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IMPEDEANCE PROPERTIES OF AN INFINITE ARRAY OF NON-PLANAR RECTANGULAR LOOP ANTENNAS EMBEDDED IN A GENERAL STRATIFIED MEDIUM

The Ohio State University

Ph.D. 1984

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IMPEDANCE PROPERTIES OF AN INFINITE
ARRAY OF NON-PLANAR RECTANGULAR LOOP ANTENNAS EMBEDDED
IN A GENERAL STRATIFIED MEDIUM

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By

Brian Michael Kent, B.S., M.S.

*****

The Ohio State University

1984

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To my loving wife, Linda
ACKNOWLEDGEMENTS

I wish to extend my sincere gratitude to the many people who in some way contributed to the completion of this research effort.

First, I would like to thank the National Science Foundation for their generous Fellowship support during my graduate education at the Ohio State University. I also wish to thank the Department of the Air Force and the ElectroScience Laboratory for their help and cooperation throughout the completion of this research effort. Special recognition should go to co-workers Dr. Robert Puskar, Dr. David Berrie and Dr. Errol English. I especially wish to recognize my brother, William J. Kent, for his many helpful comments and suggestions.

Secondly, I wish to sincerely thank my academic adviser, Professor Benedikt Munk, for his patience, assistance and guidance throughout this research effort. Reading committee members, Professors Jack Richmond and Royer Rudduck, should also be cited for their careful review of this manuscript.

Finally, I wish to extend a special thank you to the editorial and drafting staff of the ElectroScience Laboratory for their timely preparation of this dissertation.
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iv
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CHAPTER I
INTRODUCTION

A. BACKGROUND AND TECHNICAL MOTIVATION

In recent years, there has been a sustained effort to develop phased array antenna technology. Engineers quickly discovered that if a radar system employed a phased array antenna, its performance could be significantly improved. Some of these improved properties include the following.

First of all, phased arrays provide a means of instantaneously scanning by electronic means one or more highly directive antenna beams. This "inertialess scanning" greatly decreases the acquisition and tracking response of a typical target tracking radar. The ability to form several simultaneous antenna beams also allows a single radar to track many targets. For instance, the U.S. Navy's AEGIS phased array radar can simultaneously track 18 incoming targets.

Second, phased array antennas can be designed to produce very low sidelobes. This important feature reduces ground clutter due to sidelobes illuminating objects close to the radar antenna. It also makes the radar easier to install from an electromagnetic compatibility standpoint.
Third, phased array antennas can significantly improve the performance of an Electronic Countermeasure (ECM) jamming system. Such a system can send directive jamming energy toward one or more hostile radar threats instead of wastefully spewing energy in every direction. In a similar manner, a phased array system can steer a null (in the antenna pattern) in the direction of incoming jamming energy, reducing its effect.

Although phased array antennas have many positive aspects, several problems still exist with the present technology. They include the following:

1. Most phased array antennas are narrowband. This makes it difficult to build frequency diverse radars which employ phased array antenna systems. Furthermore, ECM phased array antenna systems will require large bandwidths to cover the expected threat spectrum.

2. The active scan impedance of a typical phased array radiating element is extremely difficult to control versus scan angle or frequency. Hence, when a beam is scanned 50°-70° off normal, the rising element impedance makes it difficult to efficiently radiate energy at these high scan angles.

3. In order to overcome the impedance problem in (2), the array interelement spacing (the distance between adjacent array elements) is sometimes increased in order to help control the active scan impedance. If the interelement spacing is λ/2 or greater, undesired grating lobes can result.

4. Current phased array radiating systems are extremely costly. Usually, precise waveguide or coax feed networks are custom designed for each phased array antenna. If a simple element can be designed and cheaply manufactured, these expenses may be reduced.

5. Protective antenna covers (radomes) can seriously degrade the performance of a phased array antenna. Usually a protective cover must be used to shield the antenna from the external environment. Since the radome is typically excluded from the phased array design process, the radome may ultimately determine the total antenna system performance.
Clearly, these problems pose a major challenge to the phased array designer. Unlike past research efforts, this investigation will concentrate on new design methods which may simultaneously improve many of these unfavorable phased array characteristics.

Taking a closer look at the problem, one soon discovers that the most important quantity to control in the phased array design procedure is the active scan impedance, also known as the array self impedance. If the scan impedance varies little when the antenna beam is scanned, the array performance can be significantly improved. Furthermore, if the scan impedance can be held fairly constant with scan angle and frequency, the array will be more broadband. Any phased array which maintains a small impedance variation as the antenna beam is scanned is said to be "scan independent". It is the specific goal of this research effort to develop a candidate "scan independent" phased array antenna element which also solves many of the previously discussed design problems.

Several early research efforts investigated possible analysis techniques for modeling the behavior of potential "scan independent" phased arrays. Munk, Kornbau, and Fulton [1] presented some initial design ideas using doubly infinite arrays of slots or dipoles embedded in a multilayered (stratified) planar dielectric medium. Although their solution appeared to be general, only a few simple array examples were presented. Later, Shubert [2] expanded these results to include finite sized arrays of planar slots or dipoles. (In this context, "Planar"
implies that the dipoles were restricted to the plane of the array, as shown in Figures 1.1 and 1.2. Note that the element orientation vector \( \hat{p} \) has no \( y \) component. Although both research efforts laid the groundwork for designing scan independent phased arrays, many questions remained. What is the ideal radiating element? Can interelement spacing be kept small? How broadband can radiating elements be made?

Several new research efforts have focused on the design of different radiating elements. Planar slots and dipoles had been chosen in the past because these structures were fairly straightforward to analyze. Munk et al. [3] and Shubert [4] thus chose half-wave dipole

\[
\hat{p}^{(1)} = p_x \hat{x} + o \hat{y} + p_z \hat{z}
\]

Figure 1.1 Infinite array of \( \hat{p} \) oriented dipoles embedded in a stratified dielectric medium. (Plane of the array is the x-z plane).
or slot structures. Unfortunately the interelement spacing of a planar half-wave dipole is limited to $0.45\lambda-0.50\lambda$, even when the element is loaded to reduce the resonant frequency. New elements which reduced interelement spacing had to be developed.

English [5] analyzed infinite arrays of dipole antennas with arbitrary orientation. For the first time, infinite arrays whose elements were projected out of the plane of the array could be analyzed using the plane wave expansion technique of Munk, Kornbau, and Burrell [6]. As Figure 1.3 suggests, the interelement spacings $D_x$ and $D_z$ can
Figure 1.3. Infinite periodic array of dipole elements with arbitrary orientation.
be reduced by interlacing the tilted dipole arrays. On the other hand, the tilted half-wave dipole is still a somewhat band limited element due to its resonant behavior. Nonetheless, English's work laid the basis for analyzing several new types of periodic arrays.

Three ongoing research efforts are exploring new types of candidate radiating elements. These efforts include the following:

1. Lin [7] in a dissertation to be published is analyzing the radiation properties of a periodic array of non-planar VEE dipoles, as shown in Figure 1.4. Again "non-planar" refers to the fact that the VEE dipole is partially projected in the y direction, out of the plane of the array. Lin's model also includes the effects of the feed lines.

![Figure 1.4. Infinite periodic array of non-planar VEE dipole elements embedded in a stratified dielectric media.](image-url)
(2) Ng [8] in a second dissertation to be published is investigating the radiation properties of periodic arrays of dipoles projected between a periodic array of slots cut in a metal plane, as shown in Figure 1.5. (This element is called the "Clavin" element.)

(3) Last, this dissertation will investigate the radiation properties of periodic arrays of non-planar piecewise linear rectangular loop antennas.

The candidate non-planar rectangular loop phased array radiating element is illustrated in Figure 1.6. From this point forward, the "non-planar loop" implies that the plane containing all sides of this loop (the x-y plane) will be oriented perpendicular to the plane of the array (the x-z plane). A doubly infinite array of rectangular loop elements is shown in Figure 1.7.

![Figure 1.5. Infinite periodic array of straight dipole elements interlaced with a periodic array of slots.](image)
Figure 1.6. Rectangular loop candidate phased array element.
Figure 1.7. An array of non-planar loop elements in free space.
This candidate radiating element has many possible advantages. First, it is well known that loop antennas tend to resonate when the circumference of the loop is roughly one wavelength. If the loop is square, one notices that the elements can be placed just over one quarter wavelength apart. This will delay the onset of grating lobes to a frequency of almost twice the resonant frequency. Second, the one wavelength resonant loop is more broadband than the half wavelength resonant dipole. Third, the current modes on a rectangular radiating filament are fairly well understood, which simplifies the analysis of this phased array element. The only negative feature of this element is the pattern null which occurs broadside to the antenna center axis at resonance for the dominant current mode [9]. For this reason, the plane of the loop is turned 90° relative to the plane of phased array.

An array of piecewise linear wire filaments floating in free space is clearly not a practical design. Recall that one main objective is to design a practical scan independent phased array radiating structure which incorporates the radome into the design process. This "integrated antenna/radome" idea is illustrated in the conceptual design of Figure 1.8. Basically, for structural support, the array of loops are mounted in a stratified planar dielectric medium. The thicknesses of dielectric slabs two and three and their respective dielectric constants \( \varepsilon_2 \) and \( \varepsilon_3 \) will be chosen such that the overall array scan impedance varies as little as possible. Also, a protective layer of "rain erosion coating"
Figure 1.8. Conceptual design of an integrated antenna/radome phased array antenna with potential scan independence properties.
with a given thickness and dielectric constant can be added on the outer layer, if required. Finally, the array is connected to any available phased array feed network. (The feed network will not be addressed in this investigation.)

Now that a conceptual design is visualized, it is time to formulate a technique for analyzing the radiation and impedance properties of this antenna/radome structure.

B. ANALYSIS TECHNIQUE: THE PLANE WAVE EXPANSION METHOD

In order to theoretically analyze the structure in Figure 1.8, a modified version of the Plane Wave Expansion Technique of Munk, Kornbau, and Burrell [10] will be used. The fundamental idea behind this technique is as follows.

Consider the doubly infinite, doubly periodic array of rectangular wire elements illustrated in Figure 1.9. To remain consistent with past work, the following definitions are assumed. The number of dielectric layers is denoted by the variable $M$. The $m^{th}$ slab has thickness $d_m$, and constitutive parameters $(\varepsilon_m, \mu_m)$. The slab which contains the infinite array of rectangular loops is denoted $m'$.

The individual array elements are separated from each other by $D_x$ and $D_z$ in the $x$ and $z$ direction, respectively. These quantities are referred to as the interelement spacings. Each identical rectangular element has dimension $2a$ and $2b$ in the $y$ and $x$ direction respectively.

The array centering within slab $m'$ is specified by $d_m'$ and $d_m'$ as shown by Figure 1.9.
Projected in the x-y Plane

Projected in the y-z Plane

Figure 1.9. Infinite periodic array of non-planar rectangular loop antennas embedded in a stratified dielectric medium.
Each array element will have an assumed pair of terminals. Since
the elements are identical, the variable \( y(n) \) represents the y-distance
between the origin and the x-z plane containing the antenna terminals.
Therefore, the centering parameters \( d_{m'} \) and \( d_{m''} \) are given in terms of
\( y(n) \), \( b_{m'} \) and \( b_{m'-1} \) according to the following equations.

\[
d_{m'} = b_{m'} - y(n) \quad (1.1)
\]

\[
d_{m''} = y(n) - b_{m'-1} \quad (1.2)
\]

Looking carefully at one representative element of this infinite
periodic surface, one notices that the element has four piecewise
linearly oriented segments. From Figure 1.10, side 1 contains the
antenna terminals. The orientation of side one is given by the assumed
direction of current flow \( \hat{p}^{(1)} \). The remaining three sides are labeled
sequentially in the counterclockwise direction. The orientation unit
vectors \( \hat{p}^{(j)} \) are defined in terms of the \( \hat{x} \) and \( \hat{y} \) unit vectors as
follows.

\[
\hat{p}^{(3)} = -\hat{p}^{(1)} = \hat{x} \quad (1.3)
\]

\[
\hat{p}^{(2)} = -\hat{p}^{(4)} = +\hat{y} \quad (1.4)
\]

Note that each element of the array can be loaded with an arbitrary
impedance \( Z_L \). Finally, the wire radius of each leg is denoted by \( w \).
Figure 1.10. Representative element of the periodic array of non-planar loops embedded in a stratified dielectric medium.
With the geometry established, it is time to specify the method of solution. As stated earlier, the structure of Figure 1.9 is analyzed using a modified version of the plane wave expansion technique of Munk et al. [11]. English refers to this method as the "Generalized Ohms Law" technique. Actually, we will find that Munk's technique is a modified moment method technique; this is included in a detailed theoretical treatment of the plane wave expansion method which can be found in Appendix A.

The array active scan impedance does not depend on whether the array is transmitting or receiving. Therefore, we find it convenient to treat the array in the same manner as a passive infinite periodic array. We illuminate the array with an externally impressed plane wave and solve for the fields resulting from the induced currents. The array impedance is a by-product of this method of solution. If the impedance calculation is performed correctly, energy will be conserved.

Suppose the slab geometry of Figure 1.9 is illuminated by an externally impressed plane wave, as shown in Figure 1.11. The incident Poynting vector is called \( \text{s}_{0+} \). The specularly reflected and transmitted Poynting vectors are denoted \( \text{s}_{0-} \) and \( \text{s}_{0+} \), where

\[
\hat{s}_{0\pm} = s_{ox} \hat{x} \pm s_{oy} \hat{y} + s_{oz} \hat{z}
\]

(1.5)

and

\[
s_{ox} = - \cos \alpha \sin \eta, \quad s_{oz} = - \sin \alpha \sin \eta,
\]

(1.6)

(1.7)
Figure 1.11. Slab geometry under externally impressed planewave illumination.

with

$$s_{0y} = \sqrt{1 - s_{0x}^2 - s_{0z}^2}$$ \hspace{1cm} (1.8)

The angles $\alpha$ and $\eta$ specify the plane of incidence, illustrated in Figure 1.12. Note that the plane of incidence contains the incident Poynting vector $s_{0}^\pm$ and the $-y$ axis. The angle of incidence $\eta$ is measured between $s_{0}^+$ and the $-y$ axis. This plane of incidence is important to understand, since the incident electric field is typically decomposed into components perpendicular to ($\perp E$) or parallel to ($\parallel E$) the plane of incidence. A thorough treatment of the propagation vector and the field decomposition can be found in Appendix B.
The externally impressed electric field will penetrate into the stratified dielectric medium, inducing currents on every array element. Since every element is identical and periodically spaced in the $x$ and $z$ direction, Floquet's theorem can be used to model the behavior of the induced currents. Basically, Floquet's theorem requires the induced current on each element to be identical in magnitude. Furthermore, the theorem requires the phase of the current (from element to element) to vary linearly across the array, exactly matching the phase of the externally impressed plane wave. These "Floquet currents" will reradiate a scattered field, $E^s$, such that for every $R$ along the surface of each perfectly conducting array element, the total tangential electric field vanishes except at the element terminals. That is to say
\[
\mathbf{E}_{\text{total}}(\mathbf{R}) = \mathbf{E}_{\text{tan}}^{S}(\mathbf{R}) + \mathbf{E}_{\text{tan}}^{\text{inc}}(\mathbf{R}) = Z_L I_0 \delta(\mathbf{R}\cdot\mathbf{r}^{(n)}) \tag{1.9}
\]

where \(I_0\) is the current at the antenna terminals, \(\mathbf{R}\) is a point on the loop, and \(Z_L\) is some arbitrary load impedance. If \(Z_L = 0\), \(\mathbf{E}_{\text{tan}}^{\text{total}}(\mathbf{R})\) is zero everywhere along the conductor. The plane wave expansion technique, like the moment method, uses Equation (1.9) as the initial starting place.

If this problem were attacked with the usual moment method solution, the scattered field would have to be expressed in terms of the current on each and every array element. Assuming for the moment that only one entire basis function were used for the current expansion, the scattered field might look like this;

\[
\mathbf{E}^S(\mathbf{R}) = \sum_{m=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} I_{mq} e^{imq(\mathbf{R})} \tag{1.10}
\]

Obviously since we have an infinite number of elements, there are an infinite number of unknown current amplitudes \(I_{mq}\) to find. Clearly, this is not the way to pursue this problem. Instead, we make use of the implicit assumptions in Floquet's theorem to recast the problem. As shown in Appendix A Equations (A.24) and (A.32), we expand the scattered field \(\mathbf{E}^S(\mathbf{R})\) in terms of a plane wave spectrum scattered field. Since all currents differ only by a phase constant, we can expand \(\mathbf{E}^S\) in terms of a single "entire plane wave expansion function" as shown below;

20
In Equation (1.11), $\bar{e}_s$ is given in Equation (A.32) and $I_0$ is the only unknown.

Substituting Equation (1.11) into Equation (1.9) yields the following

$$\mathbf{E}^{\text{total}}(\bar{r}) = I_0 \bar{e}_0(\bar{r}) + \mathbf{E}^{\text{inc}}(\bar{r}) = Z_L I_0 \delta(\bar{r} - \bar{r}^{(n)}) (1.12)$$

It is unrealistic to believe that Equation (1.12) will be true at every point, since Equation (1.11) is an approximation. If we relax the requirement in Equation (1.12) slightly, perhaps we can force the tangential electric field to vanish in some average sense. This is precisely what is done in most moment method solutions. From Equation (A.25) we write

$$I_0 \left[ Z_L - \frac{1}{\text{ref, loop}} \int_{\text{ref, loop}} \mathbf{E}(\bar{r}) \cdot \hat{\mathbf{r}}_0(\bar{R}') d\bar{z}' \right] = \frac{1}{\text{ref, loop}} \int_{\text{ref, loop}} \mathbf{E}^i \cdot \hat{\mathbf{r}}_0 d\bar{z} (1.13)$$

where $\hat{\mathbf{r}}_0(\bar{R})$ is some normalized weight function. Notice that in Equation (1.13), the line integral is performed around the four sides of one particular "reference loop". Since the element currents obey Floquet's theorem, we only need to solve for the current on any single element of the array. This single element is referred to as the reference element,
and is usually taken as the array element closest to the y-axis. The terminals of this reference element are located by the position vector \( \mathbf{R}(n) \). The variable \( I_0 \) in Equation (1.13) is defined as the current at the terminals of the reference element.

\[
I_0 = I(\mathbf{R}(n)) \quad (1.14)
\]

In Equation (1.13), we annotate the two integrals in the following manner. The first integral is identified as the forcing function.

\[
V_o = \int_{\text{ref tan loop}} E^i \cdot \mathbf{f}_0 d\ell \quad (1.15)
\]

It is strictly a function of the impressed incident field and the chosen weight function, \( \mathbf{f}_0 \). Equation (1.15) is also known as the induced voltage. The second integral is identified as the loop self impedance

\[
Z_{00} = Z_{\text{self}} = -\int_{\text{ref o,tan loop}} e^S(\mathbf{R}') \cdot \mathbf{f}_0(\mathbf{R}') d\ell' \quad (1.16)
\]

Remember that \( e^S_{o,tan} \) is the scattered field expansion due to the entire array, not just one element. (Again, see Appendix A.) Adopting these definitions, Equation (1.13) can be written as follows;

\[
I_o = \frac{V_o}{(Z_L + Z_{00})} \quad (1.17)
\]
It is evident why English refers to this as the "Generalized Ohms Law" method.

Earlier in this chapter, we said that the impedance term \( Z_{00} \) was the quantity of interest. If this is so, why compute \( V_0 \) and \( I_0 \) in Equation (1.17)? The answer is quite simple. If we illuminate the array with an externally impressed plane wave, we should be able to calculate \( V_0 \), \( Z_{00} \), and \( I_0 \). Once \( I_0 \) is known, the reradiated fields can be calculated. If the technique is done properly, and all dielectric slabs are lossless, the reflected energy squared plus the transmitted energy squared should equal the incident energy squared. That is to say

\[
\sqrt{|R|^2 + |T|^2} = 1 \quad ,
\]

where \( R \) is the overall reflection coefficient and \( T \) is the overall transmission coefficient. Therefore, energy conservation gives one confidence in the calculated value of the self impedance term, \( Z_{00} \). Past experience has demonstrated that if \( Z_{00} \) is in error, energy will not be conserved.

One final note regarding the weight function \( f^0 \) in Equation (1.15). How is this chosen? From Appendix A, we chose the weight function to equal the normalized array antenna current taken under transmitting conditions. If we make this choice, Schellkunoff [12] states that the induced voltage expression of Equation (1.15) will be exact.
C. SUMMARY OF REMAINING CHAPTERS

In the chapters to follow, the quantities in Equation (1.17) will be calculated. Chapter II concentrates on the calculation of the induced voltage. Chapter III discusses the difficult self impedance calculation. Chapter IV derives expressions for the scattered fields from an infinite array of non-planar loops given the terminal current $I_0$. Chapter V presents calculated results and Chapter VI summarizes the results and concludes. The main report is followed by eight appendices which explore in detail several specific aspects of the problem.
CHAPTER II
THE INDUCED VOLTAGE

In order to analyze the impedance properties of a periodic loop array embedded in a stratified dielectric medium, Chapter I indicated that three important quantities must be calculated. This chapter concentrates on the first parameter, namely the induced voltage which appears across the terminals of the reference element of the non-planar loop array due to an externally impressed incident plane wave. In this problem, the induced voltage plays the part of the forced response. Since this doubly periodic surface is infinite in extent, symmetry simplifies the induced voltage calculation. Specifically, the magnitude of the induced voltage will be identical for every loop element in the array. Furthermore, the phase of the induced voltage will vary linearly across the array elements, exactly matching the phase of the externally impressed plane wave. Therefore, by calculating the induced voltage on one given element (hereafter referred to as the array reference element), the voltage will be known for every element.

Before calculating the induced voltage for the case of the array embedded in a stratified dielectric medium, we need to calculate the externally impressed electric field for any point within any dielectric slab. As an illustration, suppose we are given the lossless planar
stratified dielectric medium shown in Figure 2.1. Assume that this layered dielectric is illuminated by an externally impressed incident plane wave traveling in the \( s_{0+} \) direction, where:

\[
\hat{s}_{0\pm} = s_{0X} \hat{x} \pm s_{0Y} \hat{y} + s_{0Z} \hat{z}
\]  

(2.1)

with

\[
s_{0X} = -\cos(\alpha)\sin(\eta),
\]  

(2.2)

\[
s_{0Z} = -\sin(\alpha)\sin(\eta),
\]  

(2.3)

and

\[
s_{0Y} = \sqrt{1-(s_{0X})^2-(s_{0Z})^2}.
\]  

(2.4)

![Figure 2.1. Electric field at a point \( \hat{R} \) in slab \( m \).](image)
(The angles \((\alpha, \eta)\) define the plane of incidence as shown in Figure 1.12.) Naturally, some of this incident energy will be reflected in the \(s_0^-\) direction to the left of the first dielectric layer. The rest is transmitted through the MM dielectric layers, eventually traveling in the \(s_0^+\) direction to the right of the final dielectric layer. Using the concept of effective reflection and transmission coefficients from Appendix C, we can easily calculate how much energy is reflected and transmitted. At the present, however, we are more interested in what happens within each individual dielectric layer. The present goal is to obtain an expression for the electric field at any point within this layered dielectric medium.

Looking closely at Figure 2.1, we note that within any \(m\)'th dielectric slab, one plane wave propagates in the \(-y\) direction \((s_m^-)\) and the other propagates in the \(+y\) direction \((s_m^+)\). The expression for \(s_m^\pm\) is given by

\[
\hat{s}_m^\pm = x_s^{mx} \pm y_s^{my} + z_s^{mz},
\]

(2.5)

where \(s^{mx}\) and \(s^{mz}\) are found by matching boundary conditions at each dielectric interface as explained in Appendix B. The quantity \(s^{my}\) is found from the following.

\[
s^{my} = \sqrt{1-(s^{mx})^2-(s^{mz})^2}
\]

(2.6)
In order to calculate the total electric field in slab \( m \) due to the externally impressed plane wave, the contributions due to the \( \hat{s}_m^+ \) and \( \hat{s}_m^- \) plane waves must be summed at every point \( \mathcal{R} \), where \( \mathcal{R} \) lies within slab \( m \). Fortunately Munk, et al. [13] and English [14] have already derived the expression for the total electric field in slab \( m \). It is given by

\[
E^{\text{inc}}(\mathcal{R}) = \left\{ \begin{array}{l}
\left( E^i(0,0,0) \cdot \hat{n}_o^+ \right) \tau_0^+ + \left( -j\beta_m(\mathcal{R} - y_{b_m-1}) \cdot \hat{s}_m^+ \right) e^{j\beta_m(\mathcal{R} - y_{b_m-1}) \cdot \mathbf{s}_m^-} \\
+ \left( E^i(0,0,0) \cdot \hat{n}_o^- \right) (S_{my} \cdot \mathbf{S}_m^-) + \left( -j\beta_m(\mathcal{R} - y_{b_m-1}) \cdot \hat{s}_m^- \right) \phi_{1,m-1}
\end{array} \right.
\]

where we define

\[
\tau_{0,m}^+ = \prod_{i=0}^{m-1} \tau_i^+
\]

and

\[
\phi_{1,m-1} = \prod_{i=1}^{m-1} e^{j\beta_i d_i s_{iy}}
\]

\[ (2.7) \]

\[ (2.8) \]

\[ (2.9) \]
In Equations (2.8) and (2.9) $\tilde{T}^+_i$ is the effective transmission coefficient for the orthogonal (or parallel) electric field from slab $i$ to slab $i+1$ traveling in the right-going (+y) direction. Similarly, $\tilde{\Gamma}^+_i$ is the effective reflection coefficient at the interface between slab $i$ and slab $i+1$ for a wave traveling in the right-going direction.

The effective reflection and transmission coefficient are derived in Appendix C. Finally, the quantity $\phi_{1,m-1}$ represents the total phase delay incurred from traveling through the first $(m-1)$ dielectric layers.

The vector direction of the incident electric field in slab $m$ is given in Equation (2.5) in terms of the orthonormal unit vectors $\hat{n}^m_+$. Decomposing the vector field in this fashion allows us to use the modified Fresnel effective reflection and transmission coefficients $\tilde{\Gamma}^+_m$ and $\tilde{T}^+_m$. Without this decomposition, there would be no hope of analyzing the integrated antenna/radome structure shown in Figure 1.8.

Note that Figure 2.2 illustrates the relative orientation of the $\hat{n}^m_+$ and $\hat{n}^m_-$ unit vectors.

Now that the incident field is known at a point $R$ in slab $m$, introduce the doubly infinite array of non-planar rectangular loops into slab $m$. Our task is to calculate the induced voltage which appears across the terminals of the reference element due to the externally impressed plane wave. An expression for this induced voltage is investigated in Appendix A. From Equation (A.26)
Figure 2.2. Relative orientation of vectors $\hat{s}_{m\pm}$, $\perp \hat{n}_{m\pm}$, and $\parallel \hat{n}_{m\pm}$.

\[
V_{\text{ind}}^{(n)} = \sum_{j=1}^{4} \int_{\text{reference loop side}} E_m^{\text{inc}}(R) \cdot \hat{p}(j) \cdot w_j(R) dR ,
\]

(2.10)

where $\hat{p}(j) \cdot E_m^{\text{inc}}(R)$ is the tangential component of the incident electric field along side (j) of the rectangular loop, $\hat{p}(j)$ is the vector orientation of the jth loop segment, and $w_j(R)$ is a suitable weight function. Schellkunoff [15] indicates that if the weight function is equal to the normalized antenna current under transmitting conditions

\[
w_j(R) = \frac{I_j^{(n)t}(R)}{I^{(n)t}(R(n))}
\]

(2.11)
the induced voltage expression in Equation (2.10) will be exact. The quantity $I^{(n)T}(\mathbf{R}^{(n)})$ represents the terminal current at the point $\mathbf{R} = \mathbf{R}^{(n)}$ for the $n$th array located in slab $m$. Substituting Equation (2.11) into Equation (2.9) yields

\[
v^{(n)} = \sum_{j=1}^{4} \int_{\text{loop side } j} \varepsilon^{\text{inc}}(\mathbf{R}') \cdot \hat{p}(j) \frac{I^{(n)T}(\mathbf{R}')}{I^{(n)T}(\mathbf{R}^{(n)})} \, d\mathbf{R}'.
\]  

(2.12)

In order to evaluate Equation (2.12), we need to establish a local coordinate system relative to the reference element terminals located at $\mathbf{R} = \mathbf{R}^{(n)}$. (Remember $\mathbf{R}^{(n)}$ is the position vector which locates the antenna terminals of the reference element.) This element coordinate system is shown in Figure 2.3. Given this coordinate system, any point on the

Figure 2.3. Element based local coordinate system.
rectangular loop can be located in terms of the reference vector $\mathbf{R}^{(n)}$. For instance, along one of the reference element, the position vector and element orientation vector are given by

\[
\mathbf{R}_1 = \mathbf{R}^{(n)} - x'x (-b \leq x' \leq b)
\]  

and

\[
\mathbf{p}(1) = -\hat{x}.
\]  

Similarly, for the remaining three sides the position vectors and element orientation vectors are given by the following:

\[
\mathbf{R}_2 = \mathbf{R}^{(n)} - bx + ay + y'y (-a \leq y' \leq a)
\]  

\[
\mathbf{p}(2) = \hat{y}
\]  

\[
\mathbf{R}_3 = \mathbf{R}^{(n)} + 2ay + x'x (-b \leq x' \leq b)
\]  

\[
\mathbf{p}(3) = \hat{x}
\]  

\[
\mathbf{R}_4 = \mathbf{R}^{(n)} + bx + ay - y'y (-a \leq y' \leq a)
\]  

\[
\mathbf{p}(4) = -\hat{y}.
\]

With this information, we are prepared to calculate the induced voltage.
The first step in calculating the induced voltage is to write down the electric field along each segment of the reference element using Equation (2.7) and one of Equations (2.13), (2.15), (2.17) or (2.19). For example, along side one of the reference loop, substitute Equation (2.13) into Equation (2.7).

\[ E_{1}^{\text{inc}}(\hat{R}_{1}) = \left\{ (E_{(0,0,0)}^{i} \cdot \hat{n}_{0+}) \sum_{0,m} \left[ -j \beta_{m}(R^{(n)} - x' \hat{x} - y_{b_{m-1}}) \cdot \hat{s}_{m} + \right. \right. \]

\[ \left. \left. \hat{n}_{m-} \cdot \hat{l}^{+} \cdot e^{-j \beta_{m}d_{s_{my}}} e^{-j \beta_{m}(R^{(n)} - x' \hat{x} - y_{b_{m-1}}) \cdot s_{m-}} \right] + (E_{(0,0,0)}^{i} \cdot \hat{n}_{0+}) \sum_{0,m} \left[ -j \beta_{m}(R^{(n)} - x' \hat{x} - y_{b_{m-1}}) \cdot s_{m-} + \right. \right. \]

\[ \left. \left. \hat{n}_{m-} \cdot \hat{m}_{m} \cdot e^{-j \beta_{m}d_{s_{my}}} e^{-j \beta_{m}(R^{(n)} - x' \hat{x} - y_{b_{m-1}}) \cdot s_{m-}} \right] \right\} \phi_{1, m-1} \cdot \]

(2.21)

The expressions for \( E_{2}^{\text{inc}}(\hat{R}_{2}) \), \( E_{3}^{\text{inc}}(\hat{R}_{3}) \), and \( E_{4}^{\text{inc}}(\hat{R}_{4}) \) can be found in a similar manner. Once these four field expressions are written down, simply substitute them into Equation (2.12), and collect common factors. Performing this operation produces the following equation for the induced voltage.
In Equation (2.22), the generalized pattern factors under transmitting conditions are given by the following expressions.

\[
V_{\text{inc}}^{(N)} = \left\{ \begin{array}{l}
-\frac{j2\beta_{m}(b_{m}-y(n))\text{sg}\left(y(n)_{m-1}\right)}{\text{sgy}} \sum_{m} \left\{ i P_{1,m}^{(n)} + i P_{2,m}^{(n)} + i P_{3,m}^{(n)} + i P_{4,m}^{(n)} \right\} \\
+ \sum_{m} \left\{ i P_{1,m}^{(n)} + i P_{2,m}^{(n)} + i P_{3,m}^{(n)} + i P_{4,m}^{(n)} \right\} \\
+ \sum_{m} \left\{ i P_{1,m}^{(n)} + i P_{2,m}^{(n)} + i P_{3,m}^{(n)} + i P_{4,m}^{(n)} \right\} \\
+ \sum_{m} \left\{ i P_{1,m}^{(n)} + i P_{2,m}^{(n)} + i P_{3,m}^{(n)} + i P_{4,m}^{(n)} \right\} \\
-\frac{j\beta_{m}(y(n)_{m-1})\text{sg}\left(y(n)_{m-1}\right)}{\text{sgy}} \sum_{m} \left\{ i P_{1,m}^{(n)} + i P_{2,m}^{(n)} + i P_{3,m}^{(n)} + i P_{4,m}^{(n)} \right\} \\
\end{array} \right.
\tag{2.22}
\]

\[i P_{1,m}^{(n)} = p(1) \sum_{m} \frac{b}{2} \int_{-b}^{b} \frac{e^{j\beta_{m}s_{m}x'}}{x'} \, dx' \tag{2.23}\]

\[i P_{2,m}^{(n)} = p(2) \sum_{m} \int_{-a}^{a} \frac{e^{j\beta_{m}s_{m}y'}}{y'} \, dy' \tag{2.24}\]

\[i P_{3,m}^{(n)} = p(3) \sum_{m} \int_{-b}^{b} \frac{e^{j\beta_{m}s_{m}x'}}{x'} \, dx' \tag{2.25}\]

\[i P_{4,m}^{(n)} = p(4) \sum_{m} \int_{-a}^{a} \frac{e^{j\beta_{m}s_{m}y'}}{y'} \, dy' \tag{2.26}\]
Note that the superscript $(n)$ in Equations (2.22) - (2.26) refers in general to the $n^{th}$ array (located in slab $m$). Also $\hat{R}(n)$ is the position vector which specifies the location of the terminals of array $n$. The variable $y(n)$ is the $y$ component of $\hat{R}(n)$. Later, we will specialize $n$ to be one, since our integrated antenna/radome structure will only have one active array present.

In order to simplify Equations (2.22), we introduce the concept of the composite pattern factor under transmitting conditions as the sum of the transmitting pattern factors for the four individual sides of the loop. Notationally, the composite pattern factor is defined as follows.

$$c(n) = \frac{4}{\pi} \sum_{j=1}^{4} p_{j,m}^{(n)} \cdot \hat{R}(n)$$

Substituting Equation (2.27) into Equation (2.22) yields the desired expression for the induced voltage which appears at the terminals of array $n$ (located in slab $m$) due to the externally impressed plane wave.

$$V_{\text{ind}}(\hat{R}(n)) = \left[ (E(0,0,0,0 \cdot \hat{n}_{o+}) \cdot \hat{R}(n)_{o,m-1} \left[ \begin{array}{c} c(n) \\ + \end{array} \right] + \right.$$  

$$+ j2\beta_{m} s_{my} (b_{m} - y(n)) \right]$$  

$$+ (E(0,0,0,0 \cdot \hat{n}_{o+}) \cdot \hat{R}(n)_{o,m-1} s_{my} \left[ \begin{array}{c} c(n) \\ - \end{array} \right] + \right.$$  

$$+ j2\beta_{m} s_{my} (b_{m} - y(n)) \right]$$  

$$+ j2\beta_{m} s_{my} (y(n) - b_{m-1}) \right]$$

(2.28)
In Equations (2.27) and (2.28) there is only one unknown quantity, namely the normalized antenna current under transmitting conditions. In this solution, the normalized current distribution under transmitting conditions is assumed to be of the following form

\[ I^{(n)}_t(\ell) = \frac{\cos B_m \ell}{\cos 2B_m (a+b)} \]  

(2.29)

In Equation (2.29), the quantity \( \ell \) is measured symmetrically opposite of the terminals of the reference loop, as shown in Figure 2.4. More will

Figure 2.4. Assumed normalized transmitting current on the rectangular loop element.
be said regarding this assumed current distribution in Chapter III. One should note that at the antenna terminals, $I = 2a + 2b$. Hence from Equation (2.29),

$$I^{(n) t}(R(n)) = I^{(n) t}(2a + 2b) = 1.0$$

Therefore, $I^{(n) t}$ is normalized with respect to the terminal current $I^{(n) t}(R(n))$ as desired.

The selection of an even cosinusoidal current function for $I^{(n) t}(R)$ was based on the assumption that this current mode is dominant when the loop circumference is roughly one wavelength. Comparisons between this assumed current mode and currents obtained through moment method calculations revealed that Equation (2.29) was a reasonable assumption even when the circumference of the loop varied from $\lambda/2$ to $3\lambda/2$. Details can be found in Appendix F.

Given the assumed current in Equation (2.29), the integrals in Equations (2.23)-(2.24) can be evaluated. These results can be found in Appendix D.

In conclusion, this chapter developed an expression for the induced voltage which appear at the terminals of array $n$ located in slab $m$ due to an externally impressed plane wave. In the present analysis, the entire non-planar loop array must be completely contained within a single dielectric layer. In other words, loop elements cannot cross the boundary between any two adjacent dielectric layers.
CHAPTER III
THE SELF AND MUTUAL IMPEDANCES

This chapter will concentrate on calculating the self impedance $Z_{11}$ of the infinite non-planar loop array in the presence of a stratified dielectric medium. As you recall, we introduce dielectric layers for two important reasons. First, the dielectric provides structural support for the antenna array. Second, the dielectric slabs can help stabilize the self impedance of the array with scan angle, if the dielectric constants and slab thicknesses are carefully chosen. The potential scan independence characteristic is the motivating force behind this entire investigation. Therefore, let us proceed with the definition and derivation of the array self impedance.

Before investigating the self impedance it is desirable to understand the mutual impedance concept first. The mutual impedance between array $n$ and array $n'$, $Z_{n'n'}^n$, is defined by English [16] as "the open circuit voltage induced at the terminals of the reference element of array $n$ due to the field radiated by the currents on the entire $n'$th array."
\[ z^{n,n'} = -\frac{V^n(R^{n})}{I^{n'}(R^{n'})} \quad \text{(3.1)} \]

Since the \( n^{th} \) array is infinite in extent, we assume that the currents on the \( n^{th} \) array are Floquet type currents. This implies that element current amplitudes are identical while the phase of the element currents follow a linear progression across the array. This linear phase progression depends exclusively on the externally impressed incident plane wave field. Munk, et al. [17] have demonstrated that under these circumstances, array \( n' \) radiates a discrete spectrum of plane waves.

A. THE SCATTERED FIELD FROM AN INFINITE ARRAY OF NON-PLANAR LOOPS IN AN INFINITE MEDIUM

The mutual impedance calculation requires knowledge of the scattered field from array \( n' \). Before introducing the stratified dielectric medium, we need to initially understand how the loop array radiates in an infinite media \( m' \) with impedance \( Z_{m'} \). Munk, et al. [18] have demonstrated that any infinite array of piecewise linear elements can be mathematically fabricated from a summation (integral) of many infinitesimal Hertzian dipole radiating arrays.

Let us introduce the equations for the radiated field to either side of an infinite array of Hertzian dipole elements. These elements are \( dp' \) long oriented in the \( p \) direction as shown in Figure 3.1.
Figure 3.1. Infinite array of infinitesimal dipole elements
The reference element is located by \( \vec{R}' \), and the current on the reference element is identified by \( I(\vec{R}') \). From Munk et al. [19], the radiated field to the left of this infinite array is given by

\[
dE_-(R) = I(R') dp' \frac{Z_{m'}}{2D_x D_z} \sum_{k=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{-j \beta_{m'}(R-R') \cdot \hat{r}_{m'}}{r_{m'y}} \hat{e}_- \tag{3.2}
\]

where \( \vec{R}' \) is the position of the reference Hertzian element.

When the field point \( \vec{R} \) is to the right of the array, the scattered field is given by

\[
dE_+(R) = I(R') dp' \frac{Z_{m'}}{2D_x D_z} \sum_{k=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{-j \beta_{m'}(R-R') \cdot \hat{r}_{m'}}{r_{m'y}} \hat{e}_+ \tag{3.3}
\]

where

\[
\hat{r}_{m'} = \hat{x} r_{m'x} + \hat{y} r_{m'y} + \hat{z} r_{m'z} \quad ,
\]

\[
r_{m'x} = s_{m'x} + \frac{k \lambda_{m'}}{D_x} \quad ,
\]

\[
r_{m'y} = s_{m'y} + \frac{n \lambda_{m'}}{D_y} \quad ,
\]

\[
r_{m'z} = s_{m'z} + \frac{n \lambda_{m'}}{D_z} \quad ,
\]

\[
r_{m'y} = \sqrt{1 - (r_{m'x})^2 - (r_{m'z})^2} \quad ,
\]

\[41\]
In Equations (3.2) and (3.3) note that $\bar{\mathbf{e}}_{m'}^{\pm}$ is the actual orientation vector of the scattered electric field, and $\hat{r}_{m'}^{\pm}$ is the unit vector describing the direction of propagation of the $(k,n)^{th}$ plane wave. Note that when $k=n=0$, the radiated plane wave direction $\hat{r}_{m'}^{\pm}$ is identical to the incident plane wave direction $\hat{s}_{m'}^{\pm}$. The $k=n=0$ plane wave is called the principal propagating wave. The (+) subscript on $\hat{r}_{m'}^{+}$ indicates that the scattered plane waves are travelling in the $+y$ direction, while the (-) subscript indicates the $-y$ direction. Also note that in Equation (3.7), when $\{1-(r_{m'}^{\prime \times})^2-(r_{m'}^{\prime z})^2\} < 0$, the $-j$ branch of the square root is chosen. Physically, the $-j$ branch produces evanescent plane wave components, instead of exponentially growing plane wave components.

In virtually every periodic surface design of interest, only a finite number of terms in Equations (3.2) and (3.3) propagate; the remaining terms are evanescent. If grating lobes are to be avoided, designs are further restricted to one propagating wave, namely the dominant $k=n=0$ wave.

Equations (3.2) and (3.3) generate an orientation vector $\bar{\mathbf{e}}_{m'}^{\pm}$ which is difficult to work with. Throughout the analysis to follow, we
essentially decompose $\tilde{e}_{m\pm}$ into two orthonormal components. One component is parallel to the plane of incidence, $\hat{\mathbf{n}}_{m\pm}$, while the second is perpendicular to the plane of incidence, $\perp \hat{\mathbf{n}}_{m\pm}$. More detail regarding this decomposition can be found in Appendix B. Using Equation (B.14), we write

$$
\tilde{e}_{m\pm} = -i \hat{\mathbf{n}}_{m\pm} (\perp \hat{\mathbf{n}}_{m\pm} \cdot \mathbf{p}) - \hat{\mathbf{n}}_{m\pm} (\hat{\mathbf{n}}_{m\pm} \cdot \mathbf{p})
$$

Substituting Equation (3.10) into Equations (3.2) and (3.3) yields the following:

$$
dE_{\pm}(R) = \frac{-i(\mathbf{R}'\mathbf{p}'z_{m\pm})}{2\mathbf{xD}z} \sum_{k,n} e^{\frac{-j\beta_{m\pm}(\mathbf{R}-\mathbf{R}') \cdot \mathbf{r}_{m\pm}}{r_{m\pm} y}} \left[ -\hat{\mathbf{n}}_{m\pm} (\perp \hat{\mathbf{n}}_{m\pm} \cdot \mathbf{p}) + \hat{\mathbf{n}}_{m\pm} (\hat{\mathbf{n}}_{m\pm} \cdot \mathbf{p}) \right]
$$

where $\mathbf{p}' = (\mathbf{R}-\mathbf{R}') > 0$. Equation (3.11) contains all of the necessary information to construct the scattered field from our infinite non-planar loop array. For notational simplicity, we allow $(\sum_{k,n})$ to represent $(\sum_{k=-\infty}^{\infty} \sum_{n=-\infty}^{\infty})$ in all equations to follow.

Now that the preliminaries are done, we shall concentrate on the formulation of the scattered field from the entire non-planar loop array. From Figure 3.2, we note that the rectangular loop element can be constructed from many infinitesimal current elements placed end to end. Since the nature of the scattered field for each infinitesimal element depends on the sign of $\hat{\mathbf{n}} \cdot (\mathbf{R}-\mathbf{R}')$, the space around the loop
Figure 3.2. Infinite array of rectangular loops embedded in an infinite medium with $Z_m' = \sqrt{\frac{\mu_m'}{\epsilon_m'}}$. 
array is broken into three distinct regions. Region I lies totally to
the left of the array, where all radiating current elements appear to
scatter in the \( \hat{r}_{m'} \) direction. Region III lies totally to the right of
the array, where all radiating current elements appear to scatter in the
\( \hat{r}_{m''} \) direction. In Region II some of the infinitesimal radiating
current elements scatter in the \( \hat{r}_{m'} \) direction, while others scatter in
the \( \hat{r}_{m''} \) direction. Clearly Region II will have the most complicated
scattered field equation.

In Region I, if we mathematically "sum" or integrate the current
along all four sides of the rectangular \((2a \times 2b)\) loop, we obtain the
following scattered field:

\[
\vec{E}_I(R_1) = \frac{-Z_m I(R_1^{(1)})}{2D_x D_z} \sum_{k,n} \frac{1}{r_{m'y}} \left[ \int_{-b}^{b} I_1(x') e^{-jB_{m'}(R_1-R_1) \cdot \hat{r}_{m'} - j\theta_{m'} \cdot \hat{p}(1)} dx' + \int_{-a}^{a} I_2(y') e^{-jB_{m'}(R_1-R_2) \cdot \hat{r}_{m'} - j\theta_{m'} \cdot \hat{p}(2)} dy' \right]
\]

\[
+ \int_{-b}^{b} I_3(x') e^{-jB_{m'}(R_1-R_3) \cdot \hat{r}_{m'} - j\theta_{m'} \cdot \hat{p}(3)} dx'
\]

\[
+ \int_{-a}^{a} I_4(y') e^{-jB_{m'}(R_1-R_4) \cdot \hat{r}_{m'} - j\theta_{m'} \cdot \hat{p}(4)} dy'
\]

(3.12)
where

\[ \hat{p}(3) = -\hat{p}(1) = \hat{x} \]  
\[ \hat{p}(2) = -\hat{p}(4) = \hat{y} \]  
\[ \bar{R}_1 = R^{(1)} - x'\hat{x} \ , \ -b < x' < b \]  
\[ \bar{R}_2 = R^{(1)} - bx + (a+y')\hat{y} \ , \ -a < y' < a \]  
\[ \bar{R}_3 = R^{(1)} + 2ay + x'\hat{x} \ , \ -b < x' < b \]  
\[ \bar{R}_4 = R^{(1)} + bx + (a-y')\hat{y} \ , \ -a < y' < a \]

and

Note that \( R^{(1)} \) is the position vector from the origin to the terminals of the reference element for array (1); \( I(R^{(1)}) \) is the value of the terminal current, and \( I_j \) is the scattering current along the jth side of the loop normalized to the terminal current \( I(R^{(1)}) \).

Equation (3.12) looks very complicated. When quantities independent of the integrations are factored out, however, it reduces to:
\[
\vec{E}_1(\vec{R}_1) = \frac{-1(\vec{R}(1))Z_{m'}}{2DxDz} \sum_{k,n} \frac{-j\beta_{m'}(\vec{R}_1-\vec{R}(1)) \cdot r_{m'} \cdot e^{\frac{-j\beta_{m'}(\vec{R}_1-\vec{R}(1)) \cdot r_{m'} \cdot x}}}{r_{m'}y}.
\]

\[
\int_{-b}^{a} \hat{n}_{m'} \cdot (-x) \int_{-b}^{b} I_1(x')e^{-j\beta_{m'}x'r_{m'}x} dx' +
\]

\[
\int_{-b}^{a} \hat{n}_{m'} \cdot (y)e^{-j\beta_{m'}(ar_{m'}y+br_{m'}x)} a \int_{-a}^{a} I_2(y')e^{-j\beta_{m'}r_{m'}y'y'} dy' +
\]

\[
\int_{-b}^{a} \hat{n}_{m'} \cdot (x)e^{j\beta_{m'}x'r_{m'}x} -b \int_{-b}^{b} I_3(x')e^{-j\beta_{m'}x'r_{m'}x} dx' +
\]

\[
\int_{-b}^{a} \hat{n}_{m'} \cdot (-y)e^{j\beta_{m'}(br_{m'}x-ar_{m'}y)} a \int_{-a}^{a} I_4(y')e^{j\beta_{m'}r_{m'}y'y'} dy'.
\]

(3.19)

If we now define \(\hat{I}_{p,j,m'}\) as the perpendicular (or parallel) component of the pattern factor due to side \(j\) in medium \(m'\) scattering in the minus (-) direction, we can write

\[
\hat{I}_{1,j,m'} = \hat{n}_{m'} \cdot (-x) \int_{-b}^{b} I_1(x')e^{-j\beta_{m'}x'r_{m'}x} dx',
\]

(3.20)

\[
\hat{I}_{2,j,m'} = \hat{n}_{m'} \cdot (y)e^{-j\beta_{m'}(ar_{m'}y+br_{m'}x)} a \int_{-a}^{a} I_2(y')e^{-j\beta_{m'}r_{m'}y'y'} dy' .
\]

(3.21)
\[ l_{m'}^{(1)} = \frac{1}{2} \int_{-b}^{b} I_{3}(x') e^{j \beta_{m'} x r_{m'} x} dx' \]

(3.22)

and

\[ l_{m'}^{(1)} = \frac{1}{2} \int_{a}^{b} I_{4}(y') e^{j \beta_{m'} r_{m'} y} dy' \]  

(3.23)

For notational simplicity, we define the perpendicular (or parallel) component of the composite pattern factor in medium \( m' \) scattering in the minus direction as the sum of Equations (3.20-3.23).

\[ c^{(1)}_{m'} = \sum_{j=1}^{4} l_{m'}^{(1)} \]

(3.24)

In Equations (3.20 - 3.23), the superscript \( (1) \) refers to array \( (1) \). Substituting Equation (3.24) into Equation (3.19) we obtain a simple equation for the scattered field in Region I.

\[ \overrightarrow{E}_{I}^{(1)}(\overrightarrow{R}_{I}) = \frac{-I(R_{I})Z_{m'}}{2Dz} \sum_{k,n} e^{-j \beta_{m'}(\overrightarrow{R}_{I} - R_{I}) \cdot r_{m'}}^{(1)} \frac{c^{(1)}}{r_{m'} y} \]

(3.25)

If we repeated this entire procedure for Region III, we obtain the following.
\[
E_{III} (\mathcal{R}_{III}) = \frac{-j^{(1)}_R Z_m'}{2 \mathcal{D}_x \mathcal{D}_z} \sum_{k,n} e^{-j \beta_m' (\mathcal{R}_{III} - \mathcal{R}(1)) \cdot r_m'^+} \cdot \frac{c(1)}{n-m'^+} (\mathcal{R}_{III} - \mathcal{R}(1)) \cdot \hat{y} > 2a
\]  

where

\[
c(1) = \frac{4}{\mathcal{P}_{m'^+}} \sum_{j=1}^{\mathcal{P}^{(1)}_j} \mathcal{P}_{j,m'^+}
\]

with

\[
\mathcal{P}_{1,m'^+} = \hat{n}_{m'^+} \cdot (-x) \int_{-b}^{b} I_1(x') e^{-j \beta_m' r_m' x' \hat{x'}} \, dx',
\]

\[
\mathcal{P}_{2,m'^+} = \hat{n}_{m'^+} \cdot (y) e^{-j \beta_m' (br_m' x - ar_m' y)} \int_{-a}^{a} I_2(y') e^{j \beta_m' y' r_m' y} \, dy',
\]

\[
\mathcal{P}_{3,m'^+} = \hat{n}_{m'^+} \cdot (x) e^{j 2 \beta_m' a r_m' y} \int_{-b}^{b} I_3(x') e^{j \beta_m' r_m' x' \hat{x'}} \, dx',
\]

and

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\[ \hat{P}_{4,m'+} = \hat{n}_{m'+} \cdot (\hat{y})e^{j\beta_{m'}(ar_{m'}+br_{m'x})}. \]

The convenience of defining the composite pattern factor is evident in Equations (3.25) and (3.26); we simply interchange \(-/\) and \(\bar{R}_{I}/\bar{R}_{III}\) when we go from Region I to Region III. It is important to note that neither equation is valid in Region II. In fact, when the field point is confined to Region II, we shall see that the composite pattern factor concept no longer applies. Further details regarding the individual and composite pattern factors can be found in Appendix D.

At this point, we are prepared to write down the expression for the scattered field in Region II. From Equations (3.14) and (B.16), we note that

\[ \hat{n}_{m'+} \cdot \hat{p}^{(2)} = \hat{n}_{m'+} \cdot \hat{p}^{(4)} = 0 \]  

Therefore, the perpendicular field in Region II is given by:

\[ \hat{E}_{II} = \frac{-I(R^{(1)})Zm'}{2Dx\partial z} \sum_{k,n} \left[ e^{-j\beta_{m'}(\bar{R}_{II}-\bar{R}(1)) \cdot r_{m'}} \frac{1}{r_{m'y}} \hat{P}_{1,m'+ \hat{n}_{m'+}}^{(1)} \hat{P}_{3,m'- \hat{n}_{m'-}} \right] \]

\[ \left( \begin{array}{c} \left(1\right) \\ 2 \end{array} \right) \]

\[ 0 < y \cdot (\bar{R}_{II}-\bar{R}(1)) < 2a \]
The parallel component is significantly more complicated in Region II.

It is as follows:

$$\hat{n}_{m'} + (y) e^{-j\beta_{m'}(br_{m'}x-\alpha_{m'}y)} \sum_{k,n} \left[ \frac{e^{-j\beta_{m'}(\mathcal{R}_{II}-\mathcal{R}(1))r_{m'}y}}{r_{m'}y} \right]_{1}^{(1)} \hat{p}_{1,m'} +$$

$$\hat{n}_{m'} + (y) e^{-j\beta_{m'}(br_{m'}x+\alpha_{m'}y)} \int_{-a}^{a} I_{2}(y') e^{j\beta_{m'}y'r_{m'}y} dy' +$$

$$\hat{n}_{m'} - (-y) e^{-j\beta_{m'}(br_{m'}x+\alpha_{m'}y)} \int_{-a}^{a} I_{4}(y') e^{j\beta_{m'}y'\alpha_{m'}y} dy'$$

$$\hat{n}_{m'} + (y) e^{-j\beta_{m'}(br_{m'}x+\alpha_{m'}y)} \int_{-a}^{a} I_{2}(y') e^{j\beta_{m'}y'r_{m'}y} dy' +$$

$$\hat{n}_{m'} - (-y) e^{-j\beta_{m'}(br_{m'}x-\alpha_{m'}y)} \int_{-a}^{a} I_{4}(y') e^{j\beta_{m'}y'\alpha_{m'}y} dy'$$

$$\hat{n}_{m'} + (y) e^{-j\beta_{m'}(br_{m'}x+\alpha_{m'}y)} \int_{-a}^{a} I_{2}(y') e^{j\beta_{m'}y'r_{m'}y} dy'$$

$$\hat{n}_{m'} - (-y) e^{-j\beta_{m'}(br_{m'}x-\alpha_{m'}y)} \int_{-a}^{a} I_{4}(y') e^{j\beta_{m'}y'\alpha_{m'}y} dy'$$

$$\hat{n}_{m'} = -z_{0} = (\mathcal{R}_{II} - \mathcal{R}(1)) \cdot y - a \quad \text{ (3.34)}$$

where from Figure 3.2

$$z_{1} = -z_{0} = (\mathcal{R}_{II} - \mathcal{R}(1)) \cdot y - a \quad \text{ (3.35)}$$
In Equations (3.33) and (3.34), the two plane wave directions \( r_{m'}^\pm \) are clearly seen. Since Equation (3.33) depends on the position parameters \( \xi_0 \) and \( \xi_1 \), it cannot be algebraically simplified.

**B. RADIATED FIELD FROM AN INFINITE ARRAY OF NON-PLANAR FLOQUET CURRENT LOOPS IN A STRATIFIED DIELECTRIC MEDIUM**

At this point in the analysis, we introduce a multilayered or stratified dielectric medium. The infinite array of non-planar loops is completely confined within the \( m' \)th dielectric slab; no side of the array may penetrate the interface between the two adjacent dielectric slabs. From Figure 3.3, the thickness of slab \( m' \) is \( d_{m'} \). Our goal is to calculate the electric field in slab \( m \) at the point indicated by \( \mathbf{R} \) due to an array located in slab \( m' \). The first case presented is for \( m > m' \).

In Figure 3.3, we draw four possible paths from the source antenna array in slab \( m' \) to the field point \( \mathbf{R} \) in slab \( m \). Each path will be given a distinct label; it is important to correctly designate these paths. Path "a-right" travels in the +y direction directly from the radiating elements to the field point. Path "b" is distinguished as the only single bounce path in Figure 3.3 to initially travel in the -y direction prior to its bounce. Path "c" is distinguished as the only single bounce path in Figure 3.3 to initially travel in the +y direction prior to its bounce. Finally, path "d" is distinguished as the only double bounce path in Figure 3.3 to initially travel in the -y direction prior to its double bounce. When these four paths are summed at the
Figure 3.3. Path contributions to the electric field in slab $m$ due to the infinite Floquet array in slab $m'$, $m > m'$. 
field point, we note that paths "a-right" and "b" travel in the \( \hat{r}_{m^+} \) direction, while paths "c" and "d" travel in the \( \hat{r}_{m^-} \) direction.

The path "a-right" contribution to the scattered electric field in slab \( m \) at \( \hat{r} \) due to the array located in slab \( m' \) is given by

\[
\mathbf{E}_{ar}(\hat{r}) = \frac{-Z_m}{2\pi \varepsilon_0} \int \sum_{k,n} e^{-j\beta_m(yb_m-\hat{r}(n'))} \hat{r}_{m'y} \left[ \begin{array}{c} \rho^c(n') \\ 1^{\hat{r}_{m^+}^+} \\ 1^{\hat{r}_{m^-}^+} \\ 1^{\hat{r}_{m^+}^-} \end{array} \right] \left[ \begin{array}{c} \rho^c(n) \\ 1^{\hat{r}_{m^+}^+} \\ 1^{\hat{r}_{m^-}^+} \\ 1^{\hat{r}_{m^+}^-} \end{array} \right] e^{-j\beta_m(yb_{m-1} - \hat{r}) \hat{r}_{m^+}} (\hat{r} \text{ in slab } m) \]

where

\[
1^{\hat{r}_{m^+}^+} = \prod_{i=m'}^{m} 1^{\hat{r}_{i^+}} 
\]

\[
\phi_{m'+1,m-1} = \prod_{i=m'+1}^{m-1} e^{-j\beta_i d_{i} r_{iy}} 
\]

\[
1^{\hat{r}_{m^-}^+} = \frac{1}{1-\prod_{i=m'}^{m} e^{-j2\beta_{m'} d_{i} r_{my'}}} 
\]

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and \( P_{m'n'} \) is the composite pattern factor for the \( n' \)th array scattering in the positive direction. Also in Equations (3.37) and (3.39), the quantities \( \Gamma^+_{m'} \) and \( \Gamma^-_{m'} \) are the effective transmission and reflection coefficients for stratified dielectric medium as derived in Appendix C. In a similar fashion, we write down the expression for the electric field due to path "b" in slab \( m; \)

\[
E_b(R) = \frac{-Z_{m'}n'(R(n'))}{2DxDz} \sum_{k,n} e^{-j\beta_{m'}(y b_m, -R(n')) \cdot \hat{r}_{m'y}}
\]

\[
\begin{align*}
- \begin{bmatrix} c(n') & - \hat{r}_{m'y} \\ \hat{r}_{m'x} & \hat{r}_{m'x} \end{bmatrix}^{\dagger} & \begin{bmatrix} n_{m+} \\ \hat{n}_{m+} \end{bmatrix}^{\dagger} + \begin{bmatrix} c(n') & - \hat{r}_{m'y} \\ \hat{r}_{m'x} & \hat{r}_{m'x} \end{bmatrix}^{\dagger} \\
& \begin{bmatrix} n_{m+} \\ \hat{n}_{m+} \end{bmatrix}^{\dagger} - j\beta_{m'}(R-y b_{m-1}) \cdot \hat{r}_{m+} \cdot \delta_{m'+1,m-1} \\
& \cdot e^{-j2\beta_{m'}(y n'-b_{m-1}) \cdot \hat{r}_{m'y}}
\end{align*}
\]

(\( \hat{R} \) in slab \( m \))

where Equations (3.37) - (3.39) apply. Notice that since path "b" initially travels to the left, the composite pattern factors in Equation (3.40) are for \( -y \) directed waves.

Next, the two leftgoing waves in slab \( m \) due to paths "c" and "d" will be investigated. The electric field due to path "c" is given by:
\[ E_c(R) = \frac{-Z_{m'} \Gamma_{n'}(R(n'))}{2\pi \Delta_x \Delta_z} \sum_{k,n} \frac{-j\beta_{m'}(y_{b_{m'}}, -R(n')) \cdot \hat{r}_{m'+}}{r_{m'y}}. \]

\[ \begin{bmatrix} 1_{m'+} \Gamma_{m',m} \Gamma_{n',m} & \hat{r}_{m'y} \end{bmatrix} + \begin{bmatrix} -p_{c}(n') \end{bmatrix} \]

\[ \begin{bmatrix} \hat{n}_{m'} \end{bmatrix} \begin{bmatrix} \hat{n}_{m'} \end{bmatrix} + \begin{bmatrix} -p_{c}(n') \end{bmatrix} \]

\[ \phi_{m'+1,m-1} \]

\[ \begin{bmatrix} -j\beta_{m'}(R-b_{m'}) \cdot \hat{r}_{m-} \end{bmatrix} \]

(R in slab m)

Since path "c" initially travels to the right before it bounces, the composite pattern factors are for \( +y \) directed waves. In a completely analogous fashion, path "d" results in the following electric field

\[ E_d(R) = \frac{-Z_{m'} \Gamma_{n'}(R(n'))}{2\pi \Delta_x \Delta_z} \sum_{k,n} \frac{-j\beta_{m'}(y_{b_{m'}}, -R(n')) \cdot \hat{r}_{m'+}}{r_{m'y}}. \]

\[ \begin{bmatrix} 1_{m'+} \Gamma_{m',m} \Gamma_{n',m} & \hat{r}_{m'y} \end{bmatrix} + \begin{bmatrix} -p_{c}(n') \end{bmatrix} \]

\[ \begin{bmatrix} \hat{n}_{m'} \end{bmatrix} \begin{bmatrix} \hat{n}_{m'} \end{bmatrix} + \begin{bmatrix} -p_{c}(n') \end{bmatrix} \]

\[ \begin{bmatrix} -j2\beta_{m'}(y_{n'} - b_{m'-1}) \cdot r_{m'y} \end{bmatrix} \]

\[ \phi_{m'+1,m-1} \]

\[ \begin{bmatrix} -j2\beta_{m'}(R - y_{b_{m'}}) \cdot \hat{r}_{m-} \end{bmatrix} \]

(R in slab m)
Once again, since path "d" initially travels to the left before it
bounces twice, the composite pattern factors are given for \( \hat{y} \) directed
waves.

In order to obtain the total field in slab \( m \) due to these four
path contributions, we simply sum Equations (3.36), (3.40), (3.41) and
(3.42);

\[
E_t(R) = \frac{-z_m i n'(R(n'))}{20 \pi D_z} \sum_{k,n} \left[ \begin{array}{c} -j 2 \beta_m' (y_n' - b_{m'-1}) \hat{r}_{m'y} \\ -j 2 \beta_m' (R - \hat{y} b_{m'-1}) \hat{r}_m \\ \end{array} \right] \cdot \left[ \begin{array}{c} j \beta_m' (\hat{y} b_m - R(n')) \hat{r}_{m'y} \\ j \beta_m (R - \hat{y} b_m) \hat{r}_m \\ \end{array} \right] \cdot \left[ \begin{array}{c} e^{j \beta_m d \hat{y} m} \\ e^{j \beta_m d \hat{y} m} \\ \end{array} \right] + \left[ \begin{array}{c} \phi_{m',m-1} \\ \phi_{m',m-1} \\ \end{array} \right]
\]

\[
e^{-j \beta_m (R - \hat{y} b_{m-1}) \hat{r}_m} + \phi_{m',m-1} \left( e^{-j \beta_m d \hat{y} m} \right) \left( e^{-j \beta_m (R - \hat{y} b_m) \hat{r}_m} \right)
\]

\[
\phi_{m'+1,m-1}
\]  

(3.43)
Equation (3.43) is the total field in slab m due to the array in slab m', for m > m'.

Although Equation (3.43) does not appear to take subsequent reflections (within slab m') into account, the parameter \( \Omega_m \) accounts for all subsequent reflections. This will be explained later in this chapter.

Next, consider the case when m < m'. As Figure 3.4 illustrates, the scattered electric field in slab m at the point \( \mathbf{R} \) is again composed of four paths. Path "a-left" travels in the -\( \hat{y} \) direction directly from the radiating elements to the field point. Path "b" is again distinguished as the single bounce path in Figure 3.4 to initially travel in the -\( \hat{y} \) direction prior to its bounce. Path "c" is again distinguished as the single bounce path in Figure 3.4 to initially travel in the +\( \hat{y} \) direction prior to its bounce. Finally, path "e" is distinguished as the only double bounce path in Figure 3.4 to initially travel in the +\( \hat{y} \) direction prior to its double bounce.

If we sum the contributions due to paths "a-left", "b", "c", and "e", we obtain an expression analogous to Equation (3.43), except it is valid only when m < m'. This expression is given by
Figure 3.4. Path contributions to the electric field in slab $m$ due to the infinite Floquet array in slab $m'$, $m < m'$. 
\[ \mathcal{E}_T(\mathbf{R}) = \frac{-Z_{m'} \Im(n')(\mathbf{R}(n'))}{2D_x D_z} \sum_{k,n} e^{-j\beta_{m'}(\hat{y} b_m - \hat{z} (n')) \cdot \mathbf{r}_{m'n'}} \]

\[ - \left[ -\text{Im} T_{m',m} \text{Im} \Omega_{m'} \cdot \left\{ \text{Im} p_{m'}^c(n') + \text{Im} p_{m'}^c(n') \cdot \text{Im} r_{m',e}^+ \right\} \right. \]

\[ + e^{-j2\beta_{m'}(b_m - y_n') \cdot \mathbf{r}_{m'nu}} \cdot \left\{ \text{Im} n_{m'}^- e^{-j\beta_{m'}(\hat{y} b_m - \hat{y} b_{m-1}) \cdot \mathbf{r}_{m'nu}} \right\} + \]

\[ - \text{Im} m_{m'} \cdot \text{Im} m_{m'} \cdot \text{Im} \Omega_{m'} \cdot \left\{ \text{Im} p_{m'}^c(n') + \text{Im} p_{m'}^c(n') \cdot \text{Im} r_{m',e}^+ \right\} \]

\[ + e^{-j2\beta_{m'}(b_m - y_n') \cdot \mathbf{r}_{m'nu}} \cdot \left\{ \text{Im} n_{m'}^- e^{-j\beta_{m'}(\hat{y} b_m - \hat{y} b_{m-1}) \cdot \mathbf{r}_{m'nu}} \right\} \]

(3.44)

where in this case

\[ - \text{Im} T_{m',m} \text{Im} \Omega_{m'} \cdot \left\{ \text{Im} p_{m'}^c(n') + \text{Im} p_{m'}^c(n') \cdot \text{Im} r_{m',e}^+ \right\} \]

\[ + e^{-j2\beta_{m'}(b_m - y_n') \cdot \mathbf{r}_{m'nu}} \cdot \left\{ \text{Im} n_{m'}^- e^{-j\beta_{m'}(\hat{y} b_m - \hat{y} b_{m-1}) \cdot \mathbf{r}_{m'nu}} \right\} \]

(3.45)

from Appendix C. Thus with Equations (3.44) and (3.43) we have the required tools to calculate the mutual impedances.
C. MUTUAL IMPEDANCE BETWEEN TWO INFINITE CO-PLANAR ARRAYS OF NON-PLANAR LOOP FILAMENTS

In Chapter II, Equation (2.5) represented the total electric field in slab \( m \) due to an incident plane wave. Equations (3.43) and (3.44) represents the radiated plane wave spectrum in slab \( m \) due to array \( n' \) in slab \( m' \). Suppose a second square loop test element is introduced in slab \( m \), as shown in Figure 3.5. This test element will represent the reference element of array \( n \) located in slab \( m \). We desire to calculate the induced voltage which appears at the terminals of the test element due to the currents of array \( n' \) radiating a plane wave spectrum from slab \( m' \) to slab \( m \). From Equation (2.8) we recall that

\[
V(n)(R(n)) = \int \frac{j\ ' l \ J}{j} \ dR 
\]

To ultimately evaluate Equation (3.46), we need to substitute Equation (3.43) into Equation (3.46) and integrate. Although the details of this operation will not be written here, it is important to understand that all four paths (a-right, b,c,d for \( m' < m \); a-left, b,c,e for \( m < m' \)) contribute to the total induced voltage.

In Equation (3.46), \( E(n')(R) \) is directly proportional to the terminal current \( I^n'(R(n')) \). Dividing Equation (3.46) by this terminal current yields an expression for the mutual impedance between array \( n' \) in slab \( m' \) and array \( n \) in slab \( m \).
Figure 3.5. Geometry for mutual impedance calculation, \( m' < m \).
With these basic definitions presented, let us proceed to develop the mutual impedance expressions, beginning with the case when \( m \) is to the right of \( m' \).

Substituting Equation (3.43) into Equation (3.46) and integrating, we obtain the mutual impedance for the case when \( m > m' \). Namely

\[
Z^{n,n'} = \frac{Z_{m'}}{20\pi B_z} \sum_{k,n} \frac{1}{r_{m'y}} \left[ \begin{array}{cc}
\frac{1}{1^m_{m',m}} & l_{q_{m'}}^1
\end{array} \right] \left[ \begin{array}{c}
p_{c(n)t}^1 + \frac{p_{c(n)t}}{1^m_{m',e}}
\end{array} \right] + e^{j2\beta_m (b_m - y^n) r_{my}} \left[ \begin{array}{c}
-p_{c(n')t}^{*} + \frac{p_{c(n')t}}{1^{m'}_{m',e}}
\end{array} \right] + e^{-j2\beta_m (y^n - b_m, -1) r_{my}} \left[ \begin{array}{c}
p_{c(n')t}^{*}
\end{array} \right] + e^{j2\beta_m (y^n - b_m, -1) r_{my}} \left[ \begin{array}{c}
-p_{c(n')t}^{*}
\end{array} \right].
\]

(3.48)
where

\[ c(n)t = 4 (n)t \]

\[ \mathcal{P}_{m} = \sum_{j=1}^{\text{side}} \mathcal{P}_{j,m} \]  (3.49)

is the perpendicular (or parallel) composite pattern factor in the \( +y \)
(or \( -y \)) direction in slab \( m \) when the current or the element is under
\( (n)t \) transmitting conditions. The equations for \( \mathcal{P}_{j,m} \) are as follows:

\[ \mathcal{P}_{1,m+} = \frac{1}{4} \mathcal{I}_{1}^{(n)t} \left( x'' \right) e^{j\beta_{m}r_{mx}x''} \]  (3.50)

\[ \mathcal{P}_{2,m+} = \frac{1}{4} \mathcal{I}_{2}^{(n)t} (y'') e^{-j\beta_{m}r_{my}y''} \]  (3.51)

\[ \mathcal{P}_{3,m+} = \frac{1}{4} \mathcal{I}_{3}^{(n)t} (x'') e^{-j2\beta_{m}ar_{my}br_{mx}} \]  (3.52)

\[ \mathcal{P}_{4,m+} = \frac{1}{4} \mathcal{I}_{4}^{(n)t} (y'') e^{j\beta_{m}r_{my}y''} \]  (3.53)

\[ \mathcal{P}_{1,m-} = \frac{1}{4} \mathcal{I}_{1}^{(n)t} \left( x'' \right) e^{j\beta_{m}r_{mx}x''} \]  (3.54)

\[ \mathcal{P}_{2,m-} = \frac{1}{4} \mathcal{I}_{2}^{(n)t} (y'') e^{j\beta_{m}r_{my}y''} \]  (3.55)
\[ \frac{1}{\lambda} P_{3,m-} = - \ln_{m-} \cdot (x) e^{j2\beta_{m} ar_{m} y} \int_{-b}^{b} I_{3}^{(n)}(x'') e^{-j\beta_{m} mx''} dx'' \]  
(3.56)

and

\[ \frac{1}{\lambda} P_{4,m-} = - \hat{n}_{m-} \cdot (-y) e^{-j\beta_{m}(br_{m} - ar_{m})} \int_{-a}^{a} I_{4}^{(n)}(y'') e^{-j\beta_{m} r_{m} y''} dy'' \]  
(3.57)

The integrals in Equations (3.50) - (3.57) are evaluated in Appendix D.

Equation (3.48) is valid when \( m > m' \). When slab \( m \) is to the left of slab \( m' \) as shown in Figure 3.6, the total mutual impedance \( Z_{n,n'} \) consists of contributions due to paths "a-left", "b", "c", and "e". Hence, when Equation (3.44) is substituted into Equation (3.46), the mutual impedance for \( m < m' \) is given by
Figure 3.6. Geometry for mutual impedance calculation, \( m < m' \).
\[ Z_{n,n'} = \frac{Z_{n'}}{2D_xD_z} \sum_{k,n} \frac{1}{r_{m'y}} \left[ -i^{m'-m} \mathbf{c}(n') + i^m \mathbf{c}(n) \right] \]

\[ \mathbf{I}_{m,e}(-j2\beta_m(y^n-b_{m-1})r_{m'y}) \cdot \left[ -i^m \mathbf{c}(n') + i^m \mathbf{c}(n') \right] \mathbf{I}_{m',e} \]

\[ = \mathbf{e}^{-j2\beta_m'(y^{n'}-b_m-1)r_{m'y}} \mathbf{I}_{m',e}(-j2\beta_{m'}(y_{m'}-b_{m'}-1)r_{m'y}) \cdot \mathbf{e}^{-j\beta_{m'}(y_{m'}-1-R_{m'}) \cdot \hat{r}_{m'} \cdot \hat{r}_{n'}-1,m+1} \]

\[ \cdot \mathbf{e}^{-j\beta_m(R^n-y_{m}) \cdot \hat{r}_{m}} \]

where the composite pattern factors are defined as before.

Throughout this section, I have maintained the same designations for the slabs \((m,m',\text{etc.})\) and arrays \((n,n',\text{etc.})\) as English [20] and Henderson [21]. Care must be taken not to confuse the array designation \((n,n')\) with the summation variable \(n\). The only quantities dependent on \([k,n]\) are \(r_{mx}, r_{m'x}, r_{my}, r_{m'y}, r_{m'z}\) and \(r_{mz}\), as defined by Equations (3.5), (3.6) and (3.7). The superscript variables on composite pattern factors and position vectors \((\mathbf{R}(n), y(n))\) strictly refer to the array number. Later in this investigation, we will specialize to the case of a single array.
D. THE ARRAY SELF IMPEDANCE IN THE PRESENCE OF A STRATIFIED DIELECTRIC MEDIUM

In the previous sections, equations were developed for the mutual impedance between two infinite square loop arrays located in different dielectric slabs. A common technique for approximating the self impedance of an infinite array involves specializing the mutual impedance concept in the following manner. Suppose the test element of array n is oriented in slab m' parallel to the reference element of array n' but separated by an equivalent wire radius w. Two views of this configuration are shown in Figures 3.7 and 3.8. The terminals of the reference element are located by \( \overline{R}(n') \), while the terminals of the test element are located by \( \overline{R}(n) \). If we calculate the induced voltage at the terminals of \( \overline{R}(n) \) due to the scattering currents of array n', Equation (3.47) can be used to calculate this "mutual impedance". Due to the close proximity of the reference and test element for \( w \ll \lambda \). Munk, et al. [22] have shown that under these circumstances, Equation (3.47) for the mutual impedance accurately approximates the array self impedance.

In order to specialize Equation (3.47) for the self impedance case, we must calculate the scattered field from array n' in Region II of Figure 3.9, at a field point represented by \( \overline{R}_{II} \). Unlike section A of this chapter, it is assumed that array n' is located in slab m' of a stratified dielectric medium. Looking at Figure 3.9, when the field
Figure 3.7. y-z plane view of test element position for self impedance calculation.

Figure 3.8. Projected view of test element position for self impedance calculation.
Figure 3.9. Six contributions to the scattered fields in Region II when the array is embedded in a stratified dielectric medium.
point is located in Region II, there are six path contributions to the scattered field. The three contributions travelling to the right are paths "a-right", "b", and "e". The three left-going contributions are paths "a-left", "c", and "d". The expressions for the two direct paths, "a-right" and "a-left" are obtained from Equations (3.33) and (3.34).

\[
E_{a\times}(\vec{R}_{II}) = \frac{-Z_{m'}I(R(n'))}{2\partial_x\partial_z} \sum_{k,n} \frac{-j\beta_{m'}(\vec{R}_{II}-\vec{R}(n'))\cdot \hat{r}_{m'}}{r_{m'y}} \left[ \frac{I_3(n')}{I_3} \right]_{\hat{n}_{m'}} - \left[ \frac{I_3(n')}{I_3} \right]_{\hat{n}_{m'}}
\]

\[
+ \left[ \frac{-I_2(n')}{I_2} \right]_{\hat{n}_{m'}} + \frac{I_4(y')}{I_4} \int_{-\infty}^{\infty} e^{-j_{m'y'}} \hat{r}_{m'y'} dy'
\]

\[
+ \left[ \frac{-I_2(n')}{I_2} \right]_{\hat{n}_{m'}} + \frac{I_4(y')}{I_4} \int_{-\infty}^{\infty} e^{-j_{m'y'}} \hat{r}_{m'y'} dy'
\]

\[
U(y, (\vec{R}_{II}-\vec{R}(n'))<2a
\]

(3.59)
In Equations (3.59) and (3.60), \( \xi_0 \) is given by

\[
\xi_0 = a - \hat{y} \cdot (R(n') - R_{II}) \quad .
\]  

Before proceeding with the other path contributions, one important point should be emphasized here. In Region II, paths "a-left" and "a-right" are the only contributions whose integrals are a function of the position parameter \( \xi_0 \). The function dependence on \( \xi_0 \) is caused by the following. As stated in Section A, the infinitesimal radiating current elements comprising the loop section to the left of \( R_{II} \) radiate \( \hat{r}_{m+} \) plane waves. Conversely, the infinitesimal radiating
current elements comprising the loop section to the right of $R_{IIy}$ radiate $\hat{r}_{m'}$. Plane waves. The $\xi_0$ parameter correctly separates the left and right going path contributions. The remaining paths "b-e" do not depend on the $\xi_0$ parameter, since paths "b" and "d" involve field expressions totally to the left of the array, while paths "c" and "e" involve field expressions totally to the right of the array. Therefore, the pattern factor integrals for paths "b-e" will have fixed limits independent of $\xi_0$.

The scattered electric field in Region II, due to the right-going single bounce path "b" is given by the following equation:

$$
E_b(\tilde{R}_{II}) = \frac{-Z_{m'}I(\tilde{R}(n'))}{2\pi D_x D_z} \sum_{k,n} e^{-j\beta_{m'}(\tilde{R}_{II}-\tilde{R}(n')) \cdot r_{m'y}^{m'}} \left[ \sum_{l \in p_{m'}} 1 \right] \\
- \left[ \hat{r}_{m'}, e \hat{\Omega}_{m'} \hat{r}_{m'}^{m'} + p_{m'}^{c}(n') \hat{r}_{m'}^{c} \hat{\eta}_{m'} \hat{\eta}_{m'}^{c} \right] \\
- e^{-j2\beta_{m'}(y^{n'} - b_{m'} - 1)r_{m'y}} 0 \leq y, (\tilde{R}_{II} - \tilde{R}(n')) < 2a \quad (3.62)
$$

Similarly, the left-going single bounce path "c" is given in Region II by Equation (3.63).
The scattered field contribution due to the right-going double bounce path "e" is given in Region II by (3.64).

Finally, the path "d" scattered field contribution due to the left-going double bounce is given in Region II by (3.65).
If the total electric field in Region II of slab $m'$ travelling to the right is desired, simply sum Equations (3.60), (3.62), and (3.64). The left-going electric field in Region II is found by summing Equations (3.59), (3.63), and (3.65).

At this point, one may inquire about contributions due to subsequent double reflections. For instance, in path "b", what about the subsequent reflections shown in Figure 3.1U? If we define the quantity $iX^q$ as

$$iX^q = \left[ \frac{-j2\beta m'dm'y}{r^-_m,e } \right]^q$$

then the path "b" contribution can be thought of as follows.

$$E_b(R_{II}) = \frac{-Zm'I(R(n'))}{2D_x U_z} \sum_{k,n} \frac{-j2\beta m'(R_{II} - R(n')) r^-_{m'y}}{r^+_{m',e}} \left[ \frac{c(n')}{\rho^-_{m'} \Pi} \right]$$

$$= e^{-j2\beta m'dm'y}$$

$$iX^q = \left[ \frac{-j2\beta m'dm'y}{r^-_m,e } \right]^q \left( 1 + iX^1 + iX^2 + iX^3 + \ldots \right)$$

(3.65)
Figure 3.10. Illustration of subsequent path "b" bounces.
Each time a double bounce occurs, q is incremented by one. Since $|\Gamma_{m,e}| < 1$, we sum Equation (3.67) using the familiar geometric series

$$\sum_{q=0}^{\infty} x^q = \frac{1}{1 - x} \quad |x| < 1 .$$

Therefore,

$$\mathbb{E}_b(\mathcal{R}_{II}) = \frac{1}{\text{Path "b" contribution}} \cdot \frac{1}{1 - \Gamma_{m,e}^{-} e^{-\epsilon \Gamma_{m,e}^{+} \Gamma_{m,e}^{-} \epsilon}}$$

From Equation (3.39), we recognize the rightmost term as

$$\frac{1}{\frac{\Omega_{m'}}{\Gamma_{m,e}^{-}} e^{-\epsilon \Gamma_{m,e}^{+} \Gamma_{m,e}^{-} \epsilon}}$$

Although this little proof has been presented before in English [23] and Munk, et al. [24], it serves to remind us that paths "b", "c", "d", and "e" have accounted for all subsequent double reflections.

In paths "a-left" and "a-right" one may ask why $\frac{\Omega_{m'}}{\Gamma_{m,e}^{-}}$ is not present in Equations (3.59) and (3.60)? The answer is simply that all subsequent reflections of "a-left" are incorporated in path "d", while subsequent reflections of "a-right" are incorporated in path "e". This also
explains why in the mutual impedance case for \( m > m' \), path "a-right" incorporated path "e"; for the case where \( m < m' \), path "a-left" incorporated path "d". (To see this, check that \( \frac{1}{2}m' \) occurred in Equation (3.36).) Hence, this path description for the electric field in Region II is fully consistent with the previous electric field expressions developed in Section C of this chapter.

Now that the contributions due to all six paths have been carefully derived, we are prepared to calculate the array self impedance.

Introducing a test loop element with its terminal located by \( \mathcal{R}(n) \), where

\[
R(n) \cdot \hat{y} = R(n') \cdot \hat{y}, \quad \text{and}
\]

\[
(\mathcal{R}(n) - \mathcal{R}(n')) \cdot \vec{r}_{m'} = w r_{m'} z
\]

we calculate the induced voltage in element \( n \) due to the currents on array \( n' \). (Note that \( w \) is the equivalent wire radius.) From (3.46)

\[
V(n)(\mathcal{R}(n)) = \sum_{j=1}^{4} \int_{\text{reference loop side}} E(n')(\mathcal{R}) \cdot \vec{p}(j) \cdot I_j(n) t(\varepsilon_j) d\varepsilon_j
\]

\[
Z_{\text{self}}^{n',n} = \frac{-V(n)(\mathcal{R}(n))}{I(n')(\mathcal{R}(n'))} \left| \begin{array}{c}
\text{One wire radius separation between array } n \text{ and } n' \\
\end{array} \right|
\]
Since there are six contributions to the scattered field in Region II, there are six contributions to the total self impedance. Breaking Equation (3.73) into these six components yields

\[ Z_{n,n'}^{\text{self}} = Z_{\text{SELF}A} + Z_{\text{SELF}B} + Z_{\text{SELF}C} + Z_{\text{SELF}D} + Z_{\text{SELF}E} \quad (3.74) \]

where we define

\[ Z_{\text{SELF}A} = Z_{A \text{ RIGHT}} + Z_{A \text{ LEFT}} \quad (3.75) \]

The remaining portion of this section will concentrate on deriving these six impedance terms. Since the self impedance contribution for paths "a-left" and "a-right" are more difficult to obtain, they will be presented after the simpler bounce path contributions are derived.

The path "b" impedance contribution is found by substituting Equation (3.62) into Equations (3.72) and (3.73). After performing the indicated integration, we obtain

\[
Z_{\text{SELF}B} = \frac{Z_{m'}^{m}}{2D_{xz}} \sum_{k,n} e^{-j\beta_{m'}'R(n)R(n')} \frac{r_{m'}+}{r_{m'}-} e^{-j2\beta_{m'}'(y(n')-y_{m-1})r_{m'}y} \left[ \mathbf{p}_{m'}^{c(n')} \mathbf{p}_{m'}^{c(n)} \mathbf{r}_{m'}^{-} \mathbf{r}_{m'}^{+} \right] \quad (3.76)
\]
For the path "c" contribution, we substitute Equation (3.63) into Equations (3.72) and (3.73). After integrating, this yields

\[
Z_{SELFC} = \frac{Z_m'}{2D_x D_z} \sum_{k,n} \frac{e^{-j\beta_m' (\rho(n) - \rho(n')) \cdot r_m'}}{r_{m'y}} e^{-j2\beta_m' (b_{m'y} - y(n')) r_{m'y}}
\]

\[
\cdot \left[ \frac{pc(n') \cdot pc(n) t \cdot r_{m'k}^+ \cdot e \cdot l_{m'}^+ \cdot \| r_{m'}^+ \cdot \| \cdot m' e \cdot Q_{m'} \cdot m'}{1_{m'}^+ \cdot 1_{m'}^+ \cdot 1_{m'}^-, e \cdot l_{m'}^- \cdot \| \cdot m' e \cdot Q_{m'} \cdot m'} \right] \cdot
\]

(3.77)

The path "d" impedance contribution is found by substituting Equation (3.65) into Equations (3.72) and (3.73). After simplification, this results in

\[
Z_{SELFD} = \frac{Z_m'}{2D_x D_z} \sum_{k,n} \frac{e^{-j\beta_m' (\rho(n) - \rho(n')) \cdot r_m'}}{r_{m'y}} e^{-j2\beta_m' d_{m'y} r_{m'y}}
\]

\[
\cdot \left[ \frac{pc(n') \cdot pc(n) t \cdot r_{m'k}^+ \cdot e \cdot l_{m'}^+ \cdot \| r_{m'}^+ \cdot \| \cdot m' e \cdot Q_{m'} \cdot m'}{1_{m'}^- \cdot 1_{m'}^- \cdot 1_{m'}^+, e \cdot l_{m'}^- \cdot \| \cdot m' e \cdot Q_{m'} \cdot m'} \right] \cdot
\]

(3.78)

Finally for the path "e" double bounce contribution, Equation (3.64) is substituted in Equations (3.72) and (3.73). After completing the integration, the path "e" contribution is given by (3.79).
In Equations (3.76) - (3.79), the quantity \( \mathcal{J}_{n}^{m',\pm} \) is the orthogonal (or parallel) composite pattern factor due to array \( n' \) in slab \( m' \) scattering in the plus (or minus) direction. Similarly, \( \mathcal{J}_{n,t}^{m',\pm} \) is the orthogonal (or parallel) composite pattern factor due to array \( n \) in slab \( m' \) transmitting in the plus (or minus) direction. For further details on the definition and evaluation of the composite pattern factor, see Appendix D. Note that Equations (3.76) - (3.79) are dimensionally correct, since the pattern factor has units of length.

Now that the bounce path contributions have been presented, it is time to investigate the self impedance due to the "a-left" and "a-right" paths. Remember that the integrals in Equations (3.59) and (3.60) are a function of position. When these equations are substituted into Equations (3.72) and (3.73) a nested double integration results. Skipping the details of this tricky operation, the total self impedance due to paths "a-left" and "a-right" can be written as follows.
\[ Z_{\text{SELFA}} = \frac{Z_m}{2D_x D_z} \sum_{k, n} e^{\frac{-j \beta_m \omega r_m z}{r_m y}} \left[ \begin{array}{c} \pi(n') \pi(n) + \pi(n') \pi(n) \\
\pi(n') \pi(n') + \pi(n') \pi(n') \\
\pi(n') \pi(n') + \pi(n') \pi(n') \\
\pi(n') \pi(n') + \pi(n') \pi(n') \end{array} \right] \]

\[ + \pi(n') \pi(n) + \pi(n') \pi(n) + \pi(n') \pi(n) + \pi(n') \pi(n) \]

\[ + \pi(n') \pi(n') + \pi(n') \pi(n') + \pi(n') \pi(n') \]

\[ + \pi(n') \pi(n') + \pi(n') \pi(n') \]

\[ + (1 - e^{-j2 \beta_m b r_m x})(1 - r_m^2) y \]

\[ \cdot \int_{-a}^{a} I_2(n') e^{j \beta_m y r_m y d_y d_y} + \int_{-a}^{a} I_2(n') e^{+j \beta_m y r_m y d_y d_y} \]

\[ \cdot \int_{y''}^{a} I_4(n') e^{-j \beta_m y r_m y d_y d_y} + (1 - e^{-j2 \beta_m b r_m x})(1 - r_m^2) y \]

\[ \cdot \int_{-a}^{a} I_4(n') e^{j \beta_m y r_m y d_y d_y} + \int_{-a}^{a} I_4(n') e^{-j \beta_m y r_m y d_y d_y} \]

\[ + \int_{-a}^{a} I_4(n') e^{+j \beta_m y r_m y d_y d_y} \]

\[ - (3.80) \]
Note that the element pattern factor integrals indicated by \((n')\) are evaluated when the normalized current on array \(n'\) is under scattering conditions. Those pattern factor integrals indicated by \((n)t\) are evaluated when the normalized current on array \(n\) is under transmitting conditions. All pattern factors are evaluated in Appendix D.

The nested integrals indicated by the braces \{\} represent the coupling between the \(y\) directed sides of the reference element with the \(y\) directed sides of the test element. In order to evaluate these integrals, we need to introduce an expression for the element current under transmitting and scattering conditions. Due to the complexity of the problem, and the difficulty of the nested integration, the following normalized current distribution under transmitting conditions from Theile, et al. [25], is assumed. (See Figure 3.11).

\[
I(\xi) = \frac{\cos \beta_m' \xi}{\cos 2\beta_m'(a+b)}
\]  

(3.81)

Since \(\xi\) is measured from the terminals, the current on each side of the loop in terms of a locally defined coordinate system can be written as follows.

\[
I_1(n)t(x) = \frac{\cos \beta_m'(2a+2b-|x|)}{\cos \beta_m'(2a+2b)} \quad -b < x < b
\]  

(3.82)

\[
I_2(n)t(y) = \frac{\cos \beta_m'(a+b-y)}{\cos \beta_m'(2a+2b)} \quad -a < y < a
\]  

(3.83)
Figure 3.11. Normalized transmitting current on the rectangular loop element.

\[ I_3(n)(x) = \frac{\cos \beta_m' x}{\cos \beta_m'(2a+2b)} \quad -b \leq x \leq b \quad (3.84) \]

\[ I_4(n)(y) = \frac{\cos \beta_m'(a+b+y)}{\cos \beta_m'(2a+2b)} \quad -a \leq y \leq a \quad (3.85) \]

Notice that all currents have been normalized with respect to the terminal current \( \cos \beta_m'(2a+2b) \).

Clearly, this current function will tend to resonate when the loop element is roughly one wavelength in circumference. Yet one may wonder if this is an accurate model for the current under transmitting conditions. After comparing this assumed current distribution with
some moment method calculations for a single isolated loop element, we find that Equation (3.81) is a good approximation at or near the antenna resonance. Comparisons are available in Appendix F.

Now that the transmitting current has been assigned, what about the current on the array under scattering conditions. Due to the resonant nature of the loops in question, and the complexity of the geometry, it will be assumed that the current under scattering conditions will be similar to the current under transmitting conditions. For resonant structures, past work has indicated that this is a reasonable approximation. Therefore, we write

\[ I_j^{(n)}(x_j) = I_j^{(n')}(-x_j) \]  

(3.86)

The remaining portion of this chapter will concentrate on the evaluation of the double integrals in Equation (3.80) assuming the current distribution of Equations (3.81) and (3.86). Notice that the given current function is symmetric with respect to a plane cutting the element through the terminals. Specifically, from Equations (3.82) and (3.84) it is noted that

\[ I_2^{(n)t}(y) = I_4^{(n)t}(-y) \]  

(3.87)

and

\[ I_2^{(n')}(-y) = I_4^{(n')}(-y) \]  

(3.88)
Making use of Equations (3.87) and (3.88), Equation (3.80) reduces to

\[
Z_{\text{SELFA}} = \frac{Z_m'}{2D_x D_z} \sum_{k,n} \frac{-j\beta_m' \omega r_m' z}{r_{m'y}} \left[ p_{3,m'}^- + p_{1,m'}^- \right] e^{-j\beta_m' x} \left[ p_{3,m'}^- + p_{1,m'}^- \right] + \left[ p_{2,m'}^- + p_{3,m'}^- + p_{4,m'}^- \right]
\]

\[
+ \frac{e^{+j\beta_m' x} (n')}{p_{3,m'}^- + p_{1,m'}^- + p_{4,m'}^-} \left[ p_{1,m'}^- + p_{2,m'}^- + p_{4,m'}^- \right] + p_{2,m'}^- \left[ p_{4,m'}^- + p_{2,m'}^- + p_{4,m'}^- \right]
\]

\[
+ \frac{4 \sin^2(\beta_m' b r_m')}{p_{3,m'}^- + p_{1,m'}^- + p_{4,m'}^-} \left[ p_{1,m'}^- + p_{4,m'}^- \right] + p_{2,m'}^- \left[ p_{4,m'}^- + p_{2,m'}^- + p_{4,m'}^- \right]
\]

\[
\cdot \int_{-a}^{a} I_2(n') (y') e^{-j\beta_m' y} d y' d y'' + \int_{-a}^{a} I_2(n') (y') e^{+j\beta_m' y} d y' d y''
\]

\[
\cdot \int_{y''}^{a} I_2(n') (y') e^{-j\beta_m' y} d y' d y'' + \int_{y''}^{a} I_2(n') (y') e^{+j\beta_m' y} d y' d y''
\]

\[
+ \int_{y''}^{a} I_2(n') (y') e^{-j\beta_m' y} d y' d y'' \cdot (1 - r_{m'y})^2 + p_{1,m'}^- + p_{1,m'}^- + p_{1,m'}^- + p_{1,m'}^-
\]

\[
+ \int_{y''}^{a} I_2(n') (y') e^{+j\beta_m' y} d y' d y'' + \int_{y''}^{a} I_2(n') (y') e^{+j\beta_m' y} d y' d y''
\]

\[
+ \int_{y''}^{a} I_2(n') (y') e^{-j\beta_m' y} d y' d y'' + \int_{y''}^{a} I_2(n') (y') e^{+j\beta_m' y} d y' d y''
\]

\[
+ \int_{y''}^{a} I_2(n') (y') e^{-j\beta_m' y} d y' d y'' + \int_{y''}^{a} I_2(n') (y') e^{+j\beta_m' y} d y' d y''
\]

\[
+ \int_{y''}^{a} I_2(n') (y') e^{-j\beta_m' y} d y' d y'' + \int_{y''}^{a} I_2(n') (y') e^{+j\beta_m' y} d y' d y''
\]

\[
+ \int_{y''}^{a} I_2(n') (y') e^{-j\beta_m' y} d y' d y'' + \int_{y''}^{a} I_2(n') (y') e^{+j\beta_m' y} d y' d y''
\]

\[
+ \int_{y''}^{a} I_2(n') (y') e^{-j\beta_m' y} d y' d y'' + \int_{y''}^{a} I_2(n') (y') e^{+j\beta_m' y} d y' d y''
\]

\[
+ \int_{y''}^{a} I_2(n') (y') e^{-j\beta_m' y} d y' d y'' + \int_{y''}^{a} I_2(n') (y') e^{+j\beta_m' y} d y' d y''
\]

\[
+ \int_{y''}^{a} I_2(n') (y') e^{-j\beta_m' y} d y' d y'' + \int_{y''}^{a} I_2(n') (y') e^{+j\beta_m' y} d y' d y''
\]

\[
+ \int_{y''}^{a} I_2(n') (y') e^{-j\beta_m' y} d y' d y'' + \int_{y''}^{a} I_2(n') (y') e^{+j\beta_m' y} d y' d y''
\]

\[
+ \int_{y''}^{a} I_2(n') (y') e^{-j\beta_m' y} d y' d y'' + \int_{y''}^{a} I_2(n') (y') e^{+j\beta_m' y} d y' d y''
\]

\[
+ \int_{y''}^{a} I_2(n') (y') e^{-j\beta_m' y} d y' d y'' + \int_{y''}^{a} I_2(n') (y') e^{+j\beta_m' y} d y' d y''
\]

\[
+ \int_{y''}^{a} I_2(n') (y') e^{-j\beta_m' y} d y' d y'' + \int_{y''}^{a} I_2(n') (y') e^{+j\beta_m' y} d y' d y''
\]

\[
+ \int_{y''}^{a} I_2(n') (y') e^{-j\beta_m' y} d y' d y'' + \int_{y''}^{a} I_2(n') (y') e^{+j\beta_m' y} d y' d y''
\]

\[
+ \int_{y''}^{a} I_2(n') (y') e^{-j\beta_m' y} d y' d y'' + \int_{y''}^{a} I_2(n') (y') e^{+j\beta_m' y} d y' d y''
\]

\[
+ \int_{y''}^{a} I_2(n') (y') e^{-j\beta_m' y} d y' d y'' + \int_{y''}^{a} I_2(n') (y') e^{+j\beta_m' y} d y' d y''
\]

\[
+ \int_{y''}^{a} I_2(n') (y') e^{-j\beta_m' y} d y' d y'' + \int_{y''}^{a} I_2(n') (y') e^{+j\beta_m' y} d y' d y''
\]

\[
+ \int_{y''}^{a} I_2(n') (y') e^{-j\beta_m' y} d y' d y'' + \int_{y''}^{a} I_2(n') (y') e^{+j\beta_m' y} d y' d y''
\]

Making use of the current symmetry has halved the number of nested integrations to perform.
At this point, we are prepared to evaluate the nested double integral in Equation (3.89). After substituting Equations (3.83) and (3.85) into Equation (3.89), we then define $I_\alpha$ and $I_\beta$ in the following manner.

\[
I_\alpha = \int_{-a}^{a} \frac{a \cos \theta_m'(a+b-y')}{\cos \theta_m'(2a+2b)} e^{-j \theta_m' y r_m'y' \frac{y}{a} \frac{y}{a}} \frac{a \cos \theta_m'(a+b-y')}{\cos \theta_m'(2a+2b)}
\]

\[
\cdot e^{+j \theta_m' y r_m'y' \frac{y}{a} \frac{y}{a}}
\]

\[
I_\beta = \int_{-a}^{a} \frac{a \cos \theta_m'(a+b-y')}{\cos \theta_m'(2a+2b)} e^{-j \theta_m' y r_m'y' \frac{y}{a} \frac{y}{a}} \frac{a \cos \theta_m'(a+b-y')}{\cos \theta_m'(2a+2b)}
\]

\[
\cdot e^{-j \theta_m' y r_m'y' \frac{y}{a} \frac{y}{a}}
\]

If we define $I_\Sigma$ as the sum of Equations (3.90) and (3.91), we obtain after integrating

\[
I_\Sigma = \frac{1}{\beta_m^2, (1-r_m^2, y)^2/(\cos 2\beta_m'(a+b))} \left[ \begin{array}{c} \cos 2\beta_m'(a+b) + \sin 2\beta_m'(2a+b) \\ -j 2 \beta_m' y \cos \beta_m'(a+b) + \cos \beta_m'(2a+b) + 2e^{+j \beta_m' y r_m'y} \\ \end{array} \right] \left[ \begin{array}{c} (j r_m'y \cos \beta_m'(a+b) + \sin \beta_m'(2a+b)) \rightarrow \Delta \\ \end{array} \right] \]

\[ (3.92) \]
where $\Delta$ is given by

$$
\Delta = \frac{2jr_m'y}{\beta_m'(1-r_m'y)(\cos 2\phi_{m'\alpha}(a+b))^2} \left[ \beta_m'a + \sin 2\beta_{m'}(a+b) - \sin 2\beta_{m'}b \right] \left( \cos 2\phi_{m'\alpha}(a+b) \right) - \frac{\sin 2\beta_{m'}(a+b) - \sin 2\beta_{m'}b}{4} \right]
$$

(3.93)

Since the remaining pattern factors are evaluated in Appendix D, one might think that the self impedance analysis is finished. Actually, Equation (3.93) requires further analysis, since it profoundly affects the convergence of the $Z_{SELF\Delta}$ impedance term of Equation (3.89). To demonstrate this, substitute Equation (3.93) alone into Equation (3.89)

$$
Z_{SELF\Delta} = \frac{Z_m}{2D_X D_Z} \sum_{k=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} e^{-j\phi_{m'}w_{m'}z} \beta_{m'}^2 \left( \cos 2\phi_{m'\alpha}(a+b) \right)^2 \left\{ 8j(\beta_{m'}a + \frac{\sin 2\beta_{m'}(a+b)}{4} - \frac{\sin 2\beta_{m'}b}{4}) \right\}
$$

(3.94)

Notice that all quantities in the brace $\{ \}$ are independent of the summation variables $k$ and $n$. Since nothing in this term decays in $k,n$
space, does it converge? The answer is no; this term does not converge at all! Yet a simple physical argument can be invoked which allows us to throw out the $\Delta$ term altogether. To show this, define the complex constant $C$ as follows.

$$C = e^{-j\beta_m' w_s m' z} \frac{\{8j(\beta_m' a + \frac{\sin 2\beta_m'(a+b)}{4} - \frac{\sin 2\beta_m' b}{4})\}}{\beta_m'(\cos 2\beta_m'(a+b))^2}$$

(3.95)

Substituting Equation (3.95) into Equation (3.94) yields

$$Z_{S E L F A, \Delta} = \frac{Z_m C}{2D_x D_z} \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} e^{-j \frac{2\pi}{D_x} k w}$$

(3.96)

Invoking the Poisson Sum Formula from Papoulis [26], the inner summation is transformed to

$$\sum_{k=-\infty}^{\infty} e^{-j \frac{2\pi}{D_x} k w} = \sum_{k=-\infty}^{\infty} \delta(w + k D_z)$$

(3.97)
where
\[ \delta(w + kD_z) = 0 \quad (3.98) \]

except when
\[ w = -kD_z \quad . \quad (3.99) \]

How are Equations (3.97) through (3.99) physically interpreted? Quite simply, it states that the separation between the test element of array (n) and the reference element of array (n') should not be an integer multiple of the interelement spacing. In other words, the test element cannot physically touch any element of array (n')! The impedance does not converge for this case, since the scattered field from array (n') in Region II is singular at any element position. As Figure 3.7 demonstrates, \( w \ll \lambda \), and certainly \( w \ll D_z \). From Equations (3.98) and (3.97), we can write
\[ A = 0 \quad , \quad w \neq kD_z \quad . \quad (3.100) \]

If Equation (3.100) is substituted into Equations (3.92) and (3.89), the resultant path "a-left" and "a-right" self impedance contribution is given by Equation (3.101).
The total self impedance of this rectangular loop antenna can be found by adding Equations (3.101), (3.76), (3.77), (3.78), and (3.79).
E. RELATION TO PAST PERIODIC SURFACE RESEARCH

It is important to relate the "path" description contained in this chapter with previous research. Recall in Figure 3.9, we identified six path contributions to the scattered field. Later, we combined the "a-left" and "a-right" impedance contribution and ended up with five distinct contributions to the element self impedance. In order to demonstrate how this path description is consistent with past work by Munk, et al. [27], I propose the following thought experiment.

Suppose we analyze an infinite array of piecewise linear elements, where the piecewise array element is strictly contained in the x-z plane. In other words,

\[ \vec{p}(j) \cdot \hat{y} = 0 \quad . \tag{3.102} \]

As Figure 3.12 suggests, when the scattered field from such a structure is calculated, we either lie totally to the left or right of the array. Since there is no equivalent Region II for this case, let us now calculate the scattered field at a point \( \mathcal{R}_{III} \) in Region III. Using the same path designation as Figure 3.9, the field is obtained by summing the contributions from paths "a-right", "b", "c", "d", and "e". This appears to be inconsistent, since we show three right-travelling waves and only two left-travelling waves!
Figure 3.12. Path description for piecewise planar elements.
The mystery is solved when we observe for the case of planar array elements, path "e" is nothing more than higher order reflections of path "a-right". Since we lie in Region III, path "a-right" can be described by a composite pattern factor, \( c(n) \), integration dependent on the position of \( R_{III} \). (In other words, path "a-right" will have no equivalent \( \lambda_0 \) parameter). Under these circumstances, path "a-right" and path "e" can be combined, since they have the same functional form. If we define the "direct path a" as the sum of "a-right" and "e", we would obtain the four path description of Munk, et al. [27]. When the self impedance of this element is calculated under these conditions, the four self impedance contributions can be called \( Z_A \), \( Z_B \), \( Z_C \), and \( Z_D \). Again, these results are identical to those of Munk.

One other important point should be made regarding the five path contributions to the total self impedance. Since one wire radius separates the reference and test element, the \( Z_{SELF A} \) impedance contribution due to path "A" will have the strongest evanescent wave coupling. As such, this contribution to the self impedance double summation will take the most terms to converse. Since the path-A contribution is independent of the surrounding dielectric layers (note the absence of \( \epsilon_{m'} \), \( \Omega_{m'} \), \( \chi_{m'}^{\pm} \), \( \psi_{m'}^{\pm} \)), it can be calculated and stored separately from the other four path contributions. As long as the dielectric surrounding the array (\( \epsilon_{m'} \)) and the interelement spacings (\( D_x \) and \( D_z \)) are not changed from one design to the next, \( Z_{SELF A} \) does not need
to be recalculated. From a design standpoint, this implies that as one varies the thicknesses and dielectric constants of the neighboring dielectric layers, only the bounce path contributions to $Z_{\text{SELF}}$ need to be calculated. The implication of this "five path" description is clear. All periodic surface design calculation codes which utilize the plane wave expansion calculation technique should be rewritten to incorporate this time savings feature.

The conclusion of this mental exercise is as follows. Munk's analysis is not general enough to handle the non-planar array element case. For this important class of periodic surface elements, the path analysis presented herein must be used.

F. SUMMARY

This chapter developed the equations for the mutual impedance between two non-planar loop arrays located in different dielectric layers $m$ and $m'$. For the case of $m' < m$ and $m' > m$, it was found that the mutual impedance was composed of four "path" contributions. When the mutual impedance concept was extended to approximate the self impedance, we found for these non-planar array elements, the self impedance was composed of six path contributions. Equations for the self impedance were derived, and the entire analysis was compared to past research in the periodic surface area.
In this section, expressions will be developed for the total forward and backscattered electric fields from an infinite array of non-planar loops embedded in slab $m'$ of a stratified dielectric medium. These expressions will be similar to those developed in Section B of Chapter III except for the location of the field point $R$. In the forward scattered case, $R$ is located on the surface of the last dielectric slab. In the backscattered case, $R$ is located on the surface of the first dielectric slab.

For the forward scattered field case, Figure 4.1 illustrates the two path contributions to the electric field at $\hat{R} = y \ b_{MM}$. As before, paths "a-right" and "b" contribute to the total field to the right of the last dielectric slab. Summing the contributions from paths "a-right" and "b" yields
Figure 4.1 Path contributions to the field at the point $R = b_{MM}$. 

$\mathbf{R} = \mathbf{y} b_{MM}$ (TRANSMISSION)
\[ E_{f.s.} (R=b_{MM}) = \frac{-I(\hat{R}(1))Z_m}{2D_x D_z} \sum_{k,n} \left[ -l_{m'}^c(1) + l_{m'}^c(1) \right]_{m',e}^{n+} \]

\[ + e^{-j2\beta_m r_{m'}y d_m} \left[ \Omega_{m',t_{m'},0}^{n+} \right] + \left[ \Omega_{m',t_{m'},0}^{n+} \right]_{m',e}^{n+} \]

\[ = -j2\beta_m r_{m'}y d_m \]

\[ \phi_{m'+1, MM} = \sum_{i=m'+1}^{MM} e^{-j\beta_i r_{i,y}} \]

\[ \phi_{m'+1, MM} = \prod_{i=m'+1}^{MM} e^{-j\beta_i r_{i,y}} \]

\[ \phi_{m'+1, MM} = \prod_{i=1}^{MM} e^{-j\beta_i r_{i,y}} \]

(4.4)

where \( R(1) \) is the position vector which defines the location of the reference element terminals, \( d_m'' = y(1) - b_{m}' \).

\[ \phi_{m'+1, MM} = \sum_{i=m'+1}^{MM} e^{-j\beta_i r_{i,y}} \]

\[ \phi_{m'+1, MM} = \prod_{i=m'+1}^{MM} e^{-j\beta_i r_{i,y}} \]

\[ \phi_{m'+1, MM} = \prod_{i=1}^{MM} e^{-j\beta_i r_{i,y}} \]

(4.4)

is the total phase shift from slab \( m'+1 \) to slab \( MM \),

\[ \Omega_{m'} = \frac{1}{1 + \left( r_{m',e} \right) - j2\beta_m d_m r_{m',y}} \]

(4.3)

98
is the conjunct in slab $m'$, and

$$\prod_{i=m'}^{M} I_{m',i}^+ = \prod_{i=m'}^{M} I_{i}^+ \quad (4.4)$$

is the effective transmission coefficient from slab $m'$ to slab $(M+1)$. The only unknown quantity in Equation (4.1) is the value of the terminal current on the reference element, $I(R(1))$. To calculate $I(R(1))$, we use Equation (A.28) from Appendix A.

$$I(R(1)) = \frac{v^{inc}(R(1))}{(Z_1^{(self)} + Z_L)} \quad (4.5)$$

Recall that the induced voltage $v^{inc}(R(1))$ was calculated in Chapter II, Equation (2.20). The self impedance was exhaustively studied in Chapter III, and the element load impedance is known. Since all quantities on the right-hand side of (4.5) are known, $I(R(1))$ can be considered a known quantity.

Equation (4.1) is the exact expression for the field at the $MM^\text{th}$ dielectric slab, since it includes all propagating and evanescent waves. As such, it is a true near field expression. If one is solely interested in the far field forward scattered waves, only the propagating modes in (4.1) need to be summed. If small interelement spacings are maintained to prevent grating lobes, only the $k=n=0$ term propagates. Thus, the far field equivalent of (4.1) is simply,
In an analogous manner, we wish to calculate the reradiated field in the \(-y\) direction at the point \(R=(0,0,0)\), as shown in Figure 4.2. The reradiated field at the surface of the first dielectric slab is found by summing the contributions due to paths "a-left" and "c". The expression for this reradiated field at \(R=(0,0,0)\) is given by

\[
\hat{E}_{bs}(R=0_{x}+0_{y}+0_{z}) = -\frac{I(\hat{R}(1))Z_{m'}}{2D_{x}D_{z}} \sum_{k,n=0}^{\text{various values}} \left[ - \hat{P}_{m'}^{+} + \hat{P}_{m'}^{-} \right] e^{-j2\beta_{m'}d_{m'}r_{m'y}} \frac{-j\beta_{m'}(b_{m'}-y(1))r_{m'y}}{r_{m'y}} \phi_{m'+1,MM} \cdot \frac{\hat{\tau}_{m'}^{+}}{r_{m'y}} \nabla_{y}
\]

(4.6)

\[
\hat{E}_{bs}(R=0_{x}+0_{y}+0_{z}) = -\frac{I(\hat{R}(1))Z_{m'}}{2D_{x}D_{z}} \sum_{k,n=0}^{\text{various values}} \left[ - \hat{P}_{m'}^{+} + \hat{P}_{m'}^{-} \right] e^{-j2\beta_{m'}d_{m'}r_{m'y}} \frac{-j\beta_{m'}(b_{m'}-y(1))r_{m'y}}{r_{m'y}} \phi_{m'+1,MM} \cdot \frac{\hat{\tau}_{m'}^{+}}{r_{m'y}} \nabla_{y}
\]

(4.7)
Figure 4.2 Path contributions to the field at the point \( \mathbf{R} = (0,0,0) \).
where once again we note that
\[
\Omega_m' = \frac{1}{1 - \frac{\sin^2 \theta}{\sin^2 \phi} \left( e^{-j \beta_m' d_{m'} r_{m'} y} \right)}
\]  \hspace{1cm} (4.8)

\[
\phi_{m'-1,1} = \prod_{i=1}^{m'-1} e^{j \beta_i d_i r_i y} = \prod_{i=1}^{m'-1} \phi_i
\]  \hspace{1cm} (4.9)

and
\[
\Gamma_{m'-0} = \prod_{i=m-1}^{1} \tau_i
\]  \hspace{1cm} (4.10)

In the backscattered case, $\Gamma_{m',0}$ is calculated from right to left. It represents the effective transmission coefficient from slab $m'$ to the free-space region left of the first slab.

Again notice that Equation (4.7) includes all evanescent waves. If one is solely interested in the reradiated fields in the far zone, the summation in (4.7) reduces to a finite sum. As with the forward scattered field case, for an array with close interelement spacing, the sum collapses to the single $k=n=0$ term.

If the current $I(\mathbf{R}^{(1)})$ is impressed by some voltage generator, Equations (4.1) and (4.7) would be the total forward scattered and backscattered fields from this array structure. If we are interested
in how much energy is scattered in the forward and backward direction
due to an externally impressed plane wave, then (4.1) and (4.7) must be
modified.

Given an incident plane wave, we have shown how to calculate
\( V(R) \) and \( Z_{11} \), therefore, \( I(R) \) is known from (4.5). The total
forward and backscattered fields are thus given by

\[
\mathbf{E}_{\text{BS}}(R=0,0,0) = \mathbf{E}^{DR}(0,0,0) + \text{[Equation (4.7)]} \tag{4.11}
\]

and

\[
\mathbf{E}_{\text{FS}}(R=0,b_{MM},0) = \mathbf{E}^{DT}(0,b_{MM},0) + \text{[Equation (4.1)]} \tag{4.12}
\]

where

\[
\mathbf{E}^{DR} = \{ \mathbf{n}_{-0} \cdot (\mathbf{E}^i(0,0,0) \cdot \mathbf{n}_{0+}) \}_{0-} \mathbf{r}_0^+ + \{ \mathbf{n}_{0-} \cdot (\mathbf{E}^i(0,0,0) \cdot \mathbf{n}_{0+}) \}_{0+} \mathbf{r}_0^+ \tag{4.13}
\]

represents the Electric field Directly Reflected from the dielectric
slab alone and

\[
\mathbf{E}^{DT} = \{ \mathbf{n}_{0+} \cdot (\mathbf{E}^i(0,0,0) \cdot \mathbf{n}_{0+}) \}_{T_{0,MM+1}} \mathbf{r}_{0+}\mathbf{n}_{0+} \tag{4.14}
\]

represents the Electric field Directly Transmitted through the
dielectric slabs as if the antenna array was not present.
By calculating (4.11) and (4.12), one can analyze the reflection and transmission properties of the entire array versus frequency and scan angle.

One further note is in order. As previously discussed in Section I, the most likely configuration for the phased array in question will have a ground plane behind the last dielectric slab. Quite clearly, this prevents any energy from being reradiated to the right, so that for a lossless medium, all incident energy must be split in some manner between reradiation in the \(-y\) direction, and energy sent towards the antenna load.

In conclusion, this Chapter presented equations which will allow one to calculate the total forward and backscattered fields of an array of loop antennas embedded in a stratified dielectric medium.
A. GENERAL COMMENTS

This chapter will examine the behavior of calculated array scan impedance (self impedance variation with scan angle \( n \) when \( Z_L=0 \)) for a variety of differing array geometries. It will be shown that an array of square loop elements in free space has large impedance variations with scan angle. This variation in the array scan impedance can be significantly improved if the square loop elements are placed in a general stratified dielectric medium backed by a ground plane.

In order to produce the results described in this chapter, a computer program was written. This computer routine calculates the array self impedance versus scan angle and frequency, for a single doubly infinite-doubly periodic array of rectangular loop elements embedded in a general stratified medium. (Figure 4.2 illustrates the array of loop antennas embedded in one of \( MM \) dielectric layers.) The number of dielectric layers (\( MM \)) is arbitrary, although no results are presented for more than two layers. An optional ground plane can be placed behind the rightmost dielectric slab. The interelement spacings \( (D_x \text{ and } D_z) \) can be varied, as well as the dimensions for the rectangular loop \( (a \text{ and } b) \).
Basically, this computer program calculates the five bounce path contributions to the array self impedance from Equation (3.74). In order to check the consistency of the impedance calculation, the induced voltage from Equation (2.28) is calculated. The value of the terminal current \( I(R(n')) \) is then calculated from Equation (1.17).

\[
I(R(n')) = \frac{V_{\text{ind}}(R(n'))}{(|Z_L| + |Z_{\text{SELF}}|)}
\]  

(5.6)

Finally, the reradiated fields are calculated from Equations (4.11) and (4.12). From these calculations, we can determine the overall specular reflection and transmission coefficients, assuming no grating lobes exist.

Suppose we assume that \( Z_{\text{SELF}} \) has been correctly calculated, and \( Z_L = 0 \). Further assume that the dielectric layers are lossless. Given these assumptions, if the array structure is illuminated with an incident plane wave, energy should be conserved. That is to say,

\[
\sqrt{|R|^2 + |T|^2} = 1
\]  

(5.2)

where \( (R,T) \) are the specular reflection and transmission coefficients, respectively.

Equation (5.2) acts as a "barometer" of the calculated results. To produce precise conservation of energy, the impedance double summation of Equation (3.74) must be accurately calculated. Although the computer
code can calculate array scan impedances to any specified accuracy, it requires large amounts of computer run time for summation accuracies under 1%. Since we are interested in the overall variation of the array scan impedance with changing geometric configurations, the summation accuracy will be relaxed somewhat. Specifically, all scan impedance results presented in this chapter are calculated with a specified accuracy of three percent. (This simply means that the scan impedance double summation of Equation (3.74) is truncated when additional "square rings" in complex \{k,n\} space do not change the impedance magnitude by more than three percent. Figure 5.1 illustrates the concept of summing in "square rings". Each ring symbolizes that every discrete \{k,n\} contribution to $Z_{\text{SELF}}$ for that ring is summed prior to summing additional \{k,n\} contributions on the next outer ring.) Despite truncating the impedance double summation, energy conservation holds quite well. For the ten cases to be presented,

$$0.98 < \sqrt{|R|^2 + |T|^2} < 1.02$$

Equation (5.3) gives additional confidence in the overall accuracy of the calculated scan impedance. One should note that a flow chart of the computer program is available in Appendix H.
Figure 5.1. Summing "square ring" contributions to the array scan impedance in the complex plane \(\{k,n\} \text{ space}\).
B. CALCULATED SCAN IMPEDANCES

At this point, we are prepared to present actual calculated scan impedance results. It is not the purpose of this document to exhaustively characterize the scan impedance behavior. Instead, we shall closely examine ten particular cases of interest. These cases will effectively demonstrate four important points:

1) The large variation in scan impedance for the loop array in free space.

2) The importance of choosing an optimum spacing between the loop array and a ground plane.

3) How to control the scan impedance variation with dielectric scan compensation.

4) How to incorporate the radome properties into the design process. (The integrated antenna/radome concept.)

Before we discuss results, let us keep in mind exactly what we are calculating. Figure (5.2) shows a side view of a loop array embedded in two "dielectric layers". (The quotations around "dielectric layers" are added since $\varepsilon_1$ and/or $\varepsilon_2$ may be one.) There are several quantities of interest in Figure 5.2; the thicknesses of the two dielectric layers ($d_1$ and $d_2$), their respective dielectric constants ($\varepsilon_1$ and $\varepsilon_2$), and the center to center spacing between the array and ground plane ($d_{cc}$). In the cases to follow, the impedance will be characterized versus these five variables.
Figure 5.2. Geometry of array for calculated results (cases 2-10).
Tables 5.1 and 5.2 summarize the data cases to be presented. These tables relate the figure numbers of the data to the case numbers as well as the particular case geometries. Both tables present the same information in different formats. Table 5.1 presents the cases in terms of the actual physical dimensions. These numbers can be inserted into the computer program to reproduce the calculated results. Table 5.2 presents the information in a completely different manner. The dielectric layer thicknesses \( d_1 \) and \( d_2 \) and center to center spacing parameter \( d_{cc} \) are tabulated in terms of the "resonant wavelength" in each respective dielectric layer. The "resonant wavelength" simply means that at some particular frequency, the loop element is precisely one wavelength in circumference. To make matters easy, we adjust the loop dimensions such that the "resonant wavelength" occurs at 10 GHz. Therefore, if the array is in free space,

\[
\lambda_{D2} = \lambda_0 = 3\text{(cm)} \quad \text{10 GHz "resonance wavelength"}
\]

(5.4)

For a square element, the side dimensions in free space are as follows.

\[
2a = 2b = \frac{\lambda_0}{4}
\]

(5.5)

\[
a = b = \frac{\lambda_0}{8} = .375\text{(cm)}
\]

(5.6)
### Table 5.1

**Summary of Case Geometries Presented in Figures 5.6 to 5.27**

(In terms of actual physical dimensions)

<table>
<thead>
<tr>
<th>FIGURE</th>
<th>CASE #</th>
<th>FREQ (GHz)</th>
<th>$\varepsilon_1$</th>
<th>$\varepsilon_2$</th>
<th>Array in Free Space (No Ground Plane Present)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.6</td>
<td>1</td>
<td>7,8,9</td>
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<td>1.0</td>
<td></td>
</tr>
<tr>
<td>6.7</td>
<td>1</td>
<td>10,11,12</td>
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<td>1.0</td>
<td></td>
</tr>
<tr>
<td>5.8</td>
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<td>7,8,9</td>
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<td>1.0</td>
<td>1.0, 1.0, 2.0625, 1.50</td>
</tr>
<tr>
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<td>10,11,12</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0, 1.0, 2.0625, 1.50</td>
</tr>
<tr>
<td>5.12</td>
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<td>7,8,9</td>
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<td>1.0</td>
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<td>1.0</td>
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<td>1.0</td>
<td>1.0, 1.0, 2.0625, 0.65625</td>
</tr>
<tr>
<td>5.15</td>
<td>4</td>
<td>10,11,12</td>
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<td>1.0</td>
<td>1.0, 1.0, 2.0625, 0.65625</td>
</tr>
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<td>1.6</td>
<td>1.0, 1.3341, 0.51881</td>
</tr>
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<td>2.0</td>
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<td>2.0</td>
<td>1.0, 1.1932, 0.46404</td>
</tr>
<tr>
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<td>2.0</td>
<td>1.0, 0.9281, 0.46404</td>
</tr>
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<td>1.0</td>
<td>2.0</td>
<td>1.0, 0.9281, 0.46404</td>
</tr>
<tr>
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<td>2.0</td>
<td>1.0, 1.0607, 0.46404</td>
</tr>
<tr>
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<td>10,11,12</td>
<td>1.0</td>
<td>2.0</td>
<td>1.0, 1.0607, 0.46404</td>
</tr>
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<td>2.0</td>
<td>1.0, 1.3258, 0.46404</td>
</tr>
<tr>
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<td>10,11,12</td>
<td>1.0</td>
<td>2.0</td>
<td>1.0, 1.3258, 0.46404</td>
</tr>
<tr>
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<td>2.0</td>
<td>1.0, 0.9713, 1.1016</td>
</tr>
<tr>
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<td>10,11,12</td>
<td>3.3</td>
<td>2.0</td>
<td>1.0, 0.9713, 1.1016</td>
</tr>
</tbody>
</table>

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TABLE 5.2
SUMMARY OF CASE GEOMETRIES PRESENTED IN FIGURES 5.6 TO 5.27
(In terms of $\lambda_{D1}$ and $\lambda_{D2}$)

| FIGURE | CASE # | FREQ (GHz) | $\varepsilon_1$ | $\varepsilon_2$ | ARRAY IN FREE SPACE
|--------|--------|------------|----------------|----------------|----------------------
| 5.6    | 1      | 7,8,9      | 1.0            | 1.0            | (No Ground Plane Present) |
| 6.7    | 1      | 10,11,12   | 1.0            | 1.0            |                      |
|        |        |            |                |                |                      |
| 5.8    | 2      | 7,8,9      | 1.0            | 1.0            | 1/3 11/16 1/2 |
| 5.9    | 2      | 10,11,12   | 1.0            | 1.0            | 1/3 11/16 1/2 |
| 5.12   | 3      | 7,8,9      | 1.0            | 1.0            | 1/3 11/16 1/4 |
| 5.13   | 3      | 10,11,12   | 1.0            | 1.0            | 1/3 11/16 1/4 |
| 5.14   | 4      | 7,8,9      | 1.0            | 1.0            | 1/3 11/16 7/32 |
| 5.15   | 4      | 10,11,12   | 1.0            | 1.0            | 1/3 11/16 7/32 |
| 5.16   | 5      | 7,8,9      | 1.0            | 1.6            | 1/3 9/16 7/32 |
| 5.17   | 5      | 10,11,12   | 1.0            | 1.6            | 1/3 9/16 7/32 |
| 5.18   | 6      | 7,8,9      | 1.0            | 2.0            | 1/3 9/16 7/32 |
| 5.19   | 6      | 10,11,12   | 1.0            | 2.0            | 1/3 9/16 7/32 |
| 5.20   | 7      | 7,8,9      | 1.0            | 2.0            | 1/3 9/16 7/32 |
| 5.21   | 7      | 10,11,12   | 1.0            | 2.0            | 1/3 9/16 7/32 |
| 5.22   | 8      | 7,8,9      | 1.0            | 2.0            | 1/3 1/2 7/32 |
| 5.23   | 8      | 10,11,12   | 1.0            | 2.0            | 1/3 1/2 7/32 |
| 5.24   | 9      | 7,8,9      | 1.0            | 2.0            | 1/3 5/8 7/32 |
| 5.25   | 9      | 10,11,12   | 1.0            | 2.0            | 1/3 5/8 7/32 |
| 5.26   | 10     | 7,8,9      | 3.3            | 2.0            | 0.0432 0.5193 7/32 |
| 5.27   | 10     | 10,11,12   | 3.3            | 2.0            | 0.0432 0.5193 7/32 |

NOTE: In all cases, $a=b=\lambda_{D2}/8$ and $D_x=D_z=(3/10)\lambda_{D2}$
Since the elements are compact in size, we can easily place the elements close together. For all cases to be presented, we choose interelement spacings $3\lambda_{D2}$ apart. In free space, this corresponds to

$$D_x = D_z = .3\lambda_0 = .9 \text{(cm)}$$  \hspace{1cm} (5.7)

Suppose the array is embedded in a dielectric layer. Under these conditions, simply modify (5.4), (5.6) and (5.7) by the square root of the appropriate dielectric constant.

$$\lambda_{D1} = \frac{3.0}{\sqrt{\varepsilon_1}} \text{ (cm)}$$ \hspace{1cm} (5.8) \hspace{1cm} 10 \text{ GHz "resonant wavelength"}

$$\lambda_{D2} = \frac{3.0}{\sqrt{\varepsilon_2}} \text{ (cm)}$$ \hspace{1cm} (5.9) \hspace{1cm} 10 \text{ GHz "resonant wavelength"}

$$a = b = \frac{.375}{\sqrt{\varepsilon_2}} \text{ (cm)}$$ \hspace{1cm} (5.10) \hspace{1cm} 10 \text{ GHz "resonant wavelength"}

$$D_x = D_z = \frac{.375}{\sqrt{\varepsilon_2}} \text{ (cm)}$$ \hspace{1cm} (5.11) \hspace{1cm} 10 \text{ GHz "resonant wavelength"}

In order to see how Equations (5.8) through (5.11) are used in Table 5.2, look at case seven.
\[ d_{cc} = \frac{7}{32} \lambda_d \left| \frac{3}{2} \right| = \frac{7}{32} \left( \frac{3}{2} \right) = 0.46404 \text{(cm)} \quad (5.12) \]

\[ d_2 = \frac{7}{16} \lambda_d \left| \frac{3}{2} \right| = \frac{7}{16} \left( \frac{3}{2} \right) = 0.9281 \text{(cm)} \quad (5.13) \]

\[ d_1 = \frac{\lambda_d l}{3} \left| \frac{3}{\sqrt{2}} \right| = \frac{1}{3} \left( \frac{3}{\sqrt{2}} \right) = 1.0 \text{(cm)} \quad (5.14) \]

Equations (5.12) through (5.14) produce the corresponding dimensions for case seven presented in Table 5.1. Although we now understand how Tables 5.1 and 5.2 relate to one another, we will exclusively refer to Table 5.2 in the discussion to follow.

With this information behind us, let us concentrate on the presentation of the scan impedance data. An example scan impedance plot is presented in Figure 5.3. This figure is an enlarged version of the data to come. The header information at the top specifies the case number, frequency, scan plane, and scan angles. Notice that there are two curves per graph. The lighter line illustrates the scan impedance variation with scan angle for the \( \alpha=0^\circ \) scan plane (see Figure 5.4). The darker line illustrates the scan impedance variation versus scan angle for the \( \alpha=90^\circ \) plane (see Figure 5.5). It is important to note that for \( n=1^\circ \), these two curves intersect. This can be explained by closely looking at Figures 5.4 and 5.5. When \( n=1^\circ \), the parallel polarized electric field (\( E_s \)) in the \( \alpha=0^\circ \) scan plane is identically oriented.
Figure 5.3. Sample scan impedance plot.

(All Impedances in Ohms)
Figure 5.4. Geometry for $\alpha=0^\circ$ scan plane.

Figure 5.5. Geometry for $\alpha=90^\circ$ scan plane.
to the orthogonal polarized electric field \( E_{\perp} \) in the \( \alpha=90^\circ \) scan plane. Since these two impedances agree for \( \eta=1^\circ \) in the two scan planes, it gives us further confidence in the results. This is true because the equations for the orthogonal and parallel polarizations are completely different and were derived independently from one another.

We are finally prepared to discuss the calculated scan impedance results. Let us begin with the most basic case. Suppose the doubly infinite doubly periodic array of square loop elements is placed in free space. The resonant wavelength is \( \lambda_0=3\text{cm} \), with an interelement spacing of \( .3\lambda_0 \) in the \( x \) and \( z \) direction, respectively. The variation of scan impedance is shown in Figure 5.6 for the two scan planes (\( \alpha=0^\circ,90^\circ \)) for frequencies of 7, 8, and 9 GHz. Case one data for frequencies of 10, 11, and 12 GHz can be found in Figure 5.7. Looking at these two figures we can immediately notice three trends. First of all, for this uncompensated array case there is a large variation of the scan impedance with scan angle in both scan planes. The variation in the \( \alpha=90^\circ \) scan plane is almost always larger than with the \( \alpha=0^\circ \) scan plane. Second, the scan impedance varies rapidly with frequency. Finally, the behavior of the real part of the scan impedance varies differently in the two scan planes. Specifically, the real part tends to increase in the \( \alpha=90^\circ \) plane and decrease in the \( \alpha=0^\circ \) plane. This problem is commonly referred to as the "divorce problem", and is inherent in almost every type of phased array radiating element. Any practical design must address this "divorce problem".
Figure 5.6. Scan impedance variation and associated geometry for Case 1, Frequency = 7, 8, 9 GHz.
Figure 5.7. Scan impedance variation and associated geometry for Case 1, Frequency = 10, 11, 12 GHz.
Next, we place the array of square loops from case one in front of an infinite ground plane. Cases two, three and four investigate the variation of scan impedance as the ground plane spacing $d_{cc}$ is varied $\lambda_0/2$, $\lambda_0/4$, and $7\lambda_0/32$, respectively. For case two ($d_{cc} = \lambda_0/2$), the scan impedance versus scan angle behaves as shown in Figures 5.8 and 5.9. Looking at these two figures reveals several interesting points. First, the impedance varies even more with scan angle for this ground plane spacing. Second, although the "divorce" problem is still evident, at some frequencies ($F = 9, 10, 11$ GHz) the real part of the impedance varies similarly for angles close to $n=1^\circ$; that is to say, the impedances are not initially heading in opposite directions in the two scan planes. The third item to notice is that as frequency is increased, the real part of the scan impedance goes to zero for some scan angles. For instance, at 10 GHz, the real part of $Z_{SELF} = 0$ at $n=1^\circ$. To understand why this happens, refer to Figure 5.10. This figure illustrates the effective composite pattern factor for an array of loops placed $\lambda_0/2$ in front of a ground plane (see Appendix E for more on the effective pattern factor). Notice that at $n=0^\circ$, the effective pattern factor has a null in the square loop element pattern. Since these loop elements have interelement spacings less than $\lambda_0/2$, no grating lobes occur. Therefore, whenever the effective pattern has a null in its antenna pattern, the real part of the scan impedance will be zero for that scan angle and frequency! We must avoid these nulls in practical antenna designs, or else the antenna will be "blind" in the
Figure 5.8. Scan impedance variation and associated geometry for Case 2, Frequency = 7,8,9 GHz.
Figure 5.9. Scan impedance variation and associated geometry for Case 2, Frequency = 10,11,12 GHz.
Figure 5.10. Effective element composite pattern factor versus \(n\) for one square loop element placed \(\lambda_0/2\) in front of a ground plane (\(F = 10\) GHz).
direction of the null. In a similar manner, looking at Figure 5.11 demonstrates that for \( F = 12 \) GHz in Figure 5.9, the real part of the scan impedance tends to zero for \( \alpha = 0^\circ, \eta = 45^\circ \).

Next, reduce the array to ground plane spacing, \( d_{cc} \), to \( \lambda_0/4 \) (case three). The scan impedance variations for this case are shown in Figures 5.12 and 5.13. In this case, at 7 GHz, the variation in the \( \alpha=0^\circ \) scan plane is much larger than the \( \alpha=90^\circ \). When the frequency is increased to 12 GHz, the situation has reversed; the \( \alpha=90^\circ \) scan plane variation has significantly reduced. Again, the behavior for this case can be explained by the effective composite pattern factor. In Figure E.10, for \( \alpha=0^\circ \) at \( F=12 \) GHz, we see the pattern factor decrease (in magnitude) with scan angle. In Figure E.17, for \( \alpha=90^\circ \) at 12 GHz, the pattern factor increases in magnitude with scan angle. The effective pattern factor shapes explain why the real part of the impedance behaves as it does in the two scan planes.

Our fourth case involves further decreasing the ground plane spacing to \( d_{cc} = 7\lambda_0/32 \). The scan impedance variations for this case are shown in Figures 5.14 and 5.15. Basically, this case has been included for comparisons with future cases where the dielectric constant surrounding the array is varied. It will be later shown that this ground plane spacing, when combined with dielectric, will produce significant reductions in the scan impedance variations.
Figure 5.11. Effective element composite pattern factor versus \( n \) for one square loop element placed \( \lambda_0/2 \) in front of a ground plane (\( F=12 \) GHz).
Figure 5.12. Scan impedance variation and associated geometry for Case 3, Frequency = 7,8,9 GHz.
Figure 5.13. Scan impedance variation and associated geometry for Case 3, Frequency = 10,11,12 GHz.

(All Impedances in Ohms)
Figure 5.14. Scan impedance variation and associated geometry for Case 4, Frequency = 7,8,9 GHz.
Figure 5.15. Scan impedance variation and associated geometry for Case 4, Frequency = 10,11,12 GHz.
To quickly summarize, these first four cases have demonstrated that the array scan impedance varies tremendously for an array in free space, even when that array is backed by a ground plane. Furthermore, we found that if the ground plane distance is poorly chosen, nulls in the element effective pattern factor will force the real part of the array scan impedance to zero for some frequencies and scan angles.

In the next two cases, we vary the dielectric constant of the layer surrounding the array, while the array ground plane thickness, the electrical circumference of the loop, and the electrical thickness of the second dielectric layer remain fixed. Specifically, for cases five and six, the following is true. From Table 5.2, for case five, we have

\[ \varepsilon_2 = 1.6 \quad , \quad (5.15) \]

\[ d_2 = \frac{9}{16} \lambda_{D2} \quad ; \quad \lambda_{D2} = 3.0/\sqrt{1.6} \quad (\text{cm}) \quad (5.16) \]

and

\[ d_{cc} = \frac{7}{32} \lambda_{D2} \quad . \quad (5.17) \]

In a similar manner, for case six, the following is true.

\[ \varepsilon_2 = 2.0 \quad (5.18) \]

\[ d_2 = \frac{9}{16} \lambda_{D2} \quad ; \quad \lambda_{D2} = 3.0/\sqrt{2.0} \quad (\text{cm}) \quad (5.19) \]

\[ d_{cc} = \frac{7}{32} \lambda_{D2} \quad (5.20) \]
Naturally, the two resonant wavelengths are different. Does the introduction of dielectric improve the scan impedance variation? To find out, look at Figures 5.16 and 5.17 for case five, and Figures 5.18 and 5.19 for case six. Considering case five first, we immediately notice the overall reduction in the impedance variation. The largest impedance remains below 600 ohms in magnitude. Furthermore, several frequencies (8, 9, and 10 GHz) demonstrate a remarkable stability in the imaginary component of the scan impedance. There still appears to be far too much variation in the real part, however.

Suppose we next look at Figures 5.18 and 5.19 where \( \varepsilon_2 = 2.0 \). Around 9 GHz we begin to see some very interesting behavior. The scan impedance begins to collapse about a point. At 10 GHz, the two scan plane variations have wrapped around each other. This candidate design would have excellent scan impedance behavior even out to 45°. At 11 GHz, the points separate somewhat, but the array is still usable. By 12 GHz, the divorce problem is again apparent.

The conclusions from cases five and six are as follows. The array scan impedance variation can be significantly improved by introducing a dielectric layer. The dielectric helps compensate for the impedance variation, hence we call this type of an array "scan compensated".

The next three cases in Table 5.2 will help us characterize the impedance behavior as the dielectric layer containing the array is thickened. Here, we maintain a constant array to ground plane spacing \([d_{cc} = (7\\lambda_2/32)]\) and dielectric constant (\(\varepsilon_2 = 2\)). In cases seven,
Figure 5.16. Scan impedance variation and associated geometry for Case 5, Frequency = 7, 8, 9 GHz.
Figure 5.17. Scan impedance variation and associated geometry for Case 5, Frequency = 10, 11, 12 GHz.
Figure 5.18. Scan impedance variation and associated geometry for Case 6, Frequency = 7,8,9 GHz.

(All Impedances in Ohms)
Figure 5.19. Scan impedance variation and associated geometry for Case 6, Frequency = 10, 11, 12 GHz.
eight and nine, the thickness of $d_2$ is varied $7\lambda_{02}/16$, $\lambda_{02}/2$ and $5\lambda_{02}/8$, respectively. (Remember, case six already covered the $9\lambda_{0}/16$ case.)

Beginning with case seven, look at the scan impedance plots in Figures 5.20 and 5.21. At 7 GHz, we see the divorce problem again, although the variation is not nearly as bad as the uncompensated case. As the frequency is increased, the $\alpha=0^\circ$ scan impedance begins to swing around until eventually, at 12 GHz, several points are again grouped together. This array is usable out to $30^\circ-40^\circ$ in the two scan planes. Nonetheless, this case is not nearly as good as case six.

Next, consider the case eight scan impedance plots in Figures 5.22 and 5.23. As with case seven, we start out at 7 GHz with the divorce problem. Yet again, as the frequency is increased, the $\alpha=0^\circ$ scan impedance swings past the $\alpha=90^\circ$ scan impedance. At 10 GHz, the scan impedance doesn't change much till $n>30^\circ$. At 11 GHz, the variation is better yet; this array would work well out to at least $n=45^\circ$ in both scan planes. This design has the potential of working over at least a 1 GHz bandwidth. Considering that many phased arrays have less than 2% bandwidth, this is already a big improvement in the bandwidth area.

Finally, consider the scan impedance graphs for case nine, shown in Figures 5.24 and 5.25. Here, the second dielectric layer is $(5\lambda_{02}/8)$ thick. For this case, it appears that the scan impedance behaves nicely at 7 and 8 GHz. Also notice that the reactive component is remarkably small for these two frequencies. It is fairly clear from cases seven, eight and nine that increasing the thickness of the dielectric layer containing the array tends to shift the array scan impedance behavior
Figure 5.20. Scan impedance variation and associated geometry for Case 7, Frequency = 7,8,9 GHz.
Figure 5.21. Scan impedance variation and associated geometry for Case 7, Frequency = 10, 11, 12 GHz.
Figure 5.22. Scan impedance variation and associated geometry for Case 8, Frequency = 7,8,9 GHz.
Figure 5.23. Scan impedance variation and associated geometry for Case 8, Frequency = 10,11,12 GHz.
Figure 5.24. Scan impedance variation and associated geometry for Case 9, Frequency = 7,8,9 GHz.
Figure 5.25. Scan impedance variation and associated geometry for Case 9, Frequency = 10,11,12 GHz.
down in frequency. Therefore, it appears that the optimum electrical thickness of this layer lies between 7λD/16 and 5λD/8.

At this point, the issue of modelling a protective radome will be addressed. In Chapter I, it was mentioned that a conceptual design for an integrated antenna/radome would include the radome properties in the calculation. Case ten is an example of modelling a protective radome rain erosion coating. As Table 5.1 indicates, a layer of dielectric material with ε1=3.3 is added in front of the layer containing the array. The outer layer models 30 mils (.0713 cm) of typical rain erosion coating material. (It could just as easily have been a multi-layered A-sandwich or B-sandwich radome.) The overall electrical thickness of layer (d1+d2) is (9λD/16). As Figures 5.26 and 5.27 indicate, we can easily calculate the scan impedances for this complicated geometry. At 9 GHz, it appears that the impedance variation is minimized. Considering that this case is strictly presented to demonstrate the feasibility of incorporating the radome into the design process, a reasonably good design has resulted.

Once again, remember that we do not wish to analyze an excessive number of numerical cases. We do have enough information here, however, to make the following general conclusions.

1. Placing the array in a dielectric with the proper choice of ε2 and d2 improves the scan impedance variation with scan angle and frequency.
Figure 5.26. Scan impedance variation and associated geometry for Case 10, Frequency = 7,8,9 Ghz.
Figure 5.27. Scan impedance variation and associated geometry for Case 10, Frequency = 10, 11, 12 GHz.

(All Impedances in Ohms)
2. For a single layer design, the optimum thickness of the dielectric layer containing the array is approximately
\[ \frac{7\lambda d_2}{16} < d_2 < \frac{5\lambda d_2}{8}. \]

3. Virtually any dielectric radome can be modelled by including additional layers of dielectric in front of the array, although careful design is still required or the overall array performance will be degraded.

Although a fair amount of array design work still needs to be done, it is clear that the analysis technique implemented by this computer program can lead to scan independent array designs.
CHAPTER VI

CONCLUSIONS

This investigation analyzed the impedance properties of a doubly infinite doubly periodic array of non-planar rectangular loop elements embedded in a general stratified dielectric medium. Specifically, analytic expressions were developed for the scan impedance of such an array. A computer program was written to implement the analysis described in this document. Computer generated scan impedance calculations demonstrated that when the non-planar loop array is placed in front of a ground plane, impedance magnitudes can easily change 600% as the beam is scanned in the two principle planes. By embedding the array in a single dielectric layer of proper dielectric constant $\varepsilon$ and thickness $d$, variations in the scan impedance magnitude can be significantly decreased. Furthermore, it was shown that a radome can be incorporated in the analysis by simply adding additional dielectric layers in front of the layer containing the loop array. A conceptual design incorporating a layer of radome rain erosion coating was modelled in this manner. Lastly, remember that these compact elements can be closely spaced, which significantly delays the onset of grating lobes in the antenna operating band. (Remember, grating lobes occur when the interelement spacings are greater than $\lambda/2$ at the antenna operating frequency.)
What is the importance of this study? The introduction briefly summarizes the need for a scan independent phased array antenna. This document demonstrated that the plane wave expansion technique can be used to analyze an array of piecewise linear elements. Although rectangular loops were used as an example, the theory described in Chapter III is quite general. If another type of non-planar piecewise radiating element replaces the loop, most, if not all of the concepts discussed in Chapter III can be applied to this new problem. (For instance, see Lin [28] and Ng [29].) Furthermore, the "bounce path" description of Chapter III corrects the plane wave expansion theory of Munk, et al. [30], when non-planar array radiating elements are analyzed.

Can this array of non-planar loop elements be built? The answer appears to be "yes". It is envisioned that rows of loop elements can be etched onto a dielectric substrate. By "stacking" these rows over a ground plane and embedding the entire structure in dielectric, it appears that the radiating portion of the antenna can be inexpensively fabricated. Apparently, this array concept is both feasible and practical.

The final question to answer is this; what should future researchers concentrate on? There are several items which were beyond the scope of this initial investigation. They include the following:

1. The inclusion of a second or "odd" current mode on the loop element. This would involve modifying the analysis to calculate four impedance quantities; the even-even, odd-odd,
and two even-odd current mode interactions. Adding this second current mode may improve the accuracy of the scan impedance calculation, especially as one departs significantly in frequency from the resonant wavelength.

2. The addition of a hybrid feed network in the array design process. This is currently being investigated by Coveyou [31].

3. Additional analysis of other promising scan independent phased array radiating elements. An element with dual polarization properties would be especially useful.

4. Finally, an experimental investigation of the measured scan impedance properties for a finite array of non planar loop elements. The technique described by Fenn [32] might be a starting point for such an investigation.
APPENDIX A
BOUNDARY CONDITIONS

For many years, researchers have analyzed the radiating and scattering properties of periodic structures (or frequency selective surfaces). Many types of resonant windows and reflectors have been built, with outstanding agreement between their respective theoretical and measured performance. The technique used by Munk, et al. [33] to analyze the radiating and scattering properties of periodic surfaces is the plane wave expansion technique. Although this is not a new technique, Munk and his associates have found efficient methods for calculating periodic surface properties based upon the method.

Although a great deal of work has been done implementing the plane wave expansion solution, and performing measurements which strongly support theoretical calculations, little has been recorded regarding the basic theoretical foundation of this technique. It is the purpose of this Appendix to offer a sound theoretical basis for the plane wave expansion technique as applied to periodic surfaces. In the discussion to follow, the plane wave expansion solution will be compared and contrasted to the moment method solution.

For the purpose of argument, let us introduce an isolated rectangular loop filament with an arbitrary known load impedance $Z_L$ as shown in Figure A.1. This particular thin wire loop will be illuminated.
Figure A.1. Geometry of Current Loop. $\hat{p}(1)$ is the vector orientation of the straight wire segment containing the terminal gap.
with an incident electric field $\mathbf{E}^i$. The terminals of the loop are located by the reference vector $\mathbf{R}^{(1)}$, while any point on the loop is denoted by the vector $\mathbf{R}'$.

According to the boundary conditions implied by Maxwell's Equations, we know the tangential field along the conductor loop must vanish, except perhaps at a "delta gap" where the terminals exist. That is to say

$$\mathbf{E}_{\text{total}}(\mathbf{R}') = \left\{ \begin{array}{ll}
0 , & R' \neq R^{(1)} \\
\lim_{\Delta d \to 0} \frac{V_0}{\Delta d} p^{(1)}(1) = p^{(1)}(1) v_0 \delta(R'-R^{(1)})
\end{array} \right.
$$

Rewriting Equation (A.2)

$$\mathbf{E}_{\text{total}}(\mathbf{R}') = \mathbf{E}^i_{\text{tan}}(\mathbf{R}') + \mathbf{E}^s_{\text{tan}}(\mathbf{R}') = \left\{ \begin{array}{ll}
0 , & R' \neq R^{(1)} \\
Z_L I(R^{(1)}) \delta(R'-R^{(1)}) p^{(1)}(1)
\end{array} \right.
$$

One may note that in Equation (A.3), $I(R^{(1)})$ is the value of the current at the terminal. Another notation for this quantity might be

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\[ I_L = I(R(1)) \quad (A.4) \]

Suppose next we desired to solve this problem using the method of moments, namely the Galerkin moment method for thin wire structures. Using this technique, let me introduce m unit dipole testing functions. The indices can be defined in any convenient fashion. (+ to -, o to m, etc.). First, we "test" Equation (A.3) with our unit dipole testing function, \( f_m \), whose vector direction is always parallel with the wire axis at each point.

\[ -E_{\text{tan}}(\vec{r}') \cdot \vec{f}_m = \int_{\text{tan}}^i \vec{E}_{\text{tan}} \cdot \vec{f}_m - Z_L I_L \delta(\vec{r}' - \vec{R}(1)) \rho(1) \cdot \vec{f}_m \quad (A.5) \]

If we integrate Equation (A.5), we obtain an integral equation of the following form.

\[ -\int E_{\text{tan}}(\vec{r}') \cdot \vec{f}_m dx = \int E_{\text{tan}}(\vec{r}') \cdot \vec{f}_m dx - Z_L I_L \int \rho(1) \cdot \vec{f}_m \delta(\vec{r}' - \vec{R}(1)) dx \quad (A.6) \]

Equations (A.5) and (A.6) are very important and deserve special attention. First of all, what is the testing function in Equation (A.5)? Naturally it depends on the geometry of the wire problem. For a straight wire, we might choose as a testing function

\[ f_m = \frac{\hat{q}(m) \sin \beta (d - \epsilon_m)}{\sin \beta d} \quad (A.7) \]
where \( \mathbf{p}(m) \) is the dipole orientation and \( z_m \) is the running variable. Figure A.2 illustrates three testing functions on a dipole antenna. Note that with the exception of the end testing functions, each testing function overlaps the testing function to either side. If Equation (A.7) were used, this method would eventually lead to the sinusoidal-Galerkin moment method, assuming that the expansion and test functions are identical.

In the loop problem the geometry is slightly different since the square loop is a piecewise linear radiating element. For argument's sake, suppose the wire loop has eight testing functions as shown in Figure A.3. In this case, it is again shown that each testing function overlaps with two adjacent testing functions on either side. Thus, when we introduce the integral in Equation (A.6), it is noted that the integral is only over the domain of the \( m \)'th testing function. That is to say, it exists only over the \( m \)'th segment. Note that in general, the segment lengths need not be identical as shown in Figure A.2. Taking this into account, the third integral in Equation (A.6) simply becomes

\[
-ZL_{LZ} \int_{z_{m-1}}^{z_{m+1}} \mathbf{p}(m) \cdot \mathbf{R}' \mathbf{e}(\mathbf{R}' - \mathbf{R}(1)) \, dz_m = ZL_{LZ} \hat{e}_m^2
\]

\( (\mathbf{R}(1) \text{ in the } l \text{'th segment}) \)

where
Figure A.2. Dipole with three testing functions.
Figure A.3. Loop with eight testing functions.
\[ \delta_m = \begin{cases} 
\pm 1, & m = \ell \\
0, & m \neq \ell 
\end{cases} \quad (A.9) \]

Here, only the \( \ell \)'th segment has a lumped load. In Figure (A.2), \( \ell = 0 \).

Substituting Equation (A.9) into Equation (A.6) yields

\[ - \int_{m}^{S} \epsilon_{\tan}(\mathcal{R'}^{m}) \cdot \vec{r}_{m} d\xi_{m} = \int_{m}^{S} \epsilon_{\tan}(\mathcal{R'}^{m}) \cdot \vec{r}_{m} d\xi_{m} - Z_L I_\ell \quad (A.10) \]

So far, we have made no statement about the amplitude of \( \vec{r}_{m} \). It is noted that \( \vec{r}_{m} \) is a "unit dipole" or normalized testing function.

At this point, we approximate the scattered field as follows. In Equation (A.10), we wish to expand the scattered field into a series of \( N \) modes, \( \epsilon_{n} \), with complex weighting factors \( I_n \) in the following manner;

\[ \vec{e}_{\text{scat}} = \sum_{n=0}^{N} I_n \epsilon_{n} \quad (A.11) \]

Substituting Equation (A.11) into Equation (A.10) yields after interchanging the order of summation and integration

\[ - \sum_{n=0}^{N} I_n \int_{m}^{S} \vec{r}_{m} \cdot \epsilon_{n} d\xi_{m} = \int_{m}^{S} \epsilon_{\tan}(\mathcal{R'}^{m}) \cdot \vec{r}_{m} d\xi_{m} - Z_L I_\ell \epsilon_{\ell} \quad (A.12) \]
Note that $I_n$ is the terminal current for mode $n$. Here, we can combine the last term on the righthand side with the left side in the following manner:

$$- \sum_{n=0}^{N} I_n \left[ - \int_{\text{segment}_m} \tilde{I}_m \tilde{e}_n \, dx \right] + Z_L I_{\delta_m}^2 = \int_{\text{segment}_m} ^i \tilde{E}_{\text{tan}(R')} \tilde{I}_m \, dx$$

(A.13)

If we now define the mutual impedance matrix $Z_{mn}$ as

$$Z_{mn} = - \int_{\text{segment}_m} \tilde{I}_m \tilde{e}_n \, dx$$

(A.14)

and the forcing voltage (sometimes referred to as the induced voltage) as

$$V_m = \int_{\text{segment}_m} ^i \tilde{E}_{\text{tan}(R')} \tilde{I}_m \, dx$$

(A.15)

then this yields the following series of linear equations, assuming $m=N$.

$$
\begin{bmatrix}
Z_{00} & Z_{01} & Z_{02} & \cdots & Z_{0N} \\
Z_{10} & Z_{11} & Z_{12} & \cdots & Z_{1N} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
Z_{N0} & Z_{N1} & \cdots & (Z_{N2} + Z_L) & \cdots & Z_{NN}
\end{bmatrix}
\begin{bmatrix}
I_0 \\
I_1 \\
\vdots \\
I_N
\end{bmatrix}
=
\begin{bmatrix}
V_0 \\
V_1 \\
\vdots \\
V_N
\end{bmatrix}
$$

(A.16)

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Clearly, the desired current amplitudes can be obtained by inverting the impedance matrix and multiplying the inverted matrix by the voltage column matrix. Notice in Equation (A.16) that the \( k \)'th testing segment had a load impedance \( Z_L \) in series with the calculated impedance \( Z_L \). If \( Z_L = 0 \), then the standard simultaneous linear equation set

\[
\sum_{n=0}^{N} I_n Z_{mn} = V_m \tag{A.17}
\]

would have resulted.

From Equation (A.17) it is clear that as the number of current segments increase (\( N \)), the number of simultaneous equations to solve also increases. For a wire structure which is electrically large, the number of equations to solve increases rapidly. If this technique is applied to an \( M \) element array of current filaments with \( N \) unknown amplitudes, then we would be forced to solve \((M \times N)\) equations. Here, I have made the subtle assumption that the number of testing domains is equal to the number of expansion modes in the assumed current. Even the largest computers have problems solving many simultaneous equations.

Before proceeding on, one important comment is in order here. Initially, in Equation (A.1), we set out to make the total electric field vanish along the wire antenna. Actually, our solution has not yielded a scattered field which when added to the incident field will yield zero along the wire filament. To see this, consider the following development from Thiele [34]. Assuming for the moment that \( Z_L = 0 \), define the residual \( \mathbf{R} \) to be the sum of the incidence and scattered field.
Substituting Equation (A.11) into Equation (A.18) yields

\[ \mathbf{R} = \sum_{n=0}^{N} \mathbf{I}_n \mathbf{e}_{n,tan} + E_{tan} \]  

(A.19)

Thiele states that by multiplying Equation (A.19) by our testing functions and integrating along the filament, one would obtain

\[ \int_{\text{filament}} \mathbf{R}(\mathbf{e}_m) \cdot \mathbf{f}_m(\mathbf{e}_m) d\mathbf{e}_m = 0 \quad m=0,1,2,\ldots,N \]  

(A.20)

This is easy to see, for Equation (A.20) is nothing more than a statement of Equation (A.13).

Although I shall not go into detail, Thiele [35] shows that by using delta testing functions and pulse expansion functions, the residual in Equation (A.19) is only zero at the N points where the point matching occurred. Equation (A.20) suggests that regardless of the choice of testing functions, the residual will be forced to zero in some average sense along the wire. Thus, the moment method yields a solution which results in a zero tangential electric field along the wire only in an average sense, as defined by the weighted integral in Equation (A.13).

So far, I have discussed a great deal about boundary conditions and the moment method, but very little about periodic surfaces. To understand how we use the above results, let us first view the geometry of an infinite array of non-planar rectangular wire elements, as shown in Figure A.4.
Figure A.4. Infinite array of loaded non-planar rectangular wire filaments with highlighted reference element. Plane of the array is the (X-Z) plane. ($D_x$, $D_z$ are the inter-element spacings in the x and z direction, respectively).
If this infinite array is illuminated by a plane electromagnetic wave, we know that the array voltages and currents from element to element will be related by a single phase delay factor (by Floquet's theorem)

\[ e^{-j\beta(qs_xD_x + is_zD_z)} \]  

(A.21)

where \((q,i)\) are integers corresponding to the element position with respect to the reference element position, \((D_x,D_z)\) corresponds to the array interelement spacing in \(x\) and \(z\), and \((s_x,s_z)\) are the incident direction cosines. Thus, in an infinite periodic surface, we need only to determine the behavior of the reference element.

Although we wish to enforce the boundary condition of Equation (A.1) for every element of the array, when the plane wave expansion technique is used, we need only enforce Equation (A.1) on the reference element of the array, and the problem is solved. Thus, we pick up the identical development as far as Equation (A.5). Here we have introduced \(m\) "testing functions. In this complicated periodic surface analysis, we only introduce 1 "entire testing function", \(\hat{\tau}_0\), so that Equation (A.5) equivalently becomes

\[ -E_{\text{tan}}^{\Sigma}(\hat{\mathbf{R}}') \cdot \hat{\tau}_0 = E_{\text{tan}}^{\Sigma} \cdot \hat{\tau}_0 - Z_l I_0 \delta(\hat{\mathbf{R}}' - \hat{\mathbf{R}}(1)) \hat{\tau}_0 \]  

(A.22)

Again, integrating along the reference loop filament, we have
By expanding the tangential scattered field in a single "entire plane wave expansion" function, we can write

$$E^S(R') = I_0 e^{S(R')}$$  \hspace{1cm} (A.24)

Hence, Equation (A.23) can be rewritten as

$$- I_o \int_{\text{loop}}^{S} e_o^{S(R')} \hat{P}_o(R') \, d\xi = \int_{\text{loop}}^{S} E_{\text{tan}} \cdot \hat{P}_o \, d\xi - Z_L I_o$$  \hspace{1cm} (A.25)

Equation (A.25) is extremely important in our plane wave expansion technique, for it contains all of the important quantities of interest.

The forcing function is what we refer to as the "induced voltage" or "forced response" and is defined by

$$V_o = V_{\text{ind}} = \int_{\text{loop}}^{S} E_{\text{tan}} \cdot \hat{P}_o \, d\xi$$  \hspace{1cm} (A.26)

The integral on the left side of Equation (A.25) is recognized as the self impedance of the loop antenna array

$$Z_{oo} = - \int_{\text{loop}}^{S} e_o^{S(R')} \hat{P}_o(R') \, d\xi$$  \hspace{1cm} (A.27)

Finally, Equation (A.25) can be rewritten as

$$I_0 \equiv I(R(1)) = \frac{V_o}{Z_{oo} + Z_L}$$  \hspace{1cm} (A.28)
which has the Thevenin equivalent circuit given in Figure A-5.

Once \( V_0 \) and \( Z_{oo} \) are known, the terminal current \( I_0 = I(\!\!R(1)\!\!) \) is calculated, followed by the calculation of the reradiated or scattered electric field.

In this array problem, the residual is given by the following equation.

\[
\dot{R} = E_{\text{tan}} + I_0 e_{0,\text{tan}}
\]  

(A.29)

There is no doubt that \( \dot{R} \) in Equation (A.29) is not zero exactly. If we multiply Equation (A.29) by our test function and integrate assuming for now that \( Z_L = 0 \), we obtain

\[
\int_{\text{ref}}^{\text{loop}} \dot{R} f_0 \, dz = \int_{\text{ref}}^{\text{loop}} E_{\text{tan}} f_0 \, dz + I_0 \int_{\text{ref}}^{\text{loop}} e_{0,\text{tan}} f_0 \, dz
\]  

(A.30)

Figure A.5. Thevenin equivalent circuit.
Rearranging Equation (A.25) we know that

\[ \int_{\text{ref}}^{} \varepsilon_{\text{tan}} \cdot \hat{F}_0 \, dz + I_0 \int_{\text{ref}}^{} \varepsilon_{\text{tan}} \cdot \hat{F}_0 \, dz = 0 \]  \hspace{1cm} (A.31)

(For the case of \( Z_L \neq 0 \), we simply combine \( Z_L \) and \( Z_{00} \) to obtain the same end result)! Equations (A.30) and (A.31) tell us that in our periodic surface analysis, we have forced the tangential electric field to zero in an average sense along the reference element. By Floquet's theorem, if we use the correct expansion function for the reradiated field from the entire array

\[
\hat{e}_o^s = \frac{Z_0}{2D_x D_z} \sum_{k=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \left( \frac{p(1) \hat{n}_x + \frac{p(1) \hat{n}_y}{\hat{r}_x}}{r} \right) e^{-j\beta (\hat{R}_j - \hat{R}(1)) \cdot \hat{r}_x} \]  \hspace{1cm} (A.32)

then the tangential electric field will be zero in the average sense on every element in this infinite array.

Equation (A.32) is presented here to demonstrate several important points. The double summation is over an infinite number of discrete plane wave components, some of which are propagating, most of which are evanescent. The "pattern factors" shown in Equation (A.32) are symbolically expressed as

\[
P_{\pm}^{C(1)} = \sum_{j=1}^{4} \int_{1}^{1} \left( \frac{1}{n} \right)^{P(1)} \int_{0}^{1} \left( \frac{1}{n} \right)^{P(1)} e^{-j\beta (\hat{R}_j - \hat{R}(1)) \cdot \hat{r}_x} \, dz \]  \hspace{1cm} (A.33)
The current function in Equation (A.33) is normalized with respect to the terminal current $I(R(1))$.

In this dissertation, only one testing and expansion function is used in Equations (A.23) and (A.24), respectively. Since the square loop will tend to resonate when the circumference is approximately one wavelength, it is thought that this approximation is valid near resonance. It should be mentioned that there is no theoretical reason why many testing (and expansion) functions cannot be used. However, since the self impedance calculation involves a truncated double infinite summation, it is clear that a tradeoff exists between accuracy and runtime. In the method of moments, much of the calculation time arises from the matrix inversion. When the plane wave expansion technique is used to analyze periodic structures, most of the calculation time is spent filling the impedance matrix. Furthermore, Henderson [36] demonstrated that electrically short current expansion modes (short with respect to $D_x$ and $D_z$) cause the self impedance term in Equation (A.27) to converge poorly in the complex plane. Therefore, a limit does exist in the number of Floquet current modes that can be used in the plane wave expansion method.

So far in this discussion, I have not mentioned how Schellkunoff fits into this development. Now that this general technique has been set on firm theoretical ground, let us consider a few specifics regarding Equation (A.25). Recall that $\varphi_0$ has not been specified,
except that it must be of unit magnitude at the antenna terminals. In our analysis, we choose \( \hat{\mathbf{t}}_0 \) to be

\[
\hat{\mathbf{t}}_0 = \sum_{j=1}^{4} I_j \left( \mathbf{e}_j \right) \mathbf{n}(t) \tag{A.34}
\]

where \( I_j \) is defined as the normalized antenna current under transmitting conditions. We make this choice of testing function since Schellkunoff [37] demonstrates that Equation (A.26) is exact when the antenna current under transmitting conditions is used. Schellkunoff's elegant, simple, and completely rigorous proof is based in part on the Lorentz reciprocity theorem and Love's field equivalence theorems.

The overall conclusion of this section is quite clear. The method of moments can be used to analyze an infinite array of piecewise linear elements. If the infinite array is periodic, Floquet's theorem can be invoked and the problem reduces to analyzing the radiating properties of any one of the array elements. When we expand the scattered field as a double infinite sum of propagating and evanescent plane waves, we call this method the plane wave expansion technique, even though it is still a moment method type solution. Thus, these two techniques are completely consistent.
APPENDIX B

DIRECTION OF PROPAGATION VECTORS AND THE DECOMPOSITION OF THE ELECTRIC FIELD VECTORS

When the plane wave expansion technique is used to analyze a periodic structure obeying Floquet's theorem, it is well known that the electric and magnetic field radiated from such a structure can be expressed as an infinite sum of discrete plane waves. Munk [38], English [39], and associates have thoroughly investigated the propagation directions of these plane waves. For the sake of theoretical and notational continuity, their results will be stated in this appendix without proof.

As stated above, a doubly infinite periodic structure with identical array elements spaced apart $D_x$ and $D_z$ in the $x$ and $z$ direction, respectively, will radiate plane waves in discrete directions given by

$$ r_{m\pm} = r_{mx} \pm y \hat{r}_{my} + z \hat{r}_{mz} \quad (B.1) $$

as shown in Figure B.1.

The positive direction is taken to the right ($+y$) of the array, and the negative direction is taken to the left ($-y$) of the array. In Equation (B.1), we note that
Figure B.1. Discrete plane waves radiating from an arbitrary doubly infinite doubly periodic array of radiating elements.
\[ r_{mx} = s_{mx} + \frac{k\lambda}{D_x} \quad , \quad (B.2) \]

\[ r_{mz} = s_{mz} + \frac{n\lambda}{D_z} \quad , \quad (B.3) \]

and

\[ r_{my} = \sqrt{1 - (r_{mx})^2 - (r_{mz})^2} \quad . \quad (B.4) \]

When

\[ 1 - r_{mx}^2 - r_{mz}^2 < 0 \quad (B.5) \]

is less than zero, physical considerations require us to choose the (-j) branch of the square root in Equation (B.4), which produces an evanescent wave instead of a growing wave.

When Equation (B.5) is positive, \( r_{my} \) is pure real, and we get a propagating wave for the corresponding value of \( \{k,n\} \). Normally in our array designs, we maintain the smallest possible interelement spacing \( D_x \) and \( D_z \) in order to hold off the onset of propagating waves for \( k \neq 0 \) and \( n \neq 0 \). These undesired propagating waves are referred to as grating lobes.

In Equations (B.2) and (B.3), the direction of the impressed plane wave field which causes the scattering to occur in the first place is given by

\[ \hat{s}_{m\pm} = s_{mx} \hat{x} \pm s_{my} \hat{y} + s_{mz} \hat{z} \quad . \quad (B.6) \]
As with previous investigators, m indicates the dielectric medium where both \( \hat{s}_{m\pm} \) and \( \hat{r}_{m\pm} \) are evaluated. From Equations (B.2) and (B.3) we see for \( k=n=0 \),

\[
\hat{s}_{m\pm} = \hat{r}_{m\pm} \bigg|_{k=n=0} \quad (B.7)
\]

In order to illustrate the direction of propagation fully, refer to Figure B.2, originally presented by English [40]. Note that \( \hat{s}_{0+} \), \( \hat{s}_{0-} \), and \( \hat{s}_{m+} \) represent the incidence Poynting vector direction, the specular reflected Poynting vector direction, and the transmitted Poynting vector direction in the \( m \)'th dielectric layer, respectively. The other directions shown are propagating modes (grating lobes) for \( k \neq 0 \) and \( n \neq 0 \). Normally if a particular wave direction \( (k,n) \) value is evanescent, we often refer to it as a plane wave "propagating in imaginary space".

Another important set of relations exist between the components of \( \hat{r}_{m\pm} \) in one dielectric medium and \( \hat{r}_{m'\pm} \) in another dielectric medium.

They are

\[
\begin{align*}
\hat{r}_{m'x} &= \frac{\beta_{m'}}{\beta_m} \hat{r}_{mx} \\
\hat{r}_{m'z} &= \frac{\beta_{m'}}{\beta_m} \hat{r}_{mz}
\end{align*}
\]

and

Of course, \( \hat{r}_{m'y} \) can be found from
Figure B.2. Plane wave directions.
\( \mathbf{r}_{m'y} = \sqrt{1 - (\mathbf{r}_{m'x})^2 - (\mathbf{r}_{m'z})^2} \) \hspace{1cm} (B.10)

as before.

Another important topic which must be addressed is the decomposition of the electrical field into vector components compatible with the stratified dielectric medium geometry discussed in Chapter I. If we decompose the electric field into components parallel to and orthogonal to the plane defined by \( \mathbf{r}_{m^+} \) and the \(-y\) axis, modified Fresnel reflection and transmission coefficients can be used to analyze the behavior of periodic surfaces embedded in a planar stratified dielectric medium.

From [41], the electric field direction \( \mathbf{e}_{m\pm} \) is defined as

\[
\mathbf{e}_{m\pm} = [\hat{\mathbf{p}} \times \mathbf{r}_{m\pm}] \times \mathbf{r}_{m\pm},
\]  

(B.11)

for a linear reference element oriented in the \( \hat{\mathbf{p}} \) direction. For piecewise linear element (like the rectangular loop), each leg (or side) of the loop will be oriented in the \( \hat{\mathbf{p}}(j) \) direction. Thus, each side will have an associated orientation vector given by

\[
\mathbf{e}_{m\pm}^{(j)} = [\hat{\mathbf{p}}(j) \times \mathbf{r}_{m\pm}] \times \mathbf{r}_{m\pm}.
\]  

(B.12)

This can be rewritten as

\[
\mathbf{e}_{m\pm}^{(j)} = (\mathbf{r}_{m\pm} \cdot \hat{\mathbf{p}}(j)) \mathbf{r}_{m\pm} - \hat{\mathbf{p}}(j)
\]  

(B.13)
In the presence of dielectric layered medium (Equation (B.12)) is decomposed into components parallel and orthogonal to the plane containing \( \mathbf{r}_{mt} \) and the \(-y\) axis. After performing this operation, we obtain

\[
\varepsilon_{m\pm}^{(j)} = -\mathbf{i} \mathbf{\hat{r}}_{m\pm} \cdot \mathbf{p}^{(j)} - \mathbf{i} \mathbf{\hat{r}}_{m\pm} \cdot \mathbf{p}^{(j)} ,
\]

where we have used the relation

\[
\mathbf{\hat{r}}_{m\pm} \cdot \mathbf{\hat{r}}_{m\pm} = 0
\]

and

\[
\mathbf{\hat{r}}_{m\pm} = \frac{-y \mathbf{\hat{r}}_{m\pm}}{|-y \mathbf{\hat{r}}_{m\pm}|} = \frac{-x \mathbf{\hat{r}}_{mz} + \mathbf{\hat{r}}_{mx}}{\sqrt{\mathbf{\hat{r}}_{mx}^2 + \mathbf{\hat{r}}_{mz}^2}}
\]

with

\[
\mathbf{\hat{r}}_{m\pm} = \mathbf{\hat{r}}_{m\pm} \times \mathbf{\hat{r}}_{m\pm} = \frac{-x \mathbf{\hat{r}}_{mx}}{\sqrt{\mathbf{\hat{r}}_{mx}^2 + \mathbf{\hat{r}}_{mz}^2}} - \frac{2 \mathbf{\hat{r}}_{mz} \mathbf{\hat{r}}_{my}}{\sqrt{\mathbf{\hat{r}}_{mx}^2 + \mathbf{\hat{r}}_{mz}^2}}
\]

Figure B.3 demonstrates the relative orientation of the above quantities for one particular \((k,n,)^{th}\) plane wave spectral component.

Decomposing the electric field in this fashion greatly simplifies the analysis, since \(\mathbf{\hat{r}}_{m\pm}\) are independent of the leg orientation \(\mathbf{p}^{(j)}\). Therefore, these orthogonal vectors will be used in place of \(\varepsilon_{m\pm}^{(j)}\) throughout this research effort.
Figure B.3. Relative orientations of $\hat{r}_\pm$, $\perp \hat{n}_\pm$ and $\parallel \hat{n}_\pm$. 
APPENDIX C

ON THE EFFECTIVE REFLECTION AND TRANSMISSION COEFFICIENT NOTATION

Munk et al. [42] have derived the regular and effective reflection coefficients and transmission coefficients through a stratified dielectric medium. Although these results are well known, it is prudent to present them here in order to clarify a recent change of notation.

Figure C.1 demonstrates a stratified dielectric medium under plane wave illumination. Obviously, each layer will have a rightgoing and leftgoing propagating plane wave. By defining the ratio of the reflected electric-field $\frac{E_m^r}{E_m^l}$ to $\frac{iE_m^l}{E_m^l}$ at the $m+1$th dielectric interface, we can define the effective reflection coefficient at the $m+1$th interface as

$$\Gamma_{m,e} = \frac{\frac{E_m^r}{E_m^l} + \frac{r_{m+1}}{\frac{r_m^+}{r_m^-} + \frac{r_{m+1}}{r_m^-} e^{-j2\beta_m+1d_m+1r_{m+1}(m+1)y}}}{1 - \frac{r_{m+1}}{r_m^-} e^{-j2\beta_m+1d_m+1r_{m+1}(m+1)y}}$$  \hspace{1cm} (C.1)

where the regular Fresnel reflection coefficients occurring between two semi-infinite media $m$ and $m+1$ are given by

$$\Gamma_m^r = \frac{Z_{m+1}r_{my} - Z_mr_{m+1}y}{Z_{m+1}r_{my} + Z_mr_{m+1}y}$$  \hspace{1cm} (C.2)
Figure C.1. Multiple reflections in a stratified medium.
\[ \Gamma_m^+ = \frac{Z_{m+1} r_{(m+1)y} - Z_m r_{my}}{Z_{m+1} r_{(m+1)y} + Z_m r_{my}} \]  

(C.3)

where the complex impedance, \( Z_m \), is related to the complex permittivity and permeability by

\[ Z_m = \sqrt{\frac{\mu_m}{\varepsilon_m}} \]  

(C.4)

Also in Equation (B.1),

\[ \Gamma_{m+1}^- = - \Gamma_m^+ \]  

(C.5)

for the regular Fresnel reflection coefficients only!

Notice the effective reflection coefficient in Equation (C.1) is for a rightgoing wave. As such, the effective reflection coefficient must be calculated in an iterative manner, beginning at the rightmost dielectric interface,

\[ \Gamma_{MM,e}^+ = \Gamma_{MM}^+ \]  

(C.6)

and working right to left using Equation (C.1).

Unlike the regular Fresnel reflection coefficients, the effective reflection coefficients for left-going waves are not in general related to the effective reflection coefficients for right-going waves in each dielectric slab. Thus, at each interface for the left-going waves, we have
where again

\[ \frac{1}{r_{m+1}} = - \frac{1}{r_m} \tag{C.8} \]

Since Equation (C.7) is also an iterative solution, we must start calculating at the first slab by equating

\[ \frac{1}{r_1} = \frac{1}{r_1} \tag{C.9} \]

then proceed with the remaining MM dielectric slabs working left to right.

In a similar fashion, we can define the effective transmission coefficient from slab \( m \) to slab \( m+1 \) as the ratio of the transmitted wave in slab \( m+1 \) to the incident wave in slab \( m \).

\[ \frac{1}{t_{m+1}} = \left( \frac{1 + \frac{1}{r_m}}{1 - \frac{1}{r_{m+1}} \left( e^{-(2\pi m+1)i + 1} r_{m+1} \right)} \right) \tag{C.10} \]

Since Equation (C.10) contains effective reflection coefficients, clearly they must be calculated first.
For leftgoing waves, another expression exists for calculating the transmitted wave into slab \( m \) due to an incident wave in slab \( m+1 \). That expression is given by

\[
\frac{1}{n} \frac{1}{\Delta x} \frac{1}{\Gamma_{m+1}^{-}} = \frac{1}{1 + \frac{1}{n} \frac{1}{\Delta x} \frac{1}{\Gamma_{m+1}^{-}} \frac{1}{\Gamma_{m}^{+}} (e^{-j2\pi d_{m} \frac{\Delta x}{\lambda}})}
\]

At this point, some comments regarding the notation are in order. Munk [43], English [44], Stosic [45], Larson [46] and others use a much bulkier notation for the effective reflection coefficients. Since I have compacted the notation, I would like to provide a table comparing the "old" notation to the "new" notation (see Table C.1).

For instance, in Equation (C.11)

\[
\Gamma_{m+1, e}^{+} (\text{NEW}) = \frac{1}{n} \frac{1}{\Delta x} \frac{1}{\Gamma_{m+1, m+2, e}^{+}}
\]

Another quantity which appears often in this analysis is the variable capital omega.

\[
\frac{1}{\pi} n_{\omega} = \frac{1}{1 + \frac{1}{n} \frac{1}{\Delta x} \frac{1}{\Gamma_{m+1}^{+}} \frac{1}{\Gamma_{m}^{-}} (e^{-j2\pi d_{m} \frac{\Delta x}{\lambda}})}
\]

It has no equivalent definition under the old notation scheme.
<table>
<thead>
<tr>
<th>OLD</th>
<th>NEW</th>
<th>MEANING</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma_{m,m+1}^+$</td>
<td>$\Gamma_m$</td>
<td>Ordinary Fresnel Reflection Coefficient, slab $m$ to slab $m+1$</td>
</tr>
<tr>
<td>$\Gamma_{m,m-1}^-$</td>
<td>$\Gamma_m$</td>
<td>Ordinary Fresnel Reflection Coefficient, slab $m$ to slab $m-1$</td>
</tr>
<tr>
<td>$\Gamma_{m,m+1,\text{eff}}^+$</td>
<td>$\Gamma_{m,e}$</td>
<td>Effective Fresnel Reflection Coefficient, slab $m$ to slab $m+1$</td>
</tr>
<tr>
<td>$\Gamma_{m,m-1,\text{eff}}^-$</td>
<td>$\Gamma_{m,e}$</td>
<td>Effective Fresnel Reflection Coefficient, slab $m$ to slab $m-1$</td>
</tr>
<tr>
<td>$\Gamma_{m,m+1}^+$</td>
<td>$\Gamma_m$</td>
<td>Effective Transmission Coefficient, slab $m$ to slab $m+1$</td>
</tr>
<tr>
<td>$\Gamma_{m,m-1}^-$</td>
<td>$\Gamma_m$</td>
<td>Effective Transmission Coefficient, slab $m$ to slab $m-1$</td>
</tr>
</tbody>
</table>
In Chapters II, III and IV, composite pattern factor expressions were developed for a doubly infinite periodic array of non-planar loop elements. From Equation (3.48) we found that

\[ c(n)t \sum_{j=1}^{4} I_{P_{j,m \pm}} (n)t \]  (D.1)

where

\[ I_{P_{1,m \pm}} (n)t = \int_{-b}^{b} (x') e^{j \beta m x x'} dx' \]  (D.2)

\[ I_{P_{2,m \pm}} (n)t = \int_{-a}^{a} (y') e^{j \beta m y y'} dy' \]  (D.3)

\[ I_{P_{3,m \pm}} (n)t = \int_{-b}^{b} (y') e^{j 2 \beta m y y'} dy' \]  (D.4)

\[ I_{P_{4,m \pm}} (n)t = \int_{-a}^{a} (y') e^{j \beta m y y'} dy' \]  (D.5)
Furthermore, from Equations (3.13) and (3.14)

\[ \hat{p}(3) = - \hat{p}(1) = \hat{x} \] (D.6)

\[ \hat{p}(2) = - \hat{p}(4) = \hat{y} \] (D.7)

Assuming the current distribution of Equation (3.70)

\[ I_n(t)(\alpha) = \frac{\cos \beta_m \alpha}{\cos 2 \beta_m (a+b)} \] (D.8)

we can write down the current function along all four sides of the reference loop element. From Equations (3.71) - (3.74) we have the following.

\[ I_1^{(n)t}(x') = \frac{\cos \beta_m (2a+2b - |x'|)}{\cos 2 \beta_m (a+b)} \quad -b < x' < b \] (D.9)

\[ I_2^{(n)t}(y') = \frac{\cos \beta_m (a+b - y')}{\cos 2 \beta_m (a+b)} \quad -a < y' < a \] (D.10)

\[ I_3^{(n)t}(x') = \frac{\cos \beta_m x'}{\cos 2 \beta_m (a+b)} \quad -b < x' < b \] (D.11)

\[ I_4^{(n)t}(y') = \frac{\cos \beta_m (a+b + y')}{\cos 2 \beta_m (a+b)} \quad -a < y' < a \] (D.12)
We are now prepared to evaluate the individual pattern factors under transmitting conditions. Substituting Equations (D.9) and (D.6) into Equation (D.2) yields after integration

\[ I_{n,m}^{(n,t)} = \frac{2r_{mz}}{\beta_m \cos(2\beta_m(a+b))(1-r_{mx}^2)\sqrt{r_{mx}^2 + r_{mz}^2}} \left[ \sin2\beta_m(a+b) \right. \\
- \sin\beta_m(2a+b)\cos\beta_mbr_{mx} \right. \\
- \left. r_{mx}\cos\beta_m(2a+b)\sin\beta_mbr_{mx} \right] \tag{D.13} \]

\[ I_{n,m}^{(n,t)} = \frac{2r_{my}r_{mx}}{\beta_m \cos(2\beta_m(a+b))(1-r_{mx}^2)\sqrt{r_{mx}^2 + r_{mz}^2}} \left[ \sin2\beta_m(a+b) \right. \\
- \sin\beta_m(2a+b)\cos\beta_mbr_{mx} - r_{mx}\cos\beta_m(2a+b)\sin\beta_mbr_{mx} \right], \tag{D.14} \]

and

\[ I_{n,m}^{(n,t)} = I_{n,m+}^{(n,t)} = I_{n,m-}^{(n,t)}. \tag{D.15} \]

Substituting Equations (D.10) and (D.7) into Equation (D.3) produces the following:

\[ I_{n,m}^{(n,t)} = 0 \tag{D.16} \]
Equations (D.16) and (D.18) are zero, since

\[ \hat{l}_{m \pm} \cdot (\hat{z} \hat{y}) = 0 \]  \hspace{1cm} (D.20)

Next, substitute Equations (D.11) and (D.6) into Equation (D.4) and integrate. The resulting equations are given by the following.
\[ \mathbb{I}^P(n)_{3,m}^+ = \frac{-2r_{m2}e}{\beta_m \cos 2\beta_m (a+b)(1-r_{mx}^2) \sqrt{r_{mx}^2 + r_{mz}^2}} \left[ \sin \beta_m b \cos (\beta_m b r_{mx}) - r_{mx} \cos \beta_m b \sin (\beta_m b r_{mx}) \right] \] (D.21)

\[ \mathbb{I}^P(n)_{3,m}^- = \frac{-2r_{m2}e}{\beta_m \cos 2\beta_m (a+b)(1-r_{mx}^2) \sqrt{r_{mx}^2 + r_{mz}^2}} \left[ \sin \beta_m b \cos (\beta_m b r_{mx}) - r_{mx} \cos \beta_m b \sin (\beta_m b r_{mx}) \right] \] (D.22)

\[ \mathbb{I}^P(n)_{3,m}^- = \frac{+2r_{m2}e}{\beta_m \cos 2\beta_m (a+b)(1-r_{mx}^2) \sqrt{r_{mx}^2 + r_{mz}^2}} \left[ \sin \beta_m b \cos (\beta_m b r_{mx}) - r_{mx} \cos \beta_m b \sin (\beta_m b r_{mx}) \right] \] (D.23)

\[ \mathbb{I}^P(n)_{3,m}^- = \frac{+2r_{m2}e}{\beta_m \cos 2\beta_m (a+b)(1-r_{mx}^2) \sqrt{r_{mx}^2 + r_{mz}^2}} \left[ \sin \beta_m b \cos (\beta_m b r_{mx}) - r_{mx} \cos \beta_m b \sin (\beta_m b r_{mx}) \right] \] (D.24)
Finally, substitute Equations (D.12) and (D.7) into Equation (D.5) and evaluate the integrals.

\[ \Pi_{4, m^+}^{(n)t} = 0 \quad (D.25) \]

\[ \Pi_{4, m^+}^{(n)t} = \frac{-jB_m r_{mx}}{\beta_m \cos 2\beta_m (a+b) \sqrt{r_{mx}^2 + r_{mz}^2}} \left[ \begin{array}{c} \frac{j r_{my} \cos \beta_m (2a+b)}{\cos \beta_m (2a+b)} \\ \frac{\sin \beta_m (2a+b) - e^{j 2B_m r_{my} \sin \beta_m b}}{-e^{j 2B_m r_{my} \sin \beta_m b}} \end{array} \right] \]

\[ \Pi_{4, m^+}^{(n)t} = \frac{e^{-jB_m r_{mx}}}{\beta_m \cos 2\beta_m (a+b) \sqrt{r_{mx}^2 + r_{mz}^2}} \left[ \begin{array}{c} \frac{j r_{my} e^{j 2B_m r_{my} \cos \beta_m b}}{e^{j 2B_m r_{my} \cos \beta_m b}} \\ \cos \beta_m (2a+b) + \sin \beta_m (2a+b) - e^{j 2B_m r_{my} \sin \beta_m b} \end{array} \right] \]

This completes the derivation for the pattern factors under transmitting conditions. One may note that for all sides, \( \Pi_{j, m^\pm}^{(n)t} \) are given in terms of the complex direction cosines \( (r_{mx}, r_{my}, r_{mz}) \). For the simple case of \( k=n=0, r_{mx}=s_{mx}, r_{mz}=s_{mz} \), and

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\[ r_{my} = s_{my} = \sqrt{1 - (s_{mx})^2 - (s_{mz})^2} \] (D.28)

In other words, all of the above results still apply if \((s_{mx}, s_{my}, \text{and } s_{mz})\) are substituted for \((r_{mx}, r_{my}, \text{and } r_{mz})\), respectively for the case where \(k=n=0\). Therefore, Equations (D.1) through (D.27) can be used to evaluate Equations (2.23)-(2.26) which appear in the induced voltage expression of Equation (2.28).

All of the pattern factors derived above assumed that the antenna current on each element was taken under transmitting conditions. When the array element currents are taken under scattering conditions, the composite pattern factor is defined in a slightly different manner.

From Equations (3.23) and (3.26) we write

\[ c(n') = \sum_{j=1}^{4} \frac{1}{|P_{j,m^\pm}|} |P_{j,m^\pm}|^2 \]

where

\[
\begin{align*}
|P_{1,m^+}| &= |\gamma_m| \int_{-\beta}^{\beta} (n') (x') e^{-j\beta_{m} r_{mx} x'} dx' , \\
|P_{2,m^+}| &= |\beta_m| \int_{-\alpha}^{\alpha} (n') (y') e^{j\beta_{m} r_{my} y'} dy' ,
\end{align*}
\] (D.30)
\[ \ell_{3,m}^{(n')} = \int_a^b e^{j2\beta_{marmy} b(n')} \int_{-1}^1 (x')e^{-j\beta_{mx}x'} dx', \quad (D.32) \]

\[ \ell_{4,m}^{(n')} = \int_a^b e^{j\beta_m(brmx-army) a(n')} -j\beta_{my}y' \int_{-1}^1 (y')e^{-j\beta_{my}y'} dy', \quad (D.33) \]

\[ \ell_{1,m}^{(n')} = \int_a^b e^{j\beta_{mx}x'} \int_{-1}^1 (x')e^{-j\beta_{mx}x'} dx', \quad (D.34) \]

\[ \ell_{2,m}^{(n')} = \int_a^b e^{-j\beta_m(brmx+army) a(n')} -j\beta_{my}y' \int_{-1}^1 (y')e^{-j\beta_{my}y'} dy', \quad (D.35) \]

\[ \ell_{3,m}^{(n')} = \int_a^b e^{-j2\beta_{marmy} b(n')} \int_{-1}^1 (x')e^{j\beta_{mx}x'} dx', \quad (D.36) \]

and

\[ \ell_{4,m}^{(n')} = \int_a^b e^{-j\beta_m(brmx-army) a(n')} -j\beta_{my}y' \int_{-1}^1 (y')e^{j\beta_{my}y'} dy'. \quad (D.37) \]

Remember that the superscript \((n')\) in Equations (D.29)-(D.37) strictly refer to array \((n')\). In our problem, \((n')\) is simply one.
As explained in Chapter III, for simplicity we choose the antenna current under scattering conditions to be of the same form as the antenna current under transmitting conditions. From Equation (3.75) we write the following:

\[ I^{(n')}_j(\xi_j) = I^{(n')}_j(\eta_j) = \frac{\cos \beta_m \xi_j}{\cos 2\beta_m(a+b)} \]  

Along the four sides of the element, the current can be derived from Equation (D.38)

\[ I^{(n')}_1(x') = \frac{\cos \beta_m(2a+2b - |x'|)}{\cos 2\beta_m(a+b)} \quad -b < x' < b \]  

\[ I^{(n')}_2(y') = \frac{\cos \beta_m(a+b-y')}{\cos 2\beta_m(a+b)} \quad -a < y' < a \]  

\[ I^{(n')}_3(x') = \frac{\cos \beta_m x'}{\cos 2\beta_m(a+b)} \quad -b < x' < b \]  

\[ I^{(n')}_4 = \frac{\cos \beta_m(a+b-y')}{\cos 2\beta_m(a+b)} \quad -a < y' < a \]  

Substituting Equation (D.39) into Equations (D.34) and (D.30) yields the following:

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Substituting Equation (D.40) into Equations (D.35) and (D.31), we obtain

\[ lP_2,m^+ = lP_2,m^- = 0 \]  \hspace{1cm} (D.46)

\[ nP_2,m^+ = e^{-j\theta_m br_{mx}} \frac{r_{my}^{r_{mx} + r_{nz}}}{\beta_m \cos \theta_m (a+b)^{r_{mx}^2 + r_{nz}^2}} \left[ \begin{array}{cc} \sin \theta_m (2a+b) - e^{j2\theta_m mr_{my} \cos \theta_m b - \cos \theta_m (2a+b)} \\ j\theta_m m \sin \theta_m b \end{array} \right] \]

\[ + \left[ \begin{array}{cc} e^{-j2\theta_m mr_{my} \cos \theta_m b - \cos \theta_m (2a+b)} \\ \sin \theta_m (2a+b) - e^{j2\theta_m mr_{my} \cos \theta_m b} \end{array} \right] \]  \hspace{1cm} (D.47)
Continuing in the same fashion, substitute Equation (D.41) into Equations (D.36) and (D.32)

\[
\begin{align*}
\I^p_{2,m-} &= \frac{-j\beta_m\rho_{mx}}{\beta_m \cos \theta_m (a+b) \sqrt{r_{mx}^2 + r_{my}^2}} \left[ j r_{my} \cos \beta_m (2a+b) \right] \\
&\quad - e^{-j2\beta_m \rho_{mx} \cos \beta_m b} \left[ \sin \beta_m (2a+b) - e^{-j2\beta_m \rho_{my} \sin \beta_m b} \right].
\end{align*}
\]

(D.48)

\[
\begin{align*}
\I^p_{3,m+} &= \frac{j2\beta_m \rho_{my}}{\beta_m \cos \theta_m (a+b) (1-r_{mx}^2) \sqrt{r_{mx}^2 + r_{my}^2}} \left[ - \frac{2r_{mx}}{r_{mx}^2 + r_{my}^2} \sin \beta_m \cos \beta_m \rho_{mx} \right] \\
&\quad - r_{mx} \cos \beta_m b \sin \beta_m \rho_{mx}.
\end{align*}
\]

(D.49)

\[
\begin{align*}
\II^p_{3,m+} &= \frac{j2\beta_m \rho_{my}}{\beta_m \cos \theta_m (a+b) (1-r_{mx}^2) \sqrt{r_{mx}^2 + r_{my}^2}} \left[ - \frac{2r_{my} r_{mx}}{r_{mx}^2 + r_{my}^2} \sin \beta_m \cos \beta_m \rho_{mx} \right] \\
&\quad - r_{mx} \cos \beta_m b \sin \beta_m \rho_{mx}.
\end{align*}
\]

(D.50)
Finally, for side four substitute Equation (D.42) into Equations (D.37) and (D.33)

\[ lP_{4,m}^+(n') = lP_{4,m}^- = 0 , \quad (D.53) \]

\[ lP_{4,m}^+(n') = \frac{-e^{-j\theta_{mb}r_{mx}}}{\beta_m \cos \beta_m (a+b) \sqrt{r_{mx}^2 + r_{mz}^2}} \left[ -e^{j\theta_{mbr_{mx}}} \right. \left. \beta_m \cos \beta_m (2a+b) \sin \beta_{m(2a+b)} - e^{j\theta_{mbr_{mx}}} \sin \beta_m (2a+b) \right] , \quad (D.54) \]
This completes the evaluation of the composite pattern factor for both polarizations under scattering conditions.

Before closing this appendix, several comments are in order. First of all, notice that every pattern factor is a function of the surrounding dielectric medium through the quantities \( \beta_m, r_{mx}, r_{mz}, \) and \( r_{my} \). Second, the pattern factors depend on the element size, through the constants "a" and "b". Third, the pattern factors are a function of the discrete plane wave summation variables \( \{k\} \) and \( \{n\} \). We see this from

\[
r_{mx} = s_{mx} + \frac{k\lambda}{D_x}, \quad (D.56)
\]

\[
r_{mz} = s_{mz} + \frac{n\lambda}{D_z}, \quad (D.57)
\]

and

\[
r_{my} = \sqrt{1 - r_{mx}^2 - r_{mz}^2} \quad (D.58)
\]

where the \((-j)\) root of Equation (D.58) is taken when
In view of Equation (D.54), this brings up our fourth point. For large \( k \) and \( n \), \( r_{m'y} \) is a negative imaginary number, so that the exponential term blows up as \( k \) and \( n \) become large.

\[
\lim_{k \to \infty} \lim_{n \to \infty} \frac{j2\beta_m r_{m'y}}{e} = (D.60)
\]

Does this mean that the solution is invalid? The answer to this question is "no", for the following reason. Each pattern factor is not calculated by itself. There is always some complex phase constant multiplying the pattern factor. The situation in Equation (D.60) does not occur since the multiplying factor always decays as fast or faster than the exponential in Equation (D.60). As an example, consider the impedance expression from Equation (3.66)

\[
Z_{\text{SELFC}} = \frac{Z_{m'}}{2 \mu \epsilon} \sum_{k',n} \frac{-j\beta_{m'}(\overline{R}(n)-\overline{R}(n'))r_{m'y}}{r_{m'y}} e^{-j2\beta_{m'}(b_{m'}-y(n'))r_{m'y}}
\]

It appears in (D.60) that \( \Pi_{P_m^c(n')} \), \( \Pi_{P_m^c(n')}^\dagger \), \( \Pi_{P_m^{c(n')}}^\dagger \), \( \Pi_{P_m^{c(n')}}^\dagger \), \( \Pi_{P_m^{c(n')}}^\dagger \), and \( \Pi_{P_m^{c(n')}}^\dagger \) all blow up for large \( k \) and \( n \). When we look closer, we note that

\[
e^{-j\beta_{m'}(b_{m'}-y(n'))r_{m'y}} e^{j\beta_{m'}2ar_{m'y}} = e^{-j\beta_{m'}(b_{m'}-y(n')-2a)r_{m'y}}
\]

(D.62)
From Figure D.1 note that

\[ b_{m'-1} - y(n') > 2a. \]  \hspace{1cm} (D.63)

Therefore, Equation (D.60) converges for large values of \( k \) and \( n \), instead of blowing up.

This concludes the composite pattern factor appendix.

Figure D.1. Geometry of loop orientation.
In Appendix D, expressions were given for the composite pattern factors $c(n)t$ and $c(n)t$. Given the sheer length of these expressions, it is difficult to conceptually grasp the physical meaning of these quantities. It is the purpose of this Appendix to give the reader some insight into what geometric quantities affect these composite pattern factors.

It is important to understand what functional variables affect the composite pattern factors. Looking at the expressions in Appendix D, we first find that the pattern factors are a function of the medium surrounding the array. In other words, $c(n)t$ depends on the intrinsic properties $(\varepsilon_m, \mu_m)$ of the medium. Next suppose that the interelement spacings $D_x$ and $D_z$ are kept less than $\lambda/2$ at the array design frequency, so that grating lobes are eliminated. This being the case, $c(n)t$ will not depend on the summation variables $(k, n)$ for the dominant $k=n=0$ propagating mode. To check this observation, simply set $k=n=0$ in Equations (D.55), (D.56), and (D.57). Finally, we note that $c(n)t$ depends on the loop dimensions $(a)$ and $(b)$, along with the direction of propagation vectors $r_{m+}$ and $r_{m-}$.

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Another important concept to understand is the relationship between the composite pattern factor and the properties of the adjacent dielectric layers. This relationship is not a simple one. Although the composite pattern factors $\frac{c(n)t}{\Gamma_{m}^{\pm}}$ do not depend individually on $\frac{\pm \sqrt{\Gamma_{m}^{\pm}}}{\Gamma_{m}^{\pm}}$ in the induced voltage (and scattered field) expressions, their combined relationships are far more complicated. For example, observe the induced voltage expression of Equation (2.28),

$$
V_{n}^{(c)}(R(n)) = \frac{1}{A} \left[ - \frac{c(n)t}{\Gamma_{m}^{+}} + \frac{c(n)t}{\Gamma_{m}^{-}} e^{-j\beta_{m}^{2}\gamma_{m}} \right]
$$

$$
+ \frac{1}{A} \left[ - \frac{c(n)t}{\Gamma_{m}^{+}} + \frac{c(n)t}{\Gamma_{m}^{-}} e^{-j\beta_{m}^{2}\gamma_{d}} \right], \quad (E.1)
$$

where $\frac{1}{A}$ are known constants. The expression in brackets is the equivalent pattern factor as seen by a plane wave travelling in the $+y$ direction. When the array is located alone in free space ($\frac{\pm \sqrt{\Gamma_{m}^{\pm}}}{\Gamma_{m}^{\pm}} = 0$), the composite pattern factor is isolated.

In this Appendix, the quantities in brackets will be investigated in some detail. For notational simplicity, define the effective composite pattern functions as follows.

$$
\frac{c(n)t}{\Gamma_{m}^{\pm}} = \frac{c(n)t}{\Gamma_{m}^{+}} + \frac{c(n)t}{\Gamma_{m}^{-}} e^{-j\beta_{m}^{2}\gamma_{d}}, \quad (E.2)
$$
Recalling from Appendix B

\[ s_{mx} = \frac{\lambda_0}{\lambda_m} s_{ox} \], \hspace{1cm} (E.3) \\
\[ s_{mz} = \frac{\lambda_0}{\lambda_m} s_{oz} \], \hspace{1cm} (E.4) \\
\[ s_{my} = \sqrt{1 - s_{mx}^2 - s_{mz}^2} \] \hspace{1cm} (E.5)

where from Chapter I

\[ s_{ox} = -\cos(\alpha) \sin(\eta) \], \hspace{1cm} (E.6)

and

\[ s_{oz} = -\sin(\alpha) \sin(\eta) \], \hspace{1cm} (E.7)

we see that the effective composite pattern factor depends strongly on the incidence angles ($\alpha$) and ($\eta$). The effective composite pattern factor also depends strongly on the distance between the terminals and the dielectric slab behind the array through the parameter $d_m$. (See Figure E.1).

A better parameter to use is the center to center spacing $d_{cc}$ from the right dielectric interface to the center of the loop. From Figure E.1, we write

\[ d_{cc} = (d_{m}^' - a) \] \hspace{1cm} (E.8)
This distance becomes especially important when a ground plane is placed in the $m$th interface directly behind the array (Figure E.2). Under the circumstances,

$$\Gamma_{m,e}^+ = -1.0,$$  \hspace{1cm} \text{(E.9)}

so that the negative directed pattern factors $\Gamma_{m,-}^+$ have a large effect on the overall effective composite pattern factor.
In the effective composite pattern factor plots to follow, the geometry of Figure E.2 is assumed. The array elements are placed \((3/10)\lambda\) apart at the design frequency of 10 GHz \((\lambda = 3\text{cm})\); thus no grating lobes should occur until 16.6 GHz. The total circumference of the loop element is one wavelength \((3\text{cm})\) at the design frequency. For simplicity, a square loop is chosen such that

\[
a = b = \lambda/8 = .375 \text{ cm} \quad .
\]  

(E.10)
The array is located in free space a set distance in front of a perfectly conducting ground plane. In the data to follow, we will look at the effective composite pattern factor for four different configurations in both principal planes of incidence. The four cases are as follows:

(a) no ground plane present (array in free space)

(b) \( d_{cc} = (1/8 + 1/60) \lambda \) at 10 GHz

(c) \( d_{cc} = 3/8 \lambda \) at 10 GHz

(d) \( d_{cc} = 1/4 \lambda \) at 10 GHz

Case (a) is included to demonstrate the simple variation of \( \sum_{m=1}^{Nm+1} c(n) t \) independent of the ground plane.

As stated above, we choose to illustrate the variations of \( \sum_{m=1}^{Nm+1} c(n) t \) versus the scan angle \( \eta \) for the two principal scan planes \( \alpha = 0^\circ \) and \( \alpha = 90^\circ \). Remember that in the \( \alpha = 0^\circ \) scan plane, the electric field is taken parallel to the plane of incidence. In the \( \alpha = 90^\circ \) scan plane, the electric field is taken perpendicular to the plane of incidence.

Before presenting the data, several comments are in order. First, for the no ground plane case, only the right hemisphere of the pattern is plotted. The identical mirror imaged left hemisphere \( (90^\circ < \eta < 270^\circ) \) has been omitted. Second, for the ground plane cases, the ground plane is assumed to be oriented parallel to the plane of the
array. Since this ground plane shields the left hemisphere, no patterns exist for scan angles in the range $+90^\circ < \theta < 270^\circ$. Third, remember that each pattern is normalized to its own maximum value. Therefore, every pattern factor grid shown is scaled to 10 dB/division, relative to the maximum value of that particular plot. Fourth, the negative scan angles $-90^\circ < \theta < 0$ are actually shown on the plots as $270^\circ < \theta < 360^\circ$. Finally, in order to illustrate in detail the geometries of the two scan planes, refer to Figures E.3 and E.4. In Figure E.3, remember that the electric field is parallel to the plane of incidence, which in this figure is the plane of the paper. Figure E.4 reminds us that for $\alpha = 90^\circ$, the electric field is oriented perpendicular to the plane of incidence (again the plane of the paper).

The first set of effective composite pattern factor plots are for the $\alpha = 0^\circ$ plane of incidence. Figure E.5 demonstrates the four cases at $f = 10$ GHz, the designed center frequency. Notice the change in the effective pattern factor as the ground plane distance, $d_{cc}$, is varied. These effects are even more profound as the frequency is varied from 7 to 9 GHz below resonance (Figures E.6 - E.8, respectively), and from 11 to 13 GHz above resonance (Figures E.9 - E.11, respectively). One surprising characteristic of this loop element is the occurrence of nulls in the effective pattern factor below resonance. For instance, in Figure E.6, note that the ground plane causes nulls to occur for configurations (b), (c), and (d). This can be explained as follows. Since the element is not oriented parallel to the plane of the array, the ground plane no longer acts as a simple doublet. The extended loop
Figure E.3. Coordinate system for the effective composite pattern factor plots in the $\alpha=0^\circ$ plane of incidence.

Figure E.4. Coordinate system for the effective composite pattern factor plots in the $\alpha = 90^\circ$ plane of incidence.
Figure E.5. Parallel polarization effective composite pattern factor variation vs η for α = 0°, F = 10 GHz.
Figure E.6. Parallel polarization effective composite pattern factor variation vs \( \eta \) for \( \alpha = 0^\circ \), \( F = 7 \) GHz.
No Ground Plane

d_{CC} = (1/8 + 1/60)\lambda_0

(\lambda_0 = 3 \text{ cm @ 10 GHz})

d_{CC} = 3\lambda_0/8

d_{CC} = \lambda_0/4

Figure E.7. Parallel polarization effective composite pattern factor variation vs \eta for \alpha = 0^\circ, F = 8 \text{ GHz}. 

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No Ground Plane

\( d_{CC} = (1/8 + 1/60)\lambda_0 \)

\( (\lambda_0 = 3 \text{ cm @ 10 GHz}) \)

\( d_{CC} = 3\lambda_0/8 \)

\( d_{CC} = \lambda_0/4 \)

Figure E.8. Parallel polarization effective composite pattern factor variation vs \( \eta \) for \( \alpha = 0^\circ, F = 9 \text{ GHz} \).
Figure E.9. Parallel polarization effective composite pattern factor variation vs η for α = 0°, F = 11 GHz.
Figure E.10. Parallel polarization effective composite pattern factor variation vs \( n \) for \( \alpha = 0^\circ \), \( F = 12 \) GHz.

\[
\begin{align*}
\text{No Ground Plane} & & \text{d}_{CC} = (1/8 + 1/60)\lambda_0 \\
\text{d}_{CC} = 3\lambda_0/8 & & \text{d}_{CC} = \lambda_0/4
\end{align*}
\]

\( \lambda_0 = 3 \) cm @ 10 GHz
Figure E.11. Parallel polarization effective composite pattern factor variation vs \( \eta \) for \( \alpha = 0^\circ \), \( F = 13 \text{ GHz} \).
element interacts with the ground plane in a more complicated manner. Also, make note of the following characteristic. Whenever a null occurs in the effective pattern factor, the real part of the array self impedance, \( \text{Re}(Z_{11}) \), goes to zero! A zero real part has no place in an array! Configurations which produce these nulls are to be avoided.

The next group of effective composite pattern factor plots are presented for the \( \alpha = 90^\circ \) plane of incidence. Figure E.12 demonstrates the variation in the effective pattern plot versus ground plane spacing at the 10 GHz design frequency. Case (a) always represents the free space (no ground plane) case. The amazing characteristic to note in Figure E.12 is the pattern variation with scan angle. In the "no ground plane" case, the pattern amplitude strictly increases with scan angle. When the ground plane is introduced, the pattern amplitude can increase and/or decrease with scan angle. This observation is supported as the frequency is varied below and above resonance. The cases of 7, 8, and 9 GHz are shown in Figures E.13, E.14, and E.15, respectively. For 11, 12, and 13 GHz, refer to Figures E.16, E.17, and E.18, respectively. It is clear that the distance separating the loop center and the ground plane will be an important design parameter in this investigation.

In conclusion, several points should be kept in mind. First, pattern nulls translate to impedances which have no real part. These must be avoided. Second, the variation in scan impedance is directly related to the composite pattern factor. Since these effective pattern functions for the dominant propagating mode are usually divided by
No Ground Plane

\[ d_{CC} = \frac{1}{8} + \frac{1}{60} \lambda_0 \]  

\( \lambda_0 = 3 \text{ cm @ 10 GHz} \)

\[ d_{CC} = \frac{3\lambda_0}{8} \]

\[ d_{CC} = \frac{\lambda_0}{4} \]

Figure E.12. Perpendicular polarization effective composite pattern factor variation vs \( \eta \) for \( \alpha = 90^\circ \), \( F = 10 \text{ GHz} \).
Figure E.13. Perpendicular polarization effective composite pattern factor variation vs $\eta$ for $\alpha = 90^\circ$, $F = 7$ GHz.
Figure E.14. Perpendicular polarization effective composite pattern factor variation vs $\eta$ for $\alpha = 90^\circ$, $F = 8$ GHz.
Figure E.15. Perpendicular polarization effective composite pattern factor variation vs $\eta$ for $\alpha = 90^\circ$, $F = 9$ GHz.

$\lambda_0 = 3$ cm @ 10 GHz

$d_{CC} = (1/8 + 1/60)\lambda_0$

$d_{CC} = 3\lambda_0/8$

$d_{CC} = \lambda_0/4$
Figure E.16. Perpendicular polarization effective composite pattern factor variation vs $n$ for $\alpha = 90^\circ$, $F = 11$ GHz.

$\lambda_0 = 3 \text{ cm} @ 10 \text{ GHz}$

$d_{CC} = (1/8 + 1/60)\lambda_0$

$d_{CC} = 3\lambda_0/8$

$d_{CC} = \lambda_0/4$
Figure E.17. Perpendicular polarization effective composite pattern factor variation vs η for α = 90°, F = 12 GHz.
Figure E.18. Perpendicular polarization effective composite pattern factor variation vs \( n \) for \( \alpha = 90^\circ \), \( F = 13 \) GHz.
1/r^y, we can make the following observation. From Equations (E.5), (E.6), and (E.7)

\[
\frac{1}{r_{my}} = \sqrt{1 - (\sqrt{e_m} \cos \alpha \sin n)^2 - (\sqrt{e_m} \sin \alpha \sin n)^2}
\]

For the principal scan plane, \( \alpha \) is either 0° or 90°. In either case, Equation (E.11) reduces to

\[
\frac{1}{r_{my}} = \sqrt{1 - \varepsilon_m \sin^2 n} \quad | \quad \alpha = 0^\circ \text{ or } \alpha = 90^\circ
\]

Therefore, a scan independent phased array must have the ratio of \( \frac{\frac{1}{r_{my}}}{s_{my}} \) vary as little as possible, in order for the changes in the element pattern factor \( \frac{1}{r_{my}} \) to track the changes in the array scan factor \( s_{my} \). Although this is not easy to do, this element offers enough diversity in \( \frac{1}{r_{my}} \) to warrant investigation as a possible scan independent array element.

Clearly, the results presented here do not optimize the spacing between the radiating loop element and the ground plane. Its purpose is to give the reader some physical insight into the composite pattern factor concept.

This concludes the appendix on the effective composite pattern factor variation with scan angle.
APPENDIX F

ON THE ASSUMPTION OF THE COSINUSOIDAL CURRENT MODE FOR THE
NON-PLANAR LOOP FILAMENT UNDER TRANSMITTING CONDITIONS

When the plane wave expansion technique of Munk, et al [47] is used
to analyze the radiation and scattering properties of doubly infinite
periodic surfaces, a subtle assumption is made regarding the choice of
the testing function \( \hat{f}_o \). From Chapter I and Appendix A, the testing
function \( \hat{f}_o \) is chosen to be the normalized antenna current under
transmitting conditions. That is to say

\[
\hat{f}_o = \sum_{i=1}^{4} i^n(t)(x_i) \hat{p}^{(i)}
\]

(loop side #)

where \( i^n(t)(x_i) \) is the antenna current along side "i" normalized to the
current amplitude at the antenna terminals, and \( \hat{p}^{(i)} \) is the unit vector
specifying the orientation of side "i". As it stands, Equation (F.1) is
not an approximation.

Further along in Chapter III, it was assumed that the normalized
antenna current amplitude under transmitting conditions had the
following form:
In Equation (F.2), $\lambda$ is measured from a point opposite of the antenna terminals, as illustrated in Figure F.1. Since Equation (F.2) is not exact, this appendix will address the validity of the assumed current distribution.

As stated in Chapter I, past work in the periodic surface area concentrated on planar resonant radiating elements. Henderson [48] and Shubert [49] analyzed several such elements, including the unloaded resonant half wave dipole, the loaded dipole, the loaded tripole, and the loaded four legged crossed dipole. These elements all have one feature in common; the current distributions under transmitting conditions were well understood, based on the analysis of one isolated
element. For instance, suppose we determined the transmitting current distribution along an isolated loaded dipole antenna at or near its resonant frequency. For the sake of argument, call this current $I_{id}(\xi)$, where "id" indicates isolated dipole. Next, let us analyze a periodic array of these same loaded dipole antennas. We allow the current distribution under transmitting conditions for the loaded dipole element in the presence of the array to equal the isolated dipole current distribution

$$I^n(t)(\xi) = I_{id}(\xi) \quad .$$

Past experience has shown that Equation (F.3) is an excellent approximation for resonant structures. This statement is based on comparisons between theoretical and measured performances of several classes of periodic structures (see [50]).

With these thoughts in mind, let us proceed with the major point of this appendix. The primary goal here is to somehow analyze a single isolated square loop filament, in order to determine its normalized transmitting antenna current distribution. For simplicity, a square loop with a circumference of one wavelength (3 cm at 10 GHz) will be analyzed using Richmond's piecewise linear wire code [51]. This code is based on Galerkin's moment method technique, where the bases and testing functions are both piecewise sinusoids. In the results to follow, Kent [52] expanded the transmitting current distribution into 64 segments.
After running the wire code, the normalized current distributions under transmitting conditions were plotted. These results will be compared to our assumed current distribution of Equation (F.2).

Before presenting the comparisons between the assumed current distribution and the moment method current distribution, examine the sample plot of Figure F.2. Notice that the current has been plotted along one linear axis. By comparing the letters shown below the abscissa with the letters on the square filament, the antenna current can be determined at any position. Also note that for all moment method calculations, the load impedance is set to zero.

The first set of comparisons are shown in Figure F.3, for frequencies of 7 and 8 GHz. Notice that the shape of the assumed current virtually overlaps the calculated moment method current, except where the assumed current goes to zero. Figure F.4 shows similar agreement at 9 and 10 GHz. Remember that the loop is one wavelength in circumference at 10 GHz. Finally, Figure F.5 presents the 11 and 12 GHz case. Again, the agreement between the assumed and the actual current distribution is quite remarkable.

So far, the theoretical calculations tend to lend credability to the assumed current distribution. One may wonder if it is realistic to feed a loop antenna in a balanced manner. Thiele, et al. [53], demonstrates that for a center fed half-wave dipole, it doesn't take much to generate unbalanced currents. For the center fed half wave dipole, the situation is illustrated in Figure F.6. Obviously, the
Figure F.2. Example current magnitude plot demonstrating the relationship between the graphed current and the rectangular current filament geometry.
Figure F.3. Comparison between the assumed current distribution from Equation (F.2) and the calculated moment method current at 7 and 8 GHz.
Figure F.4. Comparison between the assumed current distribution from Equation (F.2) and the calculated moment method current at 9 and 10 GHz.
Figure F.5. Comparison between the assumed current distribution from Equation (F.2) and the calculated moment method current at 11 and 12 GHz.
the same situation can exist with the square loop element. In this analysis, it is assumed that the cosinusoidal current distribution exists on the loop element. In reality, unless a "balun" is designed into the loop feed point, it would be difficult to obtain the cosinusoidal antenna current distribution. Thus, the loop element in question must have a balun, as shown in detail in Figure F.7. The darker black line, although hidden in reality, is a quarter wavelength balun which translates an open circuit at the end of the balun to a short at the top terminal post. This element should provide a good balanced cosinusoidal current distribution.
In closing, this appendix compared the assumed current distribution for the square loop under transmitting conditions to similar moment method calculations.
ON THE CONVERGENCE OF THE SELF IMPEDANCE DOUBLE SUMMATION

In this appendix, we will carefully look at the five contributions to the self impedance double summation, in order to verify that all contributions converge in \( \{k, n\} \) space.

The five contributions to the total self impedance in Equation (3.74) are defined as \( Z_{\text{SELF}A} \), \( Z_{\text{SELF}B} \), \( Z_{\text{SELF}C} \), \( Z_{\text{SELF}D} \), and \( Z_{\text{SELF}E} \). The expressions for these terms are given by Equations (3.90), (3.65), (3.66), (3.67) and (3.68), respectively. Let us analyze these five terms closely.

First of all, the four bounce path contributions (B, C, D, and E) will be investigated. For \( Z_{\text{SELF}B} \), we write from Equation (3.65)

\[
Z_{\text{SELF}B} = \frac{Z_m'}{2x_D D_z} \sum_{k,n} \frac{-e^{j\beta m'w r m' z}}{r_{m',y}} e^{-j2 \beta m'(d_{m'}) g_{m',y}} \left\{ c(n') c(n) t - \right. \\
\left. \frac{\hat{r}_m' - \hat{r}_m^+}{\hat{r}_m' + \hat{r}_m^+} e^{i\Omega m'} + \frac{\hat{r}_m' - \hat{r}_m^+}{\hat{r}_m' + \hat{r}_m^+} e^{i\Omega m'} \right\},
\]

where we have made the substitutions
For the self impedance calculation, we note that the test and reference elements are separated by a wire radius as shown in Figure 3.7. In Equation (G.1), as \(k\) and \(n\) increase, \(r_{m'y}\) becomes a large negative imaginary number. From Equation (C.13), this causes the following to happen.

\[
\lim_{k \to -\infty} \frac{\alpha_m}{v_m} < 1 \quad (G.4)
\]

Further note that from Equations (C.7) and (C.1), a similar event takes place with the effective reflection coefficients

\[
\lim_{k \to -\infty} \left| \frac{\alpha_{m'}}{\alpha_{n'}} \right| < 1 \quad (G.5)
\]

As \(k\) and \(n\) becomes large, what happens to the pattern factors in Equation (G.1)? In order to analyze the behavior in \([k,n]\) space of the products \(\|p^{c(n')}_{m'-} p^{c(n)}_{m'}\) and \(\|p^{c(n')}_{m'-} p^{c(n)}_{m'}\), I will leave out all quantities independent of \(k\) and \(n\). I will also make use of the following equations.
Equation (G.8) is especially important since it causes the exponential of (G.9) to decay for any positive sigma (σ), as k and n become large.

\[
\lim_{k,n \to \infty} e^{-j\sigma r_{m'y}}, \quad \sigma > 0
\]

If we turn to Appendix D and carry out the indicated pattern factor multiplications, we note that in terms of \( r_{m'x}, r_{m'y}, r_{m'z} \) the products are proportional to the following:

\[
| \|_{m'-} \|_{m'+} \|_{\mathbb{P}c(n')} \|_{\mathbb{P}c(n) t} \|_{\mathbb{P}} \frac{r_{m'y}^2}{1 - r_{m'y}^2} = -\frac{1}{r_{m'y}^2} - 1 |_{k,n \text{ large}}
\]

G.10
Although (G.11) decays as \(k,n\) increases, clearly (G.10) does not.

Fortunately, in Equation (G.1), the parameter \(\alpha > 0\) such that
\[
\lim_{k \to \infty} \prod_{k,n \to \infty} (\text{constant}) = 0 .
\]

Therefore, the \(Z_{\text{SELF}}\) term definitely converges, thanks to the decaying exponential in (G.1) and (G.12).

Next, observe the expression for \(Z_{\text{SELFC}}\). Substituting (G.2) into (3.66) yields the expression
\[
Z_{\text{SELFC}} = \frac{Z_{m'}}{2D_{x}D_{z}} \sum_{k,n} e^{-jB_{m'}w_{m'}z} e^{-j2\delta_{m'}d_{m'}r_{m'y}}
\]
\[
\left\{ \prod_{m'} c(n') e^{i\Omega_{m'}} + \prod_{m'} c(n') e^{i\Omega_{m'}} + \prod_{m'} c(n') e^{i\Omega_{m'}} \right\}
\]

where we have used the relationship
\[
d_{m'} = b_{m'} - y(n') .
\]
As \( \{k,n\} \) become large in (G.13), we again note that \( |\Pi_{m'}I_{e}| \) and \( |\Pi_{m'}I_{m}| \) are less than unity. The pattern factor products, however, are different from before. Looking at Appendix D, we determine that for large \( \{k,n\} \), the pattern factor products are proportional to the following

\[
|\Pi_{m'}I_{m'}| \cdot |\Pi_{m'}I_{m'}^*| \propto e^{+j4\beta m'a r m'y} \left( \frac{r_{m'}^2}{r_{m'}^2 + r_{m'}^2 + r_{m'}^2} \right) \quad \text{large } k,n
\]

Equations (G.15) and (G.16) look like bad news, since the exponential term blows up for large \( \{k,n\} \) values. Fortunately, if we look at Equation (G.13), a decaying exponential is also in the picture. From Figure (2.3) we note that \( d_{m'}^2 > 2a \), so that in the limit \( Z_{\text{SELFC}} \)

\[
\lim_{k \to \infty, n \to \infty} |Z_{\text{SELFC}}| < \lim_{k \to \infty, n \to \infty} e^{-j2\beta m'a r m'y(d_{m'}^2 - 2a)} \left[ \frac{-1}{r_{m'}^2} + \frac{r_{m}^2}{r_{m}^2 + r_{m}^2 + r_{m}^2} \right] \to 0
\]

vanishes for large values of \( \{k,n\} \). Hence, the summation for \( Z_{\text{SELFC}} \) converges in \( \{k,n\} \) space.
Third, let us analyze the equation for $Z_{\text{SELF}}$

$$Z_{\text{SELF}} = \frac{Z_m}{2D_x D_z} \sum_{k',n} e^{\frac{-j \beta_{m'} w r_m' z}{r_{m'y}}} e^{-j 2 \beta_{m'} d_m' r_m'y}$$

$$\{ l^c(n)t^c(n')^+ + l^e(n)t^e(n')^- \}$$

\[ + l^c(n)t^c(n')^+ l^e(n')^- e \ll m', e \ll m' \ll m' \]  \((G.18)\)

Ignoring the terms $l^c(n)l^c(n')t^e(n')$ and $l^c(n)t^e(n')$, the pattern factor products $l^c(n)t^c(n')l^e(n')$ behave in the following manner for large values of \([k,n]\).

$$|l^c(n)t^c(n')| = \alpha \frac{r_{m'Z} e^{j 2 \beta_{m'} a r_{m'y}}}{r_{m'y}^2 (r_{m'y}^2 + r_{m'z}^2)} \bigg|_{k,n \text{ large}}$$  \((G.19)\)

$$|l^c(n)t^c(n')| = \alpha e^{j 2 \beta_{m'} a r_{m'y}} \left( \frac{-1}{r_{m'y}^2} - 1 \right) \bigg|_{k,n \text{ large}}$$  \((G.20)\)

If we substitute (G.19) and (G.20) into (G.18), then take the limit as \(k\) and \(n\) become large, we obtain the following.
\[
\lim_{k \to \infty} |Z_{\text{SELF}}| < \text{constant} \cdot \lim_{n \to \infty} e^{\frac{-j2\beta_{m'}d_{m'}r_{m'y}}{r_{m'y}}}
\]

\[
\{e^{j2\beta_{m'}ar_{m'y}} \left[ -\frac{1}{r_{m'y}} + 1 + \frac{r_{m'^2}^2}{r_{m'^2}'x(r_{m'^2}'x + r_{m'^2}'z)} \right] \} + 0
\]  

Equation (G.21) is true since

\[
d_{m'} > a
\]  

causing the exponential decay to swamp out the exponential growth.

Finally, look at the last bounce mode contribution, \(Z_{\text{SELF}}\). From Equations (3.68) and (G.2), we obtain the following equation.

\[
Z_{\text{SELF}} = \frac{Z_m}{2D_x D_z} \sum_{k,n} e^{\frac{-j\beta_{m'}wr_{m'z}}{r_{m'y}}} e^{\frac{-j2\beta_{m'}d_{m'}r_{m'y}}{r_{m'y}}}
\]

\[
\{1^p c(n') P c(n)t^+ + 1^p m'+ I^m',e I^m',e I^m',e \} + \{1^p c(n') P c(n)t^+ + 1^p m'+ I^m',e I^m',e I^m',e \}
\]  

(G.23)
As \( k \) and \( n \) become large, the effective reflection coefficients \( \rho_{m,e} \) and \( \rho_{m'} \) are less than unity. The pattern factor products \( \rho_{m'+}^{c(n')} \rho_{m'+}^{c(n)} \) behave as follows for large \( k \) and \( n \) values.

\[
\frac{e^{j2\beta_m \rho_m \gamma y} 2}{r_{m'}^{z} (r_{m'}^{z} + r_{m'}^{z})} \bigg| k, n \text{ large} \tag{G.24}
\]

\[
|\rho_{m'}^{c(n')} \rho_{m'}^{c(n)}| \alpha e^{j2\beta_m \rho_m \gamma y} \left(-\frac{1}{r_{m'}^{y}} - 1\right) \bigg| k, n \text{ large} \tag{G.25}
\]

Again, taking the limit of the \([k,n]^{th}\) term as \( k \) and \( n \) approach infinity, we obtain the following result.

\[
\lim_{k \to \infty, n \to \infty} |Z_{\text{SELF}}| < \text{constant} \cdot \lim_{k \to \infty, n \to \infty} e^{-j2\beta_m \rho_m \gamma y}
\]

\[
\{e^{j2\beta_m \rho_m \gamma y} \left[\frac{r_{m'}^{z}}{r_{m'}^{z} (r_{m'}^{z} + r_{m'}^{z})} - 1 - \frac{1}{r_{m'}^{y}}\right]\} + 0 \tag{G.26}
\]

Hence, the \( Z_{\text{SELF}} \) contribution converges.
An important comment is in order here. What I have shown for the four bounce path contributions is that the \( \{k,n\} \)th term of the double summation decays to zero for large values of \( \{k,n\} \). This does not "prove" that the double sum converges. However, we do know that the series

\[
S(k,n) = \sum_{k=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{k^2 + n^2} \tag{G.27}
\]

converges. By the comparison test, we note that when \( \sigma > 0 \) and \( \{k,n\} \) are large,

\[
|a_{\{k,n\}}| \propto e^{-\sigma \sqrt{k^2 + n^2}} < \frac{1}{k^2 + n^2} \tag{G.28}
\]

Since the \( a_{\{k,n\}} \)th term is eventually less than \( 1/(k^2+n^2) \), the comparison test guarantees that the double summation will converge.

The final term to inspect is the expression for \( Z_{\text{SELFA}} \) in Equation (3.90). First, consider the perpendicular polarization.

\[
|\begin{pmatrix} c(n') \\ c(n) \end{pmatrix} |_{1P_{1,m'+1} - 1P_{1,m'+1}} \propto \frac{2}{r_{m'} z} \frac{r_{m'} z}{r_{m'}^2 x (r_{m'}^2 x + r_{m'}^2 z)} \bigg|_{k,n \text{ large}} \tag{G.29}
\]

\[
|\begin{pmatrix} c(n') \\ c(n) \end{pmatrix} |_{1P_{3,m'} - 1P_{3,m'} \prime} \propto e^{-j \beta_{m'} \omega_{m'} y} \frac{2}{r_{m'} z} \frac{r_{m'} z}{r_{m'}^2 x (r_{m'}^2 x + r_{m'}^2 z)} \bigg|_{k,n \text{ large}} \tag{G.30}
\]
\[ |I_{3,m'}^n - I_{3,m'}^n| \propto e^{-j2\beta_{m'}y} \frac{r_{m'}z}{r_{m'}^2 (r_{m'}^2 + r_{m'}^2)} \] \quad k,n \text{ large} \tag{G.31}

\[ |I_{3,m'}^n + I_{3,m'}^n| \propto e^{-j2\beta_{m'}y} \frac{r_{m'}z}{r_{m'}^2 (r_{m'}^2 + r_{m'}^2)} \] \quad k,n \text{ large} \tag{G.32}

Since Equations (G.29) to (G.32) are also divided by \( r_{m'}y \), clearly these series when summed individually will converge at least as fast as \( 1/(k^2+n^2)^{3/2} \).

Working on the parallel pattern factor components is much more difficult. There are twelve pattern factor products involving parallel components in Equation (3.90). In addition to these 12 products are quantities involving exponentials, sines, and cosines; where do they come from? From Chapter III we found that the path-A impedance consisted of contributions from every side of the reference loop reacting with every other side of the test loop. Since the loop has four sides, this creates \( 4^2 = 16 \) possible product combinations for the parallel polarization case. Since we have already identified 12 such contributions, the remaining four are buried in the nested integration of Equation (3.78). Evaluating this double integral yields the additional quantities in Equation (3.90).
The central question still remains; does the parallel component of the path-A self impedance converge? Looking at Equation (3.90), we do see one potential non-converging contribution. It is given by

\[ Z_{A, \text{divergent}} = C \sum_{k,n} \frac{e^{-j\beta_m'w r_m'z}}{r_m'y} \{ \sin^2 \beta_m' b r_m'x \left( \frac{1}{1 - r_m'y^2} \right) \} , \]

where \( C \) is a constant independent of the variables of summation. In Equation (G.33) one term varies as \( 1/r_m'y \), which does not converge in \( \{k,n\} \) space. Does this mean the entire expression diverges? The answer to this question is "no", for the following reason. Each of the 12 product terms involving parallel pattern factors are similar to the following term.

\[ a_{k,n} = C \frac{e^{-j\beta_m'w r_m'z}}{r_m'y} \sin^2 \beta_m' b r_m'x \]

Although the details are omitted, these 12 parallel polarization product terms combine with Equation (G.33) such that all \( 1/r_m'y \) terms cancel, leaving terms which converge at least as fast as \( 1/(k^2+n^2)^{3/2} \).

In conclusion, this appendix demonstrated that the double summation implied in Equation (3.74) absolutely converges in \( \{k,n\} \) space.
APPENDIX H

FLOW CHART OF COMPUTER CALCULATIONS

This appendix presents an abbreviated flow chart of the computer program NP-LOOP used to perform the calculations specified in the main body of the dissertation.

In the flow chart to follow, I have used Figure H.1 to represent functional blocks of the code. The flow chart consists of four pages, as shown in Figure H.2.
Figure H.1. Functional blocks used in the flow chart for program NP-LOOP.
Figure H.2. Flow chart of computer program NP-LOOP.
Figure H.2. (Continued).
Figure H.2. (Continued).
Figure H.2. (Continued).
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