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NON-ASSOCIATIVE ALGEBRAS
AND THEIR AUTOMORPHISM GROUPS

DISSERTATION

Presented in Partial Fulfilment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

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INTRODUCTION

A non-associative algebra, which we simply call an algebra, is a vector space with a binary product which is linear with respect to each variable. The automorphism group Aut(V) of an algebra V is defined to be the set of all elements p in GL(V) such that

\[(uv)^p = u^p v^p\]

for any vectors u, v in V (definition 1.1).

In this thesis, we consider two problems involving non-associative algebras. These are:

(i) construct a G-invariant algebra V for a given finite group G;

(ii) find the isomorphism class of Aut(V) for a given G-invariant algebra V.

Several interesting works have been appeared on this subject. Let us note the development briefly.

The original purpose for taking non-associative algebras was to show the existence of the "Monster" simple group as a subgroup of Aut(V) for a certain
algebra \( V \). In general, for any \( G \)-module \( V \) if \( V \oplus V \) contains \( V \) as a constituent, then a \( G \)-invariant algebra structure can be defined on \( V \). In particular, if the symmetric part of \( V \) contains \( V \) as a constituent, there is a commutative algebra structure on \( V \).

In Cameron, Goethals and Seidel [2], the "Norton algebra" is defined as follows:

Let \( V \) be a permutation module associated with an action of a group \( G \) on a finite set \( X \). Let \( e_1, \ldots, e_n \) be the associated permutation basis of \( V \). Let \( V \) possess an inner product defined and extended linearly by

\[
\langle e_i, e_j \rangle = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise}. \end{cases}
\]

Suppose that \( V = V_1 \oplus \cdots \oplus V_k \) is a decomposition of \( V \) into irreducible \( G \)-modules. Let \( p_j \) be the projection mapping from \( V \) onto \( V_j \). Define a binary product on \( V_j \) by

\[
uv = \sum_{i=1}^{n} \langle u, e_i \rangle \langle v, e_i \rangle p_j(e_i)
\]

for any vectors \( u, v \) in \( V_j \).

Note that the resulting algebra \( V_j \) is \( G \)-invariant: \((uv)^g = u^g v^g\) for any element \( g \) in \( G \).

We say that \( G \) admits an algebra on \( V_j \) when the
binary product is not always equal to zero and is
G-invariant.

If \( G \) is a primitive rank-3 group, then the
associated permutation module decomposes into irreducible
G-modules as
\[
V = V_1 \oplus V_2 \oplus V_3
\]
where \( V_1 \) is a trivial module. The following statement
holds:

A primitive rank-3 group of even order (other
than the dihedral group of degree 5) admits a
commutative algebra on at least one of the subspaces
\( V_2 \) or \( V_3 \) (proposition 6.8.[2]).

We say that a class \( D \) of involutions of a group
\( G \) forms a class of 3-transpositions if for any
elements \( x \) and \( y \) in \( D \) the product \( xy \) has
order 1, 2 or 3. If a group \( G \) is generated
by a class of 3-transpositions, we call \( G \) a 3-
transposition group. It is known that \( G \) acts
as a rank-3 group on \( D \).

We can form a graph on a class of 3-transpositions
\( D \) by joining commuting pairs. S.Smith[12] considered
a graph for a covering group of a 3-transposition
group. Using the graph, he provided a commutative
algebra for such a group. The automorphism group
of the smallest (12-dimensional) algebra was computed
and shown to be $3S_7$.

In [5] Griess takes a 196,883-dimensional module for a group of type $C = (2^{1+24})(1)$, a semi-direct product of an extra-special group $(2^{1+24})$ by Conway's simple group $(1)$. He defines a non-associative commutative C-invariant algebra $V$ and shows that $\text{Aut}(V)$ possesses an element $d$ such that $C$ and $d$ generate the "Monster" simple group.

In a similar fashion Frohardt[3] considers an 85-dimensional irreducible module for $J_3$, one of Janko's simple groups, and defines a $J_3$-invariant non-associative commutative algebra.

Usually, the full automorphism group of an algebra is not easy to find (see H. Suzuki[13]). However, there are algebras whose automorphism groups are completely described. Harada[6] and Allen[1] consider an algebra $V$ over a field $K$ with a basis $x_1, \ldots, x_n$ $(2 \leq n)$ satisfying

$$x_i x_i = (n - 1)x_i$$ for all $i$,

$$x_i x_j = -x_i - x_j$$ if $i \neq j$.

They determine $\text{Aut}(V)$ for all possible characteristics of the field $K$ (see also H. Suzuki[14]).
In the first two chapters of this thesis, we define a non-commutative algebra whose automorphism group is always of finite order and we consider the problem of constructing an algebra \( V \) such that \( \text{Aut}(V) \) is isomorphic to a given group \( G \). Namely we are interested in the case when \( G \) is a permutation group, a linear group, a classical or a sporadic simple group.

An algebra \( V \) over a field of characteristic zero is called an almost alternating algebra (written a.a. algebra for brevity) if \( V \) has a basis \( x_1, \ldots, x_n \) satisfying

\[
x_i x_i = x_i \quad \text{for all } i,
\]

\[
x_i x_j = -x_j x_i \quad \text{if } i \neq j.
\]

Let us list some results on a.a. algebras. The first two are analogous to those in Harada[6].

Let \( V \) be an a.a. algebra with a basis \( x_1, \ldots, x_n \). Then \( \text{Aut}(V) \) is isomorphic to a subgroup of the symmetric group \( S_n \) of degree \( n \) (theorem 1.3). Moreover, \( \text{Aut}(V) \) is isomorphic to \( S_n \) if and only if there exists a constant number \( k \) such that

\[
x_i x_i = x_i \quad \text{for all } i,
\]

\[
x_i x_j = k(x_i - x_j) \quad \text{if } i \neq j \quad \text{(theorem 1.10)}.
\]
Some simple groups are represented as automorphism groups of finite graphs. We give a general construction of an a.a. algebra defined by a given finite graph. Let \( Y = \{y_1, \ldots, y_n\} \) be a finite undirected graph, let \( d(y_s, y_t) \) be the distance between the two points \( y_s \) and \( y_t \). We define the corresponding a.a. algebra \( V \) with a basis \( x_1, \ldots, x_n \) by

\[
x_i x_i = x_i \quad \text{for all} \quad i,
\]

\[
x_i x_j = \sum_{d(y_i, y_s) = 1} x_s - \sum_{d(y_j, y_t) = 1} x_t.
\]

We show that \( \text{Aut}(Y) \) is isomorphic to \( \text{Aut}(V) \) (theorem 1.11).

In an a.a. algebra, \( \text{Aut}(V) \) induces a permutation group on the basis elements \( x_1, \ldots, x_n \) of \( V \). So let us consider permutation groups. It is shown that any permutation group has a non-trivial a.a. algebra structure on the associated permutation module (theorem 1.7). If \( G \) acts 3-transitively on a finite set \( X \), then the a.a. algebra structure on the associated permutation module \( V \) is uniquely determined and \( \text{Aut}(V) \) is isomorphic to \( S_n \) (lemma 1.8). Namely, \( V \) satisfies the same conditions as in theorem 1.12.
However, if $G$ acts sharply 3-transitively on $X$, we can define an a.a. algebra $V$ such that $\text{Aut}(V)$ is isomorphic to $G$ (theorem 2.2). In order to do this, we take a permutation module $V$ associated to the action of $G$ on the ordered pairs of $X$.

We next consider the orthogonal groups. Let $W$ be a vector space over a finite field and let $(\ ,\ )$ be an inner product on $W$. We construct an a.a. algebra $V$ such that $\text{Aut}(V)$ is isomorphic to the orthogonal group of $W$ with respect to the inner product $(\ ,\ )$ (theorem 2.6).

We then look for an a.a. algebra defined by a given abstract group $G$. Let $V$ be a vector space over the complex number field with a basis $\{v_g \mid g \in G\}$. Define a binary product by

$$v_g v_g = v_g \text{ for all } g,$$

$$v_g v_h = v_{gh} - v_{hg} \text{ if } g \neq h.$$

This is an a.a. algebra and we show that if $Z(G)$ is trivial then $\text{Aut}(V)$ is isomorphic to $\text{Aut}(G)$ (theorem 2.12).

In chapter III, we expand the concept of a.a. algebras to multi-linear mappings. Let $V$ be a vector space over the complex number field with
a basis $e_1, \ldots, e_n$. Let

$$V^k = V \times \cdots \times V,$$

$k$ copies of $V$, where $k$ is a positive integer greater than 1 and less than or equal to $n$.

A multi-linear mapping $f$ from $V^k$ to $V$ is called an almost alternating mapping (an a.a. mapping for brevity) if $f$ satisfies

1. $f(e_1, \ldots, e_i) = e_i$ and
2. if $i_s \neq i_t$ for some $s \neq t$,
   
   $$f(e_1, \ldots, e_i) = \text{sgn}(p) f(e_1^{(p)}, \ldots, e_i^{(p)})$$

where $p$ is any element in $S_k$ (definition 3.2).

In fact, when $k = 2$, such a mapping $f$ induces an a.a. algebra structure on $V$. For an a.a. multi-linear mapping $f$, the automorphism group $\text{Aut}(f)$ is the set of all elements $a$ in $\text{GL}(V)$ such that

$$f((v_1)^a, \ldots, (v_k)^a) = (f(v_1, \ldots, v_k))^a$$

for any vectors $v_1, \ldots, v_k$ in $V$ (definition 3.1).

It is shown that $\text{Aut}(f)$ is always a finite group. We give an example of a.a. mapping $f$ such that $\text{Aut}(f)$ is isomorphic to $\mathbb{Z}_{n-2} \times A_n$ (theorem 3.6).

In chapter IV, we determine a commutative algebra invariant under $M_{12}$. The group $M_{12}$ is the simple
group discovered by Mathieu which acts sharply
5-transitively on twelve points. The group $M_{12}$ has
an absolutely irreducible representation $V$ of
dimension 45 over the rational number field. Let
$V_S$ be the symmetric part of $V$. Since $V_S$ contains
$V$ as a constituent exactly once, there is a commutative
algebra structure defined on $V$, which is $M_{12}$-invariant
and is uniquely determined up to a scalar multiple.

We know that $V_{M_{11}}$, the restriction of $V$ to
$M_{11}$, is an irreducible $M_{11}$-module. If $W$ is an
irreducible $S_{11}$-module corresponding to the Young
diagram $(9,1,1)$, we have

$$V_{M_{11}} \cong W_{M_{11}}.$$  

First we describe the $M_{11}$-module $U = W_{M_{11}}$
explicitly with respect to the basis $\{e^i_j \mid 1 < i < j \leq 11\}$
where $e^i_j$ is a standard polytabloid associated to
a standard $(9,1,1)$-tableau

$$\begin{array}{cccccccc}
1 & * & * & * & * & * & * & * \\
i & & & & & & & \\
j & & & & & & & \\
\end{array}$$

Put $(e^i_j)(e^S_t) = \sum A^{isp}_{jtq} e^p_q$. As the symmetric
part $V_S$ contains $U$ as a constituent exactly
four times, the coefficients $\{A^{isp}_{jtq}\}$ which satisfy

$$(e^i_j)^g (e^S_t)^g = (e^i_j e^S_t)^g$$
for any element $g$ in $M_{11}$ are written in terms of 4 parameters. By comparing the coefficients, we obtain an $M_{11}$-invariant commutative algebra structure on $U$.

Next, we extend $U$ to an $M_{12}$-module to obtain $V$. Since $M_{12}$ is generated by $M_{11}$ and $g_5$, where $g_5$ is a certain involution, it suffices to determine the action of $g_5$ on the basis $\{e^i_j\}$. The matrix representation of $g_5$ is uniquely determined and its entries are shown to be rational numbers.

Finally, we determine the coefficients $\{A_{jstq}^{isp}\}$ uniquely up to a scalar multiple so that

$$(e^i_j)g_5 (e^s_t)g_5 = (e^i_j e^s_t)g_5.$$

In this way, we obtain the $M_{12}$-invariant commutative algebra structure on $V$. 
CHAPTER I

BASIC PROPERTIES OF

ALMOST ALTERNATING ALGEBRAS

A finite dimensional vector space $V$ over a field $K$ is said to be an algebra if a binary product on $V$ is defined and the product is $K$-linear with respect to each variable. In this chapter, we introduce a.a. algebras and investigate basic properties of their automorphism groups. When we consider algebras, we do not assume the associativity of the binary product. All vector spaces considered are of finite dimension. We assume that all algebras are defined on vector spaces over the complex number field.

Definition 1.1. For an algebra $V$, we define $\text{Aut}(V)$ to be the set of all elements $a$ in $\text{GL}(V)$, the group of non-singular linear transformations on $V$, satisfying

$$(uv)^a = u^a v^a$$

for any elements $u, v$ in $V$. 
Definition 1.2. An algebra $V$ is called an almost alternating algebra, written a.a. algebra for brevity, if $V$ has a basis $x_1, \ldots, x_n$ satisfying
\[ x_i x_i = x_i \quad \text{for all } i, \]
\[ x_i x_j = - x_j x_i \quad \text{for } i \neq j. \]

In general, it is not easy to determine the group structure of $\text{Aut}(V)$ for a given algebra $V$. But for the a.a. algebras, their automorphisms are always of finite order and we have the following result:

Theorem 1.3. Let $V$ be an a.a. algebra with a basis $x_1, \ldots, x_n$. Then $\text{Aut}(V)$ is isomorphic to a subgroup of the symmetric group $S_n$ of degree $n$.

Proof. Since $x_i x_j = - x_j x_i$ for $i \neq j$, an easy calculation shows that
\[ (\sum r_i x_i)^2 = \sum (r_i)^2 x_i. \]

For any element $a$ in $\text{Aut}(V)$, $(x_i)^a$ satisfies
\[ (x_i)^a (x_i)^a = (x_i)^a. \]

Let $(x_i)^a = \sum r_j x_j$, so we have
\[ \sum (r_j)^2 x_j = \sum r_j x_j. \]

This implies that $r_j = 1$ or $0$ for each $j$. In particular, we have
\[(x_i)^a = \sum x_k\]
where \(k\) runs through \(A_i\), a subset of the indices \(\{1, \ldots, n\}\).

We must show that each \(A_i\) consists of one index and all \(A_i\)'s are distinct. Obviously, each \(A_i\) is not empty, for \((x_i)^a \neq 0\). It suffices to show that the intersection of \(A_i\) and \(A_j\) is empty whenever \(i\) is different from \(j\). Suppose that the assertion is false. We have
\[(x_i)^a (x_j)^a = \sum x_k + \sum x_s x_t + \sum x_s x_k + \sum x_k x_t\]
where \(k\) is in \(A_i \cap A_j\), \(s\) is in \(A_i \setminus A_j\) and \(t\) is in \(A_j \setminus A_i\). By the assumption, the first summation is not zero. On the other hand,
\[(x_j)^a (x_i)^a = \sum x_k + \sum x_t x_s + \sum x_k x_s + \sum x_t x_k = \sum x_k - \sum x_s x_t - \sum x_s x_k - \sum x_k x_t\]
where \(k, s, t\) belong to the same sets as before.
Now, unless \(A_i \cap A_j\) is empty, we get
\[(x_i)^a (x_j)^a \neq (x_j)^a (x_i)^a\]
a contradiction. Thus, every element \(a\) in \(\text{Aut}(V)\) induces a permutation on the basis.

Remark. On any a.a. algebra \(V\), a basis satisfying the condition of Definition 1.2 is unique setwise.
Next, we show that any permutation group admits an a.a. algebra on the associated permutation module. Let us refer to J.P. Serre [11] and I.M. Isaacs [8] to provide some ordinary representation theory.

Let $V$ be any $\mathbb{C}[G]$-module, where $G$ is a group of finite order. Consider the tensor product $V \otimes V$. This is decomposed into the sum of the symmetric part $V_S$ and the alternating part $V_A$ as a $\mathbb{C}[G]$-module. If we fix a basis $e_1, \ldots, e_n$ of $V$, a basis of $V_S$ consists of

$$\frac{1}{2}(e_i \otimes e_j + e_j \otimes e_i)$$

for $1 \leq i \leq j \leq n$. On the other hand, a basis of $V_A$ consists of

$$\frac{1}{2}(e_i \otimes e_j - e_j \otimes e_i)$$

for $1 \leq i < j \leq n$.

Let $\chi$ be the character afforded by $V$, and let us denote the character of the symmetric part $V_S$ (resp. the alternating part $V_A$) by $\chi_S$ (resp. $\chi_A$). Define

$$\chi^{(2)}(g) = \chi(g^2)$$

for any element $g$ in $G$. We have

$$\chi_S = \frac{1}{2}(\chi^2 + \chi^{(2)})$$

and

$$\chi_A = \frac{1}{2}(\chi^2 - \chi^{(2)})$$

The following statement is known as Macshke's theorem: every $\mathbb{C}[G]$-module is completely reducible.
Let $G$ act on a finite set $X$. Let $V$ be a vector space with a basis identified with $X$. We call $V$ a permutation module where $G$ is acting on $V$ by permuting the basis elements.

**Lemma 1.4.** Let $V$ be a $\mathbb{C}[G]$-module which contains the trivial module $1_G$ as a constituent. Suppose that $V = 1_G \oplus U$ is a decomposition of $V$. Then the alternating part $V_A$ of $V$ has a constituent isomorphic to $U$.

**Proof.** Let $\chi$ be the character of $V$. We have $\chi = 1 + \psi$ where $\psi$ is the character of $U$. Since
\[
\chi_A = \frac{1}{2}(\chi^2 - \chi^{(2)})
\]
\[
= \frac{1}{2}(1 + 2\psi + \psi^2 - 1^{(2)} - \psi^{(2)})
\]
\[
= \psi + \phi_A,
\]
this implies the conclusion.

Let $e_1, \ldots, e_n$ be a basis of a vector space $V$. Define a mapping $p_A$ from $V \otimes V$ onto $V_A$ as
\[
p_A(e_i \otimes e_j) = \frac{1}{2}(e_i \otimes e_j - e_j \otimes e_i),
\]
and extend by linearity to $V \otimes V$. It is easy to see that
\[
p_A(u \otimes v) = - p_A(v \otimes u)
\]
for any elements $u, v$ in $V$. 
Lemma 1.5. Let $V$ be a $\mathbb{C}[G]$-module which contains the trivial module $1^G$ as a constituent. Suppose that $V = 1^G \oplus U$ is a decomposition of $V$ as a $\mathbb{C}[G]$-module. Then there exists a $G$-homomorphism $g$ from $V_A$ onto $U$.

Proof. By the previous lemma, the alternating part $V_A$ has a constituent isomorphic to $U$. Put $V_A = U_1 \oplus U_2$ where $U_1$ and $U_2$ are $\mathbb{C}[G]$-modules, and $U_1$ is isomorphic to $U$. To complete the proof, we choose $g$ to be the composition map of the projection from $V_A$ onto $U_1$ and an isomorphism from $U_1$ to $U$.

In the previous lemma, a $G$-homomorphism $g$ is not uniquely determined if $U_2$ contains $U$ as a constituent. Eventually, $U_2$ is isomorphic to $U_A$ as a $\mathbb{C}[G]$-module (see the proof of lemma 1.4). Later in this chapter, we determine $\text{Aut}(V)$ in case $g$ is uniquely determined.

We also note that $g$ satisfies $g(p_A(u \otimes v)) = -g(p_A(v \otimes u))$.

Lemma 1.6. Let $V$ be a permutation module associated with an action of $G$ on $X$. Let $e_1, \ldots, e_n$ be the permutation basis of $V$. Then the subspace $W$
of \( V_S \) spanned by \( \{e_i \otimes e_i\} \) for \( i = 1, \ldots, n \) is \( G \)-invariant. Moreover, \( W \) is isomorphic to \( V \) as a \( \mathbb{C}[G] \)-module under the mapping

\[
e_i \otimes e_i \mapsto e_i.
\]

**Proof.** Since \( G \) permutes the basis elements of \( V \), we have

\[
(e_i \otimes e_i)^x = (e_i^x) \otimes (e_i^x) = e_j \otimes e_j
\]

where \( j = i^x \) for an element \( x \) in \( G \). So \( W \) is \( G \)-invariant. The rest of the assertion is now obvious.

**Remark.** The conclusion of the previous lemma is not true in general, even if \( V \) is a monomial representation module for \( G \). For instance, consider \( Q_8 \), the quaternion group of order 8. It has a unique irreducible character \( \chi \) of degree 2. But \( \chi^2 \) is equal to the regular character of \( Q_8/\mathbb{Z}(Q_8) \), which is the sum of all four linear characters of \( Q_8 \).

**Theorem 1.7.** Let \( G \) act on \( X \), a finite set. Let \( V \) be the permutation module associated with the action of \( G \) on \( X \). Let \( e_1, \ldots, e_n \) be the permutation basis of \( V \). Then there exists a \( G \)-homomorphism \( f \) from \( V \otimes V \) to \( V \) satisfying
f(e_i \otimes e_i) = e_i \text{ for all } i,

f(e_i \otimes e_j) = -f(e_j \otimes e_i) \text{ for } i \neq j \text{ and }

f(e_i \otimes e_j) \neq 0 \text{ for some } i \neq j.

**Proof.** By the previous lemma, $V \otimes V = W \oplus W_1$

where $W$ is spanned by $\{e_i \otimes e_i\}$ for $i = 1, \ldots, n$.

This decomposition is given as follows: any element

in $V \otimes V$ can be written as

$$
\sum r_{ij}(e_i \otimes e_j) = \sum r_{ii}(e_i \otimes e_i) + \sum_{i \neq j} r_{ij}(e_i \otimes e_j).
$$

By lemmas 1.5 and 1.6, the mapping from

$$
\sum_{i \neq j} r_{ij}(e_i \otimes e_j)
$$

and

$$
\sum_{i \neq j} r_{ij}g(p_A(e_i \otimes e_j))
$$

is a $G$-homomorphism. So define $f$ by

$$
f(\sum r_{ij}(e_i \otimes e_j)) = \sum_{i} r_{ii}e_i + \sum_{i \neq j} r_{ij}g(p_A(e_i \otimes e_j)),
$$

then the mapping $f$ satisfies the requirements.

It is now clear to see that the mapping $f$ in

the previous theorem determines an a.a. algebra structure

on a permutation module $V$ by

$$
uv = f(u \otimes v)
$$

for any elements $u, v$ in $V$.

From now on, when we consider $G$ acting on $X$,

the action is supposed to be faithful. So by theorems
1.3 and 1.7, if \( G \) acts on \( X \), we can construct a \( G \)-invariant a.a. algebra \( V \) then \( G \) and \( \text{Aut}(V) \) are regarded as subgroups of \( S_n \) where \( n \) is the cardinality of \( X \).

Next, as we mentioned before, we show that if \( G \) acts 3-transitively on \( X \), then there is a unique \( G \)-invariant a.a. algebra structure on the associated permutation module. We describe the algebra structure explicitly.

**Lemma 1.8.** Suppose that \( G \) acts 3-transitively on \( X \). Let \( V \) be the associated permutation module with a basis \( \{x_1, \ldots, x_n\} \). Then there exists a unique \( G \)-invariant a.a. algebra structure on \( V \): there exists a constant number \( k \) such that

\[
x_i x_i = x_i \quad \text{for all } i
\]

\[
x_i x_j = k(x_i - x_j) \quad \text{for any indices } i \neq j.
\]

**Proof.** Put

\[
x_1 x_2 = \sum_{i=1}^{n} b_i x_i.
\]

Since \( G \) acts 3-transitively on \( X \), each time \( i \) is picked from 4 through \( n \), there exists an element \( g \) in \( G \) such that \( 1^g = 1, 2^g = 2 \) and \( 3^g = i \). So we get
So we have
\[ x_1 x_2 = b_1 x_1 + b_2 x_2 + b_3 (x_3 + \cdots + x_n). \]
Since \( x_2 x_1 = -x_1 x_2 \), we get \( b_3 = 0 \) and \( b_2 = -b_1 \).
So we have
\[ x_1 x_2 = b_1 (x_1 - x_2). \]
For any other pair \( i, j \) there exists an element \( s \) in \( G \) such that \( 1^s = i, 2^s = j \). Applying \( s \) to \( x_i x_j \) we get
\[ x_i x_j = b_1 (x_i - x_j) \]
as desired.

If \( G \) acts 2-transitively on \( X \), the permutation character \( \chi \) is decomposed as \( 1 + \phi \), where \( \phi \) is an irreducible character. By the proof of the previous lemma, if \( G \) acts 3-transitively on \( X \), \( \chi_A \) contains \( \phi \) as a constituent exactly once. Since \( \chi_A = \phi + \phi_A \), we have the following:

**Corollary 1.9.** Suppose that \( G \) acts 3-transitively on \( X \). Let \( 1 + \phi \) be the permutation character. Then \( (\phi_A, \phi) = 0 \).

**Theorem 1.10.** Let \( V \) be an a.a. algebra with a basis \( x_1, \ldots, x_n \). Then \( \text{Aut}(V) \) is isomorphic to \( S_n \) if and only if there exists a constant number
such that
\[ x_i x_i = x_i \text{ for all } i, \]
\[ x_i x_j = k(x_i - x_j) \text{ for any indices } i \neq j. \]

**Proof.** Suppose that an a.a. algebra \( V \) satisfies the conditions on binary product. By theorem 1.3, we only need to check that any permutation on \( x_1, \ldots, x_n \) preserves the given binary product. For instance, a permutation \( p \) sends \( x_i \) to \( x_s \) and \( x_j \) to \( x_t \). Then we have
\[
(x_i)_p (x_j)_p = x_s x_t = k(x_s - x_t) \quad \text{and}
\]
\[
(x_i x_j)_p = k(x_i - x_j)_p = k(x_s - x_t),
\]
which gives \( (x_i x_j)_p = (x_i)^p (x_j)^p \).

On the other hand, if \( \text{Aut}(V) \) is isomorphic to \( S_n \), by theorem 1.3 \( \text{Aut}(V) \) permutes \( x_1, \ldots, x_n \). When \( n = 1 \) or \( 2 \), there is nothing to prove. When \( n \) is greater than \( 2 \), \( \text{Aut}(V) \) acts 3-transitively on \( x_1, \ldots, x_n \). By lemma 1.8, the binary product satisfies the conclusion.

In the rest of the chapter, we consider a.a. algebras induced from undirected finite graphs. Let \( Y \) be an undirected graph with a finite set of vertices \( \{1, \ldots, n\} \). Let \( d(i, j) \) be the distance between the two vertices \( i \) and \( j \). The automorphism group \( \text{Aut}(Y) \) of \( Y \) is defined as:
Aut(Y) = \left\{ p \in S_Y \middle| d(i^p, j^p) = 1 \text{ for any } i, j \right\} \text{ with } d(i, j) = 1.

Theorem 1.11. Let Y be an undirected finite graph. There exists an a.a. algebra V such that Aut(V) is isomorphic to Aut(Y).

Proof. Let 1, ..., n be the vertices of Y. Let V be the permutation module of Aut(Y) with a basis \( x_1, \ldots, x_n \). If an element \( p \) in Aut(Y) acts on Y as \( i^p = j \), then the action of \( p \) on V is defined by \( (x_i)^p = x_j \).

We define a binary product on V by

\[ x_i x_j = x_i \quad \text{for all } i, \]

\[ x_i x_j = \sum_{d(i, k) = 1} x_k - \sum_{d(j, m) = 1} x_m \quad \text{for } i \neq j. \]

For any element \( p \) in Aut(Y), we have

\[ (x_i x_j)^p = \sum_{d(i, k) = 1} (x_k)^p - \sum_{d(j, m) = 1} (x_m)^p \]

\[ = \sum_{d(i^p, k^p) = 1} (x_k)^p - \sum_{d(j^p, m^p) = 1} (x_m)^p \]

for \( i \neq j \). On the other hand, we have

\[ (x_i)^p (x_j)^p = \sum_{d(i^p, s) = 1} x_s - \sum_{d(j^p, t) = 1} x_t. \]
Since $p$ permutes the vertices of $Y$, we see that

$$(x_i x_j)^p = (x_i)^p (x_j)^p.$$ 

Thus, $\text{Aut}(Y)$ is contained in $\text{Aut}(V)$.

We next check that $\text{Aut}(V)$ is contained in $\text{Aut}(Y)$.

For any element $a$ in $\text{Aut}(V)$, we have

$$(x_i x_j)^a = \sum_{d(i,k) = 1} (x_k)^a - \sum_{d(j,m) = 1} (x_m)^a$$

which is equal to

$$(x_i^a x_j^a) = \sum_{d(i^a,s) = 1} x_s - \sum_{d(j^a,t) = 1} x_t$$

for $i \neq j$. Suppose that $d(i,j) = 1$. Then in the expression of $(x_i x_j)^a$, the term $(x_j)^a$ appears only in the first summation once. This implies that in the expression of $(x_i^a x_j^a)$, the same term $(x_j)^a$ must appear in the first summation. Thus, we see that if $d(i,j) = 1$ then $d(i^a, j^a) = 1$. This completes the proof of the theorem.
CHAP'TER II

EXAMPLES OF A.A. ALGEBRAS

In this chapter, we discuss one of our main topics: Construct an algebra $V$ such that $\text{Aut}(V)$ is isomorphic to a given group $G$. In section 1, we take permutation groups. Namely, we obtain sharply 2-transitive groups and 3-transitive groups as the automorphism groups of a.a. algebras. In section 2, we consider the orthogonal groups. Let $W$ be a vector space over a finite field and let $(\cdot, \cdot)$ be a semibilinear form on $W$. We construct an a.a. algebra $V$ such that $\text{Aut}(V)$ is isomorphic to the orthogonal group of $W$ with respect to $(\cdot, \cdot)$. In section 3, we study an a.a. algebra defined by a given abstract group $G$. We show that if $Z(G)$ is trivial then the automorphism group of this a.a. algebra is isomorphic to $\text{Aut}(G)$. 
We say that $G$ acts sharply $k$-transitively on $X$ if $G$ acts $k$-transitively on $X$ and the pointwise stabilizer of any $k$-point set is the identity group. The first example is concerned with sharply 2-transitive groups. These are Frobenius groups in which the order of a one-point stabilizer is exactly one less than the cardinality of $X$.

**Theorem 2.1.** Suppose that $G$ acts sharply 2-transitively on $X$, a set of cardinality $n$. Let $V$ be the associated permutation module with a basis $\{x_1, \ldots, x_n\}$. Let $t$ be the unique element in $G$ such that $(x_1)^t = x_2$ and $(x_2)^t = x_1$. Let $m$ be the greatest integer not exceeding $n/2$.

Rearrange the numbering if necessary, define an a.a. algebra on $V$ by the following:

\[ x_i x_i = x_i \text{ for all } i, \]

\[ x_1 x_2 = \sum_{k=1}^{m} r(1,2;k)(x_k - (x_k)^t), \]
\[ x_1 x_j = (x_1 x_2)^p \] for \( i \neq j \) where \( p \) is a unique element in \( G \) such that
\[ (x_1)^p = x_i \quad \text{and} \quad (x_2)^p = x_j. \]

If all \( r(1,2;k)'s \) are non-zero and have different absolute values, then \( \text{Aut}(V) \) is isomorphic to \( G \).

Proof. We first determine the possible \( G \)-invariant a.a. algebra structures on \( V \). In a.a. algebras, the square of any basis element satisfies \( x_1 x_1 = x_1 \).

So we need to consider only \( x_i x_j \) for \( i \neq j \). Put
\[ x_1 x_j = \sum_{k=1}^{n} r(i,j;k)x_k. \]

By comparing the coefficients of \((x_i x_j)^g\) and \((x_i)^g (x_j)^g\) for an element \( g \) in \( G \), we get
\[ r(i^g,j^g;k^g) = r(i,j;k). \]

Since \( G \) acts sharply 2-transitively on \( X \), for any pair \( i \neq j \) there exists a unique element \( g \) in \( G \) such that \( 1^g = i \) and \( 2^g = j \). To describe the binary product on \( V \), it suffices to consider \( r(1,2;k) \) for \( k = 1, \ldots, n \). Namely, if a pair of indices \( i, j \) with \( i \neq j \) is given, then there is a unique element \( g \) in \( G \) such that \( 1^g = i \) and \( 2^g = j \).

Therefore, the coefficients satisfy
\[ r(i,j;k) = r(1,2;k^{g^{-1}}) \]
for \( k = 1, \ldots, n. \)
Next we investigate the interrelation of the
\( r(1,2; k) \) for \( k = 1, \ldots, n \). Consider \( G\{1,2\} \) with \( r(1,2; k) \) for \( k = 1, \ldots, n \). Consider \( G\{1,2\} \) up the global stabilizer of \{1,2\} in \( G \). The subgroup \( G\{1,2\} \) is isomorphic to \( S_n \).

Comparing the coefficients of \( x_1x_2 \) and \( x_2x_1 \), we get
\[
 r(1,2; k) = - r(2,1; k^t).
\]

Rearrange the numbering if necessary, we get
\[
 x_1x_2 = \sum_{k=1}^{m} r(1,2; k)(x_k - (x_k^t)),
\]
where \( k \) runs through a complete set of representatives of \( G\{1,2\} \)-orbits of length two.

Next, we determine the structure of \( \text{Aut}(V) \)
with a certain set of \( \{r(i,j;k)\} \). If we put
\[
 r(1,2; k) = 0
\]
for \( k = 3, \ldots, m \), then by theorem 1.10, \( \text{Aut}(V) \) is isomorphic to \( S_n \).

Let us take another set of coefficients. Suppose that all \( r(1,2; k) \)'s are non-zero and have different absolute values. We shall show that \( \text{Aut}(V) \) is isomorphic to \( G \).

For any element \( a \) in \( \text{Aut}(V) \), which induces a permutation on the basis elements of \( V \), we have
\[
 (x_1x_2)^a = x_s x_u \quad \text{where} \quad s = 1^a, \ u = 2^a.
\]
On the other
hand, there exists a unique element $g$ in $G$ such that $1^g = s$, $2^g = u$. We want to show that $i^g = i^a$ for all $i$. Compare the coefficients of $$\sum_{k=1}^{m} r(1,2;k)((x_k)^a - (x_k^t)^a)$$ and $$\sum_{k=1}^{m} r(1,2;k)((x_k)^g - (x_k^t)^g).$$

Since $a$ and $g$ permute the basis elements $x_1, \ldots, x_n$, by the assumption on the $r(1,2;k)$'s we get $$(x_i)^a = (x_i)^g$$ for all $i$.

This implies the result.

Next, we look at sharply 3-transitive groups. As we considered them in lemma 1.8, an a.a. algebra $V$ can not have the same basis as $X$ in order that $\text{Aut}(V)$ is relatively small. So we take a non-standard permutation representation of $G$.

**Theorem 2.2.** Suppose that $G$ acts sharply 3-transitively on $X$. Then there exists a $G$-invariant a.a. algebra $V$ of dimension $|X|(|X| - 1)$ such that $\text{Aut}(V)$ is isomorphic to $G$.

**Proof.** The group $G$ acts on the ordered pairs of $X$ naturally; for any element $x$ in $G$ and
(i,j) in X×X, define

\[(i,j)^X = (i^X, j^X).\]

There are two G-orbits in X×X: the diagonal set
\[\{(i,i) \mid i \in X\}\] and the off-diagonal set
\[Y = \{(i,j) \mid i \neq j\}.\]

We define a permutation module V associated with the action of G on the off-diagonal set Y. Let \[\{e(i,j) \mid (i,j) \in Y\}\] be a permutation basis of V. So we have

\[\dim(V) = |Y| = |X|(|X| - 1).\]

Define a binary product on V as follows:

(2.2.1) \[e(i,j)e(i,j) = e(i,j)\] for all (i,j) in Y.

(2.2.2)
\[e(1,2)e(1,3) = \sum r(1,2,1,3;s,t)(e(s,t) - e(s^g,t^g))\]

where \(g\) is the unique element in G such that \(1^g = 1, 2^g = 3, 3^g = 2\) and (s,t) runs through a complete set of representatives of \(\langle g \rangle\)-orbits of Y. Since \(g^2\) fixes 1, 2 and 3, we have \(g^2 = 1\). So each \(\langle g \rangle\)-orbit consists of at most two elements of Y. We see that (2.2.2) is well-defined. Note that if (s,t) = (s^g,t^g) then \[r(1,2,1,3;s,t) = 0.\] And if this happens, \(1 \in \{s, t\}\).

(2.2.3) \[e(i,j)e(i,k) = (e(1,2)e(1,3))^h\] for \(j \neq k,\)
where \( h \) is the unique element in \( G \) such that 
\[ l^h = i, \quad 2^h = j \quad \text{and} \quad 3^h = k. \]

(2.2.4) \( e(i,j)e(p,q) = 0 \) if \( i \neq p \).

Since we have defined the binary product on each \( G \)-orbit on \( Y \times Y \), it is easy to see that \( V \) has a \( G \)-invariant algebra structure. We now make some additional assumptions on the coefficients in order to make \( \text{Aut}(V) \) relatively small. Let us set

(2.2.5) \( r(1,2,1,3; s,t) \neq r(1,2,1,3; p,q) \) if 
(\( s,t \) \( \neq \) \( s^g,t^g \)), (\( p,q \) \( \neq \) \( p^g,q^g \)) and \( (s,t) \neq (p,q) \).

We shall show that \( \text{Aut}(V) \) is isomorphic to \( G \). The last condition (2.2.5) is important, as \( \text{Aut}(V) \) depends on the interrelationship of the coefficients. For example, if all \( r(1,2,1,3; s,t) \)'s are equal to zero, then by theorem 1.10, \( \text{Aut}(V) \) is isomorphic to \( S_m \) where \( m = \dim(V) \).

By theorem 1.3, \( \text{Aut}(V) \) and \( G \) are regarded as subgroups of \( S_Y \), and \( \text{Aut}(V) \) contains \( G \). For any element \( a \) in \( \text{Aut}(V) \) and \( j \neq k \), we have

\[ (e(i,j)e(i,k))^a = e(i,j)^a e(i,k)^a \neq 0. \]

By (2.2.4) if \( (i,j)^a = (u,v) \) and \( (i,k)^a = (p,w) \), then we must have \( u = p \). Thus, \( (i,j)^a = (u,v) \) and \( (i,k)^a = (u,w) \). Since \( G \) is sharply 3-transitive
on $X$, there exists a unique element $h$ in $G$ such that $i^h = u$, $j^h = v$ and $k^h = w$. To get the conclusion, we must show

$$(s,t)^a = (s^h,t^h)$$ for all $(s,t)$ in $Y$.

Compare the coefficients in $(e(1,2)e(1,3))^a$ and $(e(1,2)e(1,3))^h$:

$$(e(1,2)e(1,3))^a = \sum r(1,2,1,3;s,t)(e(s,t)^a - e(s^g,t^g)^a)$$

and

$$(e(1,2)e(1,3))^h = \sum r(1,2,1,3;s,t)(e(s,t)^h - e(s^g,t^g)^h).$$

We have $(s,t)^a = (s,t)^h = (s^h,t^h)$ if $(s,t) \neq (s^g,t^g)$. As we mentioned, unless $1 \in \{s,t\}$, we get $(s,t) \neq (s^g,t^g)$. Thus, we have to show

$$(s,t)^h = (s^g,t^g)$$ for $s = 1$ or $t = 1$.

Let us consider the coefficients in $e(i,1)e(i,2)$ for $1 \neq i \neq 2$. There exists a unique element $x$ in $G$ such that $1^x = i$, $2^x = 1$ and $3^x = 2$.

We have

$$e(i,1)e(i,2) = \sum r(1,2,1,3;s,t)(e(s,t)^x - e(s^g,t^g)^x).$$

Now compare the coefficients in $(e(i,1)e(i,2))^a$ and $(e(i,1)e(i,2))^h$:

$$(e(i,1)e(i,2))^a = \sum r(1,2,1,3;s,t)(e(s^x,t^x)^a - e(s^gx,t^gx)^a)$$

and

$$(e(i,1)e(i,2))^h = \sum r(1,2,1,3;s,t)(e(s^x,t^x)^h - e(s^gx,t^gx)^h).$$
We see that \((s^x, t^x)^a = (s^x, t^x)^h\) if the coefficient 
\(r(1,2,1,3;s,t) \neq 0\), that is, if \((s,t) \neq (s^g, t^g)\).

Since \((2,k) \neq (2^g, k^g)\) for any \(k \neq 2\), we have 
\((2^x, t^x)^a = (2^x, t^x)^h\) for any \(t \neq 2\). Therefore,
we have \((1, t^x)^a = (1, t^x)^h\). This completes the proof.

The previous trick seems to work without assuming sharp transitivity. In the following example, \(G\) acts 2-transitively but not 3-transitively on a set of seven points. The two-point stabilizer is a four-group, so \(G\) is not sharply 2-transitive.

**Theorem 2.3.** Let \(G = GL(3,2)\), a simple group of order 168. The group \(G\) acts on \(X\), the projective lines of \(PG(3,2)\), and the cardinality of \(X\) is 7. Then there exists a 42-dimensional a.a. algebra \(V\) whose automorphism group is isomorphic to \(G\).

**Proof.** The set \(X\) is identified with \(W^* = W \setminus \{0\}\), where \(W\) is a 3-dimensional vector space over the field of two elements. Let \(V\) be a vector space over the complex number field with a basis \(v_{xy}\) where \((x,y)\) is in

\[ D = \{(x,y) \mid x \neq y, x \text{ and } y \text{ are in } W^*\}, \]

the off-diagonal entries of \(W^* \times W^*\). We see that
G acts on V naturally, as

\[(v_{xy})^g = v_{xy}^g = v_{x^g y^g}\]

for any element \(g\) in \(G\).

Fix a basis \(\{a, b, c\}\) of \(W\). Define a binary product on \(V\) as follows:

(2.3.1) \(v_{xy}v_{xy} = v_{xy}'\)

(2.3.2) \(v_{xy}v_{xz} = 0\) if \(\dim_{GF(2)} \langle x, y, z \rangle = 2\) and \(y \neq z\),

(2.3.3) \(v_{xy}v_{zu} = 0\) if \(x \neq z\),

(2.3.4) \(v_{xy}v_{xz} = (v_{ab}v_{ac})^g\) if \(x, y, z\) are linearly independent in \(W\), and \(g\) is the unique element in \(GL(3,2)\) such that \(a^g = x, b^g = y, c^g = z\).

In order to complete the definition of the binary product, we only need to define \(v_{ab}v_{ac}\). Put

\[v_{ab}v_{ac} = \sum_{(x,y) \in D} r_{xy} v_{xy}'.\]

There exists a unique element \(h\) of order two in \(GL(3,2)\) such that \(a^h = a, b^h = c, c^h = b\). In order that the binary product on \(V\) is \(G\)-invariant,

\[(v_{ab}v_{ac})^h = (v_{ab})^h(v_{ac})^h = v_{ac}v_{ab} = -(v_{ab}v_{ac}).\]

By comparing the coefficients, we must have
(2.3.5) \[ v_{ab}v_{ac} = \sum r_{xy}(v_{xy} - (v_{xy})^h), \]

where \((x,y)\) runs through the complete set of representatives of \(<h>-orbits of D. Namely,

\[ r_{xy} = 0 \text{ if } (x,y)^h = (x,y). \]

Note that the fixed points of \(h\) in \(D\) are the off-diagonal entries of

\[ \{(a, a + b + c, b + c) \times (a, a + b + c, b + c)\}. \]

There are six of them, namely \((a, a + b + c), (a + b + c, a), (a, b + c), (b + c, a), (a + b + c, b + c)\) and \((b + c, a + b + c)\).

We choose a special set of coefficients. We already know that \(r_{xy} = 0\) if \((x,y)\) is fixed by \(h\). For the rest of the \(r_{xy}\)'s, we choose them all non-zero such that their absolute values are all different.

We shall show that any element \(p\) in \(\text{Aut}(V)\) induces a permutation on \(PG(3,2)\), which implies that \(p\) belongs to \(\text{GL}(3,2)\). By theorem 1.3, an element \(p\) is a permutation on the set \(D\). So we may write

\[ (v_{xy})^p = v((xy)p) \]

for \((x,y)\) in \(D\).

Apply \(p\) to (2.3.5):

(2.3.6) \[ (v_{ab}v_{ac})^p = \sum r_{xy}(v(xy)^p - v(xy)^hp). \]
Since \((a,b)^p \neq (a,c)^p\) and \((v_{ab}v_{ac})^p \neq 0\), by (2.3.1) through (2.3.4) \((a,b)^p\) and \((a,c)^p\) satisfy the conditions of (2.3.4); if we put \((a,b)^p = (x,y)\) and \((a,c)^p = (z,u)\), we have \(x = z\) and these vectors \(x, y\) and \(u\) span \(W\) over \(GF(2)\). There exists a unique element \(g\) in \(GL(3,2)\) such that \(a^g = x\), \(b^g = y\) and \(c^g = u\).

Compare (2.3.6) with

\[
(2.3.7) \quad (v_{ab})^g(v_{ac})^g = \sum_{x,y} r_{xy} (v_{xy})^g - (v_{xy})^{hg}.
\]

By the choice of the \(r_{xy}\)'s, we see that

\[
v_{xy}^p = v_{xy}^g \quad \text{and} \quad v_{xy}^{hp} = v_{xy}^{hg}
\]

if \((x,y)\) belongs to the \(\langle h \rangle\)-orbits of length two. We have shown that the actions of \(g\) and \(p\) coincide on the \(\langle h \rangle\)-orbits of length two of \(D\). Now we have to check that the actions of \(g\) and \(p\) coincide on the rest of the set \(D\), namely on the \(\langle h \rangle\)-orbits of length one of \(D\).

Consider \(v_{ad}v_{ca}\) where \(d = a + b + c\). There is a unique element \(k\) in \(GL(3,2)\) such that

\[
a^k = c, \quad b^k = d \quad \text{and} \quad c^k = a.
\]

Since \(d^h = a + c\), we have \(d^h \neq d\). Now \((c,d)\) is not fixed by \(h\). By the previous argument, we have

\[
(c,d)^p = (c,d)^g.
\]

Now we get
\[(v_{ab}v_{ac})^{kp} = (v_{cd}v_{ca})^{p} = (v_{cd})^{p}(v_{ca})^{p} = (v_{cd})^{g}(v_{ca})^{g} = (v_{cd}v_{ca})^{g} = (v_{ab}v_{ac})^{kg}.\]

We get
\[\sum r_{xy}(v_{xy})^{kp} - v_{xy}^{hkp} = \sum r_{xy}(v_{xy})^{kg} - v_{xy}^{hkp}.\]

By the choice of the \(r_{xy}\)'s, we have
\[(x, y)^{kp} = (x, y)^{kg}\]
where each \((x, y)\) belongs to an \(\langle h \rangle\)-orbit of length two in \(D\). On the other hand, all of the off-diagonal entries of
\[
\{a, a + b + c, b + c\} \times \{a, a + b + c, b + c\}
\]
are images of some elements in \(\langle h \rangle\)-orbits of length two under the mapping induced by the element \(k\).

For instance,
\[
(a, a + b + c) = (c, b)^{k},
(a, b + c) = (c, b + c)^{k},
(a + b + c, b + c) = (b, b + c)^{k}.
\]

We see that \((x, y)^{p} = (x, y)^{g}\) for all elements in \(D\).

In particular, \(p\) acts on \(W^{*}\) by
\[
(x)^{p} = (x)^{g}\quad \text{for any } x \in W^{*}.
\]

This implies the conclusion.
SECTION 2
ORTHOGONAL GROUPS

In this section, we consider the orthogonal groups of vector spaces over the finite fields. We construct an a.a. algebra whose automorphism group is isomorphic to a given orthogonal group. We follow Nagao[10] to state some basic facts on the inner products and the orthogonal groups.

Let $W$ be a vector space of finite dimension over a finite field $K$. Suppose that $W$ has an inner product corresponding to a field automorphism $t$ of $K$; that is,

$$(u,v) \in K,$$

$$(au + bv, w) = a(u,w) + b(v,w),$$

$$(u, av + bw) = a^t(u,v) + b^t(u,w)$$

for any vectors $u, v, w$ in $W$ and any elements $a, b$ in $K$. We do not assume the symmetry of the orthogonality of two vectors; $(u,v) = 0$ does not always imply $(v,u) = 0$. But we assume that the inner product is non-degenerate; if $(x,u) = 0$ for all vectors $x$ in $W$, then $u = 0$. In fact, this
condition is equivalent to that if \((u,x) = 0\) for all vectors \(x\) in \(W\), then \(u = 0\). This is also equivalent to: if \(\{w_1, \ldots, w_n\}\) is a basis of \(W\) then the Gramian matrix \((w_i^*, w_j^*)\) is non-singular.

If an element \(p\) in \(\text{GL}(W)\) satisfies
\[
(u^p, v^p) = (u, v)
\]
for any vectors \(u, v\) in \(W\), then \(p\) is called an isometry. The set of all isometries forms a subgroup of \(\text{GL}(W)\) which is denoted by \(\text{Aut}(W, (, ))\) and we call it the orthogonal group of \(W\) with respect to the inner product \((, )\).

First, we need a pair of lemmas.

**Lemma 2.4.** The set \(A(n) = \{1, 2, 4, \ldots, 2^{n-1}\}\) has the following property: if
\[
a - b = c - d \neq 0
\]
for any elements \(a, b, c, d\) in \(A(n)\), then
\[
a = c \quad \text{and} \quad b = d.
\]

**Proof.** We may assume that \(a - b = c - d\) is positive. Let \(a = 2^i, b = 2^j, c = 2^s\) and \(d = 2^t\). We have
\[
a - b = 2^j(2^{i-j} - 1) \quad \text{and} \quad c - d = 2^t(2^{s-t} - 1).
\]
Comparing the factors, we see that \(a = c\) and \(b = d\) as desired.
**Lemma 2.5.** Let $W$ be an $n$-dimensional vector space over a finite field $K$. Let $W$ have a non-degenerate inner product. Suppose that $s$ is a permutation on the vectors of $W$ such that

$$(a^s, d^s) = (a, d)$$

for any vectors $a$ and $d$ in $W$. Then $s$ is a $K$-linear mapping from $W$ to $W$.

**Proof.** First we show that $s$ is an additive mapping. By the assumption, for any vector $d$ in $W$ we have

$$((a + b)^s, d^s) = (a + b, d)$$

$$= (a, d) + (b, d)$$

$$= (a^s, d^s) + (b^s, d^s)$$

$$= (a^s + b^s, d^s).$$

Since the inner product is non-degenerate, we have

$$(a + b)^s = a^s + b^s.$$

Next, we show that $s$ is a $K$-mapping. For any element $k$ in $K$ and any vectors $a$ and $d$ in $W$, we have

$$((ka)^s, d^s) = (ka, d)$$

$$= k(a, d)$$

$$= k(a^s, d^s)$$

$$= (ka^s, d^s).$$
We get \((ka)^S = ka^S\).

Thus, \(s\) is a \(K\)-linear mapping of \(W\). This completes the proof.

In the previous lemma, the mapping \(s\) is a permutation (one-to-one and onto) on \(W\). So \(s\) belongs to \(GL(W)\). Since it preserves the inner product, we see that \(s\) is in the orthogonal group of \(W\) with respect to the inner product.

**Theorem 2.6.** Let \(W\) be an \(n\)-dimensional vector space over a finite field \(K\). Let \(W\) have a non-degenerate inner product associated with a field automorphism \(t\). There exists an a.a. algebra \(V\) such that \(Aut(V)\) is isomorphic to the orthogonal group of \(W\) with respect to the inner product.

**Proof.** Define an a.a. algebra \(V\) over the complex number field as follows: \(V\) has a basis \(\{ v_a \mid a \in W \} \) such that

\[
(v_a)(v_a) = v_a \quad \text{for all } a \in W,
\]

\[
(v_a)(v_b) = \sum_{k \in K} r_k \left( \sum_{(a,c) = k} v_c - \sum_{(b,d) = k} v_d \right)
\]

for \(a \neq b\), where the \(r_k\)'s are complex numbers. This defines an a.a. algebra structure on \(V\), since
\[(v_b)(v_a) = \sum_{k \in K} r_k \left( \sum_{(b,d)=k} v_d - \sum_{(a,c)=k} v_c \right) \]
\[= -(v_a)(v_b).\]

We note the following:

\[(v_a)(v_b) = \sum_{k \in K} \sum_{(a,c)=k} r_k v_c - \sum_{k \in K} \sum_{(b,d)=k} r_k v_d \]
\[= \sum_{c \in W} r(a,c)v_c - \sum_{d \in W} r(b,d)v_d \]
\[= \sum_{c \in W} (r(a,c) - r(b,c))v_c.\]

After defining an order on \( K \), choose the coefficients \( \{r_k\} \) to be
\[A(|K|) = \{1, 2, 4, \ldots, 2^{|K|} - 1\}\]
as in lemma 2.4. By theorem 1.3, \( \text{Aut}(V) \) is isomorphic to a subgroup of \( S_W \). For \( s \) in \( \text{Aut}(V) \) and \( v_a \) in \( V \), define \( a^s \) in \( W \) by
\[v(a^s) = (v_a)^s.\]

Similarly, if \( p \) is a permutation on the vectors of \( W \), identify \( p \) as an element in \( \text{GL}(V) \) by
\[(v_a)^p = v(a^p).\]
Let \( p \) be an element of the orthogonal group of \( W \); that is, \((a^P, b^P) = (a, b)\) for any vectors \( a \) and \( b \) in \( W \). Let us check that \( p \) preserves the a.a. algebra structure of \( V \). First, we see

\[
(v_a v_a)^P = (v_a)^P = v(a^P), \text{ while }
\]

\[
(v_a^P v_a^P) = v(a^P).
\]

Next, suppose \( a \neq b \), we have

\[
(v_a v_b)^P = \sum_{c \in W} (r(a, c) - r(b, c))v(c^P)
\]

\[
= \sum_{c \in W} (r(a^P, c^P) - r(b^P, c^P))v(c^P)
\]

\[
= v(a^P)v(b^P).
\]

So \( p \) preserves the a.a. algebra structure of \( V \).

On the other hand, for any element \( s \) in \( \text{Aut}(V) \) we check that \( s \) preserves the inner product. For two distinct elements \( a \) and \( b \) in \( W \), we have

\[
(v_a v_b)^S = \sum_{c \in W} (r(a, c) - r(b, c))v(c^S),
\]

\[
v(a^S)v(b^S) = \sum_{d \in W} (r(a^S, d) - r(b^S, d))v_d.
\]

Comparing the coefficients, we have

\[
r(a, c) - r(b, c) = r(a^S, c^S) - r(b^S, c^S)
\]
for any vector \( c \) in \( W \). By lemma 2.4, we get
\[
\rho(a, c) = \rho(a^s, c^s) \text{ if } \rho(a, c) \neq \rho(b, c).
\]
This implies that if a vector \( b \) exists in \( W \) such that
\[
(a, c) \neq (b, c) \quad \text{then} \quad (a, c) = (a^s, c^s).
\]
Since the inner product is non-degenerate, such a vector \( b \) always exists in \( W \) provided \( c \neq 0 \).

On the other hand, if \( c = 0 \) we have
\[
\rho(a, 0) - \rho(b, 0) = \rho(a^s, 0^s) - \rho(b^s, 0^s).
\]
Obviously, the left hand side is equal to zero.

Therefore, we have
\[
(a^s, 0^s) = (b^s, 0^s)
\]
for any vectors \( a \) and \( b \). We see that
\[
(a^s - b^s, 0^s) = 0.
\]
Since \( a^s - b^s \) ranges all the vectors in \( W \), we have
\[
0^s = 0.
\]
We conclude that
\[
(a, c) = (a^s, c^s)
\]
for any vectors \( a \) and \( b \) in \( W \).
By lemma 2.5, s is a K-linear mapping. We also noted that s belongs to the orthogonal group of W. This completes the proof.
SECTION 3

AUT(V) AND AUT(G)

In this section, we construct an a.a. algebra $V$ associated to a given abstract group $G$. We find some relations between Aut($V$) and Aut($G$). We assume $G \neq 1$ throughout this section.

For a given finite group $G$, let $V$ be a vector space over the complex number field with a basis \{ $v_g \mid g \in G$ \}.

Define an a.a. algebra structure on $V$ as follows:

$$(v_g)(v_h) = v_{gh},$$

$$(v_g)(v_h) = v_{gh} - v_{hg} \text{ if } g \neq h.$$ 

Let $G$ act on $V$ by conjugation;

$$(v_g)^x = v_{(x^{-1}gx)}$$

for any element $x$ in $G$. By this action, the algebra structure on $V$ is $G$-invariant.

Notice that if $g$ commutes with $h$, $g \neq h$, then $(v_g)(v_h) = 0$. Namely, if $G$ is abelian, Aut($V$) is isomorphic to the symmetric group of degree $|G|$. This is an immediate corollary to theorem 1.10.

The following is true for any group $G$:
Lemma 2.7. \textit{Aut}(G) is isomorphic to a subgroup of \textit{Aut}(V).

Proof. For any element \( t \) in \textit{Aut}(G), define a linear mapping \( a(t) \) from \( V \) to \( V \) by

\[(v_g)_{a(t)} = v_{g^t}\]

for the basis elements and we extend this linearly. Since

\[\begin{align*}
(v_g v_h)_{a(t)} &= (v_{gh} - v_{hg})_{a(t)} \\
&= (v_{gh})_{a(t)} - (v_{hg})_{a(t)} \\
&= v_{(gh)^t} - v_{(hg)^t} \\
&= v_{g^t h^t} - v_{h^t g^t} \\
&= (v_{g^t})_{(v_{h^t})} \\
&= (v_g a(t))_{(v_h)_{a(t)}},
\end{align*}\]

\(a(t)\) preserves the algebra structure. Next we see that \(a(ts) = a(t)a(s)\) for any elements \( t, s \) in \textit{Aut}(G) as follows. We have

\[\begin{align*}
(v_g)_{a(ts)} &= v_{g^{ts}} \\
&= (v_{g^t})_{a(s)} \\
&= (v_g a(t))_{a(s)}.
\end{align*}\]

If \( t \) is a non-trivial automorphism of \( G \), there exists an element \( g \) in \( G \) with \( g^t \neq g \). Therefore,
Thus, $a$ is an injective homomorphism from $\text{Aut}(G)$ to $\text{Aut}(V)$.

Conversely, any element $a$ in $\text{Aut}(V)$ induces a permutation on the basis elements \[ \{v_g \mid g \in G\}. \]
For any element $g$ in $G$, we define $g^a$ by
\[ (g^a) = (v_g)^a. \]
In this way, the element $a$ in $\text{Aut}(V)$ induces a permutation on the elements of $G$. But this may not be an automorphism of $G$. For example, as we have already seen that if $G$ is abelian then $\text{Aut}(V)$ is isomorphic to the full permutation group on the elements of $G$.

Naturally, we ask when $\text{Aut}(V)$ is reasonably small or isomorphic to $\text{Aut}(G)$. Let us consider this question.

**Lemma 2.8.** Let $a$ be an element in $\text{Aut}(V)$. For any elements $x$ and $y$ in $G$, $x$ and $y$ commute if and only if $x^a$ and $y^a$ commute. Moreover, if $x$ and $y$ do not commute, then
\[ (xy)^a = x^a y^a. \]

**Proof.** Notice that $x$ and $y$ do not commute if and only if
(v_x)(v_y) = v_{xy} - v_{yx} \neq 0.

Apply a to get

\[ (v_x v_y)^a = (v_{xy})^a - (v_{yx})^a. \]

The left hand side is

\[ (v_x v_y)^a = (v_x)^a (v_y)^a \]
\[ = (v_x^a)(v_y^a) \]
\[ = v_{x y^a}^a - v_{y x^a}^a. \]

The right hand side is

\[ (v_{xy})^a - (v_{yx})^a = (xy)^a - (yx)^a. \]

Since Aut(V) is a subgroup of GL(V), we have

\[ (v_x v_y)^a \neq 0. \]

So we get

\[ v_{x y^a}^a - v_{y x^a}^a = (xy)^a - (yx)^a \neq 0. \]

This implies that if x and y do not commute then \( x^a \) and \( y^a \) do not commute. Considering the inverse mapping \( a^{-1} \), the first assertion is true.

Now, compare both sides of the last equation. Since each vector is in the basis of V, we get

\[ v_{x y^a}^a = (xy)^a \quad \text{and} \quad v_{y x^a}^a = (yx)^a. \]

This implies the second assertion.
Lemma 2.9. Suppose that for an element $z$ in $G$ there exists an element $w$ which does not commute with $z$. Then for any element $a$ in $\text{Aut}(V)$ the following holds:

$$(z^{-1})^a = (z^a)^{-1}.$$  

Proof. The hypothesis implies that $z^{-1}$ does not commute with $zw$. By the previous lemma,

$$w^a = (z^{-1}zw)^a$$

$$= (z^{-1})^a (zw)^a$$

$$= (z^{-1})^a z^a w^a.$$  

Thus, as $(z^{-1})^a z^a = 1$ we get the result.

Lemma 2.10. Let $x$ and $y$ be commutative elements of $G$. Suppose that there exists $z$ in $G$ such that $z$ does not commute with $x$, $y$ and $xy$. Then for any element $a$ in $\text{Aut}(V)$,

$$(xy)^a = x^a y^a.$$  

Proof. The hypothesis implies that $y$ does not commute with $z^{-1}$. Furthermore,

$xy = (xz)(z^{-1}y)$ and

$$(z^{-1}y)(xz) = z^{-1}(xy)z \neq xy,$$

so $xz$ and $z^{-1}y$ do not commute.
By lemmas 2.8 and 2.9, we have
\[(xy)^a = (xzz^{-1}y)^a\]
\[= (xz)^a(z^{-1}y)^a\]
\[= xza(z^{-1})^aya\]
\[= x^{a_z}(z^{-1})^aya\]
\[= x^{a_y}a.\]

This implies the result.

**Lemma 2.11.** Suppose that $Z(G) = 1$. If $x$ and $y$ in $G$ commute, then there exists $z$ in $G$ such that $z$ does not commute with $x$, $y$ and $xy$.

**Proof.** Suppose that the assertion is false. Then $G$ is the union of $C_G(x)$, $C_G(y)$ and $C_G(xy)$. Note that these three subgroups are all distinct and proper subgroups of $G$. We have
\[C_G(x) \cap C_G(y) \subseteq C_G(xy).\]
As $C_G(g) = C_G(g^{-1})$ for any element $g$ in $G$, we also have
\[C_G(xy) \cap C_G(y) \subseteq C_G(x)\]
and
\[C_G(x) \cap C_G(xy) \subseteq C_G(y).\]
If we put $K = C_G(x) \cap C_G(y)$, we see that
\[K = C_G(x) \cap C_G(y) \cap C_G(xy)\] and
\[ K = C_G(xy) \cap C_G(y) = C_G(x) \cap C_G(xy). \]

By counting elements, we have
\[ |G| = |K| + |C_G(x) \setminus K| + |C_G(y) \setminus K| + |C_G(xy) \setminus K|. \]

We also have
\[ (2.11.1) \quad |G| = |C_G(x)| + |C_G(y)| + |C_G(xy)| - 2|K|. \]

Divide both sides of (2.11.1) by \( K \), we get
\[ (2.11.2) \quad \frac{|G|}{|K|} = \frac{|C_G(x)|}{|K|} + \frac{|C_G(y)|}{|K|} + \frac{|C_G(xy)|}{|K|} - 2. \]

Suppose that all the indices of \( C_G(x) \), \( C_G(y) \) and \( C_G(xy) \) in \( G \) are greater than 2;
\[ |G|/|C_G(x)| \geq 3, \]
\[ |G|/|C_G(y)| \geq 3 \quad \text{and} \]
\[ |G|/|C_G(xy)| \geq 3. \]

By (2.11.1), we get
\[ |G| \leq \frac{1}{3} (|G| + |G| + |G|) - 2|K| \]

a contradiction. Hence, one of the indices is equal to 2.

We may assume \( |G|/|C_G(x)| = 2 \) (the argument will proceed similarly if we assume \( |G|/|C_G(y)| = 2 \) or \( |G|/|C_G(xy)| = 2 \)). The cardinality of the conjugacy class in \( G \) containing \( x \) is
\[ |G|/|C_G(x)|. \]
The cardinality of the conjugacy class in $C_G(xy)$ containing $x$ is
\[ \frac{|C_G(xy)|}{|C_G(xy) \cap C_G(x)|} = \frac{|C_G(xy)|}{|K|}. \]
Similarly, the cardinality of the conjugacy class in $C_G(y)$ containing $x$ is
\[ \frac{|C_G(y)|}{|C_G(y) \cap C_G(x)|} = \frac{|C_G(y)|}{|K|}. \]
Comparing the sizes of the conjugacy classes, we get
(2.11.3) $|G|/|C_G(x)| \geq |C_G(xy)|/|K|$ and
(2.11.4) $|G|/|C_G(x)| \geq |C_G(y)|/|K|.$
By the assumption, we have
\[ 2 = \frac{|C_G(xy)|}{|K|} = \frac{|C_G(y)|}{|K|}. \]
By (2.11.2), we have
\[ \frac{|G|}{|K|} = \frac{|C_G(x)|}{|K|} + 2. \]
By (2.11.3), we have
\[ \frac{|G|}{|K|} \geq \frac{|C_G(xy)|}{|K|} \cdot \frac{|C_G(x)|}{|K|} = 2 \frac{|C_G(x)|}{|K|}. \]
So we get
\[ 2 \frac{|C_G(x)|}{|K|} \leq \frac{|C_G(x)|}{|K|} + 2. \]
Thus, $|C_G(x)|/|K| = 2.$

The index of $K$ in $G$ is 4, and the index of $K$ in each of the three subgroups is 2. So $K$ is a normal subgroup of $G$. Then $A = G/K$ is a four-group.
By conjugation, $A$ acts on $Z(K)$, the center of $K$, which contains $x$, $y$ and $xy$. If an element $g$ in $Z(K)$ is fixed by $A$, then $g$ must belong to $Z(G)$, which is a trivial group. So $A$ acts on $Z(K)$ fixed-point-freely. A four-group can not act on a group of even order fixed-point-freely (theorem 6.2.3,[4]), therefore $Z(K)$ is of odd order.

For any element $h$ in $C_G(xy)$ $K$, we have that $h$ acts on the normal subgroups $Z(C_G(x))$ and $Z(C_G(y))$. The quotient group $G/C_G(x)$ of order 2 is generated by $\overline{h}$, the image of $h$. The action of $h$ on $Z(C_G(x))$ is fixed-point-free, as any fixed point of $Z(C_G(x))$ is contained in $Z(G)$.

Then by theorems 6.2.3 and 10.1.4,[4], $Z(C_G(x))$ is abelian of odd order and $h$ inverts every element of $Z(C_G(x))$. Namely, we have $x^h = x^{-1}$. By a similar argument, we get $y^h = y^{-1}$. Since $x$ and $y$ commute, we get $(xy)^h = x^{-1}y^{-1} = (xy)^{-1}$.

On the other hand, $h$ belongs to $C_G(xy)$ so we have $(xy)^h = xy$. This implies that $xy$ is an element of order 2, and that $Z(K)$ is a subgroup of even order. This contradiction completes the proof of the lemma.
Theorem 2.12. If $Z(G) = 1$, then there exists an a.a. algebra $V$ such that $\text{Aut}(V)$ is isomorphic to $\text{Aut}(G)$.

Proof. Define an a.a. algebra $V$ associated to $G$ as in the beginning of this section. The theorem follows from lemmas 2.7, 2.8, 2.10 and 2.11.
CHAPTER III
ALMOST ALTERNATING MULTI-LINEAR MAPPINGS

In this chapter, we generalize the concept of a.a. algebra. Let \( V \) be a vector space over a field \( K \). Remember that we call \( V \) an algebra if a binary product is defined on \( V \). Any binary product on \( V \) canonically induces a bilinear mapping \( f \) from \( V \times V \) to \( V \) by
\[
f(u,v) = uv
\]
for any vectors \( u, v \) in \( V \). Conversely, any bilinear mapping induces a binary product on \( V \).

For a positive integer \( k (k \geq 2) \), denote the direct product of \( k \) copies of \( V \) by \( V^k \);
\[
V^k = V \times \cdots \times V.
\]

As we defined \( \text{Aut}(V) \) for any algebra \( V \), we define \( \text{Aut}(f) \) for any multi-linear mapping:

**Definition 3.1.** For any multi-linear mapping \( f \), the automorphism group \( \text{Aut}(f) \) is the set of all elements \( a \) in \( \text{GL}(V) \) satisfying
\[
f((v_1)^a, \ldots, (v_k)^a) = (f(v_1, \ldots, v_k))^a
\]
for any vectors $v_1, \ldots, v_k$ in $V$.

If $V$ is an a.a. algebra with a basis $e_1, \ldots, e_n$, then the associated bilinear mapping $f$ satisfies

$$f(e_i, e_i) = e_i \text{ for all } i,$$
$$f(e_i, e_j) = -f(e_j, e_i) \text{ for } i \neq j.$$ 

Now we define an almost alternating multi-linear mapping $f$ from $V^k$ to $V$ as follows:

**Definition 3.2.** A multi-linear mapping $f$ from $V^k$ to $V$ is called an almost alternating mapping, denoted a.a. multi-linear mapping for brevity, if $V$ has a basis $e_1, \ldots, e_n$ and $f$ satisfies

(3.2.1) $f(e_i, \ldots, e_i) = e_i$ for all $i$,
(3.2.2) $f(e_{i_1}, \ldots, e_{i_k}) = \text{sgn}(p) f(e_{i_{(1p)}}, \ldots, e_{i_{(kp)}})$

where $i_1, \ldots, i_k$ are all distinct points of $1, \ldots, n$ and $p$ is any element of $S_k$.

(3.2.3) $f(e_{j_1}, \ldots, e_{j_k}) = 0$ otherwise.

When $k = 2$, the last case (3.2.3) does not occur and such a mapping $f$ induces an a.a. algebra on $V$. We prove similar results to chapter I for a.a. multi-linear mappings.
Lemma 3.3. Let $V$ be a vector space. Let $f$ be an a.a. multi-linear mapping from $V^k$ to $V$ with a basis $e_1, \ldots, e_n$. Let

$$v_i = \sum_{j=1}^{n} r_{ij} e_j$$

be any vectors for $i = 1, \ldots, k$. Then we have

$$f(v_1, \ldots, v_k) = \sum_{j=1}^{n} r_{1j} \cdots r_{kj} e_j$$

$$+ \sum_{j_1 < \cdots < j_k} \det [(j_1, \ldots, j_k)] f(e_{j_1}, \ldots, e_{j_k})$$

where $[(j_1, \ldots, j_k)]$ is a minor of size $k \times k$ of the matrix $[r_{ij}]$ consisting of the $j_1$-th, $\ldots$, $j_k$-th columns.

Proof. Recall that the determinant of a square matrix $[c_{ij}]$ of size $m \times m$ is given by

$$\det [c_{ij}] = \sum_{p \in S_m} \text{sgn}(p) c_{1p} \cdots c_{mp}.$$ 

Our proof is a straightforward calculation:

$$f(v_1, \ldots, v_k) = f\left(\sum_{j=1}^{n} r_{1j} e_j, \ldots, \sum_{j=1}^{n} r_{kj} e_j\right)$$

$$= \sum_{j_1=1}^{n} \cdots \sum_{j_k=1}^{n} r_{j_1} \cdots r_{j_k} f(e_{j_1}, \ldots, e_{j_k})$$
= \sum_{j=1}^{n} r_{lj} \cdots r_{kj} e_j

+ \sum_{j_s \neq j_t \text{ if } s \neq t} r_{lj_1} \cdots r_{kj_k} f(e_{j_1}, \ldots, e_{j_k})

= \sum_{j=1}^{n} r_{lj} \cdots r_{kj} e_j

+ \sum_{j_1 < \cdots < j_k} \text{sgn}(p) r_{lj_1} \cdots r_{kj_k} p f(e_{j_1}, \ldots, e_{j_k})

\text{Lemma 3.4.} \text{ Let } V \text{ be a vector space. Let } f \text{ be an a.a. multi-linear mapping from } V^k \text{ to } V \text{ with a basis } e_1, \ldots, e_n. \text{ Let } v \text{ be any vector in } V;

v = \sum_{i=1}^{n} r_i e_i.

Then } f(v, \ldots, v) = v \text{ if and only if } (r_i)^k = r_i \text{ for all } i.

\text{Proof.} \text{ By lemma 3.3, we have}

f(v, \ldots, v) = \sum_{j=1}^{n} (r_j)^k e_j.

By comparing the coefficients, we get the result.
When \( k = 2 \), the above arguments are essentially the same as in theorem 1.3. However, when we consider \( \text{Aut}(f) \), it is slightly different.

**Lemma 3.5.** Let \( V \) be a vector space with a basis \( e_1, \ldots, e_n \). Let \( f \) be an a.a. multi-linear mapping from \( V^k \) to \( V \). Suppose that \( a \) belongs to \( \text{Aut}(f) \). Then for each \( i \) there exists a unique \( t \) in \( \{1, \ldots, n\} \) such that

\[
(e_i)^a = r_{it}e_t \quad \text{and} \quad (r_{it})^k = r_{it}.
\]

**Proof.** Let \( (e_i)^a = \sum_{t=1}^n r_{it}e_t \) and let

\[
A_i = \{ t \mid r_{it} \neq 0 \}.
\]

We shall show that the cardinality of each \( A_i \) is exactly one and the \( A_i \)'s are mutually disjoint.

Since \( (e_i)^a \neq 0 \) for all \( i \), \( A_i \) is not empty. Suppose that the conclusion is false. There exist \( i \) and \( j \) in \( \{1, \ldots, n\} \) such that \( A_i \cap A_j \) is not empty. By lemma 3.3, we have

\[
f((e_i)^a, \ldots, (e_i)^a, (e_j)^a) = \sum_{t \in A_i \cap A_j} (r_{it})^{k-1}r_{jt}e_t
\]

\[
+ \sum_{t_1 < \ldots < t_k} \det \begin{bmatrix} r_{it_1} & \ldots & r_{it_k} \\ \vdots & \ddots & \vdots \\ r_{it_1} & \ldots & r_{it_k} \\ r_{jt_1} & \ldots & r_{jt_k} \end{bmatrix} f(e_{t_1}, \ldots, e_{t_k}).
\]
The first summation does not vanish while the second does. This contradicts
\[ f((e_i)^a, \ldots, (e_i)^a, (e_j)^a) = (f(e_i, \ldots, e_i, e_j))^a = 0. \]
Hence, the \( A_i \)'s are mutually disjoint and each \( A_i \) consists of exactly one index.

Next we consider
\[ f((e_i)^a, \ldots, (e_i)^a) = (f(e_i, \ldots, e_i))^a = (e_i)^a. \]
By lemma 3.4, if \( (e_i)^a = r_1 e_t \) then \( (r_1)^k = r_{1t}. \)

Combining the preceding results, we can conclude that \( \text{Aut}(f) \) is isomorphic to a subgroup of the wreath product of \( \mathbb{Z}_{k-1} \), the cyclic group of order \( k-1 \), by \( S_n \); \( \mathbb{Z}_{k-1} \wr S_n \). If \( f \) vanishes except on \( V_0 \), then \( \text{Aut}(f) \) is isomorphic to \( \mathbb{Z}_{k-1} \wr S_n \). Let us consider a non-trivial case.

**Theorem 3.6.** Take the alternating group \( A_n \) of degree \( n \) and let \( V \) be the permutation module with a basis \( e_1, \ldots, e_n \). There exists an a.a. multi-linear mapping \( f \) from \( V^{n-1} \) to \( V \) such that \( \text{Aut}(f) \) is isomorphic to the direct product of \( \mathbb{Z}_{n-2} \) and \( A_n \); \( \mathbb{Z}_{n-2} \times A_n \).

**Proof.** First we describe a possible a.a. multi-linear mapping \( f \) from \( V^k \) to \( V \). Define
\[ P = \{(i_1, \ldots, i_{n-1}) \mid 1 \leq i_s \leq n, \text{ if } s \neq t \text{ then } i_s \neq i_t \}. \]
It is clear that $S_n$ and $A_n$ act on $P$ in a natural way. Since $A_n$ acts on $P$ regularly and $|P| = \frac{1}{2}(n!)$, there are two $A_n$-orbits of $P$. On the other hand, $S_n$ acts on $P$ transitively. We claim that if $f$ is an a.a. multi-linear mapping from $V_{n-1}$ to $V$, then the value of $f(e_{i_1}, \ldots, e_{i_{n-1}})$ is written in terms of $f(e_1, \ldots, e_{n-1})$, where $(i_1, \ldots, i_{n-1})$ is an element of $P$. For any element $(i_1, \ldots, i_{n-1})$ in $P$, we can find an element $x$ in $S_n$ such that

$$(i_1, \ldots, i_{n-1})^x = (1, \ldots, n-1).$$

Define that

$$f(e_{i_1}, \ldots, e_{i_{n-1}}) = \text{sgn}(x) f(e_1, \ldots, e_{n-1})^{x^{-1}}.$$

Since $S_n$ acts on $P$ regularly, such an element $x$ exists, it is unique and $f(e_{i_1}, \ldots, e_{i_{n-1}})$ is well-defined. Thus, the claim holds.

It suffices to determine the value of $f(e_1, \ldots, e_{n-1})$. For any element $g$ in the stabilizer of the letter $n$ in $A_n$, we have

$$f(e_1, \ldots, e_{n-1})^g = f(e_1, \ldots, e_{n-1}).$$

So the coefficients of $f(e_1, \ldots, e_{n-1})$ are

$$f(e_1, \ldots, e_{n-1}) = A(e_1 + \ldots + e_{n-1}) + B e_n.$$

Next we choose $A$ and $B$ to be non-zero and show that $\text{Aut}(f)$ is isomorphic to $\mathbb{Z}_{n-2} \times A_n$. For an element $a$ in $\text{Aut}(f)$, we write
\[(e_i)^a = r_i e((ia)).\]

We have

\[f((e_1)^a,\ldots,(e_{n-1})^a) = f(r_1 e(1a),\ldots,r_{n-1} e(n-1)a) = r_1 \ldots r_{n-1} \text{sgn}(a)(A(e(1a) + \ldots + e(n-1)a) + B e(na)),\]

where \(\text{sgn}(a)\) is the signature of \(a\) as a permutation on the set of 1-dimensional subspaces \(\langle e_1 \rangle,\ldots,\langle e_n \rangle\).

On the other hand, we have

\[(f(e_1,\ldots,e_{n-1}))^a = A(r_1 e(1a) + \ldots + r_{n-1} e(n-1)a) + B r_n e(na).\]

By comparing the coefficients, we get

\[A r_1 \ldots r_{n-1} \text{sgn}(a) = A r_i \text{ for } i = 1,\ldots,n-1,\]

\[B r_1 \ldots r_{n-1} \text{sgn}(a) = B r_n.\]

Since we chose \(A\) and \(B\) non-zero, we have

\[r_i = r_1 \ldots r_n \text{sgn}(a) \text{ for } i = 1,\ldots,n.\]

Namely, all \(r_i\)'s are the same, we have

\[r_i = (r_1)^{n-1} \text{sgn}(a).\]

By lemma 3.5, we get

\[r_i = r_i \text{sgn}(a).\]

This implies \(\text{sgn}(a) = 1\). If we put \(r = r_1 = \ldots = r_n\),

\[(e_i)^a = r e((ia)).\]
Now we see that $a$ is written as a product of a scalar matrix $r$, where $r^{n-2} = 1$, and a permutation matrix $T$, which is induced by the action of $a$ on the set of 1-dimensional subspaces $\langle e_1 \rangle, \ldots, \langle e_n \rangle$.

Hence $\text{Aut}(f)$ is isomorphic to $\mathbb{Z}_{n-2} \times A_n$. 
CHAPTER IV

$M_{12}$-IN Variant Commutative Algebra

In this chapter, we determine the $M_{12}$-invariant commutative algebra structure on a vector space $V$ of dimension 45 over the rational number field. The group $M_{12}$ is discovered by Mathieu which acts sharply 5-transitively on twelve points. Namely, $M_{12}$ is of order $12 \cdot 11 \cdot 10 \cdot 9 \cdot 8$. It is known that $M_{12}$ has a unique irreducible rational character $\chi$ of degree 45. Let $V$ be an $M_{12}$-module affording $\chi$ over the rational number field. The symmetric part $V_S$ of this module contains $V$ as a constituent exactly once. This implies that there exists a projection mapping $f$ from $V_S$ onto $V$. We can equip $V$ with a commutative algebra structure by putting

$$uv = f(u \otimes v)$$

for any vectors $u, v$ in $V$.

Let us list the following facts to start this project:

(1) The restriction module of $V$ to $M_{11}$ is
irreducible.

Put $U$ to be this restriction; then

(2) the symmetric part $U_S$ contains $U$ as a constituent exactly four times.

If $W$ is the representation module of $S_{11}$ corresponding to the Young diagram $[(9,1,1)]$, then

(3) the restriction module of $W$ to $M_{11}$ is isomorphic to $U$ (see James and Kerber[9], 5.5.40, p.237).

In section 1, we describe the irreducible representation $U$ of $M_{11}$. This is equivalent to giving a representation of $S_{11}$ associated to the Young diagram $[(9,1,1)]$, so we present some general representation theory of symmetric groups.

In section 2, we describe the commutative algebra structure on $U$. As mentioned before, this algebra is $M_{11}$-invariant and $\dim \mathbb{C} \text{Hom}_{M_{11}}(U_S,U) = 4$, so there should be 4 parameters to describe the algebra structure. In section 3, we extend the representation module $U$ to $M_{12}$ to obtain $V$. This is independent on the results of section 2. In section 4, using the results of the former sections, we determine the algebra structure on $V$ invariant under the action of $M_{12}$.
In this section, we follow James and Kerber [9] chapter 7 to describe the \((9,1,1)\)-representation of \(S_{11}\). Let us start with the general theory.

A sequence of non-negative integers
\[
\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots)
\]
is called a partition of \(n\) if it satisfies
\[
\alpha_i \geq \alpha_{i+1} \quad \text{for all } i \leq 1 \text{ and } \sum_{i=1}^n \alpha_i = n.
\]

A partition \(\alpha\) can be illustrated by the corresponding Young diagram \([\alpha]\), which consists of \(n\) nodes in rows. The \(i\)-th row of \([\alpha]\) consists of \(\alpha_i\) nodes, and all the rows start in the same column. For example, the Young diagram \([(9,1,1)]\) is
\[
\begin{array}{cccccccc}
* & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * \\
\end{array}
\]

An \(\alpha\)-tableau \(t\) with Young diagram \([\alpha]\) arises from \([\alpha]\) by replacing the nodes of \([\alpha]\) by the numbers \(i\) of \(\{1,2,\ldots, n\}\) without repetition.
A tableau $t$ is standard if the numbers increase along the rows from left to right and down the columns. For example, the following are standard $[(9,1,1)]$-tableaux:

$$
\begin{array}{ccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
8 & 9 & 10 & 11 & 2 & 3 \\
\end{array}
$$

To each $\alpha$-tableau $t$, we define the row stabilizer $R(t)$ of $t$ to consist of all elements of $S_n$ which preserve every row of $t$ setwise. The column stabilizer $C(t)$ of $t$ is defined similarly; $C(t)$ consists of all elements of $S_n$ which preserve every column of $t$ setwise.

Define an equivalence relation on the set of $\alpha$-tableaux by: $t_1$ is equivalent to $t_2$ if $(t_1)^p = t_2$ for some element $p$ in $R(t_1)$. The $\alpha$-tabloid $\{t\}$ containing $t$ is the equivalence class under this equivalence relation.

Let $K$ be a field of characteristic zero. Consider the $K$-vector space $M(\alpha)$ with all the $\alpha$-tabloids being a basis of $M(\alpha)$. Let $S_n$ act on the set of $\alpha$-tabloids by

$$
\{t\}_p = \{t^p\}
$$

for any element $p$ in $S_n$. The action is well-defined and $M(\alpha)$ is a $K[S_n]$-module.
For each \( \alpha \)-tableau \( t \), the signed column sum 
\[ V(t) = \sum_{p \in C(t)} \text{sgn}(p)p. \]

For each \( \alpha \)-tableau \( t \), the \( \alpha \)-polytabloid \( e(t) \) in \( M(\alpha) \) is defined by 
\[ e(t) = \{ t \} V(t). \]

We call \( e(t) \) a standard polytabloid if \( t \) is standard.

The Specht module \( S^\alpha \) associated with the
partition \( \alpha \) is defined as the subspace of \( M(\alpha) \)
spanned by \( \alpha \)-polytabloids. Let us check that \( S^\alpha \)
is in fact a \( K[S_n] \)-module.

**Lemma 4.1.** The Specht module \( S^\alpha \) is a \( K[S_n] \)-module.

**Proof.** We first show that if an element \( p \) is
in \( C(t) \), then \( p^g = g^{-1}pg \) is in \( C(t^g) \) for any
element \( g \) in \( S_n \). Since \( i \) and \( i^p \) belong to
the same column of \( t \), \( i^g \) and \( (i^p)^g \) also belong
to the same column of \( t^g \). But we have
\[ i^pg = i^gg^{-1}pg, \]
so \( p^g \) belongs to \( C(t^g) \).

Next, we have
\[ V(t)g = \sum_{p \in C(t)} \text{sgn}(p)p \]
\[= \sum_{p \in C(t)} \text{sgn}(p)g^{-1}pg\]
\[= g \sum_{p \in C(t)} \text{sgn}(p^g)p^g\]
\[= g \sum_{p^g \in C(t^g)} \text{sgn}(p^g)p^g\]
\[= g V(t^g).\]

Finally, we have
\[e(t)^g = \{t\} V(t)^g\]
\[= \{t\}_g V(t^g)\]
\[= \{t^g\} V(t^g)\]
\[= e(t^g).\]

Thus, we have the conclusion.

It is known that \(S^\alpha\) is an irreducible \(K[S_n]\)-module which corresponds to the partition \(\alpha\) of \(n\).

For each \(\alpha\)-tableau \(t\), an element \(H(t)\) in the group algebra \(K[S_n]\) is defined by
\[H(t) = \sum_{q \in R(t)} q.\]

We have the following:

Lemma (James and Kerber [9] 7.1.4). For each \(\alpha\)-tableau \(t\), the left ideal \(K[S_n]V(t)H(t)\) of the group algebra is isomorphic to the Specht module \(S^\alpha\).
Now we turn our attention to the \((9,1,1)\)-representation of \(S_{11}\). The following theorem provides a basis of the Specht module \(S^{(9,1,1)}\):

**Theorem (James and Kerber [9]7.2.7).**

\[
\{ e(t) \mid t \text{ is a standard } \alpha\text{-tableau} \} \text{ is a basis for } S^\alpha.
\]

There are 45 standard \((9,1,1)\)-tableaux. We name them by the legs. For example:

\[
t = \begin{array}{cccccccc}
1 & * & * & * & * & * & * & *\\
& i & & & & & & \\
& & j & & & & & \\
\end{array}
\]

is named \(i_j\), where \(1 < i < j \leq 11\). The standard polytabloid associated to \(t\) is denoted \(e_{i_j}^i\).

We need to describe the action of \(S_{11}\) on the standard \((9,1,1)\)-polytabloids \(\{ e_{i_j}^i \}\). We shall show how \(e(t)\) is expressed in terms of \(\{ e_{i_j}^i \}\) for any \((9,1,1)\)-tableau \(t\).

**Lemma 4.2.** For any element \(g\) in \(C(t)\), we have

\[ e(t) g = \text{sgn}(g) e(t) . \]

**Proof.** We have

\[
V(t) g = \sum_{p \in C(t)} \text{sgn}(p)p g
\]

\[= \sum_{p \in C(t)} \text{sgn}(g)\text{sgn}(pg) pg \]
\[ = \sgn(g) \sum_{p g \in \mathcal{C}(t)} \sgn(p g) p g \]
\[ = \sgn(g) V(t). \]

Hence, we have
\[ e(t)g = \{ t \} V(t) g \]
\[ = \sgn(g) \{ t \} V(t) \]
\[ = \sgn(g) e(t). \]

By the definition, \( \{ t \} g = \{ t \} \) for any element \( g \) in \( \mathbb{R}(t) \). But \( e(t)g = e(t) \) does not hold for every element in \( \mathbb{R}(t) \). We have the following:

**Lemma 4.3.** For any \((9,1,1)\)-tableau \( t \), let \( X \) be the set of numbers appearing in the first row except the one on the left end. For any element \( g \) in \( S_X \), we have
\[ e(t)g = e(t^g) = e(t). \]

**Proof.** The first equality follows from the arguments in lemma 4.1.

We see that \( S_X \) commutes with \( \mathcal{C}(t) \), we have
\[ V(t) g = g V(t) \]
for any element \( g \) in \( S_X \). Note that \( S_X \subseteq \mathbb{R}(t) \), we have
\[ e(t)g = \{ t \} V(t) g \]
\[ = \{ t \} g V(t) \]
\[ = \{ t \} V(t) = e(t). \]
Theorem 4.4. Let $S_{10}$ be the stabilizer of the number 1 in $S_{11}$. Then $S_{10}$ acts on the Specht module $s(9,1,1)$ monomially.

Proof. For a standard $(9,1,1)$-tableau $i \overline{j}$, and an element $g$ in $S_{10}$, define a $(9,1,1)$-tableau $t$ by

$$t = \begin{pmatrix} 1 & * & * & * & * & * & * \\ i \overline{j} \\ s \overline{r} \end{pmatrix} g$$

where namely $s = i \overline{g}$ and $r = j \overline{g}$.

We show that

$$(e_{i \overline{j}}^s)g = e_{r \overline{s}}^s$$ if $s < r$,

$$(e_{i \overline{j}}^r)g = -e_{s \overline{r}}^r$$ if $r < s$.

Let $X$ be the set of numbers in the first row of the tableau $t$ except 1. The numbers on the first row of $t$ are permuted by an element $h$ in $S_X$ so that they are put in order. By lemma 4.3, we have

$$e(t^h) = e(t).$$

If $s < r$, then $t^h$ is a standard tableau. Thus, we have $e(t) = e_{r \overline{s}}^s$. If $s > r$, we apply $p = (r \overline{s})$ a transposition in $C(t)$ so that $(t^h)^p$ is a standard tableau. Namely, we have $e(t^{hp}) = e_{s \overline{r}}^r$. Thus, by lemma 4.2, we get
\[ e(t^{hp}) = \text{sgn}(p)e(t^h) \]
\[ = -e(t). \]

Next we determine \((e^i_j)^g\) for any element \(g\) in \(S_{11}\). In order to do so, let us state Garnir's theorem, which is important in proving the theorem (James and Kerber [9]7.2.7).

Let \(t\) be any \(\alpha\)-tableau where \(\alpha = (\alpha_1, \alpha_2, \ldots)\) is a partition of \(n\). Let \(X\) be a subset of the \(i\)-th column, and \(Y\) be a subset of the \(j\)-th column of \(t\), with \(i < j\). Let \(p_1, \ldots, p_k\) be the coset representatives of \(S_X \times S_Y\) in \(S_{(X \cup Y)}\), and let
\[
G_{X,Y} = \sum_{r=1}^{k} \text{sgn}(p_r)p_r.
\]

We call \(G_{X,Y}\) a Garnir element for \(X \cup Y\). We have the following:

**Theorem (James and Kerber [9]7.2.3).**

If \(|X \cup Y| > \alpha_1\), then
\[
G_{X,Y} e(t) = 0.
\]

Let \(t\) be any \((9,1,1)\)-tableau;
\[
t = \begin{array}{cccccc}
\star & \star & \star & \star & \star & \star \\
\star & \star & \star & \star & \star & \star \\
\star & \star & \star & \star & \star & \star \\
\end{array}
\]
Suppose that the first column contains 1. By lemma 4.2, for some element $g$ in $C(t)$ we have

$$e(t) = \text{sgn}(g) e(t^g).$$

By lemma 4.3, for some element $h$ in $R(t)$ we see that $(t^g)^h$ becomes a standard tableau and

$$e(t^gh) = e(t^g) = \text{sgn}(g) e(t).$$

Thus, $e(t)$ is expressed in term of standard polytabloids.

Next, we assume that the first column of $t$ does not contain 1. By lemma 4.3, we may assume that

$$t = \begin{array}{|c|c|c|c|c|c|}
\hline
k & l & * & * & * & * \\
\hline
s & r & & & & \\
\hline
\end{array}$$

By lemma 4.2, we may also assume that

$$k < s < r.$$ 

By theorem (19.2.3) for $X = \{k, s, r\}$ and $Y = \{1\}$, we may take a coset representative $P_1, P_2, P_3, P_4$ such that

$$(t)_{P_1} = \begin{array}{|c|c|c|c|c|c|}
\hline
k & l & * & * & * & * \\
\hline
s & r & & & & \\
\hline
\end{array}$$

$$(t)_{P_2} = \begin{array}{|c|c|c|c|c|c|}
\hline
1 & k & * & * & * & * \\
\hline
s & r & & & & \\
\hline
\end{array}$$

$$(t)_{P_3} = \begin{array}{|c|c|c|c|c|c|}
\hline
1 & s & * & * & * & * \\
\hline
k & r & & & & \\
\hline
\end{array}$$

$$(t)_{P_4} = \begin{array}{|c|c|c|c|c|c|}
\hline
1 & r & * & * & * & * \\
\hline
k & s & & & & \\
\hline
\end{array}.$$
With this choice of coset representatives, we have

\[ G_{X,Y} = 1 - (l \ k) + (l \ k \ s) - (l \ k \ s \ r). \]

Since \( G_{X,Y} e(t) = 0 \), we have

\[ e(t) = e(t_{p2}) - e(t_{p3}) + e(t_{p4}) \]

\[ = e^s_r - e^K_r + e^K_s. \]

We summarize the results as follows: For any standard \((9,1,1)\)-polytabloid \( e^i_j \) and any element \( g \) in \( S_{11} \),

(4.5) if \( 1 \in \{1^g, i^g, j^g\} \), we put

\[ \{1^g, i^g, j^g\} = \{1, s, r\} \] where \( s < r \) and

\[ p = \begin{pmatrix} 1^g & i^g & j^g \\ l & s & r \end{pmatrix}. \]

Then \( (e^i_j)^g = \text{sgn}(g)e^s_r \).

(4.6) if \( 1 \notin \{1^g, i^g, j^g\} \), we put

\[ \{1^g, i^g, j^g\} = \{k, s, r\} \] where \( 1 < k < s < r \) and

\[ p = \begin{pmatrix} 1^g & i^g & j^g \\ k & s & r \end{pmatrix}. \]

Then \( (e^i_j)^g = \text{sgn}(p)(e^s_r - e^K_r + e^K_s) \).

Finally, we give \( M_{11} \) as a subgroup of \( S_{11} \) as follows (see Nagao [10] or Huppert and Blackburn [7] chapter XII): Let

\[ a = (4 \ 7 \ 8 \ 11)(5 \ 10 \ 9 \ 6), \]

\[ b = (4 \ 6 \ 8 \ 10)(5 \ 7 \ 9 \ 11), \]
\[ c = (3 \ 11 \ 7)(4 \ 10 \ 9)(5 \ 6 \ 8), \]
\[ d = (3 \ 4 \ 8)(5 \ 11 \ 10)(6 \ 7 \ 9), \]
\[ g_3 = (2 \ 3)(4 \ 10)(5 \ 9)(6 \ 8), \]
\[ g_4 = (1 \ 2)(4 \ 5)(6 \ 10)(8 \ 9), \]

where

1. \( \langle a, b \rangle \cong Q_8 \), the quaternion group;
2. \( \langle c, d \rangle \cong E_9 \), the elementary abelian group of order 9;
3. \( \langle a, b, c, d \rangle \cong E_9 \times Q_8 \), a Frobenius group;
4. \( \langle a, b, c, d, g_3 \rangle = M_{10} \), the stabilizer of the number 1 in \( M_{11} \);
5. \( \langle M_{10}, g_4 \rangle = M_{11} \).

We have described the \( S_{11} \)-module \( W \). By restricting the action on \( W \) to \( M_{11} \), we get \( M_{11} \)-module \( U \). We shall extend \( U \) to an \( M_{12} \)-invariant space to obtain \( V \) in section 3.
SECTION 2

\(M_{11}\)-INVARIANT ALGEBRA \(U\)

In this section, we shall determine all the coefficients \(\{A_{jrq}^{isp}\}\) defined by:

\[e_j^i e_r^s = \sum_{p < q} A_{jrq}^{isp} e_q^p\]

such that

\[(e_j^i e_r^s)g = (e_j^i)g (e_r^s)g\]

for any element \(g\) in \(M_{11}\). By theorem 4.4, we see that \(M_{10}\) acts on \(U\) monomially. And \(M_{10}\) acts 3-transitively on \(\{2, 3, \ldots, 11\}\). Hence, it suffices to determine the coefficients of

\[\{e_3^2 e_j^i \mid i < j\} .\]

Next, we consider the stabilizer of 1-dimensional subspace \(\langle e_3^2 \rangle\) in \(M_{10}\):

\[T = \{ g \in M_{10} \mid (e_3^2)g = e_3^2 \text{ or } -e_3^2 \} .\]

This group \(T\) is generated by

\[a = (4 \ 7 \ 8 \ 11)(5 \ 10 \ 9 \ 6),\]
\[b = (4 \ 6 \ 8 \ 10)(5 \ 7 \ 9 \ 11)\] and
\[g_3 = (2 \ 3)(4 \ 10)(5 \ 9)(6 \ 8) .\]
It suffices to determine the coefficients of representatives of \( T \)-orbits of \( \{ e_3^2 e_j^1 \} \). A complete set of representatives of \( T \)-orbits are the following:

\[
\begin{align*}
e_3^2 e_3^2 &= \sum A_{ij}^1 e_j^1, \\
e_3^2 e_4^2 &= \sum B_{ij}^1 e_j^1, \\
e_3^4 e_5^4 &= \sum C_{ij}^1 e_j^1, \\
e_3^4 e_6^4 &= \sum D_{ij}^1 e_j^1, \\
e_3^4 e_8^4 &= \sum F_{ij}^1 e_j^1, \\
\end{align*}
\]

where \( A_{ij}^1 = A_{33i}^{22}, B_{ij}^1 = A_{34i}^{22}, C_{ij}^1 = A_{35i}^{24}, D_{ij}^1 = A_{36i}^{24} \) and \( F_{ij}^1 = A_{38i}^{24} \). The length of each \( T \)-orbit of \( \{ e_3^2 e_j^1 \} \) is 1, 8, 16, 8 and 4 respectively.

**Lemma 4.7.** \( e_3^2 e_3^2 = 0. \)

**Proof.** Consider the subgroup generated by \( a \) and \( b \), which is isomorphic to \( Q_8 \) and fixing \( e_3^2 \). Let all elements in that subgroup act on \( e_3^2 e_3^2 \).

We get

\[
\begin{align*}
e_3^2 e_3^2 &= A_3^2 e_3^2 \\
&= A_4^2(e_4^2 + e_5^2 + e_6^2 + e_7^2 + e_8^2 + e_9^2 + e_{10}^2 + e_{11}^2) \\
&+ A_4^3(e_4^3 + e_5^3 + e_6^3 + e_7^3 + e_8^3 + e_9^3 + e_{10}^3 + e_{11}^3) \\
&+ A_5^4(e_5^4 + e_6^4 + e_7^4 + e_8^4 + e_{10}^4 + e_{11}^4) \\
&+ A_5^4(e_5^6 + e_6^5 + e_7^8 + e_{10}^9 + e_{11}^9 - e_9^6 - e_{11}^6) \\
\end{align*}
\]
Apply $g_3 = (2 \ 3)(4 \ 10)(5 \ 9)(6 \ 8)$ to get

$$(e_3^2)g_3 \ (e_3^2)g_3 = (-e_3^2)(-e_3^2) = e_3^2 e_3^2.$$  

By comparing the coefficients, we have

$$A_4^2 = A_4^3, \quad A_5^4 = -A_7^4 \quad \text{and} \quad A_5^2 = A_6^4 = 0.$$  

Next, apply $g_4 = (1 \ 2)(4 \ 5)(6 \ 10)(8 \ 9)$ to get

$$(e_3^2)g_4 \ (e_3^2)g_4 = (-e_3^2)(-e_3^2) = e_3^2 e_3^2.$$  

By comparing the coefficients with

$$(e_3^2 e_3^2)g_4 = A_4^2(-e_5^2 - e_4^2 - e_{10}^2 - e_7^2 - e_9^2 - e_8^2 - e_6^2 - e_{11}^2
\begin{align*}
  &+ e_5^3 - e_5^2 + e_3^2 + e_4^2 + e_3^3 \\
  &+ e_{10}^3 - e_3^2 + e_3^2 - e_7^2 + e_3^2 \\
  &+ e_9^3 - e_9^2 + e_8^2 + e_8^2 + e_3^2 \\
  &+ e_6^3 - e_6^2 + e_3^2 + e_{11}^2 + e_3^2 \\
  &+ A_5^4(-e_5^2 - e_4^2 + e_5^2 + e_7^2 + e_9^2 + e_8^2 - e_{10}^2 + e_6^2 + e_{11}^2 - e_6^2 - e_{11}^2 \begin{align*}
  &- e_7^2 - e_9^2 + e_8^2 + e_9^2 + e_{11}^2 + e_6^2 \\
  &- e_8^2 + e_7^2 + e_8^2 - e_{10}^2 + e_7^2 + e_5^2 - e_5^2 \\
  &+ e_{10}^2 + e_8^2 - e_5^2 + e_4^2 - e_7^2 + e_8^2 - e_{10}^2 + e_9^2 + e_7^2 + e_9^2 \\
  &- e_6^2 + e_{11}^2 + e_6^2 + e_{11}^2 - e_6^2 - e_{11}^2 \end{align*}
\end{align*}}$$

we get $A_4^2 = A_5^4 = 0.$
Lemma 4.8.

\[ e_3^2 e_4^2 = B_5^2(e_5^2 e_6^2 + e_5^3 + e_5^4 + e_5^5 + e_5^6 + e_5^7 + e_5^8 + e_5^9 + e_5^{10}) \]
\[ + B_7^2(e_7^2 e_9^2 + e_7^3 + e_7^4 + e_7^5 + e_7^6 + e_7^7 + e_7^8 + e_7^9 + e_7^{10}) \]
\[ - (B_5^2 + B_7^2)(e_9^2 e_{11}^2 + e_6^3 + e_9^3 + e_9^4 + e_9^5 + e_9^6 + e_9^7 + e_9^8 + e_9^{10}) \]
\[ + B_5^5(e_5^5 + e_5^6 + e_5^7 + e_5^8 + e_5^9 + e_5^{10} + e_5^{11} + e_5^{12} + e_6^4 + e_7^4 + e_8^4 + e_9^4 + e_{10}^4) \]
\[ + 2(B_7^2 + B_7^5)(-e_8^5 - e_8^6 + e_{11}^6 + 2e_8^7 - 2e_8^{10}). \]

Proof. Apply
\[ da^2d^{-1} = (3 4)(6 5)(7 10)(9 11) \]
to \[ e_3^2 e_4^2 = \sum B_j e_j. \] Comparing the coefficients of
\[ (e_3^2 e_4^2)da^2d^{-1} \]
and
\[ (e_3^2 da^2d^{-1})(e_4^2 da^2d^{-1}) = e_4^2 e_3^2 = e_3^2 e_4^2, \]
we get
\[ (4.8.1) e_3^2 e_4^2 = B_3^2(e_3^2 + e_4^2) + B_5^2(e_5^2 + e_6^2) + B_7^2(e_7^2 + e_10^2) + B_8^2(e_8^2) + B_9^2(e_9^2 + e_{11}^2) + B_5^3(e_5^3 + e_6^3 + e_7^3 + e_8^3 + e_{10}^3) + B_6^3(e_6^3 + e_5^4 + e_7^4 + e_8^4) + B_9^3(e_9^3 + e_{11}^3 + e_8^4 + e_7^4 + e_{10}^4) + B_7^5(e_7^5 + e_10^5) + B_8^5(e_8^5 + e_9^5 + e_{11}^5 + e_7^5) + B_9^5(e_9^5 + e_{11}^5 + e_8^5 + e_{10}^5) + B_7^7(e_7^7 + e_{11}^7 + e_8^7 + e_{10}^7) + B_9^7(e_9^7 + e_{11}^7 + e_8^7 + e_{10}^7) \]

Apply
\[ ab^{-1}e_4 = (1 2)(5 9)(6 11)(7 10) \]
to (4.8.1). The left hand side becomes

\[(e_3^2 e_4^2)ab^{-1}g_4\]

which is supposed to be equal to

\[(e_3^2)ab^{-1}g_4 (e_4^2)ab^{-1}g_4 = (-e_3^2)(-e_4^2) = e_3^2 e_4^2;\]

the right hand side becomes

\[B_3^2 (-e_3^2 - e_4^2) + B_5^2 (-e_9^2 - e_{11}^2)\]
\[+ B_7^2 (-e_{10}^2 - e_7^2) + B_8^2 (-e_6^2) + B_9^2 (-e_5^2 - e_6^2)\]
\[+ B_3^2 (e_3^2 - e_9^2 + e_3^2 + e_{11}^2 - e_{11}^2 + e_4^2)\]
\[+ B_6^2 (-e_{11}^2 - e_3^2 + e_9^2 - e_9^2 + e_4^2)\]
\[+ B_7^2 (e_3^2 - e_10^2 + e_3^2 + e_7^2 - e_7^2 + e_4^2)\]
\[+ B_8^2 (e_6^2 - e_8^2 + e_7^2 + e_8^2 - e_8^2 + e_4^2)\]
\[+ B_9^2 (e_5^2 - e_5^2 + e_3^2 + e_6^2 - e_6^2 + e_4^2)\]
\[+ B_{10}^3 (e_3^2 - e_7^2 + e_3^2 + e_{10}^2 - e_{10}^2 + e_4^2)\]
\[+ B_{11}^3 (e_6^2 - e_6^2 + e_3^2 + e_5^2 - e_5^2 + e_4^2)\]
\[+ B_7^5 (e_3^2 - e_9^2 + e_3^2 - e_{11}^2 - e_7^2 + e_{11}^2)\]
\[+ B_8^5 (-e_8^2 - e_9^2 + e_8^2 - e_{11}^2 + e_8^2 + e_{11}^2)\]
\[+ B_9^5 (-e_5^2 - e_5^2 + e_9^2 - e_{11}^2 - e_6^2 + e_{11}^2)\]
\[+ B_{10}^5 (-e_7^2 - e_7^2 - e_{10}^2 + e_{10}^2 + e_7^2 + e_{11}^2)\]
\[+ B_{11}^5 (-e_6^2 - e_6^2 - e_{11}^2 + e_5^2 + e_{11}^2 + e_{11}^2)\]
\[+ B_8^7 (-e_{10}^2 + e_9^2 + e_{10}^2 + e_8^2 + e_7^2 - e_8^2)\]
\[+ B_9^7 (-e_5^2 - e_5^2 + e_7^2 + e_6^2 + e_7^2 - e_6^2 + e_7^2)\]
+ B_{11}^7 (-e_{10}^6 - e_6^2 + e_5^2 - e_7^5 + e_7^6 - e_5^2 - e_5^2 )
+ B_{9}^8 (-e_{8}^5 - e_5^2 + e_6^2 - e_8^6 - e_6^2 + e_8^2 ).

By comparing the coefficients, we get

B_5^3 = B_9^3, B_6^3 = B_{11}^3, B_7^3 = B_{10}^3, B_7^5 = - B_{11}^7,
B_8^5 = - B_9^5, B_10^5 = - B_9^5, B_9^5 = B_{11}^5 = 0.

We also have some relations:

2B_5^2 = 2B_5^3 + 2B_6^3 + 2B_7^3 + B_8^3,
B_8^7 = 2B_7^2 + 2B_7^3 + 2B_7^5 + 2B_{10}^5,
B_7^7 = - B_8^2 - B_8^3 - 2B_8^5,
B_5^2 = - B_9^2 - B_5^3 - B_6^3 + B_{10}^5 + B_7^5 + B_8^5.

We get

(4.8.2) e_3^2 e_4^2 = B_3^2 (e_3^2 + e_4^2 ) + B_5^2 (e_5^2 + e_6^2 ) + B_7^2 (e_7^2 + e_{10}^2 )
+ B_8^2 e_8^2 + B_9^2 (e_9^2 + e_{11}^2 )
+ B_5^3 (e_5^3 + e_9^3 + e_9^4 + e_{11}^4 )
+ B_5^3 (e_9^3 + e_5^3 + e_{11}^4 + e_9^4 )
+ B_7^3 (e_7^3 + e_{10}^3 + e_{10}^4 )
+ B_5^3 (e_5^3 + e_5^4 )
+ B_7^5 (e_7^5 + e_6^5 - e_7^5 + e_{10}^9 )
+ B_8^5 (e_8^5 + e_8^6 - e_8^8 - e_8^11 )
+ B_{10}^5 (e_8^5 + e_7^6 - e_7^7 - e_{10}^{10} )
+ B_8^7 (e_8^7 - e_8^{10} ).
Next, we apply
\[ g_3 g_4 = (1\ 2\ 3)(4\ 6\ 9)(5\ 8\ 10) \]
to (4.8.2). Compare the coefficients with
\[
(e_3^2)g_3 g_4 \cdot (e_4^2) g_3 g_4 = e_3^2 (e_6^3 - e_6^2 + e_3^2)
\]
\[ = e_3^2 e_6^3 - e_3^2 e_6^2
\]
\[ = -(e_3^2 e_4^2) a_3^2 g_3 - (e_3^2 e_4^2)b
\]
where \( a^2 g_3 = (2\ 3)(4\ 6)(7\ 11)(8\ 10) \). After the computation, we get the conclusion of the lemma.

Lemma 4.9.
\[
e_3^2 e_5^4 = (B_5^2 + B_7^2)(e_3^2 + e_5^4) + (-B_5^2)(e_3^2 - e_4^3)
\]
\[ + C_6^2(e_6^2 + e_6^4) + (-C_6^2 - B_5^2 + B_7^2 + B_7^5)(e_7^2 + e_8^4)
\]
\[ + (C_6^2 + 2B_5^2 - B_7^2)(e_8^2 + e_7^4)
\]
\[ + (-C_6^2 + 2B_7^2 + B_7^5)(e_9^2 + e_{10}^4)
\]
\[ + (-B_5^2 - 2B_7^2 + B_7^5)(e_9^2 + e_9^4) + (-B_7^2)(e_9^2 + e_{10}^4)
\]
\[ + (-C_6^2 + 3B_7^2 + B_7^5)(e_9^3 + e_6^5) + (-B_7^2 - 2B_5^2)(e_7^3 + e_8^5)
\]
\[ + (C_6^2 + 2B_5^2 - B_7^2)(e_8^3 + e_7^5)
\]
\[ + (C_6^2 + B_5^2 + B_7^5)(e_9^3 + e_{10}^5)
\]
\[ + (-C_6^2 - B_5^2 + B_7^2 + B_7^5)(e_9^3 + e_9^5)
\]
\[ + (B_5^2 - B_7^2)(e_9^3 + e_{11}^5) + (2B_7^2 + B_7^5)(e_7^6 + e_8^6)
\]
\[ + (-B_7^5)(e_9^6 + e_{10}^6) + 2B_7^2 e_11^6
\]
\[ + (4B_7^2 + 5B_7^5)(e_9^7 + e_{10}^8) + (-2B_7^2 - 2B_7^5)(e_7^10 + e_8^9)
\]
\[
+ (-B_7^5)(e_{11}^7 + e_{11}^8) + (2B_7^2 + B_7^5)(e_{11}^9 + e_{11}^{10}),
\]
\[
e_3^2 e_6^4 = B_7^2(e_4^2 + e_6^2 + e_4^3 + e_6^3)
+ (-C_6^2 - 3B_5^2 + B_7^5)(e_5^2 + e_5^3 - e_5^4 + e_5^6)
+ (B_5^2 - B_7^5 - 2B_7^5)(e_7^2 + e_11^3 - e_10^4 + e_8^6)
+ (-C_6^2 + B_7^2 + B_7^5)(e_8^2 + e_10^3 - e_7^4 + e_11^6)
+ (-B_5^2)(e_9^2 + e_9^3 - e_9^4 + e_9^6)
+ (C_6^2 + B_5^2 - 2B_7^5)(e_10^2 + e_8^3 - e_11^4 - e_11^6)
+ (C_6^2 + B_5^2 - B_7^2)(e_11^2 + e_7^3 - e_8^4 - e_10^6)
+ (-2B_7^2 - 2B_7^5)(e_5^7 + e_11^{5} - e_8^5 - e_7^5 + e_9^8 + e_11^4 - e_10^9)
+ (-2B_7^2 - 3B_7^5)(e_8^7 + e_10^8 - e_7^{11} - e_11^{11}).
\]

**Proof.** Apply \( x = (dab_3)^{-1}g_3(dag_3) \) to
\[
e_3^2 e_5^4 = \sum C_j^i e_j, \text{ where the cycle decomposition of } x \text{ is } (2 4)(3 5)(8 7)(9 10). \text{ Compare the coefficients of } (e_3^2 e_5^4)x \text{ with } (e_3^2)x (e_5^4)x = e_5^4 e_3^2 = e_3^2 e_5^4.
\]
We have
\[
(4.9.1) e_3^2 e_5^4 = C_3^2 (e_3^2 + e_5^4) + C_5^2 (e_5^2 - e_4^3)
+ C_6^2 (e_6^2 + e_6^4) + C_7^2 (e_7^2 + e_8^4)
+ C_8^2 (e_8^2 + e_7^4) + C_9^2 (e_9^2 + e_10^4)
+ C_{10}^2 (e_{10}^2 + e_9^4) + C_{11}^2 (e_{11}^2 + e_{11}^4)
+ C_6^3 (e_6^3 + e_5^6) + C_7^3 (e_7^3 + e_8^5)
+ C_8^3 (e_8^3 + e_7^5) + C_9^3 (e_9^3 + e_{10}^5)
\]
Next, we apply \( \sigma_4 = (1 \, 2) (4 \, 5) (6 \, 10) (8 \, 9) \) to (4.9.1). Since \( (e_3^2 \, e_5^4) \sigma_4 = (e_3^2) \sigma_4 (e_5^4) \sigma_4 \),

\[
(e_3^2) \sigma_4 = -e_3^2 \quad \text{and} \quad (e_5^4) \sigma_4 = -e_5^4 - e_3^2 + e_5^2,
\]

we have the relation

(4.9.2) \( (e_3^2 \, e_5^4) \sigma_4 - e_3^2 \, e_5^4 = e_3^2 \, e_4^2 - (e_3^2 \, e_4^2) (a b)^{-1} \).

Comparing the coefficients of (4.9.2), we have

(4.9.3) \( C_3^2 = B_5^2 + B_7^2 \),

\( C_5^2 = -B_5^2 \),

\( C_6^2 = C_6^2 + 2B_5^2 - B_7^2 \),

\( C_9^2 = C_7^2 + B_5^2 + B_7^2 \),

\( C_{11}^2 = -2C_6^2 - 2C_7^2 - C_{10}^2 - 3B_5^2 - B_7^2 \),

\( C_6^3 = C_7^2 + B_5^2 + 2B_7^2 \),

\( C_7^3 = C_{10}^2 - B_5^2 + B_7^2 + 2B_5^2 \),

\( C_8^3 = C_6^2 + 2B_5^2 - B_7^2 \),

\( C_9^3 = C_6^2 + B_5^2 - B_7^5 \).
\[ c_{10}^3 = c_7^2, \]
\[ c_{11}^3 = -2c_6^2 - 2c_7^2 - c_{10}^2 - 2B_5^2 - B_7^2, \]
\[ c_7^6 = 2B_7^2 + B_7^5, \]
\[ c_8^6 = -B_7^5, \]
\[ c_{11}^6 = 2c_6^2 + 2c_7^2 + c_{10}^2 + 3B_5^2 + 2B_7^2, \]
\[ c_7^9 = 4B_7^2 + 5B_7^5, \]
\[ c_{10}^7 = -2B_7^2 - 2B_7^5, \]
\[ c_{11}^7 = 2c_6^2 + 2c_7^2 + c_{10}^2 + 3B_5^2 - B_7^5, \]
\[ c_{11}^9 = 2c_6^2 + 2c_7^2 + c_{10}^2 + 3B_5^2 + 2B_7^2 + B_7^5. \]

In order to determine the coefficients of \( e_3^2 e_5^4 \)
in terms of \( B_5^2, B_7^2, B_7^5 \) and \( C_6^2 \), we need to investigate
the coefficients of \( e_3^2 e_6^4 \). Apply

\[ y = g_3 a^2 = (2 \ 3)(4 \ 6)(7 \ 11)(8 \ 10) \]
to \( e_3^2 e_6^4 = \sum_{i,j} D_{ij}^i e_{ij}^i \). Compare the coefficients of
\[ (e_3^2 y)(e_6^4 y) = (-e_3^2)(-e_6^4) = e_3^2 e_6^4. \]

We have
\[
(4.9.4) \quad e_3^2 e_6^4 = D_4^2 (e_4^2 + e_6^3) + D_5^2 (e_5^2 + e_5^3) + D_6^2 (e_6^2 + e_6^3) + D_7^2 (e_7^2 + e_7^3) + D_8^2 (e_8^2 + e_8^3) + D_9^2 (e_9^2 + e_9^3) + D_{10}^2 (e_{10}^2 + e_8^3) + D_{11}^2 (e_{11}^2 + e_7^3) + D_5^4 (e_5^4 - e_6^5) + D_7^4 (e_7^4 + e_6^5) + D_8^4 (e_8^4 + e_6^5) + D_9^4 (e_9^4 + e_9^5) + D_{10}^4 (e_{10}^4 + e_7^6) + D_{11}^4 (e_{11}^4 + e_7^6) + D_7^5 (e_7^5 + e_11^5).
\]
Next, we apply
\[ z = g_3 d g_5 c a b = (2 4 3 6)(7 10 11 8) \]
to (4.9.4). Comparing the coefficients, we get

\[ (4.9.5) \quad e_3^2 e_6^4 = D_4^2 (e_4^2 + e_6^2 + e_4^3 + e_6^3) + D_5^2 (e_5^2 + e_5^3 - e_5^4 + e_6^5) \]
\[ + D_7^2 (e_7^2 + e_11^3 - e_10^4 - e_8^6) + D_8^2 (e_8^2 + e_8^3 - e_8^4 - e_7^6) \]
\[ + D_9^2 (e_9^2 + e_9^3 - e_9^4 - e_7^6) + D_{10}^2 (e_8^2 + e_8^3 - e_8^4 - e_7^6) \]
\[ + D_{11}^2 (e_7^2 + e_11^3 - e_10^4 - e_8^6) + D_7^5 (e_5^2 + e_5^3 - e_5^4 - e_7^10) \]
\[ + D_8^7 (e_8^2 + e_10^3 - e_11^4 - e_11^8) + D_9^7 (e_9^2 + e_9^3 - e_9^4 - e_11^8). \]

Now, we apply
\[ e_4 = (1 2)(4 5)(6 10)(8 9). \]
Since \((e_3^2 e_6^4)g_4 = (e_3^2)g_4 (e_6^4)g_4, (e_3^2)g_4 = - e_3^2\)
and \((e_6^4)g_4 = e_5^2 - e_10^2 + e_10^5\), we have the following:

\[ (4.9.6) \quad (e_3^2 e_6^4)g_4 = -(e_3^2 e_4^2)(ab)^{-1} + (e_3^2 e_4^2)b^{-1} \]
\[ + (e_3^2 e_5^4)abg_3, \]

where \(abg_3 = (2 3)(4 5 10 11 8 9 6 7)\). Comparing the coefficients in (4.9.6), we have

\[ (4.9.7) \quad C_{10}^2 = -B_5^2 - 2B_7^2 - 2B_7^5, \]
\[ C_{11}^7 = -B_7^5, \]
\[ D_4^2 = B_7^2. \]
We combine this with (4.9.3) to get the conclusion of the lemma.

**Lemma 4.10.**

\[
e^2_3 e^4_8 = (B^2_7 + 2B^5_7)(e^2_5 - e^2_9 + e^3_7 - e^3_{11} + e^4_5 - e^8_9 - e^8_{11} - e^8_8)
\]

\[
+ (B^2_5 + 3B^2_7 + 2B^5_7)(e^2_6 - e^2_{10} + e^3_6 - e^3_{10} + e^4_8 + e^8_{10} + e^8_8 - e^3_{10})
\]

\[
+ (-B^2_5 - 2B^2_7)(e^2_7 - e^2_{11} + e^3_5 - e^3_9 + e^4_7 - e^8_9 + e^8_{11} - e^8_{11})
\]

\[
+ B^5_7(e^5_6 + e^5_{10} - e^6_7 + e^6_{11} + e^7_{11} + e^8_9 + e^{10}_9 + e^{10}_{11})
\]

\[
+ 2B^5_7(e^5_9 + e^5_{11} + e^7_9 + e^7_{11}).
\]

**Proof.** By applying

\[
a^2 = (4 \ 8)(7 \ 11)(5 \ 9)(6 \ 10)
\]

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to $e_3^4 e_8^4 = \sum F_j^i e_j^i$, we have

(4.10.1) $e_3^2 e_8^4 = F_4^2 (e_4^2 - e_8^2) + F_5^2 (e_5^2 - e_9^2)
+ F_6^2 (e_6^2 - e_9^2) + F_7^2 (e_9^2 - e_{11}^2)
+ F_4^3 (e_4^3 - e_8^3) + F_5^3 (e_5^3 - e_9^3)
+ F_6^3 (e_6^3 - e_9^3) + F_7^3 (e_9^3 - e_{11}^3)
+ F_4^4 (e_4^4 - e_8^4) + F_5^4 (e_5^4 - e_9^4)
+ F_6^4 (e_6^4 - e_9^4) + F_7^4 (e_9^4 - e_{11}^4)
+ F_8^4 (e_9^4 + e_{11}^4) + F_9^4 (e_9^4 + e_{11}^4)
+ F_10^4 (e_{10}^4 + e_8^4) + F_11^4 (e_{11}^4 + e_8^4)
+ F_5^5 (e_5^5 - e_9^5) + F_6^5 (e_5^5 - e_9^5) + F_7^5 (e_9^5 - e_{11}^5)
+ F_8^5 (e_9^5 - e_{11}^5) + F_9^5 (e_9^5 - e_{11}^5)
+ F_10^5 (e_{10}^5 + e_9^5) + F_11^5 (e_{11}^5 + e_9^5)
+ F_6^6 (e_6^6 - e_{10}^6) + F_7^6 (e_6^6 - e_{10}^6) + F_8^6 (e_6^6 - e_{10}^6)
+ F_9^6 (e_6^6 - e_{10}^6) + F_10^6 (e_6^6 - e_{10}^6) + F_11^6 (e_6^6 - e_{10}^6) + F_{11}^7 (e_{11}^7).

We apply

$g_3 b = (2 \ 3)(5 \ 11)(6 \ 10)(7 \ 9)$ and

dg_3 bda^2 = (2 \ 8)(3 \ 4)(5 \ 7)(6 \ 10)

to (4.10.1); we get

(4.10.2) $e_3^2 e_8^4 = F_5^2 (e_5^2 - e_9^2 + e_7^3 - e_11^3 + e_5^4 - e_{11}^4 - e_8^4 - e_9^4)
+ F_6^2 (e_6^2 - e_{10}^2 + e_5^3 - e_{10}^3 + e_6^4 + e_{10}^4 + e_8^4 + e_9^4)
+ F_7^2 (e_7^2 - e_{11}^2 + e_5^3 - e_7^3 + e_9^4 - e_{11}^4 - e_8^4 - e_7^4)
+ F_8^2 (e_8^2 + e_6^3 - e_{11}^3 + e_6^4 + e_{11}^4 - e_8^4 - e_9^4)
+ F_9^2 (e_9^2 + e_{10}^3 - e_9^3 + e_{10}^4 + e_7^4 + e_{11}^4 - e_8^4 - e_{10}^4 - e_9^4)
+ F_{11}^3 (e_{11}^3 + e_9^4 + e_{11}^4 + e_7^4 + e_{11}^4 - e_8^4 - e_{10}^4 - e_9^4).$
Next, we apply
\[ g_4 = (1 \ 2)(4 \ 5)(6 \ 10)(8 \ 9) \]
to \( e_3^2 e_8^4 \). Since \((e_3^2 e_8^4)g_4 = (e_3^2)g_4 (e_8^4)g_4\),
\((e_3^2)g_4 = -e_3^2\) and \((e_8^4)g_4 = e_6^5 - e_9^2 + e_5^2\), we have
the following relation:
\[
(4.10.3) \quad (e_3^2 e_8^4)g_4 = (e_3^2 e_8^4)ab + (e_3^2 e_4^2)ab \\
- (e_3^2 e_4^2)(ab)^{-1}.
\]
Comparing the coefficients in (4.10.3), we have
\[
(4.10.4) \quad F_9^5 = 2B_7^5, \\
F_6^5 = B_7^5, \\
F_7^2 = -B_5^2 - 2B_7^2, \\
F_6^2 = B_5^2 + 3B_7^2 + 2B_7^5, \\
F_5^2 = B_7^2 + 2B_7^5.
\]
Thus, we get the conclusion of the lemma.

We have determined the \(M_{11}\)-invariant algebra on \(U\). Recall that \(U\) is obtained by restricting \(W\), the Specht module associated to the partition \((9,1,1)\). It is possible to consider \(S_{11}\)-invariant algebra on \(W\). We have the following:
**Corollary 4.11.** Let $W$ be the Specht module associated to the partition $(9,1,1)$; $W = S(9,1,1)$. The symmetric part $W_S$ of $W$ does not contain $W$ as a constituent.

**Proof.** We show that any $S_{11}$-invariant bilinear mapping $f$ from $W_S$ to $W$ is constantly zero. Note the following:

(i) such a mapping $f$ induces a commutative algebra on $W$;

(ii) the mapping $f$ is $M_{11}$-invariant.

Therefore, by the previous lemmas, the mapping $f$ is described by the four parameters; $B_5^2$, $B_7^2$, $B_7^5$, and $C_6^2$. Namely, by lemma 4.10 we have

$$f(e_3^2 \otimes e_8^4) = (B_7^2 + 2B_7^5)(e_5^2 - e_9^2 + e_7^3 - e_11^3 + e_5^4 - 4e_11^4 - e_8^2 - e_9^2)$$

$$+ (B_5^2 + 3B_7^2 + 2B_7^5)(e_6^2 - e_10^2 + e_6^3 - e_10^3 - e_6^4 + e_10^4 + e_8^6 + e_10^6)$$

$$+ (-B_5^2 - 2B_7^2)(e_7^2 - e_11^2 + e_7^3 - e_9^3 - e_7^4 + e_9^4 - e_8^5 - e_11^5)$$

$$+ B_7^5(e_6^2 + e_10^2 - e_7^2 + e_9^2 + e_11^2 - e_10^2 + e_11^2)$$

$$+ 2B_7^5(e_5^2 + e_7^2 + e_8^2 + e_11^2).$$

Applying an element $g = (5,9)$, we have

$$(f(e_3^2 \otimes e_8^4))g = f((e_3^2)g \otimes (e_8^4)g) = f(e_3^2 \otimes e_8^4).$$

Comparing the coefficients of $e_9^5$, we get $B_7^5 = 0$.

Comparing the coefficients of $e_9^2$ and $e_9^2$, we get $B_7^2 = 0$; those of $e_5^2$ and $e_9^3$, we get $B_5^2 = 0$. 
Next, we apply $h = (6 \ 8)$ to $f(e_3^2 \otimes e_8^4) = 0$. We have

$$(f(e_3^2 \otimes e_8^4))h = f((e_3^2)_h \otimes (e_8^4)_h)$$

$$= f(e_3^2 \otimes e_6^4).$$

This implies $f(e_3^2 \otimes e_6^4) = 0$. In particular, we have $C_6^2 = 0$. Hence, any $S_{11}$-invariant bilinear mapping from $W_S$ to $W$ is constantly zero.
Recall the definition of $M_{11}$ in section 1.

Let $S_{12}$ be the symmetric group on the twelve letters: $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$ and 11. Let

$$g_5 = (0 1)(4 6)(5 9)(8 10).$$

The element $g_5$ and $M_{11}$ generate $M_{12}$ in $S_{12}$ (see Nagao[10] or Huppert and Blackburn[7]). Let $V$ be an irreducible $M_{12}$-module of dimension 45. We may assume that the subgroup $M_{11}$ acts on $V$ as it acts on $U$ in section 1. In this section, we shall determine the matrix representation of $g_5$ with respect to the basis $\{e_i\}$, which we put in order lexicographically:

$$e_2, e_3, e_4, e_5, \ldots, e_8, e_9, e_{10}, e_{11}, e_{12}.$$

For any element $h$ in $M_{12}$, we have

$$(e_3^2)g_5h = (e_3^2)g_5hg_5g_5$$

and

$$(e_3^2)hg_5 = (e_3^2)g_5g_5hg_5.$$

By theorem 4.4, $M_{10}$ acts on $V$ monomially. We see that $M_{10}$ acts transitively on the set of 1-dimensional
subspaces \(<e_i^j>\).

We want to find \((e_i^j)g_5\) for all basis elements. For any basis element \(e_i^j\), we can find an element \(h\) in \(M_{10}\) such that \((e_3^2)h = e_i^j\). Therefore, we can get

\[(e_i^j)g_5 = (e_3^2)g_5(g_5h_5g_5).\]

Since \(g_5\) normalizes \(M_{10}\), \(g_5h_5g_5\) belongs to \(M_{10}\).

It suffices to determine \((e_3^2)g_5\).

Consider the stabilizer \(T\) of \(e_3^2\) in \(M_{10}\), where \(T = \langle a, b, g_3 \rangle\). Since \(g_5\) normalizes \(T\), we get

\[(e_3^2)g_5t = (e_3^2)g_5\]

for any element \(t\) in \(T\). We have

\[(e_3^2)g_5 = Ae_3^2 + B(e_4^2 + e_5^2 + e_6^2 + e_7^2 + e_8^2 + e_9^2 + e_10^2 + e_11^2) - e_4^3 - e_5^3 - e_6^3 - e_7^3 - e_8^3 - e_9^3 - e_10^3 - e_11^3)\]

\[C(e_5^4 + e_7^4 - e_9^4 + e_11^4 + e_6^5 + e_8^5 - e_10^5 + e_7^6) + e_6^7 - e_8^7 + e_10^7 + e_9^8 + e_11^8)\]

\[D(e_5^4 - e_7^4 - e_9^4 + e_11^4 + e_6^5 + e_8^5 - e_10^5 + e_7^6) - e_6^7 + e_8^7 - e_9^8 - e_10^9 - e_11^9)\]

We can compute the matrix corresponding to \(g_5\) in terms of \(A, B, C\) and \(D\) (see table 1).
Since $(g_5)^2 = 1$, the diagonal entries of $g_5$ must satisfy

$$A^2 + 16B^2 + 16C^2 + 8D^2 = 1.$$  

With the information from the character table of $M_{12}$ (see table 2), the trace of $g_5$ satisfies

$$3A + 24B = -3.$$  

According to section 1, we can compute the matrix corresponding to $g_4$ (see table 3). Compute the matrix of $g_5 g_4$. The trace of $g_5 g_4$ satisfies

$$-12B + 6D = 0.$$  

The trace of $g_3 g_5$ is computed and it satisfies

$$-5A - 20C = 5.$$  

Solving these equations simultaneously, we get two sets of solutions:

$$A = -1, B = C = D = 0 \text{ and } A = -3/11, B = -1/11, C = D = -2/11.$$  

The first solutions do not satisfy $(g_5 g_4)^3 = 1$. Hence, we conclude that the second set of the solutions gives the matrix for $g_5$. Table 4 shows the matrix corresponding to $11(g_5)$, a product with scalar matrix 11. Machine calculation confirmed $(g_5)^2 = (g_4)^2 = (g_5 g_4)^3 = 1$. 


In this section, we compare the coefficients of \((e^i_j e^s_t)g_5\) and \((e^i_j)g_5 (e^s_t)g_5\) to determine the structure of the \(M_{12}\)-invariant algebra \(V\). The computation is straightforward.

From the coefficients of \(e^2_3\) in \((e^2_3 e^2_4)g_5\) and \((e^2_3)g_5 (e^2_4)g_5\), we get

\[\frac{1}{11}( -18B^2_5 + 2B^2_7 - 12B^5_7 - 6C^2_6 ) = 0.\]

From the coefficients of \(e^2_3\) in \((e^2_3 e^4_5)g_5\) and \((e^2_3)g_5 (e^4_5)g_5\), we get

\[\frac{1}{11}( -12B^2_5 - 4B^2_7 + 6B^5_7 ) = 0.\]

From the coefficients of \(e^2_3\) in \((e^5_6 e^5_6)g_5\) and \((e^5_6)g_5 (e^5_6)g_5\), we get

\[16B^2_5 - 20B^2_7 - 24B^5_7 + 8C^2_6 = 0.\]

Solving these equations gives the ratio of the four parameters:

Up to a scalar multiple, we get

$$e_3^2 e_4^2 = 3(e_5^2 + e_6^2 + e_9^2 + e_{11}^2 + e_5^3 + e_6^3 + e_9^3 + e_{11}^3 + e_5^4 + e_6^4 + e_9^4 + e_{11}^4)$$
$$- 6(e_7^2 + e_9^2 + e_7^3 + e_9^4 + e_{10}^4)$$
$$+ 2(e_7^5 + e_{10}^6 + e_7^6 - e_7^9 - e_{11}^9 + e_{10}^9)$$
$$+ 8(e_7^8 + e_8^8 - e_9^8 - e_{10}^8)$$
$$- 16(e_8^8 - e_{10}^8),$$

$$e_3^2 e_5^2 = -3(e_3^2 + e_5^2 + e_8^2 + e_4^3 + e_5^4 + e_7^4)$$
$$- 15(e_6^2 + e_6^4)$$
$$+ 8(e_7^2 + e_9^2 + e_9^4 + e_{10}^4 + e_9^8)$$
$$+ 5(e_9^2 + e_{10}^2 + e_9^4 + e_{10}^4)$$
$$+ 6(e_{11}^2 + e_{11}^4)$$
$$- (e_6^2 + e_6^4)$$
$$- 11(e_8^2 + e_7^4)$$
$$- 14(e_9^2 + e_{10}^2 + e_9^8 + e_{10}^8)$$
$$+ 9(e_9^3 + e_{11}^8)$$
$$- 10(e_7^6 + e_8^6 + e_{11}^6 + e_{11}^8)$$
$$- 2(e_9^6 + e_{10}^6 + e_{11}^6 + e_{11}^8)$$
$$- 12 e_{11}^6,$$

$$e_3^2 e_6^4 = -6(e_4^2 + e_6^2 + e_{11}^2 + e_4^3 + e_6^3 + e_7^3 - e_8^4 - e_6^8 - e_{10}^8 - e_{10}^8 + e_8^8 + e_{11}^8)$$
APPENDIX

MATRICES AND A CHARACTER OF $M_{12}$

In tables 1, 3 and 4, matrix representations of $g_5$ and $g_4$ of $M_{12}$ on $V$ are shown. Each matrix acts on the column vectors of $V$ from the left. Negative signs are typed under the numbers; for instance, $-3$ is typed as $\text{3}$. In table 2, the first column shows cycle decomposition of each conjugacy class; the second column shows the order of each centralizer in $M_{12}$; the third column shows the character values of $\chi$ of degree 45.
Table 1. The matrix of $g_5$ in terms of A, B, C, D.
Table 2. The character values of $\chi$

| Element       | Centralizer | $|M_{12}|$ | Value |
|---------------|-------------|----------|-------|
| (1)           | $|M_{12}|$   | 45       |
| $(2)^4$       | 192         | -3       |
| $(4)^2$       | 32          | 1        |
| $(3)^3$       | 54          | 0        |
| $(5)^2$       | 10          | 0        |
| $(8)(2)$      | 8           | -1       |
| $(6)(3)(2)$   | 6           | 0        |
| (11)          | 11          | 1        |
| (11)          | 11          | 1        |
| $(2)^6$       | 240         | 5        |
| $(10)(2)$     | 10          | 0        |
| $(4)^2(2)^2$  | 32          | 1        |
| $(3)^4$       | 36          | 3        |
| $(6)^2$       | 12          | -1       |
| $(8)(4)$      | 8           | -1       |
Table 3. The matrix of $\mathcal{E}_4$.

\[
\begin{array}{cccccccccccccccc}
1&0&0&0&0&0&0&1&1&1&1&1&1&0&0&0&0
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\]
Table 4. The matrix of $g_5$.

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