
Since the displacement field has only two components in plane problems, \( \phi_1 \) may be taken to be zero. This reduces the displacements to the following form:

\[
\begin{align*}
2G u_x &= -\frac{\partial \phi_0}{\partial x} - y \frac{\partial \phi_0}{\partial x} \\
2G u_y &= (3-4v)\phi_2 - \frac{\partial \phi_0}{\partial y} - y \frac{\partial \phi_2}{\partial y}
\end{align*}
\] (5.2.16)

(5.2.17)

With the linear elasticity relations

\[
\begin{align*}
\varepsilon_x &= \frac{\partial}{\partial x} u_x \\
\varepsilon_y &= \frac{\partial}{\partial y} u_y \\
\varepsilon_{xy} &= \frac{1}{2} \left( \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right) \\
\sigma_x &= \frac{E}{1-v^2} (\varepsilon_x + v \varepsilon_y) \\
\sigma_y &= \frac{E}{1-v^2} (\varepsilon_y + v \varepsilon_x) \\
\sigma_{xy} &= 2G \varepsilon_{xy}
\end{align*}
\] (5.2.18)

(5.2.19)

(5.2.20)

(5.2.21)

(5.2.22)

(5.2.23)

and equations (5.2.16) and (5.2.17), the expressions for the stresses become

\[
\begin{align*}
\sigma_{xx} &= -\frac{2}{\partial^2 x} - 2v \frac{\partial \phi_2}{\partial y} - y \frac{\partial \phi_2}{\partial y} \\
\sigma_{xy} &= -\frac{\partial}{\partial x} \left[ \frac{\partial \phi_0}{\partial y} + y \frac{\partial \phi_1}{\partial y} \right] + (1-2v) \frac{\partial \phi_2}{\partial x} \\
\sigma_{yy} &= -\frac{\partial}{\partial y} \left[ 2(1-v)\phi_2 - \frac{\partial \phi_0}{\partial y} \right] - y \frac{\partial^2 \phi_2}{\partial y^2}
\end{align*}
\] (5.2.24)

(5.2.25)

(5.2.26)
For problems in which stresses are symmetric with respect to the plane of the crack (\(y=0\) in the present case), the displacement \(u_x\) must be an even function of \(y\) and the displacement \(u_y\) must be odd in \(y\). Furthermore, the displacements and strains are required to be continuous (on \(y=0\)) outside the surface of the crack \(|x| > c [54]\). In addition, the shear stresses along the crack plane must vanish. The last condition requires that

\[
\frac{\partial \phi_0}{\partial y} = (1-2v)\phi_2 \quad .
\]  

(5.2.27)

Substituting this relation in equations (5.2.16), (5.2.17) and (5.2.24) through (5.2.26), the stresses and displacements are reduced to

\[
2G \ u_y = 2(1-v)\phi - y \frac{\partial \phi}{\partial y} \quad .
\]  

(5.2.28)

\[
\sigma_{yy} = \frac{\partial \phi}{\partial y} - y \frac{\partial^2 \phi}{\partial y^2} \quad .
\]  

(5.2.29)

where we have omitted the subscript from the potential function \(\phi_2\).

Noting that the applied stresses are symmetric about the plane \(x=0\) (equations (5.1.1) and (5.1.3), the function \(\phi\) is represented in the following form[44]:

\[
\phi(x,y) = \int_0^\infty A(\xi)e^{-\xi y} \cos \xi x \ d\xi
\]  

(5.2.30)

where \(A(\xi)\) is an unknown function.

A combination of (5.2.28), (5.2.29), (5.2.30) and the boundary conditions (5.1.1) through (5.1.4) yields the following quadruple set of integral equations:
\[ \int_{0}^{\infty} A(\xi) \cos \xi x \, d\xi = \sigma_\infty \quad \text{for } 0 < |x| < a \quad (5.2.31) \]

\[ \int_{0}^{\infty} A(\xi) \cos \xi x \, d\xi = \overline{u}_0 \quad \text{for } a < |x| < b \quad (5.2.32) \]

\[ \int_{0}^{\infty} A(\xi) \cos \xi x \, d\xi = \sigma_\infty - \sigma_{yp} \quad \text{for } b < |x| < c \quad (5.2.33) \]

\[ \int_{0}^{\infty} A(\xi) \cos \xi x \, d\xi = 0 \quad \text{for } |x| > c \quad (5.2.34) \]

where for plane stress

\[ \overline{u}_0 = u_0 \frac{E}{2} \quad (5.2.35) \]
CHAPTER VI
ANALYSIS

A solution of quadruple integral equations with three nonhomogeneous equations such as equations (5.2.31) through (5.2.34) is currently not available and is developed in this chapter. Several dual and triple integral equations are analytically solved[43,44,48-50]. Relatively few four part mixed boundary value problems are discussed in the literature[16], however in all cases the resulting equations involve only one or two non-homogeneous components.

In this chapter an analytic solution of the quadruple integral equations, (5.2.31) through (5.2.34) is obtained using the modified finite Hilbert transform method (Chapter IV). An integral form for the unknown function \( A(\xi) \) of the integral equations is used to derive expressions for the stresses and the displacements.

6.1. Solution of Quadruple Integral Equations

Assume the function \( A(\xi) \) in a form similar to that given in [16] as follows:

\[
A(\xi) = \int_0^a g(t) J_0(\xi t) dt + \int_b^c \xi^{-1} \phi(\tau) \sin \xi \tau d\tau \quad (6.1.1)
\]

where \( g \) and \( \phi \) are functions of \( t \) and \( \tau \), respectively. A substitution of \( A(\xi) \) in equation (5.2.34) and interchanging the order of integrations gives for the left hand side
\[ \int_0^a g(t) \, dt \int_0^\infty J_0(\xi t) \cos \xi x \, d\xi \]

\[ + \int_b^c \phi(\tau) \, d\tau \int_0^\infty \xi^{-1} \sin \xi \tau \cos \xi x \, d\xi \quad \text{for } |x| > c \quad (6.1.2) \]

The values of the integrals (C.2) and (C.5) given in Appendix - C for this range of \( x \) shows that both terms in (6.1.2) are identically equal to zero. Thus with the assumed representation of \( A(\xi) \), the fourth integral equation requiring that the displacements must vanish in the remaining ligament is automatically satisfied.

A substitution of (6.1.1) in equation (5.2.32) and an interchange of the order of integrations yields the same form as (6.1.2). However it is defined on \( R^1 \), the elastic zone. For this range of \( x \), the value of the inner integral in the first term is zero (C.5) and that in the second term (C.2) is \( \frac{\pi}{2} \), therefore the function \( \phi(\tau) \) is governed by the following condition:

\[ \int_b^c \phi(\tau) \, d\tau = \frac{2}{\pi} \cdot \bar{u}_0 \quad (6.1.3) \]

Assuming that (5.2.31) and (5.2.33) may be expressed in differential forms

\[ \frac{\partial}{\partial \xi} \int_0^\infty A(\xi) \sin \xi x \, d\xi = \sigma_{\infty} \quad (6.1.4) \]

and

\[ \frac{\partial}{\partial x} \int_0^\infty A(\xi) \sin \xi x \, d\xi = \sigma_{\infty} - \sigma_{yp} \quad (6.1.5) \]

a substitution of (6.1.1) for \( A(\xi) \) in the first differential form (6.1.4) and interchanging the order of integrations yields
\[
\frac{\partial}{\partial x} \left[ \int_0^a g(t) \, dt \int_0^\infty J_0(\xi t) \sin \xi x \, d\xi \right] + \frac{\partial}{\partial x} \int_b^c \phi(\tau) \, d\tau \int_0^\infty \xi^{-1} \sin \xi \tau \sin \xi x \, d\xi
\]

\[= \sigma_\infty \text{ for } 0 < |x| < a \quad (6.1.6)\]

From the values of the inner integrals (C.1) and (C.4), the following relation between the functions \(g(t)\) and \(\phi(\tau)\) is obtained.

\[
\frac{\partial}{\partial x} \left[ \int_0^a \frac{g(t)}{\sqrt{x^2-t^2}} \, dt + \frac{\partial}{\partial x} \int_b^c \frac{1}{2} \phi(\tau) \log \left| \frac{\tau+x}{\tau-x} \right| \, d\tau \right]
\]

\[= \sigma_\infty \text{ for } 0 < |x| < a \quad (6.1.7)\]

A similar substitution of (6.1.1) in equation (6.1.5), an interchange in the order of integrations and a use of the integrals (C.1) and (C.4) leads to

\[
\frac{\partial}{\partial x} \left[ \int_0^a \frac{g(t)}{(x^2-t^2)^{1/2}} \, dt + \frac{1}{2} \int_b^c \phi(\tau) \log \left| \frac{x+r}{x-r} \right| \, d\tau \right]
\]

\[= \sigma_\infty - \sigma_{yp} \text{ for } b < x < c \quad (6.1.8)\]

Applying Abel's integral equation (C.53) and following [16], the following value for the function \(g(t)\) from equation (6.1.7) may be derived:

\[
g(t) = t \int_0^t \frac{2}{\pi} \sigma \frac{ds}{(t^2-s^2)^{1/2}} - t \int_b^c \frac{\phi(\tau) \, d\tau}{(\tau^2-t^2)^{1/2}} \quad (6.1.9)\]

A combination of equations (6.1.8) and (6.1.9) yields
\[
\frac{a}{\partial x} \int_0^a \frac{1}{(x^2-t^2)^{1/2}} \left[ t \int_0^{t/2} \frac{\sigma_{\infty}}{(t^2-s^2)^{1/2}} \, ds - t \int_0^c \frac{\phi(\tau)}{(\tau^2-t^2)^{1/2}} \, d\tau \right] \, dt
\]

\[+ \frac{b}{\partial x} \int_0^c \frac{1}{2} \phi(\tau) \log \frac{|x+\tau|}{|x-\tau|} \, d\tau = \sigma_{\infty} - \sigma_{yp} \text{ for } b < |x| < c \]

(6.1.10)

Carrying out the above differentiations with respect to \( x \), we get

\[
\int_0^a \frac{-x}{(x^2-t^2)^{3/2}} \left[ t \int_0^{t/2} \frac{\sigma_{\infty}}{(t^2-s^2)^{1/2}} \, ds - t \int_0^c \frac{\phi(\tau)}{(\tau^2-t^2)^{1/2}} \, d\tau \right] \, dt
\]

\[+ \frac{1}{2} \int_0^c \frac{2x}{(x^2-\tau^2)^{3/2}} \phi(\tau) \, d\tau = \frac{2}{\pi} (\sigma_{\infty} - \sigma_{yp}) \text{ for } b < |x| < c \]

(6.1.11)

on rearranging

\[
\int_0^a \frac{-xt}{(x^2-t^2)^{3/2}} \cdot \left[ t \int_0^{t/2} \frac{\sigma_{\infty}}{(t^2-s^2)^{1/2}} \, ds \right] \, dt
\]

\[+ \int_0^c \phi(\tau) \left[ \int_0^a \frac{xt}{(x^2-t^2)^{3/2}} \cdot \frac{1}{(\tau^2-t^2)^{1/2}} \, dt + \frac{x}{(x^2-\tau^2)^{3/2}} \right] \, d\tau
\]

\[= \frac{2}{\pi} (\sigma_{\infty} - \sigma_{yp}) \text{ for } b < |x| < c \]

(6.1.12)

A change in the order of integration of the first term and using the following elementary integrals (C.9) and (C.11):

\[
\int_0^a \frac{t \, dt}{(t^2-s^2)^{1/2} (u^2-t^2)^{3/2}} = \frac{(a^2-s^2)^{1/2}}{(u^2-s^2)(u^2-a^2)^{1/2}} \]

(6.1.13)

\[
\int_0^a \frac{t \, dt}{(x^2-t^2)^{3/2} (\tau^2-t^2)^{1/2}} = \frac{1}{\tau - x} \left[ \left( \frac{\tau^2-a^2}{x^2-a^2} \right)^{1/2} - \frac{x}{\tau} \right]
\]

(6.1.14)

results in the following relation for the unknown function \( \phi(\tau) \):
\[
\int_b^c \frac{(\tau^2 - a^2)^{1/2}}{\tau^2 - x^2} \phi(\tau) d\tau - \frac{(x^2 - a^2)^{1/2}}{x} (\sigma_\infty - \sigma_{yp})
\]

\[- \int_0^a \frac{2 \gamma (a^2 - s^2)^{1/2}}{x^2 - s^2} \sigma_\infty \cdot ds = 0 \quad (6.1.15)\]

Let \( \overline{\phi}(\tau) \) be

\[
\overline{\phi}(\tau) = \frac{(\tau^2 - a^2)^{1/2}}{\tau} \phi(\tau) \quad (6.1.16)
\]

and noting that the second and third terms may be combined into a function of \( x \), (6.1.15) may be re-expressed

\[
\int_b^c \frac{\tau \overline{\phi}(\tau)}{\tau^2 - x^2} d\tau = f(x) \quad (6.1.17)
\]

where

\[
f(x) = \frac{(x^2 - a^2)^{1/2}}{x} (\sigma_\infty - \sigma_{yp}) + \int_0^a \frac{(a^2 - s^2)^{1/2}}{x^2 - s^2} \cdot \frac{2}{\pi} \sigma_\infty ds \quad \text{on } R_3 \quad (6.1.18)
\]

using the integral (C.6), it is simple to show for \( x > s \) and \( a > s \) that

\[
f(x) = \sigma_\infty - \frac{(x^2 - a^2)^{1/2}}{x} \sigma_{yp} \quad (6.1.19)
\]

A solution of \( \overline{\phi}(\tau) \) in equation (6.1.17) may be conveniently obtained using the modified finite Hilbert transform (Chapter IV) and is given by

\[
\overline{\phi}(\tau) = - \frac{4}{\pi \tau} \left( \frac{c^2 - b^2}{c^2 - \tau^2} \right)^{1/2} \int_b^c \left( \frac{c^2 - y^2}{y^2 - \tau^2} \right)^{1/2} \frac{y f(y) dy}{y^2 - \tau^2}
\]

\[+ \frac{c}{[(\tau^2 - b^2)(c^2 - \tau^2)]^{1/2}} \quad (6.1.20)\]
where \( c \) is an arbitrary constant. The constant \( c \) in this expression is obtained from the condition (6.1.3), i.e.,

\[
\int_b^c \phi(\tau) = \frac{2}{\pi} \bar{u}_0
\]

In order to facilitate integrations over certain regions in mixed boundary value problems it is convenient to express the solution (6.1.20) in two different forms. This is accomplished by using an identity introduced by Srivastava and Lowengrub (C.40). The two solutions for \( \phi(\tau) \) using (6.1.16) are

\[
\phi(\tau) = -\frac{4}{\pi^2} \left[ \frac{\frac{2}{\pi} \bar{u}_0 \left( \frac{1}{\pi^2} \int_{\tau^2}^{b^2} \frac{f(y)dy}{y^2} \right)}{\left( \frac{\pi^2}{2\pi} \right)^2} \right]^{1/2}
\]

and

\[
\phi(\tau) = -\frac{4}{\pi^2} \left[ \frac{\frac{2}{\pi} \bar{u}_0 \left( \frac{1}{\pi^2} \int_{\tau^2}^{b^2} \frac{f(y)dy}{y^2} \right)}{\left( \frac{\pi^2}{2\pi} \right)^2} \right]^{1/2}
\]

a substitution of (6.1.21) in the condition (b.1.3) yield

\[
\int_b^c \frac{\tau^2}{\pi^2} \left[ \frac{\tau^2}{(\pi^2 - a^2)(\pi^2 - b^2)} \right]^{1/2} \frac{f(y)dy}{y^2 - \tau} + \frac{\tau^2}{\pi^2} \left[ \frac{\tau^2}{(\pi^2 - a^2)(\pi^2 - b^2)(c^2 - \tau^2)} \right]^{1/2} \]

The single integral in second term of the above equation is obtained by a simple substitution and using reference[52], its value from C.27) is
\[
\int_c^b \frac{\tau \, d\tau}{\left[\left(\tau^2 - a^2\right)\left(\tau^2 - b^2\right)\left(c^2 - \tau^2\right)\right]^{1/2}} = \frac{2}{\left(c^2 - a^2\right)^{1/2}} \left[u_1^1 \right.\left.\int_0^u \, du\right]
\]

\[
= \frac{2}{\left(c^2 - a^2\right)^{1/2}} K(k) \quad (6.1.24)
\]

where

\[
k = \left(\frac{c^2 - b^2}{c^2 - a^2}\right)^{1/2} \quad (6.1.25)
\]

and \(K\) is a complete elliptic integral of the first kind (Appendix - A).

Using (6.1.24) in (6.1.23), a following value of the constant \(C_1\) may be deduced:

\[
C_1 = \frac{2}{\pi K(k)} \cdot \left(c^2 - a^2\right)^{1/2} + \frac{4}{\pi^2} \frac{(c^2 - a^2)^{1/2}}{K(k)}
\]

\[
\cdot \int_c^b \left(\frac{y^2}{y^2 - b^2}\right)^{1/2} \cdot y \, f(y) \, dy
\]

\[
\int_b^c \left[\frac{\left(\tau^2 - b^2\right)}{(\tau^2 - a^2)\left(c^2 - \tau^2\right)}\right]^{1/2} \cdot \frac{d\tau}{\tau^2 - \tau^2} \quad (6.1.27)
\]

Following a similar procedure as for the arbitrary constant \(C_1\), an expression for \(C_2\) is

\[
C_2 = \frac{2}{\pi K(k)} \left[u_0 \cdot \left(c^2 - a^2\right)^{1/2} + \frac{4}{\pi^2} \frac{(c^2 - a^2)^{1/2}}{K(k)}\right]
\]

\[
\cdot \int_b^c \left(\frac{y^2 - b^2}{c^2 - y^2}\right)^{1/2} \cdot \frac{y \, f(y) \, dy}{y^2 - \tau^2} \quad (6.1.27)
\]

With known \(\phi(\tau)\), the other function \(g(t)\) may be obtained from (6.1.9) and since the arbitrary constants are defined by (6.1.26) and (6.1.27), the solution of the quadruple integral equations is complete.
6.2. Derivation of Stresses

From Papkovich-Neuber formulation, the stresses \( \sigma_{yy} \) (5.2.29) at \( y=0 \) are given by

\[
\sigma_{yy}(x,0) = \frac{3}{\beta y} \int_0^\infty A(\xi) e^{-\xi y} \cos\xi x \, d\xi
\]

Substituting \( A(\xi) \) from (6.1.1) the stresses are represented by

\[
\int_0^a g(t) J_0(\xi t) \sin x \, dt \, d\xi
\]

using definite integrals (6.2.1) and (C.4) for \( a < x < b \), the stress expression reduces to

\[
\sigma_{yy}(x,0) = -\frac{3}{\beta x} \int_0^a \frac{\partial}{\partial x} \left[ \int_0^\infty \frac{\xi}{1 - (x - t^2)^{1/2}} \right] \, dt
\]

expressing \( g(t) \) in terms of \( \phi(\tau) \) from Equation (6.1.9) and substituting in (6.2.3) the stresses become

\[
-\sigma_{yy}(x,0) = \int_0^t \alpha \left( \frac{1}{x^2 - t^2} \right)^{1/2} \cdot \int_0^\infty \sigma_{\xi} \left( \frac{\xi}{x^2 - t^2} \right)^{1/2} \, ds \, dt
\]

\[
\int_0^a t \cdot \frac{\partial}{\partial x} \left( \frac{1}{x^2 - t^2} \right)^{1/2} \cdot \int_b^c \phi(\tau) \left( \frac{\tau}{x^2 - t^2} \right)^{1/2} \, d\tau \, dt
\]
\[
+ \int_{c}^{d} \frac{\partial}{\partial x} \left\{ \frac{1}{2} \log \frac{r+x}{r-x} \right\} \, dr \text{ for } a < |x| < b \quad (6.2.4)
\]

A performance of necessary differentiations and an interchange of the
order of integrations in the first two terms yields

\[
\sigma_{yy}(x,0) = \int_{0}^{a} \frac{2}{\pi} \sigma_{\infty} \, ds \int_{0}^{a} \frac{x t}{(t^2-s^2)^{1/2}(x^2-t^2)^{3/2}} \, dt
\]

\[
- \int_{b}^{c} \phi(\tau) \, d\tau \int_{0}^{a} \frac{x t}{(t^2-\tau^2)^{1/2}(x^2-\tau^2)^{3/2}} \, dt
\]

\[
- \int_{b}^{c} \frac{\tau^2-\tau^2}{\tau^2-x^2} \phi(\tau) \, d\tau \text{ for } a < |x| < b \quad (6.2.5)
\]

using integrals (6.1.13) and (6.1.14), Equation (6.2.5) is reduced to

\[
\sigma_{yy}(x,0) = \frac{x}{(x^2-a^2)^{1/2}} \left[ \int_{0}^{a} \frac{2}{\pi} \sigma_{\infty} \frac{(a^2-s^2)^{1/2}}{x^2-s^2} \, ds \right.
\]

\[
- \int_{b}^{c} \frac{(\tau^2-a^2)^{1/2}}{\tau^2-x^2} \phi(\tau) \, d\tau \right] \text{ for } a < |x| < b \quad (6.2.6)
\]

with \(\phi(\tau)\) given by (6.1.22) the above expression takes the following
form:

\[
\sigma_{yy}(x,0) = \frac{2}{\pi} \frac{x}{(x^2-a^2)^{1/2}} \sigma_{\infty} \int_{0}^{a} \frac{(a^2-s^2)^{1/2}}{x^2-s^2} \, ds
\]

\[
+ \frac{4}{\pi} \cdot \frac{x}{(x^2-a^2)^{1/2}} \cdot \int_{b}^{c} \left( \frac{y^2-b^2}{c^2-y^2} \right)^{1/2} \cdot y f(y) \, dy
\]

\[
\int_{b}^{c} \frac{(c^2-\tau^2)}{[(\tau^2-b^2)(c^2-\tau^2)]^{1/2}(\tau^2-x^2)(y^2-\tau)} \, d\tau
\]

\[
- \frac{c^2x}{(x^2-a^2)^{1/2}} \cdot \int_{b}^{c} \frac{\tau}{[(\tau^2-b^2)(c^2-\tau^2)]^{1/2}} \cdot \frac{d\tau}{\tau^2-x^2}
\]

\text{for } a < x < b \quad (6.2.7)
After lengthy algebraic manipulations and the following important result [46]

\[
\int_{a}^{b} \frac{t \, dt}{[(t^2-a^2)(b^2-t^2)]^{1/2}(t^2-y^2)} = \frac{\pi}{2} [(a^2-y^2)(b^2-y^2)]^{-1/2} \quad \text{for } 0 < y < a
\]

\[
= 0 \quad \quad \quad 0 < y < b
\]

\[
= -\frac{\pi}{2} [(y^2-a^2)(y^2-b^2)]^{-1/2} \quad y > b \quad (6.2.8)
\]

when incorporated in (6.2.7) gives the following expression for stresses:

\[
\sigma_{yy}(x,0) = \frac{2}{\pi} \sigma_\infty \cdot \frac{x}{(x^2-a^2)^{1/2}} \int_{0}^{a} \frac{\left(a^2 - s^2\right)^{1/2}}{s^2 - s^2} \, ds
\]

\[
+ \frac{2}{\pi} \int_{b}^{c} \left(\frac{y^2-b^2}{b^2-y^2}\right)^{1/2} \cdot \frac{y \cdot f(y) \, dy}{y^2-x^2}
\]

\[
- \frac{\pi}{2} C_2 \left[\frac{x}{(x^2-a^2)(b^2-x^2)(c^2-x^2)}\right]^{1/2} \quad \text{for } |x| < b \quad (6.2.9)
\]

Substituting the value of the integral in the first term from (c.6), an expression for the stresses in the elastic zone in terms of applied remote stresses \(\sigma_\infty\) and the material yield strength \(\sigma_{yp}\) is obtained, i.e.,

\[
\sigma_{yy}(x,0) = \sigma_\infty \left[\frac{x}{(x^2-a^2)^{1/2}} - 1\right]
\]

\[
+ \frac{2}{\pi} \beta_1(x) (c^2-x^2) \int_{b}^{c} \left(\frac{y^2-b^2}{y^2-y^2}\right)^{1/2} \frac{y \cdot f(y) \, dy}{y^2-x^2}
\]

\[
- \frac{\pi}{2} \beta_1(x) C_2 \quad \text{for } |x| < b \quad (6.2.10)
\]
where

$$b_1(x) = x[(x^2-a^2)(b^2-x^2)c^2-x^2]^{-1/2} \quad (6.2.11)$$

and \(f(y)\) is given by \((6.1.19)\).

Similarly an expression for stresses in the remaining ligament, \((x>c)\), using \((6.2.2)\) and the integrals \((C.1)\) and \((C.4)\) is given by

$$\sigma_{yy}(x,0) = \frac{x}{(x^2-a^2)^{1/2}} \left[ \int_0^a \frac{(a^2-s^2)^{1/2}}{x^2-s^2} \cdot \frac{2}{\pi} \sigma_{\infty} \ ds \right.$$

$$\left. + \int_b^c \frac{(\tau^2-a^2)^{1/2}}{x^2-\tau^2} \psi(\tau) \ d\tau \right] \text{ for } |x| > c \quad (6.2.12)$$

where \(\psi(\tau)\) in this case is used in the form given by \((6.1.21)\).

A substitution of \((6.1.21)\) in \((6.2.12)\) and interchanging the order of integrations results in the following expression for stresses

$$\sigma_{yy}(x,0) = \frac{2}{\pi} \sigma_{\infty} \frac{x}{(x^2-a^2)^{1/2}} \left[ \int_0^a \frac{(a^2-s^2)^{1/2}}{x^2-s^2} \ ds \right.$$

$$- \frac{4}{\pi} \frac{x}{(x^2-a^2)^{1/2}} \int_b^c \left( \frac{c^2-y^2}{y^2-b^2} \right)^{1/2} y f(y) \ dy$$

$$\int_b^c \frac{\tau(\tau^2-b^2)^{1/2}}{(c^2-\tau^2)^{1/2}} \cdot \frac{d\tau}{(x^2-\tau^2)(y^2-\tau^2)}$$

$$+ \frac{x}{(x^2-a^2)^{1/2}} \cdot C_1 \left[ \int_b^c \frac{\tau \ d\tau}{[(\tau^2-b^2)(c^2-\tau^2)]^{1/2}(x^2-\tau^2)} \right.$$ \text{ for } |x| > c \quad (6.2.13)$$

Evaluating the integral in the first term by \((C.6)\) and the integral involving \(\tau\) in the second and the third terms by \((6.2.8)\), and after some algebraic manipulations similar to those in the case of stresses
in the elastic zone, the stresses in the remaining ligament are expressed by

\[
\sigma_{yy}(x,0) = \sigma_0 \left( \frac{x}{x^2 - a^2} \right)^{1/2} - \frac{\pi}{2} \beta_2(x) c_1 + \frac{2}{\pi} \beta_2(x)(x^2 - b^2) \int_b^c \left( \frac{c^2 - y^2}{x^2 - y^2} \right)^{1/2} \cdot \frac{y f(y) dy}{x^2 - y^2}
\]

for \(|x| > c\)  \hspace{1cm} (6.2.14)

where

\[
\beta_2(x) = x[(x^2-a^2)(x^2-b^2)(x^2-c^2)]^{-1/2}
\]

and \(f(y)\) is given by (6.1.19).

6.3. Derivation of Displacements

From the transverse displacement representation by Papkovich-Neuber harmonic function \(\phi(x,y)\), for the two dimensional case given by (5.2.28), the following relation may be derived:

\[
\int_0^\infty A(\xi) \cos \xi x \, d\xi = \bar{u}_y
\]

(6.3.1)

where the harmonic function \(\phi(x,y)\) is given by (5.2.30) and its reduced form at \(y=0\) is used. Also

\[
\bar{u}_y = \frac{E}{2} u_y \text{ for plane stress}
\]

(6.3.2)

A substitution of \(A(\xi)\) from (6.1.1) in (6.3.1) gives

\[
\bar{u}_y = \int_0^\infty \int_0^a g(t) J_0(\xi t) \cos \xi x \, dt \, d\xi + \int_0^c \int_b^c \left( \frac{c}{\xi} \right)^{-1} \phi(t) \sin \xi t \cos \xi x \, dt \, d\xi
\]

(6.3.3)
An interchange in the order of integration in (6.3.3) and using the 
evaluated integrals (C.2) and (C.5) for the crack zone \((0 < |x| < a)\), 
the following expression for the displacements is obtained:

\[
\overline{u}_y(x,0) = \int_0^a \frac{q(t)}{x} \frac{dt}{(t^2 - x^2)^{1/2}} + \int_b^c \frac{\pi}{2} \phi(\tau) d\tau \quad \text{for} \quad 0 < |x| < a \tag{6.3.4}
\]

where it has been assumed that the running variable \(t > x\). The second 
term in this expression is given by the relation (6.1.3) which when 
substituted in (6.3.4) yields

\[
\overline{u}_y = \int_0^a \frac{q(t)}{(t^2 - x^2)^{1/2}} \frac{dt}{x} + \overline{u}_0 \quad \text{for} \quad 0 < |x| < a \tag{6.3.5}
\]

with \(g(t)\) being given by (6.1.9). Similarly from general displacement 
expression (6.3.3) using the evaluated integrals (C.2) and (C.5) for the 
plastic zone, \(b < x < c\), the following displacement relation is obtained:

\[
\overline{u}_y = \int_b^c \frac{\pi}{2} \phi(\tau) d\tau \quad \text{for} \quad b < |x| < c \tag{6.3.6}
\]

where \(\phi(\tau)\) is given by (6.1.21) or alternatively (6.1.22) and it is 
assumed that \(x < \tau\).
The general mathematical derivations of stress and displacement fields in the last chapter involved integral expressions. An evaluation of these integrals in the stress expressions indicate that stress singularities occur at the crack-tips (|x|=a) and at both ends of plastic zones (|x|=b and |x|=c). Since the stresses must be bounded in the plastically deformed zones, a removal of these singularities yields a relation for the plastic zone size in terms of the crack length, elastic zone size, plastic zone size and applied remote tensile stresses.

The purpose of this chapter is to obtain the plastic zone size relation, the resulting crack-tip stress intensity factor, the value of the crack-tip opening displacement and the stress and displacement distributions along the crack plane.

It was shown in Chapter V that the transverse displacements within the elastic zone must be constant. The obtained results for such displacements show that they approach a value equal to CTOD at the trailing ends of the plastic zone and vanish at the leading ends. Similarly the stresses in the elastic zones and the remaining ligaments approach the yield strength as the plastic zone is approached.
7.1. Evaluation of Stresses - General Solution

The integral forms of expressions for the stresses in the elastic zones and the remaining ligaments involve two constants $C_1$ and $C_2$ which are evaluated next.

7.1.1. The Constants

The constants $C_1$ and $C_2$ are given by Equations (6.1.26) and (6.1.27) respectively. The relation for $C_1$ may be written as

$$C_1 = \frac{2}{\pi} \bar{u} \cdot (c^2-a^2)^{1/2} + \frac{4}{\pi^2} \frac{(c^2-a^2)^{1/2}}{K(k)} I_{c1}$$  \hspace{1cm} (7.1.1.1)

where

$$I_{c1} = \int_{b}^{c} \left( \frac{c^2-y^2}{y^2-b^2} \right)^{1/2} \cdot y f(y) dy \int_{b}^{c} \frac{\tau^2 (\tau^2-b^2)}{(\tau^2-a^2)(c^2-\tau^2)} \frac{1}{\sqrt{y^2-\tau^2}} d\tau$$  \hspace{1cm} (7.1.1.2)

Substituting for $f(y)$ from (6.1.19), $I_{c1}$ may be written as a sum of two integrals:

$$I_{c1} = I_{c11} + I_{c12}$$  \hspace{1cm} (7.1.1.3)

with

$$I_{c11} = \sigma_\infty \int_{b}^{c} \left( \frac{c^2-y^2}{y^2-b^2} \right)^{1/2} \cdot y dy \int_{b}^{c} \frac{\tau^2 (\tau^2-b^2)}{(\tau^2-a^2)(c^2-\tau^2)} \frac{1}{\sqrt{y^2-\tau^2}} d\tau$$  \hspace{1cm} (7.1.1.4)

$$I_{c12} = -\sigma_{yp} \int_{b}^{c} \left( \frac{(c^2-y^2)(y^2-a^2)}{y^2-b^2} \right)^{1/2} dy$$
Using worked out integral 34 (Appendix - C) the value of $I_{c11}$ is

$$I_{c11} = \frac{\pi}{2} \cdot \frac{\sigma_{\infty}}{(c^2-a^2)^{1/2}} \left[ \frac{b^2-a^2}{c^2-a^2} K(k) - E(k) \right]$$  \hspace{1cm} (7.1.1.6)

where $E(k)$ is the complete elliptical integral of the second kind.

The inner integral with respect to $\tau$ in $I_{c12}$ is obtained from integral 32 (Appendix - C) and with a substitution of its value in (7.1.1.5),

$$I_{c12} = -\sigma_{yp} \frac{(b^2-a^2)}{(c^2-a^2)^{1/2}} \cdot \int_{b}^{c} \left[ \frac{(c^2-y^2)}{(c^2-a^2)(c^2-b^2)} \right]^{1/2} dy$$

$$\Xi \left( \frac{\pi}{2}, \sigma_{y,k} \right)$$

$$+ \sigma_{yp} \cdot K(k) \cdot \frac{b^2-c^2}{(c^2-a^2)^{1/2}} \int_{b}^{c} \left[ \frac{c^2-y^2}{(y^2-a^2)(y^2-b^2)} \right]^{1/2} dy$$  \hspace{1cm} (7.1.1.7)

where

$$\sigma_{y}^2 = \frac{c^2-b^2}{c^2-a^2} \cdot \frac{a^2-y^2}{b^2-y^2}$$  \hspace{1cm} (7.1.1.8)

$$k^2 = \frac{c^2-b^2}{c^2-a^2}$$  \hspace{1cm} (7.1.1.9)

and $\Xi$ is a complete elliptic integral of the third kind.

Denoting the first term in (7.1.1.7) by $I_{c121}$ and second by $I_{c122}$, and noting that $\sigma_{y}^2$ is greater than 1 and therefore the elliptical integral of the third kind in $I_{c121}$ may be written as [52]

$$\Xi(\sigma_{y}^2,k) = -\sigma_{y} KZ(A,k) \frac{KZ(A,k)}{[(\alpha^2-1)(\alpha^2-k^2)]^{1/2}}$$  \hspace{1cm} (7.1.1.10)
where the Jacobian zeta function

\[ KZ(A, k) = K E(A, k) - E F(A, k) \quad (7.1.1.11) \]

and \( K \) and \( E \) are complete elliptic integrals. Also \( F(A, k) \) and \( E(A, k) \) are incomplete elliptic integrals of the first and second kind respectively, and

\[ A = \sin^{-1}\left( \frac{1}{\alpha_y} \right). \quad (7.1.1.12) \]

Substituting for \( \alpha_y \) and \( k \) from (7.1.1.8) and (7.1.1.9) in (7.1.1.10) and (7.1.1.11) and after lengthy algebraic manipulations, it can be shown that

\[ \pi \left( \frac{\pi}{2}, \alpha_y, k \right) = -\left( \frac{c^2-a^2}{b^2-a^2} \right)^{1/2} \cdot \left[ \frac{(y^2-a^2)(y^2-b^2)}{c^2-y^2} \right]^{1/2} \cdot [K(k) E(A, k) - E(k) F(A, k)] \quad (7.1.1.13) \]

with (7.1.1.13)

\[ I_{c121} = \sigma_{yp} \int_{b}^{c} [K(k) E(A, k) - E(k) F(A, k)] \, dy \quad (7.1.1.14) \]

using worked out integrals 36 and 37 (Appendix - C)

\[ I_{c121} = -\sigma_{yp} \left\{ \left[ \frac{a^2}{b} E(k) K(k') + \frac{b^2-a^2}{b} E(k) \right] \pi(k^2, k') + bK(k) E(k) \right\}, \quad (7.1.1.15) \]

\[ I_{c122} = \sigma_{yp} K(k) \left[ \frac{b^2-a^2}{(c^2-a^2)^{1/2}} \cdot \int_{b}^{c} \left[ \frac{c^2-y^2}{(y^2-a^2)(y^2-b^2)} \right]^{1/2} \right] \, dy \quad (7.1.1.16) \]
Let
\[
\begin{align*}
  y^2 &= t \\
  c^2 &= A \\
  b^2 &= B \\
  a^2 &= C \\
  0 &= D
\end{align*}
\]

the integral in (7.1.1.16) is reduced to
\[
\frac{1}{2} \int_{B}^{A} \frac{A-t}{(t-B)(t-C)(t-D)} \, dt \quad (7.1.1.17)
\]

From integral no. 24 (Appendix - C), its value is
\[
= \frac{1}{2} (A-B) g \int_{0}^{u_1} \frac{cn^2 u \, du}{1-\alpha^2 sn^2 u} \quad (7.1.1.18)
\]

\[
= \frac{1}{2} \cdot \frac{(c^2-b^2)}{b(c^2-a^2)^{1/2}} \cdot \left[ \frac{K(k)}{\alpha^2} + \frac{a^2-1}{\alpha^2} \Pi(a^2,k) \right] 
\]

\[
= \frac{(c^2-a^2)^{1/2}}{b} K(k) + \frac{a^2-b^2}{b(c^2-a^2)^{1/2}} \Pi \left( k^2, \overline{k} \right) \quad (7.1.1.19)
\]

with (7.1.1.19), the value of the integral
\[
I_{c12} = \frac{\sigma y_p}{b^2-a^2} \frac{b^2-a^2}{b(c^2-a^2)^{1/2}} k(k) \left[ (c^2-a^2)^{1/2} K(k) - \frac{b^2-a^2}{(c^2-a^2)^{1/2}} \Pi(k^2,\overline{k}) \right] \quad (7.1.1.20)
\]

Adding the two terms $I_{c121}$ and $I_{c122}$ of (7.1.1.7)
\[
I_{c12} = \frac{\sigma y_p}{(c^2-a^2)^{1/2}} \left[ \frac{b}{c^2-a^2} K(k) E(\overline{k}) \right. \\
- \frac{a^2}{b} (c^2-a^2)^{1/2} \left. E(k) K(\overline{k}) - \frac{(b^2-a^2)}{b} (c^2-a^2)^{1/2} \right]
\[ K(k) K(\overline{k}) + \frac{(b^2-a^2)^2}{b(c^2-a^2)^{1/2}} \cdot K(k) \Pi(k^2, \overline{k}) \]
\[ - \frac{(b^2-a^2)}{b} (c^2-a^2)^{1/2} \cdot E(k) \Pi(k^2, \overline{k}) \]  \hspace{1cm} (7.1.1.21)

With \( I_{c11} \) (7.1.1.6) and \( I_{c12} \) (7.1.1.21), \( I_{c1} \) is determined from (7.1.1.3) and the following value of \( C_1 \) (7.1.1.1) is obtained:
\[ C_1 = \frac{2}{\pi} (c^2-a^2)^{1/2} \left[ c_{11} + c_{12} + c_{13} \right] \]  \hspace{1cm} (7.1.1.22)

where
\[ c_{11} = \frac{u_0}{K(k)} \]
\[ c_{12} = (c^2-a^2)^{1/2} \left[ \frac{b^2-a^2}{c^2-a^2} - E(k) \right] \sigma \]
\[ c_{13} = \frac{2}{\pi} \frac{b^2-a^2}{b} \sigma_y p \left[ \frac{a^2}{b^2-a^2} \frac{E(k)}{K(k)} K(\overline{k}) \right. \]
\[ + \left. \frac{E(k)}{K(k)} \Pi(k^2, \overline{k}) + K(\overline{k}) - \frac{b^2-a^2}{c^2-a^2} \Pi(k^2, \overline{k}) - \frac{b^2}{b^2-a^2} E(\overline{k}) \right] \]  \hspace{1cm} (7.1.1.23)

From (6.1.27) the arbitrary constant \( C_2 \) may be written
\[ C_2 = \frac{2}{\pi} \frac{u_0}{K(k)} (c^2-a^2)^{1/2} + \frac{4}{\pi} \frac{(c^2-a^2)^{1/2}}{K(k)} I_{c2} \]  \hspace{1cm} (7.1.1.24)

where
\[ I_{c2} = \int_b^c \left[ \frac{u^2 (c^2-u^2)}{(u^2-a^2)(u^2-b^2)} \right]^{1/2} du \int_b^c \left[ \frac{y^2-b^2}{y^2-a^2} \right]^{1/2} \frac{y f(y) dy}{y^2-u^2} \]  \hspace{1cm} (7.1.1.25)

using \( f(y) \) from (6.1.19), the above integral is expressed as a sum of two integrals.
\[ I_{c2} = I_{c21} + I_{c22} \]  

where

\[ I_{c21} = \sigma \int_{b}^{c} \left[ \frac{\tau^2 (c^2 - \tau^2)}{(\tau^2 - a^2)(\tau^2 - b^2)} \right]^{1/2} d\tau \int_{b}^{c} \left( \frac{y^2 - b^2}{y^2 - c^2} \right)^{1/2} dy \cdot \frac{u}{y^2 - u^2} dy \]  

(7.1.1.27)

and

\[ I_{c22} = -\sigma y \int_{b}^{c} \left[ \frac{\tau^2 (c^2 - \tau^2)}{(\tau^2 - a^2)(\tau^2 - b^2)} \right]^{1/2} d\tau \int_{b}^{c} \left( \frac{(y^2 - b^2)(y^2 - a^2)}{c^2 - y^2} \right)^{1/2} dy \]  

(7.1.1.28)

The value of \( I_{c21} \) is given by worked out integral no. 35 (Appendix - C) i.e.,

\[ I_{c21} = \frac{\pi}{2} \cdot \frac{\sigma}{(c^2 - a^2)^{1/2}} \cdot [K(k) - E(k)] \]  

(7.1.1.29)

Assuming that the inner integral may be combined to take the following form:

\[ \int_{b}^{c} \left[ \frac{\tau^2 (c^2 - \tau^2)}{(\tau^2 - a^2)(\tau^2 - b^2)} \right]^{1/2} \frac{dr}{y^2 - \tau^2} \]  

(7.1.1.30)

its value as given by integral no. 32 (Appendix - C) is

\[ \frac{1}{(c^2 - a^2)^{1/2}} \left[ K(k) - \pi \left( \frac{a_y^2}{2} \right) \right] \]  

(7.1.1.31)

where
\[
\alpha^2_y = \frac{c^2 - b^2}{c^2 - y^2} \quad (7.1.1.32)
\]

A substitution of (7.1.1.31) in the expression (7.1.1.28) for \( I_{c22} \)
and representing it as a sum of two components, i.e.,

\[
I_{c22} = I_{c221} + I_{c222} \quad (7.1.1.33)
\]

where

\[
I_{c221} = \frac{\sigma_{yp}}{(c^2 - a^2)^{1/2}} \cdot K(k) \int_{b}^{c} \left( \frac{(y^2 - b^2)(y^2 - a^2)}{c^2 - y^2} \right)^{1/2} dy \quad (7.1.1.34)
\]

\[
I_{c222} = \frac{\sigma_{yp}}{(c^2 - a^2)^{1/2}} \cdot \int_{b}^{c} \left[ \frac{(y^2 - b^2)(y^2 - a^2)}{c^2 - y^2} \right]^{1/2} \cdot \Pi \left( \frac{\pi}{2}, \alpha^2_y, k \right) dy \quad (7.1.1.35)
\]

it is noted that the value of \( I_{c221} \) is given by integral no. 30
(Appendix - C) as given below:

\[
I_{c221} = \frac{1}{2} \sigma_{yp} \cdot (b^2 - a^2)^{1/2} \cdot K(k)
\]

\[
\left[ - \frac{b}{b^2 - a^2} E(k) - \frac{1}{b} K(k) + \frac{c^2 - b^2 - a^2}{b(c^2 - a^2)} \Pi(k^2, k) \right] \quad (7.1.1.36)
\]

Since \( \alpha^2_y > 1 \), the elliptic integral of the third kind in (7.1.1.35),
with relation 46 of Appendix - C, is expressed as follows:

\[
\Pi \left( \frac{\pi}{2}, \alpha^2_y, k \right) = \int_{0}^{K} \frac{du}{\sqrt{1 - \alpha^2_y \sin^2 u}} = \frac{\alpha^2_y KZ(A, k)}{[(\alpha^2_y - 1)(\alpha^2_y - k^2)]^{1/2}} \quad (7.1.1.37)
\]

with (7.1.1.37),
\[ I_{c222} = - \frac{\alpha_y p}{(c^2 - a^2)^{1/2}} \cdot \int_b^c \frac{(y^2 - b^2)(y^2 - a^2)}{c^2 - y^2} \, \alpha_y \, K_z(A, k) \left[ \frac{(\alpha_y^2 - 1)(\alpha_y^2 - k^2)}{(\alpha_y^2 - 1)(\alpha_y^2 - a^2)} \right]^{1/2} \]

Substituting for \( \alpha_y \) from (7.1.1.32) and using relation no. 47 (Appendix - C) \( I_{c222} \) becomes

\[ I_{c222} = - \alpha_y p \cdot \int_b^c \left[ K(k) \, E(\phi_y, k) - E(k) \, F(\phi_y, k) \right] \, dy \quad (7.1.1.39) \]

where

\[ \phi_y = \sin^{-1}\left[ \frac{(c^2 - a^2)(y^2 - b^2)}{(c^2 - b^2)(y^2 - a^2)} \right]^{1/2} \quad (7.1.1.40) \]

From integral 36 (Appendix - C)

\[ \int_b^c F(\phi_y, k) \, dy = C \, K(k) \, \frac{a^2}{b} \, K(k) - \frac{b^2 - a^2}{b} \, \Pi\left(\frac{\pi}{2}, \alpha^2, k\right) \quad (7.1.1.41) \]

and from integral 37 (Appendix - C)

\[ \int_b^c E(\phi_y, k) \, dy = C \, E(k) - b \, E(k) \quad (7.1.1.42) \]

with (7.1.1.41) and (7.1.1.42), the value of the integral

\[ I_{c222} = - \alpha_y p \left[ C \, K(k) \, E(k) - b \, K(k) \, E(k) - c \, K(k) \, E(k) \right. \]

\[ + \frac{a^2}{b} \, E(k) \, K(k) + \frac{b^2 - a^2}{b} \, E(k) \, \Pi\left(\frac{\pi}{2}, \alpha^2, k\right) \] \quad (7.1.1.43)

where
\[ \alpha^2 = k^2 = \frac{c^2 - b^2}{c^2 - a^2} \]

and

\[ k^2 = \frac{a^2}{b^2} \]

Combining \( I_{c222} \) (7.1.1.43) and \( I_{c221} \) (7.1.1.36) gives the value of the integral \( I_{c22} \) (7.1.1.33) and a combination of \( I_{c21} \) and \( I_{c22} \) from (7.1.1.29) and (7.1.1.33) respectively gives the value of the integral \( I_{c2} \) (7.1.1.25). Carrying out the above additions,

\[
I_{c2} = \frac{\pi}{2} \sigma_\infty \left( c^2 - a^2 \right)^{1/2} \left[ K(k) - E(k) \right]
\]

\[ + \frac{b^2 - a^2}{b} \sigma_\infty \left[ K(k)K(\overline{k}) + \frac{a^2}{b^2 - a^2} E(k)K(\overline{k}) \right.
\]

\[ - \frac{b^2}{b^2 - a^2} K(k)E(\overline{k}) - K(k) \frac{\pi}{2} (k^2, \overline{k}) + E(k) \frac{\pi}{2} (k^2, \overline{k}) \left] \right. \]  (7.1.1.44)

With \( I_{c2} \) known from (7.1.1.44), the value of the arbitrary constant \( C_2 \) (7.1.1.24) is given by

\[ C_2 = \frac{2}{\pi} \left( c^2 - a^2 \right)^{1/2} (c_{21} + c_{22} + c_{23}) \]  (7.1.1.45)

where

\[ c_{21} = \frac{u_0}{K(k)} \]

\[ c_{22} = (c^2 - a^2)^{1/2} \cdot \left[ 1 - \frac{E(k)}{K(k)} \right] \sigma_\infty \]

and
\[ c_{23} = \frac{2}{\pi} \frac{b^2 - a^2}{b^2 - a^2} \sigma_{y_0} X_0 \left[ \frac{K(k)}{K(k)} + \frac{a^2}{b^2} \frac{E(k)}{K(k)} \right] \]

\[
- \frac{b^2}{b^2 - a^2} E(k) - \pi(k^2,k) + \frac{E(k)}{K(k)} \pi(k^2,k) \]  \tag{7.1.1.46}

7.1.2. Stresses in the Elastic Zone

The stresses in the elastic zone, Equation (6.2.10) may be expressed as follows:

\[
\sigma_{yy}(x,0) = \sigma_{\infty} \left[ \frac{x}{(x^2 - a^2)^{1/2}} - 1 \right] - \frac{\pi}{2} B_1(x) c_2
\]

\[
+ \frac{2}{\pi} B_1(x)(c^2 - x^2) I_{R2} \]  \tag{7.1.2.1}

The arbitrary constant \( c_2 \) is given by (7.1.1.45) of the previous section and \( I_{R2} \) denotes the value of the integral

\[
\int_{c}^{c} \left[ \frac{y^2 - b^2}{c^2 - y^2} \right]^{1/2} \cdot \frac{y f(y) dy}{y^2 - x^2} \]  \tag{7.1.2.2}

Using \( f(y) \) from (6.1.19), the integral \( I_{R2} \) is expressed as a sum of two integrals, i.e.,

\[
I_{R2} = I_{R21} + I_{R22} \]  \tag{7.1.2.3}

where

\[
I_{R21} = \sigma_{\infty} \int_{c}^{c} \left[ \frac{y^2 - b^2}{c^2 - y^2} \right]^{1/2} \frac{y dy}{y^2 - x^2} \]  \tag{7.1.2.4}

and
\[ I_{R22} = - \sigma_{yp} \cdot \int_{b}^{c} \left[ \frac{(y^2-a^2)(y^2-b^2)}{c^2-y^2} \right]^{1/2} \frac{dy}{y^2-x^2} \]  

(7.1.2.5)

With identity
\[ \frac{y^2-b^2}{y^2-x^2} = 1 + \frac{x^2-b^2}{y^2-x^2} \]

the integral \( I_{R21} \) (7.1.2.4) may be written as follows:

\[ I_{R21} = \sigma_{\infty}(x^2-b^2) \int_{b}^{c} \frac{y \ dy}{[y^2-b^2](c^2-y^2)]^{1/2}(y^2-x^2)} + \sigma_{\infty} \int_{b}^{c} \frac{y \ dy}{[(y^2-b^2)(c^2-y^2)]^{1/2}} \]  

(7.1.2.6)

the value of the integral in the first term is given by (6.2.8) and
the integral in the second term is obtained from integral no. 7
(Appendix - C). Substituting their respective values in (7.1.2.6),
the above integral becomes

\[ I_{R21} = \frac{\pi}{2} \sigma_{\infty} \left[ 1 - \left( \frac{b^2-x^2}{c^2-x^2} \right)^{1/2} \right] \]  

(7.1.2.7)

The integral \( I_{R22} \) is evaluated as a sum of three integrals, i.e.,

\[ I_{R22} = I_{R221} + I_{R222} + I_{R223} \]  

(7.1.2.8)

where

\[ I_{R221} = \sigma_{yp} \int_{b}^{c} \left[ \frac{(y^2-a^2)}{(y^2-b^2)(c^2-y^2)} \right]^{1/2} \ dy \]  

(7.1.2.9)
\[
I_{R222} = \sigma_{yp} (b^2 - x^2) \int_{b}^{c} \frac{dy}{\left[ (y^2 - a^2)(y^2 - b^2)(c^2 - y^2) \right]^{1/2}} \tag{7.1.2.10}
\]

and
\[
I_{R223} = -\sigma_{yp} (b^2 - x^2)(x^2 - a^2) \int_{b}^{c} \frac{dy}{\left[ (y^2 - a^2)(y^2 - b^2)(c^2 - y^2) \right]^{1/2}(y^2 - x^2)} \tag{7.1.2.11}
\]

The following identities are used to express \(I_{R2}\) by (7.1.2.8):
\[
\frac{y^2 - a^2}{y^2 - x^2} = 1 + \frac{x^2 - a^2}{y^2 - x^2}
\]
and
\[
\frac{y^2 - b^2}{y^2 - x^2} = 1 - \frac{b^2 - x^2}{y^2 - x^2} .
\]

The value of the integral in (7.1.2.9) is given by integral no. 28 (Appendix - C),
\[
\int_{b}^{c} \left[ \frac{(y^2 - a^2)}{(y^2 - b^2)(c^2 - y^2)} \right]^{1/2} dy
\]
\[
= \frac{b^2 - a^2}{b(c^2 - a^2)^{1/2}} \cdot \int_{0}^{u_1} \frac{du}{\sqrt{1 - a^2 \text{sn}^2 u}} = \frac{b^2 - a^2}{b(c^2 - a^2)^{1/2}} \Pi(k^2, \overline{k}) \tag{7.1.2.12}
\]
from which
\[
I_{R221} = -\sigma_{yp} \cdot \frac{b^2 - a^2}{b(c^2 - a^2)^{1/2}} \Pi(k^2, \overline{k}) \tag{7.1.2.13}
\]
and the integral $I_{R222}$ is obtained by a change of the variable, $y^2 = t$
and using integral no. 22 (Appendix - C), i.e.,

$$\int_{c}^{d} \frac{dy}{b \left[(y^2-a^2)(y^2-b^2)(c^2-y^2)\right]^{1/2}}$$

$$= \frac{1}{2} \int_{c}^{d} \frac{dt}{b^2 \left[(t-a^2)(t-b^2)(t-c^2-t)\right]^{1/2}}$$

$$= \frac{1}{b(c^2-a^2)^{1/2}} \int_{0}^{1} \frac{du}{u^2} = \frac{1}{b(c^2-a^2)^{1/2}} \cdot K(k) \quad (7.1.2.14)$$

with (7.1.2.14), the integral (7.1.2.10) becomes

$$I_{R222} = c_{\gamma p} \frac{(b^2-x^2)}{b(c^2-a^2)^{1/2}} \cdot \frac{K(k)}{b(c^2-a^2)^{1/2}}. \quad (7.1.2.15)$$

The integral in (7.1.2.11) with substitutions

$t = y^2$
$A = c^2$
$B = b^2$
$C = a^2$
$D = 0$

and

$p = x^2$

takes the following form:

$$\int_{c}^{d} \frac{dy}{b \left[(y^2-a^2)(y^2-b^2)(c^2-y^2)\right]^{1/2}} \cdot \frac{1}{y^2-x^2}$$

$$= \frac{1}{2} \int_{A}^{B} \frac{dt}{\sqrt{(t-p)[(A-t)(t-B)(t-C)(t-D)]^{1/2}}} \quad (7.1.2.16)$$
assuming $x \neq B$, this integral can be evaluated from integral no. 26 (Appendix - C) with a value

\[
I_{R223} = \frac{g}{B-p} \int_{0}^{u_1} \frac{1-\frac{a_2}{2} \text{sn}^2 u}{1-a_3 \text{sn}^2 u} \, du
\]  

\[
= \frac{2}{b(c^2-a^2)^{1/2}} \cdot \frac{1}{b^2-x^2} \cdot \left[ (\alpha_3^2-\alpha^2) \Pi \left( \frac{\pi}{2}, \alpha_3, k \right) + \alpha^2 K(k) \right] \times \frac{1}{\alpha_3^2}
\]  

(7.1.2.17)

and $\alpha^2$ and $\alpha_3^2$ are defined by

\[
\alpha^2 = \frac{c^2-b^2}{c^2-a^2}
\]  

(7.1.2.19)

\[
\alpha_3^2 = \frac{(p-C)(A-B)}{(p-B)(A-C)} = \frac{(x^2-a^2)(c^2-b^2)}{(x^2-b^2)(c^2-a^2)}
\]  

(7.1.2.20)

Substituting (7.1.2.19) and (7.1.2.20) in (7.1.2.18) leads to the following value for the integral in (7.1.2.16):

\[
\frac{1}{b(c^2-a^2)^{1/2}} \left[ (b^2-a^2) \Pi (\alpha_3, k) + (x^2-b^2) K(k) \right]
\]  

(7.1.2.21)

with (7.1.2.21) the integral $I_{R223}$ becomes

\[
I_{R223} = -\frac{q_{y_p}}{b(c^2-a^2)^{1/2}} \left[ (b^2-a^2) \Pi \left( \frac{\pi}{2}, \alpha_3^2, k \right) + (x^2-b^2) K(k) \right]
\]  

(7.1.2.22)

where $\alpha_x^2 = \alpha_3^2$ given by (7.1.2.20).

Having known $I_{R21}$, $I_{R222}$ and $I_{R223}$ the value of the integral $I_{R22}$ from (7.1.2.8) is given by
\[ I_{R22} = - \sigma_{yp} \frac{b^2 - a^2}{b(c^2 - a^2)^{1/2}} \left[ \pi(k^2, \kappa) - \pi(a_x^2, \kappa) \right] \]  

(7.1.2.23)

and the integral \( I_{R2} \) (7.1.2.3) is the sum of the two integrals \( I_{R21} \) (7.1.2.7) and \( I_{R22} \) (7.1.2.23) with a value

\[ I_{R2} = \frac{\pi}{2} \sigma_\infty - \frac{\pi}{2} \left( \frac{b^2 - x^2}{c^2 - x^2} \right)^{1/2} \sigma_\infty - \sigma_{yp} \frac{b^2 - a^2}{b(c^2 - a^2)^{1/2}} \pi(k^2, \kappa) \]

\[ + \sigma_{yp} \cdot \frac{b^2 - a^2}{b(c^2 - a^2)^{1/2}} \pi(a_x^2, \kappa) \]  

(7.1.2.24)

Substituting \( I_{R2} \) (7.1.2.24) in (7.1.2.1) gives the stress distribution in the elastic zone for the mixed boundary problem (problem - A). A superposition of \( \sigma_\infty \) from the infinite plate (problem - B) gives the following distribution of stresses in the elastic zone:

\[ \sigma_{yy}(x, 0) = \left\{ \left( \frac{x(c^2 - x^2)}{(x^2 - a^2)(-x^2 + b^2)} \right)^{1/2} \sigma_\infty + \frac{2}{\pi} \sigma_{yp} \frac{b^2 - c^2}{b(c^2 - a^2)^{1/2}} \pi(a_x^2, \kappa) \right\} \left\{ \frac{\pi}{2} \sigma_x^2, \kappa \right\} - \frac{\pi}{2} \frac{c^2}{c^2 - x^2} \]  

(7.1.2.25)

7.1.3. Stresses over the Remaining Ligament

The stresses in the remaining ligament, Equation (6.2.14) may be expressed as follows:

\[ \sigma_{yy}(x, 0) = \sigma_\infty \left( \frac{x}{(x^2 - a^2)^{1/2}} - 1 \right) + \frac{\pi}{2} \beta_2(x) c_1 \]

\[ + \frac{2}{\pi} \beta_2(x)(x^2 - b^2) I_{R4} \]  

(7.1.3.1)
The value of the constant $c_1$ is given by (7.1.1.22) and $I_{R4}$ denotes the integral

$$
\int_b^c \left( \frac{c^2 - y^2}{y^2 - b^2} \right)^{1/2} \frac{y f(y) \, dy}{x^2 - y^2}
$$

(7.1.3.2)

using $f(y)$ from (6.1.19), the integral $I_{R4}$ is expressed as a sum of two integrals, i.e.,

$$
I_{R4} = I_{R41} + I_{R42}
$$

(7.1.3.3)

where

$$
I_{R41} = \sigma \cdot \int_b^c \left( \frac{c^2 - y^2}{y^2 - b^2} \right)^{1/2} \frac{y \, dy}{x^2 - y^2}
$$

(7.1.3.4)

and

$$
I_{R42} = - \sigma \cdot \int_b^c \left( \frac{(c^2 - y^2)(y^2 - a^2)}{y^2 - b^2} \right)^{1/2} \frac{dy}{x^2 - y^2}
$$

(7.1.3.5)

using the identity

$$
\frac{c^2 - y^2}{y^2 - x^2} = \frac{c^2 - x^2}{y^2 - x^2} - 1
$$

(7.1.3.6)

the integral $I_{R41}$ may be written as follows:

$$
I_{R41} = \sigma \cdot \left( \frac{x^2 - c^2}{x^2 - c^2} \right) \int_b^c \frac{y \, dy}{[(c^2 - y^2)(y^2 - b^2)]^{1/2}(x^2 - y^2)}
$$

- $\sigma \int_b^c \frac{y \, dy}{[(c^2 - y^2)(y^2 - b^2)]^{1/2}}$

(7.1.3.7)

The value of the integral in the first term as given by (6.2.8) is
and the value of the integral in the second term is given by integral no. 7 (Appendix - C)

\[
\int_{c}^{\infty} \frac{y \, dy}{b \, [(c^2 - y^2)(y^2 - b^2)]^{1/2}} = \frac{\pi}{2} \left[ (x^2 - a^2)(x^2 - b^2) \right]^{-1/2} \quad \text{for } x > c \quad (7.1.3.8)
\]

A substitution of these values in (7.1.3.7) leads to

\[
I_{R41} = \frac{\pi}{2} \sigma_c \left[ \left( \frac{x^2 - c^2}{x^2 - b^2} \right)^{1/2} \right] - 1 \quad (7.1.3.10)
\]

With the identity (7.1.3.6) the integral \( I_{R42} \) may be expressed as

\[
I_{R42} = \sigma yp \int_{b}^{c} \left[ \frac{y^2 - a^2}{(y^2 - b^2)(c^2 - y^2)} \right]^{1/2} \, dy
\]

\[
- \sigma yp \left( x^2 - c^2 \right) \int_{b}^{c} \left[ \frac{(y^2 - a^2)}{(y^2 - b^2)(c^2 - y^2)} \right]^{1/2} \cdot \frac{dy}{x^2 - y^2} \quad (7.1.3.11)
\]

and using the following identity

\[
\frac{y^2 - a^2}{x^2 - y^2} = \frac{x^2 - a^2}{x^2 - y^2} - 1 \quad (7.1.3.12)
\]

the integral \( I_{R42} \) (7.1.3.11) is re-expressed as a combination of three integrals, i.e.,

\[
I_{R42} = I_{R421} + I_{R422} + I_{R423} \quad (7.1.3.13)
\]

where
The values of the integrals in (7.1.3.14) through (7.1.3.16) are given by (7.1.2.12), (7.1.2.14) and (7.1.2.21) respectively, which are

\[
I_{R421} = 
\int_{b}^{c} \frac{y^2 - a^2}{(y^2 - b^2)(c^2 - y^2)} \, dy
\]

(7.1.3.14)

\[
I_{R422} = \sigma_{yp} \int_{b}^{c} \frac{dy}{[(y^2 - a^2)(y^2 - b^2)(c^2 - y^2)]^{1/2}}
\]

(7.1.3.15)

\[
I_{R423} = - \sigma_{yp} \cdot (x^2 - a^2)(x^2 - c^2)
\]

(7.1.3.16)

\[
\int_{b}^{c} \frac{dy}{[(y^2 - a^2)(y^2 - b^2)(c^2 - y^2)]^{1/2}} = \frac{1}{b(c^2 - a^2)^{1/2}} \pi(k^2, \overline{k})
\]

(7.1.3.17)

\[
\int_{b}^{c} \frac{dy}{[(y^2 - a^2)(y^2 - b^2)(c^2 - y^2)]^{1/2}} = \frac{1}{b(c^2 - a^2)^{1/2}} \overline{k}(k)
\]

(7.1.3.18)

and

\[
\int_{b}^{c} \frac{dy}{[(y^2 - a^2)(y^2 - b^2)(c^2 - y^2)]^{1/2}} = \frac{1}{b(c^2 - a^2)^{1/2}} \left[ (b^2 - a^2) \pi(x^2, \overline{k}) + (x^2 - b^2) \overline{k}(k) \right]
\]

(7.1.3.19)

Substituting (7.1.3.17) in (7.1.3.14)
\[ I_{R421} = \sigma_{yp} \cdot \frac{b^2-a^2}{b(c^2-a^2)^{1/2}} \pi(k^2,\bar{K}) \]  
(7.1.3.20)

and with (7.1.3.18) in (7.1.3.15)

\[ I_{R422} = \sigma_{yp} \cdot (x^2-c^2) \frac{1}{b(c^2-a^2)^{1/2}} K(k) \]  
(7.1.3.21)

A similar substitution of (7.1.3.19) in (7.1.3.16) leads to the value of the integral \( I_{R423} \) given by

\[ I_{R423} = \sigma_{yp} (x^2-a^2)(x^2-c^2) \frac{1}{b(c^2-a^2)^{1/2}} \]
\[ \left[ (b^2-a^2) \pi(\alpha_x,k) + (x^2-b^2) K(k) \right] \]  
(7.1.3.22)

Having known \( I_{R421} \), \( I_{R422} \) and \( I_{R423} \), the value of the integral \( I_{R42} \) from (7.1.3.13) is given by

\[ I_{R42} = \sigma_{yp} \frac{b^2-a^2}{b(c^2-a^2)^{1/2}} \left[ \frac{c^2-x^2}{b^2-x^2} \pi(\alpha_x,k) - \pi(k^2,\bar{K}) \right] \]  
(7.1.3.23)

Finally, the integral \( I_{R4} \) is obtained by algebraically summing \( I_{R41} \) and \( I_{R42} \). Thus (7.1.3.10) and (7.1.3.23) when substituted in (7.1.3.3) gives

\[ I_{R4} = \frac{\pi}{2} \sigma_{\infty} \left[ \frac{x^2-c^2}{x^2-b^2} \right]^{1/2} - 1 + \sigma_{yp} \cdot \frac{b^2-a^2}{b(c^2-a^2)^{1/2}} \]
\[ \left[ \frac{c^2-x^2}{b^2-x^2} \pi(\alpha_x,k) - \pi(k^2,\bar{K}) \right] \]  
(7.1.3.24)

with \( I_{R4} \) given above, the expression (7.1.3.1) gives the stress distribution in the remaining ligament zone for the mixed boundary value
problem (problem - A). Adding the stresses $\sigma_\infty$ from the uncracked infinite plate (problem B) to the stresses above, the following reduced form for the stress distributions over the remaining ligament at $y=0$ is obtained:

$$
\sigma_{yy}(x,0) = \left[ \frac{x^2(x^2-b^2)}{(x^2-a^2)(x^2-c^2)} \right]^{1/2} \left[ \sigma_\infty + \frac{2}{\pi} \sigma_{yp} \cdot \frac{b^2-a^2}{b(c^2-a^2)^{1/2}} \right]
$$

$$
\left\{ \frac{x^2-c^2}{x^2-b^2} \pi \left( \frac{\pi}{2}, \frac{a^2}{x}, k \right) - \pi \left( \frac{\pi}{2}, k^2, k \right) \right\} + \frac{c_1}{x^2-b^2} \right) \right] (7.1.3.25)
$$

where $\pi$ is a complete elliptic integral of the third kind and $c_1$ is given by (7.1.1.22).

7.2. Evaluation of Displacements - General Solution

The mathematical expressions for the transverse displacements in the crack zone and the plastic zone are given by Equations (6.3.5) and (6.3.6) respectively. These equations involve functions $g(t)$ and $\phi(\tau)$. It was found convenient to evaluate the function $\phi(\tau)$ prior to a determination of the displacements in the plastic regions.

7.2.1. Determination of the Function $\phi$

The function $\phi(\tau)$ given by (6.1.21) may be re-expressed

$$
\phi(r) = -\frac{4}{\pi^2} \left[ \frac{r^2(r^2-b^2)}{(r^2-a^2)(c^2-r^2)} \right]^{1/2} I_\phi(\tau)
$$

$$
+ c_1 \cdot \frac{r \cdot [(r^2-a^2)(r^2-b^2)(c^2-r^2)]^{-1/2}}{x^2-b^2} \right) \right] (7.2.1.1)
$$

where
Substituting for \( f(y) \) from (6.1.19) the above integral is expressed as a sum of two integrals, i.e.,

\[
I_\phi = I_{\phi 1} + I_{\phi 2}
\]

where

\[
I_{\phi 1} = \sigma_y \int_b^c \left( \frac{c^2-y^2}{y^2-b^2} \right)^{1/2} \cdot \frac{y \, dy}{y^2-t^2} \quad (7.2.1.4)
\]

and

\[
I_{\phi 2} = - \sigma_y p \cdot \int_b^c \left[ \frac{(y^2-a^2)(c^2-y^2)}{y^2-b^2} \right]^{1/2} \cdot \frac{dy}{y^2-t^2} \quad (7.2.1.5)
\]

using the identity

\[
\frac{c^2-y^2}{y^2-t^2} = \frac{c^2-t^2}{y^2-t^2} - 1
\]

The integral in the first term is evaluated by (6.2.8). Since the running variable \( \tau \) may assume values between \( b \) and \( c \) like the variable of integration \( y \), the value of this integral is zero. The second term
involves an integral whose value is found in integral no. 7 (Appendix C) which is

\[ \int_{b}^{c} \frac{y \, dy}{[(c^2 - y^2)(y^2 - b^2)]^{1/2}} = \frac{\pi}{2} \]  

(7.2.1.8)

Therefore the value of the integral \( I_{\Phi 1} \) is

\[ I_{\Phi 1} = - \frac{\pi}{2} \sigma_\infty \]  

(7.2.1.9)

The integral \( I_{\Phi 2} \) (7.2.1.5) is broken into three components using the identity

\[ \frac{(c^2 - y^2)(y^2 - a^2)}{y^2 - a^2} = (c^2 - \tau^2) - (y^2 - a^2) + \frac{(c^2 - \tau^2)(\tau^2 - a^2)}{y^2 - \tau^2} \]  

(7.2.1.10)

and the three components are given by

\[ I_{\Phi 21} = - \sigma_{yp} (c^2 - \tau^2) \int_{b}^{c} \frac{dy}{[(y^2 - a^2)(y^2 - b^2)(c^2 - y^2)]^{1/2}} \]  

(7.2.1.11)

\[ I_{\Phi 22} = \sigma_{yp} \int_{b}^{c} (y^2 - a^2)^{1/2} \cdot dy \left[ (y^2 - b^2)(c^2 - y^2) \right]^{1/2} \]  

(7.2.1.12)

\[ I_{\Phi 23} = - \sigma_{yp} \cdot (c^2 - \tau^2)(\tau^2 - a^2) \int_{b}^{c} \frac{dy}{[(y^2 - a^2)(y^2 - b^2)(c^2 - y^2)]^{1/2}.(y^2 - \tau^2)} \]  

(7.2.1.13)

The integral in (7.2.1.11) is given by Equation (7.1.3.18). With its substitution, the integral \( I_{\Phi 21} \) becomes
\[ I_{\phi 21} = - \sigma y_p \frac{c^2 - \tau^2}{b(c^2 - a^2)^{1/2}} K(k) \quad (7.2.1.14) \]

The value of the integral in \( I_{\phi 22} \) is obtained from Equation (7.1.2.12), i.e.,

\[ \int_c^b \frac{y^2 - a^2}{(y^2 - b^2)(c^2 - y^2)} \, dy = \frac{b^2 - a^2}{b(c^2 - a^2)^{1/2}} \Pi \left( \frac{\pi}{2}, \alpha^2, k^2 \right) \quad (7.2.1.15) \]

and therefore

\[ I_{\phi 22} = \frac{(b^2 - a^2)}{b(c^2 - a^2)^{1/2}} \sigma y_p \Pi \left( \frac{\pi}{2}, k^2, k \right) \quad (7.2.1.16) \]

since the parameter \( \alpha^2 = k^2 \) in this case.

To evaluate the integral \( I_{\phi 23} \), it is assumed that \( y > \tau \). With such an assumption and a following change in variable and the substitutions:

- \( y^2 = t \)
- \( r^2 = p \)
- \( a^2 = C \)
- \( c^2 = A \)
- \( b^2 = B \)
- \( 0 = D \)

(7.2.1.13) becomes
\[ I_{\phi 23} = -\frac{1}{2} \sigma_{yp} (A-p)(p-C) \]

\[ \int_{A}^{B} \frac{dt}{(t-p)[(A-t)(t-B)(t-C)(t-D)]^{1/2}} \quad (7.2.1.18) \]

A value of the integral term in (7.2.1.18) is given by integral no. 26 (Appendix - C). From which

\[ \int_{A}^{B} \frac{dt}{(t-p)[(A-t)(t-B)(t-C)(t-D)]^{1/2}} = \frac{q}{B-p} \cdot \left[ 1 - \frac{\alpha^2}{\alpha^2_3} \pi(a_3^2 K) + \frac{\alpha^2}{\alpha^2_3} K(K) \right] \quad (7.2.1.19) \]

where

\[ \alpha^2 = k^2 \]

\[ \alpha^2_3 = \frac{c^2-b^2}{c^2-a^2} \cdot \frac{c^2-a^2}{c^2-b^2} = k^2 \frac{c^2-a^2}{c^2-b^2} \quad (7.2.1.20) \]

carrying out the above algebraic additions and with (7.2.1.17), the value of the integral

\[ I_{\phi 23} = \sigma_{yp} \cdot \frac{(c^2-a^2)}{b(c^2-a^2)^{1/2}} \left\{ \frac{b^2-a^2}{c^2-b^2} \pi \left( \frac{\pi}{2}, \frac{c^2-a^2}{c^2-b^2} \right) \right\} \quad (7.2.1.21) \]

Having known the value of each of the three components of the integral

\[ I_{\phi 2} \]
its value by simple addition becomes

\[ I_{\phi 2} = \frac{\sigma_{yp}}{b(c^2-a^2)^{1/2}} \cdot (c^2-a^2) \left\{ \frac{b^2-a^2}{c^2-b^2} \pi \left( \frac{\pi}{2}, \frac{a^2}{a^2_3} \right) + \frac{b^2-a^2}{c^2-b^2} \right\} \quad (7.2.1.22) \]
and a combination of (7.2.1.22) with the value of the integral \(I_1\) (7.2.1.4) in accordance with Equation (7.2.1.3) when substituted in (7.2.1.1) leads to the following expression for the function \(\phi\):

\[
\phi(\tau) = -\frac{4}{\pi} \left[ \frac{\tau^2 (c^2 - \tau^2)}{\left(\tau^2 - a^2\right)\left(\tau^2 - b^2\right)} \right]^{1/2} \sigma_{yp} \frac{b^2 - a^2}{b(c^2 - a^2)^{1/2}}
\]

\[
\frac{c_1 \tau}{\left[\left(\tau^2 - a^2\right)\left(\tau^2 - b^2\right)(c^2 - \tau^2)\right]^{1/2}}
\]

(7.2.1.23)

It may be noted that the arbitrary function \(\phi(\tau)\) may also be evaluated, with a similar convenience, using its alternate form (7.1.22). Since such an evaluation provides a check on the relative accuracies of the arbitrary constants \(c_1\) and \(c_2\), the later form of \(\phi\) is also calculated. However, the details of necessary integrations are withheld. In its alternate form

\[
\phi(\tau) = -\frac{4}{\pi^2} \left[ \frac{\tau^2 (c^2 - \tau^2)}{\left(\tau^2 - a^2\right)\left(\tau^2 - b^2\right)} \right]^{1/2} \times \left\{ \frac{\pi}{2} \sigma_{\infty} - \sigma_{yp} \frac{b^2 - a^2}{b(c^2 - a^2)^{1/2}} \right\}
\]

\[
\frac{c_2 \tau}{\left[\left(\tau^2 - a^2\right)\left(\tau^2 - b^2\right)(c^2 - \tau^2)\right]^{1/2}}
\]

(7.2.1.24)

where \(c_2\) is given by (7.1.1.45) and the parameter \(\alpha_\tau\) is the same as in (7.2.1.23).
7.2.2. Displacements in the Plastically Deformed Regions

The displacements in the plastic zone are given by the integration of function \( \phi(\tau) \) with respect to \( \tau \) (6.3.6). With \( \phi(\tau) \) determined in the last section (7.2.1.23), these displacements may be expressed in the following form:

\[
\begin{align*}
  u_y(x,0) &= \frac{4}{\pi^2} \left\{ \frac{\pi}{2} \sigma_{\infty} - \frac{b^2-a^2}{b(c^2-a^2)^{1/2}} \sigma_{yp} \cdot \Pi \left( \frac{\pi}{2}, k^2, k \right) \right\} I_{R31} \\
  &= \frac{4}{\pi^2} \frac{b^2-a^2}{b(c^2-a^2)^{1/2}} \sigma_{yp} I_{R32} \\
  &+ C_1 I_{R33} \quad \text{for } b < |x| < c \tag{7.2.2.1}
\end{align*}
\]

where

\[
I_{R31} = \int_x^c \left[ \frac{x^2(\tau^2-b^2)}{(\tau^2-a^2)(c^2-\tau^2)} \right]^{1/2} d\tau \tag{7.2.2.2}
\]

\[
I_{R32} = -\int_x^c \left[ \frac{\tau^2(c^2-\tau^2)}{(\tau^2-a^2)(\tau^2-b^2)} \right]^{1/2} \Pi \left( \frac{\pi}{2}, \alpha^2, k \right) d\tau \tag{7.2.2.3}
\]

\[
I_{R33} = \int_x^c \frac{d\tau}{\left[ (\tau^2-a^2)(\tau^2-b^2)(c^2-\tau^2) \right]^{1/2}} \tag{7.2.2.4}
\]

The integral \( I_{R31} \) is evaluated by the following substitutions:

\[
\begin{align*}
u &= \tau^2 \\
A &= c^2 \\
B &= b^2
\end{align*}
\]
\( C = a^2 \)
\( W = x^2 \) \hfill (7.2.2.5)

which takes the form

\[
I_{R31} = \frac{1}{2} \int_{A}^{A} \frac{du}{W [A-u](u-B)(u-C)]^{1/2}} \quad \quad (7.2.2.6)
\]

A value of the above integral for \( A > W > B > C \) is obtained from integral no. 236.03 [52] given by

\[
I_{R31} = \frac{1}{2} (A-B) g \int_{0}^{u} \frac{cn^2 u du}{\left( c^2 - a^2 \right)^{1/2}} \quad \quad (7.2.2.7)
\]

which with (7.2.2.5) and

\[
g = \frac{2}{\left( c^2 - a^2 \right)^{1/2}}
\]

and 312.02 [52] becomes:

\[
I_{R31} = \frac{c^2 - b^2}{\left( c^2 - a^2 \right)^{1/2}} \left[ \frac{E(u) - k^2 F(u)}{k^2} \right] \quad \quad (7.2.2.8)
\]

where

\[
k' = (1-k^2)^{1/2} \quad \quad (7.2.2.9)
\]

\[
E(u) = E(\phi_1(x),k) \quad \quad (7.2.2.10)
\]

\[
F(u) = F(\phi_1(x),k) \quad \quad (7.2.2.11)
\]

\[
\phi_1(x) = \sin^{-1} \left( \frac{\sqrt{c^2-x^2}}{c+b} \right)^{1/2} \quad \quad (7.2.2.12)
\]
A substitution of (7.2.2.9) through (7.2.2.12) in (7.2.2.8) leads to

\[ I_{R31} = (c^2-a^2)^{1/2} \{ E(\phi_1(x),k) \} - \frac{b^2-a^2}{(c^2-a^2)^{1/2}} \{ F(\phi_1(x),k) \} \]  

(7.2.2.13)

In (7.2.2.13), \( F(\phi_1(x),k) \) and \( E(\phi_1(x),k) \) are the incomplete elliptic integrals of the first and second kind respectively. The arguments of these integrals are functions of \( x \).

The integral \( I_{R32} \) (7.2.2.3) involves a complete elliptic integral of the third kind but whose parameter depends upon the variable \( x \) of integration. Since the value of the parameter \( \alpha^2_\tau \) (7.2.1.23) varies from 1 to \( \infty \), the relation no. 45 (Appendix - C) may be used to express it in terms of Jacobian zeta function as follows:

\[ \pi \left( \frac{\pi}{2} , \alpha^2_\tau , k \right) = \frac{-\alpha_\tau}{\left( \alpha^2_\tau - 1 \right) \left( \alpha^2_\tau - k^2 \right)^{1/2}} KZ(A, \overline{k}) \]  

(7.2.2.14)

where

\[ \alpha_\tau = \left[ \frac{c^2-b^2}{c^2-a^2} \right]^{1/2} \cdot \frac{\tau^2-a^2}{\tau^2-b^2} \]

\[ A = \sin^{-1} \left[ \alpha_\tau \right]^{-1} \]

and

\[ \overline{k} = \left[ \frac{a^2}{b^2} \cdot \frac{c^2-b^2}{c^2-a^2} \right]^{1/2} \]

Substituting \( \alpha_\tau \) and \( \overline{k} \) in (7.2.2.14) leads to
\[
\Pi \left( \frac{\pi}{2}, \alpha^2, k \right) = \frac{b(c^2-a^2)^{1/2}}{b^2-a^2} \left[ \frac{(\tau^2-a^2)(\tau^2-b^2)^{1/2}}{\tau^2(c^2-\tau^2)} \right] \]

with (7.2.2.15) and the following relation for Jacobian zeta function (Appendix - C, 46)

\[
\text{KZ}(A, \kappa) = K(\kappa)E(A, \kappa) - E(\kappa)F(A, \kappa) \tag{7.2.2.16}
\]

the integral \( I_{R32} \) is simplified to

\[
I_{R32} = \frac{b(c^2-a^2)^{1/2}}{b^2-a^2} \left\{ K(\kappa) \int_{x}^{c} E(\phi_\tau, \kappa) \, d\tau - E(\kappa) \int_{x}^{c} F(\phi_\tau, \kappa) \, d\tau \right\} \tag{7.2.2.17}
\]

where \( \phi_\tau \) is given by

\[
\phi_\tau = A(\tau) = \sin^{-1} \left( \frac{\alpha_\tau}{\alpha^2} \right)
= \sin^{-1} \left( \frac{c^2-b^2}{c^2-a^2} \frac{\tau^2-a^2}{\tau^2-b^2} \right)^{-1/2} \tag{7.2.2.18}
\]

From integral no. 38 (Appendix - C)

\[
\int_{x}^{c} E[\phi(u), \kappa] \, du
\]

\[
= c \, E(\kappa) - x \, E[\phi_2(x), \kappa]
- \frac{a^2}{b} \, E[\phi_1(x), \kappa] - \frac{b^2-a^2}{b} \, F[\phi_1(x), \kappa]
+ \frac{a^2}{b} \left[ \frac{(c^2-x^2)(x^2-b^2)}{(c^2-a^2)(x^2-a^2)} \right]^{1/2} \tag{7.2.2.19}
\]
where

$$\phi_1(x) = \sin^{-1}\left( \frac{c^2 - x^2}{c^2 - b^2} \right)^{1/2} \quad (7.2.2.20)$$

and

$$\phi_2(x) = \sin^{-1}\left[ \frac{c^2 - a^2}{c^2 - b^2} \cdot \frac{x^2 - b^2}{x^2 - a^2} \right]^{1/2} \quad (7.2.2.21)$$

and the value of the integral in the second term of (7.2.2.17) is obtained from integral no. 39 (Appendix - C) which is

$$\int_x^C F[\phi(u), \kappa] \, du$$

$$= c \, K(\kappa) - b \, F[\phi_2(x), \kappa] - x \, F[\phi_1(x), \kappa] \quad (7.2.2.22)$$

where \( \phi_1 \) and \( \phi_2 \) are defined by (7.2.2.20) and (7.2.2.21) respectively.

A substitution of (7.2.2.19) and (7.2.2.22) in (7.2.2.17) gives the value of the integral \( I_{R32} \) as below:

$$I_{R32} = \frac{b(c^2 - a^2)}{b^2 - a^2} \left[ x \, E(\kappa) \, F[\phi_2(x), \kappa] \right.$$  

$$- x \, K(\kappa) \, E[\phi_2(x), \kappa]$$

$$+ b \, E(\kappa) \, F[\phi_1(x), \kappa]$$

$$- \frac{b^2 - a^2}{b} \, K(\kappa) \, F[\phi_1(x), \kappa]$$
A substitution \( r^2 = t \) in integral \( I_{R33} \) leads to an integral of the form given by integral no. 236.00 [52] from which

\[
I_{R33} = \frac{1}{(c^2-a^2)^{1/2}} \frac{1}{F[r_1(x),k]}
\]

(7.2.2.24)

Having known \( l_{R31} \) (7.2.2.13), \( I_{R32} \) (7.2.2.23) and \( I_{R33} \) (7.2.2.24), the displacements in the plastic zone are obtained from (7.2.2.1) and have the following value:

\[
U(x,0) = a_1 F(\phi_1(x),k) + a_2 E(\phi_1(x),k) + a_3 x F(\phi_2(x),k) - a_4 x E(\phi_2(x),k)
\]

\[
+ \frac{a_2^2}{b} \left[ \frac{(c^2-x^2)(x^2-b^2)}{(c^2-a^2)(x^2-a^2)} \right]^{1/2}
\]

(7.2.2.25)

where

\[
a_1 = \frac{c_1}{(c^2-a^2)^{1/2}} - \frac{b^2-a^2}{b} + b \alpha_3
\]

\[
+ \frac{4}{\pi^2} \sigma_{yp} \frac{(b^2-a^2)^2}{b(c^2-a^2)} \pi \left[ \frac{\pi}{2}, k, \theta \right] - \frac{2}{\pi} \sigma_{\infty} \frac{b^2-a^2}{(c^2-a^2)^{1/2}}
\]

(7.2.2.26)
\[ \alpha_2 = \frac{2}{\pi} \sigma_\infty (c^2 - a^2)^{1/2} - \frac{a^2}{b} \alpha_4 - \frac{4}{\pi} \sigma_{yp} \frac{b^2 - a^2}{b} \]

\[ \pi \left( \frac{\pi}{2}, k^2, \bar{k} \right) \]  \hspace{1cm} (7.2.2.27)

\[ \alpha_3 = \frac{4}{\pi} \sigma_{yp} E(\bar{k}) \]  \hspace{1cm} (7.2.2.28)

and

\[ \alpha_4 = \frac{4}{\pi} \sigma_{yp} K(\bar{k}) \]  \hspace{1cm} (7.2.2.29)

7.3. Plastic Zone Size

The stress distributions over the elastic zone (7.1.2.25) and the remaining ligament (7.1.3.25) indicate that theoretically the stress singularities exist at the crack-tip \((x=a)\), at the inner end of the plastic zone \((x=b)\) and also at the outer end \((x=c)\) of this zone.

Such mathematically indicated singularities are not uncommon in analytic solutions of crack problems. In fact, Dugdale[7] encountered such a singularity at the outer end of the plastic zone, in spite of the fact that he assumed his crack to be extended to include the plastically deformed region. The well-known plastic zone size relation in fracture mechanics was derived from a removal of this singularity at the interface of the plastic zone and the remaining ligament. A stress singularity at a point is expected to exist if the slope of the displacement changes abruptly at that point. An evaluation of displacements in the plastic zone and their graphical representation in Figure 23 seems to suggest that no stress singularity should occur at
either end of the plastic zone. Dugdale's argument on physical grounds that for a continuum material, it is not possible for the stresses to reach an infinite value also apply to the inner end of the plastic zone in this model since an elastic zone is present between the crack and the plastic region. Furthermore a distribution of dislocations given in dislocation based models[11,12] imply that the stresses must be bounded there as are the dislocations. Even though this important fact is not stated, the results derived from the anti-plane shear (mode-III) deformation models[11] suggest that an assumption for non-singularity of stresses within the plastic zone is implied.

To remove singularity at \( x = c \) from the stresses in the remaining ligament given by (7.1.3.25), the coefficient of \((x^2-c^2)^{-1/2}\) is set equal to zero which gives the condition

\[
\sigma = \frac{2}{\pi} \sigma_y \frac{b^2-a^2}{b(c^2-a^2)^{1/2}} \left\{ \frac{c^2}{c^2-a^2} \int \left( \alpha_c^2 \right) \frac{1}{2} \right\} \]

\[
+ \frac{\pi}{2} \frac{c_1}{c^2-b^2} = 0 \quad (7.3.1)
\]

It is noted in the above condition that the parameter \( \alpha_c^2 \) (7.2.2.20) of the complete elliptic integral of the third kind becomes unbounded at \( x = c \). To assure finiteness of stresses and deleting the term involving \( \alpha_c \) also, reduces (7.3.1) to

\[
\sigma = \frac{2}{\pi} \frac{b^2-a^2}{b^2(c^2-a^2)^{1/2}} \sigma_y \left( \frac{\pi}{2} , k^2 k \right) + \frac{\pi}{2} \frac{c_1}{c^2-b^2} = 0 \quad . \quad (7.3.2)
\]

Substituting for \( c_1 \) (7.1.1.22), the above conditions, after lengthy algebraic manipulations, becomes
\[ u_o = \frac{2}{\pi} \frac{b^2-a^2}{b} \cdot \sigma_{yp} \ \pi \left( \frac{\pi}{2}, k^2, \kappa \right) \left\{ K(k) - E(k) \right\} \]

\[ - (c^2-a^2)^{1/2} \sigma_\infty \ \left\{ K(k) - E(k) \right\} \]

\[ + \frac{2}{\pi b} \cdot \sigma_{yp} \ \left\{ b^2 K(k)E(\kappa) - a^2 E(k)K(\kappa) - (b^2-a^2) K(k)K(\kappa) \right\} \]

(7.3.3)

A removal of singularity at \( x=b \) from the expression (7.1.2.25) for stresses \( \sigma_{yy}(x,0) \) in the elastic zone by vanishing the coefficient of \( (x^2-b^2)^{-1/2} \) yields the following condition:

\[ \sigma_\infty + \frac{2}{\pi} \sigma_{yp} \ \frac{b^2-a^2}{b(c^2-a^2)^{1/2}} \ \left\{ \pi \left( \frac{\pi}{2}, x^2, k \right) - \pi \left( \frac{\pi}{2}, k^2, \kappa \right) \right\} \]

\[ - \frac{\pi}{2} \frac{c^2}{c^2-b^2} = 0 \]

(7.3.4)

and the parameter \( a_b^2 \) in the complete elliptic integral of the third kind in (7.3.4) becomes infinite. Since this is not permitted, the second term involving \( a_b^2 \) in (7.3.4) is deleted which reduces it to

\[ \sigma_\infty - \frac{2}{\pi} \sigma_{yp} \ \frac{b^2-a^2}{b(c^2-a^2)^{1/2}} \ \pi \left( \frac{\pi}{2}, k^2, \kappa \right) - \frac{\pi}{2} \frac{c^2}{c^2-b^2} = 0 \]

(7.3.5)

where the value of the arbitrary constant \( c_2 \) is given by (7.1.1.46) which after substitution in (7.3.5) and again solving for \( u_o \) with some rearrangement of terms leads to
\[
\bar{u}_0 = \frac{2}{\pi} \sigma_{yp} \cdot \frac{b^2 - a^2}{b} \pi \left( \frac{\pi}{2}, k^2, k \right) \left\{ \frac{b^2 - a^2}{c^2 - a^2} K(k) - E(k) \right\} \\
- (c^2 - a^2)^{1/2} \sigma_\infty \{ K(k) - E(k) \}
\]

\[
+ \frac{c^2 - b^2}{(c^2 - a^2)^{1/2}} \sigma_\infty K(k).
\]

\[
+ \frac{2}{\pi} \sigma_{yp} \frac{b^2 - a^2}{b} \pi \left( \frac{\pi}{2}, k^2, k \right) \left\{ \frac{b^2 - a^2}{c^2 - a^2} K(k) - E(k) \right\}, \tag{7.3.6}
\]

Subtracting (7.3.5) from (7.3.6) gives

\[
\frac{c^2 - b^2}{(c^2 - a^2)^{1/2}} \sigma_\infty K(k) + \frac{2}{\pi} \sigma_{yp} \frac{b^2 - a^2}{b} \left( \frac{b^2 - a^2}{c^2 - a^2} - 1 \right)
\]

\[
\pi \left( \frac{\pi}{2}, k^2, k \right) K(k) = 0, \tag{7.3.7}
\]

which simplifies to a generalized relation for the plastic-zone size as follows:

\[
\sigma_\infty - \frac{2}{\pi} \sigma_{yp} \frac{b^2 - a^2}{b(c^2 - a^2)^{1/2}} \pi \left( \frac{\pi}{2}, k^2, k \right) = 0. \tag{7.3.8}
\]

Normalizing the applied remote stress \( \sigma_\infty \) with respect to the yield strength \( \sigma_{yp} \) and defining the complete elliptic integral of the third kind in terms of Heuman's lambda function \( \Lambda_0 \) from relation 48 (Appendix - C)
the relation for the plastic-zone size is further simplified to the following form:

$$\bar{\sigma} = \Lambda_0(\theta_0, \overline{k})$$ (7.3.10)

where

$$\bar{\sigma} = \frac{\sigma_\infty}{\sigma_{yp}}$$ (7.3.11)

and \(\Lambda_0\) is the Heuman's lambda function with

$$\theta_0 = \sin^{-1}\left(\frac{\overline{k}^2}{\overline{k}^2 + \frac{a^2}{\alpha^2}}\right)^{1/2}$$ (7.3.12)

since

$$a^2 = k^2$$

$$\overline{k}^2 = \frac{\overline{k}^2}{b^2} k^2$$

and

$$\overline{k}^2 = 1 - \overline{k}^2$$

\(\theta_0\) assumes a more familiar form

$$\theta_0 = \cos^{-1}\frac{a}{c}.$$ (7.3.13)
A striking feature of the plastic zone size relation $\bar{\sigma} = \Lambda_0$ is its simplistic form as compared to even Dugdale's expression for a relatively simple problem. This is due to the fact that Heuman's lambda function is tabulated in reference handbooks[52]. Furthermore in the limit the elastic zone vanishes, i.e., as $b$ tends to $a$, $\Lambda_0$ approaches an inverse cosine function,

$$\lim_{b \to a} \Lambda_0(\theta_0, K)$$

(7.3.14)

and (7.3.10) assumes the well known Dugdale expression

$$\bar{\sigma} = \frac{2}{\pi} \cos^{-1}(\frac{a}{c})$$

(7.3.15)

7.4. Stress Distributions

With two given conditions (7.3.3) and (7.3.6), the arbitrary constants $c_1$ and $c_2$ can be further simplified. An elimination of $\bar{\mu}_0$ from the value of $c_1$ given by (7.1.1.22) using condition (7.3.3) and similarly substituting condition (7.3.6) in the expression (7.1.1.46) for $c_2$ and adding the two constants $c_1$ and $c_2$, it can be shown after lengthy algebraic manipulations that

$$c_1 + c_2 = 0$$

(7.4.1)

and from the condition (7.3.2) in terms of $c_1$, it is concluded that since the first two terms in this expression constitute precisely the plastic zone relation (7.3.8),
\[ \frac{\pi}{2} \cdot \frac{c_1}{c^2 - b^2} = 0 \]  

(7.4.2)

In other words, the arbitrary constant \( c_1 \) is equal to zero. Combining (7.4.1) with \( c_1 = 0 \), leads to the fact that \( c_2 \) must be zero also. A vanishing value of the constant \( c_2 \) may also be immediately derived from (7.3.5) since the first two terms

\[ \sigma_\infty - \frac{2}{\pi} \sigma_{yp} \cdot \frac{b^2 - a^2}{b(c^2 - a^2)^{1/2}} \cdot \pi \left( \frac{\pi}{2}, \frac{k^2}{k} \right) \]

are equal to zero by (7.3.8), it follows that \( c_2 \) must be zero also. An incorporation of the plastic zone size relation (7.3.8) in (7.1.2.25) and deleting the arbitrary constant \( c_2 \) from the latter equation leads to the following distribution of stresses in the elastic zone:

\[ \sigma_{yy}(x,0) = \left[ \frac{x^2(c^2 - x^2)}{(x^2 - a^2)(b^2 - x^2)} \right]^{1/2} \cdot \frac{2}{\pi} \sigma_{yp} \cdot \frac{b^2 - a^2}{b(c^2 - a^2)^{1/2}} \cdot \pi \left( \frac{\pi}{2}, \frac{\alpha^2}{k} \right) \quad \text{for } a < |x| < b \]  

(7.4.3)

where

\[ \alpha_x^2 = \frac{c^2 - b^2}{c^2 - a^2} \cdot \frac{x^2 - a^2}{x^2 - b^2} \]

It can be established that the value of \( \alpha_x^2 \) is negative in the elastic zone (\( a < x < b \)). For this case the elliptic integral of the third kind in (7.4.3) is related to Heuman's lambda function [52] as follows:

\[ \pi \left( \frac{\pi}{2}, \frac{\alpha^2}{k} \right) = \frac{K}{1 - \alpha} + \frac{\pi}{2} \cdot \frac{\alpha^2(\Lambda_0 - 1)}{[\alpha^2(1 - \alpha^2)(\alpha^2 - k^2)]^{1/2}} \]  

(7.4.4)
Substituting for $\alpha^2$ given by (7.1.2.20) and $k$ given by (6.1.25) in (7.4.4) and carrying out the lengthy algebraic manipulations, it can be shown that the stresses in (7.4.3) reduce to

$$
\sigma_{yy}(x,0) = \frac{2}{\pi} \left( \frac{c^2-a^2}{b} \right)^{1/2} \left[ \frac{x^2 (b^2-x^2)}{(x^2-a^2)(c^2-x^2)} \right]^{1/2} K(k) \sigma_{yp}$$

$$+ \left[ 1 - \Lambda_0(\theta_1, \kappa) \right] \sigma_{yp} \text{ for } a < |x| < b \quad (7.4.5)$$

In (7.4.4) and (7.4.5)

$$\Lambda_0 = \Lambda_0(x)$$

$$= \Lambda_0(\theta_1, \kappa) \quad (7.4.6)$$

and

$$\theta_1 = \sin^{-1} \left[ \frac{x^2 - b^2}{x^2 - c^2} \cdot \frac{c^2 - a^2}{b^2 - a^2} \right]^{1/2} \quad (7.4.7)$$

Similarly incorporating the plastic zone size relation (7.3.8) in (7.1.3.25) and deleting the arbitrary constant $c_1$, the stresses over the remaining ligament zone are distributed in the following manner:

$$\sigma_{yy}(x,0) = \frac{2}{\pi} \sigma_{yp} \cdot \frac{b^2-a^2}{b(c^2-a^2)^{1/2}} \left[ \frac{x^2 (x^2-c^2)}{(x^2-a^2)(x^2-b^2)} \right]^{1/2}$$

$$\Pi \left( \frac{\pi}{2}, \alpha_2^2, \kappa \right) \quad \text{for } |x| > c \quad (7.4.8)$$

defining the complete elliptical integral of the third kind in (7.4.8) in terms of Heuman's lambda function[52]
\[ \Pi \left( \frac{\mu}{2}, \alpha^2, k \right) = \int_0^K \frac{du}{1 - \alpha^2 \sin^2 u} \]
\[ = \frac{\alpha \Pi \Lambda_0 \left( \xi, k \right)}{2 \left( (\alpha^2 + k^2)(1 - \alpha^2) \right)^{1/2}} \quad (7.3.9) \]

and substituting for \( \alpha^2 \) given by (7.1.20), \( k \) given by (6.1.25), in (7.3.9) and carrying out the lengthy algebraic manipulations, it can be shown that the stresses in (7.4.8) reduce to

\[ \sigma_{yy}(x,0) = \sigma_{yp} \cdot \Lambda_0 (\theta_2, \bar{k}) \quad \text{for } |x| > c \quad (7.4.9) \]

where

\[ \theta_2 = \sin^{-1} \left[ \frac{x^2(c^2 - a^2)}{c^2(x^2 - a^2)} \right]^{1/2} \quad (7.4.10) \]

7.5. Crack-Tip Stress Intensity Factor

It is noted from the stress distributions (7.4.5) in the elastic zone that a stress singularity exists at the crack-tip (\( x=a \)) when an elastic zone is present between the crack and the plastically deformed zone. Such a stress singularity is possible since the stresses in the elastic zone are required to be very high in order to keep it free from pile-ups of dislocations and the assumption that the zone between the crack tip and the plastic zone follows elastic stress-strain relations.

The resulting stress intensity factor at the crack-tip is calculated using the following definition:

\[ K_1^{(a)} = \operatorname{Limit}_{x \to a^+} \left[ \frac{(2\pi(x-a))^{1/2}}{2 \sigma_{yy}(x,0)} \right] \quad (7.5.1) \]
where $\sigma_{yy}(x,0)$ is the stress distribution (7.4.5) in the elastic zone 
$(a < |x| < b)$

A substitution of (7.4.5) in (7.5.1) and noting that from Appendix - A

\[
\lim_{x\to a^+} \left[ \theta_1(x) \right] = \frac{\pi}{2} \tag{7.5.2}
\]

and

\[
\lim_{\theta_1 \to \frac{\pi}{2}} \Lambda_0(\theta_1, \bar{k}) = 1 \tag{7.5.3}
\]

yields the following relation for the stress intensity factor:

\[
K_1(a) = \sigma_{yp}(\pi a) \frac{1}{2} \left( \frac{b^2 - a^2}{b^2} \right)^{1/2} \frac{1}{\bar{k}} \tag{7.5.4}
\]

Incorporating the plastic zone size relation (7.3.10), the stress intensity factor may be expressed in terms of the applied remote stresses as follows

\[
K_1(a) = \sigma_{yp}(\pi a) \frac{1}{2} \frac{2}{\pi} \left[ \frac{(b^2 - a^2)^{1/2}}{b} \right] \frac{\Lambda_0(\theta_1, \bar{k})}{K(\bar{k})} + \left( \frac{c^2 - a^2}{b^2 - a^2} \right)^{1/2} \left\{ \bar{\sigma} - \Lambda_0(\theta_1, \bar{k}) \right\} \tag{7.5.5}
\]

7.6. Crack-Tip Opening Displacement

The crack-tip opening displacement $\delta$ is defined as

\[
\delta = 2u_0 \tag{7.6.1}
\]
where \( u_0 \) is the normal transverse displacement in the elastic zone at \( y=0 \) and is given by (5.2.35). With (5.2.35), (7.6.1) becomes

\[
\delta = \frac{4}{E} \overline{u}_0 \quad \text{for plane stress} \quad (7.6.2)
\]

In (7.6.2), a substitution of \( \overline{u}_0 \) from (7.3.3) yields the following value for the crack-tip opening displacement:

\[
\delta = \frac{8}{\pi E} \cdot \frac{b^2 - a^2}{b} \sigma_{yp} \Pi \left( \frac{\pi}{2} , \frac{k^2}{b} \right) (K(k) - E(k)) - 4 \frac{c^2 - a^2}{E} \cdot \sigma_{\infty} (K(k) - E(k))
\]

\[
+ \frac{8}{\pi b E} \sigma_{yp} \left( b^2 (K(k)E(k) - a^2 E(k)K(k)) - \frac{b^2 - a^2}{K(k)k(k)} \right)
\]

(7.6.3)

The first two terms in (7.6.3) may be expressed by re-arranging terms as follows:

\[
\frac{4}{E} \cdot \left( c^2 - a^2 \right)^{1/2} (K(k) - E(k))
\]

\[
\times \left\{ \sigma_{\infty} - \frac{2}{\pi} \sigma_{yp} \cdot \frac{b^2 - a^2}{b (c^2 - a^2)^{1/2}} \Pi \left( \frac{\pi}{2} , \frac{k^2}{b} \right) \right\}
\]

(7.6.4)

The multiplier in (7.6.4) is the generalized plastic zone size relation given by (7.3.8). Therefore the value of (7.6.4) is equal to zero. Deleting the first two terms in (7.6.3), the crack-tip opening displacement becomes
\[\delta = \frac{8}{\pi E} \cdot \frac{\sigma_{\text{yp}}}{b} \left\{ b^2 K(k)E(k) - a^2 E(k)K(k) - (b^2-a^2) K(k)k(k) \right\} \]  

(7.6.5)

In a more convenient form, (7.6.5) may be written as

\[\delta = \frac{8}{\pi} \frac{\sigma_{\text{yp}}}{E} b f_1 \]  

(7.6.6)

where \(f_1\) is a dimensionless function and the form (7.6.6) is the same as given by Goodier\[18\] and Rice\[19\] for the crack-tip opening displacement in Dugdale plasticity. The dimensionless function \(f_1\) is defined by

\[f_1 = K(k) \{E(k) - K(k)\} - \frac{a^2}{b^2} K(k) \{E(k) - K(k)\} \]  

(7.6.7)

7.7. Displacement Profile for the Plastic Zone

An expression for the displacements in the plastic zone is given by (7.2.2.25). To simplify these results, the conditions (7.3.3) and (7.3.6) derived from removal of stress singularities at \(x-b\) and \(x=c\) are incorporated in (7.2.2.25). With arbitrary constant \(c_1\) equal to zero, (7.2.2.26) reduces to

\[\alpha_1 = b \alpha_3 - \frac{b^2-a^2}{b} \alpha_4 \]

\[- \frac{2}{\pi} \frac{b^2-a^2}{(c^2-a^2)^{1/2}} \left[ \sigma_{\text{ex}} - \frac{2}{\pi} \frac{b^2-a^2}{b(c^2-a^2)^{1/2}} \Pi \left( \frac{x}{2}, k^2 \right) \right] \]  

(7.7.1)

The quantity inside the bracket in (7.7.1) vanishes on account of (7.3.8) and \(\alpha_1\) is further simplified to
\[ a_1 = b a_3 - \frac{b^2-a^2}{b} a_4 \]  
(7.7.2)

Similarly
\[ a_2 = -\frac{a^2}{b} a_4 \]  
(7.7.3)

Applying (7.7.2) and (7.7.3) for \( a_1 \) and \( a_2 \) in (7.2.2.25) the displacements in the plastic zone takes the following form:

\[ U_y(x,0) \]

\[ = \frac{4\sigma_0}{\pi E} \left[ b E(\phi_2(x), k) F(\phi_1(x), k) \right. \]
\[ - \frac{b^2-a^2}{b} K(k) F(\phi_1(x), k) \]
\[ - \frac{a^2}{b} K(k) E(\phi_1(x), k) \]
\[ - x K(k) E(\phi_2(x), k) \]
\[ + x E(k) F(\phi_2(x), k) \]
\[ + \frac{a^2}{b} \left\{ \frac{(c^2-x^2)(x^2-a^2)}{(c^2-a^2)(c^2-b^2)} \right\}^{1/2} K(k) \]  
for \( b < |x| < c \)  
(7.7.4)

where \( \phi_1(x) \) and \( \phi_2(x) \) are defined by (7.2.2.20) and (7.2.2.21) respectively and \( F(\phi_1(x), k) \) and \( E(\phi_1(x), k) \) are the incomplete elliptic integrals of the first and second kind, respectively.
CHAPTER VIII

SOME RELATED CRACK PROBLEMS AND SPECIAL CASES

The mathematical model formulated for the present problem turned out to be quite versatile. This is, in essence, due to the simplified assumption that the crack-tip elastic zone is considered to be an elastic region where the linear stress-strain law governs.

This chapter is written with a two-fold purpose. First, several important problems in fracture mechanics are solved with applications of generalized solutions obtained in Sections 7.1 and 7.2 from expressions derived in Chapter VI. Secondly the known solutions obtained using other methods or materials, for some of the problems discussed in the following sections, provide a check on the results for the problem of primary interest in this dissertation. Unfortunately, other studies conducted for the mode-III deformation do not present such a self verification of obtained solutions for all quantities of physical interest.

The problems discussed in this chapter include: a mixed boundary value problem involving three symmetric cracks with specified displacements between the cracks; a triple crack problem in an isotropic elastic medium where the cracks are subjected to internal pressures; and the problem of the Dugdale model of plasticity.
8.1. A Mixed Boundary Value Problem With Specified Displacements Between the Cracks

Consider a central crack of length 2a in an infinite plane elastic and isotropic medium. The origin of the cartesian system is located at the center of this crack with x-axis along the crack length. Two additional symmetrically located cracks occupy the region \( b \leq |x| \leq c \) at \( y=0 \) as shown in Figure 13. The central crack is subjected to a uniform internal pressure \( p_1 \) and a similar pressure \( p_2 \) is applied to the outer cracks. The transverse displacements \( u_y(x,0) \) on \( a \leq |x| \leq b \) are specified and represented by \( u_0 \).

Since this problem is similar to the problem of this dissertation, for mathematical convenience some notations are adopted as for problem \( A'(\text{Figure 12}) \) described in Chapter V. It must be noted, however, that physically the two problems differ substantially. The problem solved in Chapter V assumes a non-strain hardening elasto-plastic material while the problem to be solved in this section is for an elastic material for which linear elastic fracture mechanics apply. The material in the Dugdale plastic zones may undergo transverse displacements only when the stresses reach the yield strength there and the triple cracks, in contrast, will deform in the \( y \) direction as soon as the load is applied. Furthermore, the displacements \( u_0 \) between the cracks in the triple crack problem are specified and known while the displacements in the elastic zone (Figure 10) are unknown and must be determined from the solution of the problem. It is shown by McCartney[17] that the triple crack problem, discussed in this section, with its solution allows one to study the crack closure effects on the crack-tip plastic deformation during fatigue crack growth. Other applications include prediction of
Figure 13. A mixed boundary value problem of three cracks under uniform internal pressures with specified displacements between the cracks in an infinite isotropic medium.
stresses between a crack and a crack-like micro-void and diffusion mechanism studies at high temperature[17].

The formulation and the general solution of the mixed boundary value problem with specified displacements between cracks may be obtained from the general expressions presented in Section 7.1. If the applied remote stress $\sigma_\infty$ is replaced by pressure $p_1$ and the yield strength $\sigma_{yp}$ by $p_1-p_2$ in the results given in Section 7.1, the solution of the above problem is recovered.

With such a substitution the stresses over the region $a \leq |x| \leq b$ given by

$$
\sigma_{yy}(x,0) = \left[ \frac{x^2(c^2-x^2)}{(x^2-a^2)(b^2-x^2)} \right]^{1/2} \left[ p_1 + \frac{2}{\pi} (p_1-p_2) \right] \frac{b^2-a^2}{b(c^2-a^2)^{1/2}} \cdot \left\{ \Pi \left( \frac{\pi}{2}, \frac{(x^2-a^2)(c^2-b^2)}{(x^2-b^2)(c^2-a^2)}, k \right) - \frac{\pi}{2} \frac{c_2}{c-x^2} \right\} - p_1
$$

similarly the stresses over the remaining ligament $|x| \geq c$ becomes

$$
\sigma_{yy}(x,0) = \left[ \frac{x^2(x^2-b^2)}{(x^2-a^2)(x^2-c^2)} \right]^{1/2} \left[ p_1 + \frac{2}{\pi} (p_1-p_2) \right] \frac{b^2-a^2}{b(c^2-a^2)^{1/2}} \left\{ \frac{x^2-c^2}{x^2-b^2} \Pi \left( \frac{\pi}{2}, \frac{(x^2-a^2)(c^2-b^2)}{(x^2-b^2)(c^2-a^2)}, k \right) \right\} + \frac{\pi}{2} \cdot \frac{c_1}{x^2-b^2} - p_1
$$
In the above expressions, \( c_1 \) and \( c_2 \) are given by (7.1.1.22) and (7.1.1.46) respectively and the elliptic integral notations of Chapter V apply.

An important parameter in a linear elastic fracture mechanics analysis is the stress intensity factor. The stress distributions (8.1.1) and (8.1.2) indicate there are stress singularities at all the crack-tips in this problem.

The stress intensity factor at \( x=a \) is defined as

\[
K_I^{(a)} = \lim_{x \to a^+} \left[ (2\pi(x-a))^{1/2} \sigma_{yy}(x,0) \right]
\]  

(8.1.3)

A substitution of (8.1.1) in (8.1.3) with \( c_2 \) obtained from (7.1.1.46) leads to the following expression for the stress intensity factor:

\[
K_I^{(a)} = 2(p_1-p_2) \left( \frac{a}{\pi} \right)^{1/2} \left[ \frac{b}{(b^2-a^2)^{1/2}} \cdot \frac{E(k)}{K(k)} \cdot \frac{a^2b}{b^2(a^2-2)^{1/2}} \right] + p_1(2\pi a)^{1/2} \left[ \frac{c^2-a^2}{2(b^2-a^2)} \right]^{1/2} \cdot \frac{E(k)}{K(k)} \\
- \bar{u}_0 \left( \frac{2\pi}{a} \right)^{1/2} \left[ \frac{a^2}{2(b^2-a^2)} \right]^{1/2} \cdot \frac{1}{K(k)}
\]  

(8.1.4)

defining \( K_I \) at \( x=b \) as
\[ K_I^{(b)} = \lim_{x \to b^+} \left[ 2\pi (b-x) \right]^{1/2} \left[ \sigma_{yy}(x,0) \right] \]  

(8.1.5)

and substituting (8.1.1) in the above expression, the stress intensity factor

\[
K_I^{(b)} = p_1 (2\pi b)^{1/2} \left\{ \frac{b^2 - a^2}{2(c^2 - b^2)} \right\}^{1/2} \left\{ \frac{c^2 - a^2}{b^2 - a^2} \cdot \frac{E(k)}{K(k)} - 1 \right\}
\]

\[- \left( \frac{u_0}{K(k)} \right) \left( \frac{\pi}{b} \right)^{1/2} \left\{ \frac{b^2 (c^2 - a^2)}{(b^2 - a^2)(c^2 - b^2)} \right\} \]

\[ + 2(p_1 - p_2) \left( \frac{b}{\pi} \right)^{1/2} \left( \frac{c^2 - a^2}{c^2 - b^2} \right)^{1/2} \]

\[ \left[ \frac{b^2 - a^2}{b^2} \right]^{1/2} \left\{ \frac{b^2 - a^2}{c^2 - a^2} - \frac{E(k)}{K(k)} \right\} \]

\[ \pi \left( \frac{\pi}{2}, \frac{k^2}{k} \right) + \frac{b}{(b^2 - a^2)^{1/2}} \frac{E(k)}{K(k)} \]

\[ + \frac{a^2}{b(b^2 - a^2)^{1/2}} \left\{ \frac{b^2 - a^2}{a^2} + \frac{E(k)}{K(k)} \right\} \]  

(8.1.6)

Similarly the stress intensity factor at \( x-c \) is given by

\[ K_I^{(c)} = \lim_{x \to c^+} \left[ 2\pi (x-c) \right]^{1/2} \left[ \sigma_{yy}(x,0) \right] \]

\[ = \overline{u}_0 \left( \frac{\pi}{c} \right)^{1/2} \left( \frac{c^2 - a^2}{c^2 - b^2} \right)^{1/2} \cdot \frac{1}{K(k)} \]
+ p_1 (2\pi c)^{1/2} \left\{ \frac{c^2-a^2}{2(c^2-b^2)} \right\}^{1/2} \cdot \frac{(K(k)-E(k))}{K(k)}

+ (p_1 - p_2) \frac{2}{\pi} \cdot (\pi c)^{1/2} \left[ \left( \frac{b^2}{c^2-b^2} \right)^{1/2} \right] \cdot \left\{ (K(k)-E(k)) \right\}

- \left\{ \frac{1}{b^2(c^2-b^2)} \right\}^{1/2} \cdot \left\{ \frac{(K(k)-E(k))}{K(k)} \right\}

\left\{ a^2 k(k) + (b^2-a^2) \pi \left( \frac{k}{2}, k^2, k \right) \right\} \quad (8.1.8)

The stresses, \( \sigma_{yy}(x,0) \) in (8.1.7) above represent the stress distributions over the remaining ligament given by (8.1.2).

A special case of this problem, when the acting internal pressures in all the three cracks are the same, is of interest. In such a case

\[ p_1 = p_2 = p_o \quad (8.1.9) \]

and the stress intensity factors are reduced to the following values:

\[ K_{I}^{(a)} = p_o (2\pi a)^{1/2} \left\{ \frac{c^2-a^2}{2(b^2-a^2)} \right\}^{1/2} \cdot \frac{E(k)}{K(k)} \]

\[ - \frac{u_o}{K(k)} \left( \frac{2\pi}{a} \right)^{1/2} \left\{ \frac{a}{2(b^2-a^2)} \right\}^{1/2} \cdot \frac{1}{K(k)} \quad (8.1.10) \]

\[ K_{I}^{(b)} = p_o (2\pi b)^{1/2} \left\{ \frac{b^2-a^2}{2(b^2-a^2)} \right\}^{1/2} \cdot \left\{ \frac{(c^2-a^2)}{b^2-a^2} \right\} \cdot \frac{E(k)}{K(k)} - 1 \]

\[ - \frac{u_o}{K(k)} \left( \frac{2\pi}{b} \right)^{1/2} \left\{ \frac{b^2(c^2-a^2)}{(b^2-a^2)(c^2-b^2)} \right\}^{1/2} \quad (8.1.11) \]
and

\[ K_I^{(c)} = p_0 (2\pi c)^{1/2} \left\{ \frac{c^2 - a^2}{2(c^2 - b^2)} \right\}^{1/2} \left\{ \frac{K(k) - E(k)}{K(k)} \right\} \]

\[ + \frac{\pi}{c} \left( \frac{c^2}{c^2 - b^2} \right)^{1/2} \frac{1}{K(k)} \]  \hspace{1cm} (8.1.12)

This is the first time that the solutions of the stress distributions and the stress intensity factors for this problem are presented. A formulation of this problem is found in [17], however, no results for the above quantities are obtained except for a simpler case which is discussed in the next section. These results for the later case in [17] are obtained using the method of singular integral equations while the solution of the problem described below is obtained as a special case of this work.

It may be further noted that the stress intensity factors given by (8.1.5) through (8.1.7) and (8.1.10) through (8.1.12) are also the valid solutions for non-isotropic bodies since anisotropy does not influence these quantities.

8.2. Three Coplanar Griffith Cracks in an Isotropic Elastic Medium

Another new problem of interest in linear elastic fracture mechanics and whose solution is not available in the literature is that of a triple crack system in an isotropic elastic medium. For simplicity, two cracks are assumed to be located symmetrically to an existing central crack as shown in Figure 13 and an infinite plane medium is
assumed. The acting internal pressure in the central crack is \( p_1 \) and each of the symmetric cracks is subjected to an internal pressure \( p_2 \). The displacements, however, in this case between the cracks are zero for an elastic medium.

A solution of this problem may be obtained from the general solutions given in Section 7.1 or from problem discussed in Section 8.1, with vanishing displacements

\[
\overline{u}_0 = 0 \quad (8.2.1)
\]

and the constants \( c_1 \) and \( c_2 \) are given by

\[
c_1 = c_{12} + c_{13} \quad (8.2.2)
\]

and

\[
c_2 = c_{22} + c_{23} \quad (8.2.3)
\]

where \( c_{12} \) and \( c_{13} \) are given by (7.1.1.23) and \( c_{22} \) and \( c_{23} \) by (7.1.1.46). A further substitution of

\[
\sigma_\infty = p_1 \quad (8.2.4)
\]

and

\[
\sigma_{yp} = p_1 - p_2 \quad (8.2.5)
\]

in the components of constants in (8.2.2) and (8.2.3) gives the arbitrary constants for this problem.

The stress distributions then for the three coplanar Griffith crack problem for \( a \leq |x| \leq b \) and \( |x| \geq c \) are given by (8.1.1) and
(8.1.2) provided that the proper constants as discussed above are used.

The stress intensity factor $K_1$ at $x = a$ defined by (8.1.3) becomes

$$K_1(a) = 2(p_1 - p_2) \left( \frac{a}{\pi} \right)^{1/2} \left[ \frac{b}{b^2 - a^2} E(k) \right]$$

$$- \frac{a^2 b}{b^2 (b^2 - a^2)^{1/2}} \cdot \frac{E(k)}{K(k)} K(k)$$

$$- \frac{(b^2 - a^2)^{1/2}}{b} \frac{E(k)}{K(k)} \pi \left( \frac{\pi}{2}, k^2 \right)$$

$$+ p_1 (2\pi a)^{1/2} \left[ \frac{c^2 - a^2}{2(b^2 - a^2)} \right]^{1/2} \frac{E(k)}{K(k)}$$

(8.2.6)

similarly

$K_1$ at $x = b$ (8.1.5) is given by

$$K_1(b) = 2(p_1 - p_2) \left( \frac{b}{\pi} \right)^{1/2} \left( \frac{c^2 - a^2}{c^2 - b^2} \right)^{1/2}$$

$$\left[ \frac{(b^2 - a^2)^{1/2}}{b^2} \right]^{1/2} \left[ \frac{b^2 - a^2}{c^2 - a^2} E(k) - \frac{E(k)}{K(k)} \right]$$

$$\pi \left( \frac{\pi}{2}, k^2 \right)$$

$$+ \frac{b}{(b^2 - a^2)} E(k) + \frac{a^2}{b(b^2 - a^2)^{1/2}} K(k) \left\{ \frac{b^2 - a^2}{a^2} + \frac{E(k)}{K(k)} \right\}$$

$$+ p_1 (2\pi b)^{1/2} \left[ \frac{b^2 - a^2}{2(c^2 - b^2)} \right]^{1/2} \left\{ \frac{c^2 - a^2}{b^2 - a^2} \frac{E(k)}{K(k)} - 1 \right\}$$

(8.2.7)
The solution for a simplified case of this problem when $p_1 = p_2 = p_0$, is discussed by Dhawan and Dhaliwal[16] for a transversely isotropic medium and McCartney[17] using a singular integral equations approach. In this case from (8.2.2) and (7.1.1.23)

$$c_1 = c_{12}$$

$$= (c^2 - a^2)^{1/2} \left[ \frac{b^2 - a^2}{c^2 - a^2} - \frac{E(k)}{K(k)} \right] p_0$$

(8.2.9)

and from (8.2.3) and (7.1.1.46)

$$c_2 = c_{22}$$

$$= (c^2 - a^2)^{1/2} \left[ 1 - \frac{E(k)}{K(k)} \right] p_0$$

(8.2.10)
and the stress distributions over the region \( a \leq |x| \leq b \) reduce to

\[
\sigma_{yy}(x,0) = p_0 \left[ \left( \frac{x^2(c^2-x^2)}{(x^2-a^2)(b^2-x^2)} \right)^{1/2} - 1 \right] - \frac{\pi}{2} \frac{c_2}{c^2-x^2} \tag{8.2.11}
\]

where \( c_2 \) is given by (8.2.10).

Similarly the stress distributions over the remaining ligaments are given by

\[
\sigma_{yy}(x,0) = p_0 \left[ \left( \frac{(x-b^2)x^2}{(x-a^2)(x-c^2)} \right)^{1/2} - 1 \right] + \frac{\pi}{2} \frac{c_1}{x^2-b^2} \tag{8.2.12}
\]

for \( |x| > c \)

with \( c_1 \) defined by (8.2.9).

The resulting stress intensity factors takes the following form:

\[
k_1^{(a)} = p_0 (2\pi a)^{1/2} \left[ \frac{c^2-a^2}{2(b^2-a^2)} \right]^{1/2} \frac{\varepsilon(k)}{K(k)} \tag{8.2.13}
\]

and

\[
k_1^{(b)} = p_0 (2\pi b)^{1/2} \left[ \frac{b^2-a^2}{2(c^2-b^2)} \right]^{1/2} \left[ \frac{c^2-a^2}{b^2-a^2} \frac{\varepsilon(k)}{K(k)} - 1 \right] \tag{8.2.14}
\]

also
\[ K_I(c) = p_0 (2\pi c)^{1/2} \left[ \frac{c^2 - a^2}{2(c^2 - b^2)} \right]^{1/2} \left[ 1 - \frac{E(k)}{K(k)} \right] \quad (8.2.15) \]

The values of the stress intensity factors given above are in complete agreement with [17] and also agree with [16] for a transversely isotropic medium.

8.2. A Limiting Case - Dugdale Model

Though the physical model of the problem discussed in this dissertation is different from the Dugdale model in fracture mechanics, the mathematical model suggests that in the absence of dislocation free zone at the crack tip, the analytic results should reduce to those obtained for the Dugdale model.

The two quantities that are of most importance in practical problems and which are derived from Dugdale model of plasticity are: the plastic zone size and the crack-tip opening displacement. The plastic zone size finds its applications in linear elastic fracture mechanics where the crack-tip stress intensity factor is modified by adding this quantity to the primary crack length. The crack-tip opening displacement is used as a fracture criterion in ductile fractures of elastic-plastic materials since it represents the plastic strains between the micro-voids at the crack-tip.

The plastic zone size relation from (7.3.10) is

\[ \sigma = \sigma_0 \quad (7.3.10) \]
where \( \Lambda_0(\theta_0, k) \) is the Heuman's lambda function which relates the plastic zone size with crack length and the elastic zone size. As \( b \) tends to \( a \), the modulus

\[
\kappa = \left( \frac{c^2 - b^2}{c^2 - a^2} \cdot \frac{a^2}{b^2} \right)^{1/2}
\]

tends to 1 and

\[
\Lambda_0(\theta_0, \kappa) \rightarrow \Lambda_0(\theta_0, 1) = \frac{2}{\pi} \theta_0 \tag{8.3.1}
\]

where the limiting value of \( \Lambda_0 \) in (8.3.1) is used from (A.22). With (7.3.10) this result reduces to

\[
\frac{\sigma}{\sigma_y} = \frac{2}{\pi} \cos^{-1} \frac{a}{c} \tag{8.3.2}
\]

which is a well known plastic zone size relation from Dugdale model [7] for the case of large scale yielding.

In the case of small scale yielding, expressing \( c \) as a sum of the semi-crack length \( a \) and the plastic zone size \( r_p \), (8.3.2) takes the form

\[
\left( \frac{a + r_p}{a} \right)^{-1} = \cos \left( \frac{\pi}{2} \frac{\sigma_y}{\sigma_y} \right) \tag{8.3.3}
\]

for small plastic zone length \( r_p \) as compared to \( a \) and for small applied remote stresses in comparison with the material yield strength, the above expression may be reduced to the following plastic zone size approximation under small scale yielding:
The crack-tip opening displacement for the Dugdale model of plasticity may be derived from (7.6.6) which is

\[
\delta = \frac{8}{\pi} \frac{\sigma_{yp}}{E} \cdot b f_1
\]  

(7.6.6)

the above formulae as b=a, using (7.6.7) and relation 50 (Appendix - C) reduces to

\[
\delta = \frac{8}{\pi} \cdot \frac{\sigma_{yp}}{E} a \ln \frac{C}{a}
\]  

(8.3.5)

which agrees with the solutions obtained by Goodier[18] and Rice[19] employing Muskhelishvilli and Westergaard methods.

The stress intensity factor at the crack-tip when the dislocation free zone vanishes, becomes zero since no stress singularity is permitted in the Dugdale model of plasticity (7.5.4).
CHAPTER IX
RESULTS AND DISCUSSIONS

Exact solutions for the stress and displacement fields, plastic zone size, the crack-tip stress intensity factor and the crack-tip opening displacements were obtained in Chapter VII. A check on the accuracy of these results was based on a comparison of solutions for limiting cases of the generalized model with available solutions.

With a view to gaining a further insight into the trends and patterns of variations of these quantities along the crack plane and with increasing applied loads, they are exhibited graphically in this chapter. Such illustrations of results also serve as an additional check on their variations and accuracy. More importantly, a phenomenon of stress relaxation in the elastic zone has been observed which was not obviously suggested by the exact form of the results. The concluding remarks in this chapter include a discussion on the influences of assumptions in the physical model on the obtained results and the roles of crack-tip stress intensity factor and the crack-tip opening displacement on the fracture of crack-tip material.

9.1. Results
The boundedness of stresses within the plastic zone yields an important relation for the plastic zone size (7.3.10). This relation
is plotted in Figure 14. The plastic zone size $\gamma_p$ and the elastic zone and plastic zone size parameters $b$ and $c$ are all normalized with respect to the crack half length such that

$$\frac{\gamma_p}{a} = b$$  \hspace{1cm} (9.1.1)

$$\frac{b}{a} = \frac{b}{a}$$  \hspace{1cm} (9.1.2)

and

$$\frac{c}{a} = \frac{c}{a}$$  \hspace{1cm} (9.1.3)

It is noted that the plastic zone size is predominantly a function of the applied remote stress in the higher range of the applied loads. As the applied loads increase, the entire crack plane ahead of the dislocation free zone may be yielded approaching the state of plastic collapse. It is further noted from (7.3.10) that Heuman's lambda function depends upon two parameters, the dislocation free zone size and the plastic zone size. The Heuman function $\Lambda_0$ for some selected values of $\overline{b}$ and $\overline{c}$ is listed in Table 1. With $\overline{b}$ approaching unity which corresponds to absence of DFZ, the function $\Lambda_0$ turns into an inverse cosine function (7.3.14) and the plastic zone size in this case becomes the Dugdale result as shown by the extreme left curve ($\overline{b}$=1). Based upon experimental observations[21] there is always some lateral expansion of the plastic region but the lateral dimension is small for thin, non-hardenable materials and their shapes resemble candle flames. Past studies[25] have shown that the effect of strain-hardening on plastic zone lengths and the crack tip plastic strains is small if their lateral spread does not exceed the material thickness. The plastic zones of the
Figure 14. Dimensionless plastic zone size as a function of normalized applied remote stress for varying elastic zone size.
### TABLE 1

Heuman's Lambda Function $\Lambda_0(\theta_o, k)$

(Equation 7.1.10)

<table>
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<th>1.005</th>
<th>1.02</th>
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<td>.3156</td>
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<td>1.300</td>
<td>.4439</td>
<td>.4516</td>
<td>.4729</td>
</tr>
<tr>
<td>1.400</td>
<td>.4979</td>
<td>.5047</td>
<td>.5236</td>
</tr>
<tr>
<td>1.500</td>
<td>.5377</td>
<td>.5437</td>
<td>.5606</td>
</tr>
<tr>
<td>2.000</td>
<td>.6680</td>
<td>.6722</td>
<td>.6840</td>
</tr>
</tbody>
</table>
type for which the results in Figure 14 and Table 1 are valid, have been observed in plane carbon, structural, stainless and silicon steels [25] and many other materials including polycarbonate[31]. It is instructive to note that all Dugdale measurements of plastic zones fell slightly below the curve in Figure 14 for b=1 (also see Figure 3).

The stresses in the elastic zone (7.4.5) when normalized with respect to the yield stress becomes

\[ \bar{\sigma}_{yy} = \frac{2}{\pi} \left( \frac{c^2 - a^2}{b} \right)^{1/2} \left\{ \frac{x^2(b^2-x^2)}{(x^2-a^2)(c^2-x^2)} \right\}^{1/2} K(k) + 1 - \Lambda_0(\theta, \bar{k}) \] for \( a < |x| < b \) and \( y = 0 \) \hspace{1cm} (9.1.4)

where

\[ \bar{\sigma}_{yy} = \frac{\sigma_{yy}}{\sigma_{yp}} \] \hspace{1cm} (9.1.5)

The plots of stress distributions (9.1.4) with dimensionless distance \( \bar{x} = \frac{x}{a} \) for a constant elastic zone size (b=1.005) is shown in Figure 15. It is observed that the effect of the plastic zone length on these stresses is small. However if the plastic zone size is kept relatively constant (c=1.2) and the length of the elastic zone is allowed to vary, the normal stresses show a marked variation (Figure 16). Thus it is concluded that the stresses in this zone are primarily a function of the length of the elastic zone. It was shown in Figure 14 that the plastic zone size is governed by the applied remote stress. As the dislocation free zone size reduces the stresses are relaxed (Figure 16).
Figure 15. Stress distribution in the elastic zone. Normalized normal stress plotted as a function of dimensionless distance ahead of the crack-tip for varying plastic zone size.
Figure 16. Stress distributions in the elastic zone. Normalized normal stress plotted as a function of dimensionless distance ahead of the crack-tip for varying elastic zone size.
with applied stresses essentially constant. A similar observation of relaxation of stresses was observed for anti-plane shear deformation [11]. It should be noted that in the solution for mode-III deformation, the normal stresses in the elastic zone do not drop to a value of the material yield strength at the trailing edge of the plastic zone[11]. Based on the continuum solution in this dissertation these stresses, as required by the model, assume a value of the yield stress at this point. This difference is perhaps due to a numerical error in their calculations since the physical problem required that the normal stresses over the yielded region must remain constant including its two ends.

A normalization of the stresses over the remaining ligament (7.4.9) with respect to the yield stress gives the following distribution in this region:

$$\bar{\sigma}_{yy} = \Lambda_0(\theta_2, \kappa)$$  \hspace{1cm} (9.1.6)

for $|x| > c$

The stresses (9.1.6) are plotted in Figure 17 for two values of dimensionless elastic zone parameter, $\bar{b}=1.001$ and 1.056 and a constant plastic zone size parameter $\bar{c}=1.2$. These stresses approach a value equal to the yield stress at the leading edge of the plastic zone and initially show a sharp drop and then maintain a constant value equal to the applied remote stress $\sigma_\infty$ beyond distances approximately five times the semi-crack length. The effect of the elastic zone size on these stresses is relatively small and it is expected that the stresses over the remaining ligament would be more sensitive to applied loads.
Figure 17. Normalized normal stress distributions over the remaining ligament as a function of the normalized distance ahead of the elastic zone.

<table>
<thead>
<tr>
<th></th>
<th>( \bar{b} )</th>
<th>( \bar{c} )</th>
<th>( \bar{\sigma} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.001</td>
<td>1.20</td>
<td>.37</td>
</tr>
<tr>
<td>2</td>
<td>1.050</td>
<td>1.20</td>
<td>.43</td>
</tr>
</tbody>
</table>

Eq. 9.1.6
and the plastic zone size than to a variation in the elastic zone size.

It is important to note that our results are based on the assumption that the elastic core zone which follows the constitutive law of an elastic material, allows no stress singularity at $|x|=b$, the point near the plastic zone. Thus $K^{(b)}_I$ is equal to zero. Since $K^{(c)}_I$ must be zero at $|x|=c$, the stress singularity occurs only at $|x|=a$. This point at the crack-tip acts as a source of generation of dislocations. The stress intensity factor at the crack-tip is given by two alternate forms (7.5.4) and (7.5.5). After normalization with respect to $\sigma_p(a)^{1/2}$, the stress intensity factor may be expressed by the following forms

$$K_I^{(a)} = \left(\frac{b^2-a^2}{b}\right)^{1/2} \frac{K(k)}{\ell(a)}$$  (9.1.7)

in terms of dimensionless parameters and

$$\bar{K}_I^{(a)} = \frac{2}{\pi} \left(\frac{b^2-a^2}{b}\right)^{1/2} K(k) + \left(\frac{c^2-a^2}{b^2-a^2}\right)^{1/2} \{\bar{\sigma} - \Lambda_\delta(\theta_1, k)\}$$  (9.1.8)

in terms of the applied remote stress. The plots of dimensionless crack-tip stress intensity factor $\bar{K}_I^{(a)}$ versus $\bar{\sigma}$, the normalized applied remote stress for different values of dimensionless elastic zone parameter $b$, ranging from 1.001 to 1.050 are presented in Figure 18. As expected, the stress intensity factor shows a strong dependence upon the dislocation free zone size.

At small elastic zone lengths, $\bar{K}_I^{(a)}$ has small values and practically remains constant for increasing applied remote stresses. This is perhaps the most striking and important observation in these results.
Figure 18. Normalized stress intensity factor as a function of normalized applied remote stress for varying elastic zone size.
**TABLE 2**

Complete Elliptic Integral of the First Kind $K(\sqrt{\bar{c}})$
as a Function of Elastic Zone Size
and Length of the Plastic Zone
(Equation 7.5.4)

<table>
<thead>
<tr>
<th>$\bar{c}$</th>
<th>1.001</th>
<th>1.005</th>
<th>1.020</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.050</td>
<td>3.3248</td>
<td>2.5478</td>
<td>1.9309</td>
</tr>
<tr>
<td>1.100</td>
<td>3.6441</td>
<td>2.8453</td>
<td>2.2064</td>
</tr>
<tr>
<td>1.200</td>
<td>3.9090</td>
<td>3.1213</td>
<td>2.4663</td>
</tr>
<tr>
<td>1.300</td>
<td>4.0528</td>
<td>3.2580</td>
<td>2.6024</td>
</tr>
<tr>
<td>1.400</td>
<td>4.1528</td>
<td>3.3440</td>
<td>2.6881</td>
</tr>
<tr>
<td>1.500</td>
<td>4.2157</td>
<td>3.4130</td>
<td>2.7478</td>
</tr>
<tr>
<td>2.000</td>
<td>4.3386</td>
<td>3.5600</td>
<td>2.8926</td>
</tr>
</tbody>
</table>
As the crack-tip elastic zone is allowed to shrink to a vanishing limit, the stress intensity factor assumes a constant value of zero regardless of the applied load as expected from results for Dugdale plasticity. The complete elliptic integral of the first kind \( K(k) \) used in Equations (9.1.7) and (9.1.8) also depend upon the two dimensionless parameters \( \widetilde{b} \) and \( \widetilde{c} \). Table 2 lists the values of the elliptical integral \( K(k) \) for various values of \( \widetilde{b} \) and \( \widetilde{c} \). It should be pointed out that with our definitions of the parameters \( b \) and \( c \) (Figure 10), the later includes the effects of the elastic zone in analytic as well as plotted results but since the dislocation free zone is assumed to be much smaller as compared to the plastic zone length, \( c \) is primarily the plastic zone size parameter. The variation of \( \frac{\widetilde{K}^{(a)}}{\widetilde{K}_{I}^{(a)}} \) with parameters \( \widetilde{b} \) and \( \widetilde{c} \) is more visible in Figure 19. The effect of plastic zone size is much smaller in the practical range of \( \widetilde{b} \). This may also be deduced from \( k(k) \) given in Table 2. The value of \( \frac{\widetilde{K}^{(a)}}{\widetilde{K}_{I}^{(a)}} \), the crack-tip stress intensity factor plays an important role at the onset of fracture which is discussed in the next section.

Another important fracture parameter derived from these results is the crack-tip opening displacement \( \delta \). The CTOD (7.6.6) was derived from the condition of boundedness of stresses in the plastic zone. In a non-dimensional form it may be expressed as

\[
\bar{\delta} = b f_{I}/a \quad (9.1.9)
\]

where

\[
\bar{\delta} = \delta \cdot \frac{\pi}{8} \cdot \frac{E}{\sigma_{yp}} \cdot \frac{1}{a} \quad (9.1.10)
\]
Figure 19. Normalized stress intensity factor plotted as a function of dimensionless elastic zone size for various plastic zone lengths.
Figure 20. Normalized crack-tip opening displacement plotted as a function of normalized applied remote stress for various elastic zone sizes.
Figure 21. A plot of crack-tip opening displacement as a function of normalized applied remote stress for varying plastic zone lengths.
and $f_1$ is a dimensionless crack-tip displacement function given by (7.6.7). The variation of $\tilde{\sigma}$ with $\tilde{\sigma}$ for varying values of $\tilde{b}$ is exhibited in Figure 20. These plots show that the crack-tip opening displacement strongly depend upon the applied remote stress. A similar plot in Figure 21 for varying values of parameter $\tilde{c}$ illustrate its dependence on the plastic zone size. Such a dependence may also be noticed from the tabulated values of function $f_1$ in Table 3. The crack-tip opening displacement is also modestly influenced by the elastic zone size as shown in Figure 22. This effect becomes more pronounced as the applied stress increases. The CTOD's dependence on parameters $\tilde{b}$ and $\tilde{c}$ is not surprising. This is perhaps because the crack-tip opening displacement is a more universal parameter for fracture in linear elastic as well as elastic-plastic materials. In a limiting case as the elastic zone vanishes and $b$ tends to $a$, the value of the dimensionless crack-tip displacement function $f_1$ reduces to $\ln \frac{c}{a}$, a result obtained by Goodier[18] and Rice [19] for the Dugdale model of plasticity. This limiting case is also demonstrated graphically (Figure 20) where Dugdale CTOD is shown to correspond to value of the parameter $\tilde{b}=1$.

The general expressions for the displacements in the crack region ($0 < |x| < a$) and the plastic zone ($b < |x| < c$) are given by Equations (6.3.5) and (6.3.6) respectively. It is noted that at $x=a$, the displacements (6.3.5) reduce to $\tilde{u}_0$, the displacements in the crack-tip elastic zone since the integral term vanishes. Similarly the displacements in the plastic zone (6.3.6) becomes $\tilde{u}_0$ at $x=b$ from the condition (6.1.3). It is further demonstrated by plots in Figure 23 based on numerical calculations of the evaluated expression (7.7.4) of the integral (6.3.6). In a rearranged form (7.7.4) may be written as follows:
### TABLE 3
Normalized Displacement Function $f_1$
(Equation 7.6.6)

<table>
<thead>
<tr>
<th>$\bar{c}$</th>
<th>1.001</th>
<th>1.005</th>
<th>1.020</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.050</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>1.100</td>
<td>.09537</td>
<td>.06581</td>
<td>.04120</td>
</tr>
<tr>
<td>1.200</td>
<td>.18093</td>
<td>.14802</td>
<td>.10300</td>
</tr>
<tr>
<td>1.300</td>
<td>.24824</td>
<td>.21730</td>
<td>.16140</td>
</tr>
<tr>
<td>1.400</td>
<td>.3330</td>
<td>.2817</td>
<td>.2254</td>
</tr>
<tr>
<td>1.500</td>
<td>.41890</td>
<td>.36020</td>
<td>.29000</td>
</tr>
<tr>
<td>2.000</td>
<td>.7457</td>
<td>.64240</td>
<td>.53260</td>
</tr>
</tbody>
</table>
Figure 22. Normalized crack-tip opening displacement plotted as a function of dimensionless plastic zone size for various elastic zone lengths.
\[
\ddot{u}_y(x) = b E_2(x) F_1(x) - \frac{b^2-a^2}{b} K F_1(x) - \frac{a^2}{b} K E_1(x) - x K E_2(x) + x E F_2(x) + \frac{a^2}{b} \frac{(c^2-x^2)(x^2-b^2)}{(x^2-a^2)(x^2-b^2)}^{1/2} K \quad (9.1.11)
\]
for \(b < |x| < c\)

where

\[
\ddot{u}_y(x) = \frac{\pi E}{4 \sigma_0 \epsilon_p} u_y(x)
\]

and \(F_1, F_2, E_1\) and \(E_2\) are dimensionless incomplete elliptic integrals of the first and second kind respectively. The values of these integrals depend on \(x\), the distance along the crack plane (7.7.4). The plots for \(\ddot{u}_y\) versus the dimensionless distance ahead of the crack-tip are given in Figures 23 and 24. The effect of parameter \(b\) on \(\ddot{u}_y\) is demonstrated by Figure 23 while the effect of \(c\) is displayed on Figure 24. It is observed that the displacements in the plastic zone have a maximum value, equal to the crack-tip opening displacement at the trailing edge of the plastic zone and smoothly drops to a zero value at its leading edge in compliance with the requirement of symmetry of the elastic field with respect to the crack plane.

9.2. Discussion

The results presented in the last section may be considered to provide a more realistic assessment of the effects of an elastic zone formation at the crack-tip. This is due to a number of reasons. It must be recalled that the experimental observations of dislocation free zones at the crack-tip are reported for thin ductile materials. Dugdale is a most appropriate model for such a material. Furthermore the presence of the dislocation-free zones was observed under tensile mode of
Figure 23. Normalized normal transverse displacements as a function of dimensionless distance ahead of the crack-tip for varying elastic zone size.

<table>
<thead>
<tr>
<th></th>
<th>$\overline{D}$</th>
<th>$\overline{C}$</th>
<th>$\overline{\sigma_{\infty}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.005</td>
<td>1.200</td>
<td>.385</td>
</tr>
<tr>
<td>2</td>
<td>1.010</td>
<td>1.200</td>
<td>.391</td>
</tr>
</tbody>
</table>

Eq. 7.7.4
Figure 24. Normalized normal transverse displacements as a function of dimensionless distance ahead of the crack-tip for varying plastic zone size.

| 1  | 1.005 | 1.200 | .385 |
| 2  | 1.005 | 2.000 | .672 |

Eq. 7.7.4

The results obtained in this dissertation for mode-I deformations are based on certain assumptions which the proposed mathematical model embody. To what extent do these assumptions influence the results? It is assumed that the plastically deformed region is a perfectly plastic continuum material. For materials that strain-harden, it is shown that the restraining stresses required to close the end faces of an extended crack (crack with added plastic zones) based on the Barenblatt-Dugdale approach, vary with the displacements in the plastic zone as shown in Figure 4. For a perfectly plastic continuum these stresses are assumed to be uniformly distributed as shown by the horizontal line. It has been shown[25] that the effect of strain hardening on Dugdale results is not significant provided that the state of plane stress prevails and the lateral spread of the plastic zones does not exceed the material thickness. For a thin specimen required for microscopic observation, an existence of a plane stress condition is a reasonable assumption. For highly strain-hardened materials, a common practice to account for the hardening effects is by using material flow stress instead of the yield strength[21].

Another key assumption requires the material in the elastic zone to follow the stress-strain relation of a linearly isotropic elastic material. Such an assumption may seem questionable at least from an engineering stress-strain concept. However Weertman[14] has shown that for a small region at the crack-tip where high stresses occur, the concept of an elastic material as shown by the upper linear curve in
Figure 5 is plausible. It is interesting to note that the slope of the line referred above is equal to the Young's modulus of the material. An incorporation of this assumption in the mathematical model made it possible to obtain an analytic solution of the problem. Two recent studies[11,12] previously cited, are based upon a similar assumption for their corresponding crack-tip dislocation free zone. For other forms of stress-strain behavior, the crack-tip stress singularity may not be of the "inverse square root of the distance ahead of the crack-tip" type.

A condition of boundedness of the stresses within the plastic zone was applied to obtain physical results from mathematical solutions. It is shown in Figures 7 and 8 that the distribution of dislocations, derived from models based on the theory of dislocations[11,12] are bounded within the plastic zone including its two end points. It follows that the stresses must also be bounded at the corresponding points.

Based upon the results obtained in this dissertation, the presence of an elastic zone at the crack-tip causes a singularity of the normal stress there and the value of the resulting stress intensity factor is controlled by the length of the elastic zone (Figure 18). It is shown by Chang and Ohr[11] based upon studies by Thomson[13] that a dislocation free zone is anticipated when this stress intensity factor exceeds $K_g$, a material property. According to Thomson the material property $K_g$ is a function of the dislocation core radius and the image forces at the crack. The materials with a small $K_g$ value, allow dislocations to emit readily and an existence of $K_g$ leads to the formation of the dislocation free zone. It is noted from Figure 16 that when the elastic zone reduces, which means more dislocations pile up in the plastic zone, a relaxation
of stresses in the elastic zone occurs which tend to lower the stress intensity factor (Figure 18) and consequently the emission of dislocations may halt.

For materials where $K_g$ exists and the dislocation free zone is formed, it is shown in [11] that if $K_g < K_{1c}$, the fracture toughness, a ductile fracture occurs. Such a fracture occurs at a critical value of the crack-tip opening displacement. It is important to note that studies reported by Thomson[13] from which Chang and Ohr[11] have derived brittle ductile nature of fracture, assume an atomically sharp crack imbedded in the plastic zone.

It is shown in the last section that the crack-tip opening displacement depends upon both the size of the elastic zone as well as the length of the plastically yielded region. This seems to suggest that whether the crack-tip fracture is brittle or ductile, the CTOD must account for the local conditions of the onset of failure.

Unfortunately the mechanism of transformation of an elastic zone into a crack is not well understood. The dimensions of this zone may be so small that a microscopic model may be required to fully explore the effects of a dislocation free zone region.

Two important observations from the overall results obtained are noted. Though the physical model of the problem differs from the conventional Dugdale model, the mathematical solutions are shown to be the generalized results from which the solution of the Dugdale model may be obtained as a special case of vanishing size of the elastic zone. A comparison of the important results from the crack-tip elastic zone model with those of the Dugdale model is given in Table 4. Another observation is that even though the solutions of anti-plane shear deformation (mode-III) models are not completely analytic, a comparison with the available exact forms for some of the results are qualitatively similar.
## TABLE 4
Comparison of Results with Dugdale Model

<table>
<thead>
<tr>
<th>Physical Quantity</th>
<th>Elastic Zone Model</th>
<th>Dugdale</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) Plastic zone size relation</td>
<td>( \bar{\sigma} = \Lambda_0(\theta_0, \kappa) )</td>
<td>( \bar{\sigma} = \frac{2}{\pi} \cos^{-1} \left( \frac{a_l}{c} \right) )</td>
</tr>
<tr>
<td>(2) Crack-tip stress intensity factor</td>
<td>( \bar{F}_{IC} = \frac{2}{\pi} \left( \frac{b^2 - a^2}{b} \right)^{1/2} K(\kappa) )</td>
<td>0</td>
</tr>
<tr>
<td>(3) crack-tip displacement</td>
<td>( \bar{\delta} = \left[ K(\kappa) E(\kappa) - \frac{a^2}{b^2} K(\kappa) E(K - K) \right] )</td>
<td>( \bar{\delta} = \frac{\pi}{a} )</td>
</tr>
<tr>
<td>(4) stresses</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(a). dislocation free zone</td>
<td>( \bar{\sigma}_{yy} = \frac{2}{\pi} \left( \frac{b^2 - a^2}{b} \right)^{1/2} \left[ \frac{x^2 (b - x^2)}{(x^2 - a^2)(c^2 - x^2)} \right]^{1/2} K(\kappa) )</td>
<td>( \bar{\sigma}_{yy} = 1 - \frac{1}{2} \left( \cos^{-1} \frac{a_l}{c} \right)^{-1} )</td>
</tr>
<tr>
<td>(b). zone of elasticity</td>
<td>( \bar{\sigma}_{yy} = \Lambda_0(\theta_2, \kappa) )</td>
<td>( \bar{\sigma}_{yy} = \frac{2}{\pi} \tan^{-1} \left( \frac{\sin 2\theta}{\cos 2\theta - \epsilon} \right) )</td>
</tr>
<tr>
<td>(5) displacements</td>
<td>( \bar{u}_y = F(\theta_x, \kappa) \left{ b \left( E(\kappa) - \frac{b^2 - a^2}{b} \right) K(\kappa) \right} )</td>
<td>( \bar{u}_y = \cos \epsilon \sin \left( \frac{\pi - \theta}{\sin \theta + \theta} \right) )</td>
</tr>
<tr>
<td>(Plastic Continuum Region)</td>
<td>( - E(\theta_x, \kappa) \frac{a^2}{b} K(\kappa) - \frac{a^2}{b} K(\kappa) )</td>
<td>( \bar{u}_y = \cos \epsilon \sin \left( \frac{\pi - \theta}{\sin \theta + \theta} \right) )</td>
</tr>
<tr>
<td></td>
<td>( K(\kappa) + x F(\theta_x, \kappa) E(K) + \frac{a^2}{b^2} K(K) )</td>
<td>( \cos \epsilon \sin \left( \frac{\pi - \theta}{\sin \theta + \theta} \right) )</td>
</tr>
<tr>
<td></td>
<td>( [(c^2 - x^2)(k^2 - b^2)/(c^2 - a^2)x^2 - a^2)]^{1/2} )</td>
<td>( \frac{\sin^2 \theta}{\sin \theta - \sin \theta} )</td>
</tr>
</tbody>
</table>
The key contributions of this research to the fields of fracture mechanics and elasticity are as follows:

* The stress intensity factor and crack-tip opening displacement results for a crack-tip elastic zone model with a realistic simulation of the recent microscopic observations of the crack-tip region[11].

* A completely analytic solution of the problem based on a continuum mechanics approach for a structurally applied mode of loading. Such a solution has been sought for a number of years[13].

* An extension of the method of finite Hilbert transform to solve four-part mixed boundary value problems such as those solved in Chapter 7 and 8.

In specific terms, an analytic model for an isotropic, plane infinite medium with an internal crack accompanied by a dislocation free zone and a plastically deformed region at each end under an opening mode deformation due to uniformly applied remote stresses, is proposed. With an application of a simple superposition scheme, the above problem is treated as two individual problems: a mixed boundary value problem with all boundary conditions written on the crack plane and a problem of uncracked, linearly elastic homogeneous medium under remote tensile
loading. The material in the elastic zone is assumed to obey elastic stress-strain relations and the transverse displacements in this region are uniformly distributed. The plastically deformed region is a perfect plastic continuum, with yield stresses uniformly distributed and constant throughout the region, and thus the strain-hardening for the material is assumed to be negligible.

Taking advantage of the dual symmetry of the problem; a symmetry about the crack plane along the x-axis and another about a plane through the crack-center along the y-axis, a formulation using Papkovich-Neuber harmonic functions for a two dimensional medium, the four part mixed boundary conditions are represented by a quadruple set of integral equations involving cosine kernels. An exact solution of these integral equations is derived using a modified finite Hilbert transform technique. The relations for the normal stress distributions, transverse displacement patterns, plastic zone size, crack-tip stress intensity factor and the crack-tip opening displacement in the elastic zone, are presented.

10.1. Conclusions

Some important conclusions based upon the obtained results are as follows:

1. A completely analytic (closed form) solution for the crack-tip elastic zone model of Dugdale plastic yielding is obtained. The physical quantities of interest in the field of fracture mechanics including the plastic zone size, the crack-tip stress intensity factor and the crack-tip opening displacement are derived in exact forms.
2. The results show that the normal stresses in the elastic zone becomes unbounded and an inverse square root, of the distance ahead of the crack, singularity occurs at the crack-tip due to the presence of an elastic zone.

3. A relaxation of normal stress in the elastic zone is observed. As the size of this zone reduces during emission of dislocations in the plastic zone, the stresses in the elastic zone are decreased while the applied remote stresses are held stationary.

4. Two important fracture parameters, a crack-tip stress intensity factor and a crack-tip opening displacement, are derived. A critical value of the stress intensity factor is responsible for the formation of elastic zone and onset of fracture is governed by a critical value of the crack-tip opening displacement[11].

5. As the elastic zone shrinks to a zero value, all quantities including the stress intensity factor, the plastic zone size and the crack-tip opening displacement approach the corresponding values for the Dugdale model of plasticity.

6. The application of the method of finite Hilbert transform is extended to solutions of four-part mixed boundary value crack problems with nonvanishing fields specified on three parts of its boundaries. Based upon this work, it may be concluded that the modified finite Hilbert transform technique is relatively simple and efficient to solve such problems especially the evaluations of stress fields and the stress intensity factors.

7. The available analytical results for some quantities from anti-plane shear deformation (mode-III) models[11,12] are mathematically analogous to those for the tensile or opening mode of deformation.
8. An application of these results to predict failure, requires an additional material property $K_g$ (9.2) which must be experimentally determined[11].

10.2. Recommendations For Further Research

In many structural applications, the state of affairs at the crack-tip is three-dimensional. It is known that even for a state of plane stress, a triaxial field exists at distances approximately the thickness of the material. A solution of a three-dimensional problem may require development of new analytic techniques. However, due to its practical merit, this problem needs to be solved.

Several practical materials exhibit significant strain-hardening. Consequently the restraining stresses in the plastic zones will vary non-linearly with the displacements in these regions as shown in Figure 4. An analysis for the effects of work-hardening on the present results is a natural follow-up of the problem.

The effects of thickness on the formation and the size of the dislocation free zone is an excellent area of exploration, both experimentally as well as analytically. At present a scanning electron microscope requires a limited thickness of specimen for observations. In more realistic structures, the plane stress and the plane strain may develop simultaneously. For example a state of plane stress on the exterior and plane strain in the interior. Also the plane stress tensile fractures spread under the combined modes of deformation, an opening mode and a parallel shear deformation mode applied to the crack. A modification of these results for a mixed mode deformation would be a realistic development of this work.
The elastic zone may be considered as the length of each step for a slow stable discretely growing crack. A combination of the length of the elastic zone and the crack-tip opening displacement may be used to predict slow stable crack growth behavior using an approach similar to that used by Wnuk[57]. An extension of these results to include finite bodies will be a contribution to the state of the art. However, the methods of integral transforms including finite Hilbert transform, have not yet been advanced to handle all finite body problems.

With the demonstrated applicability of the finite Hilbert transform method to four-part mixed boundary value problems, this technique may be applied to problems of cracked bodies such as the triple crack problems solved in Chapter VIII. The potential areas of applications of the physical model are numerous including fatigue, fatigue crack closure studies, fracture at elevated temperatures, hydrogen embrittlement, short cracks and slow stable crack growth. An application of this solution to nuclear pressure vessels failure assessment diagrams based on a unified view of fracture was invited at a recent fracture mechanics conference, ASTM 1983[51].
LIST OF REFERENCES


36. C.E. Inglis, "Stresses in a Plate Due to the Presence of Cracks and Sharp Corners," Transactions, Institute of Naval Architects, 55, p. 219-241, 1913.


A brief introduction to elliptic integrals and functions is given in this appendix since such non-elementary integrals were frequently encountered in the analysis and evaluations of stress and displacement fields.

A.1. Elliptic Integrals

The integrals of the form
\[ \int R[t,\{P(t)^{1/2}] \, dt \]  
where \( P \) is a polynomial of the third or fourth degree and \( R \) is a rational function are known as elliptic integrals. These integrals are called Legendre's canonical elliptic integrals of the first, second and third kind and are usually denoted by the symbols \( F(\phi,k) \), \( E(\phi,k) \) and \( \Pi(\phi,\alpha^2,k) \) respectively. \( \phi \), \( k \) and \( \alpha^2 \) are known as the argument, the modulus and the parameter of elliptic integrals.

These three basic elliptic integrals are defined as follows:

**Elliptic Integral of the First Kind**
\[ \int_0^y \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = \int_0^\phi \frac{d\theta}{(1-k^2\sin^2\theta)^{1/2}} \]
\[ du = u_1 = \text{sn}^{-1}(y,k) = F(\phi,k) \quad (A.1.2) \]

Elliptic Integral of the Second Kind

\[ \int_0^1 \left[ \frac{1-k^2t^2}{1-t^2} \right]^{1/2} dt = \int_0^\phi (1-k^2\sin^2 \theta)^{1/2} d\theta \]

\[ \int_0^{u_1} \frac{du}{\sqrt{1-\alpha^2 \sin^2 u}} \quad (A.1.3) \]

Elliptic Integral of the Third Kind

\[ \int_0^1 \frac{dt}{(1-\alpha^2 t^2)[(1-t^2)(1-k^2t^2)]^{1/2}} = \int_0^\phi \frac{d\theta}{(1-\alpha^2 \sin^2 \theta)(1-k^2 \sin^2 \theta)^{1/2}} \]

\[ = \int_0^{u_1} \frac{du}{1-\alpha^2 \sin^2 u} = \Pi(u_1, \alpha^2) = \Pi(\phi, \alpha^2, k) \quad (A.1.4) \]

In these integrals \( \text{sn} u \) and \( \text{dn} u \) are called Jacobian elliptic functions. The Jacobian elliptic functions are similar to trigonometric functions and possess single values as a function of the argument \( u \). They are defined by

\[ \text{sn} u = y = \sin \phi \quad (A.1.5) \]

and

\[ \text{dn} u = (1-k^2y^2)^{1/2} = (1-k^2\sin^2 \phi)^{1/2} \quad (A.1.6) \]

Another commonly known Jacobian elliptic function is \( \text{cn} u \), with a definition
\[ \text{cn} \ u = (1-y^2)^{1/2} = \cos \phi \] (A.1.7)

These functions have notable properties. The function \( \text{sn} \ u \) is finite for all real or complex values of \( y \) including infinity, \( \text{dn} \ u \) has a simple pole of order 1 at \( y = \infty \) and \( \text{cn} \ u \) is logarithmically infinite at \( y = 1/\alpha \). It may be readily verified from above relations that as the argument tends to zero, all the three elliptic integrals assume a zero value. In the vanishing limits of their modulus and the parameter, their values tend to be equal to their respective arguments.

An important special case results at a constant value of \( \pi/2 \) of the arguments of the three elliptic integrals. For this value of the argument, these integrals depend only on the modulus and the parameter. Such integrals are called complete elliptic integrals. Note that the value of the upper limit \( y \) for complete elliptic integrals is 1. These integrals are denoted by

\[ F\left(\frac{\pi}{2}, k\right) = K(k) = K, \text{ complete elliptic integral of the first kind} \] (A.1.8)

Similarly

\[ E\left(\frac{\pi}{2}, k\right) = E(k) = E, \text{ complete elliptic integral of the second kind} \] (A.1.9)

and

\[ \Pi\left(\frac{\pi}{2}, \alpha^2, k\right) = \Pi(\alpha^2, k), \text{ the complete elliptic integral of the third kind} \] (A.1.10)
It follows from the previous discussion that the complete elliptic integrals approach a value of $\frac{\pi}{2}$ as the parameter $a^2$ and the modulus $k$ approach zero.

If the modulus $k$ is replaced by its complementary value $k'$, the complete elliptic integrals are called associated complete elliptic integrals with the following useful relations:

\[ K(k') = K'(k) = K' \quad (A.1.11) \]
\[ E(k') = E'(k) = E' \quad (A.1.12) \]
\[ k' = (1-k^2)^{1/2} \quad (A.1.13) \]

A.2. Special Elliptic Functions

The values of the elliptic integrals of the first and second kind are commonly available in standard reference handbooks of special functions. For this reason, it is often necessary to express the elliptic integrals of the third kind in terms of the first two integrals. This is accomplished by introducing two special elliptic functions: a Heuman's lambda function $\Lambda_0$ and a Jacobian zeta function $Z$. The particular function to be used depends upon the relative values of the parameter $a^2$ and the modulus $k$. The first two cases, Case 1 and 2, given below are called the circular cases and the complete elliptical integral of the third kind in these cases may be expressed in terms of the Heuman's lambda function. Cases 3 and 4 are referred to as the hyperbolic cases and Jacobian zeta function is used to express the complete elliptic integrals of the third kind.
Case 1. $0 < -\alpha^2 < \infty$

$$\pi(\alpha^2, k) = \int_0^K \frac{du}{1-\alpha^2 \text{sn}^2 u} = \frac{K}{1-\alpha^2} + \frac{\alpha^2}{\pi} \cdot \frac{\Lambda_0(\beta, k') - 1}{[\alpha^2 (1-\alpha^2) (\alpha^2 - k'^2)]^{1/2}} \quad (A.2.1)$$

and the Heuman's lambda function $\Lambda_0$ is defined as

$$\Lambda_0(\beta, k) = \frac{2k'^2 \text{sn} \beta \text{cos} \beta}{\pi (1-2 \sin^2 \beta)^{1/2}} \int_0^K \frac{du}{1-k'^2 (1-k'^2 \sin^2 \beta)^{-1} \text{sn}^2 u}$$

$$= \frac{2}{\pi} [\text{EF}(\beta, k') + \text{KE}(\beta, k') - \text{KF}(\beta, k')] \quad (A.2.2)$$

where

$$\beta = \sin^{-1} \frac{1}{(1-\alpha^2)^{1/2}} \quad (A.2.3)$$

and $K$ and $E$ are complete elliptic integrals of the first and second kind respectively.

Case 2. $k^2 < \alpha^2 < 1$

$$\pi(\alpha^2, k) = \frac{\alpha \pi \Lambda_0(\xi, k)}{2[(\alpha^2-k^2)(1-\alpha^2)]^{1/2}} \quad (A.2.4)$$

where $\Lambda_0$ is defined by (B.2.2) and

$$\xi = \sin^{-1} \left[ \frac{\alpha^2 - k^2}{\alpha^2 k'^2} \right]^{1/2} \quad (A.2.5)$$

The four particular values of the Heuman's lambda function are of special interest. They are
\[ A_o(\beta,0) = \sin \beta \]  \hspace{1cm} (A.2.6)

\[ A_o(0,k) = 0 \]  \hspace{1cm} (A.2.7)

\[ A_o(\beta,1) = 2\beta/\pi \]  \hspace{1cm} (A.2.8)

\[ A_o(\frac{\pi}{2},k) = 1 \]  \hspace{1cm} (A.2.9)

Case 3. \( 0 < \alpha^2 < k^2 \)

\[ \pi(\alpha^2,k) = \int_0^K \frac{du}{1-\alpha^2 \sin^2 u} = K + \frac{\alpha KZ(\beta,k)}{[1-\alpha^2(k^2-\alpha^2)]^{1/2}} \]  \hspace{1cm} (A.2.10)

where the Jacobian zeta function \( Z(\beta,k) \) is defined by

\[ KZ(\beta,k) = KE(\beta,k) - EF(\beta,k) \]  \hspace{1cm} (A.2.11)

where

\[ \beta = \sin^{-1} \left( \frac{\alpha}{k} \right) \]  \hspace{1cm} (A.2.12)

Case 4. \( \infty > \alpha^2 > 1 \)

\[ \pi(\alpha^2,k) = \frac{\alpha KZ(A,k)}{[(\alpha^2-1)(\alpha^2-k^2)]^{1/2}} \]  \hspace{1cm} (A.2.13)

where \( KZ \) is defined by (B.2.11) and

\[ A = \sin^{-1} \left( \frac{1}{\alpha} \right) \]  \hspace{1cm} (A.2.14)

The above definitions, notations and relations for elliptic integrals and functions are extracted from Byrd and Friedman "Handbook of
Elliptic Integrals for Engineers and Scientists"[52]. A detailed discussion of this subject can be found in [52], [53] and [58].
APPENDIX - B

EVALUATION OF ELLIPTIC INTEGRALS

The elliptic integrals whose values are not available in the literature including some authentic reference handbooks [52, 53 and 58] are evaluated in this appendix. In most cases, these evaluations were managed using well known techniques such as simplification of arguments, integration by parts and various substitutions. Again the most frequently cited reference is Byrd and Friedman[52] in its revised form. To simplify the lengthy expressions involving elliptic integrals, several addition theorems and limiting values of these integrals are employed. These and most frequently recalled elliptic and simple integrals with their principal values are listed in Appendix C.

B.1

\[ I_1 = \int_{b}^{c} \left[ \frac{(y^2-a^2)}{(y^2-b^2)(c^2-y^2)} \right]^{1/2} \, dy \quad (B.1.1) \]

with a change of variable and the substitutions as follows:

\[ t = y^2 \]
\[ A = c^2 \]
\[ B = b^2 \]
\[ C = a^2 \]
\[ D = 0 \]  \hspace{1cm} (B.1.2)

The above integral may be expressed in the form,
\[ I_1 = \frac{1}{2} \int_{A}^{B} \left[ \frac{t-C}{(A-t)(t-B)(t-D)} \right]^{1/2} \, dt \]  \hspace{1cm} (B.1.3)

Since in (B.1.3)
\[ A \geq t \geq B > C > D \]  \hspace{1cm} (B.1.4)

a value of \( I_1 \) may be obtained from integral no. 256.02[52], which is
\[ I_1 = \frac{1}{2} (B-C)g \cdot \int_{0}^{u_1} \frac{du}{1-\alpha^2 sn^2 u} \]  \hspace{1cm} (B.1.5)

where
\[ g = \frac{2}{b(c^2-a^2)^{1/2}} \]  \hspace{1cm} (B.1.6)

and
\[ \alpha^2 = \frac{c^2-b^2}{c^2-a^2} \]  \hspace{1cm} (B.1.7)

Noting that \( I_1 \) is a complete elliptic integral and from the definition of the elliptic integral of the third kind (A.1.4) we find that
\[ I_1 = \frac{b^2-c^2}{b(c^2-a^2)^{1/2}} \pi \left( \frac{\pi}{2}, \alpha^2, k \right) \]  \hspace{1cm} (B.1.8)

where
with a change of variable and substitutions (B.1.2), the above integral may be written in the following form:

\[ I_2 = \int_0^a \left( \frac{A-t}{(t-B)(t-C)(t-D)} \right)^{1/2} dt \]  

A value of the above integral is found in [52], integral no. 256.14, with which,

\[ I_2 = \frac{1}{2} (A-B) \cdot g \cdot \int_0^{u_1} \frac{\frac{c_n^2 u}{2}}{1-A^2 \sin^2 u} du \]  

using recursion formulae 338.01 [52],

\[ \int_0^{u_1} \frac{\frac{c_n^2 u}{2}}{1-A^2 \sin^2 u} du = \frac{1}{A^2} \int_0^{(\alpha^2-1)\pi} (\alpha^2 \cdot k) \]  

For a complete elliptic integral,

\[ \phi = \frac{\pi}{2} \]

and \( \alpha^2 \) and \( \kappa \) are given by (B.1.7) and (B.1.9) respectively with these substitutions,
\[ I_2 = \frac{(c^2 - a^2)^{1/2}}{b} \left[ \frac{K(k) - \frac{b^2 - a^2}{c^2 - a^2} \Pi(a^2, k)}{c^2 - a^2} \right] \]  \hspace{1cm} (B.2.5)

\[ I_3 = \int_b^c \left[ \frac{(y^2 - a^2)(y^2 - b^2)}{c^2 - y^2} \right]^{1/2} dy \]  \hspace{1cm} (B.3.1)

Using (B.1.2), the integral \( I_3 \) takes the following form:

\[ I_3 = \int_B^A \left[ \frac{(t-C)(t-B)}{(A-t)(t-D)} \right]^{1/2} dt \]  \hspace{1cm} (B.3.2)

With integral no. 257.18 [52], the value of the above integral is

\[ I_3 = \frac{1}{2} (A-B)(A-C) \frac{1}{2} \int_0^1 \frac{c_n u}{1 - a^2 s_n^2 u^2} du \]  \hspace{1cm} (B.3.3)

and using recurrence formulae 362.20 [52],

\[ I_3 = \frac{1}{2a_4} \left[ -a^2 E(u) + (a^2 - k^2)u + (a^4 - k^2)\Pi(u, a^2) + \frac{a^4 s_n u c_n u d u}{1 - a^2 s_n^2 u^2} \right] \times \frac{1}{2} (A-B)(A-C)g \]  \hspace{1cm} (B.3.4)

where \( a^2 \), \( g \) and \( k \) are defined previously. Since \( I_3 \) is a complete elliptic integral, the function \( c_n u \) may be taken to be zero. A performance of algebraic manipulations in (B.3.4) leads to the following value of \( I_3 \):
\[ I_3 = \left( \frac{c^2-a^2}{2b} \right)^{1/2} \left[ (c^2-a^2) E(k) + \frac{b^2+a^2}{b^2} (c^2-a^2) K(k) \right] \]
\[ + \frac{b^2-a^2}{b^2} (c^2-b^2-a^2) \Pi(a^2,k^2) \]  
\[ (B.3.5) \]

\[ * \quad * \quad * \]

B.4

\[ I_4 = \int_b^c \left[ \frac{(y^2-a^2)(c^2-y^2)}{y^2-b^2} \right]^{1/2} dy \]  
\[ (B.4.1) \]

with substitutions (B.1.2),

\[ I_4 = \frac{1}{2} \int_B^A \left[ \frac{(A-t)(t-C)}{(t-B)(t-D)} \right]^{1/2} dt \]  
\[ (B.4.2) \]

Usint integral No. 257.17 [52]

\[ I_4 = \frac{1}{2} (A-C)(C-D) a^2 g \int_0^{u_1} \frac{sn^2 u \ dn^2 u}{(1-a^2 sn^2 u)^2} \ du \]  
\[ (B.4.3) \]

and with recurrence formulae 362.19 [52] for the integral in (B.4.3),

\[ I_4 \] is given by

\[ I_4 = \frac{1}{2} (A-C)(C-D) a^2 g \cdot \frac{1}{2a^4(\alpha^2-1)} \]
\[ \left[ (\alpha^2+k^2-2\alpha^2 k^2)u - \alpha^2 E(u) + (2\alpha^2 k^2 - \alpha^2 - k^2) \Pi(u,\alpha^2) \right] \]
\[ + \frac{a^4 sn^2 u \ cn^2 u \ dn^2 u}{1-\alpha^2 sn^2 u} \]  
\[ (B.4.4) \]

Noting that \( I_4 \) is a complete elliptic integral and performing necessary algebraic manipulations, the value of the integral (B.4.1) is given by
\[ I_4 = \frac{c^2(c^2-a^2)^{1/2}}{2b} \left[ \frac{c^2+a^2}{b^2} K(\overline{k}) + \frac{c^2-a^2}{a^2-b^2} E(\overline{k}) \right. \\
\left. + \frac{b^2-a^2-c^2}{b^2} \Pi(\alpha^2, \overline{k}) \right] \quad \text{(B.4.5)} \]

where \( K, E \) and \( \Pi \) are complete elliptic integrals of the first, second and third kind respectively and \( \alpha^2 \) and \( \overline{k} \) are previously defined.

* * *

B.5

\[ I_5 = \int_b^c \frac{\frac{u^2(u^2-b^2)}{(u^2-a^2)(c^2-u^2)}}{y^2-u^2} \frac{du}{y^2-u^2} \quad \text{(B.5.1)} \]

with the identity
\[
\frac{u^2-b^2}{y^2-u^2} = \frac{y^2-b^2}{y^2-u^2} - 1 \quad \text{(B.5.2)}
\]

the integral \( I_5 \) is expressed as a sum of two integrals, i.e.

\[ I_5 = I_{51} + I_{52} \quad \text{(B.5.3)} \]

where

\[ I_{51} = (y^2-b^2) \int_b^c \frac{u}{(u^2-a^2)(u^2-b^2)(c^2-u^2)} \frac{du}{y^2-u^2} \quad \text{(B.5.4)} \]

and

\[ I_{52} = - \int_b^c \frac{u \, du}{(u^2-a^2)(u^2-b^2)(c^2-u^2)^{1/2}} \quad \text{(B.5.5)} \]

Using the following change of variable and the substitutions:
The integral $I_{51}$ becomes

$$I_{51} = \frac{p-B}{2} \int_{b}^{A} \frac{dt}{(p-t)[(A-t)(t-B)(t-C)]^{1/2}} \quad (B.5.7)$$

A value of the integral in (B.5.7) is obtained from a special case of integral no. 235.17 [52] for $m=1$ and its value is given by

$$I_{51} = \frac{g}{2} \int_{0}^{u} \frac{dn^2u}{du} \frac{du}{\sin^2u} \quad (B.5.8)$$

Applying recurrence formulae 339.04 [52], the value of

$$I_{51} = \frac{g}{2} \cdot \frac{1}{\alpha^2} \left[ k^2 u + (\alpha^2 - k^2) \Pi(\phi, \alpha^2, k) \right] \quad (B.5.9)$$

where

$$g = \frac{2}{(c^2 - a^2)^{1/2}} \quad (B.5.10)$$

$$\alpha^2 = \frac{(A-B)(C-p)}{(B-p)(A-C)} = \frac{(c^2 - b^2)(a^2 - y^2)}{(c^2 - a^2)(b^2 - y^2)} \quad (B.5.11)$$

$$k = \left( \frac{c^2 - b^2}{c^2 - a^2} \right)^{1/2} \quad (B.5.12)$$

and
\phi = \frac{\pi}{2} \quad (B.5.13)

for a complete elliptic integral. With a substitution of definitions (B.5.10) through (B.5.13) in (B.5.9), the value of the integral \( I_{51} \) is simplified to

\[
I_{51} = \frac{1}{(c^2-a^2)^{1/2}} \cdot \left[ \frac{b^2-y^2}{a^2-y^2} K(k) - \frac{b^2-a^2}{a^2-y^2} \Pi(a^2,k) \right] \quad (B.5.14)
\]

Similarly a substitution of (B.5.6) in (B.5.5) given

\[
I_{52} = -\frac{1}{2} \int_{B}^{A} \frac{dt}{[\sqrt{(A-t)(t-B)(t-C)}]^1/2} \quad (B.5.15)
\]

and with integral no. 235.00 [52], its value may be derived as

\[
I_{52} = -\frac{1}{(c^2-a^2)^{1/2}} K(k) \quad (B.5.16)
\]

Combining (B.5.41) and (B.4.16), we find that

\[
I_5 = \frac{(b^2-a^2)}{(a^2-y^2)(c^2-a^2)^{1/2}} \left\{ K(k) - \Pi(a^2,k) \right\} \quad (B.5.17)
\]

\[
* \quad * \quad * \quad *
\]

B.6.

\[
I_6 = \left[ \frac{u^2 (c^2-u^2)}{L(u^2-a^2)(u^2-b^2)} \right]^{1/2} \frac{du}{y^2-u^2} \quad (B.6.1)
\]

using the identity

\[
\frac{c^2-u^2}{y^2-u^2} = 1 + \frac{c^2-y^2}{y^2-u^2} \quad (B.6.2)
\]

the integral \( I_6 \) may be expressed as a sum of two integrals, i.e.,
\[ I_6 = I_{61} + I_{62} \] \hspace{1cm} (B.6.3)

where

\[ I_{61} = \int_{b}^{c} \frac{u \, du}{(u^2-a^2)(u^2-b^2)(c^2-u^2)^{1/2}} \] \hspace{1cm} (B.6.4)

and

\[ I_{62} = (c^2-y^2) \int_{b}^{c} \frac{u \, du}{(y^2-u^2)[(u^2-a^2)(u^2-b^2)(c^2-u^2)^{1/2}]^{1/2}} \] \hspace{1cm} (B.6.5)

With substitutions (B.5.6), (B.6.4) takes the following form:

\[ I_{61} = \frac{1}{2} \int_{b}^{A} \frac{dt}{(A-t)(t-B)(t-C)^{1/2}} \] \hspace{1cm} (B.6.6)

and its value from integral no. 236.00 [62] is

\[ I_{61} = \frac{K(k)}{(c^2-a^2)^{1/2}} \] \hspace{1cm} (B.6.7)

Similarly with (B.5.6), (B.6.5) becomes

\[ I_{62} = \frac{1}{2} (A-p) \cdot \int_{b}^{A} \frac{dt}{(p-t)[(A-t)(t-B)(t-C)^{1/2}]} \] \hspace{1cm} (B.6.8)

A value of the above integral is found in [52] given by integral no. 236.14, from which

\[ I_{62} = -\frac{A-p}{2} \cdot \frac{g}{A-p} \int_{0}^{1} \frac{du}{1-a^2 \sin^2 u} \] \hspace{1cm} (B.6.9)

\[ = -\frac{1}{(c^2-a^2)^{1/2}} \pi(a^2,y,k) \] \hspace{1cm} (B.6.10)
where
\[ \alpha^2_y = \frac{c^2 - b^2}{c^2 - y^2} \] (B.6.11)

and the value of the modulus \( k \) is given by (B.5.12). A combination of (B.6.7) and (B.6.10) yields the following value of the integral \( I_6 \):
\[ I_6 = \frac{1}{(c^2 - a^2)^{1/2}} [K(k) - \Pi(\alpha^2_y, k)] \] (B.6.12)

* * *

B.7.
\[ I_7 = \int_b^c \left( \frac{c^2 - y^2}{y^2 - b^2} \right)^{1/2} y \, dy \int_b^c \left[ \frac{u^2 (u^2 - b^2)}{(u^2 - a^2)(c^2 - u^2)} \right]^{1/2} \frac{du}{y^2 - u^2} \] (B.7.1)

Assuming that the above integral may be expressed in the form
\[ \int_b^c \left[ \frac{u^2 (u^2 - b^2)}{(u^2 - a^2)(c^2 - u^2)} \right]^{1/2} du \int_b^c \left( \frac{c^2 - y^2}{y^2 - b^2} \right)^{1/2} \frac{dy}{y^2 - u^2} \] (B.7.2)

with the identity
\[ \frac{c^2 - y^2}{y^2 - u^2} = \frac{c^2 - u^2}{y^2 - u^2} - 1 \] (B.7.3)

the inner integral of \( I_7 \) becomes
\[ I_7(y) = \frac{c^2 - u^2}{y^2 - u^2} \left[ \int_b^c \frac{dy}{(y^2 - u^2)[(c^2 - y^2)(y^2 - b^2)]^{1/2}} \right] - \left[ \int_b^c \frac{dy}{[(c^2 - y^2)(y^2 - b^2)]^{1/2}} \right] \] (B.7.4)
for \( b < u < c \) we find that from integral no. 10 (Appendix-C), the first integral term in (B.7.4) vanishes and a value of the integral in the second term is given by integral no. 7 (Appendix-C). With these values

\[
I_7(y) = - \frac{\pi}{2}
\]  
\text{ (B.7.5)}

and from (B.7.2)

\[
I_7 = - \frac{\pi}{2} \int_b^c \left[ \frac{u^2 (u^2 - b^2)}{(u^2 - a^2)(c^2 - u^2)} \right]^{1/2} \, du
\]  
\text{ (B.7.6)}

With substitutions (B.5.6), the integral \( I_7 \) takes the following form:

\[
I_7 = - \frac{\pi}{4} \int_B^A \left[ \frac{t-B}{(A-t)(t-C)} \right]^{1/2} \, dt
\]  
\text{ (B.7.7)}

A value of this integral is given by integral no. 236.03 [52] which is

\[
I_7 = - \frac{\pi}{4} (A-B) g \cdot \int_0^{u_1} \text{cn}^2 u \, du
\]  
\text{ (B.7.8)}

From recurrence formulae 312.02 [52]

\[
I_7 = - \frac{\pi}{2} (c^2 - a^2)^{1/2} \cdot \left[ E(k) - \frac{b^2 - a^2}{c^2 - a^2} K(k) \right]
\]  
\text{ (B.7.9)}

the modulus \( k \) is defined by (B.5.12).

\* \* \* \* \* \*

B.8.

\[
I_8 = \int_b^c \left[ \frac{u^2 (c^2 - u^2)}{(u^2 - a^2)(u^2 - b^2)} \right]^{1/2} \, du \int_b^c \left( \frac{y^2 - b^2}{c^2 - y^2} \right)^{1/2} \frac{y \, dy}{y^2 - u^2}
\]  
\text{ (B.8.1)}

Using the identity
the inner integral with respect to \( y \) may be expressed as a superposition of two integrals, i.e.,

\[
I_8(y) = (u^2 - b^2) \int_{b}^{c} \frac{y\,dy}{(y^2 - u^2)[(c^2 - y^2)(y^2 - b^2)]^{1/2}} \\
+ \int_{b}^{c} \frac{y\,dy}{[(c^2 - y^2)(y^2 - b^2)]^{1/2}}
\]

(B.8.3)

From integral numbers 8 and 11 in Appendix C,

\[
I_8(y) = \frac{\pi}{2}
\]

(B.8.4)

and from (B.8.1)

\[
I_8 = \frac{\pi}{2} \int_{b}^{c} \left[ \frac{u^2 (c^2 - u^2)}{(u^2 - a^2)(u^2 - b^2)} \right]^{1/2} \, du
\]

(B.8.5)

or with substitutions (B.5.6)

\[
I_8 = \frac{\pi}{4} \int_{b}^{A} \left[ \frac{A-t}{(t-B)(t-C)} \right]^{1/2} \, dt
\]

(B.8.6)

From 236.04 [52]

\[
I_8 = \frac{\pi}{4} \cdot (A-B) \cdot g \int_{0}^{u_1} \, sn^2u \, du
\]

(B.8.7)

and using the recurrence formulae 310.02 [57]

\[
I_8 = \frac{\pi}{4} \cdot (c^2 - b^2) \left[ \frac{2}{(c^2 - a^2)^{1/2}} \right]^{1/2} \frac{1}{k} \left[ u - E(u) \right]
\]

(B.8.8)

From which the value of integral
\[
I_B = \frac{\pi}{2} (c^2-a^2)^{1/2} \left[ K(k) - E(k) \right]
\]  
(B.8.9)

with modulus \( k \) being defined by (B.5.12).

* * *

B.9.

\[
I_g = \int_{b}^{c} F(\phi(y), k) \, dy
\]  
(B.9.1)

where \( F \) is an incomplete elliptic integral of the first kind whose argument \( \phi \) is a function of the variable of integration \( y \) and

\[
\phi(y) = \sin^{-1} \left[ \frac{(c^2-b^2)(y^2-a^2)}{(c^2-a^2)(y^2-b^2)} \right]^{1/2}
\]  
(B.9.2)

\[
k = \left( \frac{c^2-b^2}{c^2-a^2} \right)^{1/2}
\]  
(B.9.3)

And integration of (B.9.1) by parts leads to

\[
I_g = y \left. F(y) \right|_{y=b}^{y=c} - \int_{b}^{c} \left\{ \frac{d}{dy} F(y) \right\} y \, dy
\]  
(B.9.4)

where

\[
F(y) = F(\phi(y), k)
\]  
(B.9.5)

The first term in (B.9.4) with (B.9.2) and (B.9.3) may be reduced to

\[
y \left. F(y) \right|_{y=b}^{y=c} = c K(k)
\]  
(B.9.6)

and the elliptic integral \( F9y) \) in the second term, by definition (A.1.2) may be written in the follow ng form:
\[ F(y) = \int_0^\phi(y) \frac{d\phi(y)}{[1-k^2\sin^2\phi(y)]^{1/2}} \]  
(B.9.7)

Denoting
\[ F'(y) = \frac{dF}{dy} = \frac{dF}{d\phi} \cdot \frac{d\phi}{dy} \]  
(B.9.8)

From (B.9.2) and (B.9.7)
\[ \frac{dF}{d\phi} = \frac{1}{[1-k^2\sin^2\phi]^{1/2}} \]  
(B.9.9)

and
\[ \frac{d\phi}{dy} = \frac{\sin^{-1}\left[ \frac{(c^2-b^2)(y^2-a^2)}{(c^2-a^2)(y^2-b^2)} \right]}{1/2} \]  
(B.9.10)

A performance of the above differentiation gives
\[ \frac{d\phi}{dy} = \left[ (b^2-a^2)(c^2-a^2)^{1/2} \cdot \frac{y}{(y^2-a^2)[(y^2-b^2)(c^2-y^2)]^{1/2}} \right] \]  
(B.9.11)

and \( F' \) is now obtained from (B.9.8) together with (B.9.9) and (B.9.11) yielding
\[ F'(y) = \left[ \frac{y^2(c^2-a^2)}{(y^2-a^2)(y^2-b^2)(c^2-y^2)} \right]^{1/2} \]  
(B.9.12)

and the integral term in (B.9.4) now may be written in the following form:
\[ - (c^2-a^2)^{1/2} \int_b^c \frac{y^2 dy}{[(c^2-y^2)(y^2-a^2)(y^2-b^2)]^{1/2}} \]  
(B.9.13)

Let
\[ t = y^2 \]
\[ A = c^2 \]
\[ B = b^2 \]
\[ C = a^2 \]
\[ D = 0 \]  

(B.9.14)

With above substitutions (B.9.13) becomes

\[ - \frac{(c^2 - a^2)^{1/2}}{2} \int_{B}^{A} \left[ \frac{t - D}{(t - A)(t - B)(t - C)} \right]^{1/2} \]  

(B.9.15)

From 256.13 [52], (B.9.15) takes the value

\[ - \frac{(c^2 - a^2)^{1/2}}{2} \cdot (B - D) \int_{0}^{u_1} \frac{dn^2 u}{1 - a^2 sn^2 u} \ du \]  

(B.9.16)

using recurrence formulae 339.01 [52] and carrying out the algebraic manipulations, the value of the second term in (B.9.4) is given by

\[ - \frac{a^2}{b} K(\kappa) = \frac{b^2 - a^2}{b} \Pi(a^2, \kappa) \]  

(B.9.17)

and the value of the integral \( I_g \) is readily obtained, giving

\[ I_g = C K(\kappa) - \frac{a^2}{b} K(\kappa) - \frac{b^2 - a^2}{b} \Pi(a^2, \kappa) \]  

(B.9.18)

where

\[ \kappa = \left( \frac{c^2 - b^2}{c^2 - a^2} \right)^{1/2} \]  

(B.9.19)

and

\[ \alpha^2 = \frac{c^2 - b^2}{c^2 - a^2} \]  

(B.9.20)

* * *
where $E$ is an incomplete elliptic integral of the second kind whose argument $\phi$ is a function of the variable of integration $y$ and

$$\phi(y) = \sin^{-1} \left[ \frac{(c^2-b^2)(y^2-a^2)}{(c^2-a^2)(y^2-b^2)} \right]$$ (B.10.2)

and

$$k = \left( \frac{c^2-b^2}{c^2-a^2} \right)^{1/2}$$ (B.10.3)

An integration of (B.10.1) by parts leads to

$$I_{10} = I_{10A} - I_{10B}$$ (B.10.4)

where

$$I_{10A} = y E(\phi(y)) \bigg|_{y=b}^{y=c}$$ (B.10.5)

$$= c E(k)$$ (B.10.6)

$$I_{10B} = \int_{b}^{c} y \left\{ \frac{d}{dy} E(y) \right\} dy$$ (B.10.7)

where

$$E(y) = E(\phi(y),k)$$ (B.10.8)

Denoting $E'$ by

$$E' = \frac{dE}{dy} = \frac{dE}{d\phi} \cdot \frac{d\phi}{dy}$$ (B.10.9)
By definition (A.1.3)

\[ E = \int_0^\phi (1-k^2 \sin^2 \phi)^{1/2} \, d\phi \]  
(B.10.10)

\[ \frac{dE}{d\phi} = (1-k^2 \sin^2 \phi)^{1/2} \]  
(B.10.11)

and \( \frac{d\phi}{dy} \) is given by (B.9.11), which when substituted in (B.10.9) yields the following value of \( E' \):

\[ E'(y) = \frac{(b^2-a^2)(c^2-a^2)^{1/2} y}{(y^2-a^2)[(y^2-a^2)(y^2-b^2)(c^2-y^2)]^{1/2}} \]  
(B.10.12)

A substitution of (B.10.12) in (B.10.7) gives the following value:

\[ I_{10B} = \int_b^c \frac{(b^2-a^2)(c^2-a^2)^{1/2} y^2}{(y^2-a^2)[(y^2-a^2)(y^2-b^2)(c^2-y^2)]^{1/2}} \, dy \]  
(B.10.13)

and with substitutions (B.9.14) becomes

\[ I_{10B} = \frac{1}{2} (B^2-a^2)(c^2-a^2)^{1/2} \int_B^A \left[ \frac{t-D}{(A-t)(t-B)(t-C)} \right]^{1/2} \, \frac{dt}{t-C} \]  
(B.10.14)

From 256.01 [52]

\[ I_{10B} = \frac{1}{2} (B^2-a^2)(c^2-a^2)^{1/2} \cdot \frac{B-D}{B-C} \cdot \int_0^u \, dn^2 u \, du \]  
(B.10.15)

which simplifies to

\[ I_{10B} = b \, E(k) \]  
(B.10.16)

A combination of (B.10.6) and (B.10.16) leads to

\[ I_{10} = c \, E(k) - b \, E(\overline{k}) \]  
(B.10.17)
where
\[ \kappa = \left[ \frac{c^2 - b^2}{c^2 - a^2} \cdot \frac{a^2}{b^2} \right]^{1/2} \]  
(B.10.18)

\* \* \* \*

B.11.

\[ I_{11} = \int_x^c E(\phi(u), \kappa) \, du \]  
(B.11.1)

where
\[ \phi(u) = \sin^{-1} \left[ \frac{(c^2 - a^2)(u^2 - b^2)}{(c^2 - b^2)(u^2 - a^2)} \right]^{1/2} \]  
(B.11.2)

and
\[ \kappa = \left[ \frac{c^2 - b^2}{c^2 - a^2} \cdot \frac{a^2}{b^2} \right]^{1/2} \]  
(B.11.3)

Integrating by parts and writing

\[ I_{11} = I_{11A} - I_{11B} \]  
(B.11.4)

where
\[ I_{11A} = u \, E(u) \bigg|_{u=c}^{u=x} \]  
(B.11.5)

\[ = c \, E(\kappa) - x \, E(\phi_1(x), \kappa) \]  
(B.11.6)

and
\[ \sin \phi_1(x) = \left[ \frac{c^2 - a^2}{c^2 - b^2} \cdot \frac{x^2 - b^2}{x^2 - a^2} \right]^{1/2} \]  
(B.11.7)
and the integral \( B_{11B} \) in (B.11.4) is given by
\[
\int_{x}^{c} u \left\{ \frac{d}{du} E(u) \right\} du \quad (B.11.8)
\]

Since
\[
E'(u) = \frac{d}{du} \{ E(u) \} = \frac{dE}{d\phi} \cdot \frac{d\phi}{du} \quad (B.11.9)
\]

and
\[
\frac{dE}{d\phi} = [1 - k^2 \sin^2 \phi]^{1/2} \quad (B.11.10)
\]

with (B.11.2) and (B.11.3), we find
\[
\frac{dE}{d\phi} = \left[ \frac{(b^2 - a^2) u^2}{b^2 (u^2 - a^2)} \right]^{1/2} \quad (B.11.11)
\]

and
\[
\frac{d\phi}{du} = \left[ (c^2 - a^2)(b^2 - a^2) \right]^{1/2} \cdot \frac{u}{(u^2 - a^2)[(c^2 - u^2)(u^2 - b^2)]^{1/2}} \quad (B.11.12)
\]

so that
\[
E'(u) = \frac{b^2 - a^2}{b} \cdot (c^2 - a^2)^{1/2} \cdot \frac{u^2}{(u^2 - a^2)[(c^2 - u^2)(u^2 - a^2)(u^2 - b^2)]^{1/2}} \quad (B.11.13)
\]

and
\[
I_{11B} = (b^2 - a^2) \left( \frac{c^2 - a^2}{b^2} \right)^{1/2} \int_{x}^{c} \frac{u^3}{(y^2 - a^2)[(c^2 - u^2)(u^2 - a^2)(u^2 - b^2)]^{1/2}} \frac{du}{du} \quad (B.11.14)
\]

With the identity
\[
\frac{u^3}{u^2-a^2} = u \left[ 1 + \frac{a^2}{u^2-a^2} \right] \tag{B.11.15}
\]

I_{11B} given by (B.11.14) is expressed as a sum of two integrals, i.e.,

\[
I_{11B} = I_{11B1} + I_{11B2} \tag{B.11.16}
\]

where

\[
I_{11B1} = \frac{(b^2-a^2)}{b} \left( \frac{c^2-a^2}{2} \right)^{1/2} \int_{x}^{c} \frac{u \, du}{[(c^2-u^2)(u^2-a^2)(u^2-b^2)]^{1/2}} \tag{B.11.17}
\]

and

\[
I_{11B2} = \frac{(b^2-a^2)a^2(c^2-a^2)^{1/2}}{b} \int_{x}^{c} \frac{u \, du}{(u^2-a^2)[(c^2-u^2)(u^2-a^2)(u^2-b^2)]^{1/2}} \tag{B.11.18}
\]

For \(c > x > b\) and a change of variable \(t = u^2\) it can be shown that the integral \(I_{11B1}\) is of the type 236.00 [52] thus

\[
I_{11B1} = \frac{b^2-a^2}{b} \phi(x), k, \tag{B.11.19}
\]

where \(\phi\) is an incomplete elliptic integral with an argument

\[
\phi(x) = \sin^{-1} \left( \frac{c^2-x^2}{c^2-b^2} \right)^{1/2} \tag{B.11.20}
\]

To evaluate \(I_{11B2}\), the following substitutions are made:

\[
y = x^2
\]

\[
t = u^2
\]
\[ A = c^2 \]
\[ B = b^2 \]
\[ C = a^2 \]  \hspace{1cm} (B.11.21)

so that \( I_{11B2} \) becomes
\[
\frac{(b^2-a^2)a^2(c^2-a^2)^{1/2}}{2b} \int_{y}^{A} \frac{dt}{(t-C)((A-t)(t-B)(t-C))^{1/2}}
\]  \hspace{1cm} (B.11.22)

and the integral no. 236.10 [52] gives its value
\[
I_{11B2} = \frac{a^2(b^2-a^2)(c^2-a^2)^{1/2}}{2b} \cdot \frac{g_{A-C}}{A-C} \int_{0}^{u_1} \text{nd}^2 u \, du
\]  \hspace{1cm} (B.11.23)

From recurrence formulae 315.02 [52]
\[
\int_{0}^{u_1} \text{nd}^2 u \, du = \frac{1}{k^2} [E(u) - k^2 \, \text{sn} u \, \text{cd} u]
\]  \hspace{1cm} (B.11.24)

where the Jacobian elliptic function
\[
\text{cd} u = \text{cn} u / \text{dn} u
\]  \hspace{1cm} (B.11.25)

From the relations (A.1.5) through (A.1.7)
\[
\text{sn} u = \frac{c^2-x^2}{x^2-b^2}^{1/2}
\]  \hspace{1cm} (B.11.26)

and
\[
\text{cd} u = \left[ \frac{(x^2-b^2)(c^2-a^2)}{(x^2-a^2)(c^2-b^2)} \right]^{1/2}
\]  \hspace{1cm} (B.11.27)

with which (B.11.24) is given by
\[
\int_0^{u_1} n d^2 u \, du = \frac{c^2 - a^2}{b^2 - a^2} \left[ E(u) - \frac{(c^2 - x^2)(c^2 - b^2)}{(x^2 - a^2)(c^2 - a^2)} \right]^{1/2} \quad (B.11.28)
\]

and from (B.11.23)
\[
I_{11B2} = \frac{a^2}{b} E(u) - \frac{2}{b^2} \left[ \frac{(c^2 - x^2)(c^2 - b^2)}{(x^2 - a^2)(c^2 - a^2)} \right]^{1/2} \quad (B.11.29)
\]

Combining (B.11.19) and (B.11.29)
\[
I_{11B} = \frac{b^2 - a^2}{b} F\{\phi(x),k\} + \frac{a^2}{b} E\{\phi(x),k\}
\]
\[- \frac{a^2}{b^2} \left[ \frac{(c^2 - x^2)(c^2 - b^2)}{(x^2 - a^2)(c^2 - a^2)} \right]^{1/2} \quad (B.11.30)
\]

Finally the value of the integral \(I_{11}\) from (B.11.6) and (B.11.30) is shown to be
\[
I_{11} = c \, E(k) - \pi \, E\{\phi_1(x),k\} - \frac{b^2 - a^2}{b} F\{\phi(x),k\}
\]
\[- \frac{a^2}{b} E\{\phi(x),k\} + \frac{a^2}{b} \left[ \frac{(c^2 - x^2)(c^2 - b^2)}{(c^2 - a^2)(c^2 - a^2)} \right]^{1/2} \quad (B.11.31)
\]

**C.12.**

\[
I_{12} = \int_x^c F\{\phi(u),\bar{k}\} \, du \quad (B.12.1)
\]

where \(\phi(u)\) and \(\bar{k}\) are defined by (B.11.2) and (B.11.3).

Integrating by parts and writing
\[
I_{12} = I_{12A} - I_{12B} \quad (B.12.2)
\]
where

\[ I_{12A} = \int_{u=x}^{u=c} F(u) \, du = c \, K(k) - x \, F(\phi_1(x), k) \]  \hspace{1cm} (B.12.3)

with \( \phi_1(x) \) given by (B.11.7), and

\[ I_{12B} = \int_{x}^{c} u \, F'(u) \, du \]  \hspace{1cm} (B.12.4)

since

\[ F'(u) = \frac{dF}{d\phi} \cdot \frac{d\phi}{du} \]  \hspace{1cm} (B.12.5)

and

\[ \frac{dF}{d\phi} = \frac{1}{(1-k^2 \sin^2 \phi)^{1/2}} = \left[ \frac{b^2 \left( \frac{u^2-a^2}{u^2-b^2} \right)}{u^2 \left( \frac{b^2-a^2}{u^2-b^2} \right)} \right]^{1/2} \]  \hspace{1cm} (B.12.6)

using (B.11.12) for \( \phi'(u) \) it can be shown that

\[ F'(u) = \left[ \frac{b^2 \left( \frac{c^2-a^2}{u^2-a^2} \right)}{(u^2-a^2)(u^2-b^2)(c^2-u^2)} \right]^{1/2} \]  \hspace{1cm} (B.12.7)

and

\[ I_{12B} = b(c^2-a^2)^{1/2} \int_{x}^{c} u \, du \]  \hspace{1cm} (B.12.8)

with substitutions (B.11.21), the above integral may be written

\[ I_{12B} = \frac{b(c^2-a^2)^{1/2}}{2} \int_{y}^{A} \frac{dt}{((A-t)(t-B)(t-C))^{1/2}} \]  \hspace{1cm} (B.12.9)

a value of which is given by 236.00 [57] so that
\[ I_{12B} = b F(\phi(x),k) \]  

(B.12.10)

Having obtained \( I_{12A} \) (B.12.3) and \( I_{12B} \) (B.12.10), the value of the integral \( I_{12} \) is obtained from (B.12.2).

\[ I_{12} = c K(k) - x F(\phi_1(x), k) - b F(\phi(x), k) \]  

(B.12.11)
APPENDIX - C
A LIST OF SELECTED INTEGRALS AND FUNCTIONS

1. \( \int_{0}^{\infty} e^{-\xi} \sin \xi t \sin \xi x \, d\xi \)
   \[ = \frac{1}{2} \log \left| \frac{t+x}{t-x} \right| \]  
   \[ \text{[53]} \]

2. \( \int_{0}^{\infty} e^{-\xi} \sin \xi t \cos \xi x \, d\xi \)
   \[ = 0 \quad x > t \]
   \[ = \frac{\pi}{2} \quad x < t < 0 \]
   \[ = \frac{\pi}{4} \quad x = t \]  
   \[ \text{[53]} \]

3. \( \int_{0}^{\infty} e^{-\xi} \cos \xi t (1-\cos \xi x) \, d\xi \)
   \[ = \frac{1}{2} \log \left| \frac{t^2-x^2}{t^2} \right| \]
   \[ \text{[53]} \]
4. \[ \int_0^\infty J_0(\xi t) \sin \xi x \, d\xi \]

\[= (x^2-t^2)^{-1/2} \quad x > t > 0 \]

\[= 0 \quad t > x \] \[53\]

5. \[ \int_0^\infty J_0(\xi t) \cos \xi x \, d\xi \]

\[= 0 \quad x > t > 0 \]

\[= (t^2-x^2)^{-1/2} \quad t > x \] \[53\]

6. \[ \int_0^a \frac{(a^2-s^2)^{-1/2}}{x^2-s^2} \, ds \]

\[= \frac{\pi}{2} \left[ 1 - \left\{ \frac{x^2-a^2}{x^2} \right\}^{1/2} \right] \] \[54\]

7. \[ \int_a^b \frac{t \, dt}{\sqrt{(t^2-a^2)(b^2-t^2)}} = \frac{\pi}{2} \] \[53\]

8. \[ \int_x^b \frac{dt}{\sqrt{t^2(t^2-a^2)(t^2-b^2)}} \]

\[= \frac{1}{ab} \tan^{-1} \left[ \frac{a^2(b^2-x^2)}{b^2(x^2-a^2)} \right]^{1/2} \]

for \( x > a \) \[53\]
9. \[ \int_{s}^{a} \frac{t \, dt}{(t^2 - s^2)^{1/2}(u^2 - t^2)^{3/2}} \]

\[ = \frac{1}{u^2 - s^2} \cdot \frac{1}{u^2 - s^2} \quad [51] \]

10. \[ \int_{a}^{b} \frac{t \, dt}{a \left[ (t^2 - a^2)(b^2 - t^2) \right]^{1/2} : \frac{1}{t^2 - y^2}} \]

\[ = \pi \frac{1}{2} \left[ (a^2 - y^2)(b^2 - y^2) \right]^{1/2} , \quad y < a \]

\[ = 0 \quad a < y < b \]

\[ = \frac{\pi}{2[ (y^2 - a^2)(y^2 - b^2) ]^{1/2}} , \quad y > b \quad [46] \]

11. \[ \int_{0}^{a} \frac{t \, dt}{(y^2 - t^2)^{1/2}(x^2 - t^2)^{3/2}} \]

\[ = \frac{1}{y^2 - x^2} \left[ \left( \frac{y^2 - a^2}{x^2 - a^2} \right)^{1/2} - \frac{y}{x} \right] \quad [53] \]

12. \[ \int_{b}^{c} \left( \frac{c^2 - 2}{y^2 - b^2} \right)^{1/2} y \, dy \]

\[ = - \frac{b^2 \pi}{2} \quad [53] \]
13. \[ \int_{b}^{c} \left( \frac{y^2 - b^2}{c^2 - y^2} \right)^{1/2} y \, dy \]
\[ = \frac{c^2 \pi}{2} \]
\[ * * * \]

14. \[ \int_{c}^{b} \frac{dt}{[(t^2 - a^2)(b^2 - t^2)]^{1/2}} \]
\[ = K(k) \]
\[ \text{with } k = \frac{(b^2 - a^2)^{1/2}}{b} \]
\[ [52] \]

15. \[ \int_{a}^{b} \frac{t^2 \, dt}{[(t^2 - a^2)(b^2 - t^2)]^{1/2}} \]
\[ = b \, E(k) \]
\[ \text{with } k = \frac{(b^2 - a^2)^{1/2}}{b} \]
\[ [52] \]

16. \[ \int_{a}^{b} \log \left[ \frac{t+x}{t-x} \right] \frac{dt}{[(t^2 - a^2)(b^2 - t^2)]^{1/2}} \]
\[ = \frac{F(\phi_1, k)}{b} \quad 0 < y < a \]
\[ = \frac{\pi}{b} \, K(k) \quad a < y < b \]
\[ = \frac{\pi}{b} \, F(\phi_2, k) \quad y \geq b \]

where
\[ \phi_1 = \sin^{-1} \frac{y}{a} \]
\[ \phi_2 = \sin^{-1} \frac{b}{y} \]

\[ k = \frac{a}{b} \]

17. \[ \int_c^b \frac{dt}{\sqrt{(a-t)(t-b)(t-c)}} \]

\[ = \frac{2}{(a-c)^{1/2}} K(k) \]

where \( k^2 = \frac{a-b}{a-c} \), \( a > b > c \)

18. \[ \int_y^a \left[ \frac{t-b}{(a-t)(t-c)} \right]^{1/2} dt \]

\[ = \frac{2}{k^2(a-c)^{1/2}} \left[ E(u) - k^2 u \right] \]

where \( \phi = \sin^{-1} \left( \frac{a-y}{y-b} \right)^{1/2} \)

\[ k^2 = \frac{a-b}{a-c} \]

19. \[ \int_y^a \frac{dt}{\sqrt{(t-p)[(a-t)(t-b)(t-c)]}} \]

\[ a > y > b > c \text{ and } p \neq a \]

\[ = \frac{2}{(a-c)^{1/2}} \cdot \pi(\phi, \alpha^2, k) \]

where
\[ \phi = \sin^{-1} \left( \frac{a-y}{y-b} \right)^{1/2} \]

\[ \lambda^2 = \frac{a-b}{a-p} \]

\[ k^2 = \frac{a-b}{a-c} \]  \[52\]

20. \[ \int_{b}^{y} \frac{dt}{(t-p)[(a-t)(t-b)(t-c)]^{1/2}} \]

with \( a > y > b > c \) and \( p \neq b \)

\[ = \frac{2[k^2 \mu + (\alpha^2 - k^2) \pi(\phi, \alpha^2, k)]}{(b-p)\alpha^2(a-c)^{1/2}} \]

where

\[ \phi = \sin^{-1} \left( \frac{(a-c)(y-b)}{(a-b)(y-c)} \right)^{1/2} \]

\[ \alpha^2 = \frac{(a-b)(c-p)}{(a-c)(b-p)} \]

\[ k^2 = \frac{a-b}{a-c} \]  \[52\]

21. \[ \int_{y}^{a} \frac{dt}{[(a-t-b)(t-c)(t-d)]^{1/2}} \]

with \( a > y > b > c > d \)

\[ = \frac{2F(\phi, k)}{[(a-c)(b-d)]^{1/2}} \]
where

\[ \phi = \sin^{-1} \left[ \frac{(b-d)(a-y)}{(a-b)(y-d)} \right]^{1/2} \]

\[ k^2 = \frac{(a-b)(c-d)}{(a-c)(b-d)} \]

For integrals 22 through 26, the argument, the parameter and the modulus are defined as follows:

\[ \phi = \sin^{-1} \left[ \frac{(a-c)(y-b)}{(a-b)(y-c)} \right]^{1/2} \]

\[ \alpha^2 = \frac{a-b}{a-c} \]

\[ k^2 = \frac{(a-b)(c-d)}{(a-c)(b-d)} \]

and \( a > y > b > c > d \)

22. \[ \int_b^y \frac{dt}{\sqrt{[(a-t)(t-b)(t-c)(t-d)]^{1/2}}} \]
   \[ = \frac{2}{\sqrt{(a-c)(b-d)}} F(\phi, k) \]  \[\text{[52]}\]

23. \[ \int_b^y \left[ \frac{t-c}{(a-t)(t-b)(t-d)} \right]^{1/2} dt \]
   \[ = \frac{2(b-c)}{\sqrt{(a-c)(b-d)}} \pi(\phi, \alpha^2, k) \]  \[\text{[52]}\]
24. $\int_{y}^{y} \left[ \frac{a-t}{(t-b)(t-c)(t-d)} \right]^{1/2} dt$

\[= \frac{2(a-b)}{(a-c)(b-d)]^{1/2}} \cdot \frac{1}{a^2} \left[ u+(a^2-1)^{\frac{1}{2}}(\phi, a^2, k) \right] \tag{52} \]

25. $\int_{b}^{y} \left[ \frac{(t-c)(t-b)}{(a-t)(a-d)} \right]^{1/2} dt$

\[= \frac{2(b-c)^2}{(a-c)(b-d)]^{1/2}} \cdot \frac{1}{2a^2(a^2-1)(k^2-a^2)} \left[ a^2 E(u)+(k^2-a^2)u+(a^4-k^2) \pi(u, a^2) - \frac{\alpha^4 sn u cn u dn u}{1-a^2 sn^2 u} \right] \tag{52} \]

26. $\int_{b}^{y} \frac{dt}{(t-p)[(a-t)(t-b)(t-c)(t-d)]^{1/2}}$

\[= \frac{2}{(b-p)[(a-c)(b-d)]^{1/2}} \cdot \frac{1}{a^3} \left[ a^2 u+(a^2-\alpha^2)^{\frac{1}{2}}(\phi, a^2, k) \right] \]

where

\[\alpha^2 = \frac{(p-c)(a-b)}{(p-b)(a-c)} \tag{52}\]

\[p \neq b \]

\[* \quad * \quad * \quad * \]

For the integrals 27 through 39, the parameter and the modulus are defined as follows:
\[ \alpha^2 = \frac{c^2-b^2}{c^2-a^2} \]

\[ k^2 = \alpha^2 \]

\[ \bar{k} = \left[ \frac{a^2 (c^2-b^2)}{b^2 (c^2-a^2)} \right]^{1/2} \]

27. \[ \int_{b}^{c} \frac{t \, dt}{[(t^2-a^2)(t^2-b^2)(c^2-t^2)]^{1/2}} \]

\[ = \frac{2 \cdot K(k)}{(c^2-a^2)^{1/2}} \quad [52] \]

28. \[ \int_{b}^{c} \left[ \frac{y^2-a^2}{(y^2-b^2)(c^2-y^2)} \right]^{1/2} \, dy \]

\[ = \frac{b^2-a^2}{b(c^2-a^2)^{1/2}} \cdot \pi(\alpha^2, k) \quad (B.1) \]

29. \[ \int_{b}^{c} \left[ \frac{c^2-y^2}{(y^2-a^2)(y^2-b^2)} \right]^{1/2} \, dy \]

\[ = \frac{(c^2-a^2)^{1/2}}{b} \left[ k(k) - \frac{b^2-a^2}{c^2-a^2} \pi(\alpha^2, k) \right] \quad (B.2) \]
30. \[ \int_{y}^{c} \left[ \frac{(y^2-a^2)(y^2-b^2)}{c^2-y^2} \right]^{1/2} dy \]

\[ = \frac{(c^2-a^2)^{1/2}}{2b} \left[ (c^2-a^2)E(k) + \frac{b^2-a^2}{b^2} (c^2-a^2)K(k) \right. \]

\[ + \frac{b^2-a^2}{b^2} (c^2-b^2-a^2)\pi(a^2,k) \]  \hspace{1cm} (B.3)

31. \[ \int_{y}^{c} \left[ \frac{(y^2-a^2)(c^2-y^2)}{y^2-b^2} \right]^{1/2} dy \]

\[ = \frac{c^2(c^2-a^2)^{1/2}}{2b} \left[ \frac{c^2+a^2}{b^2} K(k) + \frac{c^2-a^2}{a^2-b^2} E(k) \right. \]

\[ + \frac{b^2-c^2-a^2}{b^2} (c^2-a^2)\pi(a^2,k) \]  \hspace{1cm} (B.4)

32. \[ \int_{b}^{c} \left[ \frac{u^2(u^2-b^2)}{u^2-a^2(u^2-b^2)} \right]^{1/2} du \frac{du}{y^2-u^2} \]

\[ = \frac{b^2-a^2}{(a^2-y^2)(c^2-a^2)^{1/2}} \left[ K(k) - \pi(a^2_y,k) \right] \]

where \[ a^2_y = \frac{(c^2-b^2)(a^2-y^2)}{(c^2-a^2)(b^2-y^2)} \]  \hspace{1cm} (B.5)
33. \[
\int_{c}^{b} \left[ \frac{u^2 (c^2-u^2)}{(u^2-a^2)(u^2-b^2)} \right]^{1/2} \frac{du}{y^2-u^2}
\]

= \frac{K(k) - \pi(\alpha_y^2, k)}{(c^2-a^2)^{1/2}}

where

\[
\alpha_y^2 = \frac{c^2-b^2}{c^2-y^2}
\] (B.6)

34. \[
\int_{c}^{b} \left[ \frac{y^2}{(y^2-b^2)} \right]^{1/2} y dy \int_{b}^{c} \left[ \frac{u^2 (u^2-b^2)}{(u^2-a^2)(c^2-u^2)} \right]^{1/2} \frac{du}{y^2-u^2}
\]

= \frac{\pi}{2} (c^2-a^2)^{1/2} \left[ \frac{b^2-a^2}{c^2-a^2} K(k) - E(k) \right]
\] (B.7)

35. \[
\int_{c}^{b} \left[ \frac{u^2 (c^2-u^2)}{(u^2-a^2)(u^2-b^2)} \right]^{1/2} du \int_{b}^{c} \left[ \frac{y^2-b^2}{c^2-y^2} \right]^{1/2} \frac{dy}{y^2-u^2}
\]

= \frac{\pi}{2} (c^2-a^2)^{1/2} [K(k) - E(k)]
\] (B.8)

36. \[
\int_{b}^{c} F[\phi(y), k] dy
\]

where

\[
\phi(y) = \sin^{-1} \left[ \frac{(c^2-b^2)(y^2-a^2)}{(c^2-a^2)(y^2-b^2)} \right]^{1/2} = CK(k) - \frac{a^2}{b} K(k) - \frac{b^2-a^2}{b} \pi(\alpha_y^2, k)
\] (B.9)
37. \[ \int_b^c E[\phi(y),k] \, dy \]

where

\[ \phi(y) = \sin^{-1} \left[ \frac{(c^2-a^2)(y^2-b^2)}{(c^2-b^2)(y^2-a^2)} \right]^{1/2} = c \, E(k) - bE(\overline{k}) \quad (B.10) \]

38. \[ \int_x^c E[\phi(u),\overline{k}] \, du \]

where

\[ \phi(u) = \sin^{-1} \left[ \frac{(c^2-a^2)(u^2-b^2)}{(c^2-b^2)(u^2-a^2)} \right]^{1/2} \]

\[ = cE(\overline{k}) - xE(\phi_2,\overline{k}) - \frac{a^2}{b} \, E(\phi_1,k) \]

\[ - \frac{b^2-a^2}{b} \, F(\phi_1,k) + \frac{a^2}{b} \left[ \frac{(c^2-x^2)(x^2-b^2)}{(c^2-a^2)(x^2-a^2)} \right]^{1/2} \]

where

\[ \phi_1 = \sin^{-1} \left( \frac{c^2-x^2}{c^2-b^2} \right)^{1/2} \]

\[ \phi_2 = \sin^{-1} \left[ \frac{(c^2-a^2)(x^2-b^2)}{(c^2-b^2)(x^2-a^2)} \right]^{1/2} \quad (B.11) \]

39. \[ \int_x^c F[\phi(u),\overline{k}] \, du \]
where

\[ \phi(u) = \sin^{-1} \left[ \frac{(u^2 - b^2)(c^2 - a^2)}{(u^2 - a^2)(c^2 - b^2)} \right]^{1/2} \]

\[ = cK(k) - x F(\phi_1, \overline{k}) - bF(\phi_2, k) \]

where

\[ \phi_1 = \sin^{-1} \left[ \frac{(c^2 - a^2)(x^2 - b^2)}{(c^2 - b^2)(x^2 - a^2)} \right]^{1/2} \]

\[ \phi_2 = \sin^{-1} \left( \frac{c^2 - x^2}{c^2 - b^2} \right)^{1/2} \]

(B.12)

* * *

\[ \left[ \frac{(t^2 - a^2)(b^2 - y^2)}{(b^2 - t^2)(y^2 - a^2)} \right]^{1/2} \left[ 1 + \frac{y^2 - t^2}{t^2 - a^2} \right] \]

\[ = \left[ \frac{(b^2 - t^2)(y^2 - a^2)}{(t^2 - a^2)(b^2 - y^2)} \right]^{1/2} \left[ 1 - \frac{y^2 - t^2}{b^2 - t^2} \right] \]

41. \[ E(k)k(k') + k(k)E(k') - K(k)K(k') = \frac{\pi}{2} \] [52]

42. \[ (1 - \alpha^2)(k^2 - \alpha^2)\pi(\alpha^2, k) + \alpha^2 k^2 \left( \frac{k^2 - \alpha^2}{1 - \alpha^2}, k \right) \]

\[ = k^2(1 - \alpha^2)K(k) \] [52]
\begin{align*}
43. & \quad \frac{2}{b(c^2-a^2)^{1/2}} \left[ (a^2-c^2)k(k) + (b^2-a^2)\pi(a^2,k) \right. \\
& \quad \left. + c^2 \pi \left( \frac{-a^2}{c^2-a^2},k \right) \right] = \pi \\
44. & \quad \frac{c^2(b^2-a^2)}{b^2[(c^2-a^2)(c^2-b^2)]^{1/2}} K(k) - \frac{b}{(c^2-b^2)^{1/2}} E(k) \\
& \quad = \frac{c(b^2-a^2)}{(c^2-a^2)(c^2-b^2)^{1/2}} K(k) - \frac{c}{(c^2-b^2)^{1/2}} E(k) \\
45. & \quad \pi(a^2,k) = \frac{-\pi KZ(A,k)}{[(\alpha^2-1)(\alpha^2-k^2)]^{1/2}} \\
\text{where} \\
& \quad A = \sin^{-1} \frac{1}{\alpha} \\
\text{and} \\
& \quad \infty > \alpha^2 > 1 \quad \text{[52]} \\
46. & \quad KZ(A,k) = KE(A,k) - EF(A,k) \quad \text{[52]} \\
47. & \quad \pi(a^2,k) = \frac{K}{1-\alpha^2} + \frac{\alpha^2[L_\alpha(\theta,k)-1]}{2\left[\alpha^2(1-\alpha^2)(\alpha^2-k^2)\right]^{1/2}} \\
\text{where}
\[ \beta = \sin^{-1} \frac{1}{(1-\alpha^2)^{1/2}} \]

and

\[ 0 < -\alpha^2 < \infty \]  \hspace{1cm} [52]

48. \[ \pi(\alpha^2, k) = \frac{\alpha \pi \Lambda_0(\xi, k)}{2[(\alpha^2-k^2)(1-\alpha^2)]^{1/2}} \]

where

\[ \xi = \sin^{-1} \left[ \frac{2-k^2}{\alpha^2} \right]^{1/2} \]

and

\[ k^2 < \alpha^2 < 1 \]  \hspace{1cm} [52]

49. \[ \Lambda_0(\beta, k) = \frac{2}{\pi} [EF(\beta, k') + KE(\beta, k') - KF(\beta, k')] \]  \hspace{1cm} [52]

\[ \ast \hspace{1cm} \ast \hspace{1cm} \ast \hspace{1cm} \ast \]

50. \[ \lim_{k \to 1} \left[ K(k) - \log \frac{4}{k'} \right] = 0 \]  \hspace{1cm} [52]

51. \[ \lim_{k \to 0} \left[ \frac{K(k) - E(k)}{k^2} \right] = \frac{\pi}{4} \]  \hspace{1cm} [52]

52. \[ \lim_{k \to 0} [E(k) - K(k)] K(k') = 0 \]  \hspace{1cm} [52]
53. \[ \int_{0}^{x} \frac{f(t) \, dt}{(x^2 - t^2)^{1/2}} = g(x), \quad a < c < b \]

\[ f(t) = \frac{2}{\pi} \frac{d}{dt} \int_{a}^{t} \frac{u \cdot g(u) \, du}{(t^2 - u^2)^{1/2}}, \quad a < t < b \] [43]

54. \[ \frac{b}{x} \frac{f(t) \, dt}{(t^2 - x^2)^{1/2}} = g(x), \quad a < b \]

\[ f(t) = -\frac{2}{\pi} \cdot \frac{d}{dt} \int_{t}^{b} \frac{u \cdot g(u) \, du}{(u^2 - t^2)^{1/2}}, \quad a < t < b \] [43]