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ON THE NONEXISTENCE OF PERFECT $E$-CODES AND TIGHT $2E$-DESIGNS IN HAMMING SCHEMES $H(N,Q)$ WITH $E$ GREATER THAN OR EQUAL TO 3 AND $Q$ GREATER THAN OR EQUAL TO 3

The Ohio State University

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ON THE NONEXISTENCE OF PERFECT $e$-CODES
AND TIGHT $2e$-DESIGNS IN HAMMING SCHEMES
$H(n,q)$ WITH $e > 3$ AND $q > 3$

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By

Yiming Hong, B.S.

* * * * *

The Ohio State University

1984

Reading Committee: Approved By

Eiichi Bannai

Bogden Baishanski

Thomas Dowling

Eiichi Bannai

Adviser
Department of Mathematics
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VITA

1954 . . . . . . . . . . Born, Chaiyi City, Taiwan, R.O.C.

1972-1976 . . . . . . . . B.S., National Taiwan University, Taipei, Taiwan, R.O.C.

1979-1984 . . . . . . . . Graduate Teaching Associate, The Ohio State University, Columbus, Ohio

PUBLICATIONS

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The theory of codes (in Hamming schemes $H(n,q)$) and designs (in Johnson schemes $J(n,k)$) has been studied for a long time and from several angles. One of the major mathematical approaches is to study them through the framework of association schemes. Delsarte [5] is the first important paper in which many concepts of coding theory and design theory are generalized to classes of association schemes, in particular, schemes which are $P$-polynomial and/or $Q$-polynomial. By applying the method of linear programming, Delsarte obtained natural bounds on the sizes of codes in $P$-polynomial schemes and of designs in $Q$-polynomial schemes. Those reaching the bounds are called perfect codes and tight designs, respectively. In his Ph.D. thesis [5], Delsarte gave generalizations of the classical Lloyd theorem for binary perfect codes to some theorems for perfect codes in $P$-polynomial schemes and for tight designs in $Q$-polynomial schemes. We will simply call those theorems Lloyd's theorem and its dual in this dissertation (§1.1). Lloyd's theorem (its dual) says that if there exist perfect codes (tight designs) in a $P$-polynomial ($Q$-polynomial) scheme, then the set of the zeros of the Lloyd polynomial (Wilson polynomial) of the scheme must be contained in the set of the zeros of the sum polynomial associated with the scheme.

In this dissertation, we will use Lloyd's theorem and its dual, some results of Best [3], and some results and ideas of Bannai [1] to prove the nonexistence of unknown perfect $e$-codes and unknown tight
2e-designs (also understood as orthogonal arrays achieving Rao's bound) in the Hamming schemes $H(n,q)$, for $e > 3$ and $q > 3$ (here $q$ is not necessarily a prime power). We prove the nonexistence of perfect codes and tight designs in $H(n,q)$ in a unified way because of the coincidence that the Lloyd polynomial and the Wilson polynomial of each $H(n,q)$ are the same. Those polynomials are in fact some family of Krawtchouk polynomials. One important feature of this unified proof is that we do not use the sphere packing condition (which is usually used to show the nonexistence of perfect codes). We only assume Lloyd's theorem. Although the result is not new for the nonexistence of most of the perfect $e$-codes in $H(n,q)$ (cf. [3], [9], etc.), it is for perfect 6-codes and 8-codes, also for tight 2e-designs in $H(n,q)$ with $e > 3$ and $q > 3$.

We conclude the introduction by listing the main theorems of the dissertation as follows.

**THEOREM AB.** For $q > 3$ and $3 < e < n-1$, each Lloyd polynomial in $H(n,q)$ has at least one nonintegral zero.

**THEOREM A.** For $e > 3$ and $q$ arbitrary, the only perfect $e$-codes in $H(n,q)$ are the trivial codes (of one codeword), the binary repetition codes (of two codewords), and the binary Golay code.

**THEOREM B.** For $e > 3$ and $q > 3$, there exist no tight 2e-designs in $H(n,q)$.
Chapter I

HAMMING SCHEMES, PERFECT CODES, AND TIGHT DESIGNS

In the first chapter, we will define Hamming schemes, perfect e-codes, and tight 2e-designs through the language of association schemes, then list some important known results about them and the main theorems of the dissertation.

§1.1 Definitions and Lloyd's Theorem

The definitions and notations used in this section generally follow those of Bannai and Ito [2].

Let \( X \) be a nonempty set and \( \{ R_i \} \) be a partition of \( X \times X \) into \( d+1 \) classes. Then \( \mathcal{A} = (X, \{ R_i \} \) of class \( d \) if it satisfies the following properties.

(A1) \( R_0 = \{(x,x) \mid x \in X\} \).

(A2) For every \( R_i \), its transpose \( R_i^t = \{(y,x) \mid (x,y) \in R_i\} \) is equal to some \( R_j \).
(A3) For every pair \((x,y) \in R_k\), the number of \(z \in X\) such that \((x,z) \in R_i\) and \((z,y) \in R_j\) is a constant \(p_{ij}^k\) depending only on \(i, j, k\).

\(p_{ij}^k = p_{ji}^k\) for all \(i, j, k\).

An association scheme is called symmetric if it satisfies the additional property

\( (A2') \quad R_i^t = R_i \) for all \(i\).

REMARK: \((A2')\) implies \((A4)\).

The Hamming schemes \(H(n,q)\) are important examples of symmetric association schemes of class \(n\). In \(H(n,q)\), \(X = F^n\), the set of all \(n\)-tuples over a finite set \(F\) of cardinality \(q > 2\), and for any two \(n\)-tuples \(\hat{x}\) and \(\hat{y}\), \((\hat{x}, \hat{y}) \in R_i\) if and only if they differ at exactly \(i\) coordinates. In fact, the Hamming schemes are both \(P\)- and \(Q\)-polynomial schemes, which we will define in the next few paragraphs.

The adjacency matrix \(A_i\) with respect to \(R_i\) of an association scheme \(\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq d})\) is defined to be the \(|X| \times |X|\) matrix whose rows and columns are indexed by elements of \(X\) and whose \((x,y)\)-entry is

\[
A_i(x,y) = \begin{cases} 
1 & \text{if } (x,y) \in R_i, \\
0 & \text{otherwise.}
\end{cases}
\]

If \(\mathcal{X}\) is a symmetric association scheme, then the adjacency matrices \(A_0, A_1, \ldots, A_d\) are symmetric (by \((A2')\)) and they commute with each other (by \((A4)\)). Thus they span a (semisimple) \((d+1)\)-dimensional real
algebra admitting a unique basis of mutually orthogonal idempotent matrices \( E_0, E_1, \ldots, E_d \), with \( E_0 = \frac{1}{|X|} J \), where \( J \) is the all-one matrix. This algebra is called the Bose-Mesner algebra of the scheme.

Let \( p_i(j) \) be the eigenvalue of \( A_i \) associated with the eigenspace spanned by the columns of \( E_j \). Then we can write

\[
A_i = \sum_{j=0}^{d} p_i(j)E_j .
\]

The first eigenmatrix \( P \) of the scheme is the \((d+1) \times (d+1)\) real matrix whose \((j,i)\) entry is \( p_i(j) \). Similarly, by writing

\[
E_i = \frac{1}{|X|} \sum_{j=0}^{d} q_i(j)A_j ,
\]

we define the second eigenmatrix \( Q \) of the scheme to be the \((d+1) \times (d+1)\) matrix whose \((j,i)\) entry is \( q_i(j) \). The two eigenmatrices are related by

\[
PQ = QP = |X|I .
\]

Among the symmetric association schemes, there are two classes of schemes that stand out; namely, the P-polynomial schemes and the Q-polynomial schemes. They are important because they are the natural setting for studying codes and designs. We define them as follows.

A symmetric association scheme \( \mathcal{X} = (X, \{R_i\}_{i=0}^d) \) is called a P-polynomial scheme (with respect to the ordering of the \( A_i \)'s) if there exist polynomials \( v_i(x) \) of degree \( i \) \((0 \leq i \leq d)\) such that
the adjacency matrices $A_i = v_i(A_i)$ with respect to the usual matrix multiplication; this is the same as requiring that the eigenvalues $p_i(j) = v_i(p_i(j))$. Dually, a symmetric association scheme is called a Q-polynomial scheme (with respect to the ordering of the $E_i$'s) if there exist polynomials $v_i^*(x)$ of degree $i$ ($0 < i < d$) such that

\[ q_i(j) = v_i^*(q_i(j)). \]

The Hamming scheme $H(n,q)$ is both P- and Q-polynomial with

$P = Q$ and $v_i(x) = v_i^*(x) = K_i(x)$ where

\[ K_i(x) = \sum_{k=0}^{i} (-1)^{i-k} \binom{i}{k} \binom{n-x}{i-k} x^k \quad (0 < i < n) \]

is a Krawtchouk polynomial of degree $i$. (Theorem 4.2 in [5])

We are nearly ready to define perfect codes (in P-polynomial schemes) and tight designs (in Q-polynomial schemes). We first investigate subsets of a symmetric association scheme $(X, \{R_i\}_{i=0}^{d})$. Let $Y$ be a nonempty subset of $X$. The inner distribution of $Y$ is defined to be the $(d+1)$-tuple $\mathbf{a} = (a_0, a_1, \ldots, a_d)$, where

$\mathbf{a}_i = |Y^2 \cap R_i|/|Y|$ is the average number of points of $Y$ $i$th related to a point of $Y$. The outer distribution of $Y$ is defined to be the $|X| \times (d+1)$ matrix $B$ whose $(x,i)$-entry is given by $B(x,i) = |\{(x) \times Y \cap R_i\}$, the number of points of $Y$ $i$th related to the fixed point $x$ in $X$. The inner distribution and the outer distribution are related by the following formula. (Theorem 3.1, [5])
where $\Delta_\alpha$ is the diagonal matrix whose entries in the main diagonal are the components of $\alpha$. It follows that $\alpha$ has all its components (entries) non-negative. This fact suggests a linear programming approach to obtain bounds on the sizes of subsets $Y$ whose inner distributions satisfy certain conditions; for example, codes in $P$-polynomial schemes and designs in $Q$-polynomial schemes.

Let $Y$ be a nonempty subset of a $P$-polynomial scheme $X$. Define the minimum distance $f$ of $Y$ to be the index of the first nonzero component of its inner distribution $\alpha$ after $a_0$, and the external distance $r$ of $Y$ to be the number of nonzero components of $\alpha$, excluding the 0th component $(\alpha)_0$. Now, by applying linear programming techniques, we have the following bounds for $|Y|$.

(Thorem 5.14 and Theorem 5.5, [5])

$$
e \sum_{i=0}^{e} k_i \leq \frac{|X|}{|Y|} \leq \sum_{i=0}^{x} k_i$$

(in particular, $e \leq r$)

where $e = \left\lfloor \frac{f-1}{2} \right\rfloor$ and $k_i = p_{ii}^0$ is the valency of $R_i$. Moreover, if one of the two bounds is achieved, so is the other. A bound-achieving subset $Y$ is called a perfect $e$-code in $X$.

One of the most important theorems concerning the existence of perfect $e$-codes in a $P$-polynomial scheme $(X, \{R_i\}_{i=0}^{d})$ is the (generalized) Lloyd theorem. To state the theorem, we need to define
the Lloyd polynomials $F_i(x)$ first. Let $v_i(x)$ ($0 < i < d$) be the polynomials associated with the $P$-polynomial scheme $X$. The Lloyd polynomials $F_i(x)$ of $X$ are defined by

$$F_i(x) = \sum_{k=0}^{i} v_k(x) \quad (0 < i < d).$$

$F_i(x)$ is of degree $i$ and has $i$ distinct real roots [5].

(Generalized) Lloyd Theorem. (Theorem 5.14 [5]), [4], [8].

If there exists a perfect $e$-code in $X$, then the zeros of $F_e(x)$ are all contained in the zeros of $F_d(x)$.

Similarly, if $X$ is a $Q$-polynomial scheme and $Y$ a nonempty subset of $X$ with inner distribution $a^+$, we define the maximum strength $t$ of $Y$ to be one less than the index of the first nonzero components of $a_Q$ after $(a_Q)_0^+$, and the degree $s$ of $Y$ to be the number of nonzero components of $a^+$, excluding $a_0^+$. In this case, we also obtain linear programming bounds for $|Y|$ (Theorem 5.21, [5]):

$$\sum_{i=0}^{s} m_i < |Y| < \sum_{i=0}^{e} m_i$$

where $e = \lfloor t/2 \rfloor$ and $m_i$ are the multiplicities of the scheme, i.e., the ranks of the idempotent matrices $E_i$. If any one of the two bounds is reached, so is the other. A subset attaining these bounds is called a tight $2e$-design in $X$. In this case, $t = 2e = s$. (See the remark on p.77, [5].) There is also a dual theorem (to Lloyd's Theorem) for the existence of tight $2e$-designs in a $Q$-polynomial scheme $(X,\{R_i\}_{0 \leq i \leq d})$. (Theorem 5.21, [5]) It is as follows.
If there exists a tight 2e-design in \( X \), then the zeros of \( F_e^*(x) \) are contained in the zeros of \( F_d^*(x) \), where \( F_i^*(x) \) are the Wilson polynomials of \( X \) defined by

\[
F_i^*(x) = \sum_{k=0}^{i} v_i^*(x) \quad (0 < i < d)
\]

and \( v_i^*(x) \) are the polynomials associated with the Q-polynomial scheme \( X \). 

As we mentioned before, the Hamming schemes \( H(n,q) \) are both P- and Q-polynomial. Therefore, we can talk about perfect codes and tight designs in them. Perfect e-codes and tight 2e-designs in \( H(n,q) \) are also well-known combinatorial objects under several names, such as perfect (q-nary) e-error correcting codes (of length \( n \)) and (q-nary) orthogonal arrays (of length \( n \)) of strength 2e achieving Rao's bounds, respectively. We will discuss them in the next two sections.

§1.2 Perfect e-Codes in \( H(n,q) \).

A perfect e-code in \( H(n,q) \) is a subset \( C \) of n-tuples in \( H(n,q) \) such that, as \( c \) runs through \( C \), the collection of closed balls \( B_e^+(c) = \{ x \in X | d_H(x,c) < e \} \) forms a partition of \( X \). Here \( d_H(x,y) \) is the Hamming distance defined by

\[
d_H^+(x,y) = i \quad \text{if and only if} \quad (x,y) \in R_i \quad (0 < i < n).
\]
In coding theory, a perfect e-code in $H(n,q)$ is the most economical e-error-correcting code of length $n$ using $q$ letters. When $e = 1$, there are many examples of perfect codes. For example, the linear single-error-correcting Hamming codes are perfect, and so are many other nonlinear ones. (Cf. Chapter 6, §10 in [17].) Unfortunately when $e > 2$, very few perfect codes are known. Besides the trivial codes (of one codeword), only the following codes are known perfect.

(i) Binary repetition codes ($q = 2$, $n = 2e+1$, $e$ is arbitrary, $|C| = 2$),
(ii) Binary Golay code ($q = 2$, $n = 23$, $e = 3$), and
(iii) Ternary Golay code ($q = 3$, $n = 11$, $e = 2$).

Codes having the same parameters as (i), (ii), or (iii) are unique up to isomorphism. If $n < e$, a code is automatically trivial.

There are many papers concerning the nonexistence of perfect e-codes in $H(n,q)$. Here is a list of the major results.

No unknown perfect e-codes exist when:

(i) $q = p^s$ where $p$ is a prime, and $e > 2$, Tietavainen–vanLint (cf. [9], [14], [15]);

(ii) $q = p_1 p_2^t$, where $p_1$ and $p_2$ are distinct primes, and $e > 3$, Tietavainen [16];

(iii) $e = 3, 4, \text{ or } 5$, Reuvers [12];

(iv) $e = 7$ or $e > 9$, Best [3].

Also (v) for each $e > 3$ with $q > 3$ arbitrary, there are only finitely many nontrivial perfect e-codes, Bannai [1].
In this dissertation, we will prove the nonexistence of nontrivial perfect e-codes for \( i = 6 \) or 8, under the assumption that \( q \geq 30 \). (§3.1, §3.3) Thus with the above results (i), (ii), (iii), and (iv), we get

**THEOREM A.** (Also see Theorem AB in §1.4) For \( e \geq 3 \) and \( q \) arbitrary, the only perfect e-codes in \( H(n,q) \) are the trivial codes (of one codeword), the binary repetition codes (of two codewords), and the binary Golay code.

**REMARK:** For \( e = 2 \), there is a good chance to prove the nonexistence of perfect codes. But the method used in this dissertation does not work well. For \( e = 1 \), there are many perfect codes, linear or nonlinear, and the classification seems difficult.

We now turn our attention to tight designs in \( H(n,q) \).

§1.3 **Tight 2e-Designs in** \( H(n,q) \).

In connection with linear codes in \( H(n,q) \), designs in \( H(n,q) \) are perhaps better understood as the dual of linear codes. Suppose \( q \) is a prime power. Then \( H(n,q) \) can be thought of as an \( n \)-dimensional vector space over \( \text{GF}(q) \), and a linear e-code in \( H(n,q) \) is a subspace of minimum distance \( 2e+1 \). If \( C \) is a linear e-code in \( H(n,q) \), then the orthogonal complement \( C^\perp \) is a 2e-design in \( H(n,q) \). (See §11.8 and p.139 in [17].)
Designs in $H(n, q)$ do not always arise from linear codes. A $t$-design in $H(n, q)$ is usually known by the name orthogonal array. (Cf. Theorem 4.4 in [5].) An orthogonal array of strength $t$ and index $k$ with parameters $(M, n, q)$ is an $M \times n$ matrix $A$, with entries from a set of $q$ elements, such that any set of $t$ columns of $A$ contains all the $q^t$ possible $1 \times t$ row vectors exactly $k$ times. Thus $M = kq^t$.

In 1947, C.R. Rao first gave a bound on the size of $M$ of an orthogonal array in his paper [12], and the bound was reproved by P. Delsarte (Theorem 5.21 in [5]). It is as follows.

$$M \geq 1 + \binom{n}{1}(q-1) + \binom{n}{2}(q-1)^2 + \ldots + \binom{n}{e}(q-1)^e$$

where $e = \lfloor \frac{t}{2} \rfloor$ and $t$ is the strength of the array.

A tight $2e$-design $Y$ in $H(n, q)$ is an orthogonal array of strength $2e$ (with parameters $(|Y|, n, q))$ achieving Rao's bound. For the rest of this dissertation, we will use the name tight designs in $H(n, q)$ instead of orthogonal arrays achieving Rao's bound. It is clear from the definition and the above inequality that $n > 2e+1$ if a tight $2e$-design in $H(n, q)$ exists. In fact, when $q = 2$ and $n = 2e+1$, we have some examples for tight $2e$-designs. They are formed by taking the orthogonal complements of the all-one vectors in $H(n, q)$. These are actually the dual of binary repetition codes. Besides those arising from binary repetition codes, we also have those arising from (being the dual of) Golay codes. We list the parameters of them as follows.

(i) $q = 2$, $n = 2e+1$, $M = 2^{2e}$, and $e$ arbitrary.

(ii) $q = 2$, $n = 23$, $M = 2048$, and $e = 3$.

(iii) $q = 3$, $n = 11$, $M = 243$, and $e = 2$. 
Like perfect e-codes, tight 2e-designs in H(n,q) are very rare. In the case e = 2, R. Noda [10] showed that if there exists a tight 2e-design of size M in H(n,q), then

(i) \( q = 2 \), \( n = 5 \), \( M = 16 \), or

(ii) \( q = 3 \), \( n = 11 \), \( M = 243 \), or

(iii) \( q = 6 \), \( n = \frac{1}{5}(9a + 1) \), \( M = \frac{9}{2}a^2(9a - 1) \) for some integer \( a \) with \( a \equiv 0 \pmod{3} \), \( a \equiv \pm 1 \pmod{5} \), and \( a \equiv 5 \pmod{16} \).

In this dissertation, we will use Lloyd's Theorem and Best's result [3] to prove

**Theorem B.** [Also see Theorem AB in §1.4.] For \( e > 3 \) and \( q > 3 \), there exist no tight 2e-designs in H(n,q).

**Remark:** (i) As we mentioned, to assure the existence of tight designs, \( n \) must be \( > 2e+1 \). (ii) When \( e > 3 \) and \( q = 2 \), the author knows only that there are no tight 2e-designs in H(n,q) when \( e \) is odd and \( n \) is even, which is a direct consequence of Corollary 1.4.2.

§1.4 Important Theorems Used to Prove Theorems A and B.

The Hamming schemes H(n,q) are not only distinguished by being P- and Q-polynomial schemes, but also by having identical eigenmatrices; namely, \( P = Q \) for any H(n,q). Consequently, for each H(n,q), the Lloyd polynomial and the Wilson polynomial are the
same. In the following theorem, we combine the Lloyd theorem in $H(n,q)$ and its dual theorem into one theorem. (Also see Remark on p.64, [5].)

**Theorem 1.4.1.** Suppose there exists a perfect $e$-code or a tight $2e$-design in $H(n,q)$. Then the Lloyd polynomial

$$ F_e(x) = \sum_{i=0}^{e-1} (-q)^i (q-1)^{e-i} (n-1-i)^{e-1} x^{-i} $$

has $e$ distinct integral zeros in the interval $[1,n]$. Here the binomial coefficient $\binom{x}{k}$ is defined by

$$ \binom{x}{k} = \begin{cases} \frac{x(x-1)...(x-k+1)}{k!} & \text{if } k > 1, \\ 1 & \text{if } k = 0 \end{cases} $$

for any real number $x$ and any non-negative integer $k$.

For convenience, we define another polynomial $G_e(x)$ which is the scaled $F_e(x)$ with leading coefficient $1$. That is

$$ G_e(x) = \frac{e!}{(-q)^e e^e} F_e(x) $$

$$ = (x-1)(x-2)...(x-e) + \left( \sum_{k=1}^{e-1} b_k (x-1)(x-2)...(x-k) \right) + b_0 $$

where

$$ b_k = \binom{1-q}{e-k} \binom{e-k-1}{k} (n-k-1)(n-k-2)...(n-e) $$

for $0 < k < e-1$. 
From the above coefficients $b_k$ and Theorem 1.4.1, we immediately get the following helpful corollary.

**COROLLARY 1.4.2.** (cf. Lemma 8.1.1 in [3]) Suppose there exists a perfect $e$-code or a tight $2e$-design in $H(n,q)$. Then, for $0 < k < e-1$, the expressions

$$
\frac{1}{q^k} \binom{e}{k} (n-k-1)(n-k-2)\ldots(n-e)
$$

are integers.

**Proof:** By Theorem 1.4.1, $G_e(x)$ is a polynomial with integral coefficients. This implies that $G'_e(x) = G_e(x) - (x-1)(x-2)\ldots(x-e)$ is a polynomial with integral coefficients. Since the leading coefficient of $G'_e(x)$ is $b_{e-1}$, we get that $b_{e-1}$ is an integer. Now repeat the argument, we have that $b_{e-2}, \ldots, b_0$ are integers.

Since $q-1$ and $q$ are relatively prime and $b_k = \frac{1-q}{q} \frac{e-k}{e} \binom{e}{k} (n-k-1)(n-k-2)\ldots(n-e)$ are integers, we conclude that

$$
\frac{1}{q^k} \frac{e}{k} (n-k-1)(n-k-2)\ldots(n-e)
$$

are integers.

The following two theorems (1.4.3 and 1.4.4) are key steps for the nonexistence of perfect $e$-codes as well as tight $2e$-designs in $H(n,q)$. The first one gives the asymptotic positions of zeros of Lloyd's polynomials. The second concerns about the nonexistence of perfect codes.
THEOREM 1.4.3. (E. Bannai [1], 1977, Proposition 15) Let the zeros of $F(x)$ be

$$
\alpha_i = \alpha + \beta \xi_i + \lambda_i \quad (i = \pm 1, \pm 2, \ldots, \pm \frac{e}{2}) \quad \text{and} \quad i = 0 \quad \text{if} \quad e \text{ is odd}
$$

where

$$
\alpha = \frac{1}{e} (\alpha_1 + \alpha_2 + \ldots + \alpha_e) = \frac{(n-e)(q-1) + e+1}{q},
$$

$$
\beta = \frac{(n-e)(q-1)}{q}, \quad \text{and}
$$

$$
\xi_i \quad \text{the zeros of the Hermite polynomial} \quad H_e(x) \quad \text{defined}
$$

by

$$
H_e(x) = (-1)^e \exp\left(\frac{x^2}{2}\right) \frac{d}{dx} \left\{ \exp\left(-\frac{x^2}{2}\right) \right\}.
$$

Then

$$
\lambda_i \rightarrow \left(1 - \frac{2}{q}\right) \left(\frac{1 - \xi_i^2}{6}\right) \quad \text{as} \quad \beta \rightarrow \infty.
$$

The family of Hermite polynomials $H_e(x)$ defined above has the following recurrence relation.

$$
H_e^{k+1}(x) = xH_e^k(x) - kH_e^{k-1}(x) \quad (k \geq 1)
$$

with $H_e(0) = 1$ and $H_e(x) = x$.

We will use this recurrence relation to obtain $H_e(x)$.

THEOREM 1.4.4. (E. Bannai [1], 1977, Proposition 16) There exists a number $\beta_0(e)$ such that if $\beta > \beta_0(e)$, then no perfect $e$-codes exist in $H(n,q)$ for $q > 3$ and $n > e+1$. 

We will use Lloyd's theorem (1.4.1), its corollary (1.4.2), Theorem 1.4.3, and the ideas given by E. Bannai in his proof of Theorem 1.4.4 to prove the nonexistence of perfect codes and tight designs in $H(n,q)$. One important thing in the proofs of this dissertation is that we avoid using the sphere packing condition for the existence of perfect codes (which says $F(0)$ divides $F_n(0)$ if there is any perfect $e$-code), because it need not be true for the existence of tight $2$-$e$ designs.

The rest of the dissertation is devoted to showing the following theorem, Theorem AB. Theorem AB and Theorem 1.4.1 imply Theorems A and B immediately.

**THEOREM AB.** For $q > 3$ and $3 < e < n-1$, each Lloyd polynomial $F(x)$ (in $H(n,q)$) has at least one nonintegral zero.

To finish this chapter, the author would like to point out that Best [3] has done most of the work to prove Theorem AB. We will see this at the beginning of the next chapter.
Chapter II

THE RESULTS OF BEST AND THE CASES $e = 7$ AND $e > 9$

Throughout this chapter, we assume that $q > 3$, $e < n-1$, $e = 7$ or $e > 9$, and that there exist perfect $e$-codes or tight $2e$-designs in $H(n,q)$, if not otherwise stated.

§2.1 The Results of Best and Some Results for the case $e > 10$.

In chapters 6 and 7 of [3], Best proved that, under the assumptions above, if $\omega < 1/2e$ where $\omega = q(\frac{2e+1}{2(q-1)(n-1)})^{1/2}$, then the zeros of every Lloyd polynomial $F_e(x)$ cannot be all integral; and in this way, he concluded the nonexistence of most of the perfect $e$-codes. Since his result assumes Lloyd's Theorem only, the nonexistence is also true for tight $2e$-designs. We rephrase this result as follows.

**THEOREM 2.1.1.** (C.f. section 5.1 and chapters 6 and 7 in [3].)

Theorem AB is proved if $n > \frac{2q^2 e (2e+1)}{q-1} + 1$.

In other words, if there exist perfect $e$-codes or tight $2e$-designs in $H(n,q)$ for $q > 3$, $e < n-1$, and $e = 7$ or $e > 9$, then $n < \frac{2q^2 e (2e+1)}{q-1} + 1$. 

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By assuming \( n < \frac{2^q 2^e (2e+1)}{q-1} + 1 \) and using the divisibility conditions in Corollary 1.4.2 (Lemma 8.1.1 in his thesis [3]), Best proved the following results.

**Lemma 2.1.2.** (Lemma 8.1.4 in [3]) For each prime power \( p \) dividing \( q, \) let \( \lambda = p^{\alpha-1} \). Then \( \lambda < (13e^{-5.8})^{1/e} \).

**Lemma 2.1.3.** (Lemma 8.2.1 in [3]) If \( e = 7 \) or \( 9 \), then \( q \mid 2520 \).

**Lemma 2.1.4.** (Lemma 8.2.2 in [3]) If \( e > 10 \), then \( q \mid 120 \).

Lemma 2.1.2 limits the existence of perfect codes and tight designs to a great degree. We investigate the existence problem for the case \( e > 10 \) by using this lemma.

- If \( e > 45 \), then \( \lambda < 1.73 \) and \( p < 3 \); hence, \( q = 2 \).
- If \( e > 33 \), then \( \lambda < 2 \) and \( p < 4 \); hence, \( q \mid 6 \).
- If \( e > 16 \), then \( \lambda < 3.3 \) and \( p < 5 \); hence, \( q \mid 12 \).
- If \( e > 13 \), then \( \lambda < 4 \) and \( p < 8 \); hence, \( q \mid 60 \). (See Lemma 2.1.4)

The results of this section are summarized in Theorem 2.1.1, Lemma 2.1.3, and the following lemma.

**Lemma 2.1.5.** Assume \( q > 3 \) and \( e > 10 \). If there exist nontrivial perfect \( e \)-codes or tight \( 2e \)-designs in \( H(n,q) \), then either
(i) $33 < e < 44$ and $q | 6$,
(ii) $16 < e < 32$ and $q | 12$,
(iii) $13 < e < 15$ and $q | 60$,
or (iv) $10 < e < 12$ and $q | 120$.

§2.2 The Case $33 < e < 44$.

For the rest of this chapter, we will use the divisibility conditions in Corollary 1.4.2 and the bound in Theorem 2.1.1 to rule out most of the remaining cases stated in Lemma 2.1.5 and Lemma 2.1.3. The divisibility conditions are

$$(1) \quad q \mid \left( \begin{array}{c} e \\ i \end{array} \right)(n-e)(n-e+1)\ldots(n-e+i-1) \quad \text{for} \quad 1 \leq i \leq e. \quad \text{The bound is}$$

$$(2) \quad n < \frac{2^2 q \cdot e \cdot (2e+1)}{q-1} + 1. \quad \text{Assume} \quad 33 < e < 44 \quad \text{Then} \quad q = 2, 3 \text{ or } 6 \quad \text{by Lemma 2.1.5.}

Suppose $3 | q$. Then by (1), $3^3 \mid \left( \begin{array}{c} e \\ 3 \end{array} \right)(n-e)(n-e+1)(n-e+2)$. Since $33 < e < 44$, $3^2 \mid \left( \begin{array}{c} e \\ 3 \end{array} \right)$ (i.e., at best $3 \mid \left( \begin{array}{c} e \\ 3 \end{array} \right)$). Therefore, $3^2 \mid (n-e)(n-e+1)(n-e+2)$. Similarly, if we further consider $3^6 \mid \left( \begin{array}{c} e \\ 6 \end{array} \right)(n-e)(n-e+1)\ldots(n-e+5)$ and $3^2 \mid \left( \begin{array}{c} e \\ 6 \end{array} \right)$, we get $3^5 \mid (n-e)(n-e+1)\ldots(n-e+5)$. Since $3^2 \mid (n-e)(n-e+1)(n-e+2)$, $3^4 \mid (n-e)(n-e+1)(n-e+2)$.

Now consider $3^e \mid (n-e)(n-e+1)\ldots(n-1)$. Since one of $(n-e)$, $(n-e+1)$, and $(n-e+2)$ is divisible by $3^4$, the product
\[(n-e+3)(n-e+4)\ldots(n-1)\] contains exponent of 3 no bigger than
\[\frac{e-1}{3} + \frac{e-1}{2} + \frac{e-1}{3} < \frac{e-1}{2}.\] Therefore, the exponent of 3 in
\((n-e)(n-e+1)(n-e+2)\) is \(e - \frac{e-1}{2} = \frac{e+1}{2}\); that is
\[\frac{e+2}{2} \mid (n-e)(n-e+1)(n-e+2).\] This implies
\[n > 3^{\frac{e+2}{2}} > 3^{\frac{33+2}{2}} = 3^{17} > 10^8.\]

But this contradicts the bound in (2)
\[n < \frac{2^2 e(2e+1)}{q-1} + 1 < \frac{2^2 \cdot 44 \cdot 89}{5} + 1 = 2481178.6 < 10^8.\]

Therefore \(3 \nmid q\). We conclude

**Lemma 2.2.1.** If \(33 < e < 44\), then \(q = 2\).

§2.3 The Case \(16 < e < 32\).

Assume \(16 < e < 32\). Then, by Lemma 2.1.5, \(q \mid 12\).

**Lemma 2.3.1.** If \(3 \mid q\), then
\[n > 3^{e-1} \cdot 9^{(e-1)/27} > 3^{\frac{e+3}{2}}.\]
Proof: Suppose \(3 \mid q\). Then by taking \(i = e\) in (1),
\[
3 \mid (n-e)(n-e+1)...(n-1) .\]
Let \((n-k)\) be one having maximum exponent
of 3 among \((n-e), (n-e+1), \ldots, (n-1)\). Then the exponent of \(e\)
in \[
\frac{(n-e)(n-e+1)...(n-1)}{(n-k)}
\]
is less than \[
\left[ \frac{e-1}{3} \right] + \left[ \frac{e-1}{3} \right] + \left[ \frac{e-1}{3} \right] < \frac{e-1}{2} .
\]
Therefore, the exponent of 3 in \((n-k)\) is
\[
> e - \left[ \frac{e-1}{3} \right] - \left[ \frac{e-1}{9} \right] - \left[ \frac{e-1}{27} \right] .
\]
Thus, \(n > 3\). Since \(e - \left[ \frac{e-1}{3} \right] - \left[ \frac{e-1}{9} \right] - \left[ \frac{e-1}{27} \right]\)
is an integer \(> e - \frac{e-1}{2} = \frac{e+1}{2}\), it is an integer \(> \frac{e+3}{2}\).

**Lemma 2.3.1.** If \(4 \mid q\), then
\[
2^{e-\left[ \frac{e-1}{2} \right]} \left[ \frac{e-1}{4} \right] \left[ \frac{e-1}{8} \right] \left[ \frac{e-1}{16} \right] \leq n > 2^{e+2} .
\]
Proof: The proof of Lemma 2.3.1 is similar to that of Lemma 2.3.1. We
just have to consider \(4 \mid (n-e)(n-e+1)...(n-1)\).

**Lemma 2.3.3.** If \(e = 18\) or \(20 \leq e \leq 32\), then \(q = 2\).

Proof: First we show that \(3 \mid q\). From (2), we have
\[
\begin{align*}
 n &< \frac{2 \cdot 2^2}{q (2e+1)} + 1 < \frac{2 \cdot 12 \cdot 32 \cdot 65}{14} < 1742662 .
\end{align*}
\]
Suppose \(3 \mid q\) and \(e \geq 25\). Then by Lemma 2.3.1,
\[
\left[ \frac{e+3}{2} \right] \left[ \frac{25+3}{2} \right]
\]
\[
= 3^{14} = 4782967 > 1742662 ,
\]
a contradiction. Thus, when \(25 < e < 32\), \(3 \mid q\), or equivalently,
\(q \mid 4\). If \(e < 24\), then
\[
\begin{align*}
 n &< \frac{2 \cdot 12 \cdot 24 \cdot 49}{11} + 1 < 738957 \text{ by (2).}
\end{align*}
\]
Suppose \(21 \leq e < 24\) and \(3 \mid q\). Then by Lemma 2.3.1,
a contradiction. Again, \(3^e \mid q\) when \(21 \leq e < 24\). Suppose \(e = 20\) and \(3^e \mid q\). Then by Lemma 2.3.1,

\[
3^{\frac{e-1}{2}} \cdot \frac{e-1}{2} \cdot \frac{e-1}{27} \
\]

\[
> \frac{3^{13}}{3} = 1594232 > 738957,
\]

On the other hand, by (2), \(n < \frac{2 \cdot 2 \cdot 20 \cdot 41}{11} < 313868\) by (2) and

\[
3^{18-6} = 3^{12} = 531441
\]

by Lemma 2.3.1, showing that \(3^e \mid q\) if \(e = 18\).

We sum up the previous arguments as follows.

If \(e = 18\) or \(20 < e < 32\), then \(q \mid 4\). (Because \(3^e \mid q\).)

Our next step is to prove \(q \neq 4\) by using Lemma 2.3.2 and by

using the bound \(n < \frac{2 \cdot 2}{q} + 1\). \(e < 32\) and \(q = 4\) imply

that \(n < \frac{2 \cdot 2 \cdot 32}{3} + 1 < 709975\). On the other hand, if \(e > 18\)

and \(q = 4\), then, by Lemma 2.3.2, \(n > \frac{2^{18+2}}{2} = 2^{20} = 1048576 > 709975\),

a contradiction. Thus, \(q \neq 4\) if \(e = 18\) or \(20 < e < 32\).

Therefore, \(q = 2\). Lemma 2.3.2 is proved.

\[\text{Lemma 2.3.4. If } e = 17 \text{ or } 19 \text{, then } q = 2.\]
Proof: Let $e = 17$ or $19$. Assume $4 \mid q$. By Lemma 2.3.2, if $e = 17$, then $n > 2^{34-15} = 2^9 = 524288$; and if $e = 19$,

$$n > 2^{38-26} = 2^{12} > 524288.$$ 

By (2), $n < \frac{2^{12} \cdot 19^2}{11} + 1 < 366814 < 524288$, a contradiction. Thus $4 \nmid q$; i.e., $q \nmid 6$ if $e = 17$ or $19$.

We now prove that $3 \nmid q$, which will finish the proof of the lemma.

Assume $3 \mid q$ and $e = 17$. Then by Lemma 2.3.1, we have

$$n > 3^{16-11} = 3^5 = 177147.$$ 

However, by (2) and $q \nmid 6$,

$$n < \frac{2 \cdot 6 \cdot 17 \cdot 19^2}{5} + 1 < 145657,$$ 

a contradiction. Thus $3 \nmid q$ if $e = 17$.

Assume $3 \mid q$ and $e = 19$. Then $q = 3$ or $6$ and by Lemma 2.3.1,

$$n > 3^{19-8} = 3^{11} = 177147.$$ 

If $q = 3$, then by (2),

$$n < \frac{2 \cdot 3 \cdot 3 \cdot 19^2}{2} + 1 < 126712,$$ 

a contradiction. If $q = 6$, then (2) says $n < \frac{2 \cdot 6 \cdot 19^2}{5} + 1 < 202738$. However, by taking $i = 1$,

$q = 6$, $e = 19$ in (1), we have $6 \mid 19(n-19)$; i.e., $6 \mid (n-19)$.

Further, by taking $i = 2, 3, ..., 19$ in (1), we get $3^{11} \mid (n-19)$.

Thus, $2 \cdot 3^{11} \mid (n-19)$, which implies $n > 2 \cdot 3^{11} + 19 > 202738$, again a contradiction. Therefore, $3 \nmid q$ when $e = 19$. The lemma is proved.

Lemma 2.3.5. If $e = 16$, then either $q = 2$ or $q = 3$ and $(n-16) = 3^{10}$.

Proof: First we show that $4 \nmid q$.

Suppose $4 \mid q$. Then by Lemma 2.3.2,

$$n > 2^{32-11} = 2^{21}.$$ 

However, by (2), $n < \frac{2 \cdot 12 \cdot 16^2}{11} + 1 < 221185 < 2^{21}$, a
contradiction. Therefore \( q = 2, 3, \) or \( 6 \). Now we ignore the case \( q = 2 \).

Suppose \( 3 | q \) (i.e., \( q = 3 \) or \( 6 \)). Then by taking \( i = 1, 2, \ldots, 16 \)
in \((1)\),

\[
q \mid \prod_{i=1}^{16} (n-16)(n-15) \ldots (n-16+i-1),
\]

and we get \( 3^{10} | (n-16) \).

If \( q = 3 \), then by \((2)\),

\[
 n < \frac{2 \cdot 2^2 \cdot 16 \cdot 33}{2} + 1 = 76033.
\]

Therefore \( 3^{10} | (n-16) \) and \( n < 76033 \) imply that \( n-16 = 3^{10} \).

If \( q = 6 \), then by \((2)\),

\[
 n < \frac{2 \cdot 6 \cdot 2^2 \cdot 33}{5} + 1 < 121653.
\]

So \( 3^{10} | (n-16) \) and \( n < 121653 \) imply that \( n-16 = 3^{10} \) or \( 2 \cdot 3^{10} \). Since \( 2 | q \), we have, by \((1)\), \( 2^{16} | (n-16)(n-15) \ldots (n-1) \). However in either case \( n-16 = 3^{10} \) or \( n-16 = 2 \cdot 3^{10} \), the exponent of 2 in the product \((n-16)(n-15) \ldots (n-1)\) is 15, a contradiction. Therefore \( q \neq 6 \). We have proved the lemma.

**Lemma 2.3.6.** The case \( e = 16, q = 3, n-16 = 3^{10} \) is impossible.

**Proof:** We will investigate the rescaled Lloyd polynomial (see §1.4) \( G(x) \) in detail to prove this lemma.

Assume that there exist perfect codes or tight designs in the case \( e = 16, q = 3, n-16 = 3^{10} \). Then, by Theorem 1.4.1, the rescaled Lloyd polynomial \( G(x) \) has 16 distinct integral zeros between 1 and \( n = 3^{10} + 16 \).

We claim that if \( x_0 \) is an integral zero of \( G(x) \), then

\[
x_0 \equiv 356 \pmod{3^8}.
\]

However, there are less than 16 (precisely 9) integers \( x_0 \equiv 356 \pmod{3^8} \) such that \( 1 < x_0 < 3^{10} + 16 \). This contradicts Theorem 1.4.1. Thus, the case \( e = 16, q = 3, \) and \( n-16 = 3^{10} \) is impossible.
Now we prove the claim. Let $x_0$ be an integral zero of $G_{16}(x)$.

Then $G_{16}(x) \equiv 0 \pmod{3^2}$. Calculation of the exponent of 3 in each (unexpanded) term of $G_{16}(x)$ shows that $G_{16}(x) \equiv$ the sum of the last two terms (mod $3^2$). Thus

$$\binom{2}{3} \binom{15}{3} \binom{10}{3} \binom{10}{3+1} \ldots \binom{10}{3+14} [-16(x-1) + \frac{2}{3}(3^{10}+15)] \equiv 0 \pmod{3^2}.$$

Since

$$\binom{2}{3} \binom{15}{3} \binom{10}{3} \binom{10}{3+1} \ldots \binom{10}{3+14} \not\equiv 0 \pmod{3},$$

we have

$$-16(x-1) + \frac{2}{3}(3^{10}+15) \equiv 0 \pmod{3^2}.$$

This implies that $x_0 \equiv 5 \pmod{3^2}$.

Now we assume $x_0 \equiv 5 \pmod{3^2}$ and consider $G_{16}(x_0) \equiv 0 \pmod{3^3}$. Again, calculation of the exponent of 3 in each term of $G_{16}(x_0)$ shows that $G_{16}(x_0) \equiv$ the sum of the last two terms (mod $3^3$), which quickly implies that $x_0 \equiv 5 \pmod{3^3}$.

Now we count the exponent of 3 in each term of $G_{16}(x)$ by assuming that $x_0 \equiv 5 \pmod{3^3}$. It turns out that all the terms of $G_{16}(x)$, except the last five terms and the eighth term from the end, are divisible by $3^8$.

Now we let $x_0 = 5 + 27(k + 3 \ell)$ and consider $G_{16}(x_0) \equiv$ the sum of the last five terms and the eighth term from the end \(\equiv 0 \pmod{3^8}\). After calculation, we get $k \equiv 13 \pmod{5^8}$. Thus, $x_0 = 5 + 27 \cdot 13 = 356 \pmod{3^8}$. The lemma is proved.

*) As a reference for the calculation of exponents $k$ of 3 in the terms of $G_{16}(x)$, we include the following page
We summarize the conclusions of this section by:

**LEMMA 2.3.7.** If $16 < e < 32$, then $q = 2$.

The first column from the right side gives lower bounds on $k$, assuming $x \equiv 5 \pmod{3^3}$; the second column from the right side assumes $x$ to be any integer.

$$e = 16, \quad q = 3, \quad n-16 = 10$$

The rescaled Lloyd polynomial

$$G(x) = \frac{1}{16} (x-16)(x-15)...(x-1)$$

- \( - \frac{2}{3} \left( \begin{array}{c} 16 \\ 1 \end{array} \right) (x-15)(x-14)...(x-1) \)

\[ \begin{array}{c|c|c} & x = \text{any integer} & x \equiv 5 \pmod{27} \\ \hline k > & k > \\ \hline 6 & 8 \\ 15 & 17 \\ 14 & 17 \\ 12 & 13 \\ 12 & 13 \\ 11 & 13 \\ 9 & 10 \\ 9 & 10 \\ 8 & 11 \\ \end{array} \]

[This equation is continued on the next page.]
The rescaled Lloyd polynomial (continued)

\[ \begin{align*}
- \binom{2}{9} 16^1 \binom{10}{3} (3^{10} +1)\ldots(3^{10} +8)(x-7)\ldots(x-1) \\
+ \binom{2}{10} 16^1 \binom{10}{3} (3^{10} +1)\ldots(3^{10} +9)(x-6)\ldots(x-1) \\
- \binom{2}{11} 16^1 \binom{10}{3} (3^{10} +1)\ldots(3^{10} +10)(x-5)\ldots(x-1) \\
+ \binom{2}{12} 16^1 \binom{10}{3} (3^{10} +1)\ldots(3^{10} +11)(x-4)\ldots(x-1) \\
- \binom{2}{13} 16^1 \binom{10}{3} (3^{10} +1)\ldots(3^{10} +12)(x-3)(x-2)(x-1) \\
+ \binom{2}{14} 16^1 \binom{10}{3} (3^{10} +1)\ldots(3^{10} +13)(x-2)(x-1) \\
- \binom{2}{15} 16^1 \binom{10}{3} (3^{10} +1)\ldots(3^{10} +14)(x-1) \\
+ \binom{2}{15} 16^1 \binom{10}{3} (3^{10} +1)\ldots(3^{10} +15)
\end{align*} \]

\[ x = \text{any integer} \]

<table>
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<tr>
<th>x = 5</th>
<th>x \equiv 5 \text{ (mod 27) }</th>
</tr>
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<td>8 6</td>
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<tr>
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<td>10 10</td>
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<tr>
<td>0 0</td>
<td>10 10</td>
</tr>
</tbody>
</table>

\section*{2.4 The Case 13 < e < 15}

Let 13 < e < 15. Then, by Lemma 2.1.5, \( q | 60 \).

**Lemma 2.4.1.** If 5 | q, then \( n > 5 \).

**Proof:** The proof is similar to that of Lemma 2.3.1.
LEMMA 2.4.2. If $e = 15$, then $q = 2$.

Proof: First, $5|q$. By (2) (in §2.3), we have

$$n < \frac{2 \cdot 60^2 \cdot 15^2 \cdot 31}{59} + 1 < 851188.$$

Suppose $5|q$. Then by Lemma 2.4.1, $n > 5^{13} > 10^9 > 851188$, a contradiction. Therefore, $5|q$; i.e., $q|12$.

But $q|12$ implies another bound on $n$; i.e.,

$$n < \frac{2 \cdot 12^2 \cdot 15^2 \cdot 31}{11} + 1 < 182620.$$

Now we show that $4|q$. Suppose $4|q$. Then by Lemma 2.3.2, $n > 2^{19} = 524288 > 182620$, a contradiction. Therefore $4|q$. That is $q|6$, and another new bound on $n$ is

$$n < \frac{2 \cdot 6 \cdot 15^2 \cdot 31}{5} + 1 < 100441.$$

Suppose $3|q$. By considering the divisibility conditions in (1) (§2.3), namely, $q | \prod_{i=1}^{e} (n-e) \cdots (n-e+i-1)$ for $1 \leq i \leq e$, we get

$3^{10} | (n-15)$ or $3^{10} | (n-14)$. Furthermore, if we assume

$G(z_0) \equiv 0 \pmod{9}$ for some integer $z_0$, then we get $3^{12} | (n-15)$ or $3^{12} | (n-14)$. This implies that $n > 3^{12} = 531441 > 100441$, in contradiction to the bound. Thus $3|q$; i.e., $q = 2$. We have proved the lemma.

LEMMA 2.4.3. If $e = 14$, then $q = 2$.

Proof: Since $q|60$, we have the following bound for $n$,
If $5|q$, then by Lemma 2.4.1, $n > \frac{5^{12}}{59} > 693642$, a contradiction. Thus $5 \nmid q$ and $q|12$. Now, the bound for $n$ is

$$n < \frac{2 \cdot 60 \cdot 2^{14} \cdot 29}{59} + 1 < 693642.$$

If $4|q$, then by Lemma 2.3.2, $n > \frac{4^{18}}{2} = 262144 > 693642$, a contradiction. Thus $4 \nmid q$ and $q|6$.

Suppose $q = 6$. Then

$$n < \frac{2 \cdot 12 \cdot 2^{14} \cdot 29}{11} + 1 < 118819.$$

By considering the divisibility conditions in (1), we get $3 \mid (n-14)$ and $4 \mid (n-14)$ or $2 \mid (n-13)$. If $2 \mid (n-14)$, then $n > \frac{2 \cdot 3 \cdot 2^{14} \cdot 29}{5} + 1 > 314928 > 81851$, a contradiction. If $2 \mid (n-13)$, then $(n-14) \equiv 5 \cdot 3^9 \pmod{3 \cdot 2^4}$. This forces $n > 5 \cdot 3^9 = 98415 > 81851$, again a contradiction. Therefore $q \neq 6$.

Suppose $q = 3$. Then by the divisibility conditions, we have $3^9 \mid (n-14)$. Now consider that $G_{14}(x_0) \equiv 0 \pmod{3}$ for some integer $x_0$. We get $3^{10} \mid (n-14)$. Thus $n > 3^{10} = 59049$. Since $q = 3$, we have the bound

$$n < \frac{2 \cdot 3 \cdot 14 \cdot 29}{2} + 1 = 51156 < 59049,$$

again a contradiction. Thus, $q \neq 3$. That is $q = 2$ and the lemma is proved.
LEMMA 2.4.4. If \( e = 13 \), then either

(i) \( q = 6 \) and \( (n-13) = 8 \cdot 3 \) or \( 6 \cdot 8 \),

(ii) \( q = 3 \) and \( (n-13) = k \cdot 8 \) for \( 1 < k < 6 \), or

(iii) \( q = 2 \).

Proof: Again, the divisibility conditions in (1) (§2.3) are the key tools for the proof. By considering the divisibility conditions for \( 1 < i < 13 \), we have:

- If \( 5 | q \), then \( 5^{11} | (n-13) \);
- If \( 4 | q \), then \( 2^{16} | (n-13) \);
- If \( 3 | q \), then \( 3^{8} | (n-13) \);
- If \( 2 | q \), then \( 8 | (n-13) \) or \( 8 | (n-11) \).

\( q | 60 \) implies the following bound on \( n \). (See (2) in §2.3.)

\[
n < \frac{2 \cdot 60 \cdot 13 \cdot 27}{59} + 1 < 556842 .
\]

Suppose \( 5 | q \). Then \( n > 5^{11} > 556842 \), a contradiction. Therefore, \( q \neq 12, 6, 4, 3, \) or \( 2 \).

If \( q = 12 \) then

\[
n < \frac{2 \cdot 12^2 \cdot 13 \cdot 27}{11} + 1 < 119469 ,
\]

and \( 3 | q \) and \( 4 | q \) imply that \( 8 \cdot 2^{16} | (n-13) \). These imply that \( n | 3 \cdot 2^{16} > 119469 \), a contradiction. Thus, \( q \neq 12 \).

Suppose \( q = 4 \). Then

\[
n < \frac{2 \cdot 4 \cdot 13 \cdot 27}{3} + 1 < 48673 .
\]
But $2^{16} | (n-13)$ implies $n > 2^{16} = 65536 > 48673$, a contradiction. Therefore $q \neq 4$.

Suppose $q = 6$. Then

$$n < \frac{2^6 \cdot 13^2 \cdot 27}{5} + 1 < 65709.$$ 

If $8 | (n-13)$ and $3^8 | (n-13)$, we get $(n-13) = 8 \cdot 3^8$. If $8 | (n-11)$ and $3^8 | (n-13)$, we get $(n-13) = 6 \cdot 3^8$.

Suppose $q = 3$. Then

$$n < \frac{2 \cdot 3^2 \cdot 13 \cdot 27}{2} + 1 < 41068.$$ 

Since $3^8 | (n-13)$, we have $(n-13) = k \cdot 3^8$ for $1 < k < 6$. Thus the lemma is proved.

**Lemma 2.4.5.** If $13 < e < 15$, then $q = 2$.

**Proof:** To eliminate the remaining cases,

(i) $e = 13$, $q = 6$, $(n-13) = 8 \cdot 3^8$ or $6 \cdot 3^8$,

(ii) $e = 13$, $q = 3$, $(n-13) = k \cdot 3^8$ for $1 < k < 6$,

we refer the reader to the Appendix. There, it is shown that, in each of the considered cases, the Lloyd polynomial has a zero in between two consecutive integers. Thus, Theorem AB is proved for each of those cases.
§2.5 The Case $10 < e < 12$.

Let $10 < e < 12$. Then, by Lemma 2.1.5, $q|120$.

**Lemma 2.5.1.** If $10 < e < 12$, then $q|12$.

**Proof:** We know $q|120$. Therefore $n$ has a bound:

$$n < \frac{2 \cdot 12 \cdot 2 \cdot 25}{119} + 1 < 871270.$$ 

Suppose $5|q$. Then, by Lemma 2.4.1, $n > 5^9 = 1953125 > 871270$, a contradiction. Therefore $5|q$ and $q|24$. Again, we have a new bound on $n$,

$$n < \frac{2 \cdot 24 \cdot 2 \cdot 25}{23} + 1 < 180315.$$ 

Suppose $8|q$. Then by using the divisibility conditions in (1), we get $2^{23} | (n-e)$, $(e = 10, 11, 12)$, $n > 2^{23} > 180315$, a contradiction. Therefore $8|q$ and $q|12$.

**Lemma 2.5.2.** if $e = 12$, then $q = 2$.

**Proof:** First, from the last lemma, we have

$$n < \frac{2 \cdot 12^2 \cdot 2 \cdot 25}{11} + 1 < 94256.$$ 

Suppose $3|q$. Then, by using the divisibility conditions in (1), we get $3^8 | (n-12)$ or $(n-11)$. Further, by considering $G_{12} (x_0) \equiv 0 \pmod{3}$ for some integer $x_0$, and then $G_{12} (x_0) \equiv 0 \pmod{9}$, then $G_{12} (x_0) \equiv 0 \pmod{27}$ and $(\pmod{81})$, we have $3^{12} | (n-12)$ or
(n-11) . This implies that \( n > 3^{12} = 531441 > 94256 \), a contradiction. Thus \( 3 \mid q \), and \( q = 2 \) or \( 4 \).

Suppose \( q = 4 \). Then

\[
n < \frac{2^2 \cdot 4 \cdot 12 \cdot 25}{3} + 1 = 38401 .
\]

Now, by considering the divisibility conditions for \( i = 2, 3, 4, 12 \), we get \( 2^{16} \mid (n-12) \) or \( (n-11) \). In either case, \( n > 2^{16} = 65536 > 38401 \), and we get a contradiction. Therefore, \( q \neq 4 \) and \( q = 2 \).

**Lemma 2.5.3.** If \( e = 11 \), then either

(i) \( q = 6 \) and \( (n-11) = 2 \cdot 3 \cdot t \) for \( t = 1 \) or \( 2 \),

(ii) \( q = 3 \) and \( (n-11) = 3 \cdot t \) for \( 1 < t < 11 \), or

(iii) \( q = 2 \).

**Proof:** Since \( q \mid 12 \), \( n < \frac{2^2 \cdot 12 \cdot 11 \cdot 23}{11} + 1 = 72865 \).

Suppose \( 4 \mid q \). Then from the divisibility conditions in (1), we have \( 2^{14} \mid (n-11) \). Then considering, for some integer \( x_0 \),

\[
G_1(x_0) \equiv 0 \pmod{2}, \text{ then } (\mod 4), \ldots, (\mod 2^8), \text{ we have } 2^{22} \mid (n-11) .
\]

This gives the contradiction that \( n > 2^{22} > 72865 \).

Thus, \( 4 \mid q \) and \( q = 2, 3, \) or \( 6 \).

The divisibility conditions also give

(i) if \( 3 \mid q \), then \( 3^7 \mid (n-11) \);

(ii) if \( 2 \mid q \), then \( 2^3 \mid (n-11) \).

Suppose \( q = 6 \). Then \( n < \frac{2 \cdot 6 \cdot 11 \cdot 23}{5} + 1 < 40077 \). Thus \( 3^7 \mid (n-11) \) and \( 2^3 \mid (n-11) \) imply that \( (n-11) = 2 \cdot 3 \cdot t \) for \( t = 1 \) or \( 2 \).
Suppose \( q = 3 \). Then \( n < \frac{2 \cdot 3^2 \cdot 11 \cdot 23}{2} + 1 = 25048 \). Therefore, 
\[ 3^7 \mid (n-11) \] gives \( (n-11) = 3^7 \) for \( 1 < t < 11 \).

**Lemma 2.5.4.** If \( e = 10 \), then \( q = 2 \).

**Proof:** First, \( n < \frac{2 \cdot 12 \cdot 10 \cdot 21}{11} + 1 < 54983 \).

Suppose \( 4 \mid q \). Then by the divisibility conditions, we have 
\[ 2^{13} \mid (n-10) \]. Then, by considering, step by step, \( G(x) \equiv 0 \pmod {2^i} \) for \( 1 < i < 8 \), we get \( 2^{21} \mid (n-10) \). Now we get the contradiction \( n > 2^{21} > 54983 \). Therefore \( 4 \mid q \); i.e., \( q \mid 6 \).

Suppose \( 3 \mid q \). Then by the divisibility conditions, we get 
\[ 3^7 \mid (n-10) \]; and by further consideration of the equations 
\( G(x) \equiv 0 \pmod {3^i} \) for \( 1 < i < 4 \), we have \( 3^{11} \mid (n-10) \). Thus, if \( 3 \mid q \), then \( n \mid 3^{11} = 177147 > 54983 \), again a contradiction. 
Therefore \( 3 \mid q \) and \( q = 2 \). The lemma is proved.

**Lemma 2.5.5.** If \( 10 < e < 12 \), then \( q = 2 \).

**Proof:** The impossibilities of the remaining cases

(i) \( e = 11, \ q = 6, \ (n-11) = 2^3 \cdot 3^7 \) or \( 2^4 \cdot 3^7 \), and

(ii) \( e = 11, \ q = 3, \ (n-11) = t \cdot 3^7 \) for \( 1 < t < 11 \),

are left to be proved in the Appendix.
§2.6 The Cases \( e = 7 \) and \( 9 \)

For these two cases, we have \( q|2520 = 2^3 \cdot 3^2 \cdot 5^1 \cdot 7^1 \) by Lemma 2.1.3.

**Lemma 2.6.1.** If \( e = 9 \), then either

(i) \( q = 4 \) and \( (n-9) = t \cdot 2^{12} \) for \( 1 < t < 4 \), or

(ii) \( q = 2 \).

**Proof:** First, \( q|2520 \) gives the bound

\[
 n < \frac{2 \cdot 2520 \cdot 2^2 \cdot 19}{2519} + 1 < 7759641
\]

Suppose \( 7|q \). Then by the divisibility conditions, we get

\( 7^8|(n-9) \); and by considering \( G(x) \equiv 0 (\text{mod } 7) \) for some integer \( x_0 \)

we get \( 7^9|(n-9) \). Thus, \( n > 7^9 > 7759641 \), a contradiction.

Therefore \( 7|q \) and \( q|360 \). Now the new bound on \( n \) is

\[
 n < \frac{2 \cdot 360 \cdot 2^2 \cdot 19}{359} + 1 < 1111168
\]

Suppose \( 5|q \). Again, by using the divisibility conditions, we have \( 5^8|(n-9) \); and then by considering \( G(x) \equiv 0 (\text{mod } 5) \), we have

\( 5^9|(n-9) \). We get a contradiction:

\[
 n > 5^9 = 1953125 > 1111168
\]

Thus, \( 5|q \) and \( q|72 \), and the new bound becomes

\[
 n < \frac{2 \cdot 72 \cdot 2^2 \cdot 19}{71} + 1 < 224739
\]

Now by the divisibility conditions, we get

(i) if \( 8|q \), then \( 2^{20}|(n-9) \), which implies \( n > 2^{20} > 224739 \);

(ii) if \( 9|q \), then \( 3^{16}|(n-9) \) or \( (n-8) \), which implies \( n > 3^{16} > 224739 \).
In either case, we get a contradiction to the bound. Therefore $8 \mid q$ and $9 \mid q$; i.e., $q \mid 12$. Now the bound is

$$n < \frac{2 \cdot 12 \cdot 9 \cdot 19}{11} + 1 < 40295.$$ 

Suppose $3 \mid q$. Then $7 \mid (n-9)$, $(n-8)$, or $(n-7)$ by using the divisibility conditions. Considering the equations $G(x_0) \equiv 0 \pmod{3}$ for $1 < i < 4$, we have $3^{11} \mid (n-9)$, $(n-8)$, or $(n-7)$. Thus $n > 3^{11} = 177147 > 40295$, a contradiction. So $3 \nmid q$ and $q = 2$ or $4$.

Suppose $q = 4$. Then $n < \frac{2 \cdot 4 \cdot 9 \cdot 19}{3} + 1 = 16417$. From the divisibility conditions, we get $2^{12} \mid (n-9)$. Combining $n < 16417$ and $2^{12} \mid (n-9)$, we have that

$$(n-9) = 2^{12} \cdot t \quad \text{for} \quad 1 < t < 4.$$

Thus, the lemma is proved.

**Lemma 2.6.2.** If $e = 7$, then either

(i) $q = 6$ and $(n-7) = 8 \cdot 3 \cdot t$ for $1 < t < 5$,

(ii) $q = 3$ and $(n-7) = 3^5 \cdot t$ for $1 < t < 27$, or

(iii) $q = 2$.

**Proof:** Again the divisibility conditions in (1) (§2.3) play important roles in the proof.

First, $q \mid 2520$ and we get the bound

$$n < \frac{2 \cdot 2520 \cdot 7 \cdot 15}{2519} + 1 < 3705872.$$
Suppose $7 \mid q$. Then $7 \mid (n-7)$ or $(n-6)$. By considering $G(x_0) \equiv 0 \pmod{7}$ for some integer $x_0$, we get $7 \mid (n-7)$ or $(n-6)$. Thus $n > 7^8 > 3705872$, a contradiction. So $7 \nmid q$ and $q \mid 360$. Now the new bound is
\[
 n < \frac{2 \cdot 360 \cdot 7 \cdot 2 \cdot 15}{359} + 1 < 530676.
\]

Suppose $9 \mid q$. Then $3 \mid (n-7)$ and $n > 3^8 > 530676$, a contradiction again. Therefore, $9 \nmid q$ and $q \mid 120$, and the bound is
\[
 n < \frac{2 \cdot 120 \cdot 7 \cdot 2 \cdot 15}{119} + 1 < 177884.
\]

Suppose $8 \mid q$. Then $2 \mid (n-7)$. Considering the equations $G(x_0) \equiv 0 \pmod{2^i}$ for $1 < i < 4$, we have $2^{21} \mid (n-7)$, which gives the contradiction that $n > 2^{21} > 177884$. Therefore, $8 \nmid q$ and $q \mid 60$.

If $5 \mid q$, then by the divisibility conditions and $G(x_0) \equiv 0 \pmod{5}$, we get $5 \mid (n-7)$.

If $4 \mid q$, then by the divisibility conditions and the equation $G(x_0) \equiv 0 \pmod{2^i}$ for $1 < i < 4$, we get $2^{14} \mid (n-7)$.

Suppose $q = 60$. Then $n < \frac{2 \cdot 60 \cdot 7 \cdot 2 \cdot 15}{59} + 1 < 89696$. But then $5 \mid q$ and $4 \mid q$ imply that $5 \cdot 2^{14} \mid (n-7)$. Thus, $n > 5 \cdot 2^{14} > 89696$, a contradiction. Thus, $q < 30$ and $n < \frac{2 \cdot 30 \cdot 7 \cdot 2 \cdot 15}{29} + 1 < 45622$.

Suppose $5 \mid q$. Then $5 \mid (n-7)$ and we get the contradiction that $n > 5^7 = 78125 > 45622$. Therefore, $5 \nmid q$ and $q \mid 12$.

Suppose $q = 12$. Then $n < \frac{2 \cdot 12 \cdot 7 \cdot 2 \cdot 15}{11} + 1 < 19245$. So $3 \mid q$ and $4 \mid q$ imply that $3 \cdot 2^{14} \mid (n-7)$, which gives the contradiction...
that \( n > 3 \cdot 2^{14} > 19245 \). Therefore, \( q \neq 12 \). That is \( q = 2, 3, 4, \) or 6. Now the bound is \( n < \frac{2 \cdot 6 \cdot 7 \cdot 15}{5} + 1 < 10585 \).

Suppose \( q = 4 \). Then \( 2^{14} | (n-7) \) and hence \( n > 2^{14} = 16384 > 10585 \), a contradiction. Thus, \( q \neq 4 \).

Suppose \( q = 6 \). Then \( 3 \cdot 8 | (n-7) \) and therefore \( (n-7) = 3 \cdot 8 \cdot t \) for \( 1 < t < 5 \).

Suppose \( q = 3 \). Then \( n < \frac{2 \cdot 3 \cdot 7 \cdot 15}{2} + 1 = 6616 \) and \( 5 | (n-7) \).

In this case, we get \( (n-7) = 3 \cdot 5 \cdot t \) for \( 1 < t < 27 \). The lemma is proved.

**Lemma 2.6.3.** If \( e = 7 \) or 9, then \( q = 2 \).

**Proof:** The impossibilities of the remaining cases

(i) \( e = 7, \ q = 3 \), \( (n-7) = 3 \cdot t \) for \( 1 < t < 27 \),

(ii) \( e = 7, \ q = 6 \), \( (n-7) = 8 \cdot 3 \cdot t \) for \( 1 < t < 5 \),

(iii) \( e = 9, \ q = 4 \), \( (n-9) = 2 \cdot 12 \cdot t \) for \( 1 < t < 4 \),

are proved in the Appendix.

The previous six sections complete the proof of Theorem AB for the cases \( e = 7 \) and \( e > 9 \).
Chapter III

THE NONEXISTENCE PROBLEM FOR THE CASES \( e = 6 \) AND \( e = 8 \)

In this chapter, we will use Bannai's idea from his proof of Theorem 1.4.4 (which is based on Theorem 1.4.3) and the divisibility conditions in Corollary 1.4.2 to prove Theorem AB for the cases \( e = 6 \) and \( e = 8 \) (under the assumption that \( q > 3 \)). We rephrase Theorem 1.4.4 as follows.

For each fixed \( e \) with \( q > 3 \) and \( \eta > e+1 \), there exists a number \( \beta_0(e) \) such that if \( \beta = \frac{\sqrt{(n-e)(q-1)}}{q} > \beta_0(e) \), then Theorem AB is true; namely, the Lloyd polynomial \( F(x) \) has at least one nonintegral zero.

§3.1 The Case \( e = 6 \) and \( q > 30 \)

In the following, we first show that we can take \( \beta_0 = 15 \) when \( e = 6 \) and \( q > 30 \). This part of the discussion is basically (but not completely) the same as the first half of the author's paper [6]. We begin with some discussion of Hermite polynomials and their zeros. For every positive integer \( n \), we define the Hermite polynomial \( H_n(x) \) by

\[ H_n(x) = \frac{1}{2^n} \frac{d^n}{dx^n} (2x^2 - 1)^n \]
This family of polynomials has the following recurrence relation.

\[ H_{n+1}(x) = xH_n(x) - nH_n(x) \quad (n > 1) \]

with \( H_0(x) = 1 \) and \( H_1(x) = x \).

By using the recurrence relation, we can easily obtain

\[ H_6(x) = x^6 - 15x^4 + 45x^2 - 15. \]

Let the zeros of \( H(x) \) be \( \xi_{-3}, \xi_{-2}, \xi_{-1}, \xi_1, \xi_2, \) and \( \xi_3 \) in increasing order. Then, by applying the Cardano formula, we can get

\[ 0.3803274 < \xi_1^2 < 0.3803276, \]
\[ 3.5689847 < \xi_2^2 < 3.5689849, \text{ and} \]
\[ 11.050687 < \xi_3^2 < 11.050689. \]

Notice that \( \xi_i = -\xi_{-i} \) for \( i = 1, 2, \) or \( 3 \).

Following Theorem 1.4.3, we express the roots \( \alpha_i \) (\( i = \pm 1, \pm 2, \pm 3 \)) of \( F_6(x) \) as follows.

\[ \alpha_i = \alpha + \beta \xi_i + \lambda_i \quad \text{where} \quad \alpha = \frac{(n-6)(q-1)}{q} + \frac{7}{2}, \]

and \( \beta = \frac{\sqrt{(n-6)(q-1)}}{q} \).
We also know that
\[ \lambda_1 \Rightarrow (1 - \frac{2}{q})(\frac{5 - \xi^2}{6}) \quad \text{as } \beta \to \infty. \]

By Theorem 1.4.1, we know if there exist perfect 6-codes or tight 12-designs in \( H(n,q) \), then
\[ (\alpha_2 + \alpha_{-2}) - (\alpha_3 + \alpha_{-3}) = (\lambda_2 + \lambda_{-2}) - (\lambda_3 + \lambda_{-3}) \]
is an integer. We also have
\[ (\lambda_2 + \lambda_{-2}) - (\lambda_3 + \lambda_{-3}) \Rightarrow (1 - \frac{2}{q})(\frac{\xi^2_3 - \xi^2}{3}) \quad \text{as } \beta \to \infty. \]

Calculation shows
\[ 2.3276406 < (1 - \frac{2}{q})(\frac{\xi^2_3 - \xi^2}{3}) < 2.4939015 , \]
if we assume \( q > 30 \). Therefore if we can get \( \beta_0 \) such that, when \( \beta > \beta_0 \) and \( q > 30 \),
\[ 2 < (\lambda_2 + \lambda_{-2}) - (\lambda_3 + \lambda_{-3}) < 3 , \]
then the nonexistence problem will be solved for \( \beta > \beta_0 \). Suppose, for \( i = \pm 2 \) and \( \pm 3 \),
\[ \lambda_1 \in B((1 - \frac{2}{q})(\frac{5 - \xi^2}{6}); \varepsilon) . \]

Here \( B(a;\varepsilon) \) means \( \{x \in \mathbb{R} : |x-a| < \varepsilon\} \). Then
\[ (\lambda_2 + \lambda_{-2}) - (\lambda_3 + \lambda_{-3}) \in B((1 - \frac{2}{q})(\frac{\xi^2_3 - \xi^2}{3}); 4\varepsilon) . \]
It is easy to see that if we choose $\varepsilon = 0.08191$, then

$$2 < (\lambda_2 + \lambda_{-2}) - (\lambda_3 + \lambda_{-3}) < 3.$$ 

**Lemma 3.1.1.** If $\beta > 15$ and $q > 30$, then

$$\lambda_i \in B((1 - \frac{2}{q})(\frac{5 - \xi^2}{6}); 0.08191)$$

for $i = \pm 2$ and $\pm 3$.

**Corollary 3.1.2.** Theorem AB is true if $\beta > 15$ and $q > 30$.

**Proof of Lemma 3.1.1:** It is enough to show that $F_6(x)/q^6$ changes its sign at $x = \alpha + \beta \xi + (1 - \frac{2}{q})(\frac{5 - \xi^2}{6}) \pm 0.08191$ for $i = \pm 2$, $\pm 3$.

First, we rewrite $F_6(x)$ by the substitutions

$$x = (\alpha + \beta \xi + \lambda) \beta^2 q + \frac{7}{2} + \beta \xi + \lambda \quad \text{and} \quad n - 6 = \frac{\beta^2 2}{q^2}.$$ 

Thus, $F_6(x)$ can be written in terms of $q$, $\beta$, $\xi$, and $\lambda$. Let

$$\frac{F_6(x)}{q^6} = \sum_{k=0}^{\infty} A_k \beta^k$$

where the $A_k$'s are expressions in $q$, $\xi$, and $\lambda$. Then, by straightforward calculation, we have $A_k = 0$ for $k > 7$. If we further replace $\lambda$ by $(1 - \frac{2}{q})(\frac{5 - \xi^2}{6}) \pm \varepsilon$ in the expressions $A_5$ and $A_4$, we get
\[ A_6 = \frac{1}{720} H_6(\xi), \]

\[ A_5 = (1 - \frac{2}{q}) \frac{-\xi}{720} H_6(\xi) \pm \frac{1}{8} \xi - \frac{1}{12} \xi^3 + \frac{1}{120} \xi^5, \]

\[ A_4 = \left[ \frac{1}{48} (3-6 \xi^2 + \xi^4) \frac{5-\xi^2}{6} + \frac{1}{12} (\xi - 1)^2 (\frac{5-\xi^2}{6})^2 - \frac{1}{576} (18 \xi^2 - 26) \right] (1 - \frac{2}{q})^2 \]

\[ \pm \frac{1}{24} (3 - 6 \xi^2 + \xi^4) \frac{5 - \xi^2}{6} + \frac{1}{12} (\xi - 1)^2 \] \(\frac{1}{12} (\xi - 1)^2\) \(1 - \frac{2}{q}\)

\[ + \frac{1}{48} \frac{2}{(3-6 \xi^2 + \xi^4)} - \frac{1}{576} (7 \xi^4 - 48 \xi^2 + 27), \]

\[ A_3 = \frac{1}{144^1 \xi \lambda} \left[ -12 \lambda^2 + 4 \xi^2 \lambda^2 + 12 (1 - \frac{2}{q}) \lambda - 7 \xi^2 + 3 (5 + \frac{12}{q} - \frac{12}{q^2}) \right] \]

\[ - \frac{1}{720} (29 + \frac{14}{q} - \frac{216}{q^2} + \frac{144}{q^3}) \xi, \]

\[ A_2 = \frac{1}{48} \lambda^4 (\xi - 1) + \frac{1}{36} \lambda^3 (1 - \frac{2}{q}) + \frac{1}{96} \lambda^2 (5 + \frac{12}{q} - \frac{12}{q^2} - 7 \xi^2) \]

\[ - \frac{1}{720} \lambda (29 + \frac{14}{q} - \frac{216}{q^2} + \frac{144}{q^3}) + \frac{259}{11520} \xi^2 \]

\[ - \frac{1}{768} (9 + \frac{24}{q} + \frac{104}{q^2} - \frac{256}{q^3} + \frac{128}{q^4}), \]

\[ A_1 = \frac{1}{5760} \xi \lambda (48 \lambda^4 - 280 \lambda^2 + 259), \]

\[ A_0 = \frac{1}{11520} \xi \lambda^2 (16 \lambda^4 - 140 \lambda^2 + 259) - \frac{5}{1024}. \]
By using

\[ 3.5689847 < \xi_2^2 < 3.5689849 , \]

\[ 11.050687 < \xi_3^2 < 11.050689 , \]

\[ q > 30 , \quad \epsilon = 0.08191 , \]

\[ |\lambda| < \begin{cases} 
0.3204127 & \text{if } \lambda = (1 - \frac{2}{q})(\frac{5 - \xi_2^2}{6}) \pm \epsilon \\
1.090357 & \text{if } \lambda = (1 - \frac{2}{q})(\frac{5 - \xi_3^2}{6}) \pm \epsilon 
\end{cases} \]

we get

\[ A_6 = 0 \text{ if } \xi = \xi_{\pm 2} \text{ or } \xi_{\pm 3} ; \]

\[ A_5 = \pm \epsilon \left( \frac{1}{8} - \frac{1}{12} + \frac{1}{120} \xi^5 \right) \text{ if } \xi = \xi_{\pm 2} \text{ or } \xi_{\pm 3} ; \]

\[ |A_5| > \begin{cases} 
0.0102545 & \text{if } \xi = \xi_{\pm 2} \\
0.0638226 & \text{if } \xi = \xi_{\pm 3} 
\end{cases} \]

\[ |A_4| < \begin{cases} 
0.08778691 & \text{if } \xi = \xi_{\pm 2} \\
0.67601801 & \text{if } \xi = \xi_{\pm 3} 
\end{cases} \]

\[ |A_3| < \begin{cases} 
0.267625 & \text{if } \xi = \xi_{\pm 2} \\
3.784096 & \text{if } \xi = \xi_{\pm 3} 
\end{cases} \]
When $\beta > 15$, we get

$$|A_5| > |A_4/\beta| + |A_3/\beta^2| + |A_2/\beta^3| + |A_1/\beta^4| + |A_0/\beta^5|.$$  

Therefore,

$$\frac{F_6(x)}{q^6} = A_6\beta^6 + A_5\beta^5 + \ldots + A_0$$

(and hence $F_6(x)$ changes its sign at

$$x = a + \beta\xi_i + (1 - \frac{2}{q})(\frac{5 - \xi_i^2}{6}) \pm 0.08191$$

for $i = \pm 2, \pm 3$.

And Lemma 3.1.1 is proved.
Now we discuss the remaining case $\beta < 15$, $e = 6$, and $q > 30$.

Suppose there exist perfect 6-codes or tight 12-designs in $H(n,q)$.

Then by Corollary 1.4.2, we have

1. $q | 6(n-6)$
2. $q^2 | 15(n-6)(n-5)$
3. $q^3 | 20(n-6)(n-5)(n-4)$
4. $q^4 | 15(n-6)(n-5)(n-4)(n-3)$
5. $q^5 | 6(n-6)(n-5)(n-4)(n-3)(n-2)$
6. $q^6 | (n-6)(n-5)(n-4)(n-3)(n-2)(n-1)$

For the rest of this section, we assume that $\beta < 15$, $q > 30$, $n > 7$, and that there exist perfect 6-codes or tight 12-designs in $H(n,q)$.

**Lemma 3.1.3.** Under the assumption above, we have

1. $p^s | q$, $s > 1$ and $p$ is a prime $> 7$ $\Rightarrow$ $p^6s | (n-6)$
2. $5^s | q$ and $s > 1$ $\Rightarrow$ $5^{6s-1} | (n-6)$
3. $3^s | q$ and $s > 2$ $\Rightarrow$ $3^{6s-1} | (n-6)$
4. $3 | q$ $\Rightarrow$ $3^5 | (n-6)(n-5)$
5. $2^s | q$ and $s > 2$ $\Rightarrow$ $2^{6s-3} | (n-6)$
6. $2 | q$ $\Rightarrow$ $8 | (n-6)(n-5)$
Proof: We get (7) by using (1) and (6),
(8) by using (1), (3) and (6),
(9) by using (1), (2) and (6),
(10) by using (2), (3) and (6),
(11) by using (1), (2) and (6), and
(12) by using (2) and (4).

Lemma 3.1.4. Theorem AB is true if \( \beta < 15 \) and \( q > 30 \).

Proof: First, \( \beta = \sqrt{(n-6)(q-1)} / q < 15 \) and \( q > 30 \) imply that
\[
\frac{(n-6)}{q} < 15 \left( \frac{q}{q-1} \right) < 15 \left( \frac{30}{29} \right) < 233
\]
and hence, by (1),
\[
\frac{6(n-6)}{q} \text{ is an integer } < 1400.
\]

Suppose there is a prime number \( p > 7 \) such that \( p^s \mid q \) and \( p^{s+1} \mid q \) for some \( s > 1 \). Then by (7), \( p^s \mid (n-6) \) and hence \( \frac{6(n-6)}{q} \) is an integer \( > p^s > 5 < 1400 \), which is a contradiction.

Suppose \( 5^s \mid q \) and \( 5^{s+1} \mid q \) for some \( s > 2 \). Then by (8), we get
\[
5^{s-1} \mid (n-6) \text{ and hence } \frac{6(n-6)}{q} > 5^{s-1} > 5 > 1400, \text{ a contradiction.}
\]

Similarly, if \( 3^s \mid q \) and \( 3^{s+1} \mid q \) for some \( s > 2 \), then by (9) we have \( 3^{s-1} \mid (n-6) \) and hence \( \frac{6(n-6)}{q} > 3^{s-1} > 3 > 1400 \), again a contradiction.

If \( 2^s \mid q \) and \( 2^{s+1} \mid q \) for some \( s > 3 \), then by (11),
\[
2^{s-3} \mid (n-6) \text{ and hence } \frac{6(n-6)}{q} > 2^{s-3} > 2 > 1400, \text{ a contradiction.}
\]

From the above discussions, we get \( q \mid 2^2 \cdot 3 \cdot 5 = 60 \). Since \( q > 30 \), \( q = 30 \) or \( 60 \).
Suppose $q = 60$. Then by (8) and (11), $5^2 2^9 | (n-6)$ and hence $\frac{6(n-6)}{q} > \frac{6 \cdot 5 \cdot 2}{60} > 1400$, a contradiction.

Suppose $q = 30$. Then by (8), $5 \mid (n-6)$ and hence $\frac{6(n-6)}{q}$ is a multiple of $5 = 625$. Since $\frac{6(n-6)}{q} < 1400$, $(n-6) = 5$ or $2 \cdot 5^4$. Either $(n-6) = 5^5$ or $(n-6) = 2 \cdot 5^5$, $8 | (n-6)(n-5)$, which contradicts (12).

We have proved the lemma.

We summarize this section by

**Lemma 3.1.5.** Theorem AB is proved if $e = 6$ and $q > 30$.

§3.2 The Case $e = 6$ and $3 < q < 29$.

In this section, we will use the same notation as and parallel discussions to those of Section 3.1 to get $\beta_0 = 64.15$ for $13 < q < 29$ and $\beta_0 = 29.1$ for $3 < q < 12$, then use the divisibility conditions (1) - (6) listed in Section 3.1 and a computer search to rule out the possibilities of the remaining cases (for the existence of perfect codes and tight designs).
First by calculation, we have

(i) $2.110224 < \left(1 - \frac{2}{q}\right)\left(\frac{\xi_3^2 - \xi_2^2}{3}\right) < 2.321910$ if $13 < q < 29$, and if we choose $\varepsilon = 0.027555$ in this case, then

$$2 < (\alpha_2 + \alpha_2) - (\alpha_3 + \alpha_3) < 3.$$  

(ii) $0.354295 < \left(1 - \frac{2}{q}\right)\left(\frac{\xi_2^2 - \xi_1^2}{3}\right) < 0.885739$ if $3 < q < 12$, and if we choose $\varepsilon = 0.028565$ in this case, then

$$0 < (\alpha_1 + \alpha_1) - (\alpha_2 + \alpha_2) < 1.$$  

**Lemma 3.2.1.** If $\beta > 64.15$ and $13 < q < 29$, then

$$\lambda_i \in B((1 - \frac{2}{q})(\frac{5 - \xi_i^2}{6}); 0.027555) \text{ for } i = \pm 2, \pm 3.$$  

**Corollary 3.2.2.** Theorem AB is proved if

$$\beta > 64.15 \text{ and } 13 < q < 29.$$  

**Lemma 3.2.3.** If $\beta > 29.1$ and $3 < q < 12$, then

$$\lambda_i \in B((1 - \frac{2}{q})(\frac{5 - \xi_i^2}{6}); 0.028565) \text{ for } i = \pm 1, \pm 2.$$  

**Corollary 3.2.4.** Theorem AB is proved if

$$\beta > 29.1 \text{ and } 3 < q < 12.$$
The proofs of Lemmas 3.2.1 and 3.2.3 are the same as that of Lemma 3.1.1.

Proof of Lemma 3.2.1: We show that $F_6(x)/q^6$ changes its sign at
\[ x = \alpha + \beta \xi_1 + \left(1 - \frac{2}{q}\right)(\frac{5 - \xi_2}{6}) \pm 0.027555 \quad \text{for } i = \pm 2, \pm 3. \]

By using $13 < q < 29$, $\epsilon = 0.027555$, and
\[ |\lambda| < \begin{cases} 0.2497 & \text{if } \lambda = (1 - \frac{2}{q})(\frac{5 - \xi_2}{6}) \pm \epsilon \\ 0.9665 & \text{if } \lambda = (1 - \frac{2}{q})(\frac{5 - \xi_3}{6}) \pm \epsilon \end{cases}, \]
we get
\[ A_6 = 0 \quad \text{and} \]
\[ A_5 = \pm \epsilon \left(\frac{1}{8} - \frac{1}{12} \xi_3 + \frac{1}{120} \xi_5\right) \quad \text{if } \xi = \xi_{\pm 2} \text{ or } \xi_{\pm 3}; \]
\[ |A_5| > \begin{cases} 0.00344932 & \text{if } \xi = \xi_{\pm 2} \\ 0.0203129 & \text{if } \xi = \xi_{\pm 3} \end{cases}; \]
\[ |A_4| < \begin{cases} 0.094089 & \text{if } \xi = \xi_{\pm 2} \\ 1.244 & \text{if } \xi = \xi_{\pm 3} \end{cases}; \]
\[ |A_3| < \begin{cases} 0.27492 & \text{if } \xi = \xi_{\pm 2} \\ 3.72363 & \text{if } \xi = \xi_{\pm 3} \end{cases}; \]
\[ |A_2| < \begin{cases} 0.13603 & \text{if } \xi = \xi_{\pm 2} \\ 1.282 & \text{if } \xi = \xi_{\pm 3} \end{cases} \]

\[ |A_1| < 0.7 \quad \text{if } \xi = \xi_{\pm 2} \text{ or } \xi_{\pm 3} \]

\[ |A_0| < 0.03 \quad \text{if } \xi = \xi_{\pm 2} \text{ or } \xi_{\pm 3}. \]

Now, when \( \beta > 64.15 \), we have

\[ |A_5| > \frac{|A_4|}{\beta} + \frac{|A_3|}{\beta^2} + \frac{|A_2|}{\beta^3} + \frac{|A_1|}{\beta^4} + \frac{|A_0|}{\beta^5} \]

and the lemma is proved.

**Proof of Lemma 3.2.3:** By using \( 3 < q < 12 \), \( \epsilon = 0.028565 \), and

\[ \lambda = \left(1 - \frac{2}{q}\right) \left(\frac{5 - \xi_1^2}{6}\right) \pm \epsilon \]

\[ |\lambda| < \begin{cases} 0.6702 & \text{if } \lambda = \left(1 - \frac{2}{q}\right) \left(\frac{5 - \xi_1^2}{6}\right) \pm \epsilon \\ 0.2274 & \text{if } \lambda = \left(1 - \frac{2}{q}\right) \left(\frac{5 - \xi_2^2}{6}\right) \pm \epsilon \end{cases} \]

we get

\[ |A_5| > \begin{cases} 0.00166493 & \text{if } \xi = \xi_{\pm 1} \\ 0.00357575 & \text{if } \xi = \xi_{\pm 2} \end{cases} \]
When $\beta > 29.1$, 

$$|A_4| < \begin{cases} 
0.01532144 & \text{if } \xi = \xi_{\pm 1} \\
0.0941080 & \text{if } \xi = \xi_{\pm 2} 
\end{cases} \; .$$

$$|A_3| < \begin{cases} 
0.15065 & \text{if } \xi = \xi_{\pm 1} \\
0.27492 & \text{if } \xi = \xi_{\pm 2} 
\end{cases} \; .$$

$$|A_2| < \begin{cases} 
0.11773 & \text{if } \xi = \xi_{\pm 1} \\
0.13603 & \text{if } \xi = \xi_{\pm 2} 
\end{cases} \; .$$

$$|A_1| < 0.03 \quad \text{if } \xi = \xi_{\pm 1} \; \text{or} \; \xi_{\pm 2} \; .$$

$$|A_0| < 0.02 \quad \text{if } \xi = \xi_{\pm 1} \; \text{or} \; \xi_{\pm 2} \; .$$

When $\beta > 29.1$, 

$$|A_5| > \frac{|A_4|}{\beta} + \frac{|A_3|}{\beta^2} + \frac{|A_2|}{\beta^3} + \frac{|A_1|}{\beta^4} + \frac{|A_0|}{\beta^5} \; .$$

**Lemma 3.2.5.** Theorem AB is true for $13 < q < 29$ and $\beta < 64.15$. 

**Proof:** Since $\beta = \frac{\sqrt{(n-6)(q-1)}}{q} < 64.15$, we have

$$(13) \quad (n - 6) < 4115.2225 \left(\frac{q^2}{q - 1}\right) \; .$$
Suppose $q = 13, 14, 17, 19, 21, 22, 23, 26, \text{ or } 29$. Then $q$ contains a prime $p > 7$. By (7), we have $p^6 | (n-6)$ and hence

\[
\frac{n-6}{p} = k(n-6) > 5 > 7 > 3(4115.2225)(\frac{q}{q-1}) \quad \text{where} \quad k = 1, 2, \text{ or } 3
\]

depending on the value of $q$. But this contradicts the fact that

\[
k(n-6) < 3(n-6) < 3(4115.2225)(\frac{q}{q-1}).
\]

Suppose $q = 28$. Then by (7) and (11), $2^9 7 | (n-6)$, which immediately gives a contradiction that $(n-6) > 2^7 > 4115.2225(\frac{28}{27})$.

Suppose $q = 27$ or $18$. Then by (9), we have $3^{11} | (n-6)$, and we get $(n-6) > 3^{11} > 4115.2225(\frac{q}{q-1})$, again a contradiction.

If $q = 25$, then $(n-6) < 107168$. By (8), we have $5^{11} | (n-6)$ and hence $(n-6) > 5^{11} > 107168$, a contradiction.

If $q = 24$, then $(n-6) < 103060$. By (11), $2^{15} | (n-6)$, and by (10), $3^5 | (n-6)(n-5)$. Suppose $2^3 3^5 | (n-6)$. Then

\[
(n-6) > 2^3 3^5 > 103060, \text{ a contradiction.}
\]

Suppose $2^{15} | (n-6)$ and $3^5 | (n-5)$. Then $(n-6) \equiv 46 \cdot 2^{15} \pmod{2^3 3^5}$. Thus $(n-6) > 46 \cdot 2^{15} > 10306$, a contradiction again.

If $q = 20$, then $(n-6) < 86637$. By (8) and (11), $2^9 5^5 | (n-6)$.

Hence $(n-6) > 2^9 5^5 > 86637$, a contradiction.

If $q = 16$, then $(n-6) < 70234$. By (11), $2^{21} | (n-6)$. Thus

\[
(n-6) > 2^{21} > 70234, \text{ a contradiction.}
\]

If $q = 15$, then $(n-6) < 66138$. By (8), $5^5 | (n-6)$, and by (10), $3^5 | (n-6)(n-5)$. In either the case $5^5 3^5 | (n-6)$ or the case $5^5 | (n-6)$ and $3^5 | (n-5)$, which implies $(n-6) \equiv 193 \cdot 5 \pmod{5^5 3^5}$, we have $(n-6) > 66138$, a contradiction.

The lemma is proved.
LEMMA 3.2.6. Theorem AB is true for $7 < q < 12$ and $\beta < 29.1$.

Proof: Since $\beta = \sqrt{(n-6)(q-1)} < 29.1$, we have

\[(n - 6) < (29.1)^2 \left(\frac{q}{q - 1}\right).
\]

If $q = 12$, then $(n-6) < 11086$. By (11), $2^9 | (n-6)$; and by (10), $3^5 | (n-6)$ or $(n-5)$. Suppose $2^3 | (n-6)$. Then $2^9 > 2^3 > 11086$, a contradiction. Suppose $2^9 | (n-6)$ and $3^5 | (n-5)$. Then $(n-6) \equiv 28 \cdot 2 \pmod{2^3}$. Thus, $(n-6) > 28 \cdot 2 = 14336 > 11086$, again a contradiction.

If $q = 11$, then $(n-6) < 10247$. By (7), $11^6 | (n-6)$. But $11^6 > 10247$.

If $q = 10$, then $(n-6) < 9409$. By (8), $5^5 | (n-6)$. Thus, $(n-6) = 5^k$ for $1 < k < 3$. But by (12), we have $8 | (n-6)$ or $(n-5)$, which is impossible for $(n-6) = 5^k$ ($1 < k < 3$).

If $q = 9$, then $(n-6) < 8574$ and, by (9), $3^{11} | (n-6)$. Thus, $(n-6) > 3^1 > 8574$, a contradiction.

If $q = 8$, then $(n-6) < 7743$ and by (11), $2^{15} | (n-6)$. But $2^{15} = 32768 > 7743$.

If $q = 7$, then $(n-6) < 6916$ and by (7), $7^6 | (n-6)$. But $7^6 > 6916$.

The lemma is proved.

LEMMA 3.2.7. Theorem AB is true for $3 < q < 6$ and $\beta < 29.1$ with the following possible exceptions.

(i) $q = 6$ and $(n-6) = 3^5 \cdot 2^3 \cdot k$ for $1 < k < 3$,

(ii) $q = 6$ and $(n-5) = 3^5 \cdot 2^3 \cdot k$ for $1 < k < 3$,
\[
q = 6 \quad \text{and} \quad (n-6) = 3^5 (5 + 8k) \quad \text{for} \quad 0 < k < 2 ,
\]
\[
q = 6 \quad \text{and} \quad (n-5) = 3^5 (3 + 8k) \quad \text{for} \quad 0 < k < 2 .
\]

(ii) \( q = 5 \) and \( (n-6) = 5^5 \).

(iii) \( q = 4 \) and \( (n-6) = 2^9 k \) for \( 1 < k < 8 \).

(iv) \( q = 3 \) and \( (n-6) = 3^5 k \) for \( 1 < k < 15 \),
\[
q = 3 \quad \text{and} \quad (n-5) = 3^5 k \quad \text{for} \quad 1 < k < 15 .
\]

**Proof:** Suppose \( q = 6 \). Then \( (n-6) < 6098 \). By (10) and (12), we have \( 3^5 \mid (n-6)(n-5) \) and \( 2^3 \mid (n-6)(n-5) \). Hence we get (i).

Suppose \( q = 5 \). Then \( (n-6) < 5293 \). By (8), we get \( 5^5 \mid (n-6) \). Hence \( (n-6) = 5^5 \).

Suppose \( q = 4 \). Then \( (n-6) < 4517 \). By (11), we get \( 2^9 \mid (n-6) \). Hence \( (n-6) = 2^9 k \) for \( 1 < k < 8 \).

Suppose \( q = 3 \). Then \( (n-6) < 3811 \). By (10), we have \( 3^5 \mid (n-6)(n-5) \). Thus \( (n-6) \) or \( (n-5) = 3^5 k \) for \( 1 < k < 15 \).

**Lemma 3.2.8.** Theorem AB is true if \( e = 6 \) and \( 3 < q < 29 \).

**Proof:** See Corollaries 3.2.2 and 3.2.4, Lemmas 3.2.5, 3.2.6 and 3.2.7. The remaining cases listed in Lemma 3.2.7 are treated in the Appendix.
$\S 3.3$ The Case $e = 8$ and $q > 30$.

The discussion for the case $e = 8$ is basically similar and parallel to that for $e = 6$. (Also see [6].)

Again we start with the Hermite polynomial

$$H_8(x) = x^8 - 28x^6 + 210x^4 - 420x^2 + 105.$$

By applying the Cardano formula, the zeros $\xi_i$ ($i = \pm 1, \pm 2, \pm 3, \pm 4$) of $H_8(x)$ are located as follows.

$$0.2906070 < \xi_1 < 0.2906071,$$

$$2.6781945 < \xi_2 < 2.6781946,$$

$$7.8539270 < \xi_3 < 7.8539271,$$

$$17.177271 < \xi_4 < 17.177272,$$

and

$$\xi_{-i} = -\xi_i \text{ for } i = 1, 2, 3, 4.$$

The zeros of $F_8(x)$ can be expressed as

$$\alpha_i = \alpha + \beta \xi_i + \lambda_i \quad (i = \pm 1, \pm 2, \pm 3, \pm 4)$$

where

$$\alpha = \frac{(n-8)(q-1)}{q} + \frac{9}{2} = \beta q + \frac{9}{2},$$

$$\beta = \sqrt{\frac{(n-8)(q-1)}{q}},$$

and

$$\lambda_i \rightarrow \left(1 - \frac{2}{q}\right)\left(\frac{7 - \xi_i^2}{6}\right) \text{ as } \beta \rightarrow \infty.$$
If there exist perfect 8-codes or tight 16-designs in \( H(n,q) \), then

\[
(\alpha + \alpha_{-1}) - (\alpha + \alpha_{-3}) = (\lambda + \lambda_{-1}) - (\lambda + \lambda_{-3})
\]

where

\[
(\lambda + \lambda_{-1}) - (\lambda + \lambda_{-3}) \implies (1 - \frac{2}{q})(\xi_{3}^{2} - \xi_{1}^{2}) \text{ as } \beta \to \infty,
\]

and

\[
2.3530329 < (1 - \frac{2}{q})(\xi_{3}^{2} - \xi_{1}^{2}) < 2.5211067,
\]

assuming that \( q > 30 \). Suppose \( \lambda_{i} \in B((1 - \frac{2}{q})(\xi_{i};\epsilon)) \) for \( i = \pm 1 \), \( \pm 3 \). Then

\[
(\lambda + \lambda_{-1}) - (\lambda + \lambda_{-3}) \in B((1 - \frac{2}{q})(\xi_{3}^{2} - \xi_{1}^{2});4\epsilon).
\]

If we choose \( \epsilon = 0.088258 \), then \( 2 < (\lambda + \lambda_{-1}) - (\lambda + \lambda_{-3}) < 3 \).

Now, by fixing \( \epsilon = 0.088258 \), we are ready to obtain \( \beta_{0} \).

**Lemma 3.3.1.** If \( \beta > 18 \) and \( q > 30 \), then

\[
\lambda_{i} \in B((1 - \frac{2}{q})(\frac{7 - \xi_{i}^{2}}{6});0.088258) \text{ for } i = \pm 1 \text{ and } \pm 3.
\]

**Corollary 3.3.2.** Theorem AB is true if \( \beta > 18 \) and \( q > 30 \).

**Proof of Lemma 3.3.1:** We will show that \( F_{8}(x) \) changes its sign at

\[
x = \alpha + \beta \xi_{i} + (1 - \frac{2}{q})(\xi_{i};\epsilon) \pm 0.088258 \text{ for } i = \pm 1 \text{ and } \pm 3.
\]

First, we rewrite \( F_{8}(x) \) by the substitutions

\[
x = (\alpha + \beta \xi + \lambda) \quad \beta q + \frac{9}{q} + \beta \xi + \lambda \quad \text{and} \quad n - 8 = \frac{2}{q - 1}.
\]

Then \( F_{8}(x) \) is rewritten in terms of \( q, \beta, \xi, \) and \( \lambda \).
Let \( F(x) \xi/q = \sum_{k=0}^{\infty} A_k \xi^k \) where \( A_k \)'s are expressions in \( \xi \), and \( \lambda \). Also, replace \( \lambda \) by \((1-\frac{2}{q})(\frac{7-\xi^2}{6})\pm\epsilon\) in the expressions \( A_7 \) and \( A_6 \). Then, by straightforward calculation, we get

\[
A_k = 0 \text{ for } k > 9 , \\
A_8 = H_8(\xi) , \\
A_7 = (1-\frac{2}{q})(\frac{4}{3})\xi H_7(\xi) \pm \epsilon \left[ 8\xi(\xi^6 - 21\xi^4 + 105\xi^2 - 105) \right] , \\
A_6 = (1-\frac{2}{q})^2 \left[ (\frac{7-\xi^2}{6})^2 (2\xi^6 - 420\xi^4 + 1260\xi^2 - 420) + \right. \\
\left. \frac{7-\xi^2}{6} (280\xi^4 - 1680\xi^2 + 840) - 105\xi^4 + 910\xi^2 - 595 \right] \\
\pm (1-\frac{2}{q})\epsilon \left[ (\frac{7-\xi^2}{6})(56\xi^6 - 840\xi^4 + 2520\xi^2 - 840) + 280\xi^4 - 1680\xi^2 + 840 \right] \\
+ \epsilon^2 \left[ 28\xi^6 - 420\xi^4 + 1260\xi^2 - 420 \right] \\
- 21\xi^6 + 350\xi^4 - 1155\xi^2 + 420 , \\
A_5 = (840 - 560\xi^2 + 56\xi^4)\xi^3 + 90(\xi^6 - 1680 + 56\xi^2)\xi^2 \\
- 70(7 + \frac{104}{q} - \frac{104}{q^2})\xi\lambda + 140(7 + \frac{12}{q} - \frac{12}{q^2})\xi^3 \lambda - 126\xi^5 \lambda \\
+ 28(1-\frac{2}{q})(29 + \frac{132}{q} - \frac{132}{q^2})\xi - 28(1-\frac{2}{q})(13 + \frac{24}{q} - \frac{24}{q^2})\xi^3 ,
\]
\[ a_4 = 70\lambda^4 (3 - 6\xi^2 + \xi^4) + (1 - \frac{2}{q})\lambda^3 (-560 + 560\xi^2) \\
+ 35\lambda^2 \left[-(7 + \frac{104}{q} - \frac{104}{q^2}) + (42 + \frac{72}{q} - \frac{72}{q^2})\xi^2 - 9\xi^4\right] \\
+ 28(1 - \frac{2}{q})\lambda \left[(29 - \frac{132}{q} - \frac{132}{q^2}) - (39 + \frac{72}{q} - \frac{72}{q^2})\xi^2\right] \\
+ \left[-35.875 + (1 - \frac{1}{q})^2 (\frac{574}{q} - \frac{7308}{q^2})\right] \\
- 7\left[64.75 + (1 - \frac{1}{q}) (\frac{150}{q} + \frac{480}{q} - \frac{480}{q^2})\xi^2 + 123.375\xi^4\right], \\
\]

\[ a_3 = 56\xi^5 (-3 + \xi^2) + 280\xi^4 (1 - \frac{2}{q}) + 140\xi^3 (7 + \frac{12}{q} - \frac{12}{q^2} - 3\xi^2) \\
- 84\xi^2 (1 - \frac{2}{q}) (13 + \frac{24}{q} - \frac{24}{q^2}) \\
- 14\xi \left[64.75 + (1 - \frac{1}{q}) (\frac{150}{q} + \frac{480}{q} - \frac{480}{q^2})\right] \\
+ 493.5\xi^3 \lambda + (375.5 + \frac{281}{q} - \frac{216}{q^2} - \frac{9456}{q^3} + \frac{14400}{q^4} - \frac{5760}{q^5})\xi^2, \\
\]

\[ a_2 = 28\lambda^6 (-1 + \xi^2) + 56\lambda^5 (1 - \frac{2}{q}) + 35\lambda^4 (7 + \frac{12}{q} - \frac{12}{q^2} - 9\xi^2) \\
- 28\lambda^3 (1 - \frac{2}{q}) (13 + \frac{24}{q} - \frac{24}{q^2}) \\
- 7\lambda^2 \left[64.75 + (1 - \frac{1}{q}) (\frac{150}{q} + \frac{480}{q} - \frac{480}{q^2})\right] \\
+ 740.25\xi^2 \lambda^2 + \lambda (375.5 + \frac{281}{q} - \frac{216}{q^2} - \frac{9456}{q^3} + \frac{14400}{q^4} - \frac{5760}{q^5}) \\
+ 98.4375 + \frac{236.25}{q} + 840(1 - \frac{1}{q}) (1 + \frac{6}{q} - \frac{6}{q^2}) - 201.8125\xi^2, \\
\]
and
\[ A_1 = 8\xi^7 - 126\xi^5 + 493.5\xi^3 - 403.625\xi \]

\[ A_0 = \lambda - 21\lambda^6 + 123.375\lambda^4 - 201.8125\lambda^2 + 43.06640625 \]

Now, by using
\[ 0.2906070 < \xi_1^2 < 0.2906071 , \]
\[ 7.8539270 < \xi_3^2 < 7.8539271 , \]

\[ \eta > 30 , \quad E = 0.088258 , \quad \text{and} \]

\[ |\lambda| < \begin{cases} 
1.2065 & \text{if } i = 1 \\
0.23058 & \text{if } i = 3 , 
\end{cases} \]

we get
\[ A_8 = 0 \quad \text{if } \xi = \xi_{1\pm} \quad \text{or} \quad \xi_{3\pm} ; \]
\[ A_7 = \pm E [8\xi(\xi^6 - 21\xi^4 + 105\xi^2 - 105)] \quad \text{if } \xi = \xi_{1\pm} \quad \text{or} \quad \xi_{3\pm} ; \]

\[ |A_7| > \begin{cases} 
29.0211 & \text{if } \xi = \xi_{1\pm} \\
180.544 & \text{if } \xi = \xi_{3\pm} , 
\end{cases} \]
\[ |A_6| < \begin{cases} 
100.80861 & \text{if } \xi = \xi_{1\pm} \\
2619.7319 & \text{if } \xi = \xi_{3\pm} , 
\end{cases} \]
\[ |A_5| < \begin{cases} 
2578 & \text{if } \xi = \xi_{1\pm} \\
7291 & \text{if } \xi = \xi_{3\pm} ; 
\end{cases} \]
When $\beta \geq 18$, 

$$|A_7| > \frac{|A_6|}{\beta} + \frac{|A_5|}{\beta^2} + \ldots + \frac{|A_0|}{\beta^7}.$$ 

Therefore 

$$G_8(x) = F_8(x)8!/q^8 = A_8\beta^8 + A_7\beta^7 + \ldots + A_0,$$

and hence $F_8(x)$ changes its sign at 

$$x = \alpha + \beta \xi_1 + \left(1 - \frac{2}{q}\right)(\frac{1}{6}) \pm 0.088258 \text{ for } i = \pm 1 \text{ and } \pm 3.$$ 

This completes the proof of Lemma 3.3.1.

Now, we consider the remaining case. For the rest of this section, we assume that $\beta < 18$, $q > 30$, $e = 8$, $n > 9$, and
that there exist perfect 8-codes or tight 16-designs in $H(n,q)$.

Then by Corollary 1.4.2, we have

(1) $q | 8(n-8)$
(2) $q^2 | 28(n-8)(n-7)$
(3) $q^3 | 56(n-8)(n-7)(n-6)$
(4) $q^4 | 70(n-8)(n-7)(n-6)(n-5)$
(5) $q^5 | 56(n-8)(n-7)(n-6)(n-5)(n-4)$
(6) $q^6 | 28(n-8)(n-7)(n-6)(n-5)(n-4)(n-3)$
(7) $q^7 | 8(n-8)(n-7)(n-6)(n-5)(n-4)(n-3)(n-2)$
(8) $q^8 | (n-8)(n-7)(n-6)(n-5)(n-4)(n-3)(n-2)(n-1)$

**Lemma 3.3.3.**

(9) $p | q$, $s > 1$ and $p$ is a prime $> 5$ $\implies$ $p^{8s-1} | (n-8)$
(10) $3^s | q$ and $s > 1$ $\implies$ $3^{8s-2} | (n-8)$
(11) $2^s | q$ and $s > 4$ $\implies$ $2^{8s-4} | (n-8)$

**Proof:** We get (9) by using (1), (3), and (8),
(10) by using (1), (3), and (8), and
(11) by using (1), (2), and (8).

**Lemma 3.3.4.** Theorem AB is true if $\beta < 18$ and $q > 30$. 
Proof: Since

\[ \beta = \frac{(n-8)(q-1)}{q} < 18 \quad \text{and} \quad q > 30 , \]

we have

\[ \frac{(n-8)}{q} < 18 \left( \frac{q}{q-1} \right) < 18 \frac{30}{29} < 336 \]

and hence by (1), \( \frac{8(n-8)}{q} \) is an integer < 2888.

Suppose there is a prime \( p > 5 \) such that \( p^s | q \) and \( p^{s+1} \mid q \) for some \( s > 1 \). Then by (9), \( p^{8s-1} \mid (n-8) \) and hence

\[ \frac{8(n-8)}{q} > \frac{7s-1}{5} > 6 > 2888 \], which contradicts the above bound.

Suppose \( 3^s \mid q \) and \( 3^{s+1} \mid q \) for some \( s > 2 \). Then by (10), \( 3^{8s-2} \mid (n-8) \) and hence \( \frac{8(n-8)}{q} > \frac{7s-2}{3} > 12 > 2888 \), a contradiction.

Suppose \( 2^s \mid q \) and \( 2^{s+1} \mid q \) for some \( s > 4 \). Then by (11), \( 2^{8s-4} \mid (n-8) \) and hence \( \frac{8(n-8)}{q} > \frac{7s-4}{2} > 24 > 2888 \), again a contradiction.

Thus, \( q \mid 2^3 \cdot 3 = 24 \), which contradicts the assumption that \( q > 30 \).

The lemma is proved.

We summarize this section by stating the following lemma.

**Lemma 3.3.5.** Theorem AB is proved if \( e = 8 \) and \( q > 30 \).
§3.4 The Cases $e = 8$ and $3 < q < 29$.

The notation in this section is the same as that in section 3.3. We will first obtain $\beta_0 = 16$ for $3 < q < 29$ and then use the divisibility conditions (1) - (8) listed in section 3.3 and a computer search (see the Appendix) to rule out the remaining cases. First, by calculation, we obtain

$$0.265287 < \left(1 - \frac{2}{q}\right)\left(\frac{\xi_2 - \xi_1}{3}\right) < 0.7409755 \quad \text{if} \quad 3 < q < 29,$$

and

$$0 < (\alpha_1 + \alpha_2) - (\alpha_1 + \alpha_2) < 1 \quad \text{if we take} \quad \epsilon = 0.064756.$$

**Lemma 3.4.1.** If $\beta > 16$ and $3 < q < 29$, then

$$\lambda_i \in B((1 - \frac{2}{q})\left(\frac{7 - \xi_i}{6}\right); 0.064756) \quad \text{for} \quad i = \pm 1, \pm 2.$$

**Corollary 3.4.2.** Theorem AB is true if $\beta > 16$ and $3 < q < 29$.

**Proof of Lemma 3.4.1:** We show that $G(x)$, and hence $F(x)$, changes its sign at

$$x = \alpha + \beta \xi_i + \left(1 - \frac{2}{q}\right)\left(\frac{7 - \xi_i}{6}\right) \pm 0.064756 \quad \text{for} \quad i = \pm 1, \pm 2.$$

Using

$$3 < q < 29, \quad \epsilon = 0.064756, \quad \text{and}$$

$$|\lambda_i| < \begin{cases} 1.105869 & \text{if} \quad i = \pm 1 \\ 0.735381 & \text{if} \quad i = \pm 2, \end{cases}$$
we get

\[ A_8 = 0 \text{ if } \xi = \xi_{\pm 1} \text{ or } \xi_{\pm 2}, \]

\[ A_7 = \pm \varepsilon \left[ 8 \alpha \xi (\xi - 21 \xi^4 + 105 \xi^2 - 105) \right] \text{ if } \xi = \xi_{\pm 1} \text{ or } \xi_{\pm 2}, \]

and

\[ |A_7| > \begin{cases} 
21.2901 & \text{if } \xi = \xi_{\pm 1} \\
37.9754 & \text{if } \xi = \xi_{\pm 2} 
\end{cases}, \]

\[ |A_6| < \begin{cases} 
113.4669 & \text{if } \xi = \xi_{\pm 1} \\
568.4435 & \text{if } \xi = \xi_{\pm 2} 
\end{cases}, \]

\[ |A_5| < \begin{cases} 
721 & \text{if } \xi = \xi_{\pm 1} \\
260 & \text{if } \xi = \xi_{\pm 2} 
\end{cases}, \]

\[ |A_4| < \begin{cases} 
3600 & \text{if } \xi = \xi_{\pm 1} \\
5000 & \text{if } \xi = \xi_{\pm 2} 
\end{cases}, \]

\[ |A_3| < 8000 \text{ if } \xi = \xi_{\pm 1} \text{ or } \xi_{\pm 2}, \]

\[ |A_2| < 5000 \text{ if } \xi = \xi_{\pm 1} \text{ or } \xi_{\pm 2}, \]

\[ |A_1| < 2000 \text{ if } \xi = \xi_{\pm 1} \text{ or } \xi_{\pm 2}, \]

\[ |A_0| < 500 \text{ if } \xi = \xi_{\pm 1} \text{ or } \xi_{\pm 2}. \]
When $\beta > 16$,

$$|A_7| > \frac{|A_6|}{\beta} + \frac{|A_5|}{\beta^2} + \ldots + \frac{|A_0|}{\beta^7} .$$

Thus, the lemma is proved.

Now we use the divisibility conditions (1) - (8) in the last section to investigate the remaining cases $\beta < 16$ and $3 < q < 29$.

First, $\beta < 16$ implies the bound

$$\frac{2}{2} \frac{q}{(n-8)} < 256 \frac{2^9}{q-1} .$$

**Lemma 3.4.3.** Theorem AB is true for $\beta < 16$ and $5 < q < 29$, except possibly $q = 12$ or $6$.

**Proof:** Since $q < 29$, we have

$$(n-8) < 256 \frac{2^9}{2^8} < 7690 .$$

For any prime $p > 5$, if $p|q$, then by (9), we get $p^7|(n-8)$. Thus $p^7 > 5^7 = 78125 > 7690$, a contradiction. Therefore, if a prime $p|q$, then $p = 2$ or $3$.

Suppose $3^2|q$. Then by (10), $3^{14}|(n-8)$ and hence $(n-8) > 3^{14} > 7690$, a contradiction.

Suppose $2^3|q$. Then by (2) and (8), we get $2^{20}|(n-8)(n-7)$. Thus either $(n-8)$ or $(n-7) > 2^{20} > 7690$, again a contradiction.

Thus $q$ can only be $12, 6, 4, or 3$.

**Lemma 3.4.4.** Theorem AB is true for $\beta < 16$ and $q = 12, 6, 4, 3$ with the following exceptions.
(i) \( q = 3 \) and \( (n-8) = 3^6 \),

(ii) \( q = 6 \) and \( (n-8) = 3^6 \).

Proof: Suppose \( q = 12 \) or \( 4 \). Then \( (n-8) < 3352 \). Since \( 4|q \), by (2), (4), and (8), we have \( 2^{12}|(n-8)(n-7) \). Therefore, either
\[
(n-8) \text{ or } (n-7) > 2^{12} = 4096 > 3352 ,
\]
a contradiction.

Suppose \( q = 6 \). Then \( (n-8) < 1844 \). By (10), we get \( 3^6|(n-8) \). Thus \( (n-8) = 3^6 \) or \( 2\cdot3^6 \). If \( (n-8) = 2\cdot3^6 \), then
\[
2^6|(n-8)(n-7)(n-6)...(n-1) ,
\]
which contradicts (8). Thus, \( (n-8) = 3^6 \).

Suppose \( q = 3 \). Then \( (n-8) < 1152 \). Again \( 3^6|(n-8) \). Using \( (n-8) < 1152 \), we have \( (n-8) = 3^6 \).

The lemma is proved.

**Lemma 3.4.5.** Theorem AB is true if \( e = 8 \) and \( 3 < q < 29 \).

Proof: The remaining cases listed in Lemma 3.4.4 are treated in the Appendix.
Chapter IV

THE NONEXISTENCE PROBLEM FOR
THE CASES  e = 3, 4, and 5

Throughout the whole chapter, we will assume that there exist perfect e-codes or tight 2e-designs in H(n,q) if not otherwise indicated.

§4.1  The Case  e = 5 and  q > 3 .

The divisibility conditions in Corollary 1.4.2 become

(1)  \( q \mid 5(n-5) \),

(2)  \( q^2 \mid 10(n-5)(n-4) \),

(3)  \( q^3 \mid 10(n-5)(n-4)(n-3) \),

(4)  \( q^4 \mid 5(n-5)(n-4)(n-3)(n-2) \),

(5)  \( q^5 \mid (n-5)(n-4)(n-3)(n-2)(n-1) \).

We improve them in the following lemma.

**Lemma 4.1.1.** In the following statements (6) - (12), \( q = p^s N \) means that \( q \) has a nontrivial prime power \( p^s \); i.e., \( p \) is a prime number, \( s > 1 \), and \( p \mid N \). We have
(6) \( q = 2N \implies 2^6 \mid (n-5) \) or \( 2^4 \mid (n-3) \),

(7) \( q = 4N \implies 2^7 \mid (n-5) \),

(8) \( q = 2^N \) and \( s > 3 \implies 2^{5s} \mid (n-5) \),

(9) \( q = 3^N \implies 3^{5s} \mid (n-5) \),

(10) \( q = 5N \implies 5^6 \mid (n-5) \) or \( 5^6 \mid (n-4) \),

(11) \( q = 5^N \) and \( s > 2 \implies 5^{5s+1} \mid (n-5) \),

(12) \( q = p^N \) and \( p > 7 \implies p^{5s} \mid (n-5) \).

**Proof:** Suppose \( q = 2N \). Then by (1) and (4), \( 2^3 \mid (n-5) \) or \( 2^3 \mid (n-3) \). Considering \( G_2(x) \equiv 0 \pmod{2} \), \( \pmod{4} \), and \( \pmod{8} \) for some integer \( x_0 \), we get the desired result.

Suppose \( q = 4N \). Then by (1) - (5), we get \( 2^7 \mid (n-5) \).

Suppose \( q = 2^N \) and \( s > 3 \). Then by (1) - (5), \( 2^{5s-3} \mid (n-5) \).

Considering \( G_2(x) \equiv 0 \pmod{2} \), \( \pmod{4} \), and \( \pmod{8} \) for some integer \( x_0 \), we get \( 2^{5s} \mid (n-5) \).

Suppose \( q = 3^N \). Then by (1) - (5) and by considering \( G_3(x) \equiv 0 \pmod{3} \) for some integer \( x_0 \), we get the desired result.

Suppose \( q = 5^N \). Then by (1) - (5), we get \( 5^{5s} \mid (n-5) \) if \( s > 2 \), and \( 5^5 \mid (n-5) \) or \( 5^5 \mid (n-4) \) if \( s = 1 \). We obtain the desired result by further considering \( G_5(x) \equiv 0 \pmod{5} \) for some integer \( x_0 \).

Suppose \( q = p^N \) and \( p > 7 \). Then by (1) - (5), we get \( p^{5s} \mid (n-5) \).

We have finished the proof of the lemma.
Following the notations used in Theorem 1.4.3, we can express the central zero \( a \) of \( G(x) \) as follows.

\[
\alpha_0 = \alpha + \lambda_0 \quad \text{(because } \xi_0 = 0) \]

where

\[
\alpha = (n - 5)(1 - \frac{1}{q}) + 3
\]

and

\[
\lambda \rightarrow \frac{2}{3} \left(1 - \frac{2}{q}\right) \quad \text{as } \beta \to \infty.
\]

From the divisibility conditions (6) - (12) in Lemma 4.1.1 and (1), we know that

\[
q \mid (n-5) \quad \text{except when } q = 5N, 5 | N, \text{ and } 5^6 \mid (n-4).
\]

**COROLLARY 4.1.2.**

\[
\left\{ \begin{array}{l}
\frac{1}{5} \mathbb{Z} & \text{if } q = 5N, 5 \nmid N, \text{ and } 5^6 \mid (n-4) \\
\mathbb{Z} & \text{otherwise}
\end{array} \right.
\]

**LEMMA 4.1.3.** If \( q > 3 \), then \( \alpha < \alpha_0 < \alpha + \frac{2}{3} \).

**Proof:** Let \( m = n - 5 \). Then

\[
G_5(\alpha + \lambda) = G(m \left(1 - \frac{1}{q}\right) + 3 + \lambda)
\]

\[
= \frac{1}{2} (1 - \frac{1}{q}) \left[ 15\lambda - 10 + \frac{20}{q} \right] m^2
\]

\[
+ \frac{1}{q} (1 - \frac{1}{q}) \left[ -10\lambda^3 + 10\lambda^2 + 10\lambda - 4 + \frac{-20\lambda^2 + 30\lambda - 4}{q} + \frac{-30\lambda + 36}{q^2} \right] m + (\lambda - 2)(\lambda - 1)\lambda (\lambda + 1)(\lambda + 2).
\]
The lemma is proved.

**Lemma 4.1.4.**

\[
\begin{align*}
&\alpha + \frac{3}{5} < \alpha_0 < \alpha + \frac{4}{5} \quad \text{if } q = 5N \text{ for } N > 6 \text{ and } 5 \nmid N. \\
&\alpha + \frac{2}{5} < \alpha_0 < \alpha + \frac{3}{5} \quad \text{if } q = 10, 15, \text{ or } 20. \\
&\alpha + \frac{1}{5} < \alpha_0 < \alpha + \frac{2}{5} \quad \text{if } q = 5. 
\end{align*}
\]

**Proof:** Again, we let \( m = (n-5) \). Then by the divisibility conditions (6) – (12), we have that:

If \( q = 5N \) and \( 5 \nmid N \), then

\[
m = (n-5) > \begin{cases} 
q^4 & \text{for } q = 20M \text{ with } 2 \nmid M \\
5 & \text{otherwise.}
\end{cases}
\]

By using these facts, we can prove the following inequality.
\[ G_5(\alpha + \frac{3}{5}) = (1 - \frac{1}{q})^2 \left( \frac{1}{q^2} \right)(-1 + \frac{20}{q})^2 \]

\[ + (1 - \frac{1}{q})\left( \frac{1}{q} \right) \left[ \frac{86}{25} + \frac{34}{5} \right] + \frac{18}{2} - \frac{24}{3} + \frac{24}{4} \]

\[ + \frac{4368}{3125} \]

\[ \begin{cases} < 0 & \text{if } q > 30 \\ > 0 & \text{if } q < 20 \end{cases} \]

\[ G_5(\alpha + \frac{2}{5}) = (1 - \frac{1}{q})^2 \left( \frac{1}{q^2} \right)(-4 + \frac{20}{q})^2 \]

\[ + (1 - \frac{1}{q})\left( \frac{1}{q} \right) \left[ \frac{24}{25} + \frac{24}{5} \right] + \frac{24}{2} - \frac{24}{3} + \frac{24}{4} \]

\[ + \frac{4032}{3125} \]

\[ \begin{cases} < 0 & \text{if } q = 10, 15, \text{ or } 20 \\ > 0 & \text{if } q = 5 \end{cases} \]

If \( q = 5 \), then

\[ G_5(\alpha + \frac{1}{5}) = \frac{-48}{5} m + \frac{-984}{5} m + \frac{2376}{3125} < 0 \text{ because } m > 5. \]

Therefore, we have

\[ \alpha + \frac{3}{5} < \alpha < \alpha + \frac{2}{3} < \alpha + \frac{4}{5} \text{ if } q > 30, \]

\[ \alpha + \frac{2}{5} < \alpha < \alpha + \frac{3}{5} \text{ if } q = 10, 15, 20, \]

\[ \alpha + \frac{1}{5} < \alpha < \alpha + \frac{2}{5} \text{ if } q = 5. \]
**LEMMA 4.1.5.** Theorem AB is true for the cases $e = 5$ and $q > 3$.

**Proof:** By Corollary 4.1.2 and Lemmas 4.1.3 and 4.1.4.

---

§4.2 The Case $e = 4$ and $q > 3$.

The divisibility conditions in Corollary 1.4.2 become

1. $q | 4(n-4)$
2. $q^2 | 6(n-4)(n-3)$
3. $q^3 | 4((n-4)(n-3)(n-2)$
4. $q^4 | (n-4)(n-3)(n-2)(n-1)$

**LEMMA 4.2.1.**

5. $p^s | q$ with $p$ being a prime $> 5$ \(\implies\) $p^4s | (n-4)$
6. $3^t | q$ with $t > 2$ \(\implies\) $3^4t | (n-4)$
7. $3 | q$ and $3^2 | q$ \(\implies\) $3^3 | (n-4)$
8. $2^u | q$ with $u > 3$ \(\implies\) $2^{4u+2} | (n-4)$
9. $4 | q$ and $8 | q$ \(\implies\) $2^{10} | (n-4)$ or $2^8 | (n-3)$
10. $2 | q$ and $2^2 | q$ \(\implies\) $2^6 | (n-4)$, $2^4 | (n-3)$

**Proof:** The proof is similar to that of Lemma 4.1.1. We use (1) - (4) and the fact that $G(x) \equiv 0 \pmod{\text{any number}}$ for some integer $x$ to prove the lemma.
For (6), we consider $G(x) \equiv 0 \pmod{3}$.

For (8), we do $\pmod{8}$.

For (9), we use $\pmod{8}$ to get $2^{10} | (n-4)$ and $\pmod{2}$ to get $2^6 | (n-3)$.

For (10), we use $\pmod{8}$ to get $2^6 | (n-4)$.

The rest are done by simply using (1) - (4).

For the rest of the section, the notation used is the same as in Theorem 1.4.3. For example, $\beta = \frac{\sqrt{(n-4)(q-1)}}{q}$. By using Lemma 4.2.1, we immediately get some lower bounds on $\beta$.

**Lemma 4.2.2.**

(i) If $q > 5$ and $q \not\equiv 4 \pmod{8}$, then $\beta > 5$.

(ii) If $q = 4k$ and $k$ is odd, then

$$\beta > \begin{cases} 
22 & \text{if } k > 3 \\
6.9 & \text{if } k = 1 
\end{cases}$$

(iii) If $q = 3$, then $\beta > \sqrt{6}$.

**Proof:** Let $q = 2^u 3^t N$ where $2|N$, $3|N$, $t > 0$, $u > 0$.

(i) If $u \neq 1$ or 2, then by Lemma 4.2.1, we have $4q | (n-4)$ if $t \neq 1$ and $\frac{4^4}{3} | (n-4)$ if $t = 1$. Therefore $(n-4) > q^4$.

We get

$$\beta = \frac{\sqrt{(n-4)(q-1)}}{q} > q\sqrt{q-1} > 5 \quad \text{if } q > 5.$$ 

Suppose $u = 1$. Then by (10) in Lemma 4.2.1,
\[
(n-4) > \begin{cases} 
\frac{4}{16} & \text{if } t \neq 1 \\
\frac{4}{48} & \text{if } t = 1 
\end{cases}
\]

Thus

\[
\beta = \frac{\sqrt{(n-4)(q-1)}}{q} > \begin{cases} 
\frac{q}{\sqrt{16}} & > 5 \text{ if } t \neq 1 \text{ and } q > 8 \\
\frac{q}{\sqrt{48}} & > 5 \text{ if } t = 1 \text{ and } q > 11 
\end{cases}
\]

To finish the proof of part (i), we only have to consider the remaining case \( q = 6 \). Assuming \( q = 6 \), then by (7) and (10), we have

\[
(n-4) \equiv 0 \pmod{12}, \ 351 \pmod{32}, \ 270 \pmod{32}, \ 189 \pmod{6},
\]

Therefore

\[
\beta = \frac{\sqrt{5(n-4)}}{6} > \frac{\sqrt{5 \cdot 189}}{6} > 5.
\]

(ii) Now we consider the cases when \( u = 2 \); i.e., \( q = 4k \) for some odd integer \( k \). By (5), (6), (7), we have

\[
(n-4) > \begin{cases} 
\frac{4}{k} & \text{if } 3 \mid k \text{ or } 3^2 \mid k \\
\frac{4}{3} & \text{if } 3 \mid k \text{ and } 3^2 \mid k \\
2 & \text{if } k = 1
\end{cases}
\]
(a) Suppose \( k > 13 \), \( 3 \mid k \) or \( 2 \mid k \). Then

\[
\beta = \frac{\sqrt{(n-4)(q-1)}}{q} > \frac{k\sqrt{4k-1}}{4} > 23.
\]

(b) Suppose \( k > 21 \) and \( 3 \mid k \). Then

\[
\beta > \frac{k}{4} \sqrt{\frac{4k-1}{3}} > 27.
\]

The remaining cases for \( k \) are \( k = 15, 11, 9, 7, 5, 3, 1 \). By \( (9) \), we also have that \( 2^{10} \mid (n-4) \) or \( 2^8 \mid (n-3) \).

(c) If \( k > 3 \) and \( 2^{10} \mid (n-4) \), then

\[
\beta > 8k \sqrt{\frac{4k-1}{3}} > 24.
\]

(d) Suppose \( k = 15, 11, 9, 7, 5, \) or \( 3 \), and \( 2^8 \mid (n-3) \). Then

for \( k = 15 \), \( (n-4) \equiv 3^4 \cdot 61 \pmod{3^4 \cdot 5 \cdot 2} \), hence

\( (n-4) > 3^4 \cdot 61 \) and \( \beta > 129 \);

for \( k = 11 \), \( (n-4) \equiv 47 \cdot 11^4 \pmod{11^4 \cdot 2} \), hence

\( (n-4) > 47 \cdot 11^4 \) and \( \beta > 120 \);

for \( k = 9 \), \( (n-4) \equiv 159 \cdot 9^4 \pmod{9^4 \cdot 2} \), hence

\( (n-4) > 159 \cdot 9^4 \) and \( \beta > 167 \);

for \( k = 7 \), \( (n-4) \equiv 95 \cdot 7^4 \pmod{7^4 \cdot 2} \), hence

\( (n-4) > 95 \cdot 7^4 \) and \( \beta > 88 \);
for \( k = 5 \), \((n-4) \equiv 111 \cdot 5 \pmod{5 \cdot 2}\), hence
\[
(n-4) > 111 \cdot 5 \quad \text{and} \quad \beta > 57
\]
for \( k = 3 \), \((n-4) \equiv 6399 \pmod{3 \cdot 2}\), hence
\[
(n-4) > 6399 \quad \text{and} \quad \beta > 22
\]

(e) Suppose \( k = 1 \). Then \( q = 4 \) and \((n-4) \equiv \beta - 1\). Thus \( \beta > 6.9 \).

(iii) The last case is that \( q = 3 \). By (7), \((n-4) \equiv 27\) and hence \( \beta > \frac{\sqrt{27 \cdot 2}}{3} = \sqrt{6} \).

The lemma is proved.

Now we rewrite \( G_4(x) \) in terms of \( \beta, q, \lambda, \xi, \varepsilon \) by setting
\[
x = a + \beta \xi + \lambda = \beta \frac{q}{2} + \beta \xi + \lambda,
\]
and, occasionally,
\[
\lambda = (1 - \frac{2}{q})(\frac{3 - \xi^2}{6}) + \varepsilon.
\]
We have
\[
G_4(x) = A_4 \beta^4 + A_3 \beta^3 + A_2 \beta^2 + A_1 \beta + A_0
\]
where
\[
A_4 = H_4(\xi),
\]
\[
A_3 = (1 - \frac{2}{q})(\frac{-2}{3}H_4(\xi) + \varepsilon(4\xi^3 - 12\xi)),
\]
\[
A_2 = 6\lambda^2(\xi^2 - 1) + 4\lambda(1 - \frac{2}{q}) + \frac{5}{2} \xi^2 + 3 - \frac{3}{2}(1 - \frac{2}{q})^2,
\]
\[
A_1 = 4\lambda^3 \xi - 5\lambda \xi^2,
\]
\[
A_0 = \lambda^4 - \frac{5}{2} \lambda^2 + \frac{9}{16},
\]
and

\[ H_4(x) = x^4 - 6x^2 + 3 \]

with zeros

\[ \xi_{\pm 1} = \sqrt{3 + \sqrt{6}} \]

and

\[ \xi_{\pm 2} = \pm \sqrt{3 + \sqrt{6}} \] .

Since \( \alpha = \frac{(n-4)(q-1)}{q} + 5 \), by Lemma 4.2.1, we have

**Lemma 4.2.3.** 2\( \alpha \in \mathbb{Z} \) if \( q \not\equiv 4 \pmod{8} \).

4\( \alpha \in \mathbb{Z} \) if \( q = 4k \) for some odd integer \( k \).

**Lemma 4.2.4.** Assume \( q > 5 \) and \( q \not\equiv 4 \pmod{8} \). Then

\[ 2\alpha - 1 < a_2 + \alpha_{-2} < 2\alpha - \frac{2}{q} \sqrt{\frac{1}{6}} \]

and hence

\[ a_2 + a_{-2} \quad \text{(by Lemma 4.2.3).} \]

**Proof:** We will show that

\[ a + \beta \xi_2 - (1 - \frac{2}{q}) \frac{1}{\sqrt{6}} < a_2 < a + \beta \xi_2 \]

and

\[ a + \beta \xi_{-2} + (1 - \frac{2}{q}) \frac{1}{\sqrt{6}} - 1 < a_{-2} < a + \beta \xi_{-2} - (1 - \frac{2}{q}) \frac{1}{\sqrt{6}} \] .

Using \( \xi_2 = \sqrt{3 + \sqrt{6}} \) and straightforward calculation, we get
\[ G_4 (\alpha + \beta \xi_{\pm 2} - (1 - \frac{2}{q})^{1-\frac{1}{\sqrt{6}}} \]

\[ = \left[-10.623724 + 1.3164966(1 - \frac{2}{q})^2\right] \beta^2 \]

\[ \pm (1 - \frac{2}{q}) \left[4.7651031 - 0.63534706(1 - \frac{2}{q})^2\right] \beta \]

\[ + \frac{1}{36} (1 - \frac{2}{q})^4 - \frac{5}{12} (1 - \frac{2}{q})^2 + \frac{9}{16} \]

\[ < 0 \quad \text{since } \beta > 5 \quad \text{by Lemma 4.2.2.} \]

\[ G_4 (\alpha + \beta \xi_{-2} - (1 - \frac{2}{q})^{1-\frac{1}{\sqrt{6}}} \]

\[ = 9.3376569(1 - \frac{2}{q})^{1-\frac{1}{\sqrt{6}}} \]

\[ < 0 \quad \text{since } \beta > 5 . \]

\[ G_4 (\alpha + \beta \xi_{-2} + (1-\frac{2}{q})^{1-\frac{1}{\sqrt{6}}} - 1) = \left[22.872495 - 18.675314(1 - \frac{2}{q})\right] \beta^3 \]

\[ + \left[16.073214 - 25.797959(1 - \frac{2}{q}) + 4.5824829(1 - \frac{2}{q})^2\right] \beta^2 \]

\[ + \left[2.3344142 - 0.95302016(1 - \frac{2}{q})\right] \]

\[ [-1 - 3.2659863(1 - \frac{2}{q}) + \frac{2}{3} (1 - \frac{2}{q})^2] \beta \]

\[ + \left[\frac{1}{4} + \frac{\sqrt{6}}{3} (1 - \frac{2}{q}) + \frac{1}{6} (1 - \frac{2}{q})^2\right] - 1 \]

\[ > 0 \quad \text{since } \beta > 5 . \]
**Lemma 4.2.5.** Let \( q = 4 \). Then

\[
4a - 1 < 2(a_2 + a_{-2}) < 4a - \frac{1}{\sqrt{6}}
\]

and hence

\[
2(a_2 + a_{-2}) \notin \mathbb{Z} \quad (\text{by Lemma 4.2.3}).
\]

**Proof:** We will show that

\[
\alpha + \beta \xi_2 - (1 - 2) \frac{1}{q \sqrt{6}} < a_2 < \alpha + \beta \xi_2,
\]

\[
\alpha + \beta \xi_{-2} + (1 - 2) \frac{1}{q \sqrt{6}} - \frac{1}{2} < a_{-2} < \alpha + \beta \xi_{-2} - (1 - 2) \frac{1}{q \sqrt{6}},
\]

and therefore

\[
4a - 1 < 2(a_2 + a_{-2}) < 4a - (1 - 2) \frac{1}{q \sqrt{6}} = 4a - \frac{1}{\sqrt{6}}.
\]

First, by Lemma 4.2.2, \( \beta > 6.9 \). Thus, from the proof of Lemma 4.2.4, we immediately have

\[
G_4 (\alpha + \beta \xi_{\pm 2} - (1 - 2) \frac{1}{q \sqrt{6}}) < 0
\]

and

\[
G_4 (\alpha + \beta \xi_{2}) > 0.
\]

Now, by taking \( q = 4 \) and using a calculator, we get

\[
G_4 (\alpha + \beta \xi_{-2} + (1 - 2) \frac{1}{q \sqrt{6}} - \frac{1}{2})
\]

\[
= 2.0985905\beta^3 - 9.2533584\beta^2 - 3.2116226\beta + 0.35130739
\]

\[> 0 \quad \text{since } \beta > 6.9.\]
**Lemma 4.2.6.** Let $q = 3$. Then

$$2\alpha - 1 < \alpha_2 + \alpha_{-2} < 2\alpha$$

and hence

$$(\alpha_2 + \alpha_{-2}) \notin \mathbb{Z} \quad \text{(by Lemma 4.2.3)}.$$

**Proof:** We show that

$$\alpha + \beta\xi_2 - (1 - \frac{2}{q^{q/6}}) < \alpha_2 < \alpha + \beta\xi_2 + (1 - \frac{2}{q^{q/6}})$$

and

$$\alpha + \beta\xi_{-2} - (1 - \frac{2}{q^{q/6}}) - 1 < \alpha_{-2} < \alpha + \beta\xi_{-2} + (1 - \frac{2}{q^{q/6}}).$$

First, we have $\beta > \sqrt{6}$ from Lemma 4.2.2. It is easy to see that, from the proof of Lemma 4.2.4,

$$G_4(\alpha + \beta\xi_2 - (1 - \frac{2}{q^{q/6}})) < 0$$

and

$$G_4(\alpha + \beta\xi_{-2} + (1 - \frac{2}{q^{q/6}}) - 1) > 0 \quad \text{if } \beta > \sqrt{6}.$$

We only need to show that

$$G_4(\alpha + \beta\xi_2 - (1 - \frac{2}{q^{q/6}})) > 0.$$ 

Since $\beta > \sqrt{6}$,

$$G_4(\alpha + \beta\xi_2 - (1 - \frac{2}{q^{q/6}})) = G_4(\alpha + \beta\xi_2 + \frac{\sqrt{6}}{18})$$

$$= 6.2251045\beta^3 - 10.11456\beta^2 - 1.5648363\beta + 0.51654664 > 0.$$
**Lemma 4.2.7.** If \( q = 4k \) for some odd integer \( k > 3 \), then

\[
1 < (a_1 + a_{-1}) - (a_2 + a_{-2}) < 2.
\]

**Proof:** Assume that \( q = 4k \) for some odd integer \( k > 3 \). Then \( q > 12 \), \( \beta > 22 \), and

\[
1.0886621 < (1 - \frac{2}{q}) \left( \frac{2 - \xi_2}{3} \right) < 1.6329932.
\]

If

\[
\lambda_i B((1 - \frac{2}{q}) \left( \frac{2 - \xi_2}{3} \right); 0.0221655) \quad \text{for} \quad i = \pm 1, \pm 2,
\]

then

\[
(a_1 + a_{-1}) - (a_2 + a_{-2}) = (\lambda_1 + \lambda_{-1}) - (\lambda_2 + \lambda_{-2})
\]

\[
\in B((1 - \frac{2}{q}) \left( \frac{2 - \xi_2}{3} \right); 0.088662),
\]

and therefore

\[
1 < (a_1 + a_{-1}) - (a_2 + a_{-2}) < 2.
\]

The rest of the proof is similar to that of Lemma 3.1.1.

First by setting \( \xi = \xi_{\pm 1} \) or \( \xi_{\pm 2} \), and \( \epsilon = \pm 0.022165 \), we get

\[
|\lambda_1| < 0.4304138,
\]

\[A_4 = 0,\]

\[A_3 = \pm 0.022165(4\xi_i^3 - 12\xi_i) \quad \text{for} \quad i = \pm 1, \pm 2,\]

and

\[
|A_3| > \begin{cases} 
0.162 & \text{if} \quad \xi = \xi_{\pm 1} \\
0.507 & \text{if} \quad \xi = \xi_{\pm 2}
\end{cases}
\]
\[ |A_2| < \begin{cases} 
2.35 & \text{if } \xi = \xi_{\pm 1} \\
8.90 & \text{if } \xi = \xi_{\pm 2} 
\end{cases} \]

\[ |A_1| < 4.28 \quad \text{if } \xi = \xi_{\pm 1} \text{ or } \xi_{\pm 2} ; \]

\[ |A_0| < 1.06 \quad \text{if } \xi = \xi_{\pm 1} \text{ or } \xi_{\pm 2}. \]

Since \( \beta > 22 \),

\[ |A_3| > \frac{|A_2|}{\beta} + \frac{|A_1|}{\beta^2} + \frac{|A_0|}{\beta^3}. \]

The Lemma is proved.

Summing up Lemmas 4.2.4 - 4.2.7, we have proved

**LEMMA 4.2.8.** Theorem AB is true if \( e = 4 \) and \( q \geq 3 \).

---

**§4.3 The Case \( e = 3 \) and \( q \geq 3 \).**

The divisibility conditions in Corollary 1.4.2 become

1. \( q | 3(n-3) \),
2. \( q^2 | 3(n-3)(n-2) \),
3. \( q^3 | (n-3)(n-2)(n-1) \).
Again we employ the same notation as in Theorem 1.4.3. First, we notice that

\[ a_0 \rightarrow (n-3)(1 - \frac{1}{q}) + 2 + \frac{1}{3}(1 - \frac{2}{q}) \quad \text{as } \beta \rightarrow \infty. \]

Also, by calculation, we have

\[ G_3 ((n-3)(1-\frac{1}{q}) + 2) = (n-3)(1 - \frac{1}{q})(1 - \frac{2}{q}(-\frac{1}{q})) > 0 \quad \text{if } q > 3, \]

and

\[ G_3 ((n-3)(1-\frac{1}{q}) + 2 + \frac{1}{3}) = -2(n-3)(\frac{q-1}{q}) - \frac{8}{27} < 0. \]

Thus

\[ (n-3)(1-\frac{1}{q}) + 2 < a_0 < (n-3)(1-\frac{1}{q}) + 2 + \frac{1}{3} \quad \text{if } q > 3, \]

and therefore

\[ \frac{3(n-3)(q-1)}{q} + 6 < 3a_0 < \frac{3(n-3)(q-1)}{q} + 7 \quad \text{if } q > 3. \]

**Lemma 4.3.1.** Theorem AB is true if \( e = 3 \) and \( q > 3 \).

**Proof:** Since \( \frac{3(n-3)}{q} \in \mathbb{Z} \), \( 3a_0 \notin \mathbb{Z} \).
APPENDIX

In the following computer output, we list 110 cases which we did not prove in the previous chapters. In each case, we locate a zero of Lloyd's polynomial \( F(x) \) between two consecutive integers by using the Intermediate Value Theorem. Thus, Theorem AB is true in these 110 cases. The "function" we use in the program is a rescaled Lloyd polynomial, which is

\[
q_F(x) = \frac{e^{qF}(x)}{(q-1)(n-e)(n-e+1)...(n-1)}
\]

\[
= \left[ \ldots \left[ \frac{(x-e)q}{(n-e)(q-1)} - \frac{e}{1} \right] \frac{(x-e+1)q}{(n-e+1)(q-1)} + \frac{e}{2} \right] \ldots
\]

\[
= \left[ \ldots \left[ \frac{(x-1)q}{(n-1)(q-1)} + (-1)^e \right] \right].
\]

The program was written in the language C and was run on a VAX 11-780 at The Ohio State University.

The 110 cases follow the order:

(i) \( e = 6, \ q = 3, \ n = 5k + 5 \) or \( 5k + 6, \ 1 < k < 15 \).

(ii) \( e = 6, \ q = 4, \ n = 2k + 6, \ 1 < k < 8 \).

(iii) \( e = 6, \ q = 5, \ n = 3131 \).
This program locates the zero of Lloyd's polynomial F(e,n,q,x)

```c
#include "stdio.h"

#include "stdio.h"

int cnt, z;

int main()
{
    double i,e,n,q,x,r,s1,t1,a,b;
    FILE *fp, *fopen();
    double fact();

    fp = fopen("data.dat","r");
    fscanf(fp, "%e%e%e%e", &e,&q,&n,&x);
    while (e != 0)
    {
        r = 1; cnt = 1;

        /* Calculating the functional value S of Lloyd's polynomial at x */
```

(iv) \[ e = 6, \quad q = 5, \quad n = \begin{cases} 
3^{8k} + (5 \text{ or } 6), & 1 < k < 3, \\
3^{5(5 + 8k)} + 6, & 0 < k < 2, \\
3^{5(3 + 8k)} + 5, & 0 < k < 2. 
\end{cases} \]

(v) \[ e = 8, \quad q = 3, \quad n = 737. \]

(vi) \[ e = 8, \quad q = 6, \quad n = 737. \]

(vii) \[ e = 7, \quad q = 3, \quad n = 3^{5k + 7}, \quad 1 < k < 27. \]

(viii) \[ e = 7, \quad q = 6, \quad n = 3^{8k + 7}, \quad 1 < k < 5. \]

(ix) \[ e = 9, \quad q = 4, \quad n = 2^{12k + 9}, \quad 1 < k < 4. \]

(x) \[ e = 11, \quad q = 3, \quad n = 3^{7k + 11}, \quad 1 < k < 11. \]

(xi) \[ e = 11, \quad q = 6, \quad n = 3^{3k + 11}, \quad 1 < k < 2. \]

(xii) \[ e = 13, \quad q = 3, \quad n = 3^{8k + 13}, \quad 1 < k < 6. \]

(xiii) \[ e = 13, \quad q = 6, \quad n = 3^{6 + 13}, \quad 3^{8 + 13}. \]
double fact(e)
{
    if (e == 0)
        return(1);
    else
        return (e * fact(e-1));
}

for (i = e; i != 0; i--)
{
    a = r - (x-i) / (n-i) * (q / (q-1));
    b = fact(e) / (fact(e - i + 1) * fact(i-1));
    if ((cnt % 2) == 0)
        r = a + b;
    else
        r = a - b;
    cnt++;
}
sl = r;
r = 1; x++; cnt = 1;

/*
Calculating the functional value T of Lloyd's polynomial at (x + 1)
*/
for (i = e; i != 0; i--)
{
    a = r - (x-i) / (n-i) * (q / (q-1));
    b = fact(e) / (fact(e - i + 1) * fact(i-1));
    if ((cnt % 2) == 0)
        r = a + b;
    else
        r = a - b;
    cnt++;
}
tl = r;

/*
Checking to see if there is a zero between x and (x + 1)
*/
if ( (si * tl) < 0 )
    z = 1;
else
    z = 0;

printf("The input values are e = %4.0f q = %4.0f n = %4.0f x = %4.0f", e,q,n,x);
printf("The functional values are \n");
printf("%4.15e for x = %4.0f\n", sl, (x-1));
printf("%4.15e for x = %4.0f\n", tl, x);
printf("Is there a zero between %4.0f and %4.0f ?");
if (z == 1)
    printf("TRUE\n");
else
    printf("FALSE\n");
scanf(fp, "%e%e%e%e", &e,&q,&n,&x);
}
The input values are \( e = 6 \ q = 3 \ n = 248 \ x = 169 \)
The functional values are
-3.022084e-08 for \( x = 169 \)
1.708134e-08 for \( x = 170 \)
Is there a zero between 169 and 170? TRUE

The input values are \( e = 6 \ q = 3 \ n = 249 \ x = 170 \)
The functional values are
-3.515350e-08 for \( x = 170 \)
3.257893e-08 for \( x = 171 \)
Is there a zero between 170 and 171? TRUE

The input values are \( e = 6 \ q = 3 \ n = 491 \ x = 333 \)
The functional values are
-2.195291e-09 for \( x = 333 \)
2.096851e-09 for \( x = 334 \)
Is there a zero between 333 and 334? TRUE

The input values are \( e = 6 \ q = 3 \ n = 492 \ x = 334 \)
The functional values are
-8.141130e-10 for \( x = 334 \)
3.513972e-09 for \( x = 335 \)
Is there a zero between 334 and 335? TRUE

The input values are \( e = 6 \ q = 3 \ n = 734 \ x = 496 \)
The functional values are
-9.740983e-10 for \( x = 496 \)
4.679560e-11 for \( x = 497 \)
Is there a zero between 496 and 497? TRUE

The input values are \( e = 6 \ q = 3 \ n = 735 \ x = 497 \)
The functional values are
-6.430974e-10 for \( x = 497 \)
3.911293e-10 for \( x = 498 \)
Is there a zero between 497 and 498? TRUE

The input values are \( e = 6 \ q = 3 \ n = 977 \ x = 660 \)
The functional values are
-6.405288e-11 for \( x = 660 \)
3.236284e-10 for \( x = 661 \)
Is there a zero between 660 and 661? TRUE

The input values are \( e = 6 \ q = 3 \ n = 978 \ x = 660 \)
The functional values are
-3.144609e-10 for x = 660
6.173027e-11 for x = 661
Is there a zero between 660 and 661? TRUE

The input values are e = 6 q = 3 n = 1220 x = 823
The functional values are
-4.134526e-11 for x = 823
1.359997e-10 for x = 824
Is there a zero between 823 and 824? TRUE

The input values are e = 6 q = 3 n = 1221 x = 823
The functional values are
-1.561540e-10 for x = 823
1.636455e-11 for x = 824
Is there a zero between 823 and 824? TRUE

The input values are e = 6 q = 3 n = 1463 x = 986
The functional values are
-7.971555e-11 for x = 986
1.836656e-11 for x = 987
Is there a zero between 986 and 987? TRUE

The input values are e = 6 q = 3 n = 1464 x = 986
The functional values are
-7.971555e-11 for x = 986
1.836656e-11 for x = 987
Is there a zero between 986 and 987? TRUE

The input values are e = 6 q = 3 n = 1706 x = 1149
The functional values are
-4.071812e-11 for x = 1149
4.997246e-11 for x = 1150
Is there a zero between 1149 and 1150? TRUE

The input values are e = 6 q = 3 n = 1707 x = 1149
The functional values are
-4.071812e-11 for x = 1149
1.299942e-11 for x = 1150
Is there a zero between 1149 and 1150? TRUE

The input values are e = 6 q = 3 n = 1949 x = 1311
The functional values are
-3.080952e-11 for x = 1311
2.796666e-12 for x = 1312
Is there a zero between 1311 and 1312? TRUE
The input values are $e \times 6 q = 3 n \times 1950 \times x = 1312$
The functional values are
- $1.9820864e-11$ for $x = 1312$
- $1.406639e-11$ for $x = 1313$
Is there a zero between 1312 and 1313?  TRUE

The input values are $e \times 6 q = 3 n \times 2192 \times x = 1474$
The functional values are
- $1.552375e-11$ for $x = 1474$
- $6.910209e-12$ for $x = 1475$
Is there a zero between 1474 and 1475?  TRUE

The input values are $e \times 6 q = 3 n \times 2193 \times x = 1475$
The functional values are
- $8.175599e-12$ for $x = 1475$
- $1.442506e-11$ for $x = 1476$
Is there a zero between 1475 and 1476?  TRUE

The input values are $e \times 6 q = 3 n \times 2435 \times x = 1637$
The functional values are
- $1.674078e-12$ for $x = 1637$
- $0.962137e-12$ for $x = 1638$
Is there a zero between 1637 and 1638?  TRUE

The input values are $e \times 6 q = 3 n \times 2436 \times x = 1638$
The functional values are
- $1.550066e-12$ for $x = 1638$
- $1.418458e-11$ for $x = 1639$
Is there a zero between 1638 and 1639?  TRUE

The input values are $e \times 6 q = 3 n \times 2678 \times x = 1800$
The functional values are
- $1.440931e-12$ for $x = 1800$
- $9.840531e-12$ for $x = 1801$
Is there a zero between 1800 and 1801?  TRUE

The input values are $e \times 6 q = 3 n \times 2679 \times x = 1800$
The functional values are
- $8.928521e-12$ for $x = 1800$
- $2.258013e-12$ for $x = 1801$
Is there a zero between 1800 and 1801?  TRUE

The input values are $e \times 6 q = 3 n \times 2921 \times x = 1962$
The functional values are
- $6.512682e-12$ for $x = 1962$
1.692787e-12 for $x = 1963$

Is there a zero between 1962 and 1963?  TRUE

The input values are $e = 6$  $q = 3$  $n = 2922$  $x = 1963$  
The functional values are  
-3.832970e-12 for $x = 1963$  
4.425946e-12 for $x = 1964$

Is there a zero between 1963 and 1964?  TRUE

The input values are $e = 6$  $q = 3$  $n = 3164$  $x = 2125$

The functional values are  
-2.717410e-12 for $x = 2125$  
3.530037e-12 for $x = 2126$

Is there a zero between 2125 and 2126?  TRUE

Is there a zero between 2125 and 2126?  TRUE

The input values are $e = 6$  $q = 3$  $n = 3407$  $x = 2288$

The functional values are  
-6.617659e-12 for $x = 2288$  
5.617659e-12 for $x = 2289$

Is there a zero between 2288 and 2289?  TRUE

The input values are $e = 6$  $q = 3$  $n = 3650$  $x = 2450$

The functional values are  
-3.450585e-12 for $x = 2450$  
1.334585e-12 for $x = 2451$

Is there a zero between 2450 and 2451?  TRUE

The input values are $e = 6$  $q = 3$  $n = 3651$  $x = 2451$

The functional values are  
-1.198269e-12 for $x = 2451$  
2.594702e-12 for $x = 2452$

Is there a zero between 2451 and 2452?  TRUE
The input values are $e = 6$, $q = 4$, $n = 518$, $x = 393$

The functional values are
- $-1.052755e-09$ for $x = 393$
- $5.358593e-11$ for $x = 394$

Is there a zero between 393 and 394? TRUE

The input values are $e = 6$, $q = 4$, $n = 1030$, $x = 780$

The functional values are
- $-4.54561e-11$ for $x = 780$
- $5.690461e-11$ for $x = 781$

Is there a zero between 780 and 781? TRUE

The input values are $e = 6$, $q = 4$, $n = 1542$, $x = 1166$

The functional values are
- $-9.04473e-12$ for $x = 1166$
- $1.590622e-11$ for $x = 1167$

Is there a zero between 1166 and 1167? TRUE

The input values are $e = 6$, $q = 4$, $n = 2054$, $x = 1551$

The functional values are
- $-8.808843e-12$ for $x = 1551$
- $1.517259e-13$ for $x = 1552$

Is there a zero between 1551 and 1552? TRUE

The input values are $e = 6$, $q = 4$, $n = 2566$, $x = 1937$

The functional values are
- $-1.696338e-12$ for $x = 1937$
- $2.489699e-12$ for $x = 1938$

Is there a zero between 1937 and 1938? TRUE

The input values are $e = 6$, $q = 4$, $n = 3078$, $x = 2322$

The functional values are
- $-1.539974e-12$ for $x = 2322$
- $6.740935e-13$ for $x = 2323$

Is there a zero between 2322 and 2323? TRUE

The input values are $e = 6$, $q = 4$, $n = 3590$, $x = 2707$

The functional values are
- $-1.125136e-12$ for $x = 2707$
- $1.522810e-13$ for $x = 2708$

Is there a zero between 2707 and 2708? TRUE

The input values are $e = 6$, $q = 4$, $n = 4102$, $x = 3092$

The functional values are
- $-7.878975e-13$ for $x = 3092$
1.217082e-14 for x = 3093
Is there a zero between 3092 and 3093? TRUE

The input values are e = 6 q = 5 n = 3131 x = 2517
The functional values are
-7.160661e-13 for x = 2517
2.246814e-13 for x = 2518
Is there a zero between 2517 and 2518? TRUE

The input values are e = 6 q = 6 n = 1949 x = 1633
The functional values are
-8.847922e-13 for x = 1633
1.861178e-12 for x = 1634
Is there a zero between 1633 and 1634? TRUE

The input values are e = 6 q = 6 n = 1950 x = 1634
The functional values are
2.314593e-12 for x = 1635
Is there a zero between 1634 and 1635? TRUE

The input values are e = 6 q = 6 n = 3893 x = 3257
The functional values are
-1.253991e-13 for x = 3257
1.176559e-13 for x = 3258
Is there a zero between 3257 and 3258? TRUE

The input values are e = 6 q = 6 n = 3894 x = 3258
The functional values are
-8.573697e-14 for x = 3258
1.561251e-14 for x = 3259
Is there a zero between 3258 and 3259? TRUE

The input values are e = 6 q = 6 n = 5837 x = 4880
The functional values are
-4.315992e-14 for x = 4880
1.561251e-14 for x = 4881
Is there a zero between 4880 and 4881? TRUE

The input values are e = 6 q = 6 n = 5838 x = 4881
The functional values are
-3.358425e-14 for x = 4881
2.502165e-14 for x = 4882
Is there a zero between 4881 and 4882? TRUE
The input values are \( e = 6 \ q = 6 \ n = 1221 \ x = 1024 \)
The functional values are
- \( 7.533807e-12 \) for \( x = 1024 \)
- \( 6.476028e-12 \) for \( x = 1025 \)
Is there a zero between 1024 and 1025? \( \text{TRUE} \)

The input values are \( e = 6 \ q = 6 \ n = 3165 \ x = 2649 \)
The functional values are
- \( 2.04348e-13 \) for \( x = 2649 \)
- \( 2.68864e-13 \) for \( x = 2650 \)
Is there a zero between 2649 and 2650? \( \text{TRUE} \)

The input values are \( e = 6 \ q = 6 \ n = 5109 \ x = 4272 \)
The functional values are
- \( 6.7333e-14 \) for \( x = 4272 \)
- \( 5.31993e-14 \) for \( x = 4273 \)
Is there a zero between 4272 and 4273? \( \text{TRUE} \)

The input values are \( e = 6 \ q = 6 \ n = 734 \ x = 616 \)
The functional values are
- \( 3.4377e-13 \) for \( x = 616 \)
- \( 8.01899e-13 \) for \( x = 617 \)
Is there a zero between 616 and 617? \( \text{TRUE} \)

The input values are \( e = 6 \ q = 6 \ n = 2678 \ x = 2242 \)
The functional values are
- \( 1.10480e-13 \) for \( x = 2242 \)
- \( 3.68652e-13 \) for \( x = 2243 \)
Is there a zero between 2242 and 2243? \( \text{TRUE} \)

The input values are \( e = 6 \ q = 6 \ n = 4622 \ x = 3866 \)
The functional values are
- \( 4.04676e-14 \) for \( x = 3866 \)
- \( 9.38416e-14 \) for \( x = 3867 \)
Is there a zero between 3866 and 3867? \( \text{TRUE} \)

The input values are \( e = 8 \ q = 3 \ n = 737 \ x = 497 \)
The functional values are
- \( 4.11946e-12 \) for \( x = 497 \)
- \( 1.34012e-12 \) for \( x = 498 \)
Is there a zero between 497 and 498? \( \text{TRUE} \)

The input values are \( e = 8 \ q = 6 \ n = 737 \ x = 618 \)
The functional values are
- \( 3.42842e-14 \) for \( x = 618 \)
Is there a zero between 618 and 619?  TRUE

The input values are $e = 7$, $q = 3$, $n = 250$, $x = 166$.
The functional values are $1.784914e-09$ for $x = 166$ and $-3.584221e-09$ for $x = 167$.
Is there a zero between 166 and 167?  TRUE

The input values are $e = 7$, $q = 3$, $n = 493$, $x = 328$.
The functional values are $1.145404e-10$ for $x = 328$ and $-2.295480e-10$ for $x = 329$.
Is there a zero between 328 and 329?  TRUE

The input values are $e = 7$, $q = 3$, $n = 736$, $x = 490$.
The functional values are $2.282641e-11$ for $x = 490$ and $-4.571532e-11$ for $x = 491$.
Is there a zero between 490 and 491?  TRUE

The input values are $e = 7$, $q = 3$, $n = 979$, $x = 652$.
The functional values are $7.254530e-12$ for $x = 652$ and $-1.452440e-11$ for $x = 653$.
Is there a zero between 652 and 653?  TRUE

The input values are $e = 7$, $q = 3$, $n = 1222$, $x = 814$.
The functional values are $2.9791456e-12$ for $x = 814$ and $-5.963785e-12$ for $x = 815$.
Is there a zero between 814 and 815?  TRUE

The input values are $e = 7$, $q = 3$, $n = 1465$, $x = 976$.
The functional values are $1.439099e-12$ for $x = 976$ and $-2.880585e-12$ for $x = 977$.
Is there a zero between 976 and 977?  TRUE

The input values are $e = 7$, $q = 3$, $n = 1708$, $x = 1138$.
The functional values are $7.7837774e-13$ for $x = 1138$ and $-1.5566444e-12$ for $x = 1139$.
Is there a zero between 1138 and 1139?  TRUE
<table>
<thead>
<tr>
<th>Input Values</th>
<th>Functional Values</th>
<th>Zero Between?</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>( e = 7 )</td>
<td>( q = 3 )</td>
<td>( n = 1951 )</td>
<td>( x = 1300 )</td>
</tr>
<tr>
<td>( 4.569013e-13 ) for ( x = 1300 )</td>
<td>(-9.134915e-13 ) for ( x = 1301 )</td>
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<td></td>
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</tbody>
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<td>( e = 7 )</td>
<td>( q = 3 )</td>
<td>( n = 2194 )</td>
<td>( x = 1462 )</td>
</tr>
<tr>
<td>( 2.85241e-13 ) for ( x = 1462 )</td>
<td>(-5.700995e-13 ) for ( x = 1463 )</td>
<td></td>
<td></td>
</tr>
</tbody>
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<td>( e = 7 )</td>
<td>( q = 3 )</td>
<td>( n = 2439 )</td>
<td>( x = 1624 )</td>
</tr>
<tr>
<td>( 1.874612e-13 ) for ( x = 1624 )</td>
<td>(-3.743533e-13 ) for ( x = 1625 )</td>
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</tbody>
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<td>( q = 3 )</td>
<td>( n = 2680 )</td>
<td>( x = 1786 )</td>
</tr>
<tr>
<td>( 1.278144e-13 ) for ( x = 1786 )</td>
<td>(-2.560174e-13 ) for ( x = 1787 )</td>
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<tr>
<td>( e = 7 )</td>
<td>( q = 3 )</td>
<td>( n = 3166 )</td>
<td>( x = 2110 )</td>
</tr>
<tr>
<td>( 6.589174e-14 ) for ( x = 2110 )</td>
<td>(-1.321165e-13 ) for ( x = 2111 )</td>
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<td></td>
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</tbody>
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<td>( n = 3409 )</td>
<td>( x = 2272 )</td>
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<tr>
<td>( 4.887757e-14 ) for ( x = 2272 )</td>
<td>(-9.824066e-14 ) for ( x = 2273 )</td>
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<td>( q = 3 )</td>
<td>( n = 3652 )</td>
<td>( x = 2434 )</td>
</tr>
<tr>
<td>( 3.722033e-14 ) for ( x = 2434 )</td>
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<td></td>
</tr>
<tr>
<td>x</td>
<td>f(x)</td>
<td>Is there a zero between 2596 and 2597?</td>
<td></td>
</tr>
<tr>
<td>------------</td>
<td>--------------</td>
<td>-------------------------------------</td>
<td></td>
</tr>
<tr>
<td>2435</td>
<td>-7.392699e-14</td>
<td>TRUE</td>
<td></td>
</tr>
<tr>
<td>2758</td>
<td>2.192690e-14</td>
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</tr>
<tr>
<td>2920</td>
<td>1.740275e-14</td>
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</tr>
<tr>
<td>3082</td>
<td>9.742207e-15</td>
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</tr>
<tr>
<td>3244</td>
<td>8.160139e-15</td>
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<td>3406</td>
<td>9.742207e-15</td>
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</tr>
<tr>
<td>3568</td>
<td>8.160139e-15</td>
<td>TRUE</td>
<td></td>
</tr>
</tbody>
</table>
The input values are $e = 7$, $q = 3$, $n = 5596$, $x = 3730$
The functional values are
$5.494800e-15$ for $x = 3730$
$-1.3b4472e-14$ for $x = 3731$
Is there a zero between 3730 and 3731? TRUE

The input values are $e = 7$, $q = 3$, $n = 5839$, $x = 3892$
The functional values are
$5.745404e-15$ for $x = 3892$
$-1.121325e-14$ for $x = 3893$
Is there a zero between 3892 and 3893? TRUE

The input values are $e = 7$, $q = 3$, $n = 6082$, $x = 4054$
The functional values are
$5.023759e-15$ for $x = 4054$
$-9.547918e-15$ for $x = 4055$
Is there a zero between 4054 and 4055? TRUE

The input values are $e = 7$, $q = 3$, $n = 6325$, $x = 4216$
The functional values are
$4.218847e-15$ for $x = 4216$
$-8.215650e-15$ for $x = 4217$
Is there a zero between 4216 and 4217? TRUE

The input values are $e = 7$, $q = 3$, $n = 6568$, $x = 4378$
The functional values are
$3.219647e-15$ for $x = 4378$
$-7.216450e-15$ for $x = 4379$
Is there a zero between 4378 and 4379? TRUE

The input values are $e = 7$, $q = 6$, $n = 1951$, $x = 1624$
The functional values are
$4.665712e-14$ for $x = 1624$
$-2.373102e-14$ for $x = 1625$
Is there a zero between 1624 and 1625? TRUE

The input values are $e = 7$, $q = 6$, $n = 3895$, $x = 3244$
The functional values are
$2.859824e-15$ for $x = 3244$
$-1.693090e-15$ for $x = 3245$
Is there a zero between 3244 and 3245? TRUE

The input values are $e = 7$, $q = 6$, $n = 5839$, $x = 4864$
The functional values are
$5.273554e-16$ for $x = 4864$
-5.273559e-16 for x = 4865
Is there a zero between 4864 and 4865? TRUE

The input values are e = 7 q = 6 n = 7783 x = 6484
The functional values are
-9.714451e-17 for x = 6484
2.775558e-16 for x = 6485
Is there a zero between 6484 and 6485? TRUE

The input values are e = 7 q = 6 n = 9727 x = 8103
The functional values are
6.661338e-16 for x = 8103
-4.579670e-16 for x = 8104
Is there a zero between 8103 and 8104? TRUE

The input values are e = 9 q = 4 n = 4105 x = 3077
The functional values are
1.831859e-15 for x = 3077
-7.216450e-16 for x = 3078
Is there a zero between 3077 and 3078? TRUE

The input values are e = 9 q = 4 n = 8201 x = 6150
The functional values are
9.436896e-16 for x = 6150
-7.216450e-16 for x = 6151
Is there a zero between 6150 and 6151? TRUE

The input values are e = 9 q = 4 n = 12297 x = 9221
The functional values are
-6.800116e-16 for x = 9221
7.494005e-16 for x = 9222
Is there a zero between 9221 and 9222? TRUE

The input values are e = 9 q = 4 n = 16393 x = 12296
The functional values are
2.442491e-15 for x = 12296
-8.326673e-16 for x = 12297
Is there a zero between 12296 and 12297? TRUE

The input values are e = 11 q = 3 n = 2198 x = 1463
The functional values are
-3.889781e-16 for x = 1463
1.193490e-15 for x = 1464
Is there a zero between 1463 and 1464? TRUE
The input values are $e = 11 \, q = 3 \, n = 4385 \, x = 2923$
The functional values are
5.412337e-15 for $x = 2923$
-3.386180e-15 for $x = 2924$
Is there a zero between 2923 and 2924? TRUE

The input values are $e = 11 \, q = 3 \, n = 6572 \, x = 4380$
The functional values are
-5.689893e-15 for $x = 4380$
4.468648e-15 for $x = 4381$
Is there a zero between 4380 and 4381? TRUE

The input values are $e = 11 \, q = 3 \, n = 8759 \, x = 5838$
The functional values are
-1.901257e-15 for $x = 5838$
1.110223e-16 for $x = 5839$
Is there a zero between 5838 and 5839? TRUE

The input values are $e = 11 \, q = 3 \, n = 10946 \, x = 7296$
The functional values are
9.575674e-15 for $x = 7296$
-1.762479e-15 for $x = 7297$
Is there a zero between 7296 and 7297? TRUE

The input values are $e = 11 \, q = 3 \, n = 13133 \, x = 8754$
The functional values are
2.914336e-15 for $x = 8754$
-8.354428e-15 for $x = 8755$
Is there a zero between 8754 and 8755? TRUE

The input values are $e = 11 \, q = 3 \, n = 15320 \, x = 10212$
The functional values are
6.133938e-15 for $x = 10212$
-5.676015e-15 for $x = 10213$
Is there a zero between 10212 and 10213? TRUE

The input values are $e = 11 \, q = 3 \, n = 17507 \, x = 11670$
The functional values are
2.164939e-15 for $x = 11670$
-5.606626e-15 for $x = 11671$
Is there a zero between 11670 and 11671? TRUE

The input values are $e = 11 \, q = 3 \, n = 19694 \, x = 13129$
The functional values are
-2.678418e-15 for $x = 13129$
3.663736e-15 for x = 13130
Is there a zero between 13129 and 13130 ? TRUE

The input values are e = 11 q = 3 n = 21881 x = 14587
The functional values are
-1.110233e-16 for x = 14587
4.38586e-15 for x = 14588
Is there a zero between 14587 and 14588 ? TRUE

The input values are e = 11 q = 3 n = 24068 x = 16043
The functional values are
4.080070e-15 for x = 16043
-1.734723e-15 for x = 16044
Is there a zero between 16043 and 16044 ? TRUE

The input values are e = 11 q = 6 n = 17507 x = 14587
The functional values are
-1.984524e-15 for x = 14587
7.188694e-15 for x = 14588
Is there a zero between 14587 and 14588 ? TRUE

The input values are e = 11 q = 6 n = 35003 x = 29166
The functional values are
-7.771561e-16 for x = 29166
2.775558e-15 for x = 29167
Is there a zero between 29166 and 29167 ? TRUE

The input values are e = 13 q = 3 n = 6574 x = 4381
The functional values are
-2.636780e-15 for x = 4381
1.026956e-14 for x = 4382
Is there a zero between 4381 and 4382 ? TRUE

The input values are e = 13 q = 3 n = 13134 x = 8755
The functional values are
3.166911e-14 for x = 8755
-2.459144e-14 for x = 8756
Is there a zero between 8755 and 8756 ? TRUE

The input values are e = 13 q = 3 n = 19696 x = 13131
The functional values are
-1.064426e-14 for x = 13131
1.934644e-14 for x = 13132
Is there a zero between 13131 and 13132 ? TRUE
The input values are $e = 13, q = 3, n = 26257, x = 17503$
The functional values are
- $1.243450e-14$ for $x = 17503$
- $-1.214306e-14$ for $x = 17504$
Is there a zero between 17503 and 17504? $\text{TRUE}$

The input values are $e = 13, q = 3, n = 32818, x = 21878$
The functional values are
- $1.865176e-14$ for $x = 21878$
- $-1.285083e-14$ for $x = 21879$
Is there a zero between 21878 and 21879? $\text{TRUE}$

The input values are $e = 13, q = 3, n = 39379, x = 26251$
The functional values are
- $-1.398881e-14$ for $x = 26251$
- $1.942890e-15$ for $x = 26252$
Is there a zero between 26251 and 26252? $\text{TRUE}$

The input values are $e = 13, q = 6, n = 39379, x = 32814$
The functional values are
- $-1.398881e-14$ for $x = 32814$
- $1.942890e-15$ for $x = 32815$
Is there a zero between 32814 and 32815? $\text{TRUE}$

The input values are $e = 13, q = 6, n = 52501, x = 43750$
The functional values are
- $-1.285083e-14$ for $x = 43750$
- $9.353629e-15$ for $x = 43751$
Is there a zero between 43750 and 43751? $\text{TRUE}$
BIBLIOGRAPHY


