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DISTANCE-REGULAR GRAPHS AND HALVED GRAPHS

The Ohio State University Ph.D. 1984

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DISTANCE-REGULAR GRAPHS AND HALVED GRAPHS

DISSertation

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By

Joseph Hemmeter, B.S., M.S.

* * * * *

The Ohio State University
1984

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PUBLICATIONS


FIELDS OF STUDY

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Let $G$ be a simple, connected, finite graph with vertex set $V(G)$. If $x \in V(G)$, then $G_i(x)$ will denote the set of vertices at distance $i$ from $x$. $G_i(x)$ is also written $G(x)$.

Now fix $x \in V(G)$, $y \in G_i(x)$. Define

$$a_i = |G_i(x) \cap G_i(y)|,$$

$$b_i = |G_{i+1}(x) \cap G_i(y)|$$

and

$$c_i = |G_{i-1}(x) \cap G_i(y)|.$$

If $a_i$, $b_i$ and $c_i$ never depend on $x$ and $y$, but only on $i$, then $G$ is called a distance-regular graph. Note that, if $i = 0$, then $x = y$ and $b_0 = |G_1(y)|$. So a distance-regular graph is regular.

An important property of distance-regular graphs is that, if $x, y \in V(G)$ with $x \in G_i(y)$, then $|G_j(x) \cap G_k(y)|$ depends only on $i, j$ and $k$ (see [1]).
Suppose $G$ is a bipartite distance-regular graph with bipartition $V(G) = X \cup Y$. We define a graph $G'$ as follows. Let $V(G') = X$ and, for $x_1, x_2 \in X$, let $x_1 \in G'(x_2)$ if and only if $x_1 \in G_2(x_2)$. $G'$ is called the halved graph of $G$.

Let $x_1, x_2 \in V(G')$ with $x_1 \in G'_1(x_2)$. Then

$$|G'_1(x_1) \cap G'_2(x_2)| = |G_1(x_1) \cap G_2(x_2)|.$$ 

Since these numbers do not depend on the choice of $x_1$ and $x_2$, $G'$ is itself distance-regular.

The questions I address in this dissertation are the following.

Given a distance-regular graph $H$, is there a bipartite distance-regular graph $G$ such that $G' = H$. If there is, what is $G$?

There are about 22 known families of large-diameter distance-regular graphs (see [1], [2]). For all but one of these, I am able to answer these questions. The results can be summarized by saying that I found no new bipartite distance-regular graphs.

Why study halved graphs? One of the major open problems in this area of combinatorics is the classification of all large-diameter distance-regular graphs. Since halving a graph is one of the two ways known to get one large-diameter distance-regular graph from another (the other way is folding graphs: See Section 2.2), it is thus a natural process to study. Also, if you know $G'$, you may be able to say something about $G$. See, for example, Corollary 2.3. Finally, bipartite distance-regular graphs and their halved graphs are of
intrinsic interest. For example, they in some sense generalize symmetric designs (Lemma 1.5).

Chapter I gives some basic facts about distance-regular graphs and halved graphs. Lemmas 1.2 and 1.3 are the crucial ones. In the succeeding chapters, each of the families of large-diameter distance-regular graphs is considered, and the question of whether it is a halved graph addressed.

The basic technique consists of characterizing the maximal cliques of the family under consideration, and then using Lemma 1.3 and parametric constraints to put conditions on any bipartite distance-regular graph whose halved graph is in the family. The most difficult part of the process usually is characterizing the maximal cliques. This was the sticking point in the graph of quadratic forms, the case I was unable to finish.
Throughout the dissertation, unless otherwise stated, every graph will be simple, finite and connected. $G$ will refer to a bipartite distance-regular graph with bipartition $V(G) = X \cup Y$. $G'$ will refer to its halved graph with vertex set $X$. $a_i$, $b_i$ and $c_i$ will be the parameters of $G$, with $k = b_0$ being the valency. The corresponding parameters of $G'$ will be $a_i'$, $b_i'$, $c_i'$ and $k'$.

Lemma 1.1. Let $G$ be a distance-regular graph.

(i) For $1 < i < \text{diameter of } G$, $a_i + b_i + c_i = k$.

(ii) If $G$ is also bipartite, then $a_i = 0$, so $b_i + c_i = k$.

Proof. Let $x, y \in V(G)$ with $x \in G_i(y)$.

(i) If $z \in G(y)$, then the distance between $z$ and $x$ must be $i - 1$, $i$ or $i + 1$. So $a_i + b_i + c_i = |r_i(x) \cap r_i(y)| + |r_{i+1}(x) \cap r_i(y)| + |r_{i-1}(x) \cap r_i(y)| = |r_i(y)| = k$. 

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(ii) Since $G$ is bipartite, it has no odd circuits. But if $z \in r_i(x) \cap r(y)$, we have a circuit of length $2i + 1$. (See Figure 1.) So $a_i = |r_i(x) \cap r(y)| = 0$.

![Figure 1. An odd circuit](image)

Two other useful facts about distance-regular graphs, without proof, are:

1. If $1 < i < j$, then $1 < c_i < c_j$ (see [1]);

2. If $c_3 = c_2$, then $c_3 = c_2 = 1$ (see [5]).

The most important tools for dealing with halved graphs are Lemmas 1.2 and 1.3.

**Lemma 1.2.** If $G$ is a bipartite distance-regular graph, then

(i) $b_i^{'} = b_{2i}^{'}b_{2i+1}^{'} / c_2$;

(ii) $c_j^{'} = c_{2j}^{'}c_{2j-1}^{'} / c_2$. 
Proof. (i) Let \( x_1, x_2 \in V(G') \), \( x_1 \in G_i(x_2) \). Count the number of pairs \((v, z)\) with \( v \in G_1(x_2) \cap G_{2i+1}(x) \) and \( z \in G(v) \cap G_1(x_2) \). (See Figure 2.) There are \( b_{2i} \) such \( v \)'s, each of which has \( b_{2i+1} \) \( z \)'s. This gives \( b_{2i} b_{2i+1} \) pairs.

![Figure 2. Diagram for Lemma 1.2. (i)](image)

Now suppose we have a pair \((v_0, z_0)\) satisfying the above conditions. Since \( v_0 \in G(x_2) \), the distance between \( z_0 \) and \( x_2 \) is less than or equal to 2. But it cannot be less than 2, since \( x \in G_1(x_2) \) and \( x \in G_1(z) \). So \( z \in G_1(x_2) \cap G_1(z) \).

So pick an arbitrary \( z \in G_1(x_2) \cap G_1(z) = G_i'(x_2) \cap G_{i+1}(x) \). There are \( b_{i+1} \) such \( z \)'s, since \( x_1 \in G_i'(x_2) \).

For each, it is possible to pick \( c_{2i} \) \( v \)'s such that \((v, z)\) works.

Hence \( b_{2i} c_{2i+1} = b_{2i} b_{2i+1} \) and we have (i).
(ii) Let $x_1, x_2$ be as above, and count the pairs $(v, z)$ with $v \in G_1(x_1) \cap G_{2j-1}(x_1)$, $z \in G_1(v) \cap G_{2j-2}(x_1)$ (see Figure 3).

Counting $v$'s first yields $c_{2j} c_{2j-1}$.

![Figure 3. Diagram for Lemma 1.2. (ii)](image)

But if $(v_0, z_0)$ is such a pair, then an argument exactly analogous to that of (i) gives $z_0 \in G_2(x_2)$. Thus there are

$$|G_{2j-2}(x_1) \cap G_2(x_2)| = |G'_j(x_1) \cap G'_j(x_2)| = c'_j$$

candidates for $z$. Each of them has $c_j$ $v$'s. So $c'_j c_j = c_j^2 c_{2j-1}$, which gives (ii).

**Lemma 1.3.** Assume that $G'$ is not a complete graph. Then for every $y \in Y$, $G(y)$ is a maximal clique in $G'$. If $y_1, y_2 \in Y$ are distinct, then $G(y_1) \neq G(y_2)$. 
Proof. Let \( y \in Y \), \( x_1, x_2 \in G(y) \). Then \( x_1 \in G_2(x_2) = G_1(x_2) \). So \( G(y) \) is a clique in \( G' \). Suppose it is not maximal. That is, there is some \( x \in X - G(y) \) which is adjacent in \( G' \) to every vertex of \( G(y) \). Then \( x \in G_3(y) \) and \( c_3 = |G(y) \cap G_2(x)| = |G(y)| = k \). So by Lemma 1 (ii), \( b_3 = k - c_3 = 0 \). But this implies that the diameter of \( G \) is at most 3. To see why this is true, suppose the contrary and let \( x_1, x_2 \in V(G) \) with \( x_1 \in G_4(x_2) \). Then there must be a vertex \( x_3 \in G_1(x_1) \cap G_3(x_2) \). So \( x_1 \in G_4(x_2) \cap G_1(x_3) \). Hence \( b_3 = |G_4(x_2) \cap G_1(x_3)| > 0 \), a contradiction.

But if the diameter of \( G \) is less than or equal to 3, then \( G' \) is a complete graph, contradicting our original assumption. So \( G(y) \) is indeed a maximal clique.

Now suppose that \( y_1, y_2 \in Y \) are distinct, and that \( G(y_1) = G(y_2) \). Then \( y_1 \in G_2(y_2) \) and \( c_2 = |G(y_1) \cap G(y_2)| = k \). Hence \( b_2 = k - c_2 = 0 \), and the above arguments apply a fortiori.

Corollary 1.4. Let \( G \) be a bipartite distance-regular graph with more than 4 vertices. If \( a_1' = 0 \), then \( G \) is a polygon.

Proof. Since \( a_1' = 0 \), a maximal clique in \( G' \) has size 2 or less. Since \( G \) has more than 4 vertices, \( G' \) has more than 2 and cannot be a complete graph. So Lemma 1.3 applies. That means that for
any \( y \in Y \), \(|G(y)| = 2\). Since \( G \) is connected, it must then be a polygon.

**Remark.** Let \( n \) be a positive integer, and \( P_i \), for \( i = 0,1,2,\ldots \), be the polygon with \( n2^i \) vertices. Then for \( i > 1 \), \( P_i = P_{i-1} \). A simple consequence of Corollary 1.4 is that this is the only chain of halved graphs with more than two graphs. For suppose that \( G_3 = G_2 \) and \( G_2 = G_1 \). Since \( G_2 \) is bipartite, its \( a_1 = \emptyset \) so \( G_3 \) is a polygon.

What happens when \( G^i \) is a complete graph?

**Lemma 1.5.** If \( G^i \) is a complete graph \( K \), then \( G \) corresponds to the incidence structure of a \( 2-(n,k,c_2) \)-symmetric design. Conversely, if there exists a symmetric design with \( n \) points, then \( K \) is a halved graph.

**Proof.** Assume that \( K = G^i \) and define an incidence structure with point set \( X \), block set \( \{G(y)|y \in Y\} \), and the obvious incidence. Because \( G \) is regular, each point lies on \( k \) blocks and each block has \( k \) points. Since \( G^i \) is complete, every pair of points corresponds to a pair of vertices of \( X \), say \( x_1, x_2 \), such that \( x_1 \in G(x_2) \). Then \( |G(x_1) \cap G(x_2)| = c_2 \). That is \( x_1 \) and \( x_2 \)
are in \( c_2 \) common blocks, and the incidence structure is a design. It is symmetric since \(|X| = |Y|\).

Now assume that we have a 2-(n,k,\( \lambda \))-symmetric design \( D \). Let \( \{ P : x \in X \} \) be the set of points, \( \{ B : y \in Y \} \) the set of blocks of \( D \). Then \(|X| = |Y| = n\) of course. Define a bipartite graph \( G \) with \( V(G) = X \cup Y \) and, for \( x \in X \), \( y \in Y \), with \( x \in G(y) \) if and only if \( P \) is incident to \( B \). The claim is that \( G \) is a distance-
regular graph. \( G \) is obviously connected, so we need check only \( a_i, b_i \) and \( c_i \) for \( i \) up to the diameter of \( G \).

Clearly \( b = k \), and since \( G \) is bipartite, all the \( a_i \)'s are 0. So we have to check only the \( c_i \)'s, and use Lemma 1 (ii). \( c_1 = 1 \) for any graph. Since \( D \) is a symmetric design, \( c_2 = \lambda \). Now since every two points lie in \( \lambda \) common blocks, and every two blocks intersect in \( \lambda \) points, we know that the diameter of \( G \) is at most 3. So pick, if possible, 2 vertices at distance 3 from each other. One, say \( x \), must be in \( X \), and one, \( y \), in \( Y \). We need to check that \(|G(x) \cap G(y)| = |G(y) \cap G(x)| \) is a constant \( c_3 \). But since \( x \) and \( y \) are not adjacent, \( G(x) \subseteq G_2(y) \) and \( G(y) \subseteq G_2(x) \). So \( c_3 = k \) and we are done.
CHAPTER II
The Johnson and Related Graphs

2.1. The Johnson Graph.

The Johnson graph $J(n,d)$ has as vertices the $d$-subsets of a fixed set $S$ of cardinality $n$, where $n > 2d$. Two vertices $x_1$ and $x_2$ are adjacent if $|x_1 \cap x_2| = d - 1$. $J(n,d)$ is distance-regular.

$2U_d$ is the bipartite graph whose vertices are the $d$-subsets and the $(d + 1)$-subsets of a fixed $(2d + 1)$-set, with adjacency by inclusion. It too is distance-regular. Its halved graph is $J(2d + 1,d)$.

Lemma 2.1. There are two types of maximal clique in $J(n,d)$. Type 1 consists of the $d$-subsets of a fixed $(d + 1)$-subset of $S$. Type 2 consists of the $d$-sets containing a fixed $(d - 1)$-subset of $S$.

Proof. Suppose there is a maximal clique $C$ that is not of Type 1, and $x_1, x_2 \in C$. Since $x_1$ and $x_2$ are adjacent, $|x_1 \cup x_2| = d + 1$. So there must be some $x_3 \in C$ such that $x_3 \notin x_1 \cup x_2$. That means that at most $d - 1$ elements of $x_3$ are in $x_1 \cup x_2$. 
Now let \( x_4 \in C \). I claim that \( x_4 \supsetneq x_1 \cap x_2 \). Suppose not. Since \(|x_4 \cap x_1| = d - 1\), \(|x_4 \cap x_1 \cap x_2| > d - 2\) and \(x_1 - (x_1 \cap x_2) \subseteq x_4\). Likewise \(x_2 - (x_1 \cap x_2) \subseteq x_4\) and \(x_3 - (x_1 \cap x_2) \subseteq x_4\). So far we have \(d + 1\) elements in \(x_4\), but \(|x_4| = d\). So the claim is established, and every vertex in \(C\) contains \(x_1 \cap x_2\). Thus \(C\) is of Type 2.

**Theorem 2.2.** Let \(G\) be a bipartite distance-regular graph whose halved graph is the Johnson graph \(J(n,d)\), \(d > 2\). Then \(n = 2d + 1\) and \(G\) is isomorphic to \(2U_d\).

**Proof.** For \(G' = J(n,d)\), \(b_i = (d - i)(n - d - i)\) for \(0 < i < d - 1\), and \(c'_j = j^2\) for \(1 < j < d\). (See [3], also [2].)

Lemma 1.2 (ii), with \(j = 2\), now gives \(c_4 = 4c_2/c_3\). In view of (1) then, \(c_4 < 4\). In fact the only possibilities are: \(c_4 = 4\) and \(c_2 = c_3 = w\) for \(w = 1, 2, 3, 4\); or \(c_4 = 2 = c_3\) and \(c_2 = 1\).

Case 1: \(c_4 = 4\), \(c_2 = c_3 = w\). By (2), \(w = 1\).

Subcase 1a: \(d > 3\). Lemma 1.2 (ii), with \(j = 3\), yields \(9 = c_5 c_6\). But \(c_6 > c_5 > c_4 > 4\), a contradiction.
Subcase 1b: $d = 2$. We know that $b_0 = k$ and $c_1 = 1$, in general. By Lemma 1.1 (ii), $b_1 = k - c_1 = k - 1$. Using Lemma 1.2 (i) with $i = 0$, we get $b_1' = \frac{b_0 b_1}{c_2} = k(k - 1)$. The same lemma, with $i = 1$, gives $b_1' = b_2 b_3 = (k - c_2)(k - c_3)$. Plugging in the values for $b_0', b_1', c_2, c_3$ and $d$, the last two equations become $2(n - 2) = k(k - 1)$ and $n - 3 = (k - 1)^2$. These together give $k = n$. Plugging this in, we get $n - 3 = (n - 1)^2$, which has no real solutions.

So Case 1 is impossible.

Case 2: $c_4 = 2 = c_3$ and $c_2 = 1$. Using Lemma 1.2 (i) with $i = 0$, we find that $d(n - d) = b_1' = \frac{b_0 b_1}{c_2} = k(k - 1)$. With $i = 1$, the same lemma yields $(d - 1)(n - d - 1) = b_1' = \frac{b_2 b_3}{c_2} = \frac{(k - c_2)(k - c_3)}{c_2} = (k - 1)(k - 2)$. Subtracting the 2nd equation from the first gives $n - 1 = 2k - 2$ or $k = \frac{n + 1}{2}$. Plugging this into $d(n - d) = k(k - 1)$ and solving for $n$, we get $n = 2d + 1$. By assumption $n > 2d$, so $n = 2d + 1$. Note that $k = \frac{n + 1}{2} = d + 1$. 
Now we need to see that \( G \) is isomorphic to \( 2U_d \). By Lemma 1.3, \( k \) must be the size of one of the maximal cliques of \( J(n,d) \). By Lemma 2.1 then, \( k = d + 1 \) or \( k = n - (d - 1) = (2d + 1) - (d - 1) = d + 2 \). But we know that \( k = d + 1 \), so only a clique of Type 1 can appear as \( G(y) \) for any \( y \in Y \).

We have then \( V(G) = X \cup Y \) with \( X = V(G') = V(J(n,d)) \), the elements of \( Y \) corresponding to some \((d + 1)\)-subsets of \( S \), and adjacency by inclusion. To conclude that \( G \) is isomorphic to \( 2U_d \) we need only that all \((d + 1)\)-subsets of \( S \) are represented in \( Y \). That is, that \( |Y| = \binom{2d + 1}{d + 1} \). But, since \( G \) is regular and bipartite,

\[
|Y| = |X| = \binom{2d + 1}{d} = \binom{2d + 1}{d + 1}.
\]

Corollary 2.3. Let \( G \) be a distance-regular graph with the same parameters \( a, b \) and \( c \), as \( 2U_d \). Then \( G \) is isomorphic to \( 2U_d \).

Proof. Lemma 1.2 shows that \( G' \) has the same parameters as \( J(2d + 1,d) \). In [8], Moon showed that no other graph has these parameters. So \( G' \) must be isomorphic to \( J(2d + 1,d) \). Then Theorem 2.2 applies, and \( G \) must be isomorphic to \( 2U_d \).
Remark. The arguments of Theorem 2.2, for \( n \neq 2d + 1 \), depend only on the parameters of \( J(n,d) \). So it is possible to say that no distance-regular graph with the same parameters as \( J(n,d) \), \( n \neq 2d + 1 \), can be a halved graph. However, Terwilliger \([9]\) has recently proved that no other graph with the parameters of \( J(n,d) \) exists.

2.2. Folded Graphs, \( \overline{J}(2d,d) \).

Let \( H \) be a distance-regular graph of diameter \( D \). For \( x, y \in V(H) \), say that \( x \) is opposite to \( y \) if \( x = y \) or \( x \in H(y) \). If being opposite is an equivalence relation, \( H \) is said to be antipodal. In this case we can define a graph \( \overline{H} \) whose vertices are the equivalence classes of \( G \). If \( \overline{x,\overline{y}} \in V(\overline{H}) \), we say that \( \overline{x} \in \overline{H}(\overline{y}) \) if, for some \( x \in \overline{x} \) and \( y \in \overline{y} \), \( x \in H(y) \). \( \overline{H} \) is called the folded graph of \( H \). It is distance-regular.

From \([6]\) we get the following facts. If \( D \) is even, the diameter of \( \overline{H} \) is \( D/2 \). If \( D \) is odd, the diameter of \( \overline{H} \) is \( (D-1)/2 \). For all \( 0 \leq i < \text{the diameter of } \overline{H} \), the parameters \( a_i \), \( b_i \) and \( c_i \) are the same for \( H \) and \( \overline{H} \).

\( J(2d,d) \) is antipodal, with each equivalence class having two elements, disjoint \( d \)-subsets of \( S \) whose union is \( S \).
Lemma 2.4. If $d > 3$, every maximal clique $C$ of $\overline{J}(2d,d)$ looks like
\[ \{x_i | 1 < i < |C|\}, \]
where $\overline{x}_i = \{x_i, y_i\}$ with $x_i, y_i \in V(J(2d,d))$, and
\[ \{x_i | 1 < i < |C|\} \] is a maximal clique of $J(2d,d)$.

Proof. Let $\overline{x}_1, \overline{x}_2$ be adjacent vertices of $\overline{J}(2d,d)$ with
$\overline{x}_1 = \{x_1, y_1\}$. Then either $x_1$ or $y_1$ is adjacent in $J(2d,d)$ to
something in $\overline{x}_2$. Suppose $|y_1 \cap y_2| = d - 1$. Then $|x_1 \cap x_2| = d - 1$. So $x_1$ is always adjacent (in $J(2d,d)$) to something in $x_2$.

Now let $C = \{\overline{x}_i | 1 < i < |C|\}$. By the above remarks, we can write
$\overline{x}_i = \{x_i, y_i\}$ with $|x_1 \cap x_i| = d - 1$ for all $i > 1$. For distinct
$i, j \neq 1, |x_i \cap x_j| = 1 \text{ or } d - 1$. But $|x_i \cap x_j \cap x_1| > d - 2 > 1$
since $d > 3$. This means that $|x_i \cap x_j| = d - 1$ for all $i \neq j$.
Hence $\{x_i | 1 < i < |C|\}$ is a clique in $J(2d,d)$.

Theorem 2.5. Let $G$ be a bipartite distance-regular graph whose
halved graph is $\overline{J}(2d,d)$. Then $d < 3$ and $\overline{J}(2d,d)$ is a complete
graph.

Proof. Assume $d > 3$. The size of the maximal cliques given in
Lemma 2.4 is $d + 1$. By Lemma 1.3 then, $k = d + 1$. Now $c_1 = 1,$
$b_1 = k - c_1 = k - 1$ and $k' = d^2$. So by Lemma 1.2 with $i = 0$,
\[ d^2 = k' = \frac{b}{c_2} = \frac{k(k-1)}{c_2} = \frac{(d+1)d}{c_2} \quad \text{and} \quad c_2 = \frac{(d+1)d}{d^2}. \]

But this is an integer only when \( d = 1 \).

So \( d < 3 \). In this case, the facts cited before Lemma 2.4 tell us that \( \overline{J}(2d,d) \) is a complete graph.

### 2.3. The Odd Graph.

The odd graph \( U_d \) has the same vertices as \( J(2d + 1,d) \). But two vertices \( x_1, x_2 \) are adjacent in \( U_d \) if \( x_1 \cap x_2 = \emptyset \).

**Theorem 2.6.** Let \( G \) be a bipartite distance-regular graph whose halved graph is \( U_d \). Then \( d = 1 \).

**Proof.** Clearly \( a'_1 = 0 \). If \( d > 1 \), then \( U \) has \( \binom{2d + 1}{d} > 2 \) vertices, so Corollary 1.4 applies. That is, \( G \) is a polygon. But the halved graph of a bipartite polygon is a polygon. Hence \( k' \) must be two. But \( k' = d + 1 \), so \( d \) must be 1.
2.4. The q-analogue of the Johnson Graph.

Let $V$ be an $n$-dimensional vector space over $GF(q)$, and $d < \frac{n}{2}$. The q-analogue of the Johnson graph, $J_q(n,d)$, has as vertices the $d$-dimensional subspaces of $V$. Two subspaces are adjacent if their intersection has dimension $d - 1$. $J_q(n,d)$ is distance-regular.

$2 \circ_d^q$ is the bipartite graph whose vertices are the $d$-dimensional subspaces and the $(d + 1)$-dimensional subspaces of a $(2d + 1)$-dimensional vector space over $GF(q)$, with adjacency by inclusion. $2 \circ_d^q$ is distance-regular, and its halved graph is $J_q(2d + 1,d)$.

Lemma 2.7. There are two types of maximal clique in $J_q(n,d)$. Type 1 consists of all $d$-subspaces of a fixed $(d + 1)$-subspace of $V$. Type 2 consists of all $d$-subspaces of $V$ containing a fixed $(d - 1)$-subspace.

Proof. Let $C$ be a maximal clique of $J_q(n,d)$, and suppose that no $(d + 1)$-space contains every vertex of $C$. Let $x_1, x_2 \in C$. We want to show that every vertex of $C$ contains $x_1 \cap x_2$, making $C$ a clique of Type 2. $\dim (x_1 \cap x_2) = d - 1$, so $\dim (\langle x_1, x_2 \rangle) = d + 1$, where $\langle x_1, x_2 \rangle$ is the span of $x_1$ and $x_2$. Since $C$ is not of Type 1, there is an $x_3 \in C$ with $x_3 \notin \langle x_1, x_2 \rangle$. The claim is that $x_1 \cap x_2 \subset x_3$. If not, since
there is some \( u \in (x_3 \in \langle x_1, x_2 \rangle) - (x_1 \cap x_2) \). Also we can pick \( v \in \langle x_3 - x_1, x_2 \rangle \).

Now either \( u \notin x_1 \) or \( u \notin x_2 \). Without loss of generality, say \( u \notin x_2 \). Then \( \dim (\langle x_2 \cap x_3, u \rangle) = \dim (x_2 \cap x_3) + 1 = d \). Since \( \langle x_2 \cap x_3, u \rangle \subseteq \langle x_1, x_2 \rangle \) and \( v \notin \langle x_1, x_2 \rangle \),
\[
\dim (\langle x_2 \cap x_3, u, v \rangle) = d + 1.
\]
But \( \langle x_2 \cap x_3, u, v \rangle \subseteq x_3 \) and \( \dim (x_3) = d \), a contradiction.

Hence if \( x_3 \notin \langle x_1, x_2 \rangle \), then \( x_1 \cap x_2 \subseteq x_3 \). Suppose \( x_4 \in C \) and \( x_4 \subseteq \langle x_1, x_2 \rangle \). Then \( x_4 \cap x_3 \), \( \langle x_1, x_2 \rangle \cap x_3 = x_1 \cap x_2 \). Since \( \dim (x_4 \cap x_3) = d - 1 = \dim (x_1 \cap x_2) \), we have \( x_1 \cap x_2 \subseteq x_4 \), and so \( C \) is a clique of Type 1.

**Theorem 2.8.** Let \( G \) be a bipartite distance-regular graph whose halved graph is \( J(n,d) \), and \( d > 1 \). Then \( n = 2d + 1 \) and \( G \) is isomorphic to \( 2^{\alpha} \).

**Proof.** In general, the number of \( r \)-subspaces of an \( n \)-space is
\[
\begin{bmatrix} n \\ r \end{bmatrix} = \pi (q^n - q^i)/(q^{r-i} - q^i). \quad \text{These are thus} \begin{bmatrix} n \\ d+1 \end{bmatrix}_q \quad \text{and} \quad \begin{bmatrix} n \\ d-1 \end{bmatrix}_q
\]
maximal cliques of Types 1 and 2, respectively. A clique of
Type 1 has size \[
\begin{bmatrix}
\frac{d+1}{d}
\end{bmatrix}_q.
\]
Suppose \(C\) is a maximal clique of Type 2 corresponding to \((d - 1)\)-subspace \(W\). What is \(|C|\)?

Every element of \(C\) looks like \(<v,W>\) where \(v \in V - W\). There are \(q^n - q^{d-1}\) choices for \(v\). For a fixed \(x \in C\), there are \(q^d - q^{d-1}\) \(v's\) such that \(x = <v,W>\). So \(|C| = (q^n - q^{d-1})/(q^d - q^{d-1})\).

Now suppose that, for every \(y \in Y\), \(G(y)\) is a clique of Type 2. Then

\[
\begin{bmatrix}
\frac{n}{d-1}
\end{bmatrix}_q > |Y| = |X| = \left[\begin{bmatrix}
\frac{n}{d}
\end{bmatrix}_q
\right].
\]

But \(\left[\begin{bmatrix}
\frac{n}{d-1}
\end{bmatrix}_q = \left[\begin{bmatrix}
\frac{n}{d}
\end{bmatrix}_q \left(\frac{\frac{\frac{d}{2}}{\frac{d-1}{2}}}{q - q^{d-1}}\right)^\frac{\frac{n}{d}}{q - q^{d-1}}\right] = \left[\begin{bmatrix}
\frac{n}{d}
\end{bmatrix}_q
\right]\), since \(n > 2d\). So not every \(G(y)\) can be of Type 2.

Suppose that, for some \(y_1, y_2 \in Y\), \(G(y_1)\) is of Type 1 and \(G(y_2)\) is of Type 2. Then

\[
\frac{q^{d+1} - 1}{q - 1} = \left[\begin{bmatrix}
\frac{d+1}{d}
\end{bmatrix}_q = \left[\begin{bmatrix}
\frac{d+1}{d}
\end{bmatrix}_q = |G(y_1)| = k = \left[\begin{bmatrix}
\frac{n}{d}
\end{bmatrix}_q
\right]\right.
\]

\[
|G(y_2)| = \frac{q^n - q^{d-1}}{q^d - q^{d-1}} = \frac{n^{d+1} - 1}{q - 1}.
\]
So \( n = 2d \). In this case, \( k = \frac{q^{d+1} - 1}{q - 1} \) and \( k' = q^{\binom{d}{1}} \) (see [2]). So Lemma 1.2 (i), with \( i = 0 \), becomes

\[
q^{\binom{d}{1}} = k' = \frac{b_0 b_1}{c_2} = \left( \frac{q^{d+1} - 1}{q - 1} \right) \left( \frac{q^{d+1} - q}{q - 1} \right) / c_2.
\]

This gives \( c_2 = (q^{d+1} - 1)/(q^d - 1) \). But \( \frac{q^{d+1} - 1}{q - 1} = q + \frac{q - 1}{q - 1} \), which is an integer only when \( d = 1 \).

Thus every \( G(y) \) is of Type 1. To prove the theorem, there remain to show that every clique of Type 1 appears as \( G(y) \) for some \( y \), and that \( n = 2d + 1 \). Let \( W \) be an \((d + 1)\)-subspace of \( V \).

Pick \( x_1, x_2 \in W \) of dimension \( d \), \( x_1 \neq x_2 \). Then \( x_1 \in G'(x_2) \), so there is some \( y \in Y \) such that \( x_1, x_2 \in G(y) \). That is, the clique of Type 1 corresponding to \( W \) must be used. Since \( W \) was arbitrary, we have \( [n^d]_q = |X| = |Y| = \left[ \begin{array}{c} n \\ d + 1 \end{array} \right]_q \). Now

\[
\left[ \begin{array}{c} n \\ d \end{array} \right]_q = \prod_{i=0}^{d-1} \frac{(q^n - q^i)/(q^d - q^i)}{q^{d-1}} = q^d \prod_{i=0}^{d-1} \frac{(q^n - q^i)/(q^{d+1} - q^{i+1})}{q^{d-1}}
\]

\[
= q(q^{d+1} - 1)/(q^n - q^d) \prod_{i=0}^{d} \frac{(q^n - q^i)/(q^{d+1} - q^i)}{q^{d-1}}
\]
\[ \frac{d}{d+1} W, n = q (q - i)(q - q) \]

So \( n = 2d + 1 \) and \( G \) is isomorphic to \( 2^q \).
3.1. The Dual Polar Spaces.

There are six dual polar spaces, arising from nondegenerate forms on a vector space $V$ over $GF(q)$.

<table>
<thead>
<tr>
<th>Name</th>
<th>dim $(V)$</th>
<th>Form</th>
<th>$e$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B_d(q)$</td>
<td>$2d + 1$</td>
<td>quadratic</td>
<td>0</td>
</tr>
<tr>
<td>$C_d(q)$</td>
<td>$2d$</td>
<td>symplectic</td>
<td>0</td>
</tr>
<tr>
<td>$D_d(q)$</td>
<td>$2d$</td>
<td>quadratic, Witt index $d$</td>
<td>-1</td>
</tr>
<tr>
<td>$^2D_{d+1}(q)$</td>
<td>$2d + 2$</td>
<td>quadratic, Witt index $d$</td>
<td>1</td>
</tr>
<tr>
<td>$^2A_{2d}(q)$</td>
<td>$2d + 1$</td>
<td>hermitian ($q = r^2$)</td>
<td>$1/2$</td>
</tr>
<tr>
<td>$^2A_{2d-1}(q)$</td>
<td>$2d$</td>
<td>hermitian ($q = r^2$)</td>
<td>$-1/2$</td>
</tr>
</tbody>
</table>

An isotropic subspace of $V$ is one on which the form vanishes. The vertices of the dual polar space are the maximal isotropic subspaces, which have dimension $d$ in each case. Two vertices are of
distance $i$ if their intersection has dimension $d-i$.

General information on these forms may be found in [7].

Lemma 3.1. Let $C$ be a maximal clique of one of the dual polar spaces. Then $C$ consists of all maximal isotropic subspaces of $V$ containing a fixed $(d-1)$-dimensional isotropic subspace.

Proof. Each dual polar space is an induced subgraph of $J(n,d)$, where $n = \dim(V)$. By Lemma 2.6 then, the only other possible type of clique consists of the maximal isotropic subspaces of a fixed $(d+1)$-subspace of $V$.

Suppose $C$ is a clique of the latter sort. Let $x_1, x_2 \in C$. Then we can pick $u \in x_1$, $v \in x_2$ such that $(u,v) \neq 0$, where $(\ ,\ )$ is the bilinear form of the dual polar space (in the quadratic case, the bilinear form associated with the quadratic form). Define $A = \langle u \rangle^\perp \cap \langle v \rangle^\perp \cap \langle x_1, x_2 \rangle$. Then $\dim(A) = \dim(\langle u \rangle^\perp \cap \langle x_1, x_2 \rangle) + \dim(\langle v \rangle^\perp \cap \langle x_1, x_2 \rangle) - \dim(\langle x_1, x_2 \rangle) = \dim(x_1) + \dim(x_2) - \dim(\langle x_1, x_2 \rangle) = d + d - (d+1) = d - 1$. But $x_1 \cap x_2 \subseteq A$, so $A = x_1 \cap x_2$.

Now let $x_3 \in C$. $x_3 \subset \langle x_1, x_2 \rangle$ since we assumed that all the vertices of $C$ are in a $(d+1)$-space. So $x_3 \cap \langle u, v \rangle \neq \{0\}$. Pick $w \neq 0$, $w \in x_3 \cap \langle u, v \rangle$. Because $w \in \langle u, v \rangle$, $w \notin A$. But
A = \langle w \rangle^\perp$, and since $w$ is in an isotropic subspace, $(w, w) = 0$.

Thus $\langle w, A \rangle \subseteq \langle w \rangle^\perp \cap \langle x_1, x_2 \rangle$. But $\dim (\langle w, A \rangle) = d = \dim (\langle w \rangle^\perp \cap \langle x_1, x_2 \rangle)$, so $\langle w, A \rangle = \langle w \rangle^\perp \cap \langle x_1, x_2 \rangle$. In addition, $x_3 \subseteq \langle w \rangle^\perp \cap \langle x_1, x_2 \rangle$. Hence $x_3 = \langle w, A \rangle$. This means that $x_1 \cap x_2 = A \subseteq x_3$.

Since $x_3$ was arbitrary, every vertex of $C$ contains $A$, and the lemma is proved.

The number of isotropic $k$-spaces in $V$ is

$$
(4) \quad \left[ \begin{array}{c} d \\ k \end{array} \right] \pi^\frac{k - 1}{d - 1} (q^{d+e-i} + 1). \quad (\text{See [2].})
$$

**Theorem 3.2.** Let $G$ be a bipartite distance-regular graph whose halved graph is a dual polar space of diameter $d$. Then $d < 2$.

**Proof.** Assume $d > 1$. By (4), the number of maximal cliques available is $\left[ \begin{array}{c} d \\ d - 1 \end{array} \right] \pi^\frac{d - 2}{q^{d+e-i} + 1}$. We want to show that each of them must be used as $G(y)$ for some $y \in \mathcal{Y}$.

Let $A$ be a $(d - 1)$-dimensional isotropic subspace of $V$, and consider $A^\perp$. It is a space of Witt index 1. According to (4), we can find two isotropic 1-spaces, $\langle u \rangle$ and $\langle v \rangle$, in $A^\perp$. Since the Witt index is 1, $\langle u, v \rangle$ cannot be isotropic. Hence $(u, v) \neq 0$. 
Define $x_1 = \langle u, A \rangle$ and $x_2 = \langle v, A \rangle$. $x_1$ and $x_2$ are maximal isotropic subspaces of $V$, and $x_1 \neq x_2$ since $(u, v) \neq 0$. Of course $x_1$ and $x_2$ are of distance 1 in the dual polar space. Thus $x_1, x_2 \in G(y)$ for some $y \in Y$. But then $G(y)$ must be the maximal clique of $G'$ corresponding to $A$.

Since $A$ was arbitrary, $|Y| = \text{the number of maximal cliques of the dual polar space}$. So

$$
\begin{bmatrix}
  d \\
  d-1
\end{bmatrix}_{q}^{d-2} = (q^{d+e-1} + 1) = |Y| = |X| =
$$

$$
\begin{bmatrix}
  d \\
  d-1
\end{bmatrix}_{q}^{d-1} = (q^{d+e-1} + 1) = |Y| = |X| =
$$

Thus $\frac{q^d - 1}{q - 1} = q^{e+1} + 1$.

Assume that $e \in \{0, 1, -1\}$. Then $\frac{q^d - 1}{q - 1} = q^{d-1} + q^{d-2} + \ldots + q + 1 = q^{e+1} + 1$. This can happen only if $d = 2$ and $e = 0$. 
If $e \in \{\pm 1/2\}$, replace $q$ by $r$. Then $r^{2d-2} + r^{2d-4} + \ldots + r^2 + 1 = r^{2e+2} + 1$. Since $2e + 2$ is odd, this is impossible.

3.2. The Halved Graph of $D_d(q)$.

One of the dual polar spaces, $D_d(q)$, is bipartite. Is it possible that its halved graph, $D'_d(q)$, is also the halved graph of another bipartite distance-regular graph?

Lemma 3.3. Let $C$ be a maximal clique of $D'_d(q)$. Then either $C$ is the neighborhood of some vertex $y$ in $D_d(q)$, or $|C| < 2(q^2 + 1)(q + 1)$.

Proof. Two vertices of $D'_d$ are adjacent if their intersection has dimension $d - 2$. Pick distinct $x_1, x_2$ and $x_3$ in $C$ in such a way that $A = x_1 \cap x_2 \cap x_3$ has minimum dimension. Clearly $\dim A < d - 2$. Since $d + 2 < \dim (\langle x_1, x_2, x_3 \rangle) =$ \[ \dim (x_1) + \dim (x_2) + \dim (x_3) - \dim (x_1 \cap x_2) - \dim (x_1 \cap x_3) - \dim (x_2 \cap x_3) + \dim (A) = 6 + \dim (A), \] we have $\dim (A) > d - 4$. 

Case 1: \( \dim(A) = d - 2 \). Let \( x_4 \in C \). Then
\[
A = x_1 \cap x_2 \supset x_1 \cap x_2 \cap x_4.
\]
Since \( \dim(x_1 \cap x_2 \cap x_4) > d - 2 \),
\( A \subset x_4 \).

Now let \( A^\perp = A \perp B \). That is, \( B \) is orthogonal to \( A \) and \( A^\perp \) is the direct sum of \( A \) and \( B \). Since \( A \subset x_4 \) and \( x_4 \) is isotropic, \( x_4 \subset A^\perp \). So \( x_4 = A \perp D \) with \( D \subset A^\perp \), \( \dim(D) = 2 \).

Suppose \( u \in U \). Then \( u = a + b \) for some \( a \in A \), \( b \in B \). Since \( u \) and \( a \) are in \( x_4 \), \( b \) is also. So we may assume, without sacrificing generality, that \( D \subset B \). This means that every vertex of \( C \) looks like \( A \perp D \), where \( D \) is a 2-dimensional isotropic subspace of \( B \). But \( B \) is isomorphic to \( U_2(q) \), so by (4), \( |C| < 2(q + 1) \).

Case 2: \( \dim(A) = d - 4 \). Choose \( u_1 \in (x_1 \cap x_2) - A \).
\( u_1 \notin x_3 \), so we can pick \( u_1' \in x_3 \) such that \( \langle u_1', u_1 \rangle \neq 0 \). Let \( D_1 = \langle A, u_1, u_1' \rangle \).

In general, if \( U \subset V \), then \( \dim(U) + \dim(U^\perp) = \dim(V) \). So if \( \langle u_1 \rangle^\perp = \langle u_1 \rangle_\perp U \), then \( \dim(W = 2d - 2 = \dim\langle u_1, u_1' \rangle^\perp \). Since \( \langle u_1, u_1' \rangle^\perp \subset \langle u_1 \rangle^\perp \) and \( \langle u_1, u_1' \rangle \neq 0 \), \( \langle u_1' \rangle = \langle u_1 \rangle \perp U \). Thus, although \( u_1 \notin U_1^\perp \), \( \dim(x_1 \cap U_1^\perp) = d - 1 \). So if \( U_1^\perp = A \perp E_1 \), we can find \( F_1 \subset E_1 \) such that \( x_1 = \langle A, u_1 \rangle \perp F_1 \). Likewise \( x_2 = \langle A, u_1 \rangle \perp F_2 \) and \( x_3 = \langle A, u_1 \rangle \perp F_3 \), with \( F_2, F_3 \subset E_1 \).
Now we can choose $u_2 \in x_1 \cap x_2 \cap E_1$ and $u'_2 \in x_3 \cap E_1$ so that $(u_2, u'_2) \neq 0$. Let $D_2 = \langle A, u_1, u'_1, u_2, u'_2 \rangle$ and $D'_2 = A \perp E_2$.

As above, $x_1 = \langle A, u_1, u'_1, u_2 \rangle \perp F_4$, $x_2 = \langle A, u_1, u'_1, u_2 \rangle \perp F_5$, and $x_3 = \langle A, u_1, u'_1, u_2 \rangle \perp F_6$, with $F_4, F_5, F_6 \subseteq E_2$.

Remember that $\dim (x_1 \cap x_2) = \dim (x_1 \cap x_3) = \dim (x_2 \cap x_3) = d - 2$ and $\dim (A) = d - 4$. So $\dim (x_3 \cap E_2) = 2$ and $\dim (x_1 \cap x_3 \cap E_2) = \dim (x_2 \cap x_3 \cap E_2) = 2$. That means that $x_3 \cap E_2 \subseteq x_1 \cap x_2 \cap x_3 = A$, a contradiction.

Case 3: $\dim (A) = d - 3$. Let $A' = A \perp F$. In this case, we can choose $v_i \in F$, $i = 1, 2, 3$, such that $v_i \in x_j$ if and only if $i \neq j$, $j = 1, 2, 3$.

Subcase 3a: All vertices of $C$ contain $A$. Then each looks like $A \perp U$ with $U$ a maximal isotropic subspace of $F$. Since $F$ is isomorphic to $D_3(q)$, (4) yields $|C| < 2(q^2 + 1)(q + 1)$.

Subcase 3b: There is some $x_4 \in C$ with $A \not\subset x_4$. Now $\dim (x_1 \cap x_2 \cap x_4) > d - 3$, so $\dim (A \cap x_4) = \dim (x_1 \cap x_2 \cap x_3 \cap x_4) = \dim [(x_1 \cap x_3 \cap x_4) \cap (x_1 \cap x_2 \cap x_4)] > \dim (x_1 \cap x_2 \cap x_4) + \dim (x_1 \cap x_2 \cap x_3) - \dim (x_1 \cap x_2) > d - 4$.

Hence $\dim (x_4 \cap A) = d - 4$.

Arguing as in Case 2, we see that $\dim (x_4 \cap F) = 3$ and $\dim (x_1 \cap x_4 \cap F) = 2 = \dim (x_2 \cap x_4 \cap F)$. So
\[ \dim (x_1 \cap x_2 \cap x_3 \cap F) > 1. \] Since \( x_1 \cap x_2 \cap F = \langle v_3 \rangle \), we have \( v_3 \in x_4 \). Likewise \( v_1 \) and \( v_2 \) are in \( x_4 \). All told, 
\[ \langle A \cap x_4, v_1, v_2, v_3 \rangle \subseteq x_4. \]

Now define \( y = \langle A, v_1, v_2, v_3 \rangle \). \( y \) is a vertex of \( D_d(q) \), and \( x_i \) is adjacent in \( D_d(q) \) to \( y \) for \( i = 1, 2, 3, 4 \). Note also that, for distinct \( i_1, i_2, i_3 \in \{1, 2, 3, 4\} \), \( \dim (x_{i_1} \cap x_{i_2} \cap x_{i_3}) = d - 3 \) and 
\[ y = \langle x_{i_1} \cap x_{i_2} \cap x_{i_3} \cap x_{i_4} \cap x_{i_5} \rangle. \]

The above argument tells us that if \( x_5 \in C \), and if \( \dim (x_{i_1} \cap x_{i_2} \cap x_{i_3} \cap x_{i_4}) = d - 4 \) for any distinct \( i_1, i_2, i_3 \in \{1, 2, 3, 4\} \), then \( x_5 \) is adjacent to \( y \) in \( D_d(q) \). The lemma will thus be proven if we can show that such \( i_1, i_2, i_3 \) exist for any \( x_5 \in C \).

Suppose, to the contrary, that for some \( x_5 \in C \), 
\[ \dim (x_{i_1} \cap x_{i_2} \cap x_{i_3} \cap x_{i_4}) > d - 3 \] for all distinct \( i_1, i_2, i_3 \in \{1, 2, 3, 4\} \). Then \( (x_{i_1} \cap x_{i_2} \cap x_{i_3}) \subseteq x_5 \). For \( \{i_1, i_2, i_3\} = \{1, 2, 3\} \), this means that \( A \subseteq x_5 \). From \( \{2, 3, 4\} \), we get \( v_1 \subseteq x_5 \). \( \{1, 3, 4\} \) and \( \{1, 2, 4\} \) give \( v_2, v_3 \subseteq x_5 \). So \( x_5 = y \). But this can't be, since \( y \) is not in the vertex set of \( D'_d(q) \).
Theorem 3.4. Let $G$ be a bipartite distance-regular graph whose halved graph is $D'_d(q)$. If $d > 8$, then $G$ is isomorphic to $D_d(q)$.

Proof. Assume $d > 1$.

$$k' = \frac{\left[\begin{array}{c} d \\ 1 \end{array}\right] q \left(\left[\begin{array}{c} d \\ 1 \end{array}\right] q - 1\right)}{\left[\begin{array}{c} 2 \\ 1 \end{array}\right] q} = \frac{(q - 1)(d - 1)}{(q - 1)^2(q + 1)}$$

(from [2] and Lemma 1.2) By Lemma 1.2, $C_2 = k(k - 1)/k'$. $C_2 > 1$, so $k(k - 1) > k'$. Using this, and assuming that $k < 2(q^2 + 1)(q + 1)$, we get

$$\frac{(q^d - 1)(q^d - q)}{(q - 1)^2(q + 1)} < (2q^3 + 2q^2 + 2q + 2)(2q^3 + 2q^2 + 2q + 1) < 4(q^3 + q^2 + q + 1) < q^{10}.$$  

So $(q^d - 1)(q^d - q) < q^{13}$. Since $2^{d-2} < (q^d - 1)(q^d - q)$ for $d > 1$, we get $2d - 2 < 13$ and $d < 7$. 
So if \( d > 7 \), then the size of the maximal cliques which appear as \( G(y) \), for some \( y \in Y \), must be larger than \( 2(q^2 + 1)(q + 1) \).

By Lemma 3.3, they must each be \( D_d(y) \) for some \( y \) in \( V(D_d(q)) - V(D'_d(q)) \). There are just \( |V(D'_d(q))| \) of these available, so all must be used. That is, \( G \) must be isomorphic to \( D_d(q) \).
CHAPTER IV
The Hamming and Related Graphs

4.1. The Hamming Graph.

Let $S$ be a set of cardinality $n$. The vertices of the Hamming graph $H(n,d)$ are the $d$-tuples of elements from $S$. Two vertices are adjacent if their Hamming distance, the number of coordinates in which they differ, is one.

Theorem 4.1. Let $G$ be a bipartite distance-regular graph whose halved graph has the same parameters as the Hamming graph $H(n,d)$. Then $d < 2$.

Proof. Assume $d > 2$. From [2], we get that $c'_j = j$. Lemma 1.2 with $j = 2$ gives $2 = \frac{c_4 c_3}{c_2}$. Using $c_2 < c_3 < c_4$ yields two cases: $c_2 = c_3 = c_4 = 2$, and $c_2 = c_3 = 1$, $c_4 = 2$. (2) of Section 1.2 rules out the first case. Now since $d > 2$, we can use Lemma 1.2 with $j = 3$, giving us $3 = \frac{c_6 c_5}{c_2} = c_6 c_5$. But $c_6 > c_5 > c_4 = 2$, hence $d < 2$.
so this is impossible.

Note: Egawa [4] proved that if \( n \neq 4 \), no other graph has the same parameters as \( H(n,d) \), and that if \( n = 4 \), there are exactly \([d/2]\) isomorphism classes of such graphs, other than \( H(n,d) \).

4.2. The Halved Graph of \( H(2,d) \).

Consider \( H(2,d) \), with \( S = \{U,1\} \). No two vertices with an even (resp. odd) number of 1's as coordinates will be adjacent. Thus \( H(2,d) \) is bipartite.

Lemma 4.3. \( H'(2,d) \), \( d > 4 \), has two types of maximal clique. Type 1 is a neighborhood of a vertex in \( H(2,d) \). Type 2 has size 4.

Proof. Let \( x = (x_1) \), \( y = (y_1) \), \( u = (u_1) \) be vertices of a clique \( C \) of \( H'(2,d) \). Without loss of generality, \( x = (1,1,0,0,\ldots) \), \( y = (0,0,0,0,\ldots) \) and \( u = (1,0,1,0,\ldots) \), where \( x_1 = y_1 = u_1 \) for \( i = 5,6,\ldots,d \).

Any other vertex in \( C \) will have exactly one 1 in its first two coordinates. Suppose there is some \( v = (v_1) \) in \( C \) such that
\( v_1 = U, v_2 = 1 \). Then \( v_i = u_i \) for \( i = 3, 4, \ldots, d \). Furthermore, 
\[ C = \{x, y, u, v\}, \] 
since the first two coordinates of a vertex adjacent to each of \( x, y, u \) and \( v \) determine all the others. \( C \) is a clique of Type 2.

It remains to consider the case when every vertex of \( C - \{x, y\} \) starts out \( (1, U, \ldots) \). But then every vertex of \( C \) is adjacent, in \( H(2,d) \), to \( w = (1, U, U, U, \ldots) \), where \( w_i = x_i = y_i = u_i \) for \( i = 5, 6, \ldots, d \). In this case, \( C \) is a clique of Type 1.

**Theorem 4.4.** Let \( G \) be a bipartite distance-regular graph whose halved graph is \( H'(2,d) \), with \( d > 4 \). Then \( G \) is isomorphic to \( H(2,d) \).

**Proof.** The valence of \( H(2,d) \) is \( d \), so \( k' = d(d-1)/2 \) from Lemma 1.2 with \( i = U \). If a maximal clique of Type 2 appears as \( G(y) \) for some \( y \), then \( k = 4 \). By Lemma 1.2, again with \( i = U \),
\[
\frac{d(d - 1)}{2} = \frac{4 \cdot 3}{c_2},
\] 
But this is impossible with \( d > 4 \).

Hence \( G(y) \), for \( y \in Y \), is always a maximal clique of Type 1. Since there are \( |X| = |Y| \) of these available, all must be used, and \( G \) must be isomorphic to \( H(2,d) \).
4.3. $H(2,d)$.

Any vertex $x$ in $H(2,d)$ has a unique vertex at distance $d$ from it, the vertex obtained by changing every coordinate of $x$. Thus $H(2,d)$ is an antipodal graph.

Theorem 4.5. Let $G$ be a bipartite distance-regular graph whose halved graph is $H(2,d)$. Then $d < 3$.

Proof. Suppose $d > 3$. Let $\overline{x}_1 = \{x_1, y_1\}$ and $\overline{x}_2 = \{x_2, y_2\}$ be two vertices of $\overline{H}(2,d)$. If $y_1$ and $y_2$ are adjacent in $H(2,d)$, then $x_1$ and $x_2$ are as well. So if $\{\overline{x}_i | 1 < i < |C|\} = C$ is a maximal clique of $\overline{H}(2,d)$, we can without surrendering generality write $\overline{x}_i = \{x_i, y_i\}$ for $1 < i < |C|$, with $x_i$ adjacent to $x_1$ in $H(2,d)$ for $i = 2, 3, \ldots, |C|$.

Suppose $i, j \in \{2, 3, \ldots, |C|\}$ with $i \neq j$. Since both $x_i$ and $x_j$ are adjacent to $x_1$, they differ in at most 2 coordinates. Since both $\overline{x}_i$ and $\overline{x}_j$ are in $C$, $x_i$ and $x_j$ differ in 1 or $d-1$ coordinates. Since $d > 3$, $x_i$ and $x_j$ are adjacent in $H(2,d)$. So $\{x_i | 1 < i < |C|\}$ is in fact a clique of $H(2,d)$.

$H(2,d)$ is bipartite. Consequently Lemma 1.4 applies, and $G$ must be a polygon. However, since $k' = d > 2$, $G'$ is not a polygon, so $G$ cannot be a polygon.
4.4. $\overline{H}'(2,2s)$.

If $d$ is even, then $\overline{H}(2,d)$ is bipartite. Alternatively $H'(2,d)$ is antipodal, and $(\overline{H}') = (H)' = \overline{H}'(2,d)$.

Theorem 4.6. Let $G$ be a bipartite distance-regular graph whose halved graph is $\overline{H}'(2,2s)$. Then $s < 3$ or $G$ is isomorphic to $\overline{H}(2,2s)$.

Proof. The diameter of $H'(2,2s)$ is $s$, so if $s < 3$, $\overline{H}'(2,2s)$ is a complete graph. Assume $s > 4$. Let $C = \{x_i | 1 < i < |C|\}$ be a maximal clique, with $\overline{x}_i = \{x_i, y_i\}$, $x_i, y_i \in V(H'(2,2s))$. By the same argument as in Theorem 4.5, we can, while preserving generality, assume that $\{x_i | 1 < i < |C|\}$ is a maximal clique of $H'(2,2s)$. We can now invoke Lemma 4.3. Either $|C| = 4$ or $C$ is the neighborhood of a vertex of $\overline{H}(2,2s)$.

Assume $|C| = 4$. Then $k = 4$. By Lemma 1.2, with $i = 0$, $k' = k(k - 1)c_2$. But $k' = \binom{2s}{2}$ and $s > 4$, so this is impossible. Hence $C$ is the neighborhood of a vertex in $\overline{H}(2,2s)$, for every maximal clique $C$. There are just $|X| = |Y|$ of these neighborhoods available. So all must be used, and $G$ must be isomorphic to $\overline{H}(2,2s)$. 
4.5. The $q$-anologue of the Hamming Graph.

The vertices of the $q$-anologue of the Hamming graph, $H_q(n,d)$, are the $d \times (n + d)$ matrices over $F = GF(q)$. Two matrices are adjacent if their difference has rank one. So, for $x_1, x_2 \in V(H_q)$, $x_1$ is adjacent to $x_2$ if $x_1 - x_2 = uv^T$ for some $u \in F^d$, $v \in F^{n+d}$.

For fixed vertex $x_1$ and $u_1 \in F^d \setminus \{0\}$, let $D(x_1, u_1) = \{x_1 + u_1 v^T | v \in F^{n+d}\}$. For fixed vertex $x_1$ and $v_1 \in F^{n+d}$, write $E(x_1, v_1) = \{x_1 + u v_1^T | u \in F^d\}$. These are both cliques of $H_q(n,d)$.

Lemma 4.7. Every maximal clique of $H_q(n,d)$ is one of these types.

Proof. Let $x_1, x_2, x_3$ be in a maximal clique $C$. Then $x_2 = x_1 + u_1 v_1^T$, $x_3 = x_1 + u_2 v_2^T$, and $x_3 = x_2 + u_3 v_3^T$, for some $u_i \in F^d$ and $v_i \in F^{n+d}$, $i = 1, 2, 3$. Claim: Either $\{u_1, u_2\}$ or $\{v_1, v_2\}$ is dependent. Suppose that $\{u_1, u_2\}$ is independent. Let $u_1 = (\alpha_1, \ldots, \alpha_d)$, $u_2 = (\beta_1, \ldots, \beta_d)$, $u_3 = (\gamma_1, \ldots, \gamma_d)$. Then we can pick independent vectors $(\alpha_i, \beta_i)$ and $(\alpha_j, \beta_j)$. Since $x_2 + u_3 v_3^T = x_3 = (x_2 - u_1 v_1^T) + u_2 v_2^T$, we have $\gamma_i v_3 = -\alpha_i v_1 + \beta_i v_2$.
and \( j_1 v_3 = -\alpha j_1 + \beta j_2 \). These equations are independent, so \( \{v_1, v_2\} \) is dependent and the claim is proved.

Now suppose the clique is of neither the form \( D(x, u) \) nor \( E(x, v) \). Then there exists some \( x_1 + u_4 v_4^T \) with \( \{u_4, u_1\} \) independent, and some \( x_2 + u_5 v_5^T \) with \( \{v_1, v_5\} \) independent. By the claim, \( x_1 + u_4 v_4^T \neq x_2 + u_5 v_5^T \) and both \( \{v_1, v_4\} \) and \( \{u_1, u_5\} \) are dependent. This means, however, that \( \{u_4, u_5\} \) and \( \{v_4, v_5\} \) are independent, in contradiction to the claim.

**Theorem 4.8.** Let \( G \) be a bipartite distance-regular graph whose halved graph is \( H(n, d) \). Then \( d = 1 \).

**Proof.** Assume \( d > 1 \). Suppose \( D(x_1, u_1) = D(x_2, u_2) \) with \( x_1 \neq x_2 \). Then \( x_1 = x_2 + u_2 v_2^T \) and \( x_2 = x_1 + u_1 v_1^T \) for some \( v_1, v_2 \in F^{n+d} - \{0\} \). Hence \( u_2 v_2^T = -u_1 v_1^T \). This implies that \( u_2 = \lambda u_1 \) for some \( \lambda \in F - \{0\} \). Similarly, if \( D(x_1, u_1) = D(x_1, u_2) \), then \( u_1 = \delta u_2 \) for some \( \delta \in F - \{0\} \). So in general, \( D(x_1, u_1) = D(x_2, u_2) \) if and only if \( x_2 \in D(x_1, u_1) \) and \( u_2 = \lambda u_1 \) for some \( \lambda \in F - \{0\} \).
This allows us to count the number of distinct cliques of type D(x,u). There are \( q^{d(n+d)}(q^d - 1) \) choices for \( x \) and \( u \), and each clique can be represented by \( q^{n+d}(q - 1) \) such choices. So the number of cliques of this type is \( \frac{q^{d(n+d)}(q^d - 1)}{q^{n+d}(q - 1)} \). Likewise the number of cliques of type E(x,v) is \( \frac{q^{d(n+d)}n^d(q - 1)}{q^d(q - 1)} \).

Since \( |Y| = |X| = q^{d(n+d)} \), there are not enough cliques of type D(x,u) to account for every \( G(y) \). So either each \( G(y) \) is of the form E(x,v), or both types of clique occur.

Suppose that every \( G(y) \) is of type E(x,v). Pick \( x_1 \in X \), \( v \in F^{n+d} \setminus \{U\} \), \( u \in F^d \). Let \( x_2 = x_1 + uv^T \). Then \( x_1 \) and \( x_2 \) are in some \( G(y) \), say \( E(x_3,v_0) \). So \( x_1 = x_3 + u_1v_0^T \) and \( x_2 = x_3 + u_2v_0^T \) for some \( u_1, u_2 \in F^d \). Then \( x_1 + (u_2 - u_1)v_0^T = x_2 = x_1 + uv^T \). Thus \( (u_2 - u_1)v_0^T = uv^T \) and \( v_0 = \lambda v \) for some \( \lambda \in F \setminus \{0\} \). That is \( E(x_1,v) = E(x_3,v_0) \). Since \( x_1 \) and \( v \) were arbitrary, all cliques of type E(x,v) must occur. Hence

\[
\frac{q^{d(n+d)}n^d(q - 1)}{q^d(q - 1)} = |Y| = |X| = q^{d(n+d)} \]
But \( \frac{q^{n+d} - 1}{d(q - 1)} \neq 1 \), so this is impossible.

The only possibility left is that both types of clique occur. Then they must have the same size. So \( n = 0 \) and \( b_0 = k = |G(y)| = q^d \).

From [2] we get \( b_i^* = (q - 1) \binom{d}{1}^2 \). Lemma 1, with \( i = 0 \), then gives

\[
(q - 1) \binom{d}{1}^2 = b_0^* = \frac{b_0 b_1}{c_2} = \frac{q^d(q^d - 1)}{c_2}.
\]

So

\[
c_2 = \frac{q^d(q^d - 1)}{(q - 1) \binom{d}{1}^2} = \frac{q^d(q - 1)}{d - 1},
\]

which, since \( (q, q^d - 1) = 1 \), is an integer only if \( d = 1 \).
CHAPTER V
Alternating, Hermitian and Quadratic Forms

5.1. The Graph of Alternating Bilinear Forms.

Let $V$ be an $n$-dimensional vector space over $GF(q)$. The vertex set of the graph $\text{Alt}$ is the vector space of alternating bilinear forms on $V$. That is $x_1 \in V(\text{Alt})$ if $x_1$ is a bilinear form on $V$ such that $x_1(v,v) = 0$ for all $v \in V$. Define $\text{Rad } x_1 = \{u \in V | x_1(u,v) = 0 \text{ for all } v \in V\}$ and $\text{rk}(x_1) = \dim (V/\text{Rad } x_1)$. Then $x_1$ and $x_2$ in $V(\text{Alt})$ are adjacent when $\text{rk}(x_1 - x_2) = 2$.

Lemma 5.1. Let $x_1$ and $x_2$ be bilinear forms on $V$. If $\text{Rad } x_1 + \text{Rad } x_2 = V$, then $\text{Rad } x_1 \cap \text{Rad } x_2 = \text{Rad } (x_1 - x_2)$.

Proof. Certainly $\text{Rad } x_1 \cap \text{Rad } x_2 \subseteq \text{Rad } (x_1 - x_2)$. Suppose that $u \in \text{Rad } (x_1 - x_2)$. We need only show that $x_1(u,v) = 0$ for all $v \in V$. Now any $v \in V$ can be written $v = w_1 + w_2$ with $w_1 \in \text{Rad } x_1$, by assumption. But $x_1(u,w_1) = 0$ and $x_1(u,w_2) = x_2(u,w_2) = 0$, so $x_1(u,v) = 0$ and we are done.

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Lemma 5.2. The maximal cliques of $\text{Alt}$ containing $0$ come in two types. Type 1 consists of all forms whose radicals contain a fixed $(n - 3)$-subspace of $V$. Type 2 consists of all forms of rank 2 whose radicals are contained in a fixed $(n - 1)$-subspace of $V$, plus $0$.

Proof. (i) Let $x_1, x_2 \in \text{Alt}(0)$, $x_1 \neq x_2$. Then

$$\dim (\text{Rad } x_1) = n - 2 = \dim (\text{Rad } x_2).$$

So $n - 4 < \dim (\text{Rad } x_1 \cap \text{Rad } x_2) < n - 2$. Suppose $\dim (\text{Rad } x_1 \cap \text{Rad } x_2) = n - 4$. Then

$$\text{Rad } x_1 + \text{Rad } x_2 = V.$$ By Lemma 5.1, $\dim (\text{Rad } (x_1 - x_2)) = n - 4$. Thus $x_1 \notin \text{Alt}(x_2)$.

Suppose $\dim (\text{Rad } x_1 \cap \text{Rad } x_2) > n - 3$. Then

$$\dim (\text{Rad } (x_1 - x_2)) > n - 3.$$ Since the rank of an alternating form must be even, this means that $\dim (\text{Rad } (x_1 - x_2)) = n - 2$, and

$$x_1 \in \text{Alt}(x_2).$$

So for $x_1, x_2 \in \text{Alt}(0)$, $x_1 \neq x_2$, $x_1 \in \text{Alt}(x_2)$ if and only if $\dim (\text{Rad } x_1 \cap \text{Rad } x_2) = n - 2$ or $n - 3$. Hence Types 1 and 2 are actually cliques.

(ii) Let $x_1, x_2, x_3$ and $0$ have distinct radicals and be pairwise disjoint. By (i), $\dim (\text{Rad } x_1 \cap \text{Rad } x_2) = \dim (\text{Rad } x_2 \cap \text{Rad } x_3) = n - 3$. So $2n - 6 = \dim (\text{Rad } x_1 \cap \text{Rad } x_3) + \dim (\text{Rad } x_2 \cap \text{Rad } x_3) = \dim [(\text{Rad } x_1 \cap \text{Rad } x_3) + (\text{Rad } x_2 \cap \text{Rad } x_3)] + \dim (\text{Rad } x_1 \cap \text{Rad } x_2 \cap \text{Rad } x_3)$. Suppose that $\{x_1, x_2, x_3, 0\}$ is not
contained in a clique of Type 1. Then
\[ \dim (\text{Rad} x_1 \cap \text{Rad} x_2 \cap \text{Rad} x_3) < n - 4, \text{ and } 2n - 6 < \]
\[ \dim [(\text{Rad} x_1 \cap \text{Rad} x_3) + (\text{Rad} x_2 \cap \text{Rad} x_3)] + n - 4. \]
So \[ \dim [(\text{Rad} x_1 + \text{Rad} x_2) \cap \text{Rad} x_3] > \dim [(\text{Rad} x_1 \cap \text{Rad} x_3) + \]
\[ (\text{Rad} x_2 \cap \text{Rad} x_3)] > n - 2. \] This means that \( \text{Rad} x_3 \subseteq \text{Rad} x_1 + \text{Rad} x_2. \) Since \( \dim (\text{Rad} x_1 + \text{Rad} x_2) = n - 1, \) \( \{x_1, x_2, x_3, 0\} \)
is contained in a clique of Type 2.

(iii) Suppose \( C \) is a maximal clique containing \( 0, \) and is not of
Type 2. Then we can find \( x_1, x_2, x_3 \in C, \) all nonzero, such that
\( \text{Rad} x_1 + \text{Rad} x_2 + \text{Rad} x_3 = V. \) Let \( W = \text{Rad} x_1 \cap \text{Rad} x_2 \cap \text{Rad} x_3. \)
By (ii), \( \dim (W) = n - 3. \) We also have that \( W = \\
[(\text{Rad} x_1 + \text{Rad} x_2) \cap \text{Rad} x_3]. \) But \( \dim [(\text{Rad} x_1 + \text{Rad} x_2) \cap \)
\( \text{Rad} x_3] = \dim (\text{Rad} x_1 + \text{Rad} x_2) + \dim (\text{Rad} x_3) - \dim (\text{Rad} x_1 + \text{Rad} x_2 + \\
\text{Rad} x_3) = (n - 1) + (n - 2) - n = n - 3. \) Hence \( W = \\
(\text{Rad} x_1 + \text{Rad} x_2) \cap \text{Rad} x_3. \)

Now pick any \( x_4 \in C, \) \( x_4 \neq 0. \) The claim is that \( W \subseteq \text{Rad} x_4. \)
Suppose not. By (ii), \( \{x_1, x_2, x_3, 0\} \) is in a clique of Type 1 or 2.
Since \( W = \text{Rad} x_1 \cap \text{Rad} x_2, \) Type 1 means \( W \subseteq x_4. \) So \( \{x_1, x_2, x_3, 0\} \)
must be in a clique of Type 2. That is, \( \text{Rad} x_4 \subseteq \text{Rad} x_1 + \text{Rad} x_2. \)
So \( \text{Rad} x_4 \cap \text{Rad} x_3 \subseteq [(\text{Rad} x_1 + \text{Rad} x_2) \cap x_3] = W. \) Since \( x_3 \)
and \( x_4 \) are adjacent, \( \dim (\text{Rad} x_3 \cap \text{Rad} x_4) > n - 3 = \dim W. \) So
\( W \subseteq \text{Rad} x_4 \) after all, and \( C \) is a clique of Type 1.
Theorem 5.3. Let $G$ be a bipartite distance-regular graph whose halved graph is $Alt$. Then $n < 3$.

Proof. If $n < 3$, then for any $x_1, x_2 \in V(Alt)$,
$$\dim (\text{Rad } x_1 \cap \text{Rad } x_2) > n - 3.$$ By Part (i) of the proof of Lemma 5.2, $x_1$ and $x_2$ are adjacent. So $Alt$ is a complete graph if $n < 3$. Assume then that $n > 3$. How big are the maximal cliques?

Let $C$ be a maximal clique of Type 1, corresponding to $(n - 3)$-subspace $W$. Let $\{v_1, \ldots, v_n\}$ be a basis of $V$ such that $\{v_1, \ldots, v_n\}$ is a basis of $W$. Any vertex $x_1$ in $C$ is completely determined once we choose $x_1(v_1, v_2), x_1(v_1, v_3)$ and $x_1(v_2, v_3)$. These three elements of $GF(q)$ may be freely chosen, so $|C| = q^3$.

Let $C$ be a maximal clique of Type 2, corresponding to $(n - 1)$-subspace $U$. Let $\text{Rad } x_1 \subset U$ be fixed, for $x_1 \in C$, and let $\{u_1, \ldots, u_n\}$ be a basis of $V$ with $\{u_1, \ldots, u_n\}$ a basis of $\text{Rad } x_1$. Then $x_1$ will be fixed once $x_1(u_1, u_2)$ is. There are $q - 1$ choices for $x_1(u_1, u_2)$ if $x_1$ is not to be $U$. So the number of nonzero vertices of $C$ is $(q - 1) \binom{n - 1}{n - 2}$, where $\binom{n - 1}{n - 2}$ is the number of $(n - 2)$-subspaces of $U$ available to serve as radicals. $U$ is also in $C$, so $|C| = (q - 1) \binom{n - 1}{n - 2} + 1 = \ldots$
\[(q - 1) \binom{n - 1}{1} + 1 = (q - 1) \frac{q^{n-1} - 1}{q - 1} + 1 = q^{n-1} \cdot\]

Now Alt is vertex-transitive. So any maximal clique of Alt has size \(q^3\) or \(q^{n-1}\). Hence \(k\) is \(q^3\) or \(q^{n-1}\). From [2] we get

\[k' = (q - 1) \binom{n}{2} = \frac{(q - 1)(q^{n-1} - 1)}{q^2 - 1} \cdot\]

Suppose \(k = q^3\). Then Lemma 1.2 with \(i = 0\) gives

\[c_2 = \frac{k(k - 1)}{k'} = \frac{3(q^3 - 1)(q^2 - 1)}{(q^n - 1)(q^{n-1} - 1)} \cdot\]

\(c_2\) is an integer, so \((q^n - 1)(q^{n-1} - 1)\) divides \((q^3 - 1)(q^2 - 1)\).
Hence \(n < 3\), a contradiction.

Suppose \(k = q^{n-1}\). Lemma 1.2 gives

\[c_2 = \frac{n-1}{(q^n - 1)(q^{n-1} - 1)} \cdot\]

which is an integer only if \(n < 2\).
5.2. The Graph of Hermitian Forms.

Let $V$ be a $d$-dimensional vector space over $GF(q)$, where $q = r^2$. The vertices of the graph $\text{Her}$ are the Hermitian forms on $V$. Two forms are adjacent if the rank of their difference is 1.

Lemma 5.4. Let $C$ be a maximal clique of $\text{Her}$ containing $U$. Then $C$ is the set of scalar multiples of some rank 1 form.

Proof. Clearly the set of scalar multiples of a rank 1 form is a clique. Let $x_1, x_2 \in C$. Then $\dim(\text{Rad } x_1) = \dim(\text{Rad } x_2) = d - 1$. If $\text{Rad } x_1 \neq \text{Rad } x_2$, then $\text{Rad } x_1 + \text{Rad } x_2 = V$. So by Lemma 5.1 if $\text{Rad } x_1 \neq \text{Rad } x_2$, then $\dim(\text{Rad } (x_1 - x_2)) = \dim(\text{Rad } x_1 \cap \text{Rad } x_2) = d - 2$, which is impossible since $x_1$ and $x_2$ are adjacent. So $\text{Rad } x_1 = \text{Rad } x_2$.

Now let $\{v_1, \ldots, v_d\}$ be a basis of $V$ with $\{v_2, \ldots, v_d\}$ a basis of $\text{Rad } x_1$. $x_1$ will be determined once we know $x_1(v_1, v_1)$. Likewise for $x_2$. Hence $x_2 = \frac{x_2(v_1, v_1)}{x_1(v_1, v_1)} x_1$, and $C$ is the set of scalar multiples of $x_1$.

Theorem 5.5. Let $G$ be a bipartite distance-regular graph whose halved graph is $\text{Her}$. Then $d = 1$. 
Proof. From Lemma 5.4 we yet that the size of a maximal clique of Her is \( q \). So \( k = q \). From [2], \( k' = (r - 1)^d \). Using Lemma 1.2,

\[
C_2 = \frac{k(k - 1)}{k'} = \frac{r^2(r^2 - 1)(r + 1)}{r^{2d} - 1},
\]

which is an integer only when \( d = 1 \).

5.3 The Graph of Quadratic Forms.

Let \( V \) be an \( n \)-dimensional vector space over \( F = \text{GF}(q) \). A map \( x: V \to F \) is quadratic if, for every \( u, v \in V \) and \( a, b \in F \),

\[
x(au + bv) = a^2x(u) + b^2x(v) + abB(u,v),
\]

for some fixed bilinear form \( B \). Define \( \text{Rad} x = \{ u \in \text{Rad} B \mid x(u) = 0 \} \) and \( \text{rk}(x) = \dim (V/\text{Rad} x) \).

The graph \( Q \) has as vertices the quadratic forms on \( V \). Two forms \( x_1 \) and \( x_2 \) are adjacent if \( \text{rk}(x_1 - x_2) = 1 \) or \( 2 \). In other words, \( x_1 \in Q(x_2) \) if \( \dim (\text{Rad} (x_1 - x_2)) = n - 1 \) or \( n - 2 \).

Information on \( Q \) and on quadratic forms can be found in [2] and [7]. In particular, we need the following facts, presented without proof.
Lemma 5.6. Let $x_1$ and $x_2$ be quadratic forms with associated bilinear forms $B_1$ and $B_2$.

(i) $u \in \text{Rad } x_1$ if and only if $x_1(u + v) = x_1(v)$ for all $v \in V$.

(ii) If $\text{char } (F) \neq 2$, then $\text{Rad } x_1 = \text{Rad } B_1$.

(iii) If $\text{char } (F) = 2$, then $\dim (\text{Rad } B_1)$ is $\dim (\text{Rad } x_1)$ or $\dim (\text{Rad } x_1) + 1$. In either case, $B_1$ is an alternating form.

(iv) The bilinear form associated with $x_1 - x_2$ is $B_1 - B_2$.

To find out about the maximal cliques of $Q$, we need only look at the ones in $Q(0)$, since $\text{Aut } (Q)$ is vertex-transitive. $x_1 \in Q(0)$ if and only if $\dim (\text{Rad } x_1) = n - 1$ or $n - 2$.

Lemma 5.7. Let $x_1, x_2 \in Q(0)$, $x_1 \neq x_2$.

(i) If $\dim (\text{Rad } x_1) = n - 1 = \dim (\text{Rad } x_2)$, then $x_1 \in Q(x_2)$.

(ii) If $\dim (\text{Rad } x_1) = n - 1$ and $\dim (\text{Rad } x_2) = n - 2$, then $x_1 \in Q(x_2)$ if and only if $\text{Rad } x_1 \supset \text{Rad } x_2$.

(iii) If $\dim (\text{Rad } x_1) = n - 2 = \dim (\text{Rad } x_2)$ and $\dim (\text{Rad } x_1 \cap \text{Rad } x_2) = n - 4$, then $x_1 \notin Q(x_2)$.

(iv) If $\text{Rad } x_1 = \text{Rad } x_2$, then $x_1 \in Q(x_2)$.
Proof. (i) \( \dim (\text{Rad} (x_1 - x_2)) > \dim (\text{Rad} x_1 \cap \text{Rad} x_2) > n - 2 \).

(ii) If \( \text{Rad} x_1 \supset \text{Rad} x_2 \), then \( \dim (\text{Rad} (x_1 - x_2)) > \dim (\text{Rad} x_2) = n - 2 \), and \( x_1 \notin U(x_2) \). Suppose \( \text{Rad} x_1 \not\supset \text{Rad} x_2 \), and let \( B_1, B_2 \) be the bilinear forms associated with \( x_1 \) and \( x_2 \).

Consider the case where \( \text{char} (F) \neq 2 \). Then by Lemma 5.6,

\[ \text{Rad} x_1 = \text{Rad} B_1 \quad \text{and} \quad \text{Rad} x_2 = \text{Rad} B_2. \]

Now \( \text{Rad} B_1 + \text{Rad} B_2 = \text{Rad} x_1 + \text{Rad} x_2 = V \), so by Lemma 5.1, \( \text{Rad} (B_1 - B_2) = \text{Rad} B_1 \cap \text{Rad} B_2 \).

\[ \text{Rad} B_2 = \text{Rad} x_1 \cap \text{Rad} x_2. \]

But \( \text{Rad} (x_1 - x_2) \subseteq \text{Rad} (B_1 - B_2) \) by Lemma 5.6 (iv). So \( \text{Rad} (x_1 - x_2) \subseteq \text{Rad} x_1 \cap \text{Rad} x_2 \). That is,

\[ \dim (\text{Rad} (x_1 - x_2)) < n - 3. \]

Thus, \( x_1 \notin U(x_2) \).

Now consider the case \( \text{char} (F) = 2 \). Let \( \{u_1, \ldots, u_n\} \) be a basis for \( V \), with \( \{u_3, u_4, \ldots, u_n\} \) a basis of \( \text{Rad} x_2 \) and

\[ \{u_1, u_2, u_4, \ldots, u_n\} \]

a basis of \( \text{Rad} x_1 \). Let \( a = x_1(u_3) \), \( b = x_2(u_1) \), \( c = x_2(u_2) \) and \( d = B_2(u_1, u_2) \). Since \( u_1, u_2 \notin \text{Rad} x_2 \) and \( u_3 \notin \text{Rad} x_1 \), \( a, b \) and \( c \) are nonzero. If \( d \) were 0, then \( u_1 \) and \( u_2 \) would be in \( \text{Rad} B_2 \), since \( B_2 \) is symplectic. But then \( \text{Rad} B_1 \) would be \( V \), which it is not by Lemma 5.6 (iii). So \( d \neq 0 \).

Now let \( v = \sum_{i=1}^{n} r_i u_i \in \text{Rad} (x_1 - x_2) \). We want to show that \( r_i = 0 \) for \( i = 1, 2, 3 \). Let \( w = \sum_{i=1}^{n} s_i u_i \) be any vector in \( V \).

Now \( v \in \text{Rad} (x_1 - x_2) \subseteq \text{Rad} (B_1 - B_2) \), so \( B_1(v, w) = B_2(v, w) \). Thus
\[ x_1(v + w) - x_2(v + w) = [x_1(v) + x_1(w) + B_1(v, w)] - \\
[x_2(v) + x_2(w) + B_2(v, w)] = [x_1(v) - x_2(v)] + [B_1(v, w) - B_2(v, w)] + \\
[x_1(w) - x_2(w)] = x_1(w) - x_2(w). \] In terms of coordinates, this is

\[
(r_3 + s_3)^2 a - [(r_1 + s_1)^2 b + (r_2 + s_2)^2 c + (r_1 + s_1)(r_2 + s_2)d] = \\
x_1(v + w) - x_2(v + w) = x_1(w) - x_2(w) = s_3^2 a - \\
[s_1^2 b + s_2^2 c + s_1 s_2 d]. \] This simplifies to

\[
r_3^2 a - r_1^2 b - r_2^2 c - r_1 r_2 d = (s_1 r_2 + r_1 s_2)d.\] which holds for all \( w \in V \). Since \( d \neq 0 \), either \( r_1 = r_2 = 0 \) or we can choose \( w \) to make the right-hand side anything we want. For fixed \( v \), however, the left-hand side is fixed. So \( r_1 = r_2 = 0 \). But then \( r_3^2 a = 0 \), so \( r_3 = 0 \). We know then that if \( v \in \text{Rad} (x_1 - x_2) \), it must be in the span of \( \{u_4, \ldots, u_n\} \). Thus \( \dim (\text{Rad} (x_1 - x_2)) < n - 3 \), and \( x_1 \notin Q(x_2) \).

(iii) If \( \text{char} (F) \neq 2 \), then \( \text{Rad} x_i = \text{Rad} B_i \) for \( i = 1, 2 \). If \( \text{char} (F) = 2 \), then \( \text{Rad} B_i \) has dimension \( n - 2 \) or \( n - 1 \) by Lemma 5.6. But \( B_i \) is symplectic and must have even rank. Hence \( \text{Rad} x_i = \text{Rad} B_i \) for \( i = 1, 2 \) in this case also. Because

\[ \dim (\text{Rad} x_1 \cap \text{Rad} x_2) = n - 4 \], we have \( V = \text{Rad} x_1 + \text{Rad} x_2 = \text{Rad} B_1 + \text{Rad} B_2 \). Now we can use Lemma 5.1. \( \text{Rad} (x_1 - x_2) \subseteq \text{Rad} (B_1 - B_2) = \text{Rad} B_1 \cap \text{Rad} B_2 = \text{Rad} x_1 \cap \text{Rad} x_2 \). So \( \dim (\text{Rad} (x_1 - x_2)) < n - 4 \).
and \( x_1 \not\in Q(x_2) \).

(iv) This is trivial.

The missing case in Lemma 5.7 is taken care of in Lemma 5.8.

Now for any bilinear form \( B \) on \( V \) and ordered basis \( \{ v_1, \ldots, v_n \} \) of \( V \), we can associate a matrix whose \((i,j)\)-entry is \( B(v_i,v_j) \).

**Lemma 5.8.** Let \( x_1, x_2 \in Q(0) \) with \( \text{rk } x_1 = \text{rk } x_2 = 2 \), \( \dim (\text{Rad } x_1 \cap \text{Rad } x_2) = n - 3 \) and \( x_1 \in Q(x_2) \). If \( B_1 \) and \( B_2 \) are the associated bilinear forms for \( x_1 \) and \( x_2 \), then we can find a basis of \( V \) such that the matrices for \( B_1 \) and \( B_2 \) look like one of the following.

\[
\begin{align*}
\text{(i) } (B_1) &= \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & a & 0 & 0 \\
0 & 0 & b & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \\
\text{(ii) } (B_1) &= \begin{pmatrix}
a & 0 & b & 0 \\
0 & 0 & 0 & 0 \\
b & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \\
\text{and } (B_2) &= \begin{pmatrix}
c & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & b & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\end{align*}
\]

\( a, b \) and \( c \) all nonzero.
Proof. Since \( \text{rk } x_1 = \text{rk } x_2 = 2 \), then \( \text{Rad } x_i = \text{Rad } B_i \) for \( i = 1,2 \), whatever char \( (F) \) is. Define \( W = \text{Rad } B_1 \cap \text{Rad } B_2 \). \( \dim (W) = n - 3 \). We can find \( u_1, u_2, u_3 \in V - W \) such that \( u_1 \in \text{Rad } B_1 \), \( u_2 \in \text{Rad } B_2 \) and \( B_1 (u_3,v) = B_2 (u_3,v) \) for all \( v \in V \).

Case 1: \( \langle u_1, u_2, u_3 \rangle + W = V \). Pick \( w_1, \ldots, w_{n-3} \in W \) so that \( \{u_1, u_2, u_3, w_1, \ldots, w_{n-3}\} \) is a basis for \( V \). Since \( x_i (v_1 + v_2) = x_i (v_1 + v_2) \) for \( i = 1,2 \) and all \( v_1, v_2 \in V \), the matrices associated with \( B_1 \) and \( B_2 \) are symmetric. Using this fact, and the definition of \( u_1, u_2 \) and \( u_3 \), we can immediately fill in the entries of the matrices with respect to the basis \( \{u_1, u_2, u_3, w_1, \ldots, w_{n-3}\} \). We get the matrices of (i). \( a, b \) and \( c \) are nonzero since \( \text{rk } B_1 = \text{rk } B_2 = 2 \).

Case 2: \( \langle u_1, u_2, u_3 \rangle + W \neq V \). \( \dim (\langle u_1, u_2 \rangle + W) = n - 1 \), since \( \text{Rad } B_1 \neq \text{Rad } B_2 \). Pick \( u' \in V - W \), \( w_1, \ldots, w_{n-3} \in W \) such that \( \{u', u_1, u_2, w_1, \ldots, w_{n-3}\} \) is a basis for \( V \). With respect to this basis,

\[
(B_1) = \begin{pmatrix}
a & 0 & b \\
0 & 0 & 0 \\
b & 0 & c \\
0 & 0 & 0
\end{pmatrix}, \quad (B_2) = \begin{pmatrix}
d & f & 0 \\
f & g & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]
Now $u_3 = r_1 u_1 + r_2 u_2 + \sum_{i=1}^{n-3} s_i w_i$ for some $r_1, r_2, s_i \in F$.

Recall that $B_1(u_3, v) = B_2(u_3, v)$ for all $v \in V$. So $r_2 b = B_1(u_3, u') = B_2(u_3, u') = r_1 f$, $u = B_1(u_3, u_1) = B_2(u_3, u_1) = r_1 y$ and $r_2 c = B_1(u_3, u_2) = B_2(u_3, u_2) = 0$. Rewriting,

(*)

$r_2 b = r_1 f$,

$0 = r_1 g$,

$r_2 c = 0$.

Now $u_3 \notin W$, so $r_1$ and $r_2$ cannot both be $0$. Thus at least one of $c, y$ must be $0$. Suppose $c \neq 0$. Then $r_2 = 0$, $r_1 \neq 0$ and $y = 0$. By (*) then, $f = 0$. But $f, y = 0$ imply that $\text{rk } B_2 = 1$, a contradiction. Hence $c = 0$. Likewise $y = 0$, and we have (ii).

Suppose that $x_1, x_2, x_3 \in Q(0)$ with $\text{rk } x_1 = \text{rk } x_2 = \text{rk } x_3 = 2$ and $\dim (\text{Rad } x_i \cap \text{Rad } x_j) = n - 3$ for all $i \neq j$. If $\{0, x_1, x_2, x_3\}$ is in some clique, what can we say about $x_1$, $x_2$, and $x_3$?

To start with, because of the way the radicals intersect, we can invoke Lemma 2.7 to say that either all the $\text{Rad } x_i$ are contained in a
fixed \((n - 1)\)-space, or they contain a fixed \((n - 3)\)-space. For the next two lemmas let us suppose the latter, and call the \((n - 3)\)-space \(W\). Further, let \(V = U + W\), with \(\dim(U) = 3\).

Let \(B_i\) be the bilinear form of \(x_i\). \(\text{Rad} x_i = \text{Rad} B_i\) even if \(\text{char } F = 2\), since \(\text{rk} B_i\) must be even. So we can find \(u_1, u_2, u_3, v_1, v_2, v_3 \in U\) such that \(u_i \in \text{Rad} B_i\), for \(i = 1, 2, 3\), and \(v_i \in \text{Rad} (B_i - B_i)\) for \(j, k \neq i\).

**Lemma 5.9.** At least one of the following must happen.

1. \(\{v_1, v_2, v_3\}\) is independent.
2. \(\{v_1, v_2, v_3\}\) is dependent and \(\{u_i, v_j, v_k\}\) is independent for some distinct \(i, j, k\).
3. \(\dim(\langle u_1, u_2, u_3, v_1, v_2, v_3 \rangle) = 2\).
4. \(\dim(\langle v_1, v_2, v_3 \rangle) = 1\).

**Proof.** Assume that (1), (2) and (4) don't happen. Suppose that \(\dim(\langle v_i, v_j \rangle) = 1\) for some \(i \neq j\). Let \(i \neq k \neq j\). We have, for all \(w \in V\), \(B_i(v_i, w) = B_j(v_j, w)\). Since \(v_i\) and \(v_j\) are dependent, \(B_k(v_i, w) = B_j(v_j, w)\). So \(B_i(v_i, w) = B_i(v_j, w) = B_j(v_j, w)\). Also \(B_i(v_k, w) = B_j(v_k, w)\). But by Lemma 5.8, \(B_i\) and \(B_j\) can agree on only only a 1-dimensional subspace of \(U\). So \(\dim(\langle v_1, v_2, v_3 \rangle) = 1\) and
we are in (4), contrary to assumption.

Hence each pair \( \{v_i, v_j\} \) is independent.

Now we are not in (1), so \( v_3 \in \langle v_1, v_2 \rangle \). We are not in (2), so \( u_i \in \langle v_1, v_2, v_3 \rangle = \langle v_1, v_2 \rangle \), for \( i = 1, 2, 3 \). Hence
\[ \langle u_1, u_2, u_3, v_1, v_2, v_3 \rangle = \langle v_1, v_2 \rangle, \] and we are in (3).

Lemma 5.10. Let \( U, W, x_i, B_i, u_i \) and \( v_i \), for \( i = 1, 2, 3 \), be as in Lemma 5.9. Then at least one of the following happens.

(i) There is a quadratic form \( x_4 \) such that \( \text{rk}(x_i - x_4) = 1 \) for \( i = 1, 2, 3 \).

(ii) There is an \( (n - 1) \)-dimensional subspace \( Z \) such that \( B_1, B_2, \) and \( B_3 \) are identical when restricted to \( Z \).

If \( \text{char} F \neq 2 \), (ii) implies that \( x_1, x_2 \) and \( x_3 \) agree on \( Z \).

Proof. Suppose we are in Case (1) of Lemma 5.9. Define a symmetric bilinear form \( B_4 \) by \( B_4(v_i, z) = B_j(v_i, z) = B_k(v_i, z) \) for all \( z \in V, i, j, k \) distinct, and \( W \subseteq \text{Rad} B_4 \). Certainly \( \langle W, v_i, v_j \rangle \subseteq \text{Rad} (B_4 - B_i) \), for \( i, j, k \) distinct. So \( \text{rk} (B_4 - B_i) < 1 \). But, since we are in Case (1), \( v_i \notin \text{Rad} (B_j - B_i) \) if \( i \neq j \). Since \( v_i \in \text{Rad} (B_4 - B_j) \), we know that \( B_4 \neq B_i \). Hence \( \text{rk} (B_4 - B_i) = 1 \), for \( i = 1, 2, 3 \).
Now if \( \text{char } F \neq 2 \), there is only one quadratic form \( x_4 \) corresponding to \( B \) and \( \text{rk}(B_4 - B_i) = 1 \), for \( i = 1,2,3 \). This gives (i). What if \( \text{char } F = 2 \)? \( B_4 \) and \( B_i \) are symmetric. So \( B_4 - B_i \) is alternating. Hence \( \text{rk}(B_4 - B_i) \neq 1 \), contradicting the previous paragraph. This means that Case (1) does not occur if \( \text{char } F = 2 \).

Suppose we are in Case (2) of Lemma 5.9. Without loss of generality, \( \{u_1, v_2, v_3\} \) is independent. In this basis,

\[
\begin{pmatrix}
B_1 \\
U
\end{pmatrix}
= \begin{pmatrix}
0 & 0 & 0 \\
0 & a & b \\
0 & b & c
\end{pmatrix}, \quad 
\begin{pmatrix}
B_2 \\
U
\end{pmatrix}
= \begin{pmatrix}
d & f & u \\
f & g & b \\
0 & b & c
\end{pmatrix}, \quad \text{and}

\begin{pmatrix}
B_3 \\
U
\end{pmatrix}
= \begin{pmatrix}
h & u & 1 \\
0 & a & b \\
1 & b & m
\end{pmatrix}.
\]

Now \( v_1 = s v_2 + t v_3 \), for some \( s, t \in F \). From the proof of Lemma 5.9 we have that neither \( s \) nor \( t \) is zero. \( B_2(v_1, z) = B_3(v_1, z) \) for all \( z \in V \) means that \( s g + t b = s a + t b \) and \( s b + t c = s b + t m \). So \( g = a \) and \( c = m \). This gives (ii), with \( Z = \langle v_2, v_3, W \rangle \).

Suppose we have Case (3) of Lemma 5.9. \( u_1 \) and \( u_2 \) are independent, since \( \text{Rad } x_1 \neq \text{Rad } x_2 \). Pick some \( u' \) such that \( \{u_1, u_2, u'\} \) is a basis for \( U \). In this basis,
Now $v_3 = su_1 + tu_2$ for some $s, t \in F$. Since $v_3 \in \text{Rad } (B_1 - B_2)$, $ds = at = 0$. If $t = 0$, then $v_3 = su_1$, and $\text{Rad } x_1 = \text{Rad } x_2$, a contradiction. So $t \neq 0$. Likewise, $s \neq 0$.

Hence $a = d = 0$.

Since $\text{Rad } x_1 \neq \text{Rad } x_3$, $v_2$ and $u_3$ are independent vectors inside $\langle u_1, u_2 \rangle$. Say $v_2 = s'u_1 + t'u_2$ and $u_3 = s''u_1 + t''u_2$.

$v_2 \in \text{Rad } (B_1 - B_3)$ implies that

$$
\begin{pmatrix}
g & h \\
h & m
\end{pmatrix}
\begin{pmatrix}
s' \\
t'
\end{pmatrix} = 0.
$$

$u_3 \in \text{Rad } x_3$ implies that

$$
\begin{pmatrix}
g & h \\
h & m
\end{pmatrix}
\begin{pmatrix}
s'' \\
t''
\end{pmatrix} = 0.
$$
Since $v_2$ and $u_3$ are independent, $g = h = m = 0$. Thus we are in (ii), with $Z = \langle u_1, u_2, v \rangle$.

Finally, suppose we have Case (4). Further, assume that 
$\{u_i, u_j, v_k\}$ is dependent for all distinct $i, j, k$. Now $u_1$ and $v_3$ are independent, since $\text{Rad } x_1 \neq \text{Rad } x_2$. So $\{u_1, u_2, v_3\}$ dependent implies that $u_2 \in \langle u_1, v_3 \rangle$. Likewise $\{u_1, u_3, v_2\}$ dependent implies that $u_3 \in \langle u_1, v_2 \rangle = \langle u_1, v_3 \rangle$. So, in fact, we are in Case (3).

Thus we can assume, without suffering any loss of generality, that $\{u_1, u_2, v_3\}$ is independent. For this basis,

$$
\left( B_1 \right) U = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix}, \quad \left( B_2 \right) U = \begin{pmatrix} c & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & b \end{pmatrix}
$$

and

$$
\left( B_3 \right) U = \begin{pmatrix} d & f & 0 \\ f & y & 0 \\ 0 & 0 & b \end{pmatrix}.
$$

Since $\text{rk } (B_1) = 2$, $b \neq 0$. Since $\text{rk } (B_3) = 2$ and $b \neq 0$, $dg = f^2$. Define $B_4$ by $\left( B_4 \right) U = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & b \end{pmatrix}$, and $W \subseteq \text{Rad } (B_4)$.

As in Case (1), if $\text{char } F \neq 2$, $B_4$ gives a unique quadratic form $x_4$. 

Since $v_2$ and $u_3$ are independent, $g = h = m = 0$. Thus we are in
which satisfies the conditions of (i). But if \( \text{char } F = 2 \), \( B_1 \) is an alternating form. Hence \( a = b = 0 \) and \( \text{Rad } B_1 = V \), contrary to the assumption.

The last statement of the lemma is an immediate consequence of the fact that, if \( \text{char } F \neq 2 \), \( B_i \) determines \( x_i \), for \( i = 1,2,3 \).

The preceding lemmas suggest the following candidates for maximal cliques: The set of all quadratic forms of rank 0 or 1; the set of all forms whose radicals contain a fixed \((n - 2)\)-space; \( \{ x \in Q(0) | \text{rk} (x - x_0) < 1 \} \cup \{0\} \), where \( x_0 \) is some fixed quadratic form of rank 3 or less; the set of all rank 2 forms which agree on a fixed \((n - 2)\)-space, plus \( 0 \). That these are maximal cliques is easy to see. That they exhaust the possibilities is doubtful.

A full characterization of the maximal cliques of \( Q \) must await a fuller understanding.
BIBLIOGRAPHY


