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FINITE ELEMENT ANALYSIS OF BI-MATERIAL SYSTEMS
WITH APPLICATIONS IN HYDRAULIC FRACTURING

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By

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*****

The Ohio State University
1984

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1984
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FIELD OF STUDY

Fracture Mechanics and Finite Element Methods
PUBLICATIONS


"FELPA - A Finite Element Analysis Program for Linear Plane and Axisymmetric Solid Elasticity Problems," by A. Gurkok, H.U. Akay and O. Gurdogan, CAD/CAM Application Software No. 82-2, Purdue University at Indianapolis School of Engineering and Technology, March 1982.


# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>ACKNOWLEDGEMENT</td>
<td>ii</td>
</tr>
<tr>
<td>VITA</td>
<td>iii</td>
</tr>
<tr>
<td>PUBLICATIONS</td>
<td>iv</td>
</tr>
<tr>
<td>LIST OF TABLES</td>
<td>viii</td>
</tr>
<tr>
<td>LIST OF FIGURES</td>
<td>ix</td>
</tr>
<tr>
<td>NOMENCLATURE</td>
<td>xiv</td>
</tr>
</tbody>
</table>

## CHAPTER

### I. INTRODUCTION

1.1 Preliminaries                                                      | 1    |
1.2 Literature Review                                                  | 3    |
1.2.1 Finite Element Method Applications in LEFM                      | 5    |
1.2.2 Fracture Criteria                                                | 7    |
1.3 Research Objectives                                                | 9    |

### II. FINITE ELEMENTS IN LEFM

2.1 Degenerate Isoparametric Finite Elements for Linear Elastic Crack Problems in Homogeneous Media | 11   |
2.1.1 Investigation of Singular Behavior in Isoparametric Elements     | 12   |
2.1.2 Quarter Point Isoparametric Mapping in 2 and 3 Dimensions        | 17   |
2.2 Singularity Elements for Bi-Material Crack Problems.               | 28   |
2.2.1 Modified Akin's Element in Degenerate Triangular Form            | 29   |
2.2.2 Subparametric Semi-Radial Singularity Element.                  | 33   |
2.3 Stress Intensity Factor Estimation                                 | 36   |
2.3.1 Homogeneous Medium in 2 and 3 Dimensions                         | 37   |
2.3.2 Layered Plane Medium                                             | 41   |
2.3.2.1 Interfacial crack in a bi-material body                        | 42   |
2.3.2.2 Terminal crack in a bi-material body                           | 48   |
<table>
<thead>
<tr>
<th>CHAPTER</th>
<th>FRACTURE CRITERIA ........................................</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>III.</td>
<td>FRACTURE CRITERIA ......................................</td>
<td>51</td>
</tr>
<tr>
<td></td>
<td>3.1 Review of Linear Elastic Fracture Criteria for Homogeneous Media.</td>
<td>51</td>
</tr>
<tr>
<td></td>
<td>3.2 Fracture Criteria for Bi-Material Problems ......</td>
<td>54</td>
</tr>
<tr>
<td></td>
<td>3.2.1 Interface Crack Between Two Dissimilar Media</td>
<td>56</td>
</tr>
<tr>
<td></td>
<td>3.2.2 Terminal Crack Perpendicular to the Bi-Material Interface.</td>
<td>68</td>
</tr>
<tr>
<td></td>
<td>3.2.3 Path Independent Integrals for Bi-Material Media</td>
<td>83</td>
</tr>
<tr>
<td></td>
<td>3.2.3.1 Bi-material J-integral ........................</td>
<td>85</td>
</tr>
<tr>
<td></td>
<td>3.2.3.2 Bi-material M-integral ........................</td>
<td>90</td>
</tr>
<tr>
<td>IV.</td>
<td>COMPUTER IMPLEMENTATION AND TEST CASES................</td>
<td>94</td>
</tr>
<tr>
<td></td>
<td>4.1 Computer Implementation ............................</td>
<td>94</td>
</tr>
<tr>
<td></td>
<td>4.2 Test Cases ...........................................</td>
<td>96</td>
</tr>
<tr>
<td></td>
<td>4.2.1 A Deep Cantilever Beam ...........................</td>
<td>97</td>
</tr>
<tr>
<td></td>
<td>4.2.2 Single Edge Notch Specimen .......................</td>
<td>99</td>
</tr>
<tr>
<td></td>
<td>4.2.3 Bi-Material Interfacial Crack Model ............</td>
<td>100</td>
</tr>
<tr>
<td></td>
<td>4.2.4 Bi-Material Terminal Crack Model ...............</td>
<td>105</td>
</tr>
<tr>
<td></td>
<td>4.2.5 Embedded Flat Crack in 3D Medium ...............</td>
<td>114</td>
</tr>
<tr>
<td>V.</td>
<td>HYDRAULIC FRACTURE MODEL SIMULATIONS ..................</td>
<td>118</td>
</tr>
<tr>
<td></td>
<td>5.1 Preliminary Remarks ..................................</td>
<td>118</td>
</tr>
<tr>
<td></td>
<td>5.2 Bi-Material Terminal Crack Model, Infinite Medium</td>
<td>121</td>
</tr>
<tr>
<td></td>
<td>5.3 Fracture Penetration Through an Interface ........</td>
<td>125</td>
</tr>
<tr>
<td></td>
<td>5.3.1 Crack Penetrating an Interface ..................</td>
<td>125</td>
</tr>
<tr>
<td></td>
<td>5.3.2 Through Cleavage Crack Opening Pressure ........</td>
<td>127</td>
</tr>
<tr>
<td></td>
<td>5.4 Hydraulic Fracturing Field Examples ...............</td>
<td>131</td>
</tr>
<tr>
<td></td>
<td>5.4.1 Dendritic Fracturing Simulations ................</td>
<td>132</td>
</tr>
<tr>
<td></td>
<td>5.4.2 Pressurized Crack Intersecting a Joint ..........</td>
<td>138</td>
</tr>
<tr>
<td></td>
<td>5.4.3 Hydraulic Fracture Propagation in Layered Media.</td>
<td>142</td>
</tr>
<tr>
<td></td>
<td>5.4.3.1 Crack arrest at the interface ..................</td>
<td>142</td>
</tr>
<tr>
<td></td>
<td>5.4.3.2 Fracture height and propagation/containment pressures</td>
<td>146</td>
</tr>
<tr>
<td>VI.</td>
<td>CONCLUSIONS AND RECOMMENDATIONS ........................</td>
<td>164</td>
</tr>
<tr>
<td>BIBLIOGRAPHY .............................................</td>
<td>175</td>
<td></td>
</tr>
</tbody>
</table>

APPENDICES

A. STRESS AND DISPLACEMENT FIELDS NEAR THE CRACK TIP ........ 184

<table>
<thead>
<tr>
<th>Subsection</th>
<th>Title ........................................</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>A.1</td>
<td>Homogeneous Media in Two Dimensions ....</td>
<td>184</td>
</tr>
<tr>
<td>A.2</td>
<td>Homogeneous Media in Three Dimensions ...</td>
<td>186</td>
</tr>
<tr>
<td>A.3</td>
<td>Interfacial Crack in Two Dimensions .....</td>
<td>188</td>
</tr>
<tr>
<td>A.4</td>
<td>Terminal Crack in Two Dimensions .......</td>
<td>190</td>
</tr>
</tbody>
</table>
APPENDICES (continued)

B. SOME SINGULAR ELEMENTS .......................... 194
   B.1 Akin's Variable Singularity Element .......... 194
   B.2 Semi-Radial Singularity Mapping Element ... 195

C. FRACTURE PROCESS ZONE. .......................... 199
   C.1 Plastic Zone for a Bi-Material Terminal Crack .. 199
   C.2 Microcrack Zone for Rock .................... 203
   C.3 Effect of Friction on the Microcrack Zone .... 204

D. BI-MATERIAL J-INTEGRAL CONSIDERATIONS. ....... 208
## LIST OF TABLES

<table>
<thead>
<tr>
<th>Table</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.1 Selected Material Properties and Singularity Power, ( p ), for Bi-Material Terminal Crack Fracture Envelopes.</td>
<td>74</td>
</tr>
<tr>
<td>4.1 Comparison of Normalized Deflections and Stresses ([100]) in the Cantilever Beam.</td>
<td>97</td>
</tr>
<tr>
<td>4.2 Normalized Stress Intensity Factor for the Single Edge Notch Plate ([100]).</td>
<td>100</td>
</tr>
<tr>
<td>4.3 Stress Intensity Factors and Energy Release Rates for the Interfacial Crack.</td>
<td>105</td>
</tr>
<tr>
<td>4.4 Normalized Stress Intensity Factors for the Pressurized Terminal Crack, ( K/qa^{\sqrt{\pi}} a^p ).</td>
<td>112</td>
</tr>
<tr>
<td>4.5 Normalized Bi-Material ( J^n ) Integral, ( J^n/qa(\times10^{-5}) ).</td>
<td>112</td>
</tr>
<tr>
<td>5.1 Material Properties, Normalized ( J^n ) and Stress Intensity Factors for the Plexiglass-Resin Model.</td>
<td>130</td>
</tr>
<tr>
<td>5.2 Infinite Bi-Material Model Data and Response Results for ( E_1=5 \times 10^5 ) psi and ( \nu_1=\nu_2=0.3 ).</td>
<td>131</td>
</tr>
<tr>
<td>5.3 Normalized Stress Intensity Factors for Dendritic Fracture Model.</td>
<td>134</td>
</tr>
<tr>
<td>5.4 Interface Joint Parameters for Lower Crack Tip Closure.</td>
<td>142</td>
</tr>
<tr>
<td>5.5 Normalized Stress Intensity Factors, ( K/qa^{\sqrt{\pi}} ) for Crack Approaching and Penetrating the Interface.</td>
<td>150</td>
</tr>
<tr>
<td>5.6 Treatment Pressure to Minimum Horizontal In-Situ Stress Ratios ( (q//<em>H) ) for Fracture Propagation/Containment ( (\sigma_H=1589 \text{ psi}, K</em>{1c}=1000 \text{ psi}-\text{in}). )</td>
<td>162</td>
</tr>
<tr>
<td>C.1 Shear Strength and Friction Coefficient Values for Some Materials.</td>
<td>206</td>
</tr>
<tr>
<td>D.1 Non-Dimensional J-Integral Results for a Crack Approaching, Terminating and Crossing an Interface ( (J/qa \times 10^{-6}) ).</td>
<td>210</td>
</tr>
<tr>
<td>D.2 Plane Strain Infinite Plate Bi-Material Data and Response Results for J-Integral Criterion.</td>
<td>211</td>
</tr>
</tbody>
</table>
## LIST OF FIGURES

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>4n-Node Isoparametric Quadrilateral Element</td>
<td>14</td>
</tr>
<tr>
<td>2.2</td>
<td>Degenerate Eight Node Elements</td>
<td>18</td>
</tr>
<tr>
<td>2.3</td>
<td>Shape Function Variation for Quadratic Triangle</td>
<td>20</td>
</tr>
<tr>
<td>2.4</td>
<td>Shape Function Variation for Collapsed Triangle</td>
<td>20</td>
</tr>
<tr>
<td>2.5</td>
<td>Straight and Curved Collapsed Singular Triangular Elements</td>
<td>23</td>
</tr>
<tr>
<td>2.6</td>
<td>Collapsed Quarter Point Singularity Pie Element</td>
<td>26</td>
</tr>
<tr>
<td>2.7</td>
<td>Possible Degeneration of a Brick Element</td>
<td>26</td>
</tr>
<tr>
<td>2.8</td>
<td>Collapsed Singular Elements, Crack Tip Configuration for Unsymmetric Loading and/or Geometry</td>
<td>40</td>
</tr>
<tr>
<td>2.9</td>
<td>Orientation of Isoparametric Pie Singularity Elements Along Crack Border</td>
<td>40</td>
</tr>
<tr>
<td>2.10</td>
<td>Crack Tip Coordinates and Geometry for a Crack Lying Along a Bi-Material Interface</td>
<td>43</td>
</tr>
<tr>
<td>2.11</td>
<td>Crack Tip Coordinates and Geometry for a Crack Perpendicular to Bi-Material Interface</td>
<td>43</td>
</tr>
<tr>
<td>3.1</td>
<td>Comparison of Mixed Mode Fracture Theories</td>
<td>55</td>
</tr>
<tr>
<td>3.2</td>
<td>Fracture Envelopes for Interfacial Crack</td>
<td>59</td>
</tr>
<tr>
<td>3.3</td>
<td>Initial Fracture Angle Variation with $K_1/K_2$.</td>
<td>60</td>
</tr>
<tr>
<td>3.4</td>
<td>Effect of Characteristic Distance &quot;r&quot; on Initial Fracture Angle for m=0.1</td>
<td>60</td>
</tr>
<tr>
<td>3.5</td>
<td>Effects of Characteristic Distance &quot;r&quot; on Fracture Envelopes for m=0.1 and m=0.2</td>
<td>62</td>
</tr>
<tr>
<td>3.6</td>
<td>Poisson's Ratio Effect for m=0.1 (r=0.002)</td>
<td>63</td>
</tr>
<tr>
<td>3.7</td>
<td>Fracture Envelopes for Debonding ($\mu_f=0$)</td>
<td>63</td>
</tr>
<tr>
<td>3.8</td>
<td>Compressive Debonding Criteria with Friction Effects for m=0.1 and m=0.2 (r=0.002)</td>
<td>66</td>
</tr>
<tr>
<td>3.9</td>
<td>Effect of Characteristic Distance &quot;r&quot; on Debonding Envelopes</td>
<td>67</td>
</tr>
<tr>
<td>Figure</td>
<td>Description</td>
<td>Page</td>
</tr>
<tr>
<td>--------</td>
<td>------------------------------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>3.10</td>
<td>Poisson's Ratio Effect on Debonding Envelopes for $m=0.1$ ($r=0.002$)</td>
<td>67</td>
</tr>
<tr>
<td>3.11</td>
<td>Fracture Envelopes for Perpendicular Cracks Terminating at Bi-Material Interfaces</td>
<td>75</td>
</tr>
<tr>
<td>3.12</td>
<td>Poisson's Ratio Effects for $m=0.05$ and $m=5$ (Terminal Crack)</td>
<td>75</td>
</tr>
<tr>
<td>3.13</td>
<td>Variation of Initial Fracture Angle for Terminal Cracks with $K_1/K_2$</td>
<td>77</td>
</tr>
<tr>
<td>3.14</td>
<td>Comparison of Fracture Envelopes for Propagating and Reflecting Cracks</td>
<td>77</td>
</tr>
<tr>
<td>3.15</td>
<td>Poisson's Ratio Effect for Reflecting Crack ($m=0.05$)</td>
<td>78</td>
</tr>
<tr>
<td>3.16</td>
<td>Poisson's Ratio Effect for Reflecting Crack ($m=0.10$)</td>
<td>78</td>
</tr>
<tr>
<td>3.17</td>
<td>Fracture Initiation Angle for Reflecting Crack ($m=0.05$ and $m=0.10$)</td>
<td>79</td>
</tr>
<tr>
<td>3.18</td>
<td>Fracture Envelopes for Tensile Debonding</td>
<td>80</td>
</tr>
<tr>
<td>3.19</td>
<td>Fracture Envelopes for Combined Shear and Tensile Debonding</td>
<td>80</td>
</tr>
<tr>
<td>3.20</td>
<td>Fracture Envelopes for Compressive Debonding ($\mu_f=0.5$ and 0.7)</td>
<td>81</td>
</tr>
<tr>
<td>3.21</td>
<td>Effect of $m$ and $\mu_f$ on Compressive Debonding Envelopes ($m=2.5$ and 0.50)</td>
<td>82</td>
</tr>
<tr>
<td>3.22</td>
<td>Bi-Material Body Configuration Without Cracks</td>
<td>86</td>
</tr>
<tr>
<td>3.23</td>
<td>Bi-Material Media with (a) Interfacial, (b) Terminal Cracks</td>
<td>86</td>
</tr>
<tr>
<td>3.24</td>
<td>Bi-Material (a) $J^c$ and (b) $J^n$ Integral Applications</td>
<td>93</td>
</tr>
<tr>
<td>4.1</td>
<td>Deep Cantilever Beam and Cases of Degeneracy</td>
<td>98</td>
</tr>
<tr>
<td>4.2</td>
<td>Single Edge Crack Plate in Tension</td>
<td>98</td>
</tr>
<tr>
<td>4.3</td>
<td>Interfacial Crack Test Problem</td>
<td>101</td>
</tr>
<tr>
<td>4.4</td>
<td>FEM Model for Interfacial Crack Problem</td>
<td>101</td>
</tr>
<tr>
<td>4.5</td>
<td>Strain Energy Release Rate Variation</td>
<td>103</td>
</tr>
<tr>
<td>4.6</td>
<td>Interfacial Crack Surface Normal Displacement Profile</td>
<td>103</td>
</tr>
<tr>
<td>4.7</td>
<td>Stress Intensity Factor Estimation</td>
<td>104</td>
</tr>
<tr>
<td>Figure</td>
<td>Description</td>
<td>Page</td>
</tr>
<tr>
<td>--------</td>
<td>-----------------------------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>4.8</td>
<td>Variation of $K_1$ and $K_2$ with $m$</td>
<td>104</td>
</tr>
<tr>
<td>4.9</td>
<td>Bi-Material Plate with a Terminal Crack</td>
<td>106</td>
</tr>
<tr>
<td>4.10</td>
<td>FEM Model and $M^n$ Integral Evaluation Paths for the Terminal Crack Problem</td>
<td>106</td>
</tr>
<tr>
<td>4.11</td>
<td>Study of the Shape Function Modification and Degenerating Schemes for Variable Singularity Elements</td>
<td>108</td>
</tr>
<tr>
<td>4.12</td>
<td>Crack Opening Displacement Profiles for $m=0.043$ and 23.08</td>
<td>110</td>
</tr>
<tr>
<td>4.13</td>
<td>Interface Tip Angular Stress Variation for $m=0.043$ and 23.08</td>
<td>110</td>
</tr>
<tr>
<td>4.14</td>
<td>Bi-Material $J^R$ Integral Variations with (a) $r_o$ and (b) different paths for fixed $r_o$</td>
<td>113</td>
</tr>
<tr>
<td>4.15</td>
<td>Pressurized Elliptical Crack Model in 3D Medium</td>
<td>115</td>
</tr>
<tr>
<td>4.16</td>
<td>A Typical Plane Section of the FEM Model for the Elliptical Crack</td>
<td>116</td>
</tr>
<tr>
<td>4.17</td>
<td>Displacement Profiles Along the Semi-Major (a) and Semi-Minor (b) Axis of the Elliptical Crack</td>
<td>117</td>
</tr>
<tr>
<td>4.18</td>
<td>Stress Intensity Factor Variation Along the Elliptical Boundary</td>
<td>117</td>
</tr>
<tr>
<td>5.1</td>
<td>Types of Hydraulically Induced Fractures</td>
<td>119</td>
</tr>
<tr>
<td>5.2</td>
<td>A Schematic Representation of Different Horizontal In-Situ Stress Magnitudes Along Formation Layers</td>
<td>119</td>
</tr>
<tr>
<td>5.3</td>
<td>Stress Intensity Factor Variations at the Homogeneous (a) and Bi-material (b) Crack Tips for Infinite Plane Strain Plate Model</td>
<td>122</td>
</tr>
<tr>
<td>5.4</td>
<td>Variation of $K$ at the Bi-Material Crack Tip with $m$ (a) and $\sigma_2/\sigma_1$ (b) for $q/\sigma_1=1.0$</td>
<td>123</td>
</tr>
<tr>
<td>5.5</td>
<td>Variation of $K$ at the Bi-Material Crack Tip with $m$ (a) and $\sigma_2/\sigma_1$ (b) for $q/\sigma_2=1.0$</td>
<td>124</td>
</tr>
<tr>
<td>5.6</td>
<td>Bi-Material Crack Tip Stress Intensity Factor Variation with $\sigma_2/\sigma_1$, Simulation Arrest at the Interface</td>
<td>126</td>
</tr>
<tr>
<td>5.7</td>
<td>Stress Intensity Factor Variation for a Crack Approaching and Penetrating an Interface</td>
<td>126</td>
</tr>
<tr>
<td>5.8</td>
<td>Plexiglass-Resin Sandwich Model</td>
<td>128</td>
</tr>
<tr>
<td>Figure</td>
<td>Description</td>
<td>Page</td>
</tr>
<tr>
<td>--------</td>
<td>-------------</td>
<td>------</td>
</tr>
<tr>
<td>5.9</td>
<td>Treatment Pressure Ratio Comparisons for Infinite Bi-Material Plate Model</td>
<td>128</td>
</tr>
<tr>
<td>5.10</td>
<td>Dendritic Fracture Model with Primary Fracture and Secondary Cracks Under Uniform Treatment Pressure and In-Situ Loading</td>
<td>135</td>
</tr>
<tr>
<td>5.11</td>
<td>FEM Model for Dendritic Fracturing</td>
<td>136</td>
</tr>
<tr>
<td>5.12</td>
<td>Stress Intensity Factor Variations with $\sigma_y/\sigma_x$ for (a) $c/b=1.0$ and (b) $c/b=1/2$</td>
<td>137</td>
</tr>
<tr>
<td>5.13</td>
<td>Angular Variations of the Minimum Principal Stress, $\sigma_{MIN}$ at $r/a=0.002$</td>
<td>137</td>
</tr>
<tr>
<td>5.14</td>
<td>Pressurized Crack Intersecting a Joint</td>
<td>138</td>
</tr>
<tr>
<td>5.15</td>
<td>Crack Surface Displacement Profile for $(\alpha,\beta)=(0,0)$ and $(\alpha,\beta)=(1/2,0)$</td>
<td>141</td>
</tr>
<tr>
<td>5.16</td>
<td>Vertical Hydraulic Fracture Model in a Layered Medium</td>
<td>143</td>
</tr>
<tr>
<td>5.17</td>
<td>Angular Variation of Non-Dimensional Energy Release Rate</td>
<td>144</td>
</tr>
<tr>
<td>5.18</td>
<td>Normalized Bi-Material Stress Intensity Factors for Vertical Fracture Model</td>
<td>145</td>
</tr>
<tr>
<td>5.19</td>
<td>Crack Surface Normal Displacement Profiles for $q=1.0$</td>
<td>145</td>
</tr>
<tr>
<td>5.20</td>
<td>Normalized Bi-Material Stress Intensity Factor Variations with $m$ for (a) $</td>
<td>q/\sigma_H</td>
</tr>
<tr>
<td>5.21</td>
<td>Normalized Bi-Material Stress Intensity Factor Variations with (a) $\Delta\sigma/\sigma_H$, and (b) $q/\sigma_H$ ($\sigma_H=-1.0$)</td>
<td>148</td>
</tr>
<tr>
<td>5.22</td>
<td>Loading Combinations for $K_B=0$</td>
<td>149</td>
</tr>
<tr>
<td>5.23</td>
<td>Normalized Stress Intensity Factor Variations as the Crack Approaches and Penetrates the Interface, (a) $q$-Loading, (b) $\sigma_H$-Loading</td>
<td>151</td>
</tr>
<tr>
<td>5.24</td>
<td>Normalized Stress Intensity Factor Variations with $m$ (a) $q$-Loading, (b) $\sigma_H$-Loading</td>
<td>153</td>
</tr>
<tr>
<td>5.25</td>
<td>Non-Dimensional Stress Intensity Factor Variation with $a/H_1$ for (a) $</td>
<td>q/\sigma_H</td>
</tr>
<tr>
<td>5.26</td>
<td>Treatment Pressure Versus Fracture Containment, $a/H_1$ for $m=0.1$</td>
<td>156</td>
</tr>
<tr>
<td>Figure</td>
<td>Description</td>
<td>Page</td>
</tr>
<tr>
<td>--------</td>
<td>-----------------------------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>5.27</td>
<td>Treatment Pressure Versus Fracture Containment, ( \frac{a}{H_1} ) for ( m=0.5 )</td>
<td>157</td>
</tr>
<tr>
<td>5.28</td>
<td>Treatment Pressure Versus Fracture Containment, ( \frac{a}{H_1} ) for ( m=2.5 )</td>
<td>158</td>
</tr>
<tr>
<td>5.29</td>
<td>Treatment Pressure Versus Fracture Containment, ( \frac{a}{H_1} ) for ( m=5.0 )</td>
<td>159</td>
</tr>
<tr>
<td>5.30</td>
<td>Normalized Treatment Pressure for Fracture Height Estimation Corresponding to ( m=3.0 ) and (a) ( H_1=100 ) ft, (b) ( H_1=200 ) ft</td>
<td>163</td>
</tr>
<tr>
<td>A.1</td>
<td>Crack Tip Polar Coordinates</td>
<td>185</td>
</tr>
<tr>
<td>A.2</td>
<td>Stress State Near Crack Border</td>
<td>187</td>
</tr>
<tr>
<td>B.1</td>
<td>Semi-Radial Singularity Element Local Coordinates and Geometry.</td>
<td>198</td>
</tr>
<tr>
<td>C.1</td>
<td>Bi-Material Plastic Zone</td>
<td>202</td>
</tr>
<tr>
<td>C.2</td>
<td>Bi-Material Microcrack Zone</td>
<td>202</td>
</tr>
<tr>
<td>C.3</td>
<td>Fracture Process Zone Due to Friction</td>
<td>207</td>
</tr>
<tr>
<td>D.1</td>
<td>Non-Dimensional Total Potential Energy Variation for Crack Approaching and Crossing the Interface.</td>
<td>209</td>
</tr>
<tr>
<td>D.2</td>
<td>Normalized J-Integral Variation for Crack Approaching and Crossing the Interface.</td>
<td>209</td>
</tr>
<tr>
<td>D.3</td>
<td>Normalized J-Integral Variation with ( \frac{\mu_2}{\mu_1} )</td>
<td>212</td>
</tr>
<tr>
<td>D.4</td>
<td>Bi-Material J-Integral Criterion for Infinite Bi-Material Plate Model.</td>
<td>212</td>
</tr>
<tr>
<td>D.5</td>
<td>Bi-Material Terminal Crack Configuration.</td>
<td>214</td>
</tr>
</tbody>
</table>
Latin Alphabet Symbols

\(A, B, C, b, c, d\)  Constants
\(a\)  Half crack length
\(E\)  Young's modulus
\(F(s), G(s), H(s), P(s)\)  Functions of the variable \(s\), as defined
\(f(\theta), g(\theta)\)  Functions of the variable \(\theta\), as defined
\(H(\xi, \eta)\)  Singular interpolation functions
\([\mathbf{J}]\)  Jacobian of transformation matrix
\(i, j\)  Running indices
\(J\)  Path independent \(J\) integral
\(K_1, K_2\)  Opening and sliding mode stress intensity factors, respectively
\(K_{IC}\)  Fracture toughness
\(K_d\)  Debonding strength
\(k\)  Poisson's ratio parameter as defined in Appendix A1
\(L\)  Crack tip element dimension
\(M\)  Path independent \(M\)-integral
\(M_i(\xi, \eta)\)  Semi-radial singularity mapping functions
\(m\)  Shear moduli ratio
\(N_i(\xi, \eta)\)  Isoparametric interpolation functions
\(\mathbf{n}\)  Unit outward normal
\(p\)  Power of stress/strain singularity
\(q\)  Crack surface pressure
\(r, \theta, z\)  Cylindrical coordinates
\(S_o\)  Shear strength
\(\mathbf{u}\)  Displacement vector
\( W(\xi, \eta) \)  
Local function related to singular displacement field

\( W \)  
Strain energy density

\( x, x, y, z \)  
Cartesion coordinates

**Greek Alphabet Symbols**

\( G_1, G_2 \)  
Energy release rate for mode I and mode II, respectively

\( G_c \)  
Critical value of the energy release rate

\( \alpha \)  
Running index

\( \epsilon \)  
Bi-material constant related with interfacial crack

\( \xi, \eta, \zeta \)  
Local coordinates of parent element

\( \xi_1, \eta_1, \zeta_1 \)  
Gaussian quadrature points

\( \delta \)  
Kronecker delta

\( \mu \)  
Shear modulus

\( \mu^* \)  
Equivalent shear modulus for bi-material media

\( \nu \)  
Poisson's ratio

\( \nu_f \)  
Coefficient of friction

\( \delta_c \)  
Characteristic distance measured from the bi-material crack tip

\( \delta_{12}^c \)  
Characteristic, interfacial distance

\( \theta \)  
Polar angle measured from the crack plane

\( \phi \)  
Airy's stress function

\( \tilde{\sigma} \)  
Stress vector

\( \sigma_u \)  
Tensile strength

\( \sigma_{12}^c \)  
Bond strength

\( \Gamma \)  
Boundary of a closed region

\( \tau \)  
Shear stress

\( \rho \)  
Non-dimensional radius
1.1 PRELIMINARIES

Linear elastic fracture mechanics (LEFM) theory has been an effective tool in the analysis and design of structures with material defects. Recently, with the advancement of technology, more attention is being focused on multi-layered systems. Increased applications in structural design, geomechanics and biomechanics have been evidenced. It is the designer's duty to eliminate the detrimental effects of the cracks and crack-like defects in structural components. However, in some applications, a fracture or a fracture system is induced by the designer in the geological environment for energy resource extraction such as hydraulic fracturing. The critical factor in all cases, though, is the control of the fracture process by effective design.

The stress field at the crack tip is known to be proportional to $K/r^p$, where $r$ is the radial distance measured from the crack tip, $p$ is the singularity power ($0 < p < 1$) and $K$ represents the strength of the stress singularity known as the stress intensity factor. The power of singularity has been known to be $1/2$ for homogeneous linear elastic, isotropic media, deformed under conditions of planar behavior.

The singular character of the stresses around a crack at the bondline of two elastic half-planes of dissimilar material was first
examined by Williams [1]. Although the inverse square root stress singularity is still valid as in the homogeneous material, an oscillatory character of the type $r^{-\left(1/2 + i\varepsilon\right)}$ is observed for the local stresses as the crack tip is approached. The parameter, $\varepsilon$, is a bi-material constant, determined from the elastic constants and the stress state. When a crack terminates perpendicularly at the interface, the nature of the singularity changes characteristically from the inverse square root behavior [2]. The power of singularity $p$ varies as a function of the elastic constants and the state of planar deformation (i.e. plane stress or plane strain).

Once the stress field around the crack tip is known, a realistic fracture criterion must be established to predict propagation or arrest of the crack. The nature of the problem considered plays a significant role on such a criterion due to the inherent mechanisms. For example, in a geological environment, layer interfaces, imperfections such as joints, differences in horizontal in-situ stress magnitudes on different layer formations may all act as fracture control mechanisms. Fracture of a composite can, on the other hand, be identified with matrix cracking, fiber breaking, fiber-matrix debonding or a combination of them. Cohesive or adhesive modes of joint failure, nature of fiber reinforcement and layer laminates must be considered as primary factors in the established failure criterion.

Exact solutions to engineering problems are often very difficult or nearly impossible to obtain. Thus the developed analytical solution techniques are restricted to simplified geometries and loading conditions and numerical solution techniques are sought. Among the various
competing numerical methods, the finite element method (FEM) has gained considerable importance due to its generality [3-5]. FEM also offers the advantage of performing a fracture mechanics analysis in addition to a stress analysis by incorporating appropriate crack tip finite elements in the computational procedure.

1.2 LITERATURE REVIEW

Following the pioneering work of Inglis [6] and Westergaard [7], the elastic stress distributions around cracks have been studied by many investigators. Numerous results on cracks in plates, shells and composites can be found in handbooks [8,9]. Here, particular attention will be focused on bi-material fracture mechanics problems.

Williams [1] and Zak and Williams [2] first studied the singular crack tip stress field for an interfacial and a terminal bi-material crack, respectively. The solution to the biharmonic equation was obtained by assuming the continuity of displacements and stresses across the interface and stress free crack surfaces. Cook and Erdogan [10] studied the stresses in bonded materials with a crack perpendicular to the interface by solving the integral equations obtained via Mellin transform techniques. Bogy [11] considered the plane problem of a loaded crack that terminates at an angle to the interface of a bi-material composite. Ashbaugh [12] studied the finite crack with an arbitrary orientation and distance from the straight interface. Swenson and Rau [13] complemented the plane stress solution of Zak and Williams [2] by presenting the plane strain problem of a perpendicular terminal interface crack. Hilton and Sih [14] have considered a crack contained
in a strip of one material which is bonded on each side to a second material. The crack was either parallel or perpendicular to the sides of the strip. Various other problems of cracks with different orientations to the interface in a composite have been studied by several investigators [15-17].

The problem of a crack or ray of cracks along the common interface of two dissimilar materials was considered by a number of investigators [18-20]. Goree and Venezia [21] gave the stress intensity factors, strain energy release rate, stresses and displacements for two bonded elastic half-planes having a crack along the interface as well as a perpendicular crack in one of the half planes, which may intersect the interface crack. Mak and Keer [22] considered the no-slip condition for an interface edge crack due to the oscillating singularities and material overlapping. Piva and Viola [23] studied the effects of in-plane biaxial loading of two dissimilar materials with a crack along their common interface and discussed the unrealistic nature of the oscillatory stresses in the vicinity of the crack tip. In [24] they further investigated the same problem for incompressible media and biaxial and shear loading at infinity. Atkinson [25] has reexamined the interfacial crack problem. To remove the oscillatory singularity, he inserted an artificial third material between two solids. Comninou [26-28] successfully removed the oscillatory behavior of crack tip stress and displacement by assuming that the crack is not completely open and that its surfaces are in frictionless contact. Although there are restrictions such as the crack closure size and the range of bi-material
constant, Comninou's approach has been pursued by other investigators [29-30].

1.2.1 Finite Element Method Applications in LEFM

A major limitation in applying principles of fracture mechanics is the availability of stress intensity factor solutions for physical structures. Determination of these crack tip stress intensity factors requires an exact solution of the associated elasticity problem. In light of the extensive analytical formulations for the basic problem, the finite element method application to LEFM has gained considerable momentum.

Finite element schemes for computational fracture mechanics can be divided into two broad categories. The first and the earliest scheme uses refined mesh of regular elements around the crack tip. This scheme was not sufficiently successful due to its inability to incorporate the theoretical crack tip singularity. Anderson et al. [31] and Chan et al. [3] both emphasized the importance of optimizing the mesh to obtain the best possible results with reasonable accuracy. The second scheme uses special finite elements with singular functions such that the local derivatives are represented in the desired singular form. Thus, a more accurate solution can be obtained with significantly reduced degrees of freedom. In this scheme the singularity is represented either by using a singular displacement interpolation with the standard geometrical mapping or singular geometrical mapping with the standard displacement interpolation.
The stress singularity at the tip of a crack fully embedded in a homogeneous isotropic medium is known to vary as a function of $r^{-1/2}$. A review of finite element techniques for the solution of such problems is given in Refs. [4] and [32]. To name a few, the elements due to Barsoum [33], Henshell and Shaw [34] and Pu et al. [35] incorporate the desired singularity through geometric mapping. The singularity elements developed by Blackburn [36], Stern and Becker [37] and Tracey [38] use singular displacement interpolation. For the class of problems in which the power of stress singularity is not 1/2, the above mentioned singular elements are no longer applicable. Several variable singularity power elements to represent the $r^{-p}$ stress variation for $0<p<1$ have been developed. Tracey and Cook [39], Stern [40], Akin [41] and Staab [42] have assumed displacement functions which incorporate $r^{-p}$ terms in their formulations. These elements are generally non-conforming and lack inter-element compatibility. The semi-radial singularity mapping technique, first introduced by Okabe [43-44], imposes the desired singularity through the geometric mapping. Lin and Mar [45] constructed a hybrid crack element, to account for the variable nature of the singularity for interfacial and terminal crack problems. Yamada et al [46] has reviewed the various proposed singularity elements and discussed their correlations. In all these schemes, it is assumed that the form of the singularity is known, i.e. the power of the singularity $p$ in the leading singular derivative field, $r^{-p}$, is known a priori. Staab [47,48] has proposed a simple method for estimating the singularity power by modelling the crack tip region with standard finite elements. Swedlow [49]
introduced a finite element scheme in which \( p \) is treated as an undetermined exponent in the finite element analysis and is subsequently obtained by minimization of the total potential energy functional.

A number of the aforementioned finite element techniques have been extended to their three-dimensional counterparts. These include the three-dimensional representation of the quarter point mapping element \([50,51]\), three-dimensional form of \([52]\) of Tracey's element \([39]\) and the extension of semi-radial singularity mapping to the representation of line singularities by Okabe and Kikuchi \([53]\).

Both in their planar and three-dimensional forms, isoparametric elements exhibit a special behavior peculiar to the inherent singularity representation. Degenerate elements obtained by collapsing a side in two dimensions and a plane in three dimensions demonstrate a better mesh utilization and significant ease in the incorporation of a singularity element to an existing code. But, misrepresentation of the desired singularity is possible if degeneracy is not proper.

1.2.2 Fracture Criteria

In his historical work, A.A. Griffith \([54]\) formulated a criterion for the extension of an isolated crack in a solid subjected to an applied stress using the fundamental energy theorems of classical mechanics and thermodynamics. Most of the modern fracture criteria are an extension of Griffith's energy balance concept which simply states that crack propagation will occur when the energy released during crack front advance equals the energy required to form new crack surfaces \([55]\). The energy released during co-planar crack propagation was shown
to be expressed in terms of the stress intensity factor corresponding to the stress state prior to crack propagation [56]. The maximum circumferential tensile stress theory developed by Erdogan and Sih [57] assumes that the reaching of the corresponding stress intensity factor to a critical value accompanies the start of slow crack extension.

The problem of crack extension in brittle solids under general loading has attracted several investigators. To mention a few, Palaniswamy and Knauss [55], Swedlow [58], Hussain et al. [59], Sih [60], Budiansky and Rice [61] and others have studied the problem extensively. As a result, several theories have emerged. Besides the maximum tensile stress theory [57], the minimum strain energy density theory by Sih [60] and the maximum energy release rate theory [55,59] gained popularity among researchers and engineers. Local mechanisms, such as crack closure and frictional effects due to the compressive nature of the stresses, necessitated further considerations. Swedlow [58] reconsidered the minimum strain energy density criterion. He proposed adding a corollary statement to its two hypotheses for the proper representation of compressive behavior of the stress field. Lee and Advani [62] extended the model of McClintock and Wash [63] to represent crack closure and frictional effects.

considered the containment/propagation phenomenon of hydraulically induced fractures [66-74]. For justifying the applicability of LEFM, such as fracture toughness or critical energy release rate, the definition of a process zone for rock is gaining popularity [75,76]. Such a process zone in rock is identified by the formation of microcracks around the crack tip, and closing of crack on some portions of its length. So far the available fracture criteria for heterogeneous multi-layered media are limited and controversial.

1.3 RESEARCH OBJECTIVES AND SCOPE

The main objective of this research is to develop a finite element model capable of representing the singular behavior of stresses and displacements for both homogeneous and bi-material fracture problems. The need for a proper fracture criterion for heterogeneous multi-layered media is well understood. In Chapter III the available fracture criteria developed for crack propagation in an isotropic medium are reviewed. The maximum circumferential tensile stress criterion is extended for bi-material interfacial and terminal cracks with emphasis on frictional effects and debonding of the interface. For the accurate computation of fracture parameters, such as stress intensity factors and energy release rate, efficient numerical techniques are proposed and demonstrated.

To represent the variable nature of the stress singularity, the singular displacement field element of Akin [41] and the semi-radial singularity mapping element [43] in its subparametric form are closely studied as degenerate triangular elements. For a homogeneous medium,
the inverse square root singularity and the quarter point collapsed element are modified for proper degeneracy in two and three dimensions. Sub-objectives of the research include:

  i) Development of an efficient finite element program for the analysis of linear elastic, plane homogeneous and bi-material fractured systems using improved singular elements,

  ii) Extension of the available LEFM fracture criteria to bi-material systems with cracks along and perpendicular to an interface,

  iii) Accurate computation of stress intensity factors for interfacial and terminal cracks and the study of path independent integrals for bi-material media,

  iv) Model studies and comparisons of the current results with the available analytical and numerical solutions,

  v) Hydraulic fracturing model simulations to predict the fracture containment/propagation phenomenon in large scale field problems, by considering material property mismatch, variation in horizontal in-situ stress and fracturing fluid pressure.

The microcrack process zone in rock [75] is reconsidered and the effect of friction on the microcrack zone size is included in Appendix C. The application of such a process zone to the bi-material terminal crack is also presented. The finite element computer code developed, FECAP, is used to solve all of the plane models in this study. The program structure and capabilities of FECAP are briefly summarized in Section 4.1.
CHAPTER II
FINITE ELEMENTS IN LINEAR ELASTIC FRACTURE MECHANICS

In this chapter several finite element schemes are studied for representing the LEFM singularity in crack problems. Isoparametric elements are shown to have the potential of producing strain singularities [77-79, 33-35]. Quarter point isoparametric elements [33,50] are modified for proper degeneracy, to represent inverse square root strain singularities of 2 and 3 dimensional crack problems in homogeneous media. Singularity elements proposed by Akin [41] and Okabe [43] are considered for bi-material terminal crack problems. The degenerate triangular form of the former is modified for better singularity representation. Sub-parametric form of the latter is shown to perform better. Stress intensity factors for a crack, perpendicular to a bi-material interface and along the interface are accurately computed using the proposed finite element schemes.

2.1 DEGENERATE ISOPARAMETRIC FINITE ELEMENTS FOR LINEAR ELASTIC CRACK PROBLEMS IN HOMOGENEOUS MEDIA

Henshel and Shaw [34] and Barsoum [78] independently showed that, in an eight-node quadratic displacement isoparametric element, an inverse square root singularity in the strain field is obtained by placing the midside nodes at the quarter points of the sides emanating from the singular crack tip. Pu, Hussain and Lorenson [35] obtained
the same singular behavior for the twelve-node cubic isoparametric element by placing the two mid-side nodes at locations defined by 1/9 and 4/9 times the length of the sides from the crack tip. The above findings were generalized by Nayfeh and Nassar [79] for any serendipity family of elements having (4n) nodal points, where n is the order of the interpolation function.

2.1.1 Investigation of Singular Behavior in Isoparametric Elements

Hibbitt [77] demonstrated that, with an nth order isoparametric interpolation, a singularity of order \( r^{(1-n)/n} \) can be obtained. For example, considering a one-dimensional polynomial interpolation on \( 0 \leq \xi \leq 1 \), for \( n \geq 1 \), we have

\[
F(\xi) = b_0 + b_1 \xi + b_2 \xi^2 + \ldots + b_n \xi^n
\]  

(2.1)

For the isoparametric formulation, the same interpolation function is used to represent both the position coordinate, \( r \), and the displacement, \( u \). Hence

\[
r = A_0 + A_1 \xi + A_2 \xi^2 + \ldots + A_n \xi^n
\]

(2.2)

\[
u = b_0 + b_1 \xi + b_2 \xi^2 + \ldots + b_n \xi^n
\]

where \( b_i \) and \( A_i \) are defined by the nodal displacements and positions, respectively. The nodes can be arranged such that all \( A_i \), (except \( A_0 \)) and \( A_n \), are zero. Thus

\[
r = A_0 + A_n \xi^n
\]

(2.3)
For an arbitrary choice of origin, i.e. \( A_0 = 0 \), the displacement gradient along the considered line becomes

\[
\frac{d u}{d r} = \frac{1}{n A_n} \left\{ b_1 \left( \frac{r}{A_n} \right)^{1-n} + 2 b_2 \left( \frac{r}{A_n} \right)^{2-n} + \ldots + n b_n \right\} \tag{2.4}
\]

In the above equation, the required strain singularity is maintained by the leading \( r^{(1-n)/n} \) term. The range of singularity, as \( n \) varies from 2 to \( \infty \), is from \( r^{-1/2} \) to \( r^{-1} \).

For the positioning of the nodes, consider, for simplicity, one side of a \((4n)\)-node quadrilateral isoparametric element having \((n+1)\) nodes on a side (Fig. 2.1). The position and displacement functions are

\[
x = \sum_{i=1}^{n+1} N_i x_i = A_0 + A_1 \xi + A_2 \xi^2 + \ldots + A_n \xi^n \tag{2.5}
\]

\[
u = \sum_{i=1}^{n+1} N_i u_i = b_0 + b_1 \xi + b_2 \xi^2 + \ldots + b_n \xi^n
\]

where \( N_i \) are the Lagrange interpolating polynomials defined by

\[
N_i = \sum_{j=1}^{n+1} \frac{\prod_{j \neq i} (\xi - \xi_j)}{\prod_{j=1}^{n+1} (\xi_i - \xi_j)} \quad \text{and} \quad -1 \leq \xi \leq 1 \tag{2.6}
\]

in which

\[
\xi_i = \frac{2(i-1)}{n} - 1 \quad , \quad i = 1, 2, \ldots, (n+1) \tag{2.7}
\]

Hence, the denominator of Eq.(2.6) becomes [79]:
Rewriting Eq. (2.6) after using Eq. (2.8), we obtain

\[ N_i = \frac{1}{D_i} \{(\xi - \xi_1)(\xi - \xi_2) \ldots (\xi - \xi_{i-1})(\xi - \xi_{i+1}) \ldots (\xi - \xi_n)\} \]  

(2.9)

The reduced Jacobian \(dx/d\xi\), must vanish at \(\xi = -1\), to yield singular strain at \(x=0\).

\[ \frac{dx}{d\xi} = \sum_{i=1}^{n+1} \frac{dN_i}{d\xi} x_i \]  

(2.10)

which leads to the condition

\[ \sum_{i=2}^{n+1} \frac{(-1)^{n+1-i}}{(i-1)(i-1)! (n+1-i)!} x_i = 0 \]  

(2.11)
In order to have an inverse square root singularity in \( du/dx \) at \( x=0 \), consider

\[
\frac{du}{dx} = \frac{d}{d\xi} \left( \frac{dx}{d\xi} \right) = b_1 + 2b_2\xi + 3b_3\xi^2 + \ldots + nb_n\xi^{n-1}
\]

Hence, one can deduce that \( x \) must be a quadratic function of \( \xi \), which is realized for \( A_r=0 \), for \( 3 \leq r \leq n \). This condition leads to

\[
\sum_{i=2}^{n+1} \frac{(-1)^{n+1-i} \psi_{ij} x_i}{(i-1)!(n-i)!} = 0 \quad , \quad j = 0,1,2,\ldots,(n-3)
\]

where \( \psi_{ij} \) is the coefficient of the \( \xi^r \) term in the expression

\[
(\xi-\xi_1)(\xi-\xi_2) \ldots (\xi-\xi_{i-1})(\xi-\xi_{i+1}) \ldots (\xi-\xi_{n+1})
\]

and is given by

\[
\psi_{ij} = (-1)^j \left\{ B_j - B_{j-2}\xi_1^2 + B_{j-2}\xi_1^3 - B_{j-2}\xi_1^2 + \ldots + (-1)^j B_2\xi_1^{j-2} \right\}
\]

where

\[
j = n-r
\]

\[
B_\ell = \sum_{1 \leq p < q < s < \cdots < t} (\xi_p \xi_q \xi_s \ldots \xi_t) \quad , \quad \ell = 1,2,3,\ldots,j
\]

As revealed in Fig. (2.1), the nodal points are located according to the relation

\[
x_i = \alpha_{i-1} L_i \quad , \quad i = 1,2,3,\ldots,(n+1)
\]

where \( \alpha_0=0 \) and \( \alpha_n=1 \)
Inspection of Eqs.(2.11), (2.13) and (2.17) provides the solution for

$$a_{i-1} = \left(\frac{i-1}{n}\right)^2, \quad i = 1, 2, 3, \ldots, (n-1) \quad (2.18)$$

The preceding discussion indicates that the distorted quadratic and cubic isoparametric singularity elements for linear elastic fracture mechanics problems represent special cases of the \((4n)\)-node isoparametric element with midside nodes of the sides meeting at the singular tip located according to the relation given by Eq.(2.19) for an \(r^{-1/2}\) crack tip strain singularity.

As stated in References [35,79] and shown here, the position coordinates must be a quadratic function of the local coordinates, to ensure an \(r^{-1/2}\) singularity. Since 8-node quadratic isoparametric elements are readily available in most finite element programs, these elements are often utilized in their singular forms for linear elastic fracture problems involving homogeneous media. For such quadrilateral isoparametric singularity elements, singular conditions prevail only along the edges of such elements and not on an arbitrary ray emanating from the singular crack tip. To remedy this situation Barsoum [33], collapsed one side of this element to the singular node and placed the midside nodes at quarter points from the crack tip, and so the degenerate triangular isoparametric singularity element is formed. In Reference [35], the same idea was successfully used for a cubic version of the element. In this case, the singularity prevails on all rays emanating from the vertex. However, when a quadrilateral degenerates into a triangle, the shape functions of the midside nodes
at the edge opposite to the collapsed corner create undesirable effects [80]. A slight perturbation in placing the mid-side node opposite to the crack tip for a collapsed 8-node quadrilateral element increases the error involved in stress intensity factors [81]. The same type of behavior was observed in the collapsed 12-node quadrilateral element if one or both side nodes of the side opposite to the crack tip are slightly perturbed from their nominal positions. In addition, the results are sensitive to the particular way a quadrilateral element is degenerated. In general, the collapsed nodes must be at the crack tip [80] to obtain a correct singularity.

For non-singular problems, a modification has been suggested [82] to remedy the situation related with the troublesome shape function. This modification entails multiplication of the particular shape function by a dimensionless measure of the distance from the degenerate node. In the present investigation, it is shown that when the original shape function is properly modified, the degenerate elements behave considerably better both in singular and non-singular problems. This improvement is demonstrated in Chapter IV by considering a deep cantilever beam and a single edge notch plate as non-singular and singular problems, respectively.

2.1.2 Quarter Point Isoparametric Mapping in Two and Three Dimensions

A typical 8-node isoparametric element in two dimensions is shown in Fig. 2.2. The corresponding shape functions are [83]

a) At the midside nodes
Figure 2.2 Degenerate Eight Node Elements
\[ N_i = \frac{(1-\xi_i^2)(1+\eta_i)}{2} \quad ; \quad i = 5,7 \]
\[ N_i = \frac{(1-\eta_i^2)(1+\xi_i)}{2} \quad ; \quad i = 6,8 \]  
\[ \text{b) At the corner nodes} \]
\[ N_i = N_i^* - \frac{(N_{i+3} + N_{i+4})}{2} \quad ; \quad i = 2,3,4 \]
\[ N_i = N_i^* - \frac{(N_5 + N_8)}{2} \]
\[ N_i^* = \frac{(1+\xi_i)(1+\eta_i)}{4} \quad ; \quad i = 1,2,3,4 \]

where \( \xi_i, \eta_i \) represent the coordinates of node \( i \) with \(-1 < \xi < 1 \) and \(-1 < \eta < 1 \).

A collapsed triangular element obtained by superimposing the nodes 1, 4 and 8 of the parent square element is shown in Fig. 2.2(a). In general, such an element can be formed by superimposing the three nodes on any one side of the parent element. All shape functions in this case degenerate to those of a regular six-node triangle (Fig. 2.3), except the shape function of the midside node opposite the collapsed corner. Since this shape function varies linearly along the radial lines from the collapsed node and changes quadratically in the other direction, a nonpolynomial surface with indeterminate slopes is formed at the collapsed node as shown in Fig. 2.4. To maintain a quadratic variation in the other direction, the following modification is realized. For midside nodes, opposite to the collapsed corner, on the local coordinate axis, \( \eta = 0 \), the corrected shape function is
\[ N_i^* = \frac{1}{2} N_i (1+\xi_i) = \frac{1}{4} (1-\eta_i) (1+\xi_i)^2 \]  
and for nodes on the \( \xi = 0 \) axis, we have
Figure 2.3 Shape Function Variation for Quadratic Triangle

Figure 2.4 Shape Function Variation for Collapsed Triangle
\[ N_i^* = \frac{1}{2} N_i (1 + \eta \eta_i) = \frac{1}{4} (1 - \xi_i^2) (1 + \eta \eta_i)^2 \] (2.22)

where \( N_i \) is the original shape function of the midside node across the collapsed corner. To complete the modifications, \( N_i \) appearing in the shape function expressions for the corner nodes given by Eqs.(2.20) is replaced by \( N_i^* \). For the remaining nodes we have \( N_i = N_i^* \). The new shape functions still satisfy the constant strain condition, namely

\[ \sum_{i=1}^{8} N_i^* = 1 \] (2.23)

and the usual interpolation property

\[ N_i^*(\xi_j, \eta_j) = \delta_{ij} . \] (2.24)

At the same time, the indeterminate slope situation at the collapsed corner is prevented. Hence, the quadratic isoparametric quadrilateral degenerates successfully into a quadratic triangular element by assigning the same coordinates for the nodes at the collapsed corner, and by modifying the related midside node as suggested above.

The singular version of the collapsed triangular isoparametric element is obtained by placing the midside nodes at quarter points from the singular crack tip, and by employing the suggested modification to maintain the correct degeneracy.

For the element shown in Fig. 2.2(b), following the isoparametric transformations for the coordinate vector, \( \tilde{x} \) and displacement vector, \( \tilde{u} \)

\[ \tilde{x} = \sum_{i=1}^{8} N_i \tilde{x}_i ; \quad \tilde{u} = \sum_{i=1}^{8} N_i \tilde{u}_i \] (2.25)
we obtain
\[ x = L_1(1+\xi)^2/4 \]
\[ y = L_2(1+\eta)(1+\xi)^2/8 \]  \hspace{1cm} (2.26)

The displacement gradients in global coordinates can be expressed as
\[
\begin{bmatrix}
\ddot{u}_x \\
\ddot{u}_y 
\end{bmatrix}
= [J]^{-1}
\begin{bmatrix}
N_{1,\xi} \ddot{u}_1 \\
N_{1,\eta} \ddot{u}_1 
\end{bmatrix}
\]  \hspace{1cm} (2.27)

where the inverse of the Jacobian matrix is
\[
[J]^{-1} = \frac{2}{L_1L_2(1+\xi)^2}
\begin{bmatrix}
L_2(1+\xi) & -2L_1(1+\eta) \\
0 & 4L_1 
\end{bmatrix}
\]  \hspace{1cm} (2.28)

The position coordinate \( r \) of any point \( P \) on the radial line \( R \) (Fig. 2.2(b)) is
\[
r = \left( x^2 + y^2 \right)^{1/2} = \frac{L_2(1+\xi)^2}{8} \left[ \frac{1}{(2L_1/L_2)^2} + (1+\eta)^2 \right]^{1/2} \]  \hspace{1cm} (2.29)

Requiring \( \ddot{u}_1 = \ddot{u}_4 = \ddot{u}_8 \) and rewriting Eq.(2.27) along with Eq.(2.29) and taking \( \eta \) as constant along a radial line \( R \), the displacement gradients in global coordinates are obtained as
\[
\frac{\partial u}{\partial x} = \tilde{A}_0/\sqrt{r} + \tilde{A}_1 \\
\frac{\partial u}{\partial y} = \tilde{B}_0/\sqrt{r} + \tilde{B}_1 
\]  \hspace{1cm} (2.30)

where \( \tilde{A}_0, \tilde{B}_0, \tilde{A}_1, \tilde{B}_1 \) are constants for any given set of nodal displacements.
Consequently, we see that the triangular isoparametric singularity element possesses the required inverse square root singularity both at the element boundaries and at the interior of the element. Also a constant term for representing the constant strain term is present.

To study how the finite element results for the collapsed triangular element are affected by the perturbation of the midside node across the collapsed corner, we consider Fig. 2.5. For the unmodified element...

Figure 2.5 Straight and Curved Collapsed Singular Triangular Elements
The radial lines (n=constant) in the parent element are mapped into second order curves in the real element except when a=1. For the modified element

\[ y/x = (y_o/x_o) \frac{n\sqrt{y/y_0}}{n\sqrt{y/y_0} + (\alpha-1)(1-n^2)} \]  

(2.31)

It is seen that, for all a values, the n=constant lines in the parent element are mapped into straight lines in the real element.

Furthermore, the strain energy for the triangular singularity element is observed to be bounded. Although the strain energy density in any singular problem can become unbounded, the strain energy must be finite [77].

For a three-dimensional singularity element we will consider the twenty-node isoparametric "Brick" element which employs the quadratic shape functions [83]

\[ \mathbf{x} = \sum_{i=1}^{20} N_i(\xi, \eta, \zeta) \mathbf{x}_i \]  

(2.33)

\[ \mathbf{u} = \sum_{i=1}^{20} N_i(\xi, \eta, \zeta) \mathbf{u}_i \]  

(2.34)

where \( \mathbf{x} \) and \( \mathbf{u} \) are coordinate and displacement vectors. For the corner nodes
and the midside nodes

\[ N_i = \frac{1}{4} (1-\xi^2) (1+\eta_i) (1+\zeta_i) \quad ; \quad i = 9,11,13,15 \]

\[ N_i = \frac{1}{4} (1-\eta^2) (1+\xi_i) (1+\zeta_i) \quad ; \quad i = 10,12,14,16 \]

\[ N_i = \frac{1}{4} (1-\zeta^2) (1+\xi_i) (1+\eta_i) \quad ; \quad i = 17,18,19,20 \]

with \(-1 \leq \xi, \eta, \zeta \leq 1\). A three-dimensional isoparametric singularity element can be obtained (Fig. 2.6) by collapsing one face (3,4,8,7) and placing the midside nodes (10,12,14,16) at quarter points. The interelement compatibility condition is still satisfied both in the regular and singular forms of the element. As a counterpart of the plane isoparametric singularity element, here the crack tip is replaced by the crack front and the crack edge by the crack face. Following the same argument for the collapsed plane triangle and the results in Reference [82], for successful degeneration of the brick element to the crack front pie element, a shape function modification is performed for the midside nodes across the collapsed sides. Irons [82] outlines the proper modification scheme for degenerate brick elements in their nonsingular form (Fig. 2.7). The typical quadratic unmodified shape function at a midside node, \((\xi, \eta, \zeta) = (0, \eta_i, \zeta_i)\) is of the form

\[ N_i = \frac{1}{4} (1-\xi^2) (1+\eta_i) (1+\zeta_i) \]
Figure 2.6  Collapsed Quarter Point Singularity Pie Element

Figure 2.7  Possible Degeneration of a Brick Element
To illustrate possible degenerating schemes, consider the two nodes $(+1, -\eta_1, \xi_1)$ to coalesce, i.e. the edge opposite to the $i$th node disappears. Then the corrective factor of $\frac{1}{2}(1+\xi_1)$ is applied to $N_i$. If, however, the two nodes $(+1, \eta_1, -\xi_1)$ coalesce, then the corrective factor of $\frac{1}{2}(1+\eta_1)$ is applied to $N_i$. Thus any triangular section parallel to the base of the wedge in Fig. 2.7(c) now contains the correct function basis. For the quarter point collapsed singularity element, it is sufficient to modify $N_{17}$ and $N_{18}$ by multiplying them with the corrective factor of $\frac{1}{2}(1+\eta)$, for proper degeneracy.

To observe the singular behavior of the displacements and strains in Fig. 2.6b, we let $\xi=1$ be one of the crack faces, i.e. $\eta = -1$ corresponds to the crack front with $r,s,t$ as the triply-orthogonal crack front coordinates. The displacement vector $\tilde{u}$ along any line on the crack face normal to the crack line can be expressed from Eqs.(2.34) as

$$
\tilde{u} = \frac{1}{4} \left[ (1-\xi^2) \left[ (1+\eta) (2\tilde{u}_9 - \tilde{u}_1 - \tilde{u}_2) + (1-\eta) (2\tilde{u}_{11} - \tilde{u}_4 - \tilde{u}_3) \right] 
+ (1+\xi) \left[ 2(1-\eta^2) \tilde{u}_{12} + \eta (1+\eta) \tilde{u}_1 - \eta (1-\eta) \tilde{u}_4 \right] 
+ (1-\xi) \left[ 2(1-\eta^2) \tilde{u}_{10} + \eta (1+\eta) \tilde{u}_2 - \eta (1-\eta) \tilde{u}_3 \right] \right] 
$$

(2.38)

For any section $\xi=$constant, perpendicular to the crack front, the radial coordinate $R$ is expressed from Eqs.(2.33) as

$$
R = L(1+\eta)^2 / 4 
$$

(2.39)

where Eq.(2.39) is obtained for $\theta=\pi$ (i.e. the crack surface). From Eqs.(2.38) and (2.39) we obtain

$$
\tilde{u} = (\tilde{A} + \tilde{B} \xi + \tilde{C} \xi^2) \sqrt{R} + (\tilde{D} + \tilde{E} \xi) R + (\tilde{F} + \tilde{G} \xi + \tilde{H} \xi^2) 
$$

(2.40)
In Eqs.(2.41) the constant vectors $\ddot{A}, \ddot{B}, \ddot{C}, \ddot{D}, \ddot{E}, \ddot{F}, \ddot{G}$ and $\ddot{H}$ are functions of nodal point displacements. Their elements correspond to particular displacement component under consideration, i.e.
\[
\ddot{A} = \{ A_u, A_v, A_w \}
\] (2.42)

In Section 2.3.1 it is shown that Eq.(2.40) for the planes $\xi = \pm 1$ reduces exactly to the two-dimensional counterpart of the quarter point isoparametric element. Hence, as observed from Eq.(2.40), the $1/\sqrt{r}$ linear elastic fracture mechanics singularity is correctly maintained for all three possible modes of fracture.

2.2 SINGULARITY ELEMENTS FOR BI-MATERIAL CRACK PROBLEMS

In LEFM literature it is well known that [84,85] for a crack terminating at or lying along a bi-material interface, the nature of the displacement singularity ahead of the bi-material crack tip is given by
\[
u \propto r^{(1-p)} \quad , \quad p \neq 1/2
\] (2.43)

where $p$ is the power of singularity in the strain ($0 \leq p \leq 1$) and varies as a function of the elastic constants and the state of plane stress or plane strain [10].
In finite element literature several singular finite elements have been proposed to represent the $r^{1-p}$ singularity. In most of these elements special displacement shape functions with polynomial parametric representations are suggested. Among many, the elements proposed by Akin [41] and Okabe [43] gained special attention due to their generality and ease in their implementation into an existing finite element program. Akin [41] proposed the use of the power, $W^p$, of a local function, $W(\xi, \eta) = 1 - N_1(\xi, \eta)$, where $N_1$ is the ordinary shape function associated with the singular node. Okabe [43] introduced the semi-radial singularity mapping element with guaranteed strain energy boundedness. Here, the elements proposed by Akin [41] and Okabe [43] are closely studied and extensively revised. A special technique for stress intensity factor estimation for bi-material problems is proposed. Degenerate triangular forms of these elements are shown to be more accurate in representing the proper $r^{1-p}$ singularity, provided that the degeneracy of the quadrilateral element to the triangular one is proper.

2.2.1 Modified Akin's Element in Degenerate Triangular Form

The variable singularity element introduced by Akin [41,86] is briefly explained in Appendix B1. In this study, quadratic serendipity type interpolation functions (Eqs. 2.19, 2.20) are used to derive the singular displacement shape functions and in turn degenerate triangular elements (Fig. 2.2a,c) are studied in their $(r^{1-p})$ displacement singularity formulations. To obtain the degenerate triangular element, one can proceed exactly as the quarter point element of section 2.1.2, i.e. the same coordinates are assigned for the collapsed nodes and they are
forced to have equal displacements. For the degenerating scheme II of Fig. 2.2c, using Eqs. (B7) and (B8) of Appendix B along with Eqs. (2.19) and (2.20), we obtain the singular displacement interpolation functions as

\[ H_1(\xi, \eta) = 1 - \frac{1}{w}(\xi, \eta) \]

(2.44)

\[ H_{1}(\xi, \eta) = N_1(\xi, \eta)w^{-p}(\xi, \eta), \quad i \neq 1 \]

(2.45)

\[ W_{II}(\xi, \eta) = \frac{1}{4} \{4 + (1-\xi)(1-\eta)(1+\xi+\eta)\} \]

(2.46)

where the local function \( W(\xi, \eta) \) is obtained using Eq. (B4). At the interior of the element and along the two sides, \( \xi = -1 \) and \( \eta = -1 \), meeting at the singular corner the desired singularity is obtained by proper representation of the function \( W \). Along the side, \( \xi = +1 \), across the singular corner \( W \) becomes unity, thus all singularities are extinguished and a compatible displacement field with the adjacent element is formed. However, if the collapsed corner coincides with the singular point as in Fig. 2.2(a), assigning the same coordinates for the collapsed nodes is not sufficient for the correct degeneracy. Along the side with midside node 7 (\( \eta = 1 \) side), \( W \) becomes unity. Hence, an incompatible singular displacement field is formed. As a remedy, the shape functions of the collapsed nodes can be combined as

\[ N_1 + N_4 + N_8 \Rightarrow N_1 \]

(2.47)

and then the functions \( N_4 \) and \( N_8 \) are simply set to zero. Now \( W_{I} \) corresponding to the degenerating scheme I of Fig. 2.2(a), becomes

\[ W_{I}(\xi, \eta) = \frac{1}{2}(2-\xi)(1+\xi) \]

(2.48)
Thus except along the side across the singular node, a singular displacement field is obtained everywhere within the element and at the element boundaries. Unfortunately, numerical studies have revealed that a different singular field is obtained within the element than the degenerating scheme II. To demonstrate this instability, the local function $W$ of Eqs. (2.46) and (2.48) corresponding to degenerating schemes I and II are expressed in terms of a common xy coordinate system. For $L_1 = L_2 = L$ in Fig. 2.2 we have

$$W_{II}(x,y) = \frac{(2xy^2 - 2Ly^2 - Lxy + 3xL^2)}{L^2} (L-y) \quad (2.49)$$

$$W_I(x,y) = \frac{x(3L-2x)}{L^2} \quad (2.50)$$

where $0 \leq x, y \leq L$. On the element boundaries

$$W_I(x,0) = W_{II}(x,0) = x(3L-2x)/L^2, \quad 0 \leq x \leq L$$

$$W_I(x,L) = W_{II}(x,L) = 1, \quad 0 \leq y \leq L \quad (2.51)$$

$$W_I(x,y) = W_{II}(x,y) = x(3L-2x)/L^2, \quad y=x$$

At the interior of the element, on a radial line $y=Ax$ with $0<A<1$, it is observed that $W_I$ is not equal to $W_{II}$. Different singular displacement fields are, therefore, represented within the element. For both degenerating schemes, the displacement function $u$ is observed to be capable of representing the powers of the local coordinates $\xi$ and $\eta$. In practice, it is desirable that the same powers of the geometric coordinates $x$ and $y$ be representable. If the terms $1, \xi, \eta$ are present among the powers of $\xi$ and $\eta$ in Eqs. (2.44) and (2.45), then the isoparametric concept may be utilized, i.e.

$$\tilde{x} = \sum_{i=1}^{8} H_i(\xi,\eta) \tilde{x}_i \quad (2.52)$$
If $\xi$ and $\eta$ are not present in Eqs. (2.45), then an additional set of shape functions should be used to represent the geometry [86]. Hence, one may use regular interpolation functions

$$\bar{x} = \sum_{i=1}^{8} N_{i}(\xi,\eta) \bar{x}_{i}$$  \hspace{1cm} (2.53)

As observed from Eqs. (2.19), (2.20) and (2.45), it seems more appropriate to use Eqs. (2.53) to describe the geometry. On the other hand, it will be shown that numerical investigations yield superior results using the isoparametric representation. To remedy the numerical instabilities due to different degenerating schemes, the shape function modification suggested in Section 2.1.2 is utilized for the midside node across the collapsed corner. Unfortunately, no apparent improvement is obtained for the particular scheme of clustering the nodes at the singular point. The results improved considerably for the degenerating scheme II with an isoparametric representation.

To investigate the displacement variation along the crack line, we consider the side of the collapsed triangular element of Fig. 2.2c represented by $\eta = -1$ with the standard geometric polynomial interpolation

$$r = N_{1}r_{1} + N_{2}r_{2} + N_{5}r_{5}$$

$$u = H_{1}u_{1} + H_{2}u_{2} + H_{5}u_{5}$$

where

$$N_{1} = -\xi(1-\xi)/2 \hspace{1cm} ; \hspace{1cm} H_{1} = 1 - W^{1-p}$$

$$N_{2} = \xi(1+\xi)/2 \hspace{1cm} ; \hspace{1cm} H_{2} = N_{2}W^{p}$$

$$N_{5} = (1-\xi^{2}) \hspace{1cm} ; \hspace{1cm} H_{5} = N_{5}W^{-p}$$

$$W = (1+\xi)(2-\xi)/2$$
The first of Eqs. (2.54) gives \((1+\xi)=2r/L\) and the displacement expression becomes
\[
u = u_1 + (3-2s)^{-p} s^{1-p} \left\{2s(u_1+u_2-2u_5) + (4u_5-3u_1-u_2)\right\}
\] (2.56)
where \(s=r/L\). Using the binomial expansion for the term \((3-2s)^{-p}\) we have
\[
u = u_1 + 3^{-p} s^{1-p} \left\{1 + \frac{2p}{3} s + \frac{2p(p+1)}{9} s^2 + \ldots\right\} \left\{(4u_5-3u_1-u_2) + 2s(u_1+u_2-2u_5)\right\}
\] (2.57)
Hence, Eq. (2.57) demonstrates that the \(r^{1-p}\) displacement singularity is attained along the crack line and the leading coefficient of the \(r^{1-p}\) term is
\[3^{-p} s^{p-1}(4u_5-3u_1-u_2)\] (2.58)

2.2.2 Subparametric Semi-Radial Singularity Mapping Element

The semi-radial singularity mapping technique of Okabe [43] is briefly explained in Section B2 of Appendix B. Similarities and differences with the variable power singularity element of Akin [41] and its correspondence with the isoparametric representation are also presented. This singularity mapping assumes that the terms of \(O(J^{-p})\) and \(O(r^{1-p})\) are proportional. To investigate this proportionality, we consider the side \(\eta = -1\) of an 8-node quadrilateral singularity element (Fig. 2.2c). Among three equally spaced nodal points on this side, we let node 1 correspond to the singular node. The regular quadratic Lagrange interpolation functions are
\[
N_1 = \xi(\xi-1)/2 \quad ; \quad N_2 = (1-\xi^2) \quad ; \quad N_5 = \xi(1+\xi)/2
\] (2.59)
Equations (B.14) and (2.59) yield
The nondimensional radius, $\rho$, is not monotone and does not correctly represent the behavior of the radius $r$. Considering the same line element, the two nodes describing the geometry with $N_1 = \frac{(1-\xi)}{2}$ and $N_2 = \frac{(1+\xi)}{2}$ yield

$$\rho^{1-p} = \frac{(1+\xi)}{2}$$

(2.61)

The monotone behavior of the nondimensional radius (Eq. 2.61) is required to have the proper correspondence between the $O(r^{1-p})$ and $O(\rho^{1-p})$ terms. Consequently, straight singular edges must be adopted in the derivation of the semi-radial singularity mapping elements. Moreover, the strain energy boundedness within the element is guaranteed [44]. Thus, the singular crack tip surrounded by regular isoparametric elements yields a conforming displacement field, since the singular edges are conforming for the adjacent singular elements and the outer edges conform with regular isoparametric elements.

In this study quadratic serendipity-type interpolation functions are used for the displacement field and the singular element mapping functions are derived from linear shape functions to guarantee the $r^{1-p}$ displacement singularity representation and monotone nondimensional radius. Degenerate triangular elements obtained by collapsing one side are examined in the $r^{1-p}$ displacement singularity formulations. For the element geometry and nodal numbering sequence of Fig. 2.2a,c, the obtained singularity mapping functions using Eqs. B15 and B18 are

$$M_1 = 1 - \frac{(1-N_1^{*})^{1/(1-p)}}{1}$$

(2.62)

$$M_1 = N_1^{*}(1-N_1^{*})^{1/(1-p)}$$

; $i = 2, 3, 4$
Thus, the desired singular displacement field is obtained from the parametric transformations

\[ \tilde{u} = \sum_{i=1}^{8} N_i(\xi,\eta)\tilde{u}_i \]

\[ \tilde{x} = \sum_{i=1}^{4} M_i(\xi,\eta)\tilde{x}_i \]

where \( N_i^*, N_1^* \) are given by Eqs. (2.19) and (2.20). If the collapsed node coincides with the singular node (Fig. 2.2a), for proper degeneracy, the shape functions of the collapsed nodes must be combined in the same fashion as for the element of Section 2.2.1, i.e.

\[ N_1^* + N_4^* \Rightarrow N_1^* \]  \hspace{1cm} (2.65)

and \( N_4^* \) is set to zero. It can be easily shown that for all possible degenerating schemes, \( \rho^{1-p} \) is consistently represented and no instabilities are encountered. To investigate the displacement variation along the crack line, we consider the \( \eta = -1 \) side of Fig. 2.2c, with the shape functions

\[ N_1^* = (1-\xi)/2 \quad ; \quad N_2^* = (1+\xi)/2 \]

\[ N_1 = \xi(\xi-1)/2 \quad ; \quad N_2 = \xi(\xi+1)/2 \quad ; \quad N_5 = (1-\xi^2) \]  \hspace{1cm} (2.66)

The geometric singularity mapping yields

\[ r = LM_2 = LN_2^*(1-N_1^*)p/(1-p) = L(1+\xi)/2 \]

\[ \xi = 2(r/L)^{1-p} - 1 \]

The regular displacement field becomes
\[ u = u_1 N_1 + u_2 N_2 + u_5 N_5 \]

\[ u = \frac{1}{2} \{ 2u_5 + (u_2 - u_1) \xi + (u_2 + u_1 - 2u_5) \xi^2 \} \]  

(2.68)

Eqs. (2.67) and (2.68) give

\[ u = u_1 + (4u_5 - 3u_1 - u_2)(r/L)^{1-p} + 2(u_2 + u_1 - 2u_5)(r/L)^{(1-p)} \]  

(2.69)

It is observed from Eq. (2.69) that the \( r^{1-p} \) displacement singularity is consistently attained along the crack line.

The subparametric representation guarantees the stability of this element. If quadratic geometric mapping were used, the same instability observed for the degenerate form of the Akin's element would be encountered, due to inconsistent representation of the nondimensional radius, \( \rho \).

2.3 STRESS INTENSITY FACTOR ESTIMATION

For in-plane problems, the stress state around the singular crack tip is given by [87]

\[ \sigma_{ij}(r, \theta) = \frac{1}{\sqrt{\pi \rho}} \left\{ K_{1i} f_{ij}(r, \theta) + K_{2i} f_{2ij}(r, \theta) \right\} + O(r^{1-p}) \]  

(2.70)

where \( r, \theta \) are the polar coordinates in the plane perpendicular to the crack front, and \( f_{ij}, f_{2ij} \) are bounded functions. The constants, \( \rho \), the singularity power and \( K_1, K_2 \), the stress intensity factors, fully describe the nature of the stress singularity at the crack tip. The strength of the stress singularity for a given stress component is defined by Erdogan as [10]

\[ K_{ij}(\theta) = \lim_{r \to 0} r^{p} \sigma_{ij}(r, \theta) \]  

(2.71)
Those components of Eq.(2.71) corresponding to the cleavage stress, \(\sigma_\theta\) and the shear stress \(\sigma_{r\theta}\) at \(\theta=0\) are known as the stress intensity factors, \(K_1\) and \(K_2\), respectively.

In this section, we will develop expressions for the evaluation of the stress intensity factors as an outcome of the discussed singular finite element formulations. This technique is simple and straightforward. The normal and tangential crack surface displacement components are used to evaluate the opening and sliding mode stress intensity factors, respectively, by correlating the finite element singular displacement fields with the asymptotic displacement expressions. Using such an approach the cases of a crack in two- and three-dimensional homogeneous medium, a crack terminating perpendicular to a bi-material interface and a crack along a bi-material interface are closely studied. Stress intensity factor results for finite element models corresponding to the above cases are presented in Chapter IV. For all but the last case, the results are observed to be in very good agreement with the available literature. For the case of an interfacial crack due to the coupling effect of symmetric and skew symmetric loadings, a significant loss of accuracy is observed. Thus, an alternative way of stress intensity factor evaluation for interfacial cracks is proposed, yielding very good accuracy.

2.3.1 Homogeneous Medium in Two and Three Dimensions

For plane crack problems, the stress and displacement fields in the immediate vicinity of the crack tip are given in Section A1 of Appendix A. It can be shown that the crack tip displacement field in the quarter point collapsed triangle is of the form
\[ u_i = A_i + B_i r^{1/2} + C_i r^*, \quad i = x, y \]  
(2.72)

where \( A_i, B_i, C_i \) are constants. The corresponding strains are

\[ \varepsilon_{ij} = a_{ij} + b_{ij}/r^{1/2} \]  
(2.73)

Since the \( r^{1/2} \) term in Eq.(2.72) contributes to the \( r^{-1/2} \) strain singularity in Eq.(2.73), a consistent way to determine the stress intensity factor in sufficiently small crack tip elements is to equate the coefficients of the \( r^{1/2} \) term in the analytical and numerical displacement expansion near the tip. The mode I stress intensity factor, \( K_1 \), is estimated using the normal displacements of the crack surface and the mode II stress intensity factor, \( K_2 \), is estimated using the tangential displacements of the crack surface, in each case by interpolating the displacement function along the crack surface. For the crack surface \( \Theta = \pi \), Eqs.(A.2) yield

\begin{align*}
  u_x &= K_2 (1+k) r^{1/2}/2 \mu (2\pi)^{1/2} \\
  u_n &= K_1 (1+k) r^{1/2}/2 \mu (2\pi)^{1/2}
\end{align*}

(2.74)

where \( u_n \) and \( u_t \) are normal and tangential displacement components, respectively. The discretized normal displacement expression for the collapsed triangle (Fig. 2.2b) is

\[ u_n (r) = u_n (0) + \left\{ 4u_n \left( \frac{L}{4} \right) - u_n (L) - 3u_n (0) \right\} \left( \frac{r}{L} \right)^{1/2} + 2\left\{ u_n (L) + u_n (0) - 2u_n \left( \frac{L}{4} \right) \right\} \left( \frac{r}{L} \right) \]  
(2.75)

Hence, equating the coefficients of \( (r)^{1/2} \) terms in Eqs.(2.74) and (2.75), we have
\[ K_1 = \frac{2\mu}{(1+k)} \left( \frac{2\pi}{L} \right)^{1/2} \{4u_n \left( \frac{L}{4} \right) - u_n(L) - 3u_n(0) \} \] (2.76)

The tangential displacement and \( K_2 \) expressions can be obtained by simply replacing \( u_n \) by \( u_t \) in Eqs. (2.75) and (2.76), respectively. For the case of unsymmetrical geometry and/or loading, we have \( K_1(r, \pi) \) and \( K_1(r, -\pi) \) for \( i=1,2 \). An accurate estimate for \( K_1 \) is obtained by simply averaging the two values \([35]\)

\[ K_1 = \frac{1}{2} [K_1(r, \pi) + K_1(r, -\pi)] \quad i=1,2 \] (2.77)

Equation (2.77) cancels the effects of possible rigid body displacement and rotations involved due to the finite element modelling of the problem and can be expressed in terms of the nodal displacements, (Fig. 2.8)

\[ K_j = \frac{2\mu}{(1+k)} \left( \frac{2\pi}{L} \right)^{1/2} \{4[u_j \left( \frac{L}{4} \right) - u'_j(L)] - [u_j(L) - u'_j(0)] - 3[u_j(0) - u'_j(0)] \} \] (2.78)

where \( j=n,t \) and \( K_n \) and \( K_t \) correspond to the opening and sliding mode stress intensity factors, respectively.

Crack front stress and displacement fields for a three-dimensional homogeneous body are given in Section A2 of Appendix A. Numerical experiments of Chapter IV have shown that the best results are obtained on the plane of the crack where \( \theta=\pi \) in Fig. A2 and the displacement expressions of Eqs. (A14) uncouple on this plane, thereby giving

\[ u = (1-\nu)(2R/\pi)^{1/2} K_2/\mu \]

\[ v = (1-\nu)(2R/\pi)^{1/2} K_1/\mu \]

\[ w = (2R/\pi)^{1/2} K_3/\mu \] (2.79)
Figure 2.8 Collapsed Singular Elements, Crack Tip Configuration for Unsymmetric Loading and/or Geometry

Figure 2.9 Orientation of Isoparametric Pie Singularity Elements Along Crack Border
Following the same procedure as in the two-dimensional analysis, the coefficients of the \( R^{1/2} \) term of the analytical (Eq. 2.40) and numerical (Eq. 2.79) displacement expressions near the tip are equated. The following stress intensity factors are obtained

\[
K_1 = \mu(\pi/2)^{1/2}(A_v + B_v \xi + C_v \xi^2)/(1-\nu)
\]

\[
K_2 = \mu(\pi/2)^{1/2}(A_u + B_u \xi + C_u \xi^2)/(1-\nu)
\]

\[
K_3 = \mu(\pi/2)^{1/2}(A_w + B_w \xi + C_w \xi^2)
\]

where the constants \( A_i, B_i, C_i \), \( i=u,v,w \) are functions of the corresponding nodal point displacements only and they are given by Eqs.(2.41). For unsymmetrical geometry and/or loading and to cancel the rigid body effects, the \( K_i \) values obtained for upper and lower crack surfaces are averaged [51] in terms of the finite element nodal displacements to yield (Fig. 2.9)

\[
K_1 = \mu(\pi/2)^{1/2}((A_v - A'_v) + (B_v - B'_v) \xi + (C_v - C'_v) \xi^2)/(1-\nu)
\]

\[
K_2 = \mu(\pi/2)^{1/2}((A_u - A'_u) + (B_u - B'_u) \xi + (C_u - C'_u) \xi^2)/(1-\nu)
\]

\[
K_3 = \mu(\pi/2)^{1/2}((A_w - A'_w) + (B_w - B'_w) \xi + (C_w - C'_w) \xi^2)/(1-\nu)
\]

Hence, once the finite element nodal point displacements are obtained and transformed to the crack coordinate system, Eqs.(2.80) or (2.81) can be used efficiently to compute the stress intensity factors along the crack front.

2.3.2 Layered Plane Medium

The plane elasticity solution for bi-material media can be obtained by using Muskhelishvili's [88] complex variable formulation
and the Airy stress function, \( \phi(r, \theta) \), which leads to the biharmonic equation

\[ \nabla^4 \phi(r, \theta) = 0 \]  

(2.82)

Representing the Airy stress function in terms of the two analytic complex potentials as

\[ \phi(r, \theta) = \text{Re}\{\bar{z} \Omega_\alpha(z) + \int F_\alpha(z) \, dz\} \]  

(2.83)

and writing for the displacements and stresses

\[(u_{\alpha r} + iu_{\alpha \theta}) = (e^{-i\theta}/2u_\alpha)\{k_\alpha \Omega_\alpha(z) - z\bar{\Omega}'(\bar{z}) - \bar{F}'(\bar{z})\} \]  

(2.84)

\[ (\sigma_{\alpha rr} + i\sigma_{\alpha \theta \theta}) = \Omega''_\alpha(z) + \bar{\Omega}'(\bar{z}) - z\bar{\Omega}''(\bar{z}) - (z/\bar{z})F'(\bar{z}) \]  

(2.85)

\[ (\sigma_{\alpha \theta \theta} - i\sigma_{\alpha r r}) = \Omega'_\alpha(z) + \bar{\Omega}'(\bar{z}) + z\bar{\Omega}''(\bar{z}) + (z/\bar{z})\bar{F}'(\bar{z}) \]  

(2.86)

where \( \alpha = 1, 2 \) corresponding to the material number (Figs. 2.10 and 2.11), \( k_\alpha \) is the Poisson's effect parameter, \( \Omega_\alpha(z) \), \( F_\alpha(z) \) are complex potential functions of the complex variable, \( z = x + iy = r e^{i\theta} \), \( \mu_\alpha, \nu_\alpha \) are the shear modulus and Poisson's ratio of the material \( \alpha \), respectively. The two materials are assumed to be perfectly bonded along their common interface except at the crack surfaces. Assuming the complex potential functions in the form

\[ \Omega_\alpha(z) = \Lambda_\alpha z^p \quad \text{and} \quad F_\alpha(z) = B_\alpha z^p \]  

(2.87)

the solutions to the problems characterized by Figs. 2.10 and 2.11 are sought.

2.3.2.1 Interfacial crack in a bi-material body

The problem configuration is shown in Fig. 2.10. A crack lies along a portion of the interface between two dissimilar materials. The
Figure 2.10 Crack Tip Coordinates and Geometry for a Crack Lying Along a Bi-Material Interface (Interfacial Crack)

Figure 2.11 Crack Tip Coordinates and Geometry for a Crack Perpendicular to Bi-Material Interface (Terminal Crack)
regions \( y > 0, \ (0 \leq \theta \leq \pi) \) and \( y < 0, \ (-\pi \leq \theta < 0) \) are occupied by materials 2 and 1, respectively. The boundary conditions are represented by the continuity of displacements and balance of forces. The crack surfaces are assumed to be stress free. Following the complex variable-stress function approach, the satisfaction of the boundary conditions leads to the following characteristic equation [89]

\[
(1 - e^{2i\pi})^2 \{(m+k_2) + (1+mk_1)e^{2i\pi}\} \{(m+k_1) - (m+k_2)e^{2i\pi}\} = 0
\]  

(2.88)

The obtained eigenvalues are in conjugate pairs and of the form

\[
p_n = (n + \frac{1}{2}) \pm ie \quad , \quad n=0,1,2,...
\]  

(2.89)

where

\[
e = \frac{1}{2\pi} \ln\gamma
\]  

(2.90)

\[
\gamma = (1+mk_1)/(m+k_2)
\]  

(2.91)

\[
m = \frac{\mu_2}{\mu_1}
\]  

(2.92)

For guaranteed finite displacements as \( r \) goes to zero, \( 0 \leq \text{Re}(p) < 1 \), Eq.(2.89) gives \( p=1/2 + ie \) as the first eigenvalue. Now defining the stress intensity factors in the usual manner by

\[
K_1 + iK_2 = \lim_{r \to 0} (2\pi r)^{1/2}(\sigma_{\theta\theta} + i\sigma_{r\theta}) \bigg|_{\theta=0}
\]  

(2.93)

the near tip displacement expression becomes

\[
(u_{ar} + iu_{a\theta}) = \frac{(2\pi \gamma)^{-1/2}}{4p\mu_\alpha} \left[ r^p \beta_{1\alpha} e^{i(p-1)\theta} - \beta_{2\alpha} e^{-i(p+1)\theta} \right] \\
\hspace{1cm} + \beta_{1\alpha} r^p \beta_{\alpha \theta} e^{-i(p+1)\theta} - e^{-i(p-1)\theta}
\]  

(2.94)

where

\[
\beta_{ij} = \begin{cases} 1 & \text{if } i=j \\ \gamma & \text{if } i \neq j \end{cases}
\]  

(2.95)
Rewriting Eq.(2.94) for $\theta = \pm \pi$ and defining the crack opening displacement in the form
\[ \Delta u_y = (u_y^2) - (u_y^1) \]
(2.96)
one can obtain a relationship between the stress intensity factors $K_1$, $K_2$ and the crack opening displacement $\Delta u_y$. Thus, the crack surface displacements take the form

\begin{align*}
(u_x)_{\alpha} &= -u_{\alpha r} \bigg|_{\theta = \pm \pi} = C_{\alpha} \{ f(r)K_1 + g(r)K_2 \} \\
(u_y)_{\alpha} &= -u_{\alpha \theta} \bigg|_{\theta = \pm \pi} = C_{\alpha} \{ g(r)K_1 - f(r)K_2 \}
\end{align*}
(2.97)

(2.98)
where
\[ C_{\alpha} = (-1)^{\alpha}(1+k_{\alpha})/[2(2\pi)^{1/2} \mu_{\alpha}(1+4\varepsilon^2)] \]
(2.99)

\begin{align*}
f(r) &= \{2\varepsilon \cos(\varepsilon \ln r) - \sin(\varepsilon \ln r)\}r^{1/2} \\
g(r) &= \{2\varepsilon \sin(\varepsilon \ln r) + \cos(\varepsilon \ln r)\}r^{1/2}
\end{align*}
(2.100)

And the crack opening displacement is
\[ \Delta u_y = (1+\gamma)(m+k_2)\{g(r)K_1 - f(r)K_2\}/[2(2\pi)^{1/2} \mu_2(1+4\varepsilon^2)] \]
(2.101)

As observed from the Eqs.(2.94), (2.97) and (2.98), the displacements ahead of an interface crack remain proportional to $r^{1/2}$ as in the case of a homogeneous material. However, an oscillatory character reflected by Eqs.(2.100) exists. Moreover, if the stress intensity factors $K_1$ and $K_2$ are written in terms of the crack surface displacements (at $\theta = \pm \pi$) via Eqs.(2.97) and (2.98) in the form

\begin{align*}
K_1 &= (-1)^{\alpha}2(2\pi)^{1/2} \mu_{\alpha}\{f(r)(u_x)_{\alpha} + g(r)(u_y)_{\alpha}\}/\{(1+k_{\alpha})r^{1/2}\} \\
K_2 &= (-1)^{\alpha}2(2\pi)^{1/2} \mu_{\alpha}\{g(r)(u_x)_{\alpha} - f(r)(u_y)_{\alpha}\}/\{(1+k_{\alpha})r^{1/2}\}
\end{align*}
(2.102)
(2.103)
it is observed that both the symmetric and skew symmetric loadings contribute to the stress intensity factors due to the nonhomogeneous nature of the problem. For homogeneous problems, it was possible to obtain quite accurate stress intensity factors by correlating the finite element displacement approximations with the analytical crack surface displacement expressions. This was due to the fact that Eqs. (2.102) and (2.103) were not coupled in terms of the opening and sliding displacement components. The same technique can be used for bi-material problems only if the stress intensity factor expressions are in an uncoupled form in terms of the x and y displacement components. This is possible if and only if the bielastic constant, ε, is zero. As can be observed from Eqs. (2.90) and (2.91), ε vanishes only if the bi-material constant, γ, is unity which gives us the condition

\[(1+mk_1) = (m+k_2)\]  

(2.104)

An interesting case satisfying Eq. (2.104) occurs when the two bonded media are incompressible and in plane strain conditions, i.e. \(k_1=k_2=1\). This case was studied by Viola and Piva [24]. Another case for which the oscillatory character of the solution vanishes is when \(m=(1-k_2)/(1-k_1)\), which puts several restrictions on the material pairs. Thus, to obtain the stress intensity factors \(K_1\) and \(K_2\), in general, an additional equation supplementing Eq. (2.101) is necessary. The required relationship can be obtained by considering the validity of Eshelby-Rice conservation law [90,91] for bi-material bodies with a straight bondline. The standard J-integral, which has been shown to be path independent and equal to the energy release rate of the crack [92], is given by
\[ J = \int_\Gamma (\dot{W}_n - \sigma_{jk} n_j u_{,k})\,ds \]  

(2.105)

where the path \( \Gamma \) is shown in Fig. 2.10, \( W \) is the strain energy, \( \sigma_{jk} \) is the stress component, \( u_j \) is the displacement component and \( n_j \) is the unit outward normal to the boundary of \( \Gamma \).

Selecting \( \Gamma \) as a small circle centered at the tip and using the stress field given in Section A3 of Appendix A, the J-integral of Eq. (2.105) can be related to \( K_1 \) and \( K_2 \) [92] by

\[ J = \frac{(K_1^2 + K_2^2)}{4} \sum_{\alpha=1}^{2} D_\alpha \]  

(2.106)

where

\[ D_\alpha = \begin{cases} 
\frac{(1-\nu_\alpha)}{\mu_\alpha} & \text{plane strain} \\
\frac{1}{\mu_\alpha(1+\nu_\alpha)} & \text{plane stress}
\end{cases} \]  

(2.107)

The energy release rate for the debond crack was first derived by Malyshchev and Salganik [93] as a useful parameter for the study of adhesive joints. It was also noted in Ref. [93] that the stresses at the crack tip change monotonously, tending to infinity as \( r \) tends to zero, but for \( r \) being greater than the characteristic distance

\[ r^* = 2ae^{-\pi/2\varepsilon} \]  

(2.108)

where \( a \) is the half crack length. At distances smaller than \( r^* \) the magnitude of the stresses will grow, but will change sign with infinitely increasing frequency, as the crack tip is approached. For plane strain, the maximum range for \( r^*/2a \) ratio is \( 10^{-6} \). Thus, the behavior for distances corresponding to larger ratios remains qualitatively the same as in the homogeneous material problem.
The finite element results obtained by using regular singularity elements at the interface crack tip showed good agreement with Eq. (2.101). Energy release rate computations have also yielded results within 1-2% accuracy. As expected, stress intensity factors, especially $K_2$, estimated from crack opening and sliding displacement components are observed to be very much in error. On the other hand, $K_1$ and $K_2$ obtained using Eq. (2.101) and (2.106) simultaneously yielded quite accurate results.

2.3.2.2 Terminal crack in a bi-material body

The problem configuration is shown in Fig. 2.11 with a crack originating in material 1 and terminating perpendicularly at the bi-material interface with material 2. The region $x>0$, $(-\pi/2 \leq \theta \leq \pi/2)$ is occupied by material 1, and the region $x<0$, $(\pi/2 < \theta < 3\pi/2)$ is occupied by material 2. The tractions and displacements are assumed to be continuous across the interface, the crack surfaces are stress free and symmetry exists about the $x$-axis. To satisfy the aforementioned boundary conditions, a characteristic equation in terms of the power of singularity, $p$, is obtained in the form [10]

$$bp^2 - 2bp + 2c \cos \pi p + d = 0$$

(2.109)

where

$$m = \frac{\mu_2}{\mu_1}$$

$$b = -4c(1-m)/(1+mk_1)$$

$$c = (m+k_2)(1+mk_1)$$

(2.110)

$$d = (3b/4) + c(k_2-mk_1)/(m+k_2)$$

The power of singularity $p$ is a function of the elastic properties of the materials and the state of plane strain or plane stress. The first
eigenvalue $p_1$ of Eq.(2.109) determines the behavior of the displacement singularity as $r$ goes to zero and $p_1$ is real for all possible material combinations. The values of $p_1$ for different material combinations and plane stress/strain states have been obtained [10,45].

Recalling that the material 1 is the cracked material and material 2 is the uncracked material, the strength of the singularity at $\theta=0$ (Fig. 2.11) for the cleavage stress in material 2, gives the crack opening mode stress intensity factor

$$K_1 = \lim_{r \to 0} (2\pi)^{1/2} r^{1-p} \sigma_{2\theta}(r,0) \quad (2.111)$$

where $\sigma_{2\theta}$ can be related to the displacement field [10] through the relation

$$\sigma_{2\theta}(r,0) = 2(1-p)\mu^* u_\theta(r,\pi)/r \quad (2.112)$$

and the bielastic constant $\mu^*$ is defined by

$$\mu^* = \nu_2 \{(3-2p)(1+m_k_1)-(1-2p)(m+k_2)\}/\{(m+k_2)(1+m_k_1)\sin(1-p)\pi\} \quad (2.113)$$

For the plane strain homogeneous material case ($m=1$), $\mu^*=\mu/2(1-\nu)$.

Using Eqs.(2.111), (2.112) and (2.113), $K_1$ can be obtained from the displacement field by means of the relation

$$K_1 = 2(2\pi)^{1/2}(1-p)r^{p-1} \mu^* u_\theta(r,\pi) \quad (2.114)$$

Rewriting Eq.(2.114) for $u_\theta$ as

$$u_\theta(r,\pi) = K_1 r^{1-p}/2(2\pi)^{1/2}(1-p)\mu^* \quad (2.115)$$

Since Eq.(2.115) is in uncoupled form, an accurate estimate for the stress intensity factor can be obtained by equating the coefficients of the $r^{1-p}$ terms of the finite element singular displacement field
and Eq. (2.115). For degenerate triangular singularity element of Section 2.2.1, using Eqs. (2.58) and (2.115), we have

\[ K_1 = 2(2\pi)^{1/2} L^{p-1} (1-p) \mu^*(4u_5 - 3u_1 - u_2) \]  \hspace{1cm} (2.116)

where \( L \) represents the dimension of the singular crack tip element.

For the subparametric semi-radial singularity mapping element of Section 2.2.2, from Eqs. (2.69) and (2.115), we have

\[ K_1 = 2(2\pi)^{1/2} L^{p-1} (1-p) \mu^*(4u_5 - 3u_1 - u_2) \]  \hspace{1cm} (2.117)

The finite element solution of the terminal crack model of Chapter IV indicates that the displacements in Eq. (2.116) obtained by using regular interpolation functions for the geometric mapping are larger than those in Eq. (2.117). The multiplication term, \( 3^p \), thus compensates somewhat the difference in stress intensity factors evaluated for the two different singular finite element formulations. The displacements obtained using the isoparametric representation of degenerate triangular elements with the suggested shape function modification are within a 2\% difference with the subparametric semi-radial singularity mapping element. Hence, Eq. (2.117) is used to evaluate the stress intensity factors yielding very accurate results for both the subparametric semi-radial singularity mapping element of Section 2.3.2 and the degenerate isoparametric triangular singularity element with the requested shape function modification of Section 2.2.1.
CHAPTER III
FRACTURE CRITERIA

3.1 REVIEW OF LINEAR ELASTIC FRACTURE CRITERIA FOR HOMOGENEOUS MEDIA

For a self-similar crack extending along its plane, the onset of unstable crack growth under mode I loading is predicted by the critical value of either the energy release rate, $G_c$ or the stress intensity factor $K_{ic}$. The parameters $K$ and $G$ are related by

$$G = \begin{cases} \frac{K_{Ic}^2}{E}, & \text{plane stress} \\ \frac{(1-v^2)K_{Ic}^2}{E}, & \text{plane strain} \end{cases}$$

and for combined mode I and mode II conditions, with self-similar crack growth

$$G = G_I + G_{II} = (1+v)(1+k)(K_{Ic}^2+K_{IIc}^2)/4E$$

Equation (3.2) does not provide a fracture criterion for mixed mode crack problems unless $K_{Ic}$ and $K_{IIc}$ are interpreted properly [59]. Thus, several mixed mode crack propagation theories to include the effects of unsymmetric loading and/or geometry have been developed to predict the onset of unstable crack growth load and direction. Available theories can be categorized as (i) those examining the angular behavior of a parameter in terms of the stress field around an existing sharp crack or a slender ellipse representation of the crack and (ii) those
permitting a small extension from the crack tip in some direction and examining the resulting configuration.

The maximum circumferential tensile stress theory developed by Erdogan and Sih [57] falls in the first category. Crack propagation is assumed to take place along a ray $\theta=\theta_0$ for which the hoop stress $\sigma_0$ is maximum, i.e. the quantity $\sqrt{2\pi r_0}$ reaches a critical maximum value (fracture toughness of the material). The initial fracture angle is governed by [57]

$$\left(\frac{K_1}{K_{ic}}\right)\sin\theta + \left(\frac{K_2}{K_{ic}}\right)(3\cos\theta-1) = 0 \quad (3.3)$$

and the fracture envelope is defined by

$$\left(\frac{K_1}{K_{ic}}\right)\cos(\theta/2) - 3\left(\frac{K_2}{K_{ic}}\right)\sin(\theta/2) = \sec^2(\theta/2) \quad (3.4)$$

Another theory belonging to the first category is the minimum strain energy density theory developed by Sih [60]. Here the strain energy density factor $S$ is used as the controlling parameter for analyzing crack problems. The quantity $S$ represents the product of the radial distance $r_0$, multiplied by the strain energy density function, $dW/dV$. The crack is assumed to propagate in the direction of maximum potential energy density or equivalently crack initiation will start in a radial direction along which the strain energy density is a minimum, $S_{\text{min}}$, when this minimum reaches $S_{\text{cr}}$. The derived fracture angle is given by [60]

$$2\left(\frac{K_1}{K_{ic}}\right)^2\sin\theta[\cos\theta-(1-2\nu)] + 4\left(\frac{K_1}{K_{ic}}\right)\left(\frac{K_2}{K_{ic}}\right)[\cos2\theta-(1-2\nu)\cos\theta]$$

$$-2\left(\frac{K_2}{K_{ic}}\right)^2\sin\theta[3\cos\theta-(1-2\nu)] = 0 \quad (3.5)$$
and the fracture envelope is defined by

\[
\frac{K_1}{K_{ic}}^2(3-4v-\cos \theta)(1+\cos \theta)+4\left(\frac{K_1}{K_{ic}}\right)\left(\frac{K_2}{K_{ic}}\right)\sin \theta \left[\cos \theta -(1-2v)\right] \\
+\left(\frac{K_2}{K_{ic}}\right)^2 \left[4(1-v)(1-\cos \theta)+(3\cos \theta -1)(1+\cos \theta)\right] = 4(1-2v)
\]

(3.6)

The crack closure and frictional effects theory by Lee and Advani [62] is another example of the fracture theories of the first category. In this theory, McClintock and Walsh's model [63] is extended for two- and three-dimensional thermomechanical loading and internal crack/cavity pressure problems. The fracture envelope, expressed in terms of the stress intensity factors, is governed by

\[
\frac{K_1}{K_{ic}} + \left(\frac{K_2}{K_{ic}}\right)^2 = 1
\]

(3.7)

Thus, in this theory the closure of the crack due to the compressive nature of the loading is properly incorporated. On the other hand, the minimum strain energy density criterion is insensitive to the tensile/compressive nature of the loading. Swedlow [58] gives a corollary\(^1\) to Sih's [60] hypothesis\(^2\) so that a sign change in the loading from tensile to compressive will be reflected properly throughout the analysis as it is the case for maximum hoop stress criterion.

Hussain et al. [59] and Palaniswamy and Knauss [55] studied the mixed mode propagation problem using the model of the second category. Their maximum strain energy release rate theory following the Griffith-Irwin criterion requires that the direction of propagation will be in

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\(^1\)Swedlow's corollary states that "This minimum need not be global but that it must be associated with a tensile hoop stress." (p.512)

\(^2\)Sih's hypothesis states that "Crack initiation will start in a radial direction along which the strain energy density is a minimum" (p.XXI)
the direction along which the strain energy release rate $G(\theta)$ is maximum, and the crack growth will start when this maximum reaches a critical value. The strain energy release rate, $G(\theta)$, is given as [59]

$$G(\theta) = \frac{4}{E(3+\cos^2 \theta)} \left( \frac{\pi-\theta}{\pi+\theta} \right)^{\theta/\pi} \left( K_2^2 (1+3 \cos^2 \theta) + 4K_1 K_2 \sin 2\theta + K_2^2 (9-5 \cos^2 \theta) \right)$$

(3.8)

and the fracture locus is governed by [113]

$$(K_1/K_{Ic})^4 + 6(K_1/K_{Ic})^2(K_2/K_{Ic})^2 + (K_2/K_{Ic})^4 = 1$$

(3.9)

The fracture envelopes defined by Eqs. (3.4), (3.6), (3.7) and (3.9) are plotted in Fig. 3.1. Comparison of the above theories with the available experimental data indicates that the tensile loading theories are in reasonable agreement. Keeping in mind that the experimental results happen to be quite scattered, there is no solid basis of preference of one theory on another except the completeness in their mathematical formulations [58]. For compressive loading the crack closure and frictional effects theory seems to be the most appropriate one and the minimum strain energy density theory, the least appropriate one, unless it is interpreted properly as proposed by Swedlow [58].

3.2 FRACTURE CRITERIA FOR BI-MATERIAL PROBLEMS

For layered fracture problems with a terminal or interfacial crack, available fracture criteria are limited as well as controversial. For a terminal crack, propagation may be in the form of through cleavage into the uncracked medium, debonding of the interface or a reflected crack into the first medium. For an interfacial crack, self similar
Fig. 3.1 Comparison of Mixed Mode Fracture Theories

(i) Eq.(3.4)
(ii) Eq.(3.6) (ν = 0.2)
(iii) Eq.(3.7)
(iv) Eq.(3.9)
propagation along the interface or extension into one of the adjoining materials represent possible fracture propagation modes.

Cook and Erdogan [10] appear to be the first to state a tentative fracture criterion for terminal crack problems. Viola and Piva [23] studied the interfacial fracture propagation in terms of a local bi-axiality parameter. In both of the above studies, the maximum stress criterion is used to predict the onset of unstable fracture growth. As yet no energy based approach has been reported to study bi-material crack growth. In the following subsections, fracture propagation for an interfacial and terminal crack are studied using the maximum stress criterion. Path independent J and M integrals are examined as an aid for predicting the fracture growth mechanism.

3.2.1 Interface Crack Between Two Dissimilar Media

It was discussed in subsection 2.3.2.1 that the effects of both symmetric and skew-symmetric loadings are intermixed in the expressions for the stress intensity factors, $K_1$ and $K_2$. Therefore, Rice and Sih [94] suggested the use of a function of $K_1$ and $K_2$ as an extension of the Griffith-Irwin theory of fracture as,

$$f(K_1, K_2) = f_{cr}$$

Here, the interface crack is expected to grow when $f(K_1, K_2)$ reaches a critical value $f_{cr}$, which must be determined experimentally. To the best of our knowledge, no such detailed study for bi-materials has been reported in the literature. Mulville and Mast [95] used the strain energy release rate as the fracture parameter and their experimental results yield almost constant strain energy release rate as the
propagation takes place along the interface. Piva and Viola [23] studied the effects of load bi-axially on crack branching by emphasizing the significance of non-singular terms.

Here, we will consider three possible propagation patterns for a crack along the interface of two different homogeneous, isotropic, elastic materials. These cases are (i) propagation into material 1, with a branching angle $\theta^*_1$, (ii) propagation into material 2, with a branching angle $\theta^*_2$, and (iii) extension along the interface with $\theta = 0$. The maximum circumferential tensile stress criterion is exploited to study the propagation into one of the materials. Using the near tip stress field expressions given in Appendix A3, we have for material 1

$$
\sigma_{1\theta} = \frac{e^{-\varepsilon \theta}}{2(2\pi r)^{1/2}} \left\{ (F_{12}C - G_{12}S)K_1 + (F_{12}S + G_{12}C)K_2 \right\}
$$

(3.11)

where $-\pi < \theta < 0$ and $0 < \theta < \pi$ for material 1 and 2, respectively. We rewrite Eq. (3.11) in a more convenient form

$$
\sigma_{1\theta} = \frac{e^{-\varepsilon \theta}}{2(2\pi r)^{1/2}} \left\{ g_{11}(r, \theta)K_1 + g_{12}(r, \theta)K_2 \right\}
$$

(3.12)

where

$$
g_{11}(r, \theta) = F_{12}C - G_{12}S
$$

(3.13)

$$
g_{12}(r, \theta) = F_{12}S + G_{12}C
$$

and postulate that crack propagation is expected into the medium 1, when the quantity $(2\pi r)^{1/2} \sigma_{1\theta}$ is maximum, reaching a critical material value, namely the fracture toughness of the material, $(K_{IC})_i$. Thus, the fracture angle is governed by
\[ e^{-\varepsilon \theta} \left( g_{11} e^{-\varepsilon g_{11}} \frac{K_1}{(K_{1c})_1} + (g_{12} e^{-\varepsilon g_{12}}) \frac{K_2}{(K_{1c})_1} \right) = 0 \]  

(3.14)

A strong dependence on material parameters \( \gamma \) and \( \varepsilon \) is observed. Since the \( J \) or \( G \) term is involved in the functions \( g_{11} \) and \( g_{12} \), a crack tip characteristic distance has to be specified. Equations (3.12) and (3.14) reduce to the homogeneous material counterpart for \( m=1.0 \). In Fig. 3.2, fracture envelopes corresponding to various \( m \) values for a plane strain interfacial crack and for \( r=0.002 \) are shown. Here, the elastic modulus of material 1 is taken to be unity and the elastic modulus of material 2 is adjusted with \( m \) assuming the same Poisson's ratio for both materials. For comparison, the curve with \( m=1 \) is also shown in the figure. Note that the very same curves will be generated for \( m > 1 \) and \( (K_{1c})_1 \). As the material property mismatch becomes larger, the effect of \( K_2 \) gets more significant, increasing the area enclosed by the fracture envelope, thus requiring more energy for the crack to propagate into the softer material. Although the \( K_1 \) and \( K_2 \) cannot be interpreted plainly as the crack opening and sliding mode stress intensity factors as in the homogeneous material case, a simple observation of Eqs.(2.100) through (2.103) indicates that for loading normal to the interface there is a significant dominance of \( K_1 \) over \( K_2 \). The same observation can also be made by examining the Eqs.(31) of Ref. [94]. In Fig. 3.3 the variation of the fracture angle with \( K_1/K_2 \) ratio, corresponding to Fig. 3.2 is given. For large values of \( K_1 \), smaller values of the fracture angle are approached with decreasing slope but with the effect of \( K_2 \) preserved. This behavior is observed to be less significant as the material property mismatch increases. To study the effect of the
Fig. 3.2 Fracture Envelopes for Interfacial Crack
Figure 3.3 Initial Fracture Angle Variation with $K_1/K_2$

Figure 3.4 Effect of Characteristic Distance "r" on Initial Fracture Angle for $m=0.1$
characteristic distance $r$, the fracture envelopes corresponding to three different values of $r$, namely 0.02, 0.002 and 0.0002 are considered for two different material pairs, $m=0.1$ and 0.2 (Fig. 3.5). Very small $r$ values tend to imply stronger $K_2$ dependence, and large $r$ values give similar behavior as in the homogeneous material case. This behavior becomes less significant as $m$ gets closer to unity. Figure 3.4 shows how the fracture angle is affected by different characteristic distances. As in the homogeneous material case, $r$ must be determined experimentally for different materials.

So far Poisson's ratios of the two materials are assumed to be equal. Effects of different Poisson's ratio on the fracture envelope are examined in Fig. 3.6. As observed, there is little variation between the curves A and B. However, when the material into which propagation is expected has a higher Poisson's ratio, a shift in favor of $K_1$ dominance is observed from envelope C of Fig. 3.6. An increased $K_2$ effect is expected for the case of smaller Poisson's ratio for the softer material.

The possibility of the extension of the crack along the interface is studied by considering two criteria. First, the maximum tensile stress criterion is utilized which predicts the interfacial propagation when

$$\frac{(2\pi r)^{1/2}}{\sigma_{0}} \bigg|_{\theta=0, r=r_{o}} \geq K_{d} \quad (3.15)$$

where the bonding strength parameter is defined by

$$K_{d} = (2\pi r_{o})^{1/2} \sigma_{12}^c \quad (3.16)$$
Fig. 3.5 Effects of Characteristic Distance "r" on Fracture Envelopes for m=0.1 and m=0.2
Fig. 3.6 Poisson's Ratio Effect for $m=0.1$ ($r=0.002$)

Fig. 3.7 Fracture Envelopes for Debonding ($\mu_f=0$)
in which \( c_{12}^c \) is the bond strength. Since both \( K_1 \) and \( K_2 \) are represented in the near tip \( \sigma_\theta \) expression, the fracture envelope is defined by

\[
\cos(\varepsilon \ln r)\left(\frac{K_1}{K_d}\right) + \sin(\varepsilon \ln r)\left(\frac{K_2}{K_d}\right) = \frac{2(\gamma)^{1/2}}{(1+\gamma)}
\]  

(3.17)

Secondly, the debonding criterion suggested by Erdogan [85] due to the critical combination of normal and shear stresses is considered. For tensile debonding (i.e. \( \sigma_\theta > 0 \) along the interface) we have

\[
(\sigma_\theta^2 + \sigma_{r\theta}^2)^{1/2} > c_{12}
\]  

(3.18)

and if \( \sigma_\theta \) is compressive (\( \sigma_\theta < 0 \))

\[
\sigma_{r\theta} + \mu_f \sigma_\theta > c_{12}
\]  

(3.19)

where \( \mu_f \sigma_\theta \) term represents the friction resistance due to compressive nature of the normal stress along the interface. Using the stress field of Appendix C, the fracture envelope corresponding to Eq.(3.18) is governed by

\[
\frac{K_1^2}{K_d} + \frac{K_2^2}{K_d} = \frac{4\gamma}{(1+\gamma)^2}
\]  

(3.20)

and for Eq.(3.19) we have

\[
[\mu_f \cos(\varepsilon \ln r) - \sin(\varepsilon \ln r)]\left(\frac{K_1}{K_d}\right) + [\mu_f \sin(\varepsilon \ln r) + \cos(\varepsilon \ln r)]\left(\frac{K_2}{K_d}\right) = \frac{2(\gamma)^{1/2}}{(1+\gamma)}
\]  

(3.21)

Fracture envelopes corresponding to Eqs.(3.17), (3.20) and (3.21) are plotted in Fig. 3.7. For all cases Poisson's ratios for materials 1 and 2 are assumed to be equal and the characteristic distance \( r_o \) is taken to be 0.002.
The maximum tensile stress criterion is $K_1$-controlled as $m$ tends to 1 and self similar crack growth conditions dominate for a smaller material property mismatch. For large material property mismatches, the interaction of $K_1$ and $K_2$ is strongly pronounced for the debonding of the interface. Erdogan's [85] critical normal and shear stress combination criterion (Eq. 3.20) yields a circle with radius \( \frac{4\gamma}{1+\gamma} \) as the failure envelope for tensile $\sigma_0$, in which case the radius parameter is least affected by the material property mismatch. If the frictional resistance due to the compressive nature of $\sigma_0$ is neglected in Eq.(3.21), fracture envelopes due to shear effects only are obtained, which in turn approach the pure shear mode behavior as $m$ gets closer to unity.

In Fig. 3.8 the friction effect due to compression is studied. By assuming values of $\mu_f$ ranging from 0.3 to 1.0, we observe pure shear mode behavior of certain $\mu_f$ value. For $m=0.1$ this corresponds to a value of $\mu_f=0.725$, whereas for $m=0.2$ it is 0.55. As the material property mismatch gets less significant, pure $K_2$ behavior is approached for much smaller $\mu_f$ values. Thus, we can conclude that debonding due to shear and frictional effects is more likely for smaller $m$ values.

The influences of different Poisson's ratio for the two bonded materials and the characteristic distance $r_0$ are graphically represented in Figs. 3.9 and 3.10. Similar to the earlier observations, as $r$ increases the homogeneous material response is approached. Since for a sufficiently small fracture process zone the qualitative behavior of the singularity is the same as for homogeneous case, this observation is readily justified. Figure 3.10 reveals that for certain combinations of Poisson's ratio and elastic moduli, the bielastic constant, $\varepsilon$,
Fig. 3.8 Compressive Debonding Criteria with Friction Effects for $m=0.1$ and $m=0.2$ 
($r=0.002$)
Fig. 3.9 Effect of Characteristic Distance "r" on Debonding Envelopes

Fig. 3.10 Poisson's Ratio Effect on Debonding Envelopes for m=0.1 (r=0.002)
approaches zero. Therefore, the homogeneous material behavior is clearly approached. Consequently, the critical combination of tensile and shear stresses criterion for debonding does not realistically represent the behavior due to material property mismatch. Thus, the maximum tensile stress criterion can be adopted for tensile debonding along with the shear and frictional effects criterion for compressive debonding, with the modification that for \( m \) close to unity, the term \( \mu r \sigma \) be omitted.

3.2.2 Terminal Crack Perpendicular to the Bi-Material Interface

The energy balance concept is still valid for a crack terminating at a bi-material interface. However, its application, if not impossible, is very difficult due to lack of a suitable solution to the related mechanics problem. Erdogan [10] gives a tentative local fracture criterion based on the maximum stress fracture theory, stated by

\[
\sigma_{ic}(\delta_{ic}, \theta_i^*) \geq \sigma_{ic}, \quad i=1,2
\]  
\[
\sigma_{12}(\delta_{12}^c, \pi/2) \geq \sigma_{12}^c
\]

where \( \sigma_{ic} \) is the normal stress on the maximum cleavage plane at a distance \( \delta_{ic} \) from the crack tip. It is measured at the fracture initiation load for medium \( i \). \( \theta_i^* \) is the angle for which the cleavage stress \( \sigma_{ic} \) is maximum. \( \sigma_{12}^c \) is defined in the previous section with \( \delta_{12}^c \) being the characteristic distance from the interfacial crack tip. Through cleavage of medium 2 and reflected crack propagation in medium 1 are represented by \( i=2 \) and \( i=1 \), respectively in Eq.(3.22). Equation (3.23) is the debonding criterion.
For the design and analysis of fractured layered media, we need to predict whether the material interfaces will help as an arrest mechanism or through cleavage will occur. Equations (3.22) and (3.23) seem inconvenient from the point of view that a characteristic fracture process zone, identified with $\delta_{ic}$, is involved. Thus, experimental findings for $\delta_{ic}$ corresponding to different material combinations are required. Erdogan [10] suggests the size of the process zone as $10^{-4}$ in < $\delta_c$, $\delta_{12}^c$ < $10^{-2}$ in.

For through cleavage in a self-similar manner, a simple criterion will be stated here. The bi-material stress intensity factor for a terminal crack is defined as the strength of the stress singularity at $\theta=0$ for the cleavage stress in material 2 (Eq. 2.111). The cleavage stress, $\sigma_{c2\theta}$, is maximum at $\theta=0$. Hence, the propagation across the interface for self-similar crack growth is assumed to be governed by the relation

$$\frac{K}{a^{p}} \geq \left(\frac{K_{Ic}}{a^{1/2}}\right)^2$$

Inequality (3.24) is dimensionally consistent and it is a restatement of the maximum stress criterion. We note that the characteristic distance, $\delta_{ic}$, is not involved in the application of inequality (3.24), and that the bi-material stress intensity factor for a terminal crack is readily available through the technique suggested in Section 2.3.2.2. The drawback of this criterion is that its application is limited to the propagation of the crack in its own plane.

To obtain the fracture envelopes and the direction of crack propagation under general loading, we consider the configuration illustrated
in Fig. 2.11. Following Zak and Williams [2], a stress function \( \phi_i(r, \theta) \) satisfying the biharmonic equation in each of the regions is assumed to be

\[
\phi_i(r, \theta) = r^{2-p} F_i(\theta)
\]

(3.25)

where

\[
F_i(\theta) = a_i \sin(2-p)\theta + b_i \cos(2-p)\theta - c_i \sin p\theta + d_i \cos p\theta
\]

(3.26)

The stress and displacement expressions are

\[
\sigma_{ir} = r^{-p}[F_i, \theta + (2-p)F_i]
\]

\[
\sigma_{i\theta} = (1-p)(2-p)r^{-p}F_i
\]

\[
\sigma_{ir\theta} = -(1-p)r^{-p}F_{i, \theta}
\]

(3.27)

\[
u_{ir} = \frac{-r^{1-p}}{2\mu} \{(2-p)F_i - (1+k_i)(d_i \cos p\theta - c_i \sin p\theta)\}
\]

\[
u_{i\theta} = \frac{-r^{1-p}}{2\mu} \{F_i, \theta + (1+k_i)(c_i \cos p\theta + d_i \sin p\theta)\}
\]

The overall solution is obtained by superposing the individual solutions corresponding to symmetric and antisymmetric modes. Boundary conditions are the continuity of \( \sigma_\theta, \sigma_{r\theta}, u_r \) and \( u_\theta \) along the interface \( (\theta = \pm \pi/2) \) and traction free crack surfaces \( (\theta = \pm \pi) \). For the symmetric mode, in region 2, we have

\[
\sigma_{2r\theta} = 0 \text{ and } u_{2\theta} = 0 \text{ at } \theta = 0
\]

(3.28)

Defining the stress intensity factor in the usual manner as

\[
K_1 = \lim_{r \to 0} \sqrt{2\pi} r^p \sigma_{2\theta}(r, \theta) \bigg|_{\theta = 0}
\]

(3.29)
we obtain

$$K_1 = (1-p)(2-p)\sqrt{2\pi}(b_2+d_2)$$

(3.30)

For the antisymmetric mode, denoting the constants as \(a_i', b_i', c_i', d_i'\), in region 2, we have

$$\sigma_{2\theta} = 0 \text{ and } u_{2r} = 0 \text{ at } \theta = 0$$

(3.31)

$$K_2 = \lim_{r \to 0} \sqrt{2\pi} r^p \sigma_{r\theta}(r, \theta) \bigg|_{\theta = 0}$$

Thus, for each mode we have 8 conditions in terms of 8 unknowns. The power of singularity \(p\) is not treated as an unknown for this solution. It can be evaluated through a similar analysis as determined by

Swenson and Rau [13]. Following algebraic manipulations, \(F_i(\theta)\) can be written in the form

$$F_i(\theta) = \frac{1}{(1-p)\sqrt{2\pi}} \left\{ K_{1i} f_i(\theta) + K_{2i} g_i(\theta) \right\} , \quad i = 1, 2$$

(3.33)

where the functions \(f_i(\theta)\) and \(g_i(\theta)\) are given in Appendix A4.

The maximum circumferential tensile stress criterion is exploited to study the propagation into one of the materials. Crack propagation is expected into the medium \(i\) when the quantity \(\sqrt{2\pi} \sigma_{\theta i}\) at \(r = r_0\) is maximum and when this maximum reaches a critical material value, namely the fracture toughness of that material, \((K_{ic})_i\). Thus the fracture angle is governed by

$$f_{i, \theta} \left( \frac{K_1}{(K_{ic})_i} \right) + g_{i, \theta} \left( \frac{K_2}{(K_{ic})_i} \right) = 0$$

(3.34)

and the fracture envelope is defined by
To study the interface failure, first we consider the maximum tensile stress criterion. When the quantity \( \sqrt{2\pi} r_0 \sigma_\theta \) at \( \theta = \pi/2 \) and \( r = r_0 \) reaches the bond strength, \( K_d \), debonding of the interface is expected. Here we define \( K_d \) as
\[
K_d = \sqrt{2\pi} r_0 \sigma_{12}^{c}
\]
where \( \sigma_{12}^{c} \) is the bond strength. Due to continuity of \( \sigma_\theta \) along the interface, \( f_1(\pi/2) = f_2(\pi/2) = f \) and \( g_1(\pi/2) = g_2(\pi/2) = g \). Hence, such a fracture envelope is defined by
\[
f \frac{K_1}{K_d} + g \frac{K_2}{K_d} = \frac{1}{2-p}
\]
(3.37)

Secondly, the criterion of Erdogan [85] with critical combinations of normal and shear stresses is considered. For tensile debonding \( (\sigma_\theta > 0) \), corresponding to Eq.(3.18), we have
\[
[(2-p)^2f^2 + f_\theta^2] \left( \frac{K_1}{K_d} \right)^2 + 2[(2-p)^2fg+f_\theta g_\theta, g_\theta] \left( \frac{K_1}{K_d} \right) \left( \frac{K_2}{K_d} \right) + [(2-p)^2g^2 + g^2_\theta] \left( \frac{K_2}{K_d} \right) = 1
\]
(3.38)

For compressive \( \sigma_\theta (\sigma_\theta < 0) \), corresponding to Eq.(3.19), fracture envelope is defined by
\[
[(2-p)f_\theta g - f, g_\theta] \left( \frac{K_1}{K_d} \right) + [(2-p)f_\theta g - g_\theta, g] \left( \frac{K_2}{K_d} \right) = 1
\]
(3.39)

Explicit forms of Eqs.(3.34), (3.35), (3.37) and (3.38) are given in Appendix A4. The characteristic distance \( r_0 \) in Eq.(3.35), should be within the region where LEFM conditions prevail and it must be evaluated experimentally. A description of size and shape of the plastic zone for
bi-material terminal cracks is given in Appendix C. A Von Mises yield condition is used to obtain the boundary of the plastic zone within which the stresses are greater than the yield stress. We note that these boundaries are approximate, since they only define the region where the yielding is predicted by the elastic solution. Following the argument of Schmidt [75], a similar crack tip process zone exists for rock, but the inelastic behavior at the crack tip is described as micro-cracking. The description of size and shape of such a crack tip process zone for bi-material rock media is also given in Appendix C. The friction effect due to compressive stresses is shown to decrease the size of such a process zone. Nevertheless, the need for experimental results for the characteristic distance \( r_0 \) corresponding to various material combinations is eminent.

Selected bi-material properties and corresponding singularity powers, \( p \) (Eq. 2.109) are given in Table 3.1. Fracture envelopes of propagation into the uncracked material for several \( m \) values and plane strain conditions are shown in Fig. 3.11. Here, the elastic modulus of material 1 is taken to be unity and the elastic modulus of material 2 is adjusted with \( m \), assuming the same Poisson's ratio for both materials. It can be easily shown that Eqs.(3.34) and (3.35) reduce to Eqs.(3.3) and (3.4) for homogeneous problems. Thus, the \( m=1 \) envelope is also shown in the figure for comparison. The corresponding fracture initiation angles are presented in Fig. 3.13.

The behavior of crack propagation into a stiffer material is much like the propagation in a homogeneous medium. However, for propagation into a less stiff material, a different behavior is observed. For \( m>1 \),
TABLE 3.1 Selected Material Properties and Singularity Power, \( p \), for Bi-Material Terminal Crack Fracture Envelopes

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<th>( \frac{m\nu_2}{\mu_1} )</th>
<th>( E_1 )</th>
<th>( E_2 )</th>
<th>( \nu_1 )</th>
<th>( \nu_2 )</th>
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as the material property mismatch increases, the effect of \( K_2 \) decreases, whereas for \( m<1 \), the \( K_2 \) effect is much stronger. The fracture initiation angle measured from the line of crack into a stiffer material is expected to be larger than for softer material under \( K_1 \) dominant conditions. Our analysis indicates that for \( K_1=3K_2 \), crack initiation angle...
Fig. 3.11 Fracture Envelopes for Perpendicular Cracks Terminating at Bi-Material Interfaces

Fig. 3.12 Poisson's Ratio Effects for \( m = 0.05 \) and \( m = 5 \) (Terminal Crack)
is about 20° for m=0.1, 32° for m=1 and 35° for m=5. Thus, if a fracture back through the interface into a stiffer medium under mixed mode conditions, the behavior would be far from self-similar crack growth.

The effects of the two media having different Poisson's ratios is shown in Fig. 3.12. Poisson's ratio of the stiffer material seems to have little effect on the shape of the fracture envelopes. However, an increase in softer material's Poisson's ratio is accompanied by an upward shift of the fracture envelope for m>1 and vice versa for m<1.

Fracture envelopes for a crack reflecting back into material 1 are shown in Fig. 3.14. Propagation curves are also shown on the same figure for comparison. It is much easier for the crack to propagate into the undamaged softer material, than to reflect back into the harder cracked medium. The effects of different Poisson's ratios are shown in Figs. 3.15 and 3.16. Again, stiffer material's Poisson's ratio has little effect. The shape of the fracture envelope is governed by the softer material. As the material property mismatch becomes less severe, the Poisson's ratio effect gets more pronounced.

The fracture initiation angle $\beta$, measured from the interface for the reflecting crack, is shown in Fig. 3.17. As pure $K_1$ dominance is approached, $\beta$ approaches 30°. For lower $K_1/K_2$ ratios an interfacial extension rather than a sharply reflected back fracture is expected.

Fracture envelopes for interfacial debonding are shown in Figs. 3.18 through 3.21. A pure tensile debonding criteria requires the largest stress intensity factor combinations for $m<1$. The combined shear and normal stress criterion predicts an increased $K_2$ effect as
Fig. 3.13 Variation of Initial Fracture Angle for Terminal Cracks with $K_1/K_2$

Fig. 3.14 Comparison of Fracture Envelopes for Propagating and Reflecting Cracks
Fig. 3.15 Poisson's Ratio Effect for Reflecting Crack (m=0.05)

Fig. 3.16 Poisson's Ratio Effect for Reflecting Crack (m=0.10)
Fig. 3.17 Fracture Initiation Angle for Reflecting Crack ($m=0.05$ and $m=0.10$)
Fig. 3.18 Fracture Envelopes for Tensile Debonding

Fig. 3.19 Fracture Envelopes for Combined Shear and Tensile Debonding
Fig. 3.20 Fracture Envelopes for Compressive Debonding ($\mu_f = 0.5$ and $0.7$)
Fig. 3.21 Effect of $m$ and $\mu_f$ on Compressive Debonding Envelopes ($m=2.5$ and 0.50)
the value of \( m \) gets smaller. It was observed [104] that an external compressive loading normal to the interface creates tensile radial stress at \( \theta = \pi/2 \) in material 2 along the interface in contrast to compressive radial stress at \( \theta = \pi/2 \) in material 1 for \( m > 1 \) and vice versa for \( m < 1 \). We have shown that a compressive loading greatly affects the interfacial frictional resistance. Thus, the combined shear and normal stresses criterion is superior to the maximum tensile stress debonding criterion. In Fig. 3.20 debonding envelopes for compressive \( \sigma_\theta \) and various values of \( m \) are shown. The effect of the friction coefficient \( \mu_f \) is represented in Fig. 3.21. Debonding envelopes are observed to be shifted up as \( \mu_f \) gets smaller. The behavior of the combined shear and tension criterion is approached as \( \mu_f \) increases. Pure tensile debonding and critical combination of shear and tensile stresses criteria predict that if debonding occurs, it is more likely for \( m > 1 \) than for \( m < 1 \). On the other hand, for compressive \( \sigma_\theta \), the interface with the softer material is more likely to fail (i.e. \( m < 1 \)).

3.2.3 Path Independent Integrals For Bi-Material Media

In recent years, path independent integrals have received special attention in fracture mechanics problems. Among more than sixty different path independent integrals (or conservation laws), the \( J \)-integral introduced by Rice [91] and the \( L \) and \( M \) integrals of Knowles and Sternberg [96] represent the most popular ones. The \( J \)-integral is the first component of the vector

\[
J_i = \oint_c (W_n - T_{ij} u_j) \, ds , \quad i = 1, 2
\]  

(3.40)
and the Knowles-Sternberg integrals in 2-D are represented by

\[ L = \oint_{\Gamma} \phi_{3ij} (Wx_j n_i + T_i u_j - T_k u_k, i x_j) ds \]  \hspace{1cm} (3.41)

\[ M = \oint_{\Gamma} (Wx_i n_1 - T_i u_k, i x_i) ds \]  \hspace{1cm} (3.42)

where \( W \) is the strain energy density, \( T_i = \sigma_{ij} n_j \) is the traction associated with the unit outward normal, \( \hat{n} \) on the closed contour \( \Gamma \). In this discussion the usual index (cartesian) notation is used for convenience.

The Eshelby-Rice conservation law [90,91] assures that each component of \( J_1 \) vanishes in a region bounded by a closed path in which \( W \) depends only on the strains, \( \varepsilon_{ij} \) and the stresses, \( \sigma_{ij} \) satisfy the equilibrium equations and

\[ \sigma_{ij} = \frac{1}{2}(\frac{\partial W}{\partial \varepsilon_{ij}} + \frac{\partial W}{\partial \varepsilon_{ji}}) \]  \hspace{1cm} (3.43)

Thus, \( J \) is interpreted as the energy release rate associated with an extending crack. Budiansky and Rice [61] related the \( L \) and \( M \) integrals to energy release rates associated with cavity or crack rotation and expansion, since \( L \) vanishes under the same conditions stated for \( J \). In addition, \( W \) must depend only on the scalar invariants of \( \varepsilon_{ij} \) and must be a quadratic function of the components of \( \varepsilon_{ij} \) for \( M \) to vanish.

There have been some attempts [92,97] to extend the \( J \) integral conservation law to bi-material problems for interfacial and terminal crack cases. Different approaches were reported [89,98,99] for the numerical evaluation of the interfacial crack tip stress intensity factors, using the finite element method in connection with the \( J \)-integral.
In the following subsections, extension of J-integral conservation law to bi-materials is closely studied. For pressurized interfacial and terminal cracks, J-integral expressions are derived. In a similar fashion, M-integral conservation laws for such crack configurations are presented.

3.2.3.1 Bi-material J-integrals

Consider two linearly elastic bodies, \( B_1 \) and \( B_2 \), bonded as shown in Fig. 3.22. A closed contour \( \partial R_1 + \partial R_2 \) is defined for each homogeneous body \( B_i \).

The Eshelby-Rice conservation law has been extended [92] to a bi-material body with a bondline along x-axis (c.f. Fig. 3.22a) as

\[
J^* = \oint_{\partial R_1 + \partial R_2} (W_n - \sigma_{jk} n_k u_{j,i}) ds - \oint_{\partial R_b} ([W] \delta_{j2} - \sigma_{j2} [u_{j,i}]) ds = 0 \tag{3.44}
\]

where \( \partial R_b = \partial R_1 = -\partial R_2 \), \([W] = W(x,0^+) - W(x,0^-)\) and \([u_{j,i}] = u_{j,i}(x,0^+) - u_{j,i}(x,0^-)\). Similarly, for the bondline along y-axis (c.f. Fig. 3.22b)

\[
J^* = \oint_{\partial R_1 + \partial R_2} (W_n - \sigma_{jk} n_k u_{j,i}) ds - \oint_{\partial R_b} ([W] \delta_{j1} - \sigma_{j1} [u_{j,i}]) ds = 0 \tag{3.45}
\]

where \([W] = W(0^+,y) - W(0^-,y)\) and \([u_{j,i}] = u_{j,i}(0^+,y) - u_{j,i}(0^-,y)\). The interface conditions for Eq.(3.44) are

\[
[u_i] = 0 \quad , \quad [\sigma_{iy}] = 0 \quad , \quad i=1,2 \tag{3.46}
\]

Noting that the tangential derivative of \( u_j \) along the interface must also be continuous, we have

\[
[u_{i,x}] = 0 \tag{3.47}
\]
Fig. 3.22 Bi-Material Body Configuration Without Cracks

Fig. 3.23 Bi-Material Media with (a) Interfacial, (b) Terminal Cracks
Similarly for Eq. (3.45), the interface conditions are

\[ [u_i] = 0 \quad , \quad [\sigma_{ij}] = 0 \quad , \quad i=1,2 \]  

(3.48)

\[ [u_{1,y}] = 0 \]

Using Eqs. (3.44), (3.46) and (3.47), the \( x \) or first component of \( J_1^t \) becomes

\[ J_1^t = \int_{\partial R_1 + \partial R_2} (W_n - \sigma_{jk} n_k u_j, x) ds = 0 \]  

(3.50)

Equation (3.50) is the conservation law for the bi-material body configuration shown in Fig. 3.22a. This equation is exactly the same as the corresponding homogeneous case equation. For a traction free sharp interfacial crack \( (q=0 \text{ in Fig. 3.23a}) \) we obtain, from Eqs. (3.40) and (3.50)

\[ J_1^t = J_1 = \int_{\partial R} (W_n - T_j u_j, x) ds \]  

(3.51)

The above integral is path independent and equal to the energy release rate as given by Eq. (2.106).

For the case of a pressurized interfacial crack as shown in Fig. 3.23a, the bi-material conservation law, Eq. (3.50), becomes, using Eq. (3.44)

\[ J_1^t = \int_{\partial R} (W_n - T_j u_j, x) ds - \int_{\partial R} q(x) [u_y, x] ds = 0 \]  

(3.52)

Thus, the path independent \( J \)-integral for a pressurized interfacial crack lying along a straight bondline is

\[ J_1^t = \int_{\partial R} (W_n - T_j u_j, x) ds - \int_{\partial R_{2c}} q_2(x) u_y, x ds - \int_{\partial R_{1c}} q_1(x) u_y, x ds \]  

(3.53)
where \( q_2(x) \) and \( q_1(x) \) are the loadings on the crack surfaces associated with materials 2 and 1, respectively. For the important case of constant pressure loading, the second and third integrals in Eq.(3.53) can simply be evaluated from integration by parts, i.e.

\[
\oint_{\partial R_{2c}} q_2 u_y \, dx = q_2 u_y \bigg|_{x=-b}^{x=0}
\]

where \( b \) denotes the intersection of the contours \( \partial R_2 \) and \( \partial R_{2c} \) on the negative x-axis. If \( q(x) \) is not constant, then the related integrals can be evaluated numerically.

As an alternate way of obtaining \( J_{\perp}^t \), we consider Fig. 3.23a. The Eshelby-Rice conservation law for a homogeneous medium assures that \( J_1 \) written for body \( B_1 \) and body \( B_2 \) along the contour \( \partial R_1 + \partial R_{1c} + \partial R_{1b} \) for \( i=1,2 \) vanish, and so does their sum. The bi-material conservation law is thus formulated. We now let \( \partial R^* \) be the inverse integration path of \( \partial R_1 + \partial R_2 \), i.e. \( \partial R^* = -(\partial R_1 + \partial R_2) \). As \( \partial R^* \) shrinks to the crack tip, \( J_{\perp}^t \) for the bi-material interfacial crack is represented by the line integrals on the contours \( \partial R_1 + \partial R_2 \) and \( \partial R_{1c} + \partial R_{2c} \), which yields the same result as Eq.(3.53).

We use Eqs.(3.45), (3.48) and (3.49) in Fig. 3.22b and obtain

\[
J_{\perp}^n = \oint_{\partial R_1 + \partial R_2} (W_{n,k} \sigma_{j,k} n_j u) \, ds - \oint_{\partial R_b} ([W] - \sigma_{j,k} u_j u) \, ds = 0 \tag{3.55}
\]

Using the notation in Fig. 3.23b, the Eshelby-Rice conservation law is employed for the homogeneous regions \( B_1^+ \), \( B_1^- \) and \( B_2 \) along the contours \( \partial R_1^+ + \partial R_{1c} + \partial R_{1b}^+ \), \( \partial R_1^- + \partial R_{1c}^- + \partial R_{1b}^- + \partial R_2^+ \) and \( \partial R_{2b}^- + \partial R_{2c} + \partial R_{2b}^+ + \partial R_{2c}^+ \), respectively. Now, the bi-material \( J \)-integral for a terminal crack perpendicular to an interface is defined as
\[ J^n_1 = \int_{\partial R^*} (W_n x - T_j j_j, x) ds \]  

(3.56)

where \( \partial R^* \) is the inverse integration path along a small contour surrounding the bi-material crack tip, i.e. \( \partial R^* = (\partial R'_1 + \partial R'_2 + \partial R'_3) \). Using the bi-material conservation law of Eq.(3.55) and (Eq.(3.56), J-integral for a pressurized terminal crack perpendicular to a straight bondline becomes

\[ J^n_1 = \int_{\partial R^*_1+\partial R^*_2+\partial R^*_3} (W_n x - T_j j_j, x) ds + \frac{1}{2} \int_{\partial R} (\sigma_x[u_x,x] - u_y[y] + \sigma_y[u_y,x]) ds \]

\[ - \int_{\partial R^c} q^+(x) u_y,x ds - \int_{\partial R^-} q^-(x) u_y,x ds \]  

(3.57)

We note here that by collapsing the contour \( \partial R^* \) to the tip of the bi-material terminal crack, \( J^n_1 \) either becomes unbounded or vanishes depending on the elastic properties of the two materials. To demonstrate this behavior, let \( \partial R^* \) be a small circular arc of radius \( r_o \). The integrand of Eq.(3.56) is in the form of displacement gradient products. Hence, \( \partial R^* \) is proportional to \( r_o^{1-2p} \). It has been shown [2] that for a crack in the softer material, \( p < \frac{1}{2} \). Hence, the term \( r_o^{1-2p} \) approaches zero as \( r_o \) goes to zero; consequently, \( J^n_1 \) vanishes. If the crack is in the stiffer material, \( p > \frac{1}{2} \) and the term \( r_o^{1-2p} \) approaches infinity as \( r_o \) goes to zero, making \( J^n_1 \) unbounded. Thus, the path independence of the bi-material J integral for a terminal crack perpendicular to an interface is of a different nature. As illustrated in Fig. 3.24b, \( J^n_1 \) of Eq.(3.57) corresponding to the paths \( \partial R^* \), \( \partial R_1 \) and \( \partial R_2 \) are equal for fixed \( \partial R^* \). This behavior is numerically observed and the results are presented in subsection 4.2.4.
3.2.3.2 Bi-material M integrals

The purpose of this section is to extend the Knowles-Sternberg [96] conservation law for the M-integral to bi-material bodies. We first review some features of the M-integral. For a closed path $\partial R$, enclosing a simply connected region, the value of the M-integral of Eq. (3.42) is zero. If $\partial R$ completely encloses a crack or other flaw in a solid, then the value of M for this contour is nonzero in general. For homogeneous media, Budiansky and Rice [61] have shown that the value of M gives the energy-release rate for self-similar crack growth, where the energy release rate here is with respect to the relative scale change $dL/L$, where $L$ is any characteristic length of the cavity.

A special feature of the M-integral is that the first term, $Wn \cdot x_i$ of the integrand of Eq. (3.42), vanishes on any part of a closed contour which coincides with a segment of a radial line in the coordinate system used, since $x_i n = 0$ on such radial lines. The second term, $T_{k \cdot i} u_{k,i} x_i$ of the integrand, also vanishes on a radial line segment if the tractions are zero, or if uniform displacement conditions prevail there. It is obvious from the above argument that if one component is uniform, the contribution of the second term to the value of M is zero.

Reconsidering Figs. 3.22a and b and writing M integrals for the closed contours $\partial R_1 + \partial R_1b$ and $\partial R_2 + \partial R_2b$ for the homogeneous bodies $B_1$ and $B_2$, respectively, we have

\[
M = \int_{\partial R_1 + \partial R_2} (Wn \cdot x_i - T_{k \cdot i} u_{k,i} x_i) ds + \int_{\partial R_1b} (Wn \cdot x_i - T_{k \cdot i} u_{k,i} x_i) ds
\]

\[
+ \int_{\partial R_2b} (Wn \cdot x_i - T_{k \cdot i} u_{k,i} x_i) ds = 0 \quad (3.58)
\]
When the bondline is along the x-axis (Fig. 3.22a), $W_{n,x}$ terms in the integrands of the second and third integrals vanish, since along $y=0$ the bondline is a radial line. The second terms simplify to yield

$$M^t = \int_{\partial R_1 + \partial R_2} (W_{n,x} - T_u) ds - \int_{\partial R_{1b}} x(\sigma_{ky} u_k, x) ds - \int_{\partial R_{2b}} x(\sigma_{ky} u_k, x) ds = 0$$

(3.59)

Similarly for a bondline along the y-axis (Fig. 3.22b), Eq.(3.58) becomes

$$M^n = \int_{\partial R_1 + \partial R_2} (W_{n,x} - T_u) ds - \int_{\partial R_{1b}} y(\sigma_{kx} u_k, y) n_x ds - \int_{\partial R_{2b}} y(\sigma_{kx} u_k, y) n_x ds = 0$$

(3.60)

Now, imposing the interface conditions Eqs.(3.46 and 3.47) on $M^t$ and Eqs.(3.48 and 3.49) on $M^n$, we have

$$M^t = \int_{\partial R_1 + \partial R_2} (W_{n,x} - T_u) ds = 0$$

(3.61)

$$M^n = \int_{\partial R_1 + \partial R_2} (W_{n,x} - T_u) ds = 0$$

(3.62)

Thus, the $M$-integral conservation law for a homogeneous body extends without change to the bi-material medium. For the case of a bi-material interfacial crack, the associated $M$-integral, $M^t$, can be related to $J_1^t$ of the previous section in a simple manner. Since the $M^t$-integral is path independent (Fig. 3.24a), its value along the path $\partial R_1$ will be the same as that along any path $\partial R_2$. The straight segments of $\partial R_1$, represented by traction free radial lines with respect to the coordinate axis, will have no contribution to $M^t$. The contribution of the vanishingly small path at $x=a$, initiating on one crack face surrounding the
crack tip, and terminating on the opposite face is
\[ \int_{\partial R_x} (W_{x-T} u_{k,x}) \, ds = a \int_{\partial R_x} (W_{x-T} u_{k,x}) \, ds = a J^t \] (3.63)

Noting that the same contribution is made by the path \( \partial R_x \) at \( x=-a \), the total value of \( M^t \) is
\[ M^t = 2a J^t \] (3.64)

which, in turn, can be related to the interfacial crack tip stress intensity factors by Eq.(2.106). For plane stress case, for example
\[ M^t = \frac{a(K_1^2+K_2^2)}{2} \sum_{j=1}^{n} \frac{1}{\mu_j^* \nu_j} \] (3.65)

Similar \( M^t, J^t \) relations can be obtained for the pressurized interfacial fractures by including the contributions of the path \( \partial R_1 \) due to the crack surface tractions in Eq.(3.53).

For a terminal crack, the \( M \) integral conservation law of Eq.(3.62) does not appear to provide any further information despite several efforts. It is valid specifically when the coordinate system is chosen to be on the interface only. An \( M^n, J^n \) relationship can be obtained if the origin of the coordinate system is translated at a point \( \xi \) units to the left or right of the crack tip. The interface terms do not vanish for this case, since the interface integration path is no longer a radial line. Eq.(3.58) now becomes
\[ M = \int_{\partial R_1+\partial R_2} (W_{x-T} u_{k,x}) \, ds - \int_{\partial R_b} ([W]-\sigma_{xk}[u_{k,x}]) \, dy = 0 \] (3.66)

Considering the cracked configuration of Fig. 3.23b, with the origin of the coordinate system at \( \xi \) units to the left of the crack tip, we have
Comparing with Eq.(3.56), we see that the third integral term in Eq.
(3.67) is equal to $-\xi J^n$, thereby giving

$$M^n - \xi J^n = 0 \quad (3.68)$$

with $M^n$ defined as

$$M^n = \mathcal{J} \int_{\partial R_1 + \partial R_2 + \partial R_3} (W_{1,1}^{x_1} - T_{1,1} u_{k,1}) ds - \int_{\partial R_2} ([W] - \sigma_k x_k [u_k, x]) dy \quad (3.69)$$

The second integral term of Eq.(3.69) is exactly the same as the second
term of Eq.(3.55) multiplied by the distance $\xi$. 

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Fig. 3.24 Bi-Material (a) $J^t$ and (b) $J^n$ Integral Applications
CHAPTER IV

COMPUTER IMPLEMENTATION AND TEST CASES

4.1 COMPUTER IMPLEMENTATION

The developed finite element computer program FECAP (Finite Element Crack Analysis Program) is an extended version of the computer program CAP1 [100]. It is capable of analyzing linear plane and axisymmetric solid elasticity problems using eight-node isoparametric quadrilateral elements with curved geometry. Triangular elements obtained by collapsing one side of the quadrilateral elements are also implemented with the proper shape function modification. The element stiffness is numerically integrated using nine point Gauss integration scheme [83]. For singular problems, three different elements are available; namely, quarter point singularity element, Akin's modified singular displacement element and semi-radial singularity element in its subparametric formulation. Modifications indicated in Sections 2.1 and 2.2 are incorporated for the degenerate triangular quarter point mapping and singular displacement field elements. Straight radial sides emanating from the crack tip are required for the semi-radial singularity element due to its subparametric nature.

It is a well known fact that the storage scheme for the assembled stiffness matrix significantly affects the computation time and the storage requirements. The conventional skew rectangular storage of a
banded symmetric stiffness matrix requires a storage of size \( N \times P \) where \( N \) is the number of equations to be solved and \( P \) is the upper half bandwidth. For such a scheme, since \( NP^2/2 \) number of multiplicative operations are involved in a standard Gauss elimination or Cholesky Decomposition method of solution. Since \( P \) is governed by the maximum difference of the greatest and the smallest node numbers of each element in the mesh, the bandwidth \( P \) is the critical parameter. For a more efficient mesh utilization, compact columnwise storage scheme along with the equation solver OPTSOL, proposed by Mondkar and Powell [101] is used in FECAP. The total stiffness matrix is assembled by observing the location of the coefficients \( a_{ij} \) of the full matrix as

\[
\text{Location} = i - j + \sum_{k=1}^{j} (k - L_k + 1) \tag{4.1}
\]

where \( i \) and \( j \) are the row and column numbers of the particular coefficient \( a_{ij} \), \( L_k \) is the row number of the first nonzero entry at column \( k \). Since coefficients with \( i > j \) and zero terms within a row are not stored, this scheme requires less storage. Consequently, better mesh utilization is maintained by using finer elements near the crack tip without significantly increasing the storage and the computation time. This scheme also enables the use of additional node numbers at the crack tip without renumbering the entire mesh.

Besides the nodal point displacements, the element stresses, calculated at the integration points and the nodal point reaction forces are available upon request as the output of the program. For the analysis of linear elastic solids containing cracks, fracture parameters such as homogeneous and bi-material stress intensity factors and energy
release rates are computed and displayed. The energy release rates associated with cracks in homogeneous media and cracks along bi-material interfaces are obtained through numerical evaluation of the conventional path independent J integral. For terminal cracks perpendicular to bi-material interfaces, $J^N$ or $M^N$ integrals presented in Section 3.2.3 are evaluated. Numerical schemes for evaluation of these integrals are incorporated into the finite element code FECAP. This calculation proceeds element by element once the contour of integration is chosen, i.e. along the element boundaries or an arbitrary path within an element. Since the displacement gradients evaluated at the element stiffness integration points are more accurate than those evaluated elsewhere, the J and M integral integration paths are currently chosen to pass through these points, along which the three point Gauss-Legendre integration rule is employed.

For the study of the behavior of the three-dimensional isoparametric singularity element, the program SAPIV [102] is used along with the suggested modifications of Section 2.1.2.

4.2 TEST CASES

In order to test the overall behavior of the degenerate elements, first a deep cantilever beam is treated as a nonsingular problem. Then a single edge crack plate under tension is analyzed to study the effect of shape function modification for various collapsing schemes on the quarter point singularity element. A bi-material interfacial crack problem and a bi-material terminal crack problem are studied to investigate and compare the behavior of elements of Chapter II. Finally, a
three-dimensional problem of an embedded flat crack under uniform pressure is presented. The results are discussed and compared with solutions available in the literature.

4.2.1 A Deep Cantilever Beam

To investigate the behavior of the degenerate triangular element for unmodified and modified forms, a deep cantilever beam subjected to end shear is treated as a plane stress elasticity problem [100]. Three cases of degeneracy shown in Fig. 4.1 are considered. The values obtained: a) for vertical deflection at \(x=48\) in., \(y=6\) in. position, b) for horizontal normal stress at \(x=48\) in., \(y=9\) in. position, on the beam are given in Table 4.1 where all values are normalized with respect to the plane stress elasticity solution [103]. As shown, the unmodified triangles behave poorly, and the results are sensitive to the way the elements are degenerated. The modified triangles, on the other hand, yield results which are invariant for the cases considered, and are also far more accurate while the normalized deflection obtained with the modified collapsed triangle is \(\bar{v}=0.989\) at the tip. With the same mesh

Table 4.1 Comparison of Normalized Deflections and Stresses [100] in the Cantilever Beam - Exact: \(\bar{v}=1\), \(\sigma_{xx}=1\)

<table>
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<th>Case</th>
<th>Unmodified</th>
<th>Modified</th>
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<tr>
<td></td>
<td>(\bar{v})</td>
<td>(\sigma_{xx})</td>
</tr>
<tr>
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<td>0.983</td>
<td>0.940</td>
</tr>
<tr>
<td>B</td>
<td>0.919</td>
<td>0.886</td>
</tr>
<tr>
<td>C</td>
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<td>0.850</td>
</tr>
</tbody>
</table>
Fig. 4.1 Deep Cantilever Beam and Cases of Degeneracy

Fig. 4.2 Single Edge Crack Plate in Tension
the regular six-node triangles yield $\bar{v}=0.955$. Therefore, as far as the accuracy is concerned, the modified collapsed triangles are competitive in comparison with the regular six-node triangles.

4.2.2 Single Edge Notch Specimen in Tension

The shape function modification is further studied considering a singular problem by using degenerate triangular quarter point singularity elements. A single edge crack plate under uniform tension (Fig. 4.2) is solved with the plate dimensions, $W=H=10$ in. and the crack length, $a = 4$ in. Plane stress conditions are assumed and due to symmetry, only the upper half of the plate is modelled. As shown in Fig. 4.2, three cases of local mesh configurations I, II and III are considered near the crack tip, and for each mesh, two schemes of node collapsing are employed. In scheme A, the collapsed nodes are placed at the crack tip and in scheme B the collapsed nodes are placed away from the tip. In all cases $L/a=0.1$ for the crack tip elements, where $L$ is a measure of the element length indicated in Fig. 4.2. Normalized stress intensity factors $K_1/\sigma_o (\pi a)^{1/2}$ obtained by using Eq. (2.76) are tabulated in Table 4.2. As shown, the unmodified elements are extremely sensitive both to the geometry of the side opposite to the tip and to the location of the collapsed nodes. Whereas the modified triangles are not affected by these, and even with the rather coarse mesh of 26 elements and 95 nodal points shown in Fig. 4.2, they consistently yield good results. Obtained values are even better in the collapsing scheme B.
Table 4.2 Normalized Stress Intensity Factors for the Single Edge Notch Plate [100] - Exact: $K_I/(\sigma_o \sqrt{a})=2.11$

<table>
<thead>
<tr>
<th>Mesh</th>
<th>Unmodified</th>
<th></th>
<th>Modified</th>
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</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Scheme A</td>
<td>Scheme B</td>
<td>Scheme A</td>
<td>Scheme B</td>
</tr>
<tr>
<td>I</td>
<td>2.12</td>
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<td>2.08</td>
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<tr>
<td>II</td>
<td>2.14</td>
<td>1.68</td>
<td>2.10</td>
<td>2.12</td>
</tr>
<tr>
<td>III</td>
<td>1.51</td>
<td>1.81</td>
<td>2.06</td>
<td>2.11</td>
</tr>
</tbody>
</table>

4.2.3 Bi-Material Interfacial Crack Model

The test problem considered is a 20x20 inch plate with a 2-inch crack lying along the bi-material interface, subjected to normal stresses at the outer boundaries (Fig. 4.3). Plane stress conditions are assumed. To be able to compare the results with the solution for an infinite panel, the continuity condition for the strain $\varepsilon_x$ is imposed to yield

$$\varepsilon_x = \frac{E_1}{E_2} (\sigma_x) + [\nu_1 - \frac{E_1}{E_2} \nu_2] (\sigma_y)$$

(4.2)

The assumed Poisson's ratios are $\nu_1=\nu_2=0.3$. The elastic modulus for material 2 is taken as unity and it is varied for material 1. Due to symmetry, only one half of the plate is modelled and solved (Fig. 4.4). Eight collapsed triangular subparametric semi-radial singularity elements with the power of singularity $p=0.5$ are used to surround the crack tip and 62 quadratic isoparametric elements are used elsewhere. It may be noted that quarter point elements could also be used to model the crack tip region. The energy release rate is computed by evaluating the $J$-integral given by Eq.(2.106). It is observed that there is less than 0.5% difference in $J$ calculated for different paths when the
Fig. 4.3 Interfacial Crack Test Problem

Fig. 4.4 FEM Model for Interfacial Crack Problem
integration path coincides with the finite element stiffness integration points. This difference is more than 3% if the integration path followed the element boundaries.

Sih [84] gives the exact solution for the stress intensity factors for an infinite panel as

\[
K_1 = \left[ \frac{\cos(\epsilon h 2a) + 2 \epsilon \sin(\epsilon h 2a)}{\cosh \pi \epsilon} \right] \sqrt{\pi \rho a} \sigma_y \tag{4.3}
\]

\[
K_2 = \left[ \frac{\sin(\epsilon h 2a) - 2 \epsilon \cos(\epsilon h 2a)}{\cosh \pi \epsilon} \right] \sqrt{\pi \rho a} \sigma_y \tag{4.4}
\]

The strain energy release rate variation with respect to the shear moduli ratio m shows good agreement with the exact solution (Fig. 4.5). In Fig. 4.6, the finite element vertical crack surface displacement variation for m=100 is presented. The analytical displacement variation is obtained using Eqs.(4.3) and (2.98). After obtaining good estimates for the crack opening displacement, \( \Delta u_y \) and J from the finite element analysis, Eqs.(2.101) and (2.106) are solved simultaneously for \( K_1 \) and \( K_2 \) for small increments of the radial distance r measured from the crack tip. Plotting the \( K_1 \) estimates with respect to r/a, points sufficiently far away from the crack tip reached a constant slope. Extrapolating back this constant sloped line to r=0 gives the stress intensity factor estimates (Fig. 4.7). In Fig. 4.8 the estimated stress intensity factors are shown to be quite accurate. For comparison, the \( K_2 \) values obtained using the regular stress intensity factor extraction technique of Section 2.3.1 are also shown in dotted lines. As observed, the obtained value of \( K_2 \) is very much in error. The above discussion can explain why Smelser [89] obtained such poor stress intensity factor estimates from
Fig. 4.5 Strain Energy Release Rate Variation

Fig. 4.6 Interfacial Crack Surface Normal Displacement Profile
Fig. 4.7 Stress Intensity Factor Estimation

Fig. 4.8 Variation of $K_1$ and $K_2$ with $m$
his finite element crack flank displacement data. The stress intensity factors and the energy release rates for various material and loading combinations are tabulated in Table 4.3.

Table 4.3 Stress Intensity Factors and Energy Release Rate for the Interfacial Crack

<table>
<thead>
<tr>
<th>m</th>
<th>$E_1$ (psi)</th>
<th>$(\sigma x)_1$ (psi)</th>
<th>$K_1$ (psi-in)</th>
<th>$K_2$ (psi-in)</th>
<th>$G=\partial U/\partial a$ (lb/in)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Exact</td>
<td>FEM</td>
<td>Exact</td>
<td>FEM</td>
<td>Exact</td>
</tr>
<tr>
<td>1000</td>
<td>0.001</td>
<td>0.3007</td>
<td>1.687</td>
<td>1.705</td>
<td>0.251</td>
</tr>
<tr>
<td>100</td>
<td>0.010</td>
<td>0.3070</td>
<td>1.689</td>
<td>1.708</td>
<td>0.247</td>
</tr>
<tr>
<td>10</td>
<td>0.100</td>
<td>0.3700</td>
<td>1.716</td>
<td>1.739</td>
<td>0.208</td>
</tr>
<tr>
<td>1</td>
<td>1.000</td>
<td>1.000</td>
<td>$\sqrt{\pi}$</td>
<td>1.783</td>
<td>0.000</td>
</tr>
</tbody>
</table>

4.2.4 Bi-Material Terminal Crack Model

The singularity elements presented in Section 2.2 are extensively studied by considering a plane strain model with a pressurized crack normal to and terminating at the interface (Fig. 4.9). The left end of the crack is embedded in material 1; hence, the order of strain singularity at this tip is $1/r^{1/2}$. The interface crack tip has a singularity depending on the bi-material elastic properties and it is obtained using Eq.(2.109).

The two materials considered here are aluminum and epoxy with the elastic properties

$\mu_{\text{aluminum}} = 3.846 \times 10^6 \text{ psi}$, $\nu_{\text{aluminum}} = 0.30$

$\mu_{\text{epoxy}} = 0.1667 \times 10^6 \text{ psi}$, $\nu_{\text{epoxy}} = 0.35$
Fig. 4.9 Bi-Material Plate with a Terminal Crack

Fig. 4.10 FEM Model and $M^N$ Integral Evaluation Paths for the Terminal Crack Problem
When the crack is in aluminum, $m=0.043$ and $p=0.8258$ and when the crack is in epoxy, $m=23.08$ and $p=0.3381$. From symmetry consideration, only one half of the structure is modelled incorporating the proper boundary conditions. Four degenerate triangular quarter point singularity elements are used to model the homogeneous crack tip region for all of the cases considered. The bi-material crack tip is modelled by four degenerate triangular elements with a radial dimension of $a/100$. Regular quadratic quadrilateral and degenerate triangular elements are used elsewhere.

The total mesh (Fig. 4.10) includes 251 nodal points and 74 elements. Tracey and Cook [39] solved the same problem with 429 nodes and 433 elements using power type singularity elements around the crack tips and bilinear isoparametric elements elsewhere, with the radial singularity element dimension of $a/100$. First, the singular displacement field element due to Akin [41] is used to model the bi-material crack tip region in its degenerate triangular form. It is observed that this element is very sensitive to the particular way the quadrilateral element is degenerated to obtain the triangular element. Several degenerating schemes are investigated. Collapsing the nodes to the singular point in a cluster is observed to yield the worst results. In Fig. 4.11 for $m=0.043$, the variation of crack opening displacement with the radial crack tip dimension is shown for two different collapsing schemes. The solution from the singular integral equations [10] is also presented on the same figure for comparison. As discussed in Section 2.2.1, since the displacement field yields the necessary singularities in the stress and strain, a better representation is expected by using regular
Fig. 4.11 Study of the Shape Function Modification and Degenerating Schemes for Variable Singularity Elements
polynomial shape functions for the geometric mapping. On the contrary, numerical observations of the stress and displacement fields obtained indicated that the isoparametric representation yields far better results. To obtain the proper degeneracy, shape function modification is employed. For certain schemes, the results improved considerably. However, no improvement was observed for the particular scheme of clustering the nodes at the singular point (scheme I). This instability occurred both for the isoparametric and the standard polynomial geometric mappings with a wider variation for the latter case. When the bi-material crack tip region is modelled with subparametric semi-radial singularity elements of Section 2.2.2, no such instabilities occurred. In Fig. 4.11 it is observed that the results obtained by using the subparametric semi-radial singularity mapping element and the degenerating scheme II for the singular displacement field element are in good agreement with the singular integral equation solution. Comparison of the current results with Tracey and Cook's [39] finite element results show that comparatively efficient analysis can be performed with almost half the total number of degrees of freedom. In Fig. 4.12, the normalized crack opening displacement \( u_y/a \) is shown as a function of the distance from the interface crack tip for aluminum as the cracked material and for epoxy as the cracked material. The corresponding angular variations of the normalized stress \( \sigma_{yy}/q \), for \( q=1 \) at \( r/a=0.005 \) are plotted in Fig. 4.13. As illustrated, good agreement is obtained with the singular integral equation results given at \( \theta=0, 90 \) and 180 degrees except for \( m = 0.043 \) at \( \theta=90^\circ \). A similar trend was observed in Ref. [39] with the indication that further mesh refinement in the angular sense at the
Fig. 4.12 Crack Opening Displacement Profiles for $m=0.043$ and 23.08

Fig. 4.13 Interface Tip Angular Stress Variation for $m=0.043$ and 23.08
bi-material tip would improve the results. In Table 4.4 the stress intensity factors computed for the homogeneous and bi-material crack tips are presented along with the finite element results of Ref. [39] and singular integral equation results of Ref. [10]. The results for the homogeneous tip appears to be overestimated. This is due to the use of rather coarse modelling around this crack tip. Improved results can be obtained by using a smaller quarter point element to crack length ratio. For the bi-material tip stress intensity factor, the subparametric semi-radial mapping element appears to yield good results in comparison with the results presented in Refs. [39] and [10]. Use of modified singular displacement field element along with regular polynomial geometric interpolation, Eq.(2.116) yields larger stress intensity values. This is due to the improper singularity representation in the element, as discussed in Section 2.2.1. More accurate results are obtained via the isoparametric formulation by extrapolating the K values obtained from Eq.(2.114) back to r=0.

To illustrate the application of the bi-material M integral, four different contours are considered as shown with dotted lines in Fig. 4.10. As discussed earlier, the path independence of the bi-material J-integral of Section 3.2.3.1 is of a different nature than the homogeneous case. Chen and Wu [97] considered the same problem and reported a nondimensional, $J^n/qa$ value of $0.2013 \times 10^{-4}$ for $m=0.043$. However, the analysis of Ref. [97] is based on defining the $J^n$ of Eq.(3.56) as the contour $\partial R^*$ collapses to the bi-material crack tip. Such a definition is not acceptable due to the variable nature of the stress/strain singularity at the bi-material crack tip. Here, we assume $\partial R^*$ as a small circle of radius
Table 4.4 Normalized Stress Intensity Factors for the Pressurized Terminal Crack, $K/q^{\sqrt{a}}$

<table>
<thead>
<tr>
<th></th>
<th>$m=0.043$</th>
<th></th>
<th>$m=23.08$</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Tip A</td>
<td>Tip B</td>
<td>Tip A</td>
<td>Tip B</td>
</tr>
<tr>
<td>SSME (1)</td>
<td>1.529</td>
<td>0.113</td>
<td>0.961</td>
<td>2.787</td>
</tr>
<tr>
<td>MSDE (2)</td>
<td>1.528</td>
<td>0.115</td>
<td>0.961</td>
<td>2.848</td>
</tr>
<tr>
<td>MSDE (3)</td>
<td>1.557</td>
<td>0.146</td>
<td>0.961</td>
<td>3.062</td>
</tr>
<tr>
<td>Ref.[39]</td>
<td>1.520</td>
<td>0.112</td>
<td>0.890</td>
<td>2.850</td>
</tr>
<tr>
<td>Ref.[10]</td>
<td>1.355</td>
<td>0.070</td>
<td>0.883</td>
<td>2.785</td>
</tr>
</tbody>
</table>

(1) Subparametric semi-radial singularity mapping element.
(2) Modified singular displacement field element, isoparametric form.
(3) MSDE, regular polynomial geometric mapping.

Table 4.5 Normalized Bi-Material $J^n$-Integral, $J^n/q^a$ (x10^{-5})

<table>
<thead>
<tr>
<th>$m$</th>
<th>$r_o/a$</th>
<th>Path</th>
<th>0.01</th>
<th>0.03</th>
<th>0.07</th>
<th>0.15</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.043</td>
<td>1</td>
<td>0.2379</td>
<td>0.1402</td>
<td>0.1017</td>
<td>0.0809</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.2389</td>
<td>0.1412</td>
<td>0.1027</td>
<td>0.0819</td>
<td></td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>0.2385</td>
<td>0.1407</td>
<td>0.1022</td>
<td>0.0814</td>
<td></td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>0.2395</td>
<td>0.1417</td>
<td>0.1032</td>
<td>0.0824</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Average</td>
<td>0.2387</td>
<td>0.1409</td>
<td>0.1025</td>
<td>0.0816</td>
<td></td>
</tr>
<tr>
<td>23.08</td>
<td>1</td>
<td>0.0601</td>
<td>0.1083</td>
<td>0.1488</td>
<td>0.1895</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.0480</td>
<td>0.0951</td>
<td>0.1389</td>
<td>0.1784</td>
<td></td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>0.0499</td>
<td>0.0958</td>
<td>0.1381</td>
<td>0.1817</td>
<td></td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>0.0423</td>
<td>0.0895</td>
<td>0.1309</td>
<td>0.1721</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Average</td>
<td>0.0501</td>
<td>0.0972</td>
<td>0.1392</td>
<td>0.1805</td>
<td></td>
</tr>
</tbody>
</table>
Fig. 4.14 Bi-Material $J^n$ Integral Variations with (a) $r_o$ and (b) Different Paths for Fixed $r_o$
r_o, centered at the interface crack tip. M^n-J^n relationship of Eq.
(3.68) is used. J^n values for a fixed \( \partial R^k \), corresponding to different
M^n contours shown in Fig. 4.10 are given in Table 4.5. Figure 4.14a
illustrates the variations of the average bi-material J^n integral with
r_o, for m=0.043 and m= 23.08. As expected, J^n becomes unbounded for
m=0.043 and approaches zero for m= 23.08 , as r_o goes to zero. The two
curves intersect at r_o/a=0.045. Next we consider the variation of J^n
with the interfacial path length, r', measured from the crack tip to
the outer contour for fixed r_o. The resulting behavior is shown in Fig.
4.14b. The path independence of J^n for fixed \( \partial R^k \) is clearly observed
and the previous observation of equal J^n values for m=0.043 and m =
23.08 at r_o/a=0.045 is confirmed.

4.2.5 Embedded Flat Crack in Three-Dimensional Medium

The problem considered as the test case for three-dimensional iso-
parametric degenerate singularity element is a flat elliptical crack in
an infinite solid where the crack surfaces are subjected to uniform
pressure. Due to symmetry, one octant of the structure is modelled in-
corporating the proper displacement boundary conditions (Fig. 4.15).
The ratio of semi-major to semi-minor axes is chosen to be 2. Figure
4.16 illustrates the typical mesh geometries employed in the computa-
tions. The evaluated displacement profiles along the semi-major and
semi-minor axes of the elliptical crack are plotted in Fig. 4.17 with
exact theory results [104,105]. The exact c.o.d. solution is repre-
sented by

\[
u_x = \frac{2(1-\nu^2)}{E} \frac{P_b}{E(k)} \left(1 - \frac{y^2}{a^2} - \frac{z^2}{b^2}\right)^{1/2}
\]  

(4.5)
where \( k^2 = 1 - \left( \frac{b}{a} \right)^2 \) and \( E(k) \) is the Jacobian elliptic integral of the first kind. The variation of the crack opening displacement stress intensity factor along the boundary of the elliptic crack is shown in Fig. 4.18, along with the exact solution of Ref. [104]. Again, good agreement between finite element model and the exact theory is evident.

![Pressurized Elliptical Crack Model in 3D Medium](image)

Fig. 4.15 Pressurized Elliptical Crack Model in 3D Medium
Fig. 4.16 A Typical Plane Section of the FEM Model for the Elliptical Crack
Fig. 4.17 Displacement Profiles Along the Semi-Major (a) and Semi-Minor (b) Axes of the Elliptical Crack

Fig. 4.18 Stress Intensity Factor Variation Along the Elliptical Boundary
5.1 PRELIMINARY REMARKS

Hydraulic fracturing is an established and effective well stimulation technique for oil and gas recovery. In this process a pressurized fluid is injected through a wellbore and a fracture or fracture system is induced in the zone of interest (payzone) [107]. Low permeability heterogeneous reservoirs can become more productive, since hydraulically induced fractures serve as a channel for enhanced oil and gas flow through the payzone to the recovery well. Due to the prevailing in-situ stresses, most of the induced fractures observed in the field, below a depth of 1 km. are vertical fractures (Fig. 5.1), and they propagate in a direction normal to the minimum in-situ stress [67]. Since geological media are multilayered, they are subjected to variable horizontal in-situ stress distributions across the layers. This behavior is schematically represented in Fig. 5.2. In practice, oil and gas bearing sandstone layers are usually bounded by layers of shale. It is undesirable to have fracture extending into these bounding layers, since unproductive strata would be penetrated and expensive fracturing fluid would be lost. Hence, fracture containment within the payzone layer is a critical factor in the design and planning of hydraulic fracturing processes. Material property mismatches, variable horizontal in-situ
Fig. 5.1 Types of Hydraulically Induced Fractures

Fig. 5.2 A Schematic Representation of Different Horizontal In-Situ Stress Magnitudes Along Formation Layers
stress distributions and interface slip/friction properties may prevent or facilitate the fracture propagation into the bounding layers. The inherent shear strength of the layer interfaces, for example, causes the onset of slippage between the payzone layer and the barrier formation and can lead to fracture containment. Furthermore, ductility of the bounding layers [68], microcracking [75], permeability of the reservoir formation and fluid pressure profile can be considered as local factors influencing containment phenomenon.

In this study, two-dimensional cracks in linear elastic media are considered with constant fluid pressure acting along the entire crack length. Adjoining layers are assumed to be perfectly bonded at their interfaces. The latter assumption is valid at greater overburden depths. Standard LEFM theory predicts that the stress intensity factor approaches zero (infinity) as a fracture approaches a bi-material interface with a higher (lower) stiffness bounding layer. This observation has led many investigators to the erroneous conclusion that material interfaces are strict barrier mechanisms. However, recent research [68,69] and laboratory experiments [70] indicate that fractures do break into higher stiffness barrier layers. If the stress intensity factor for the crack tip at the bi-material interface is defined by considering the variable nature of the singularity power, p (Eq. 2.111), it is observed that the stress intensity factor assumes a finite value at the interface crack tip. This behavior is demonstrated by numerical examples presented previously and in this chapter. To provide a better understanding of the containment phenomenon, several simulated models are considered.
here with the effects of material property mismatch and variable horizontal in-situ stress.

5.2 BI-MATERIAL TERMINAL CRACK MODEL, INFINITE MEDIUM

The effects of the differences in horizontal in-situ stresses and the material properties, for uniform fracturing fluid pressure, are studied by considering the same model geometry presented in Section 4.2.5 (Fig. 4.9). To approximate the infinite boundaries, it is assumed that $b/h=1$, and $a/h=1/18$. Plane strain conditions are assumed. To simulate several shear moduli ratios, the elastic modulus of material 1 is kept constant while elastic modulus of material 2 is varied for equal Poisson's ratios. Figure 5.3 illustrates the normalized stress intensity factors at the homogeneous and bi-material crack tips induced by (i) crack surface pressure, $q$, (ii) horizontal in-situ stress, $\sigma_1$ in the payzone and (iii) horizontal in-situ stress, $\sigma_2$ in the overburden, for different $\sigma_2/\sigma_1$ ratios. The effects of the material mismatch on the homogeneous tip are observed to be stronger when the crack is in the stiffer material ($m<1$) and less pronounced for $m>1$. On the other hand, the bi-material crack tip is very sensitive to the shear moduli ratio.

In Figs. 5.4 and 5.5, corresponding superposed bi-material stress intensity factor variations with respect to different $m$ and $\sigma_2/\sigma_1$ ratios are presented. First, crack opening pressure, $q$ and in-situ stress, $\sigma_1$ are kept equal and overburden stress, $\sigma_2$ is varied (Fig. 5.4). Bi-material crack tip stress intensity factor is observed to be zero for $\sigma_2/\sigma_1=m$. The linear behavior of Fig. 5.5 can be represented by the relation
Fig. 5.3 Stress Intensity Factor Variations at the Homogeneous (a) and Bi-Material (b) Crack Tips for Infinite Plane Strain Plate Model
Fig. 5.4 Variation of K at the Bi-Material Crack Tip with $m$ (a) and $\sigma_2/\sigma_1$ (b) for $q/\sigma_1 = 1.0$
Fig. 5.5 Variation of $K$ at the Bi-Material Crack Tip with $m$ (a) and $\sigma_2/\sigma_1$ (b) for $q/\sigma_2=1.0$
\[ \frac{K_p}{qa^p} = A(m - \frac{\sigma_2}{\sigma_1}) \] (5.1)

where \( A \) is a positive constant. Hence, for \( \sigma_2/\sigma_1 > m \), interface is expected to arrest the bi-material crack tip. Secondly, crack pressure and overburden stress are kept equal and in-situ stress, \( \sigma_1 \) is varied (Fig. 5.5). The observed behavior is not as simple as the latter case. As \( \sigma_2/\sigma_1 \) ratio becomes smaller than 1, interfacial crack arrest is more likely. Increasing \( \sigma_2/\sigma_1 \) tends to stabilize the stress intensity factor variation. Hence, material property mismatch and pressure magnitude becomes the controlling parameters. To simulate the crack arrest at the interface, the stress \( (\sigma_2/\sigma_1) \) and material property \( (m) \) ratios are adjusted using the above two cases and the resulting behavior is shown in Fig. 5.6.

5.3 **FRACTURE PENETRATION THROUGH AN INTERFACE**

Two cases are considered here. First, the problem of a crack going through the interface of two bonded half planes is studied. Secondly, the crack opening pressure required for through cleavage is determined.

5.3.1 **Crack Penetrating an Interface**

The variation of the stress intensity factor ahead of a pressurized crack propagating into a stiffer material is studied by simulating infinite boundaries and assuming plane strain conditions (Fig. 5.7). Aluminum-epoxy pair (\( m=0.043 \)) is chosen as the test case for comparison with the results of Ref. [15]. The crack tip 1, in the stiffer material, is assumed to be fixed at a unit distance \( b_1 \) from the interface. The
Fig. 5.6 Bi-Material Crack Tip Stress Intensity Factor Variation with \( \sigma_2/\sigma_1 \), Simulation Arrest at the Interface

Fig. 5.7 Stress Intensity Factor Variation for a Crack Approaching and Penetrating an Interface
distance $b_2$ from the interface to the crack tip 2 is kept variable. Erdogan and Biricikoglu [15] solved the problem by scaling the crack surface loading to meet the interface continuity condition as

$$
\frac{(1-\nu_1^2)q_1}{E_1} = \frac{(1-\nu_2^2)q_2}{E_2}
$$

For certain class of problems, e.g. hydraulic fracturing, imposing such a continuity condition would be unrealistic. Here, both the scaled and uniform crack surface pressure loadings are considered. Figure 5.7 illustrates the normalized stress intensity factor variations for tips 1 and 2 with respect to the crack length ratio. Stress intensity factor at tip 2 in the softer material is observed to be approaching unity from above as $b_2$ increases. Similarly, tip 1 in the stiffer material follows the same trend. Comparison of the current finite element stress intensity factors with the results reported in Ref. [15] show excellent agreement.

When a uniform pressure loading is applied at the crack surfaces, both in the stiffer and the softer materials, the normalized stress intensity factor at tip 2 approaches unity from below as $b_2$ increases, whereas for tip 1, it remains essentially constant.

5.3.2 Through Cleavage Crack Opening Pressure

As the first application, the model shown in Fig. 5.8 is considered. A crack of length $2a$ is assumed in the payzone. The upper and lower crack tips are located at the two parallel interfaces normal to the crack. The bounding layers are assumed to have the same elastic properties. Due to symmetry, only one quarter of the model is solved.
Fig. 5.8 Plexiglass-Resin Sandwich Model

Fig. 5.9 Treatment Pressure Ratio Comparisons for Infinite Bi-Material Plate Model
under plane strain conditions. A plexiglass-resin material pair is chosen for comparison purposes with the analysis of Ref. [74]. Material property parameters are given in Table 5.1. The corresponding shear modulus ratio \( m \) and the singularity power \( p \) are 0.882 and 0.5083, respectively. The effective critical bottom hole pressure ratio, \( q_2/q_1 \) (i.e., pressure for bi-material penetration/pressure for isotropic propagation) is computed using critical stress intensity factor and bi-material \( J \)-integral criteria. The former can be represented by the relation

\[
\frac{q_2}{q_1} = \frac{1}{C_k} \left( \frac{K_h}{K_b} \right)^{2p-1}
\]

(5.3)

where \( C_k = \left( \frac{K_{c,1}}{K_{c,2}} \right) \) is the critical stress intensity factor for material \( i \) (\( i=1,2 \)) and \( K_h, K_b \) are the isotropic and bi-material stress intensity factors, respectively. For this particular problem, since \( \mu_2/\mu_1 \) ratio is close to unity and the type of singularity is almost the same as in homogeneous media, the following bi-material \( J \) criterion is considered

\[
\frac{q_2}{q_1} = \frac{1}{C_k} \left( \frac{1-\nu_2}{m(1-\nu_1)} \right)^{1/2} \left( J_h/j^n \right)^{1/2}
\]

(5.4)

where \( J_h \) and \( J^n \) are the isotropic and bi-material \( J \) values, respectively. Biot et al. [74] presented a simplified elasticity solution, predicting the pressure ratio by

\[
\frac{q_2}{q_1} = \frac{1}{C_k} \left( \frac{1-\nu_2}{1-\nu_1} \right)^{1/2}
\]

(5.5)
Stress intensity factors and the $J^n$ value obtained from the current finite element analysis are given in Table 5.1. We note here that $J^n$ is computed by taking the contour $\partial R^k$ in Eq.(3.56) as the exterior boundary of the assemblage of singular elements at the bi-material crack tip.

<table>
<thead>
<tr>
<th>Material</th>
<th>$E$ (psi$\times$10$^5$)</th>
<th>$v$</th>
<th>$K_{IC} [74]$ (psi$\sqrt{in}$)</th>
<th>$K_h/q{\sqrt{a}}$</th>
<th>$K_b/q{\sqrt{a}}$</th>
<th>$J^n/q{\sqrt{a}}$ (x10$^{-4}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 (Plexiglass)</td>
<td>4.366</td>
<td>0.341</td>
<td>396.03</td>
<td>5.611</td>
<td>5.350</td>
<td>0.650</td>
</tr>
<tr>
<td>2 (Resin)</td>
<td>3.902</td>
<td>0.359</td>
<td>498.06</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Equations (5.3) and (5.4) predict $q_2/q_1$ ratio as 1.32 and 1.31, respectively. Experimental results presented in Ref. [74], however, reveal that $q_2/q_1$ is approximately 2.0 while Eq.(5.5) gives 1.24. Biot et al. explain such a difference between the expected and measured results by the many approximations in their theory and the less than ideal nature of their experiments.

As a second application, the infinite medium bi-material terminal crack model of Section 5.2 is considered for crack surface pressure loading only. Corresponding to various material combinations, $q_2/q_1$ ratios are computed using the critical stress intensity factor criterion of Eq.(5.3). A tabulation of the data and the corresponding stress intensity factors for a pressurized plane strain crack are presented in Table 5.2. Figure 5.9 illustrates the comparison of criteria for
Eqs. (5.3) and (5.5). The special case obtained by assuming fracture toughness ratio $C_k$ equal to shear moduli ratio $m$ is also presented in Fig. 5.9. Since Poisson's ratios of adjoining materials are assumed to be same, Eq. (5.5) yields a linear relationship between $q_2/q_1$ and $m$. Stress intensity factor criterion predicts larger (smaller) pressure $q_2$ for penetration through the interface when $m<1$ ($m>1$). The generalization of the $J^N$ criterion of Eq. (5.4) is not clear at this stage, due to the variable nature of the singularity at the interface crack tip. Some preliminary considerations and results related with the application of $J^N$ criterion are presented in Appendix D.

Table 5.2 Infinite Bi-Material Data and Response Results for $E_1 = 5 \times 10^5$ psi and $v_1 = v_2 = 0.3$.

<table>
<thead>
<tr>
<th>$m = \nu_2/\nu_1$</th>
<th>0.05</th>
<th>0.10</th>
<th>0.50</th>
<th>1.00</th>
<th>2.50</th>
<th>5.00</th>
<th>7.50</th>
<th>10.00</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_2$ (psix10$^5$)</td>
<td>0.25</td>
<td>0.50</td>
<td>2.50</td>
<td>5.00</td>
<td>12.50</td>
<td>25.00</td>
<td>37.50</td>
<td>50.00</td>
</tr>
<tr>
<td>$K_b$ (psi-in$^P$)</td>
<td>0.20</td>
<td>0.35</td>
<td>1.12</td>
<td>1.75</td>
<td>3.07</td>
<td>4.23</td>
<td>4.90</td>
<td>5.33</td>
</tr>
<tr>
<td>$C_k (q_2/q_1)$ (Eq. 5.3)</td>
<td>8.75</td>
<td>5.00</td>
<td>1.56</td>
<td>1.00</td>
<td>0.57</td>
<td>0.41</td>
<td>0.36</td>
<td>0.33</td>
</tr>
</tbody>
</table>

5.4 HYDRAULIC FRACTURING FIELD EXAMPLES

The heterogeneous, layered nature of geological formations provides various local disturbances which affect the behavior of the hydraulically induced fractures. In addition to layer interfaces, local joints in the rock may interact with a propagating fracture and
influence further extension, either by slippage along the joint plane and/or by causing new fracture orientations. Three cases commonly encountered in the field are simulated in this section. First, a dendritic fracture model is considered. The dendritic fracturing process can be simply defined as potentially creating additional fractures by secondary crack propagation. In this process, fracture fluid injection is stopped, the fluid is allowed to flow back, with the relieved fluid pressure more fracture surfaces are formed.

As a second application, the problem of a crack intersecting an interface joint is considered. In this model, the lower end of the main crack is assumed to be closed and the joint surfaces are allowed to undergo slip without opening. The resulting tear drop fracture profile is studied for different ratios of $\mu_2/\mu_1$.

Finally, the problem of a fractured middle layer (payzone) bounded by upper and lower layers is considered as a plane strain problem. The effects of material property mismatch, horizontal in-situ stress differences and bottom hole treatment pressure are closely studied. Bimaterial stress intensity factors, directional preference for through cleavage with respect to maximum potential energy release rate, fracture height and treatment pressure required are computed for various $\mu_2/\mu_1$ ratios.

5.4.1 Dendritic Fracturing Simulations

Dendritic fracturing process is associated with creating more fractures due to crack branching and hence increasing reservoir productivity. It has been observed [112] that the secondary fracture propagation depends largely on the difference in magnitude between the
principal horizontal in-situ stresses for the process of hydraulic frac­
turing. To obtain a better understanding of the propagation of secondary
and primary cracks, a plane strain dendritic fracture model as shown in
Fig. 5.10 is considered. A typical finite element mesh is shown in
Fig. 5.11. The loading conditions studied are (i) uniform treatment
pressure \( q \), (ii) maximum horizontal in-situ stress \( \sigma_x \) and (iii) minimum
horizontal in-situ stress \( \sigma_y \). The corresponding stress intensity factors
for the primary and the secondary crack tips (crack tips C and B in Fig.
5.10, respectively) for different c/b ratios are presented in Table 5.3.
The stress intensity factor variations with \( \sigma_y/\sigma_x \) for c/b=1.0 and c/b=
1/2 are illustrated in Figs. 5.12(a) and (b), respectively. Figure 5.13
shows the angular variation of the minimum principal stress, \( \sigma_{\text{MIN}} \) at
r/a=0.002 for both the primary and the secondary crack tips.

For zero treatment pressure, Fig. 5.12(a) and (b) reveal that
\( K_1^B > K_1^C \) when \( \sigma_y/\sigma_x \) ratio is greater than 0.44 and 0.30 for c/b ratio of
1/2 and 1, respectively. Hence, crack branching is expected by having
the primary crack tip isolated. Secondary crack propagation is further
encouraged for larger c/b ratios, since a smaller \( \sigma_y/\sigma_x \) ratio is
required for the stress intensity factor at the branching crack tip to
exceed the one at the primary crack tip. For a uniform treatment
pressure of q=1, as the horizontal in-situ stress ratio approaches to
unity, an increased tendency of secondary crack propagation is observed
for all c/b ratios. This tendency is also strengthened by the Mode II
stress intensity factor of the secondary crack tip, \( K_2^B \). These observa­
tions agree very well with the concept of dendritic fracturing, in which
case, when the fracture fluid injection is stopped, the fluid flows back
and the fluid pressure decreases gradually encouraging the branching and secondary crack propagation.

Table 5.3 Normalized Stress Intensity Factors for Dendritic Fracture Model

<table>
<thead>
<tr>
<th>$c/b$</th>
<th>Loading $Q$</th>
<th>$K_1^R/Q^\sqrt{a}$</th>
<th>$K_2^R/Q^\sqrt{a}$</th>
<th>$K_1^C/Q^\sqrt{a}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$q$</td>
<td>0.762</td>
<td>-0.510</td>
<td>---</td>
</tr>
<tr>
<td></td>
<td>$\sigma_x$</td>
<td>-0.460</td>
<td>-0.030</td>
<td>---</td>
</tr>
<tr>
<td></td>
<td>$\sigma_y$</td>
<td>-0.320</td>
<td>0.547</td>
<td>---</td>
</tr>
<tr>
<td>1/4</td>
<td>$q$</td>
<td>0.730</td>
<td>-0.592</td>
<td>1.025</td>
</tr>
<tr>
<td></td>
<td>$\sigma_x$</td>
<td>-0.465</td>
<td>-0.040</td>
<td>0.210</td>
</tr>
<tr>
<td></td>
<td>$\sigma_y$</td>
<td>-0.280</td>
<td>0.630</td>
<td>-1.310</td>
</tr>
<tr>
<td>1/2</td>
<td>$q$</td>
<td>0.672</td>
<td>-0.472</td>
<td>1.471</td>
</tr>
<tr>
<td></td>
<td>$\sigma_x$</td>
<td>-0.473</td>
<td>-0.026</td>
<td>0.193</td>
</tr>
<tr>
<td></td>
<td>$\sigma_y$</td>
<td>-0.217</td>
<td>0.500</td>
<td>-1.681</td>
</tr>
<tr>
<td>3/4</td>
<td>$q$</td>
<td>0.590</td>
<td>-0.300</td>
<td>1.710</td>
</tr>
<tr>
<td></td>
<td>$\sigma_x$</td>
<td>-0.483</td>
<td>-0.013</td>
<td>0.151</td>
</tr>
<tr>
<td></td>
<td>$\sigma_y$</td>
<td>-0.120</td>
<td>0.360</td>
<td>-1.874</td>
</tr>
<tr>
<td>1</td>
<td>$q$</td>
<td>0.489</td>
<td>-0.248</td>
<td>1.839</td>
</tr>
<tr>
<td></td>
<td>$\sigma_x$</td>
<td>-0.488</td>
<td>-0.025</td>
<td>0.107</td>
</tr>
<tr>
<td></td>
<td>$\sigma_y$</td>
<td>-0.021</td>
<td>0.257</td>
<td>-1.966</td>
</tr>
</tbody>
</table>

Since hydraulic fractures propagate in a direction perpendicular to the minimum principal compressive stress, similar observations are obtained from the results presented in Fig. 5.13. The curves in this figure correspond to $c/b=1$ and zero treatment pressure. The minimum principal compressive stress magnitudes at the primary crack tip region are larger than those at the secondary crack tip region for $\sigma_x/\sigma_y<10/3$. Hence, the secondary crack propagation is favored. As $\sigma_x/\sigma_y$ ratio increases, the minimum principal compressive stress magnitudes at the
Figure 5.10  Dendritic Fracture Model with Primary Fracture and Secondary Cracks Under Uniform Treatment Pressure and In-Situ Loading

secondary crack tip region exceeds those at the primary crack tip region isolating the secondary crack tip and encouraging the primary crack propagation. Thus the results revealed by Fig. 5.12 are confirmed. It is well understood here that the success of the dendritic fracturing process very much depends on the accurate estimation and/or in-situ measurement of the magnitudes of the principal horizontal in-situ stresses.
Fig. 5.11 FEM Model for Dendritic Fracturing
Fig. 5.12 Stress Intensity Factor Variations with $\sigma_y/\sigma_x$ for (a) $c/b = 1.0$ and (b) $c/b = 1/2$

Fig. 5.13 Angular Variations of the Minimum Principal Stress, $\sigma_{\text{MIN}}$ at $r/a = 0.002$
5.4.2 **Pressurized Crack Intersecting a Joint**

Cracks induced by hydraulic fracturing may rise to the surface by upper crack tip growth, while the lower tip is closed [108]. It was pointed out by Weertman [109] that this phenomenon is more likely when the density of rock is greater than the density of the fracturing fluid, unless there exists a geological mechanism which will stop the crack from rising. To study the behavior of a pressurized crack interacting with a joint as such a mechanism, the problem of Fig. 5.14 is considered.

![Figure 5.14 Pressurized Crack Intersecting a Joint](image)

The joint is assumed to behave as an interfacial crack and the pressurized crack runs perpendicular to the joint at its midpoint. Due to symmetry about the y-axis, only one half of the model is solved assuming plane strain conditions. Triangular quarter point elements are used to model the lower crack tip region and subparametric semi-radial singularity elements are employed at the interfacial crack tip region.
Following the argument of Keer and Chen [108], relative slip is assumed to occur in the joint, near the upper crack tip. The shear tractions in the joint which resist against the opening of the upper crack tip is assumed to have the form
\[ \zeta(x) = \zeta_0 x \] (5.6)

This linear behavior reflects the loss of frictional resistance due to the leaking of some fracturing fluid into the joint. The crack opening pressure \( q(y) \) is assumed in a linear form as
\[ q(y) = d + ey \] (5.7)

for \( 0 > y > -2a \). Here, \( d \) represents the effective pressure due to the tectonic stresses and the fracturing fluid pressure, and \( e \) reflects the relative hydrostatic pressure effect due to the difference in the densities of rock and fluid.

For certain values of \( d \) and \( e \) in Eq.(5.7), the linear part of the loading may exceed the constant part; hence, causing the lower crack tip to close for a fixed crack length of \( 2a \). Further injection of the fracturing fluid causes slipping along the joint, thus increasing the fracture volume. For nonzero lower crack tip stress intensity factors, fracture growth takes place by simultaneous action of the downward growth and slipping along the joint. In the above discussion, the possibilities of breaking into the upper layer or extension of the interfacial crack are excluded. Obviously, joint tractions of Eq.(5.6) discourages the breaking of the crack into the upper layer, since such a propagation is expected to be governed by the minimum compressive stress in material 2 around \( y=0 \).
Interfacial crack propagation can be studied by employing the analysis of Section 3.2.1 once the strength of the interface is known. In Table 5.4 the values of $\zeta_0$, $\delta$ and $e$ to produce zero lower crack tip stress intensity factor are presented [108]. The same table also shows the computed lower crack tip stress intensity factors for (i) homogeneous medium, where $(\alpha, \beta) = (0,0)$ and (ii) bi-material medium, where $(\alpha, \beta) = (1/2,0)$. The parameters $\alpha$ and $\beta$ are Dundurs' constants and they are defined as

$$\alpha = \frac{m(1+k_1)-(1+k_2)}{m(1+k_1)+(1+k_2)}$$  \hspace{1cm} (5.8)

$$\beta = \frac{m(k_1-1)-(k_2-1)}{m(1+k_1)+(1+k_2)}$$  \hspace{1cm} (5.9)

The $(\alpha, \beta)$ values mentioned above are selected to be able to compare the present finite element results with the results of Ref. [108]. It is noteworthy to mention at this point that vanishing of the constant $\beta$ implies the vanishing of the bi-elastic constant $c$ for the interfacial crack, thus eliminating the oscillatory character of the stresses in the close vicinity of the joint tip. As shown in Table 5.4, the lower crack tip stress intensity factors are quite small, thus causing the crack closure.

The corresponding crack opening displacement profiles obtained by the present method are compared with the results of Keer and Chen [108] in Fig. 5.15, which shows a relatively good agreement.
Fig. 5.15 Crack Surface Displacement Profile for \((\alpha, \beta) = (0,0)\) and \((\alpha, \beta) = (1/2,0)\)
Table 5.4 Interface Joint Parameters for Lower Crack Tip Closure

<table>
<thead>
<tr>
<th>(α, β)</th>
<th>(0,0)</th>
<th>(1/2,0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>ζ₀</td>
<td>0.5277</td>
<td>0.5920</td>
</tr>
<tr>
<td>d</td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td>e</td>
<td>1/3</td>
<td>1/3</td>
</tr>
<tr>
<td>K/d√a</td>
<td>0.0213</td>
<td>0.0161</td>
</tr>
</tbody>
</table>

5.4.3 Hydraulic Fracture Propagation in Layered Media

A two-dimensional representation of a hydraulic fracture embedded in a payzone formation bounded by upper and lower barrier layers is shown in Fig. 5.16. Since the length of the fracture is relatively large when compared to its height, plane strain conditions are assumed to prevail. Different formation in-situ stress conditions are simulated by considering various Δσ/σₕ ratios. Young's modulus for the payzone is assumed to be 10⁷ psi. Poisson's ratio for both the payzone and the barrier layer is taken as 0.3. Hence, Young's modulus for the barrier layers is given by

\[ E_2 = mE_1 \tag{5.10} \]

The assumed model geometry is illustrated in Fig. 5.16. The case of the terminal crack with possible arrest at the interfaces is considered first.

5.4.3.1 Crack arrest at the interface

For a/Hₗ=1.0 and applied crack opening pressure, the total potential energy release rate, corresponding to infinitesimal bi-material crack tip advance, is computed from the finite element analysis of the symmetric half of the model. The angular variation of the nondimensional
Figure 5.16 Vertical Hydraulic Fracture Model in a Layered Medium

Bi-material energy release rate is shown in Fig. 5.17. For all material combinations considered, self-similar through cleavage is the dominating propagation mechanism since the maximum energy release rate is attained at $\theta=0^\circ$. Hence, maximum cleavage stress occurs at $\theta=0^\circ$ ahead of the bi-material crack tip in the barrier layer for symmetric mode.

Considering the symmetry involved, only one quarter of the problem is modelled for the rest of the analysis. The variation of normalized stress intensity factors corresponding to $q, \sigma_H$, and $\Delta \sigma$ loadings are shown in Fig. 5.18. In Fig. 5.19 the fracture profile for unit pressure is presented. If propagation through the interface occurs it will be induced in a self-similar manner with the maximum cleavage stress in the barrier layer at $\theta=0^\circ$. It is assumed that when the quantity $K_D/a^p$
Fig. 5.17 Angular Variation of Non-Dimensional Energy Release Rate
Fig. 5.18 Normalized Bi-Material Stress Intensity Factors for Vertical Fracture Model

Fig. 5.19 Crack Surface Normal Displacement Profiles for $q=1.0$
is larger than $K_{ic}/\sqrt{a}$ of the bounding layers, the fracture will propagate. The effects of the variable horizontal in-situ stress and the crack opening pressure on the bi-material stress intensity factor is studied in Fig. 5.20. For sufficiently large $q/\sigma_H$ ratios, a decreasing $\Delta\sigma/\sigma_H$ effect is observed. When the barrier layers are much stiffer than the fractured layer effect of the differential stress, $\Delta\sigma$ becomes even less significant. However, differential in-situ stress values are always a critical factor for containment when $m<1$.

The maximum cleavage stress in the barrier zone ahead of the bi-material crack tip decreases with increasing $\Delta\sigma/\sigma_H$ ratio. Hence, for all cases, as $\Delta\sigma/\sigma_H$ ratio increases, fracture containment in the payzone layer is encouraged as expected. Figure 5.21 shows the stress intensity factor variations with $\Delta\sigma/\sigma_H$ (Fig. 5.21a) and with $q/\sigma_H$ (Fig. 5.21b). Loading combinations that will cause bi-material crack tip closure are represented in Fig. 5.22. It is observed that the ratio of stress intensity factors due to $\sigma_H$ and $\Delta\sigma$ is constant for all $m$ and even for nonzero fluid pressures, crack closure can happen and in turn may result in crack branching.

5.4.3.2 Fracture Height and Propagation/Containment Pressures

To determine the fracture height and the treatment pressure for the vertical hydraulic fracture model of Fig. 5.16, various $a/H_1$ ratios are considered. Table 5.5 presents the normalized stress intensity factors corresponding to $q$, $\sigma_H$ and $\Delta\sigma$ loadings. The variation of stress intensity factors as the fracture in the payzone layer propagates towards the interface and penetrates into the barrier layers is shown in Fig.
Fig. 5.20 Normalized Bi-Material Stress Intensity Factor Variations with m for (a) \( |q/o_H| = 0.8 \), (b) \( |q/o_H| = 1.0 \), (c) \( |q/o_H| = 1.6 \) \( (o_H = -1.0) \).
Fig. 5.21  Normalized Bi-Material Stress Intensity Factor Variations with (a) $\Delta \sigma / \sigma_H$, and (b) $q / \sigma_H$ ($\sigma_H = 1.0$).
Fig. 5.22 Loading Combinations for $K_b = 0$
5.23 for \( q \) and \( \sigma_H \) loadings. The qualitative behavior for \( \Delta \sigma \) loading is the same as \( \sigma_H \) loading.

Table 5.5 Normalized Stress Intensity Factors; \( K_p/q a^{\delta/\sqrt{\pi}} \) for Crack Approaching and Penetrating the Interface (\( p=1/2, a H_1^{1/4} \))

<table>
<thead>
<tr>
<th>Loading</th>
<th>( m a/H_1 )</th>
<th>0.50</th>
<th>0.70</th>
<th>0.90</th>
<th>0.95</th>
<th>1.00</th>
<th>1.05</th>
<th>1.10</th>
<th>1.30</th>
<th>1.50</th>
</tr>
</thead>
<tbody>
<tr>
<td>( q )</td>
<td>0.10</td>
<td>1.09</td>
<td>1.30</td>
<td>1.93</td>
<td>2.43</td>
<td>0.30</td>
<td>0.94</td>
<td>0.93</td>
<td>0.85</td>
<td>0.86</td>
</tr>
<tr>
<td></td>
<td>0.50</td>
<td>1.01</td>
<td>1.07</td>
<td>1.19</td>
<td>1.30</td>
<td>0.68</td>
<td>0.93</td>
<td>0.93</td>
<td>0.93</td>
<td>0.94</td>
</tr>
<tr>
<td></td>
<td>1.00</td>
<td>0.98</td>
<td>0.98</td>
<td>0.98</td>
<td>0.99</td>
<td>0.99</td>
<td>0.99</td>
<td>0.99</td>
<td>0.99</td>
<td>1.00</td>
</tr>
<tr>
<td></td>
<td>2.50</td>
<td>0.91</td>
<td>0.86</td>
<td>0.76</td>
<td>0.68</td>
<td>1.60</td>
<td>1.02</td>
<td>1.02</td>
<td>1.09</td>
<td>1.09</td>
</tr>
<tr>
<td></td>
<td>5.00</td>
<td>0.89</td>
<td>0.81</td>
<td>0.66</td>
<td>0.56</td>
<td>2.17</td>
<td>1.06</td>
<td>1.04</td>
<td>1.15</td>
<td>1.14</td>
</tr>
<tr>
<td>( \sigma_H )</td>
<td>0.10</td>
<td>3.74</td>
<td>3.34</td>
<td>6.38</td>
<td>7.97</td>
<td>1.01</td>
<td>2.55</td>
<td>2.30</td>
<td>1.69</td>
<td>1.49</td>
</tr>
<tr>
<td></td>
<td>0.50</td>
<td>1.77</td>
<td>1.86</td>
<td>2.09</td>
<td>2.23</td>
<td>1.16</td>
<td>1.43</td>
<td>1.39</td>
<td>1.24</td>
<td>1.19</td>
</tr>
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<tr>
<td></td>
<td>2.50</td>
<td>0.44</td>
<td>0.41</td>
<td>0.36</td>
<td>0.33</td>
<td>0.76</td>
<td>0.65</td>
<td>0.71</td>
<td>0.85</td>
<td>0.91</td>
</tr>
<tr>
<td></td>
<td>5.00</td>
<td>0.21</td>
<td>0.19</td>
<td>0.16</td>
<td>0.14</td>
<td>0.51</td>
<td>0.49</td>
<td>0.57</td>
<td>0.76</td>
<td>0.86</td>
</tr>
<tr>
<td>( \Delta \sigma )</td>
<td>0.10</td>
<td>2.85</td>
<td>3.34</td>
<td>4.87</td>
<td>6.08</td>
<td>0.77</td>
<td>1.95</td>
<td>1.76</td>
<td>1.30</td>
<td>1.16</td>
</tr>
<tr>
<td></td>
<td>0.50</td>
<td>1.37</td>
<td>1.44</td>
<td>1.62</td>
<td>1.73</td>
<td>0.90</td>
<td>1.11</td>
<td>1.09</td>
<td>0.97</td>
<td>0.93</td>
</tr>
<tr>
<td></td>
<td>1.00</td>
<td>0.81</td>
<td>0.81</td>
<td>0.81</td>
<td>0.81</td>
<td>0.82</td>
<td>0.82</td>
<td>0.82</td>
<td>0.82</td>
<td>0.83</td>
</tr>
<tr>
<td></td>
<td>2.50</td>
<td>0.36</td>
<td>0.34</td>
<td>0.30</td>
<td>0.27</td>
<td>0.63</td>
<td>0.53</td>
<td>0.57</td>
<td>0.68</td>
<td>0.73</td>
</tr>
<tr>
<td></td>
<td>5.00</td>
<td>0.19</td>
<td>0.17</td>
<td>0.14</td>
<td>0.12</td>
<td>0.45</td>
<td>0.40</td>
<td>0.46</td>
<td>0.62</td>
<td>0.69</td>
</tr>
</tbody>
</table>

As the interface is approached from the stiffer (softer) material side, stress intensity factor increases (decreases) for both the crack surface and exterior loading configurations. At the interface, the stress intensity factor is finite. As the crack penetrates into the barrier zone, the behavior for crack surface pressure and external loadings are observed to be different. The stress intensity factor due
Fig. 5.23 Normalized Stress Intensity Factor Variations as the Crack Approaches and Penetrates the Interface, (a) $q$-Loading, (b) $\sigma_H$-Loading
to applied pressure loading is larger for \( m>1 \) than for \( m<1 \). This behavior is reversed for external loading. Despite a sudden decrease (increase) in the stiffer (softer) barrier layer for \( q \) loading, immediately after the interface a more subtle behavior is observed for external loading. As \( a/H_1 \) becomes sufficiently large, the homogeneous medium behavior is approached with decreasing influences exerted by the material property mismatch. In Fig. 5.24 the variation of stress intensity factors with respect to \( m \) for crack surface pressure and external boundary loadings are shown. The terminal crack case \( (a/H_1=1.0) \) does not necessarily follow the response pattern of the cases corresponding to \( a/H_1 \neq 1.0 \).

In Fig. 5.25 the non-dimensional stress intensity factor variations corresponding to various \( q/\sigma_\infty \) and \( \Delta \sigma/\sigma_\infty \) ratios are shown. For stiffer barrier layers, if the penetration of the interface is realized, the differential stress \( \Delta \sigma \) always retards crack propagation and eventually leads to crack arrest. When interface penetration occurs in the softer barrier layer, an increasing \( \Delta \sigma \) decreases stress intensity factor, leading to a possibility of containment in the barrier zone. For relatively larger \( q/\sigma_\infty \) ratios, differential in-situ stress effects become less significant.

In order to be able to predict the bottom hole treatment pressure required for containment and/or corresponding fracture height, the principle of superposition is exploited. For the loading configurations, \( q, \sigma_\infty \) and \( \Delta \sigma \) of Fig. 5.16, we define

\[
f_a = \frac{K}{qa} \sqrt{\pi}, \quad f_b = \frac{K}{\sigma_\infty a} \sqrt{\pi}, \quad f_c = \frac{K}{\Delta \sigma a} \sqrt{\pi}
\]  

(5.11)
Fig. 5.24 Normalized Stress Intensity Factor Variations with $m$ (a) $q$-Loading, (b) $\sigma_H$-Loading
Fig. 5.25 Non-Dimensional Stress Intensity Factor Variation with $a/H_1$ for (a) $|q/\sigma_H|=1.2$, (b) $|q/\sigma_H|=1.6$
which correspond to the plots presented in Fig. 5.23. From the super-
position, the treatment pressure is obtained via the relation

$$q_a f_a + \sigma_H f_b + \Delta \sigma f_c = \left( \frac{K_{IC}}{a} \right)_1 \sqrt{a \pi}$$

(5.12)

where $i=1$ for $a/H_1 < 1$ and $i=2$ for $a/H_1 > 1$. Hence, the normalized treatment
pressure is given by

$$\frac{q\sqrt{H_1}}{(K_{IC})_i} = \frac{R_B}{f_a} \left\{ \sqrt{\frac{H_1}{\pi a}} \left[ \frac{1}{R_B} - f_b - \frac{(\Delta \sigma)}{\sigma_H} f_c \right] \right\}$$

(5.13)

where

$$R_B = \frac{\sigma_H \sqrt{H_1}}{(K_{IC})_i}$$

(5.14)

For various values of $R_B$, $\Delta \sigma/\sigma_H$ and $\mu_2/\mu_1$ the treatment pressure vari-
ations are shown in Figs. 5.26 through 5.29. Negative $\Delta \sigma/\sigma_H$ ratios are
also included, since for $m<1$, differential in-situ stress in the barrier
layers can be smaller than the one in the payzone layer. These figures
can be used as design curves to define the range of injection pressures
for hydraulic fracturing operations. Our analysis indicates that
within this range of treatment pressure, fracture migration into the
barrier layers is prevented and fracture propagates laterally. If,
somehow, the fracture penetrates into the barrier layers, differential
in-situ stress $\Delta \sigma$ significantly dominates the magnitude of the treatment
pressure $q$ and containment is still possible. A simple observation of
Figs. 5.26 through 5.29 indicates that when $a/H_1 > 1$, a small increment of
the treatment pressure causes a large increase of fracture heights for
low $\Delta \sigma/\sigma_H$ values. This behavior is more pronounced for $m>1$ than for
$m<1$. Thus, the fracture height corresponding to a certain treatment
Fig. 5.26 Treatment Pressure Versus Fracture Containment, a/H₁ for m=0.1
Fig. 5.27 Treatment Pressure Versus Fracture Containment, $a/H_1$ for $m=0.5$
Fig. 5.28 Treatment Pressure Versus Fracture Containment, $a/H_1$ for $m=2.5$
Fig. 5.29 Treatment Pressure Versus Fracture Containment, $a/H_1$ for $m=5.0$
pressure and horizontal in-situ stress ratio can also be obtained from Figs. 5.26 through 5.29. We emphasize here that the differential in-situ stress, along with the material property mismatch, is the primary factor governing the hydraulic fracture propagation/containment phenomenon.

In this analysis it is assumed that the interfaces are perfectly bonded. Hence, when the crack tip is in the payzone layer \((a/H_l < 1)\), the stress intensity factor is still influenced by the horizontal in-situ stress acting on the barrier layers. Chang [73] assumes that when the crack is in the payzone layer \((a/H_l < 1)\), the stress intensity factor is independent of the loading conditions on the barrier layers. Hence, the propagation is governed by the fracture toughness of the payzone and the stress intensity factor due to the effective pressure (pumping pressure minus the minimum in-situ stress in the payzone). Obviously, such an analysis reveals the effect of differential in-situ stress only after the interface is crossed. Furthermore, the stress intensity factor, when the crack tip is at the interface, is not defined in Ref. [73] and the results presented are for stiffer barrier layers only \((m>1)\).

As an example of the application of the proposed model simulations, Fig. 5.30 is presented. Another class of applications are considered next. Referring to the fracture geometry shown in Fig. 5.16 and Eq. (5.13), we consider the effects of in-situ horizontal stress differential \((\Delta \sigma)\) on the treatment pressure depending on the location of the crack tip. Selected parameters are

\[
E_1 = 1 \times 10^7 \, \text{psi} \quad , \quad (K_{Ic})_1 = 740 \, \text{psi} - \sqrt{\text{in}}
\]

\[
E_2 = mE_1 \quad , \quad (K_{Ic})_2 = 1000 \, \text{psi} - \sqrt{\text{in}}
\]
\[ v_1 = v_2 = 0.2 \]

\[ \sigma_H = 1589 \text{ psi} \]

The particular value of the horizontal stress is chosen to compare the present results with other known solutions for special cases.

Normalized treatment pressure variations for various horizontal in-situ stress differentials are shown in Fig. 5.30 for two different pay zone heights, which show a possibility of using simple extrapolation to compute the pressure for different values of \( \Delta \sigma \).

Table 5.6 shows the treatment pressure ratio (\( q/\sigma_H \)) for some special cases and compares with the results of Simonson et al. [67] and Chang [73]. The results are based on the fracture toughness of 1000 psi-\(\sqrt{\text{in}}\), which indicate an increase in the treatment pressure if the fracture extends into a higher in-situ stress formation. Apparently the present method yields higher propagation pressure than others when the crack tip is in the barrier layer with \( \Delta \sigma \neq 0 \). It is also observed that, for a larger payzone height, it is easier for the fracture to propagate toward the interface. However, stress containment is more likely as the crack tip intersects the interface. This conclusion can be confirmed by the steeper slopes of the curves in Fig. 5.30(b).
Table 5.6 Treatment Pressure to Minimum Horizontal In-Situ Stress Ratios ($q/o_H$) for Fracture Propagation/Containment ($\sigma_H = 1589$ psi, $K_{IC} = 1000$ psi-√in)

<table>
<thead>
<tr>
<th>$m$</th>
<th>$2H_1$</th>
<th>200 ft</th>
<th>400 ft</th>
<th>200 ft</th>
<th>400 ft</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$a/H_1$</td>
<td>1.0</td>
<td>1.5</td>
<td>1.0</td>
<td>1.5</td>
</tr>
<tr>
<td></td>
<td>$(\Delta\sigma=0)$</td>
<td>$(\Delta\sigma=1100$psi)</td>
<td>$(\Delta\sigma=0)$</td>
<td>$(\Delta\sigma=1100$psi)</td>
<td>$(\Delta\sigma=1100$psi)</td>
</tr>
<tr>
<td>Simonson [67]</td>
<td>1.010</td>
<td>1.252</td>
<td>1.007</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Present</td>
<td>1.010</td>
<td>1.562</td>
<td>1.007</td>
<td>1.559</td>
<td>1.256</td>
</tr>
<tr>
<td>Chang [73]</td>
<td>1.010</td>
<td>1.379</td>
<td>1.007</td>
<td>1.290</td>
<td>1.113</td>
</tr>
</tbody>
</table>
Fig. 5.30 Normalized Treatment Pressure for Fracture Height Estimation Corresponding to m=3.0 and (a) \( H_1 = 100 \) ft, (b) \( H_1 = 200 \) ft.
CHAPTER VI

CONCLUSIONS AND RECOMMENDATIONS

In this dissertation, finite element method evaluations and investigations of fracture criteria for LEFM problems in homogeneous isotropic and bi-material media are conducted. For a crack lying along the interface of two bonded half planes, a fracture criterion in terms of fracture envelopes is presented. The solution to the problem of a crack oriented perpendicular to and terminating at a bi-material interface is formulated in terms of the bi-material stress intensity factors. Corresponding fracture envelopes representing possible modes of failure are presented. Although the applications offered for consideration are primarily relevant to the analysis of hydraulic fracturing, various other structural mechanics problems can be similarly analyzed.

The singular finite elements presented in Chapter III are modified versions of the quarter point mapping [78], variable displacement singularity [41] and semi-radial singularity mapping [43] elements. For the sake of completeness, the shape function modification technique suggested in [80] for the degenerate quarter point triangular element is extended to degenerate quarter point "pie" elements in three dimensions for crack problems in homogeneous media. The variable displacement singularity element of Akin [41] is observed to behave very poorly in its degenerate triangular form. Several degenerating schemes were
examined. Considerable improvement in this element's response is obtained for a particular degenerating scheme of collapsing the nodes at points away from the singular crack tip and not having common collapsed nodes for two adjacent elements. Since the required strain singularity is obtained via the singular displacement interpolation, standard interpolation functions were expected to suffice for the geometric mapping. However, superior results are obtained by utilizing an isoparametric representation. In addition, the shape function modification discussed in Section 2.2.1 is found to be necessary to assure uniform variation of displacement fields, thereby further improving numerical results.

The semi-radial singularity mapping technique introduced by Okabe [43] is utilized to obtain the subparametric singular element of Section 2.2.2. Here the desired strain singularity is obtained through the geometric mapping, as was the case in the quarter point element [33]. Hence, standard quadratic interpolation functions are used to describe the displacement field with the restriction of straight element sides. This element is observed to be insensitive to the degenerating scheme.

The nine-point Gauss-Legendre quadrature rule [83] is used to numerically integrate the element stiffness for all singularity elements studied. The displacement solutions, both for the subparametric semi-radial singularity element and the modified variable singularity element, are found to be slightly overestimated. This behavior is attributed to the approximate nature of the numerical integration scheme used. An nth order Gauss-Legendre quadrature rule integrates a polynomial of degree (2n-1) exactly whereas in variable singularity elements, the
integrations are expressed by terms with local coordinates raised to powers p and their multiples. Since p is in the real interval [0,1], use of standard numerical integration techniques does not guarantee exact integration of the element stiffness. The sensitivity of the variable singularity element due to different schemes of degeneracy is believed to be intensified by the approximate nature of the numerical integration employed. Nevertheless, the results obtained by using both the subparametric element and the modified variable singularity element with proper degeneracy indicate that the error involved is less than 4%. Accuracy can be improved by further mesh refinement.

Fracture analysis parameters, such as stress intensity factors, are obtained by correlating the analytical and numerical displacement expressions. For the problem of a crack along the interface of two bonded materials, both the crack opening and sliding displacements are coupled in the specific expression for the stress intensity factor (e.g. $K_1$ or $K_2$). It is demonstrated in Section 4.2.3 that for $K_1$ dominant problems, the abovementioned correlation technique overestimates $K_2$ considerably. The technique presented in Subsection 2.3.2.1 overcomes this difficulty and yields very good results for both $K_1$ and $K_2$.

For a terminal crack perpendicular to a bi-material interface due to the isoparametric nature of the variable displacement singularity element, the numerical displacement expression is complicated and the coefficient of the dominant singular term has to be obtained from a power series expansion. Thus, the error in the stress intensity factor value, derived from correlating this coefficient with the analytical
expression, increases. The subparametric semi-radial singularity element is therefore used for the model simulations of Chapter V.

For bi-material crack problems, it is crucial to predict if the interface serves as an arrest mechanism. In Chapter III, fracture envelopes for propagation into the adjacent medium or interface failure with emphasis on frictional effects are presented in terms of the stress intensity factors $K_i$, the resistance of the materials $(K_{ic})_i$ (i=1,2) and interfacial strength, $K_d$. For both interfacial and terminal bi-material crack problems, the use of such criteria requires knowledge of a characteristic fracture process zone size. At the critical distance from the crack tip, characterized by the process zone boundary, the stresses or their critical combinations are checked against the strength parameters of the materials or the interface itself.

The fracture process zone ahead of the crack tip in metals is largely due to yielding [110], which can be predicted by the theory of LEFM. In rock, the process zone is believed to be due to tensile micro-cracking rather than due to plastic deformation [75]. Hence, to justify the applicability of fracture parameters, such as elastic stress intensity factors and fracture toughness, the characteristic distance considered should be beyond the process zone. Experimental observations would be highly desirable for the justification. However, no such experimental findings for bi-material terminal crack problems considering the fracture process zone have been encountered in the available literature. To shed some light on this issue, the bi-material plastic zone size and shape for metals are estimated from the elastic solution of a terminal crack problem and presented in Appendix C. Similarly, the
bi-material process zone for rock, based on microcracking is also presented in Appendix C. The process zone shapes are observed to be double-lobed regions with discontinuities along the interface.

The presented analysis indicates that both the plastic zone and the microcrack zone sizes are larger for the stiffer material than for the softer material. It is suggested here that such a characteristic distance associated with the interfacial damage must be based on the properties of the stiffer material. For propagation into one of the adjacent media, use of a characteristic distance based on the properties of a single material would yield superficial results. The process zone size of the material in which propagation is expected should be the lower limit for the characteristic dimension.

The effect of compression induced friction on the process zone in homogeneous rock is studied by considering Mohr-Coulomb criterion of failure [111]. The fracture process zone due to shear failure is observed to be much smaller than both the plastic and microcrack zones, with coefficient of friction controlling the zone size. A simple observation of the case for which the friction coefficient $\mu_f = 0$ indicates that frictional effects tend to decrease the process zone size.

It was shown [92] that the conventional J-integral conservation law is valid for a bi-material crack along a straight bondline. Also, the M-integral conservation law extends without change to the interfacial crack case. This integral is related to the J-integral in a manner similar to the homogeneous medium. For a terminal crack perpendicular to the bi-material interface, $J^n$ and $M^n$ conservation laws are given in Section 3.2.3. It was also stated that the path independence
of $J^n$ integral is of a different nature than the conventional $J$-integral. Considering a small circle $\partial R^*$ with radius $r_o$ surrounding the bi-material crack tip, we have shown that the value of $J^n$ integral, for the path $\partial R^*$ and for any other path surrounding the bi-material crack tip and connecting to $\partial R^*$ along the interface, are equal. By collapsing the contour $\partial R^*$ to the bi-material crack tip, $J^n$ becomes either unbounded ($m<l$) or vanishes ($m>l$). Numerical results (c.f. Section 4.2.4) are in good agreement with the analytical findings. It was observed in the bi-material terminal crack model of Section 4.2.4 that when the material properties of two adjacent media are interchanged, the corresponding $J^n$ values became equal for a small, fixed contour $\partial R^*$. Further preliminary numerical efforts revealed that for other material property pairs if not equal, $J^n$ for interchanged material property states become very close to each other for a distinct contour $\partial R^*$, that varies with the shear moduli ratio $m$. Encouraged with this behavior, some results related with bi-material $J^n$ integral for a small contour $\partial R^*$, are presented in Appendix D. Through the application of Green's theorem on a modified form of the M-integral, a new path independent integral for bi-material perpendicular cracks is also proposed in Appendix D.

Several model simulations for hydraulic fracturing problems are presented in Chapter V. Local fractures/interfaces in the geological environment, such as joints in rock, may serve as effective control mechanisms in the propagation of hydraulic fractures. A typical case of a crack intersecting an interfacial joint is presented in Section 5.4.2. The joint is treated as a closed interfacial crack. It is allowed to slip, under the effect of pressure in the main crack,
resisted by shear tractions on the joint surfaces. A teardrop shaped fracture results as long as the lower homogeneous crack tip does not propagate and the main fracture does not break through the joint into the bounding layer. The possibility of crack initiation from the joint tip and its orientation can be studied by using the fracture criteria of Section 3.2.1. If an inclined fracture terminates at an angle to the bi-material interface, the mixed mode fracture criteria of Section 3.2.2 can be extended to study the terminal bi-material inclined crack problem. Dendritic fracture method, associated with creating more fractures by crack branching, is considered as a means of increasing the reservoir productivity. Secondary crack propagation is observed to depend largely on the principal horizontal in-situ stress magnitudes. As the ratio of the minimum to maximum horizontal in-situ stress magnitudes approaches one, the secondary crack propagation becomes more likely, due to the isolation of the primary crack.

Along with the analysis of possible hydraulic fracture orientations in the geological environment, our primary objective was to predict the fracture height and width for a given bottom hole treatment pressure, material property mismatch and differential horizontal in-situ stress loading. On the same token, the treatment pressure required to maintain arrest at a certain fracture height or propagation beyond is also sought. In the problem of Section 5.4.3, self-similar vertical fracture extension at a relatively large depth is considered. By assuming perfect bonding of the layer interfaces, the effects of material property mismatch, differential in-situ stress magnitudes and uniform treatment pressure are coupled to obtain the design curves presented in Section
5.4.3. The perfectly bonded interface assumption does not permit interfacial slip; hence, the displacements and the normal and shear components of stresses are continuous across the interface. Due to the potential for self-similar fracture growth, the maximum tensile stress criterion is formulated in terms of the opening mode stress intensity factor at the bi-material crack tip. Thus, the possible modes of failure are fracture propagation across the layer interface or arrest at the interface by the combined effects of material mismatch and in-situ stress differences. At shallow depths, however, the possibility of interface failure can not be ignored and a combined Mode I, Mode II type failure criterion must be employed.

The fracture criterion of Subsection 5.4.3.2 implies that the loading conditions on the barrier layers affect the stress intensity factor corresponding to the fracture embedded in the payzone layer. In the design curves of Figs. 5.26 through 5.29, this behavior is incorporated. On the other hand, Chang [73] assumes that the propagation is governed only by the properties of the layer the fracture is in and the in-situ stresses acting on that layer. Hence, differential in-situ stress effect becomes significant only after the fracture crosses the interface. Simonson et al. [67] presented a similar criterion but their applications were limited to the prediction of treatment pressure magnitudes required for the fracture to propagate into the higher in-situ stress region for a homogeneous domain. Recently, Teufel and Clark [66] conducted laboratory experiments and elastic finite element studies on layered rock formations. They have reported that the two distinct geological conditions controlling the vertical fracture growth or containment
in layered rock are (i) a weak interfacial shear strength of the layers and (ii) the differential horizontal in-situ stress magnitudes in the barrier layers. Their studies indicate that the interfacial containment becomes possible when the shear strength of the layer interface is sufficiently weak relative to the tensile strength and the minimum horizontal compressive stress in the bounding layers. This condition is more likely to occur at shallow depths. At greater depths, an increase in the minimum horizontal stress in the barrier layers may contain the fracture, resulting in a vertical rectangular fracture bounded by the high stress formations. Hence, material property mismatch does not serve as an effective containment mechanism by itself but it can affect the fracture growth indirectly by affecting the minimum horizontal in-situ stresses induced in the payzone and the barrier layers.

Due to the geological discontinuities, such as joints, faults, bedding planes and the effects of porosity and permeability of the layer formations, the fracturing fluid may leak off and the proppant transport may be reduced, decreasing the fracture length. The possible creation of multiple or secondary fractures and the plasticity of the shale layers may restrict the fracture height. Hence, the present analysis is by no means complete and the hydraulic fracture propagation/containment pressures predicted can serve as a lower estimate.

For future research the following recommendations are made to increase the efficiency and accuracy of the finite element method analysis and to have a better understanding of the bi-material fracture
systems:

i) The sensitivity of the variable displacement singularity element to all possible degenerating schemes must be corrected.

ii) An efficient and accurate numerical integration scheme for the integration of singular element stiffnesses must be developed.

iii) Fracture criteria of Section 3.2 must be verified using appropriate experimental results. The characteristic distance required for the applicability of such criteria must be determined experimentally for various material combinations.

iv) Fracture criteria of Subsection 5.4.3.2 for self-similar crack growth and design curves of Figs. 5.26 through 5.29 must be justified by experiments and/or field measurements. Recent work of Teufel and Clark [66] presents such experimental results along with finite element analysis of their experimental models.

v) Development of a general, energy based failure criterion for bi-material fracture systems capable of representing local effects such as friction, joint-crack interactions, etc.

vi) Further investigation of the bi-material path independent integrals, such as extending the M-integral and preferably establishing analytical relationships between the bi-material stress intensity factors and such integrals.

vii) Improving the hydraulic fracturing analysis and design by coupling the effects of material mismatch, differential horizontal in-situ stress with frictional interfaces and variable treatment pressure along the height of hydraulic fracture.
viii) Extension of the above suggestions to handle material anisotropy and nonlinearities.


APPENDIX A

STRESS AND DISPLACEMENT FIELDS NEAR THE CRACK TIP

A.1 HOMOGENEOUS MEDIA IN TWO DIMENSIONS

In the immediate vicinity of the crack tip, for the coordinate system of Fig. A.1 and for the modes I and II, we have [110]

\[
\begin{align}
\sigma_r & = \frac{K_1}{(2\pi r)^{1/2}} A_r(\theta) + \frac{K_2}{(2\pi r)^{1/2}} B_r(\theta) \\
\sigma_\theta & = \frac{K_1}{(2\pi r)^{1/2}} A_\theta(\theta) + \frac{K_2}{(2\pi r)^{1/2}} B_\theta(\theta) \\
\sigma_{r\theta} & = \frac{K_1}{2\mu(2\pi r^{1/2})} C_r(\theta) + \frac{K_2}{2\mu(2\pi r^{1/2})} C_\theta(\theta) \\
\end{align}
\]

(A.1)

\[
\begin{align}
u_r & = \frac{K_1}{2\mu(2\pi r^{1/2})} \frac{1}{2} \cos(\theta/2) \left[ 1 + \sin^2(\theta/2) \right] \\
u_\theta & = \frac{K_2}{2\mu(2\pi r^{1/2})} \frac{1}{2} \cos^3(\theta/2) \\
\end{align}
\]

(A.2)

\[
\begin{align}
\sigma_z & = \nu(\sigma_r + \sigma_\theta) \\
\sigma_{rz} & = \sigma_{\theta z} = 0 \\
u_z & = -\frac{\nu^2}{E} (\sigma_r + \sigma_\theta) \\
\end{align}
\]

(A.3)

(A.4)

(A.5)

where

\[
\begin{align}
A_r(\theta) &= \begin{cases} \cos(\theta/2) \left[ 1 + \sin^2(\theta/2) \right] \\
A_\theta(\theta) &= \cos^3(\theta/2) \\
A_{r\theta}(\theta) &= \sin(\theta/2) \cos^2(\theta/2) \\
\end{cases} \\
B_r(\theta) &= \begin{cases} \sin(\theta/2) \left[ 1 - 3\sin^2(\theta/2) \right] \\
B_\theta(\theta) &= -3\sin(\theta/2) \cos^2(\theta/2) \\
B_{r\theta}(\theta) &= \cos(\theta/2) \left[ 1 - 3\sin^2(\theta/2) \right] \\
\end{cases} \\
\end{align}
\]

(A.6)

(A.7)
\[
\begin{align*}
\left\{ C_r(\theta) \right\} & = \left\{ \cos(\theta/2)[(k-1)+2\sin^2(\theta/2)] \right\} \\
\left\{ C_\theta(\theta) \right\} & = \left\{ \sin(\theta/2)[-(k+1)+2\cos^2(\theta/2)] \right\} \\
\left\{ D_r(\theta) \right\} & = \left\{ \sin(\theta/2)[-(k+1)+6\cos^2(\theta/2)] \right\} \\
\left\{ D_\theta(\theta) \right\} & = \left\{ \cos(\theta/2)[-(k-1)-6\sin^2(\theta/2)] \right\}
\end{align*}
\] (A.8)

Here, \( E \) is Young's modulus, \( \nu \), is Poisson's ratio, \( \mu=E/2(1+\nu) \) is shear modulus, \( k=3-4h \) and we take

\[ h=\nu \quad , \quad \nu'=\nu \quad , \quad \nu''=0 \quad \text{(plane strain)} \]

\[ h= \nu/(1+\nu), \quad \nu'=0 \quad , \quad \nu''=\nu \quad \text{(plane stress)} \]

For Mode III we have

\[
\begin{align*}
\left\{ \sigma_{rz} \right\} & = \frac{K_3}{(2\pi)^{1/2}} \left\{ \sin(\theta/2) \right\} \\
\left\{ \sigma_{\theta z} \right\} & = \frac{K_3}{(2\pi)^{1/2}} \left\{ \cos(\theta/2) \right\}
\end{align*}
\] (A.10)

\[
\begin{align*}
\sigma_r = \sigma_\theta = 0 \\
\sigma_r = \sigma_\theta = 0 \\
\sigma_r = \sigma_\theta = 0 \\
\sigma_r = \sigma_\theta = 0 \\
\sigma_r = \sigma_\theta = 0
\end{align*}
\] (A.11)

\[
\begin{align*}
u_r = u_\theta = 0 \\
u_r = u_\theta = 0 \\
u_r = u_\theta = 0 \\
u_r = u_\theta = 0 \\
u_r = u_\theta = 0
\end{align*}
\] (A.12)

\[
\begin{align*}
v_z & = \frac{K_3}{2\mu(2\pi)^{1/2}} \sin(\theta/2) \\
v_z & = \frac{K_3}{2\mu(2\pi)^{1/2}} \sin(\theta/2)
\end{align*}
\] (A.13)

\[\text{Figure A.1 Crack tip polar coordinates.}\]
A.2 HOMOGENEOUS MEDIA IN THREE DIMENSIONS

In order to study the crack propagation problem for a general state of stress, a continuum model has to be generalized in which the crack has some suitable three-dimensional shape. It was shown [104] that an elliptical surface of discontinuity was a general and realistic model shape. By changing the ellipticity, crack border curvedness of various degrees may be realized. This implies that the qualitative character of the stress state near an elliptical crack remains valid for any three-dimensional crack, bounded by a smooth curve [105].

The displacement and the stress fields around the periphery of an elliptical plane of discontinuity in an infinite solid were derived by Kassir and Sih [105]. Consider the triply orthogonal system \( x, y, z \) in Fig. A2, where the origin 0 traces the periphery of the elliptical crack. For a plane normal to the crack border, we have

\[
\begin{align*}
\sigma_x &= \frac{1}{4\sqrt{2\pi}} \left\{ K_1 \cos \frac{\theta}{2} (1-2\nu) + \sin^2 \frac{\theta}{2} \right\} + O(R) \\
\sigma_y &= \frac{K_3}{2\mu} \sqrt{\frac{2R}{\pi}} \sin^2 \frac{\theta}{2} + O(R) \\
\sigma_z &= \frac{1}{4\sqrt{2\pi}} \left\{ K_1 \sin \frac{\theta}{2} (2(1-\nu) - \cos^2 \frac{\theta}{2}) - K_2 \cos \frac{\theta}{2} (1-2\nu) - \sin^2 \frac{\theta}{2} \right\} + O(R) \\
\sigma_yz &= \frac{K_3}{\sqrt{2\pi}} \cos \frac{\theta}{2} + O(R^0)
\end{align*}
\]
Fig. A.2 Stress State Near Crack Border
\[ \sigma_{xz} = \frac{1}{4\sqrt{2\pi R}} \left( K_1 \left( \sin^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2} \right) + K_2 \left( 3\cos^2 \frac{\theta}{2} + \cos^2 \frac{\theta}{2} \right) \right) + O(R^0) \quad \text{(A.21)} \]

\[ \sigma_{xy} = \frac{-K_3}{\sqrt{2\pi R}} \sin \frac{\theta}{2} + O(R^0) \quad \text{(A.22)} \]

where \( K_1, K_2 \) and \( K_3 \) are the stress intensity factors, corresponding to opening, sliding and tearing modes of fracture, respectively and

\[ x = R \cos \theta \]

\[ y = y \quad \text{(A.23)} \]

\[ z = R \sin \theta \]

A.3 INTERFACIAL CRACK IN TWO DIMENSIONS

For the geometry and local coordinates of Fig. 2.10, equilibrium solutions to the equations of plane elasticity without body forces yield [89]

\[ \begin{pmatrix} \sigma_{1r} \\ \sigma_{1\theta} \\ \sigma_{r1} \end{pmatrix} = \frac{e^{-\epsilon \theta}}{2(2\pi \gamma)^{1/2}} \begin{bmatrix} F_{11} & G_{11} \\ F_{12} & G_{12} \\ F_{13} & G_{13} \end{bmatrix} \begin{bmatrix} C & S \\ -S & C \end{bmatrix} \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} \quad \text{(A.24)} \]

\[ \begin{pmatrix} u_{1r} \\ u_{1\theta} \end{pmatrix} = \frac{e^{-\epsilon \theta}}{4\mu_1(1+4\epsilon^2)} \left( \frac{1}{2\pi \gamma} \right)^{1/2} \begin{bmatrix} H_{11} & P_{11} \\ H_{12} & P_{12} \end{bmatrix} \begin{bmatrix} C & S \\ -S & C \end{bmatrix} \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} \quad \text{(A.25)} \]

where \( i=1,2 \) for material 1 and material 2, respectively, and

\[ F_{11} = \cos(\theta/2)[2+\gamma e^{2\epsilon \theta}+(1-2\gamma e^{2\epsilon \theta})\cos\theta-2\epsilon \sin \theta] \]

\[ F_{21} = \cos(\theta/2)[2\gamma e^{2\epsilon \theta}+(\gamma-2\epsilon e^{2\epsilon \theta})\cos\theta-2\epsilon \gamma \sin \theta] \]

\[ F_{12} = \cos(\theta/2)[2-\gamma e^{2\epsilon \theta}-(1-2\gamma e^{2\epsilon \theta})\cos\theta+2\epsilon \sin \theta] \quad \text{(A.26)} \]
\[ F_{22} = \cos(\theta/2)\left[2\gamma - e^{2\gamma} - (\gamma - 2e^{2\gamma})\cos\theta + 2\varepsilon\gamma\sin\theta \right] \]
\[ F_{13} = \sin(\theta/2)\left[\gamma e^{2\gamma} - (1 - 2\gamma e^{2\gamma})\cos\theta + 2\varepsilon\sin\theta \right] \]
\[ F_{23} = \sin(\theta/2)\left[e^{2\gamma} - (\gamma - 2e^{2\gamma})\cos\theta + 2\varepsilon\gamma\sin\theta \right] \]
\[ G_{11} = -\sin(\theta/2)\left[2\gamma e^{2\gamma} - (1 + 2\gamma e^{2\gamma})\cos\theta + 2\varepsilon\sin\theta \right] \]
\[ G_{21} = -\sin(\theta/2)\left[2\gamma e^{2\gamma} - (\gamma + 2e^{2\gamma})\cos\theta + 2\varepsilon\gamma\sin\theta \right] \]
\[ G_{12} = -\sin(\theta/2)\left[2 + \gamma e^{2\gamma} + (1 + 2\gamma e^{2\gamma})\cos\theta - 2\varepsilon\sin\theta \right] \]
\[ G_{22} = -\sin(\theta/2)\left[2 + \gamma e^{2\gamma} + (\gamma + 2e^{2\gamma})\cos\theta - 2\varepsilon\gamma\sin\theta \right] \]
\[ G_{13} = -\cos(\theta/2)\left[\gamma e^{2\gamma} - (1 + 2\gamma e^{2\gamma})\cos\theta + 2\varepsilon\sin\theta \right] \]
\[ G_{23} = -\cos(\theta/2)\left[e^{2\gamma} - (\gamma + 2e^{2\gamma})\cos\theta + 2\varepsilon\gamma\sin\theta \right] \]
\[ H_{11} = 2k_1 \left[\cos(\theta/2) - 2\varepsilon\sin(\theta/2)\right] - 2\gamma e^{2\gamma} \left[\cos(3\theta/2) - 2\varepsilon\sin(3\theta/2)\right] \]
\[ + (1 + 4\varepsilon^2) \left[\cos(3\theta/2) - \cos(\theta/2)\right] \]
\[ H_{21} = 2k_2 \gamma \left[\cos(\theta/2) - 2\varepsilon\sin(\theta/2)\right] - 2\varepsilon^2 \gamma \left[\cos(3\theta/2) - 2\varepsilon\sin(3\theta/2)\right] \]
\[ + (1 + 4\varepsilon^2) \gamma \left[\cos(3\theta/2) - \cos(\theta/2)\right] \]
\[ H_{12} = -2k_1 \left[\sin(\theta/2) + 2\varepsilon\cos(\theta/2)\right] + 2\gamma e^{2\gamma} \left[\sin(3\theta/2) + 2\varepsilon\cos(3\theta/2)\right] \]
\[ - (1 + 4\varepsilon^2) \left[\sin(\theta/2) + \sin(3\theta/2)\right] \]
\[ H_{22} = -2k_2 \gamma \left[\sin(\theta/2) + 2\varepsilon\cos(\theta/2)\right] + 2\varepsilon^2 \gamma \left[\sin(3\theta/2) + 2\varepsilon\cos(3\theta/2)\right] \]
\[ - (1 + 4\varepsilon^2) \gamma \left[\sin(\theta/2) + \sin(3\theta/2)\right] \]
\[ P_{11} = -2k_1 \left[\sin(\theta/2) + 2\varepsilon\cos(\theta/2)\right] + 2\gamma e^{2\gamma} \left[\sin(3\theta/2) + 2\varepsilon\cos(3\theta/2)\right] \]
\[ + (1 + 4\varepsilon^2) \left[\sin(\theta/2) + \sin(3\theta/2)\right] \]
\[ P_{21} = -2k_2 \gamma \left[\sin(\theta/2) + 2\varepsilon\cos(\theta/2)\right] + 2\varepsilon^2 \gamma \left[\sin(3\theta/2) + 2\varepsilon\cos(3\theta/2)\right] \]
\[ + (1 + 4\varepsilon^2) \gamma \left[\sin(\theta/2) + \sin(3\theta/2)\right] \]
\[ P_{12} = -2k_1 \cos(\theta/2) - 2\varepsilon \sin(\theta/2) + 2y e^{2\varepsilon \theta} \cos(3\theta/2) - 2\varepsilon \sin(3\theta/2) \] (A.29)
\[ + (1+4\varepsilon^2) \cos(\theta/2) - \cos(\theta/2) \]

\[ P_{22} = -2k_2 \gamma \cos(\theta/2) - 2\varepsilon \sin(\theta/2) + 2y e^{2\varepsilon \theta} \cos(3\theta/2) - 2\varepsilon \sin(3\theta/2) \]
\[ + (1+4\varepsilon^2) \gamma \cos(\theta/2) - \cos(\theta/2) \]

\[ S = \sin(\varepsilon \pi r) \]
\[ C = \cos(\varepsilon \pi r) \]

\[ k_1 = \begin{cases} 
(3-4v) & , \text{ plane strain} \\
(3-v)/(1+v) & , \text{ plane stress} 
\end{cases} \] (A.31)

\[ \gamma = (1+mk_1)/(m+k_2) \] (A.32)

\[ \varepsilon = \frac{1}{2\pi} \ln \gamma \] (A.33)

**A.4 TERMINAL CRACK IN TWO DIMENSIONS**

For the geometry and local coordinates of Fig. 2.11, equilibrium solutions to the equations of plane elasticity without body forces and traction free crack surfaces yield:

\[ \sigma_{ir} = r^{\lambda-1} F_1'(\theta) + (\lambda+1) F_1(\theta) \]

\[ \sigma_{i\theta} = r^{\lambda-1} \lambda(\lambda+1) F_1(\theta) \] (A.34)

\[ \sigma_{ir\theta} = -r^{\lambda-1} \lambda F_1'(\theta) \]

\[ u_{ir} = \frac{r^\lambda}{8\lambda^2} \left\{ (1+k_1) F_1'(\theta) + (\lambda+1) [ (1+3\lambda) + k_1 (\lambda+1) ] F_1(\theta) \right\} \] (A.35)

\[ u_{i\theta} = \frac{r^\lambda}{8\lambda^2} \left\{ [k_1 (\lambda-1) - (1+3\lambda)] F_1'(\theta) + (\lambda+1)^2 (\lambda-1) (1+k_1) \int_0^\theta F_1(\theta) d\theta \right\} \]
\[ F_1(\theta) = \frac{1}{\lambda \sqrt{2\pi}} \{ K_1 f_1(\theta) + K_2 g_1(\theta) \} \]  \hspace{1cm} (A.36)

where \( \lambda = 1-p \) and \( i = 1, 2 \) for material 1 and material 2, respectively, and

\[
\begin{align*}
\begin{pmatrix} f_1(\theta) \\ f_2(\theta) \end{pmatrix} &= c_f \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ 0 & y_2 & 0 & 1 \end{pmatrix} \begin{pmatrix} s_1 \\ c_1 \end{pmatrix} \\
\begin{pmatrix} g_1(\theta) \\ g_2(\theta) \end{pmatrix} &= c_g \begin{pmatrix} \bar{x}_1 & \bar{x}_2 & \bar{x}_3 & \bar{x}_4 \\ \bar{y}_1 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} s_1 \\ c_1 \end{pmatrix}
\end{align*}
\]  \hspace{1cm} (A.37)

where

\[
\begin{align*}
c_f &= \frac{(1+mk_1)}{(1+mk_1)(2\lambda+1)-(m+k_2)(2\lambda-1)} \hspace{1cm} (A.39) \\
c_g &= \frac{(1+mk_1)}{(1+mk_1)(2\lambda-1)-(m+k_2)(2\lambda+1)} \hspace{1cm} (A.40) \\
y_2 &= \frac{(1+mk_1)+(1-2\lambda)(m+k_2)}{(\lambda+1)(1+mk_1)} \hspace{1cm} (A.41)
\end{align*}
\]

\[
\begin{align*}
x_4 &= \frac{1}{m(1+k_1)} \{ (m+k_2)+(1-m) [ \lambda - (\lambda+1)1/2 \cos \lambda \pi ] \} \\
x_3 &= \frac{\cot(\lambda \pi/2)}{m(1+k_1)} \{ (\lambda+k_2)-m(\lambda-1)-m(1+k_1)x_4-(\lambda+1)(1-m)y_2 \} \hspace{1cm} (A.42) \\
x_2 &= \frac{1}{m(\lambda+1)} \{ -(\lambda+k_2 \cos \lambda \pi)+m k_1 \sin \lambda \pi x_3+m(\lambda+k_1 \cos \lambda \pi)x_4+(\lambda+1)y_2 \} \\
x_1 &= \frac{\tan(\lambda \pi/2)}{m(\lambda+1)} \{ (\lambda-k_2)+m(\lambda+1)\cot\frac{\lambda \pi}{2} + m(\lambda-k_1)x_3-m(\lambda-k_1)x_4-(\lambda+1)y_2 \} \\
\bar{y}_1 &= \frac{\lambda(1+mk_1)-(1+2\lambda)(m+k_2)}{(\lambda+1)(1+mk_1)} \hspace{1cm} (A.43)
\end{align*}
\]
\[ x_4 = \frac{(1-m)\sin \lambda \pi}{m(1+k_1)} \{ \lambda - (\lambda+1) \bar{y}_1 \} \]

\[ x_3 = \frac{1}{m(1+k_2)} \{ \lambda(1-m)+(m+k_2)-(1-m)(\lambda+1) \bar{y}_1 - m(1+k_1)\cot \frac{\lambda \pi}{2} \bar{x}_4 \} \]

\[ x_2 = \frac{1}{m(\lambda+1)} \{ -k_2 \sin \lambda \pi + m(\lambda+k_1) \bar{x}_4 + mk_1 \sin \lambda \pi \bar{x}_3 \} \]

\[ x_1 = \frac{1}{m(\lambda+1)} \{ (k_2-\lambda)+(\lambda+1) \bar{y}_1 - m(\lambda-k_1) \tan \frac{\lambda \pi}{2} \bar{x}_4 + m(\lambda+k_1) \bar{x}_3 + m(\lambda+1) \tan \frac{\lambda \pi}{2} \bar{x}_2 \} \]

\[ S_1 = \sin(\lambda+1) \theta \]

\[ C_1 = \cos(\lambda+1) \theta \]

\[ S_2 = \sin(\lambda-1) \theta \]

\[ C_2 = \cos(\lambda-1) \theta \]

Initial fracture angle for propagation into uncracked medium, i.e. material 2 is governed by

\[ C_f [ (\lambda+1)y_2 S_1 + (\lambda-1) S_2 ] \frac{K_1}{(K_{IC})_2} - C_g [ (\lambda+1) \bar{y}_1 C_1 + (\lambda-1) C_2 ] \frac{K_2}{(K_{IC})_2} = 0 \]  \hspace{1cm} (A.46)

The corresponding fracture envelope is defined by

\[ C_f [ y_2 C_1 + C_2 ] \frac{K_1}{(K_{IC})_2} + C_g [ \bar{y}_1 S_1 + S_2 ] \frac{K_2}{(K_{IC})_2} = \frac{1}{(\lambda+1)^2} \frac{1-2\lambda}{R_0} \]  \hspace{1cm} (A.47)

Fracture angle for a crack reflecting back into the cracked medium, i.e. material 1 is governed by

\[ C_f [ (\lambda+1)x_1 C_1 - (\lambda+1)x_2 S_1 + (\lambda-1)x_3 C_2 - (\lambda-1)x_4 S_2 ] \frac{K_1}{(K_{IC})_1} \]

\[ + C_g [ (\lambda+1) \bar{x}_1 C_1 - (\lambda+1) \bar{x}_2 S_2 + (\lambda-1) \bar{x}_3 C_2 - (\lambda-1) \bar{x}_4 S_2 ] \frac{K_2}{(K_{IC})_1} = 0 \]  \hspace{1cm} (A.48)

And the fracture envelope is defined by
Fracture envelope for pure tensile debonding is given by

$$C_f \left[ (1-y_2) \sin \frac{\lambda \pi}{2} \right] \frac{K_1}{K_d} - C_g \left[ (1-y_1) \cos \frac{\lambda \pi}{2} \right] = \frac{1}{\gamma} \left( \frac{2}{\lambda+1} \right)$$  

(A.49)

And for the critical combination of normal and shear stresses, for

$$\sigma_\theta > 0 \text{ we have}$$

$$C_f^2 \left[ (\lambda+1)^2 (1-y_2)^2 - 2(1+\cos \lambda \pi) [(\lambda+1)(1-y_2)-1] \right] \frac{K_1}{K_d}$$

$$+ 2C_f C_g \sin \lambda \pi \left[ (1+\lambda)(\bar{y}_1+y_2)-2\lambda \right] \left( \frac{K_1}{K_d} \right) \left( \frac{K_2}{K_d} \right)$$

$$+ C_g^2 \left[ (\lambda+1)^2 (1-\bar{y}_2)^2 - 2(1-\cos \lambda \pi) [(\lambda+1)(1-\bar{y}_2)-1] \right] \left( \frac{K_1}{K_d} \right)^2 = 1$$  

(A.51)

For $$\sigma_\theta < 0$$ we have

$$C_f \left[ (1+y_2)-\lambda(1-y_2) \right] \cos \frac{\lambda \pi}{2} + \mu_f (\lambda+1)(1-y_2) \sin \frac{\lambda \pi}{2} \left( \frac{K_1}{K_d} \right)$$

$$+ C_g \left[ (1+\bar{y}_1)-\lambda(1-\bar{y}_1) \right] \sin \frac{\lambda \pi}{2} - \mu_f (\lambda+1)(1-\bar{y}_1) \cos \frac{\lambda \pi}{2} \left( \frac{K_2}{K_d} \right) = 1$$  

(A.52)
B.1 AKIN'S VARIABLE SINGULARITY ELEMENT

Akin [41] proposed families of $O(r^{1-p})$ displacement singularity elements derivable from existing families of triangular and quadrilateral nonsingular finite elements with the displacement interpolation as

$$\tilde{u} = \sum_{i=1}^{n} N_i(\xi, \eta) \tilde{u}_i$$  \hspace{1cm} (B.1)

where $N_i$ satisfy the usual identity

$$N_i(\xi_j, \eta_j) = \delta_{ij}$$  \hspace{1cm} (B.2)

and embed the constant field through the relation

$$\sum_{i=1}^{n} N_i(\xi, \eta) = 1$$  \hspace{1cm} (B.3)

To impose the $r^{1-p}$ singularity in the displacement field, a local function is defined as

$$W(\xi, \eta) = 1 - N_j(\xi, \eta)$$  \hspace{1cm} (B.4)

where $N_j$ is the shape function associated with the node at which $r^{-p}$ derivative singularity is expected. Hence, the function $W$ vanishes at the singular node $j$ and attains a unit value of all other nodes. In addition, $W$ is unity at all sides which do not intersect node $j$. The singular displacement field is

$$\tilde{u} = \sum_{i=1}^{n} H_i(\xi, \eta) \tilde{u}_i$$  \hspace{1cm} (B.5)
To ensure the range of singularity constant (0 < p < 1), we consider the relationship

\[ W^{1-p}(\xi, \eta) = \sum_{i=1, i \neq j}^{n} H_i(\xi, \eta) \]  

(B.6)

Noting that the local function \( W(\xi, \eta) \) vanishes at the singular node \( j \), Eq. (B.2) is satisfied by assuming \( H_j \) as

\[ H_j(\xi, \eta) = 1 - W^{1-p}(\xi, \eta) \]  

(B.7)

and Eq. (B.3) readily yields the rest of the shape functions as

\[ H_i(\xi, \eta) = N_i(\xi, \eta)W^{-p}(\xi, \eta) , \quad i \neq j \]  

(B.8)

Hence, the singular interpolation functions \( H_i \) derived from regular shape functions \( N_i \) are capable of representing the \( r^{1-p} \) displacement singularity and they satisfy the constant field condition as

\[ \sum_{i=1}^{n} H_i = 1 \]  

(B.9)

Since the function \( W \) is unity at sides which do not intersect the singular node \( j \), this singularity element is compatible with the regular elements when they encompass the singular point in an assemblage. Hence, a compatible interpolation field is automatically formed.

### B.2 SEMI-RADIAL SINGULARITY MAPPING ELEMENT

The semi-radial singularity mapping concept was first introduced by Okabe [43]. Similarities can be observed in the derivation of this element and Akin's singularity element. However, the main difference is the manner in which the \( r^{1-p} \) displacement singularity is imposed. In Akin's [41] formulation, the displacement field is forced to represent the
desired singularity, whereas in the semi-radial singularity mapping
technique, the displacement interpolation is not singular. Rather, the
geometric mapping imposes the singular behavior on the displacement
field. It is noteworthy that the same idea is incorporated in the
inverse square root displacement representation of the quarter point
element for homogeneous medium crack problems.

We consider a general quadrilateral element in the xy global coor-
dinate system, which is mapped to a normalized square in the local coor-
dinates $\xi, \eta$ through the parametric representation (Fig. B.1).

$$\tilde{x} = \sum_{i=1}^{n} M_i(\xi, \eta) \tilde{x}_i$$  \hspace{1cm} (B.10)

To ensure that the interpolation function space includes the constant
term as well as x and y terms, the following condition is enforced

$$\sum_{i=1}^{n} M_i(\xi, \eta) = 1$$  \hspace{1cm} (B.11)

Since the desired $r^{-p}$ displacement singularity will be imposed via the
geometric mapping, the displacement field should be represented using a
different set of shape functions [43]. Hence,

$$\tilde{u} = \sum_{i=1}^{n} N_i(\xi, \eta) \tilde{u}_i$$  \hspace{1cm} (B.12)

The constant field condition is imposed on $N_i$ through the relation

$$\sum_{i=1}^{n} N_i(\xi, \eta) = 1$$  \hspace{1cm} (B.13)

Okabe [43] defines the semi-radial field condition as

$$\rho^{(1-p)} = 1 - N_j(\xi, \eta)$$  \hspace{1cm} (B.14)

where $N_j$ is the regular polynomial shape function associated with node
j, at which the $r^{-p}$ derivative singularity is expected. Equation (B.14) ensures that the range of the singularity constant is $0 < p < 1$. The term $\rho (1-p)$ vanishes at the singular node j and attains a unit value at all other nodes, as well as at the sides which do not intersect node j. Hence, the quantity $\rho$ can be interpreted as a nondimensional coordinate as illustrated in Fig. B.1. In addition to the semi-radial field condition of Eq.(B.14), a second condition, relating the geometric and displacement interpolation functions, is imposed.

$$M_j(\xi, \eta) = \rho N_j(\xi, \eta) \quad \text{if} j$$  \hspace{1cm} (B.15)

Equation (B.15) implies that for nonsingular cases (i.e. $p=0$), the singular geometric mapping and the regular displacement interpolation functions become identical and an isoparametric representation is realized. Rewriting Eq.(B.15) in the form

$$\sum_{i=1}^{n} M_i(\xi, \eta) = \rho \sum_{i=1}^{n} N_i(\xi, \eta) \quad \text{if} j$$  \hspace{1cm} (B.16)

Equations (B.14) and (B.16) together yield

$$p = \sum_{i=1}^{n} M_i(\xi, \eta) \quad \text{if} j$$  \hspace{1cm} (B.17)

and imposing the constant field condition of Eq.(B.11) on Eq.(B.17) we have

$$M_j(\xi, \eta) = 1 - p$$  \hspace{1cm} (B.18)

Hence, Eqs.(B.15) and (B.18) together with Eq.(B.14) give the singular mapping shape functions, which assure the representation of $\rho^{1-p}$ displacement singularity.
Fig. B.1 Semi-Radial Singularity Element Local Coordinates and Geometry
The elastic stress field solutions obtained from LEFM analysis exhibit a stress singularity at the tip of an elastic crack. This singularity indicates that the stresses become infinite at the crack tip. In practice, however, metals exhibit a yield stress, above which they deform plastically. Hence, the plastic region around the tip of a crack in a metal, where plastic deformation occurs is known as the crack tip plastic zone. The size and shape of the plastic zone can be estimated [110] by imposing a Tresca-Von Mises yield criterion. The boundaries obtained represent the region in which the stresses are limited by the yield stress. Due to redistribution of these stresses, the actual plastic zone size increases. In subsection C.1, the above idea is extended to a bi-material terminal crack tip model. In subsection C.2, a fracture process zone defined by tensile failure of the crack tip region in rock is discussed and its extension to layered media are given. The effects of friction on the process zone are discussed in subsection C.3.

C.1 PLASTIC ZONE FOR A BI-MATERIAL TERMINAL CRACK

We consider the geometry shown in Fig. 2.11 for a crack perpendicular to a bi-material interface. The crack tip is on the interface. Around the tip of the crack, in each material Von Mises' yield criterion is imposed, which is given [110] in terms of the principal stresses \((\sigma_1, \sigma_2, \sigma_3)\) as
\[(a_i - a_{i2})^2 + (a_{i2} - a_{i3})^2 + (a_{i3} - a_{i1})^2 = 2\sigma_{iys}^2, \quad i=1,2 \quad (C.1)\]

where \(\sigma_{iys}\) is the uniaxial yield stress of material \(i\). We define the out of plane stress \(\sigma_{i3}\) as follows

\[\sigma_{i3} = \nu_i(a_{i1} + a_{i2}) \quad \text{for plane strain} \quad (C.2)\]

\[\sigma_{i3} = 0 \quad \text{for plane stress} \quad (C.3)\]

The principal stresses are given by

\[\sigma_{i1} = \frac{1}{2}(\sigma_{ir} + \sigma_{ie}) + \sqrt{\left(\frac{\sigma_{ir}-\sigma_{ie}}{2}\right)^2 + \sigma_{ir0}^2} \quad (C.4)\]

\[\sigma_{i2} = \frac{1}{2}(\sigma_{ir} + \sigma_{ie}) - \sqrt{\left(\frac{\sigma_{ir}-\sigma_{ie}}{2}\right)^2 + \sigma_{ir0}^2} \quad (C.4)\]

Here we will consider the Mode I plane stress case only. Plane strain and Mode II cases can be treated in a similar way.

Using the expressions for stress components given by Eqs.(3.27), Eqs.(C.2), (C.3), (C.4) and imposing the yield criterion of Eq.(C.1), we obtain for a plane stress case,

\[3[F_{1,\theta} + p(2-p)F_{1}]^2 + 12(1-p)^2F_{1,\theta}^2 + [F_{1,\theta} + (2-p)F_{1}]^2 = 4r^2p\sigma_{iys}^2 \quad (C.5)\]

Substituting for \(F_1\), after algebraic manipulations, the non-dimensional representation of the bi-material plastic zone shape is obtained as follows, in region 2

\[\frac{r_2(\theta)}{(K_1/\sigma_{2ys})^{1/p}} = \left\{\frac{1}{2\pi(2-p)^2(1+y_2^2)} \left[4\cos^2 p\theta + 6y_2 p(p-2)\cos2\theta + 3(2-p)^2 y_2^2 + 3p^2 \right]\right\}^{1/2p} \quad (C.6)\]

and in region 1
\[
\frac{r_1(\theta)}{(K_1/\sigma_{ys})^{1/p}} = \left\{ \frac{1}{2\pi(2-p)^2(1+y_2)^2} \left( 4x_3^2 \sin^2 p\theta + 4x_4^2 \cos^2 p\theta \right) \right.
\]
\[
-4x_3x_4 \sin 2p\theta + 3(2-p)^2(x_1^2 + x_2^2) + 3p^2(x_3^2 + x_4^2)
\]
\[
+ 6p(p-2) \left[ (x_1 x_3 + x_2 x_4) \cos 2\theta + (x_1 x_4 - x_2 x_3) \sin 2\theta \right] \right\}^{1/2p}
\]
\[\text{(C.7)}\]

where the bi-material constants \(x_1, x_2, x_3, x_4\) and \(y_2\) are given in Appendix A4. Equations (C.6) and (C.7) both reduce to their homogeneous counterpart [110] for \(m=1\) and \(p=1/2\) as

\[
\frac{r(\theta)}{(K_1/\sigma_{ys})^2} = \frac{1}{4\pi} \left( 1 + \frac{3}{2} \sin^2 \theta + \cos \theta \right)
\]

\[\text{(C.8)}\]

Graphical representation of the bi-material crack plastic zone shapes are shown in Fig. C.1. These curves correspond to equal Poisson's ratios for both materials and plane stress conditions. We note that the magnitudes in region 1 are multiplied by the shear moduli ratio \(m\) for scaling purposes. The discontinuous behavior of the interface is peculiar. Hence, yielding of one material along the interface before the adjacent one may cause interface damage in metals. When compared to the homogeneous case \((m=1)\), the bi-material plastic zone increases for \(m<1\) and decreases for \(m>1\). Overall, the plastic zone size along the interface \((\theta=\pm\pi/2)\) is larger as the interface is approached from the stiffer material side. Hence, if a radial characteristic distance is to be used in the bi-material crack analysis of metals, it should be based on the properties of the stiffer material.
Fig. C.1 Bi-Material Plastic Zone

Fig. C.2 Bi-Material Microcrack Zone
Yielding, and consequently a plastic deformation field, is unlikely for cracks in rock unless considerable temperature or confining pressure effects are present. Instead, the inelastic behavior at the crack tip is attributed to microcracking. Since most crack propagation occurs due to tensile stresses or compressive–shear induced phenomena, a microcrack zone defined by the maximum normal stress criterion was proposed by Schmidt [75] for a homogeneous fractured medium. Here, we will extend this idea to define the bi-material microcrack process zone.

Around the crack tip in each material, the maximum tensile stress criterion is imposed as

\[ \sigma_{\text{ll}} = \sigma_{\text{u}} \quad i=1,2 \]  \hspace{1cm} (C.9)

where \( \sigma_{\text{ll}} \) and \( \sigma_{\text{u}} \) are the maximum principal stress and the ultimate tensile stress of material \( i \). Using Eqs. (3.26), (C.3) and (C.9) we obtain the non-dimensional representation of the bi-material microcrack zone as follows, in region 2

\[ \frac{r_2(\theta)}{(K_1/\sigma_{\text{2u}})^{1/p}} = \left[ \frac{2\cos\theta + \sqrt{(2-p)^2 y_2^2 + 2y_2 p(p-2) \cos 2\theta}}{\sqrt{2\pi(2-p)(1+y_2)}} \right]^{1/p} \]  \hspace{1cm} (C.10)

and in region 1

\[ \frac{r_1(\theta)}{(K_1/\sigma_{\text{1u}})^{1/p}} = \left[ \frac{2x_4 \cos\theta - 2x_3 \sin\theta + \sqrt{(2-p)^2(x_1^2 + x_2^2) + p^2(x_3^2 + x_4^2)}}{\sqrt{2\pi(2-p)(1+y_2)}} \right]^{1/p} \]

\[ \ldots \]

\[ + \frac{2p(p-2)[(x_1 x_3 + x_2 x_4) \cos \theta + (x_1 x_4 - x_2 x_3) \sin \theta]}{\ldots} \]  \hspace{1cm} (C.11)
where the constants $x_1, x_2, x_3, x_4$ and $y_2$ are given in Appendix A4. For $m=1$ and $p=1/2$, Eqs. (C.10) and (C.11) both reduce to the microcrack expression for homogeneous medium given by Schmidt [75]

$$\frac{r(\theta)}{K_1/a_0} = \frac{1}{2\pi} \left( \frac{\cos \theta (1+\sin \theta)}{2} \right)^2$$

(C.12)

In Fig. C.2 the bi-material microcrack process zones for rock, for several material combinations, are presented. Plane strain conditions and equal Poisson's ratios for both materials are assumed. We observe exactly the same qualitative behavior as for the bi-material plastic zone of the previous section. The discontinuous behavior of the interface, indicates microcracking on one side of the interface, with undamaged adjacent material. Such a behavior may cause interfacial damage in the form of normal debonding of the interface or interfacial slip. The microcrack zone size along the interface is observed to be larger as the interface is approached from the stiffer material side. Hence, if the fracture process zone for a radial characteristic distance is based on microcracking in rock, the properties of the stiffer material should be considered in defining such a characteristic distance.

C.3 EFFECT OF FRICTION ON THE MICROCRACK ZONE

To study the effect of friction due to compression, we consider the criterion of slip in the plane as

$$|\tau| = S_o - \mu_f \sigma$$

(C.13)

where $\sigma$ and $\tau$ are the normal and shear stresses across the plane, $\mu_f$ is the coefficient of friction and $S_o$ is the inherent cohesive shear strength. Following a similar derivation as in Ref. [111], we obtain
the failure criterion as

$$(\sigma_3 - \sigma_1)(1+u)^{1/2} + \mu_x(\sigma_1 + \sigma_3) = 2S_o$$ \hspace{1cm} (C.14)$$

in which $\sigma_1, \sigma_3$ are the principal stresses such that $\sigma_3$ is the least and $\sigma_1$ is the maximum compression ($|\sigma_1| > |\sigma_3|$). Following a transformation of stresses, the mode I component of Eqs. (A.1) can be written in terms of the principal stresses as

$$\sigma_1 = \frac{K_1}{\sqrt{2\pi}} \cos^2(1+\sin^2)$$

$$\sigma_2 = \frac{K_1}{\sqrt{2\pi}} \cos^2(1-\sin^2)$$ \hspace{1cm} (C.15)$$

$$\sigma_3 = \begin{cases} 0 & \text{for plane stress} \\ \frac{2\sqrt{2}K_1}{\sqrt{2\pi}} \cos^2 & \text{for plane strain} \end{cases}$$

Using Eqs. (C.15) in the failure criterion of Eq. (C.14), crack tip process zone due to friction is defined by, for plane stress,

$$\frac{r(\theta)}{(K_1/S_o)^2} = \frac{1}{2\pi} \left[ \frac{1}{2} \left( (1+u)^{1/2} - \mu_x \right) \cos^2(1+\sin^2) \right]^2$$ \hspace{1cm} (C.16)$$

and for plane strain

$$\frac{r(\theta)}{(K_1/S_o)^2} = \frac{1}{2\pi} \left[ \frac{1}{2} \cos^2 [(2\nu-1-\sin^2)(1+\mu_x)^{1/2} + \mu_x(2\nu+1+\sin^2)] \right]^2$$ \hspace{1cm} (C.17)$$

Equation (C.16) reduces to the microcrack zone expression of Eq. (C.12) for $\mu_x = 0$ and $2S_o = q_u$ (Griffith case). Table C.1 shows some values of $S_o$ and $\mu_x$ obtained for rough surfaces [111]. Equations (C.16) and (C.17) are graphically represented in Fig. C.3, corresponding to some rock types of Table C.1.
The fracture process zone due to friction is observed to be considerably smaller than the microcrack zone of Eq.(C.11) (m=1.0 curve in Fig. C.2). For plane stress, increasing $\mu_f$, decreases the zone size whereas for plane strain larger process zone sizes correspond to larger $\mu_f$ values. The shape of the process zone is sensitive to the state of stress. For plane stress, it is a somewhat circular double-lobe region. For plane strain, it is narrower and more elongated in the direction of crack. An observation of the case for which friction coefficient $\mu_f=0$ indicates that overall effect of the friction on the fracture process zone is more likely to decrease the process zone size.

Table C.1  Shear Strength and Friction Coefficient Values for Some Materials [111]

<table>
<thead>
<tr>
<th>Rock Type</th>
<th>$\mu_f$</th>
<th>$S_o$ (psi)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Granite</td>
<td>0.64</td>
<td>45</td>
</tr>
<tr>
<td>Gabbro</td>
<td>0.66</td>
<td>55</td>
</tr>
<tr>
<td>Trachyte</td>
<td>0.68</td>
<td>60</td>
</tr>
<tr>
<td>Sandstone</td>
<td>0.51</td>
<td>40</td>
</tr>
<tr>
<td>Marble</td>
<td>0.75</td>
<td>160</td>
</tr>
</tbody>
</table>
Fig. C.3 Fracture Process Zone Due to Friction
For a perpendicular crack terminating at a bi-material interface, the definition of $J^n$ integral (Eq.(3.56)) was based on an infinitesimal contour $\partial R^*$ surrounding the bi-material crack tip. It was observed that when $\partial R^*$ collapses to the crack tip, the value of $J^n$ becomes unbounded for $m<1$ and zero for $m>1$. Numerical results of Section 4.2.4 reveal that when the material properties of two adjoining media are interchanged, $J^n$ values corresponding to a specific contour $\partial R^*$ become equal. The $J^n$ integral is path independent in the sense that, for any other contour surrounding the bi-material crack tip and extending to $\partial R^*$ along the interface, $J^n$ remains unchanged. So far any relation between $J^n$ and other fracture parameters, such as the stress intensity factors, is unclear.

Here the plane strain bi-material hydraulic fracture model of Section 5.4.3 is reconsidered. For crack surface pressure as the applied loading, non-dimensional total potential energy variation as the crack approaches, terminates and crosses the interface is shown in Fig. D.1. Corresponding $J$-integral variation is illustrated in Fig. D.2. For $a/H_1=1.0$, the $J^n$ integral is evaluated based on a small circular $\partial R^*$ path centered at the bi-material crack tip, with a non-dimensional radius parameter of $r/a=0.025$. Numerical values for the plots in Fig. D.2 are presented in Table D.1.
Fig. D.1 Non-Dimensional Total Potential Energy Variation for Crack Approaching and Crossing the Interface

Fig. D.2 Normalized J-Integral Variation for Crack Approaching and Crossing the Interface
Table D.1 Non-Dimensional J-Integral Results for a Crack Approaching, Terminating and Crossing an Interface (J/qa x 10^{-6})

<table>
<thead>
<tr>
<th>m \times a/H_1</th>
<th>0.5</th>
<th>0.7</th>
<th>0.9</th>
<th>0.95</th>
<th>1.0</th>
<th>1.05</th>
<th>1.1</th>
<th>1.3</th>
<th>1.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.34</td>
<td>0.49</td>
<td>1.06</td>
<td>1.68</td>
<td>1.54</td>
<td>2.51</td>
<td>2.48</td>
<td>2.08</td>
<td>2.13</td>
</tr>
<tr>
<td>0.5</td>
<td>0.29</td>
<td>0.33</td>
<td>0.42</td>
<td>0.48</td>
<td>0.44</td>
<td>0.50</td>
<td>0.52</td>
<td>0.50</td>
<td>0.51</td>
</tr>
<tr>
<td>1.0</td>
<td>0.26</td>
<td>0.27</td>
<td>0.27</td>
<td>0.27</td>
<td>0.27</td>
<td>0.27</td>
<td>0.28</td>
<td>0.28</td>
<td>0.29</td>
</tr>
<tr>
<td>2.5</td>
<td>0.24</td>
<td>0.21</td>
<td>0.16</td>
<td>0.13</td>
<td>0.15</td>
<td>0.12</td>
<td>0.12</td>
<td>0.14</td>
<td>0.14</td>
</tr>
<tr>
<td>5.0</td>
<td>0.23</td>
<td>0.19</td>
<td>0.13</td>
<td>0.09</td>
<td>0.11</td>
<td>0.06</td>
<td>0.06</td>
<td>0.076</td>
<td>0.075</td>
</tr>
</tbody>
</table>

When the Figs. D.1 and D.2 are compared, we observe a similar qualitative trend, except that the bi-material J exhibits a discontinuous behavior at a/H_1=1.0. This discontinuity becomes more pronounced for m<<1. In general, the qualitative behavior of bi-material J integral exhibits a reasonable pattern. Figure D.3 shows that the a/H_1=1.0 curve falls in between those corresponding to a/H_1=0.9 and a/H_1=1.1. Thus, within the practical range of shear moduli ratio, the bi-material J does assume finite values compatible with the homogeneous, conventional J-integral variation.

The above preliminary findings are solely based on numerical computations. For further applicability and design considerations, analytical interpretations, such as relations with energy release rate and stress intensity factors, must be established. The numerical results presented here are obtained by considering the validity of M-integral conservation law for bi-material problems of Subsection 3.2.3.2, which in turn is related to the bi-material J-integral.
As a second application, the plane strain infinite medium bi-material terminal crack model of Section 5.2 is reconsidered. For various material combinations, the $q_2/q_1$ ratios for propagation across the interface are computed using the bi-material $J$ integral criterion of Eq.(5.4). The non-dimensional radius parameter for the circular contour $\partial R^*$ is chosen as $r/a=0.005$. For equal fracture toughness and shear moduli ratios (i.e. $C_k=C_{\infty}$), bi-material data and response results are given in Table D.2. Figure D.4 illustrates the comparison of criteria of Eqs.(5.3), (5.4) and (5.5). We observe that the approximate criterion of Ref.[57] is bounded by bi-material $J$-integral and stress intensity factor criteria. Nevertheless, for a different $\partial R^*$ contour, Eq.(5.4) would yield different results. Hence, a rationale for the size of the contour $\partial R^*$ must be established for the application of bi-material $J$-integral criterion.

Table D.2 Plane Strain Infinite Plate Bi-Material Data and Response Results for J-Integral Criterion

<table>
<thead>
<tr>
<th>$m$</th>
<th>0.05</th>
<th>0.10</th>
<th>0.50</th>
<th>1.00</th>
<th>2.50</th>
<th>5.00</th>
<th>7.50</th>
<th>10.00</th>
</tr>
</thead>
<tbody>
<tr>
<td>$J^0\times 10^{-6}$</td>
<td>263.00</td>
<td>85.16</td>
<td>17.50</td>
<td>5.57</td>
<td>1.02</td>
<td>0.43</td>
<td>0.30</td>
<td>0.26</td>
</tr>
<tr>
<td>$q_2/q_1$ (Eq.(5.4))</td>
<td>0.03</td>
<td>0.08</td>
<td>0.40</td>
<td>1.00</td>
<td>3.69</td>
<td>8.00</td>
<td>11.76</td>
<td>14.53</td>
</tr>
</tbody>
</table>

It was shown in Subsection 3.2.3.2 that the standard $M$-integral conservation law is valid for bi-material media with a terminal crack perpendicular to a straight bondline, if the origin of the coordinate axis is located at the interface crack tip. Unfortunately, no additional information related with the interface crack tip could be
Fig. D.3 Normalized J-Integral Variation with $\frac{\mu_2}{\mu_1}$

Fig. D.4 Bi-Material J-Integral Criterion for Infinite Bi-Material Plate Model
obtained from such an interpretation. Here we will propose a new, tentative bi-material path independent integral. For the configuration illustrated in Fig. D.5, we define

$$M^* = \lim_{r \to 0} \int r^{2p-1} (W_n x_i - T_k u_k, i x_i) ds$$  \hspace{1cm} (D.1)$$

where $r_o$ is the radius of a small circle around the bi-material crack tip. For any other contour around the crack tip in the same plane, it follows from Green's theorem that

$$M^* = \int r^{2p-1} (W_n x_i - T_k u_k, i x_i) ds + \lim_{r \to 0} \int r^{2p-3} (x^2 \sigma_{ky} u_k, y + y^2 \sigma_{kx} u_k, x - xy(\sigma_{kx} u_k, y + \sigma_{ky} u_k, x)) dxdy$$  \hspace{1cm} (D.2)$$

where the double integral is evaluated over the area $A$, between the contours. If the exterior contour of integration is taken to be large enough so that on $\Gamma$ stresses/strains are proportional to $1/r$, the first integral term of Eq.(D.2) becomes proportional to $r^{2p-1}$. The second integral becomes independent of $r$ on parts of the region $A$ where stresses/strains are singular and proportional to $r^{2(1-p)}$ on the rest of the region. For the special case of a single material domain ($p=1/2$) Eq.(D.2) reduces to the standard $M$-integral.
Fig. D.5 Bi-Material Terminal Crack Configuration