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Daso, Endwell Obene

THEORETICAL AND EXPERIMENTAL INVESTIGATIONS OF ASYMMETRIC TURBULENT SUPersonic FREE SHEAR LAYERS

The Ohio State University

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THEORETICAL AND EXPERIMENTAL INVESTIGATIONS OF
ASYMMETRIC TURBULENT SUPERSONIC FREE SHEAR LAYERS.

DISSERTATION

Presented in Partial Fulfillment of the Requirements to
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By

Endwell Obene Daso, B.A.E.M., M.S.

* * * * *

The Ohio State University
1984

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<td>$\tilde{G}$</td>
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\begin{itemize}
\item $P$ static pressure; Prandtl number, page 32
\item $\bar{P}$ normalized total pressure
\item $Pr$ laminar Prandtl number
\item $Pr_t$ turbulent Prandtl number
\item $p$ average velocity point in similarity coordinate
\item $p'$ fluctuating static pressure
\item $Q$ coefficient in the dimensionless inner eddy viscosity model
\item $R^*_x$ reference Reynolds number
\item $Re$ Reynolds number
\item RMS root mean square
\item $r$ $u_2/u_1$; turbulent Reynolds number
\item $S$ transformed coordinate of $Y$
\item $S'_1$ value of the derivative of $S$ with respect to $Y$ at $Y_1$
\item $S'_N$ value of the derivative of $S$ with respect to $Y$ at $Y_N$
\item $T_1$ total temperature of faster stream
\item $T_2$ total temperature of slower stream
\item $t_t$ relaxation time constant
\item $U$ non-dimensional streamwise velocity
\item $u_e$ normalized edge velocity
\item $u_e$ streamwise edge velocity; jet efflux velocity
\item $u_1$ streamwise edge velocity of faster stream
\item $u_2$ streamwise edge velocity of slower stream
\item $u_\infty$ free-stream velocity
\end{itemize}
\( u \) instantaneous streamwise velocity \\
\( u_0 \) reference velocity \\
\( \bar{u} \) streamwise mean velocity \\
\( u^* \) normalized streamwise velocity, Eq. (2.18) \\
\( u' \) fluctuating streamwise velocity \\
\( v \) instantaneous normal velocity \\
\( v_1 \) normal edge velocity of faster stream \\
\( v_2 \) normal edge velocity of slower stream \\
\( \bar{v} \) normal mean velocity \\
\( v' \) fluctuating normal velocity \\
\( w' \) fluctuating spanwise velocity \\
\( -\rho u'v' \) Reynolds stress \\
\( X, \bar{X} \) normalized streamwise coordinate, pages 89 and 29, respectively \\
\( \bar{X} \) integral of normalized eddy viscosity \\
\( x \) streamwise or axial coordinate \\
\( x^* \) reference length \\
\( x_0 \) virtual origin \\
\( Y \) normalized lateral coordinate, page 69 \\
\( Y_1 \) edge of faster stream in \( Y \) plane; first \( Y \) value on mesh \\
\( Y_2 \) edge of slower stream in \( Y \) plane \\
\( Y' \) last \( Y \) value on mesh \\
\( \tilde{Y}_N \) approximation to \( Y' \) \\
\( Y_1' \) normal or lateral coordinate \\
\( y \) edge of faster stream in \( y \) plane \\
\( y_2 \) edge of slower stream in \( y \) plane
\( \bar{y} \) normalized lateral coordinate, page 29; \( y - y \)

\( y_d \) position of the dividing streamline

\( y_e \) either edge of the free shear layer

\( z \) independent variable, page 20

**GREEK**

\( \alpha \)

\( \epsilon_{\rho} \); \( \mu \); empirical constant, Eq. (2.46)

\( \alpha_{\epsilon} \) empirical constant in Glushko's model

\( \beta \) exponent, Eq. (2.16); ratio of normalized total pressures, page 26; empirical constant, Eq. (2.46)

\( \beta_1, \beta_2, \beta_3, \beta_4 \) coefficients in the dimensionless transport equations

\( \beta_5, \beta_6, \beta_7, \beta_8 \)

\( \gamma \) ratio of specific heats

\( \Delta \) difference operator

\( \Delta(x) \) boundary layer growth rate scaling factor

\( \delta \) boundary layer thickness; half the thickness of symmetric wakes

\( \delta' \) \( |y_e - y_d| \)

\( \epsilon \) eddy viscosity

\( \epsilon_0 \) the value of the eddy viscosity at the average velocity point

\( \epsilon_0 \) proportionality constant (eddy viscosity), page 17; outer eddy viscosity of Cebeci, Smith and Monsinski model

\( \epsilon_k \) turbulent kinetic energy diffusion coefficient
\( \dot{\varepsilon} \)  
\( \gamma \)  
\( \eta \)  
\( \eta_0 \)  
\( \eta_d \)  
\( \eta' \)  
\( \eta^* \)  
\( \tilde{\eta} \)  
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\( \lambda \)  
\( \lambda_P \)  
\( \lambda' \)  
\( \mu \)  
\( \mu_0 \)  
\( \mu_N \)  
\( \nu \)  
\( \xi \)  
\( \rho \)  
\( \rho_1 \)

dimensionless eddy viscosity  
independent variable, page 20  
similarity variable, pages 12, 14, 21, 26, 65  
\( \eta \) at the average velocity point  
\( \eta \) at \( y = y_d \)  
similarity variable, page 19  
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scattering angle; momentum thickness  
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\( \sigma \)  
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\( \Phi_0 \)  
\( \Phi_\infty \)  
\( \omega \)  

censity at edge of slower stream  
density at average velocity point  
reference fluid density  
local mean density  
Goertler's spreading parameter  
\( \sigma \) for single stream mixing \( (r = 0) \)  
empirical constant of diffusion  
total shear stress  
Reynolds stress or turbulent shear stress  
integral of \( f(X) \)  
constant of integration  
turbulent kinetic energy dissipation rate  
exponent in viscosity-temperature relation  

**SUPERSCRIPT**  
\( \ast \)  
non-dimensional temperature Eq.(2.16); normalized streamwise velocity; similarity variable  
\( + \)  
"simple" layer with positive shear stress  
\( - \)  
"simple" layer with negative shear stress  
\( \bar{\text{mean values}} \)  
\( \sim \)  
approximate value  
\( \cdot \)  
fluctuation, differentiation  

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**SUBSCRIPTS**

0  zeroth order; virtual origin; zero velocity; stagnation condition; stationary state; \( Y = 0 \) point; trailing edge

1  first order; first mesh point, faster stream edge condition

2  slower stream edge condition; constant of proportionality

0, 1, 2, ..., \( i \)  the \( i \)th term of the Poincare series

\( a \)  average velocity point, \( u = -(u + u^\prime) \)

\( D \)  Doppler frequency

\( d \)  dividing streamline

\( e \)  condition at edge of shear layer

\( i \)  inner eddy viscosity

\( i, j \)  mesh point

\( j \)  node \( j \), see Appendix

\( k \)  turbulent kinetic energy transport

\( l \)  linearity of dissipation length scale, 2nd One-Equation model

\( m \)  mixing length

\( max \)  maximum

\( N \)  last mesh point

\( o \)  outer eddy viscosity

\( r \)  wake relaxation law

\( w \)  wall condition

\( X \)  differentiation with respect to \( X \)

\( Y \)  differentiation with respect to \( Y \)
reference or incompressible condition

\( \rho \)

refers to density
I. INTRODUCTION

1.1 General Introduction

In engineering fluid mechanics and heat transfer processes the flow field is, in general, turbulent. In many engineering applications such as chemical transfer and gas dynamic lasers, the flow field involves turbulent shear layers in which mixing and/or fluid dynamic interactions take place. Also, fluid mixing processes have long been known to characterize incompressible as well as subsonic flow fields of many common applications like combustion and propulsion. In such a wide range of applications the flow structure may be quite different from one flow to the other, thus exemplifying various degrees of mixing.

In recent years, the turbulent structure of free shear layers has been better understood. In low speed flows, the turbulent mixing layer is now known to consist primarily of large-scale eddies on a background of fine-grain random field. These macro-scale structures continuously feed on the mean flow energy to sustain their local motion. As the energetic ingredients of the flow field, the large eddies are
believed to be the principal agents of the mixing mechanism and of the asymptotic growth of the layer as they are convected downstream at the average velocity of the mixing layer.

Although better understanding of low speed free shear layer turbulence has been gained through experimental investigations, theoretical treatment of these flows has nonetheless remained difficult. Also the flow physics is more complex because of an inherent geometrical discontinuity at the start of mixing, as well as the absence of the no-slip boundary condition in free shear layers. Consequently a third boundary condition is generally lacking and must be specified in some manner in order to solve the Reynolds averaged conservation equations of motion for a given flow. For the problem of closure, this new picture of large-scale turbulence structure superimposed on a micro-scale background can lead to better prediction methods if the various scales are appropriately accounted for in turbulence models. This more difficult approach is beginning to receive some attention.

The discovery of the presence of coherent large-scale eddies in mixing layers is very significant in turbulence research and has provided an impetus and renewed interest in this area in the past decade or so. Much progress has been made from both the theoretical
and experimental directions. The experimental progress, which has generated much of the new wealth of information about the structure of turbulent free shear layers, is due primarily to recent advances in diagnostic tools such as the laser Doppler velocimeter (L/DV) and highly improved hot wires and flow visualization techniques. However, the experimental emphasis has clearly been placed on low speed flows, thus creating a particular paucity of high speed (supersonic) free shear layer data.

1.2 The Present Investigation

The main purpose of this study is to investigate theoretically and experimentally the fluid-mechanical character and structure of an asymmetric, two-dimensional, supersonic, turbulent, free shear layer.

The theoretical analysis involves solving the equations of motion using various turbulence models for the Reynolds stress term for closure. Four different turbulence models based on the Boussinesq's eddy viscosity concept are employed. Two of the models make use of an auxiliary transport equation for the turbulent kinetic energy to determine the eddy viscosity while the other two models are what are commonly known as zero-equation or algebraic models. Using these models, theoretical predictions are made
and compared with measurements.

The experimental investigation was conducted in a new asymmetric supersonic wind tunnel facility. This is a high Reynolds number facility, with a working unit Reynolds number in excess of 6 million per foot. The tunnel has a two-nozzle configuration with the nozzles having different contours. A thin splitter plate divides the tunnel into two, running along the axis of the tunnel and with the trailing edge extending slightly into the test section. Mixing of the two initially separated streams takes place just downstream of the trailing edge of the plate. The faster stream has a design Mach number of 3.5 while the other has a value of 1.5.

The primary flow diagnostic tool is a dual-beam fringe-mode laser Doppler velocimeter which is used to measure two-component velocities. Also, a pitot probe is used to obtain pressure and Mach number profiles in the free shear layer.

The ensuing analyses are broken down into the following chapters. In chapter II a review of previous theoretical and experimental investigations relevant to this study is undertaken. Chapter III discusses details of the theoretical analysis of the investigation. In chapter IV, the experimental apparatus and procedure are described, and chapter V
deals with results and comparisons. Finally, in chapter VI appropriate conclusions are drawn and recommendations made.
II. PREVIOUS INVESTIGATIONS

Free mixing layers (including wake flows) are commonly classified as thin shear layer flows to which Prandtl's boundary layer assumptions apply. Therefore within the framework of Prandtl's theory, the governing equations of motion for mixing layers are seen to be the same as the boundary layer equations. However, unlike boundary layer flows, the no-slip wall conditions do not apply in free shear layers due to the absence of a retarding body in the flow.

An immediate consequence of the relaxation of the no-slip conditions is a step change or discontinuity in flow properties at the trailing edge where mixing is initiated. This discontinuity or singularity, as well as the relaxation of the no-slip boundary conditions, present further complexity in solving the equations of motion for free shear layers. In view of this added analytical complexity of free shear flows, theoretical solution methods of these flows in the literature show a variety of approaches. To place this investigation in proper perspective, a review of the previous theoretical and experimental studies is undertaken in
this chapter, beginning with a discussion of the trailing edge singularity.

2.1 Theoretical Studies

2.1.1 Trailing Edge Singularity

The nature of the trailing edge discontinuity or Goldstein's singularity was first studied by Goldstein(1,2). In Ref.1, Goldstein treated the mathematical singularity in a symmetrical laminar wake by using a series expansion scheme to solve the equations of motion in the neighborhood of the trailing edge of a flat plate and matched it to the outer solution of the far field (Ref.2). His solution scheme is very similar to the present-day method of matched asymptotic expansions.

About the trailing edge, Goldstein showed the streamwise velocity to grow as the 1/3rd power of the streamwise coordinate. Comparison with the measurements of Fage and Faulkner (see Ref.2) showed good agreement very close to the trailing edge. Further downstream (greater than 2% of plate length) Goldstein's near wake solution showed pronounced disagreement with the measurements. However, at much lower plate Reynolds numbers (200-300) the measurements of Grove, Petersen and Acrivos(3) showed fair qualitative agreement with the near wake solution of
Goldstein at downstream distances much greater than the 2% plate length.

Stewartson(4) addressed the singularity problem in a laminar near wake with an external pressure distribution about the trailing edge. In his triple-deck expansion method, Stewartson divided the wake into three regions in the cross-plane. With an affine transformation of the flow variables, he then employed series expansions in each region, matching the solutions at the boundaries. Though his solution agreed with Goldstein's near wake solution, it failed to satisfy the triple-deck expansion hypothesis. To reconcile this disagreement, Stewartson argued that in the region of influence of the trailing edge singularity, which was of the order of -3/4th power of the Reynolds number, the triple-deck solution could indeed be matched to a previously obtained solution in this region.

Talke and Berger(5) also considered the trailing edge singularity in a laminar wake. Using series truncation method in an inner and outer asymptotic expansion matching procedure, they found the region of influence of the singularity to be the same as that given by Stewartson.

For turbulent flows, it is more difficult to assess the influence of the Goldstein's singularity
because of the Reynolds stresses in the Reynolds averaged momentum equation which lack constitutive relations. Eurggraf (6) investigated the singularity problem for turbulent wakes by applying series expansion in a manner similar to that of Goldstein (1). He employed the eddy viscosity concept to represent the Reynolds shear stress and imposed compatibility conditions to determine the coefficients of the expansion. Eurggraf also found the domain of influence of the singularity to be of the order of \(-3/4\)th power of the Reynolds number and emphasized the need to use a very small step size (of order \(-3/5\)th power of of the Reynolds number) in finite-difference calculations in order to obtain accurate numerical solutions in the neighborhood of the trailing edge. For laminar wakes, the solution reduces to that of Goldstein.

Alber (7) also analyzed Goldstein's singularity in the turbulent wake of a flat plate. In his analysis, the wake was divided into three regions: (1) the laminar inner wake region where the upstream sublayer predominates; (2) the turbulent inner wake and (3) the asymptotic region. In the first region the influence of turbulence was considered negligible and consequently Goldstein's solution was assumed here. In the turbulent inner region, the contributions of molecular transport (diffusion) was considered to be
much smaller than that of turbulence. For this region Alber obtained similarity solutions. He then matched these inner solutions to the asymptotic far wake solution to give a complete solution. Alber reported relatively good agreement with data.

Ames (3), on the other hand, accounted for the trailing edge singularity in his numerical solution of the laminar flat plate wake flow by applying Goldstein's solution in the immediate near wake (of the order of $3/4$th power of the Reynolds number). Beyond this range, he solved the rest of the wake numerically.

The above review shows that the handling of the trailing edge singularity problem requires specialized mathematical tools. However, in numerical solution of free shear layers the Goldstein's singularity can be considered by employing Burggraf's step size criterion.

2.1.2 Phenomenological Theories

Early theoretical treatment of free shear layer flows considered primarily mixing between parallel semi-infinite uniform incompressible free-streams with zero streamwise pressure gradients (see, for example, Abramovich (9), Pai (10) and Schlichting (11)). For such flows the equations of motion for two-dimensional shear layers are given by
where $x$ is the streamwise coordinate (axis), $y$, the normal or lateral coordinate and $t$ being time. $u$ and $v$ are the instantaneous streamwise and normal velocity components and $\rho$ and $\mu$ are the constant fluid density and absolute viscosity, respectively. For turbulent flows, applying Reynolds decomposition and averaging to the equations of motion lead to

\[ \rho\left( \frac{\partial \bar{u}}{\partial t} + \bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} \right) = \frac{\partial}{\partial y}\left( \mu \frac{\partial \bar{u}}{\partial y} - \rho \bar{u}' \bar{v}' \right) \]  

(2.3)

\[ \frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0 \]  

(2.4)

where \( \bar{u} \) and \( \bar{v} \) are the temporal mean velocities in the $x$ and $y$ directions respectively, $u'$ and $v'$ are the time dependent fluctuating components along the $x$ and $y$ axes, respectively, and $-\rho \bar{u}' \bar{v}'$ is the apparent or Reynolds stress which must be modelled for closure.

Most theoretical investigations considered the shear stress due to molecular diffusion to be much less than the Reynolds stress, thus simplifying Eq.(2.3) to

\[ \rho\left( \frac{\partial \bar{u}}{\partial t} + \bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{v}}{\partial y} \right) = \frac{\partial}{\partial y}\left( \mu \frac{\partial \bar{u}}{\partial y} - \rho \bar{u}' \bar{v}' \right) \]  

(2.3a)
In 1926, Tollmien (12) published what appears to be the first theoretical treatment of turbulent mixing layer flows. In his analysis he modeled the Reynolds stress term using Prandtl's mixing length hypothesis:

\[-\rho \overline{u'v'} = \rho l^2 \left| \frac{du}{dy} \right| \frac{du}{dy} \]

where \( l \), the mixing length, was expressed as a linear function of the streamwise coordinate, \( x \). That is

\[ l = c_m x \]

\( c_m \), being an empirical constant. Then using the similarity variable

\[ \eta = y/x \]

and the stream function

\[ \psi = \int f(\eta) dy \]

he reduced Eq. (2.3a) to the ordinary differential equation

\[ FF'' + 2c_m^2 F'''F'''' = 0 \]

where \( \tilde{u} = u + f(\eta) \) and \( F = \int f(\eta) d\eta \).

Tollmien solved Eq. (2.7) for the homogeneous mixing between free-stream and quiescent air with the following boundary conditions:
to obtain similarity solutions. $\eta_1$ and $\eta_2$ are the flow boundaries of the layer at the high speed and low (zero) speed sides, respectively, which were also determined from the solution. $u_1$ is the edge velocity of the streaming side. The measurements of Liepmann and Laufer (13) of a half jet-still air two-dimensional mixing layer showed fair agreement with Tollmien's solution.

Kuethe (14) extended Tollmien's method to the case of two-stream mixing layer. In his solution, he applied the same boundary conditions as Tollmien but for the condition

$$\nu(\eta_1) = 0$$

in place of which he used

$$u_1 \nu_1 + u_2 \nu_2 = 0$$  \hspace{1cm} (2.8)

at the suggestion of von Karman. $u_1$ and $\nu_1$ are the streamwise and normal edge velocities, respectively, on the high speed side of the shear layer while $u_2$ and $\nu_2$ correspond to those of the low speed side.

Goertler (see, for example, Refs. 9 and 11) solved Eq. (2.3a) using Boussinesq's eddy viscosity concept to
represent the Reynolds stress and modelled the eddy viscosity with Frandtl's exchange coefficient formulation. Thus,

\[-\rho \overline{uv'} = \rho \frac{\partial u}{\partial y}\]

where \( \xi \), the eddy viscosity was given by

\[\xi = K_b(u_1 - u_2) \quad (2.9)\]

\( K \) is an empirical constant. \( b \) is the width of the mixing layer which was taken to vary linearly with \( x \), as

\[b = cx\]

where \( c \) is also an empirical constant. Assuming self-similarity, he derived the ordinary differential equation

\[F''''(\eta) + 2\beta^2F(\eta)F''(\eta) = 0 \quad (2.10)\]

where

\[u_a = \frac{1}{2}(u_1 + u_2) \quad \eta = \sigma \frac{y}{x},\]

\[\lambda = (u_1 - u_2)/(u_1 + u_2) \quad \psi = x u_a F(\eta),\]

\[\sigma = \frac{1}{2}(Kc)^{\nu/4} \quad \text{and} \quad \bar{u} = u_a \sigma F'(\eta).\]

\( \sigma \) is the spreading parameter. With the asymptotic boundary conditions

\[\eta = \pm \infty \quad F''(\eta) = 1 \pm \lambda\]
and expanding $\mathcal{F}$ in power series, Goertler obtained the error function similarity solution

$$
\tilde{u} = u_0 \left[ 1 + \text{erf}(\eta) \right].
$$

Liepmann and Laufer (12) and Reichardt (see Ref. 11) compared their measurements with Goertler's solution for the single-stream mixing layer and showed good agreement for spreading parameters of 11 and 13.5 respectively.

Goertler employed only two boundary conditions even though the ordinary differential equation is third order. The solution therefore is not mathematically unique and would admit an infinite number of third boundary conditions, giving a family of parametric solutions. Experimental comparison with the solution could therefore only be made by "matching" the solution to the data at some selected reference. This reference is often taken to be the lateral position where the streamwise velocity equals $u_0$, the arithmetic average of the free-stream velocities.

After an exhaustive review of available experimental data, Reichardt (see Refs. 9, 10 and Hinze(15)) proposed an inductive theory for free shear flow turbulence. In this theory Reichardt assumed the right hand side of Eq. (2.1) to be negligible, thus reducing the momentum equation to
\[ \bar{\frac{\partial u^2}{\partial x}} + \bar{\frac{\partial u v}{\partial y}} = 0 \] \hspace{1cm} (2.11)

where the overbar denotes conventional time averaging.

He then expressed \( \bar{u v} \) as

\[ \bar{u v} = -\Lambda \frac{\partial u^2}{\partial y} \]

where \( \Lambda \), defined as a momentum transfer length, was said to be a function of \( x \) or a constant. This effectively reduced Eq. (2.11) to an analogous linear heat conduction equation, given by

\[ \bar{\frac{\partial u^2}{\partial x}} = \Lambda \bar{\frac{\partial u^2}{\partial y^2}} \] \hspace{1cm} (2.12)

which has the classical error integral solution.

Pai (16) studied the two-dimensional laminar and turbulent compressible mixing layers of jets and two semi-infinite streams. For the laminar compressible shear layer, Pai considered the steady state form of Eq. (2.1) and the continuity equation

\[ \bar{\frac{\partial (\rho u)}{\partial x}} + \bar{\frac{\partial (\rho v)}{\partial y}} = 0 \] \hspace{1cm} (2.4a)

while for the turbulent case, he treated the equations

\[ \bar{\rho u} \bar{\frac{\partial u}{\partial x}} + \bar{\rho v} \bar{\frac{\partial u}{\partial y}} = \bar{\frac{\partial}{\partial y} \left\{ -\bar{\rho u v} \right\} } \] \hspace{1cm} (2.3b)
Pai employed the eddy viscosity concept, expressing the eddy viscosity by
\[ \varepsilon = \varepsilon_0 \left( \frac{x}{L} \right)^n \]

where \( \bar{\rho} \) is the local mean density, \( \varepsilon_0 \) is the proportionality constant having dimensions of kinematic viscosity, \( L \) is a reference or characteristic length and the exponent, \( n \) lies between 0 and 1. The value of \( n \), according to Pai, indicated the degree of mixing of the shear layer. He then applied both small perturbation theory and von Mises transformation to reduce, in each case, the laminar and turbulent momentum equations to analogous heat conduction equations. Thus, for the same boundary conditions and appropriate choices of \( \varepsilon_0 \) and \( n \), Pai stated that solutions for laminar mixing layer flows could directly be applied to turbulent shear layers.

For \( n \) equal to 1, Ebershad and Pai(17) employed Pai's eddy viscosity model to solve the flow of a two-dimensional supersonic free jet. Using Goertler's similarity variable, they obtained density profiles which compared favorably with experiment for a spreading parameter of 17.
Also, Fai(18) considered density effects in the heterogeneous turbulent mixing of two free-streams in a constant temperature field in which molecular diffusion is a very important mixing process. Thus he solved, in addition, the species concentration diffusion equation after performing similar Reynolds decomposition and averaging on it. Prandtl's exchange coefficient relation was used to model the eddy viscosity and a similar expression was also adopted for the analogous species concentration eddy viscosity.

Then assuming the product $K_b$ in Eq. (2.9) to be the same for both the velocity and species concentration fields, he applied Tollmien's similarity variable and a stream function to reduce both the momentum and the species concentration equations to the single third order ordinary differential equation

$$F^{'''} + 2 \sigma F' F^{''} = -2A(F'^{''})^2/(1 - AF')$$  (2.13)

where

$$\lambda_p = (\rho_1 - \rho_2)/(\rho_1 + \rho_2) \quad \text{and} \quad \lambda = \lambda_p/(\lambda + \lambda_p).$$

$\sigma$ is Goertler's spreading parameter. Equation (2.13) is readily seen to be the non-homogeneous form of Eq. (2.10), given by Goertler.
Pai next expanded the right hand side of Eq. (2.13) binomially to first order to give

$$F'''' + 2 \partial^2 F F'' = -2 \lambda \rho(F'')^2 / \lambda.$$  \hspace{1cm} (2.14)

Making the substitution

$$F = F_0 + \lambda \rho F_1$$

he obtained the equations:

$$F''''_0 + 2 \partial^2 (F_0 F'' + F_0 F''') = 0$$  \hspace{1cm} (2.15a)

$$F''''_1 + 2 \partial^2 (F_0 F'' + F_1 F'''') = -2(F'')^2 / \lambda.$$  \hspace{1cm} (2.15b)

Equation (2.15a) is the same as Eq. (2.10). Introducing another similarity variable

$$\eta^* = \sigma \eta$$

and imposing the boundary condition

$$\eta^* = 0, \quad F'(\eta^*) = 1$$

in addition to the asymptotic conditions employed by Goertler he then solved Eq. (2.15) by the series expansion method of Goertler. Pai concluded that the effect of first order density differences on the shear layer mean velocity distribution is negligible.

Baker and Weinstein (19) employed the above analysis of Ref. 18 to both turbulent and laminar free
shear flows. However, for the laminar calculations they solved Eq. (2.13) by the "analytic continuation" method of Lesson (20).

Crane (21) reduced the momentum equations for laminar and turbulent compressible shear layers to

\[
\frac{\partial \gamma}{\partial Z} \frac{\partial \gamma}{\partial \xi} - \frac{\partial \gamma}{\partial \xi} \frac{\partial \gamma}{\partial Z} = \alpha \frac{\partial}{\partial Z} \left[ T^K \frac{\partial \gamma}{\partial Z} \right]
\]

(2.16)

where \( \gamma \) is a stream function which satisfies continuity. The independent variables, \( Z \) and \( \xi \) were given by the transformations

\[
z = \int_0^y \frac{P}{\rho_a} \, dy \quad \text{and} \quad \xi = \int_0^x f(x) \, dx
\]

where \( f(x) \) is an empirical function, which has the constant value of 1 for laminar flows. Crane expressed the eddy viscosity by

\[
\zeta = \xi_a \, f(x)
\]

where \( \xi_a \) is a constant eddy kinematic viscosity. Also, \( \xi_a \) was defined as the product \( \xi_a \rho_a \) and is equal to \( \mu_a \) for laminar flows. \( T^K \) is a dimensionless mean temperature. The subscript \( a \) denotes the point where \( \bar{u} \) is equal to \( u_a \). The exponent, \( \beta \) in Eq. (2.16) was given by

\[
\beta = \omega - 1
\]
where \( \omega \) is obtained from the viscosity-temperature relation

\[ \mu \propto T^\omega. \]

Using the stream function

\[ \psi = \left( \rho \alpha q \xi \right)^{1/2} g(\eta) \]

he converted Eq. (2.16) into the ordinary differential equation

\[ (T^\beta g^{\prime\prime\prime})^\prime + \frac{1}{2} (g g^{\prime\prime\prime}) = 0 \tag{2.17} \]

where \( \eta = z \left( \frac{u^*_p}{\alpha \xi} \right) \) and \( \bar{u} = \rho g^{\prime}(\eta) \).

The primes denote differentiation with respect to the similarity variable, \( \eta \). Further introducing the variable

\[ \xi = \frac{1}{2} (\eta - p) \]

\( p \), being the point where \( \bar{u} \) equals \( u^*_p \), he applied the boundary condition

\[ g^{\prime\prime\prime}(p) = 1 \]

and the asymptotic conditions employed by Goertler to solve Eq. (2.17), using double expansion series. For a Prandtl number of unity Crane obtained solutions to the energy equation by means of the Crocco integral relation. From his solution, he concluded that the
lateral spread of the mixing layer decreased with increasing Mach number. Also he stated that asymmetric distribution of temperature appeared to broaden the hotter side of the layer while shrinking the other. Crane pointed out that since $p$ is theoretically indeterminate the solution is therefore non-unique, like Goertler's.

Though Pai(19) did include initial conditions in his analysis, the effect of the upstream boundary layer at the trailing edge on the downstream free shear layer was first locked into by Torda, Ackermann and Burnett(22). They treated the compressible turbulent symmetric mixing layer of a flat plate. Prandtl's exchange coefficient formula was used to model the eddy viscosity while a quartic polynomial was assumed for the streamwise velocity profile. Applying the boundary conditions

$$y = \pm \delta, \quad \bar{u} = u_e, \quad \frac{\partial \bar{u}}{\partial y} = 0;$$

$$y = 0, \quad \frac{\partial \bar{u}}{\partial y} = 0,$$

$\delta$ being defined as half the shear layer thickness and $u_e$, the free-stream or edge velocity, they determined the constants of the assumed profile from Eqs.(2.3b) and (2.4b). They reported that initial conditions cause the shear layer to curve about the trailing edge,
the curvature flattening out in the far region.

The subsequent work of Chapman and Korst (23) also dealt with the effect of the initial boundary layers. For an incompressible free jet shear layer, they followed the small perturbation analysis of Pai (16), employing his eddy viscosity model as well. Assuming a power law initial velocity profile and with a value of 0.7 for the exponent, $n$ in the eddy viscosity model, they obtained solutions which showed good agreement with their measurements. Also, their data showed that the proportionality constant in the eddy viscosity expression changed with initial conditions.

Libby (24) applied Prandtl's exchange coefficient relation to the axisymmetric turbulent mixing flows of reactive gases. He treated the supersonic mixing of a hydrogen jet and an external free-stream. Using von Mises transformation Libby solved the resulting momentum equation to obtain what he called offset circular probability functions. Comparison of jet centerline velocity with experiment showed fair agreement. Energy and species concentration solutions were given by the respective Crocco integrals for unity Prandtl and Lewis numbers. Initial conditions were not accounted for.

Kleinstein (25) and Alpinieri (26) also analyzed various heterogeneous free shear flows using the
solution method of Libby. Kleinstein studied axisymmetric gas-jets exhausting into still air while the analysis of Alpinieri involved coaxial turbulent mixing of central hydrogen and carbon dioxide jets with outer air streams. They employed coordinate transformation to eliminate the eddy viscosity in the momentum equation. For a reverse transformation, a modified form of Prandtl's exchange coefficient relation was used for the product. The solutions of Kleinstein for the axial velocity, energy and species concentration compared favorably with measurements at Mach numbers of 0.6 to 2.6 for helium, nitrogen, carbon dioxide and air jets. He found the eddy viscosity to be independent of the lateral (radial) coordinate and that it was constant for most of the flow field. Alpinieri also found good agreement between his calculations and experimental results. Also the correlations showed the product of the eddy viscosity and the local density to depend only on the streamwise coordinate and also that initial conditions did dictate the length of the potential core.

Heterogeneous mixing of different gases was also studied by Conalsond and Gray(27). They investigated the turbulent mixing of helium, methane, nitrogen, carbon dioxide and freon jets with quiescent air for Mach numbers of 0.75 to 3.30. They used two
experimental velocity profiles; one for the core region and the other for the developed region, to calculate the Reynolds shear stress, based on an exchange coefficient eddy viscosity, and other integral parameters of interest. Donaldson and Gray noted that jet nozzle exit geometry did influence the initial decay and spreading rates of the shear layers and that a plane exit appeared to induce slightly higher mixing, initially, than a sharp edge nozzle.

Sabin(28) apparently first considered pressure gradients in turbulent free shear layer calculations. He theoretically and experimentally investigated the mixing of two semi-infinite incompressible streams with arbitrary free-stream pressure distributions and velocity ratios. Sabin treated the momentum equation

\[ \rho \left( \frac{u^2}{\delta x} + \frac{v^2}{\delta y} \right) = - \frac{dP}{dx} + \frac{\partial}{\partial y} \left\{ -\rho u^* w^* \right\} \]  

(2.3c)

and employed eddy viscosity formulation and von Mises variables to transform Eq.(2.3c) to

\[ \frac{d\tilde{P}}{dx} = u^* \frac{3P}{\partial \Psi^2} \]  

(2.18)

where \( \tilde{P} \), \( u^* \), and \( \Psi \) are normalized pressure, velocity and stream function, respectively. The dimensionless variable, \( \tilde{x} \) is the integral of the normalized eddy
viscosity. Next he expanded $\bar{P}$ and $u^*$ in Poincare series:

$$\bar{P} = P_0 + \beta P_1 + \beta^2 P_2 + \cdots$$

and

$$u^* = u_0 + \beta u_1 + \beta^2 u_2 + \cdots$$

where the parameter $\beta$ was defined by

$$\beta = \frac{(\bar{P}_1 - \bar{P}_2) / (\bar{P}_1 + \bar{P}_2)}.$$

$\bar{P}_1$ and $\bar{P}_2$ are normalized constant total pressures in stream [1] and stream [2], respectively. From the series expansion he obtained the zeroth and first order differential equations, giving the error function similarity solutions

$$\bar{P} = 1 + \beta \text{erf}(\eta)$$

$$u^* = \frac{1}{2}(u_1^2 + u_2^2) \left[ 1 + \frac{u_1^2 - u_2^2}{u_1^2 + u_2^2} \text{erf}(\eta) \right]$$

where

$$\eta = \frac{\psi}{2(\psi + \Theta_0)^{\frac{1}{2}}}$$

and

$$\psi = \int f(x) dx$$

$\Theta_0$, being an integration constant. In the analysis he used the eddy viscosity

$$\mathcal{E} = Kc(x + x_0)(u_1 - u_2).$$

(2.19)

$K$ and $c$ are the same constants defined by Goertler.

Sabin stated that the virtual origin, $x_0$, is due to a lateral shift of the shear layer. He compared his
solution to that of Goertler for the zero pressure gradient case and found good agreement for velocity ratios greater than 0.6. Below this value the agreement became progressively poor, being more pronounced towards the inner edge (low velocity side) of the layer, and was only fair for velocity ratios greater than 0.6. He did not consider initial boundary layer effects in the analysis.

Subsequent work on turbulent free shear layers in an external straining pressure field were undertaken concurrently by Anderson(29) and Lee(30). They both conducted theoretical and experimental studies of incompressible mixing layers of impinging and parallel free-streams for velocity ratios ranging from 0 to 1, using the same test facility for the experiment. Anderson applied the eddy viscosity concept and transformed Eq.(2.3c) into a dimensionless equation in von Mises variables which he solved numerically in the von Mises plane. For the eddy viscosity, he used both the expressions given by Goertler and Sabin. Comparison of the numerical solutions with his hot-wire measurements showed good agreement for velocity profiles, though Reynolds stress distributions showed pronounced disagreement. Anderson found that the effect of the external pressure gradient on incompressible free shear layers was negligible.
Initial boundary layer conditions were assumed at a downstream location 4% of the characteristic length of the upstream body.

In his theoretical analysis, Lee(30) numerically solved the continuity equation after eliminating the lateral velocity component by means of the momentum equation, Eq.(2.3c). He replaced the Reynolds stress with eddy viscosity relation, and assumed the eddy viscosity to be a function of the streamwise coordinate only. The calculated velocity profiles compared favorably with his data at free-stream velocity ratios of 0.35 and 1.0. Correlation of the eddy viscosity did show lateral coordinate dependency, though only slight. Initial conditions were accounted for in the numerical solution.

Sirieix and Solignac(31) performed theoretical and experimental investigations of zero pressure gradient supersonic turbulent mixing of half jet and quiescent air. In the theoretical analysis, they employed Crane's eddy viscosity relation, with a proportionality coefficient which varied linearly with the streamwise coordinate, as

$$
\varepsilon_0 = \frac{\kappa x}{(4D^2)}
$$

where $\sigma$ is Goertler's spreading parameter and $\kappa$ is the jet efflux velocity. By assuming that
\[ \bar{v} \ll \bar{u} \quad \text{and} \quad \bar{u} \frac{\partial \bar{u}}{\partial x} \approx \frac{1}{\bar{v}} \frac{\partial \bar{c}}{\partial x} \]

and introducing the independent variable

\[ \xi = \frac{1}{\bar{v}} \int_0^x \bar{x} f(\bar{x}) d\bar{x} \]

where \( f(\bar{x}) \) is an empirical function, Eq.(2.3b) was simply reduced to the dimensionless linear equation

\[ \frac{\partial \psi}{\partial \bar{y}} = \frac{\partial^2 \psi}{\partial \bar{y}^2} \quad . \tag{2.20} \]

\( \bar{x} \) and \( \bar{y} \) are dimensionless streamwise and lateral coordinates, respectively, and \( \psi \), the normalized velocity. Equation (2.20) yields the classical error integral solution. Comparison of the error integral velocity profiles with their data at a Mach number of 3.0 was generally fair. Energy and species concentration profiles for heterogeneous shear layers, calculated in the same manner, showed comparatively better agreement with their measurements. Initial conditions as well as the virtual origin were considered in the analysis.

Korst and Chow(32,33), Lilienthal(34) and Korst et al(35) also treated zero pressure gradient turbulent free shear layer flows by reducing Eq.(2.3b) to an analogous heat conduction equation. References 32 and
33 considered compressible mixing layers, Ref. 34 dealt with both theoretical and experimental analyses of a supersoronic-subsonic mixing while the work of Ref. 35 involved the theoretical investigation of compressible and incompressible free shear flows. The analysis of Korst and Chow (32) is basically the same as that of Ref. 31. They treated a non-isocientic mixing layer of two compressible free-streams, with a Prandtl number of 1. The velocity profiles were given by the error integral solution while Crocco integral relation gave the energy distribution. Initial boundary layer conditions were accounted for in the solution.

References 33, 34 and 35 arrived at the analogous dimensionless heat conduction equation by employing von Mises transformation, with all three analyses using the same solution method. In the resulting equation, the eddy viscosity was expressed only implicitly through a coordinate transformation of the streamwise coordinate. This equation was further transformed into a third order ordinary differential equation of a dimensionless stream function.

Korst and Chow (33) and Korst et al (35) solved this third order differential equation numerically, imposing the asymptotic boundary conditions. In both references a Prandtl number of 1 was assumed, thus giving solutions of the energy equation by the Crocco integral
relation. Only Ref. 35 included initial conditions in the solution. Lilienthal (34), on the other hand, solved the resulting linear "heat conduction" equation numerically in the von Mises plane. His eddy viscosity varied with the streamwise coordinate only. The theoretical velocity profiles compared favorably with his supersonic-subsonic shear layer data. For the initial velocity profiles he assumed the 1/7th power law variation.

Harsha (36) reviewed various theoretical treatments of free shear flows, including some of the work discussed above. His review showed that most studies employed Prandtl's mixing length or exchange coefficient turbulence models or various modifications thereof and that these locally-depended models cannot predict a wide range of turbulent free mixing layers satisfactorily.

Mills (37) investigated both compressible and incompressible, laminar and turbulent free mixing of two semi-infinite streams. Mills employed Crocco variables to transform the momentum and energy equations to the coupled equations:

$$\frac{\partial}{\partial x} \left[ \frac{\rho E u}{\tau} \right] + \frac{\partial^2 \tau}{\partial u^2} = 0 \quad (2.21)$$

$$\tau \left\{ \frac{d}{du} \left( \frac{1}{\rho u} \frac{du}{dx} \right) + 1 \right\} + (1 - P) \left( \frac{1}{\rho u} \frac{du}{dx} \right) \frac{\partial \tau}{\partial u} = 0 \quad (2.22)$$
where $I, \tau, \bar{u}$ and $\rho$ represent the total enthalpy per unit mass, the shear stress, the streamwise velocity and the Prandtl number respectively for laminar or turbulent flow. $\epsilon$ is either the laminar kinematic viscosity or turbulent viscosity. To obtain similarity solutions he expressed the shear stresses in the forms:

$$\tau = \left[ \frac{\rho \epsilon (u_1 - u_2)^3}{8 \lambda \chi} \right] F(u^*, \lambda, \lambda', \gamma, \omega, \rho, M_1) \quad (2.23)$$

for the laminar case and

$$\tau = \rho \sigma K_c (u_1 - u_2) F(u^*, \lambda, \lambda', \gamma, \omega, \rho, M_1) \quad (2.24)$$

for the turbulent flow, for which he used Prandtl's exchange coefficient to model the eddy viscosity.

$K, \sigma,$ and $c$ are the same Goertler parameters. In these equations

$$u^* = (\bar{u} - u_2)/(u_1 - u_2) \quad \text{and} \quad \lambda^1 = (T_1 - T) / (T_1 + T_2)$$

where $T_1$ and $T_2$ are the total temperatures of stream [1] and stream [2] respectively. $M_1$ is the Mach number of stream [1], $\gamma$, the specific heat ratio and $\omega$ is the exponent of the viscosity-temperature relation as given previously. He related the total enthalpy to the total temperature by the calorically perfect gas law. Mills then integrated the resulting equations using the method of successive approximation. The theoretical
solutions compared well with his incompressible flow measurements for various velocity ratios. Theoretical solutions for the compressible mixing layer were also presented. He did not consider initial boundary layer effects.

Turbulent free shear flows of wakes and jets have also been extensively studied (see, for example, Abramovich(9), Batchelor(38) and Townsend(39)). A few, relevant to the present investigation are discussed next.

The flow structure of turbulent wakes and jets have been extensively investigated by Townsend(39,40,41). His equilibrium or self-preservation hypothesis is most familiar and does not require further discussion here.

El-Assar and Page(42) investigated the zero pressure gradient incompressible wake of a flat plate. They used Sabin's eddy viscosity model and von Mises transformation to reduce the momentum equation to a non-linear heat conduction type which was then solved numerically in the von Mises plane. The numerical solutions showed reasonable agreement with their measurements. Initial velocity profile was assumed to have the 1/7th power law variation.

Incuye, Marvin and Sheaffer(43) considered both compressible and incompressible zero pressure gradient
symmetric wake flows. They employed the outer eddy viscosity model of Cebeci, Smith and Monsinski (44) which is given by

$$\eta_0 = K_o \int_0^\infty (u_e - \bar{u}) dy$$

(2.25)

where $K_o$ is an empirical constant and $u_e$ is the free-stream or edge velocity. They solved the equations of motion numerically, calculating the initial conditions as well. The calculated velocity profiles did compare well with the incompressible wake data of Chevray and Kovasny (45), though the shear stress showed pronounced disagreement. To improve the numerical solution they modified the outer eddy viscosity to

$$\eta_0 = K_o \left\{ 2 - X exp(1 - X - 4[X - 1]^2) \right\}$$

(2.26)

for $X$, the normalized streamwise coordinate, equal to or greater than 1. For the same value of $K_o$ of 0.0168, the modified eddy viscosity made little difference in the numerical solution.

Other more recent wake flow analyses include Toyoda and Hirayama (46, 47) and Agrawal, Pande and Prakash (48), all dealing with zero pressure gradient, flat plate wakes. The work of Refs. 46 and 48 involved both theoretical and experimental investigations of incompressible wakes, while Ref. 47 included compressible wakes also. Reichardt's inductive theory
was applied in all three studies to reduce the momentum equation into an analogous dimensionless heat conduction equation, and all considered initial conditions. References 46 and 48 used experimental initial conditions while Ref. 47 employed the 1/3rd, 1/5th, 1/7th and 1/9th power law variations. Toyoda and Hirayama (46, 47) used the error function solution of the resulting momentum equation while Ref. 48 integrated it, using a step by step numerical scheme.

For the incompressible wake, the error function solutions compared well with the maximum velocity defect, shape parameter and streamwise velocity measurements of Toyoda and Hirayama. The supersonic wake data of Ref. 47 also agreed with the error function maximum velocity defect. The comparison of Ref. 48 was fair. Reference 47 showed that the various initial profiles did influence the near wake. Asymmetric boundary conditions were also considered by Toyoda and Hirayama (47).

2.1.3 Energy Methods

Auxiliary transport equations have also been employed in the theoretical analyses of turbulent free shear flows. Two such commonly used equations are the Reynolds stress and turbulent kinetic energy conservation equations. Harsha (49) recently reviewed
various analyses that use energy methods. Here, only those analyses that employ the turbulent kinetic energy equation are of interest.

The turbulent kinetic energy transport equation for two-dimensional compressible free shear flows is

\[ \overline{\rho} \frac{\partial \overline{k}}{\partial t} + \overline{\rho} \frac{\partial \overline{k}}{\partial y} = -\overline{\rho} \overline{u} \frac{\partial \overline{v}}{\partial y} - \frac{\partial}{\partial y} \left[ \overline{\rho} \overline{v} \frac{\partial \overline{v}}{\partial y} + \overline{k} \frac{\partial \overline{v}}{\partial y} \right] - \overline{\rho} \overline{\Phi} \tag{2.7} \]

where the left hand side of the equation is the advection of turbulent kinetic energy and on the right hand side are, from left to right, the production, diffusion and dissipation terms, respectively. \( \overline{p}' \) is the fluctuating static pressure. \( \overline{k} \), the turbulent kinetic energy per unit mass is commonly defined by

\[ \overline{k} = \frac{1}{2} \left( \overline{u'^2} + \overline{v'^2} + \overline{w'^2} \right) \]

where \( \overline{w'} \) is the fluctuating velocity component in spanwise direction. The diffusion term is often expressed by the gradient diffusion model

\[ \overline{\rho} \overline{v} \frac{\partial \overline{v}}{\partial y} = -\overline{c_k} \frac{\partial \overline{k}}{\partial y} \]

and the dissipation modelled after isotropic turbulence as

\[ \overline{\rho} \overline{\Phi} = c_k \frac{\overline{k}^{\frac{3}{2}}}{\overline{k}} \]
where $c_k$ is defined as the coefficient of turbulent kinetic energy diffusion, $c_k$ is an empirical proportionality constant and $l_k$ is the dissipation length scale. Using Boussinesq's eddy viscosity relation, Eq. (2.27) becomes

$$\tilde{\rho} \frac{\partial \tilde{k}}{\partial t} + \tilde{\rho} \frac{\partial \tilde{k}}{\partial y} = \frac{2}{\partial y} \left( \frac{\partial \tilde{\rho}}{\partial y} \right) + \tilde{\rho} \left[ \frac{\partial u}{\partial y} \right]^2 - c_k \tilde{\rho} \frac{k^{3/2}}{l_k}. \quad (2.28)$$

Lee and Harsha (50), Lee and Auiler (51), Harsha and Lee (52), Lee et al. (53) and Harsha (54) analyzed turbulent free mixing layers by numerically solving the equations of motion simultaneously with Eq. (2.28) and the correlation (see Bradshaw, Ferriss and Atwell (55) and Harsha and Lee (56))

$$\tilde{\zeta} = -\tilde{\rho} \tilde{u} \tilde{v} = a_1 \tilde{\rho} \tilde{k} \quad (2.29)$$

which is also a consequence of the "structural similarity" postulate of Townsend (39, 40). $a_1$ is an empirical constant, found to be 0.15 and 0.2 by Refs. 55 and 56, respectively, for boundary layer flows.

Inspection of Eq. (2.29) readily reveals two physical inconsistencies. One is that the Reynolds stress can either be positive or negative though the turbulent kinetic energy is always positive, by definition. The other is that the turbulent kinetic energy is everywhere non-zero while the Reynolds stress may
vanish at some point in the shear layer. To circumvent these inconsistencies, the following corrections were made. The subscript, max, denotes maximum values at a given cross-section. In these studies, the diffusion coefficient, \( \frac{\varepsilon}{\kappa} \) was given by

\[
\frac{\varepsilon}{\kappa} = \frac{0.3}{\left| \frac{\partial u}{\partial y} \right|/\left| \frac{\partial u}{\partial y} \right|_{\text{max}}}
\]

were made. The subscript, max, denotes maximum values at a given cross-section. In these studies, the diffusion coefficient, \( \frac{\varepsilon}{\kappa} \) was given by

\[
\frac{\varepsilon}{\kappa} = \frac{0.3}{\left| \frac{\partial u}{\partial y} \right|/\left| \frac{\partial u}{\partial y} \right|_{\text{max}}}
\]

where \( \frac{\varepsilon}{\kappa} \) is an empirical constant, synonymous with turbulent Prandtl number. Using various algebraic expressions for the dissipation length scale, Eqs. (2.28) and (2.29) were then solved simultaneously with the equations of motion for the variables \( \overline{u}, \overline{v}, R, \overline{\varepsilon}, \) and \( \varepsilon \), using the numerical method of Spalding and Patankar (57).

The work of Ref. 50 dealt with incompressible two-dimensional and axisymmetric shear layers. The dissipation length scale was taken as the width of the layer. The two-dimensional calculations showed good agreement with the measurements of Lee (30) for a freestream velocity ratio of 1.0 while both the velocity and shear stress distributions showed marked
differences with the data at velocity ratios less than 1.3. Comparison of the numerical solutions with the data of Zawacki and Weinstein (55) showed pronounced disagreement for the axisymmetric case. Lee and Auilar (51) treated flat plate and cylinder wakes, using a dissipation length scale equal to half the layer width. Their velocity profiles compared well with the data of Chevray and Kovasnavay (45) while the shear stresses did not agree as well. The cylinder wake solutions agreed only fairly with the measurements of Townsend (39).

Harsha and Lee (52) extended the analysis of Ref. 50 to heterogeneous coaxial free mixing. The 1/7th power law variation was assumed for the initial velocity profiles. Comparison of the numerical solutions with data was fair. Lee et al (53) investigated both incompressible and compressible, isothermal and non-isothermal and homogeneous and heterogeneous mixing of two-dimensional and axisymmetric shear layers. The empirical constants used in the various conservation equations were for equilibrium turbulence. In the asymptotic region the numerical solutions compared favorably with experimental data for hydrogen-air coaxial mixing.

Harsha (36, 54) treated both homogeneous and heterogeneous two-dimensional and axisymmetric mixing
layers. In his calculations the constant $a$ was allowed to vary with the lateral coordinate, and the variation determined from turbulent kinetic energy and shear stress profiles of two-dimensional incompressible shear layer. Comparison of the solutions with measurements showed fair agreement. In the review paper, Harsha(36) showed that the energy method was superior in predicting more general class of free shear flows.

Bradshaw(59) applied the analysis of Ref.55 to calculate symmetric incompressible near wakes. In this approach, the turbulent kinetic energy diffusion was modelled in terms of the bulk convection of the large scale motion as

$$
\rho' \nu' + kv' = \rho \frac{\partial}{\partial x} (\bar{U}^{2} \frac{\partial C_{y}}{\partial y})
$$

while the dissipation term was expressed by

$$
= \rho \frac{\partial}{\partial x} \left| \bar{U}^{2} \right| \frac{1}{\lambda_{k}(y/\delta)}
$$

where $\lambda_{k}(y/\delta)$, the dissipation length scale and the diffusion function, $g(y/\delta)$ are determined experimentally. Using Eq.(2.29), Eq.(2.27) was then reduced to a hyperbolic shear stress equation which could be solved either numerically or by the method of characteristics. Bradshaw used the method of characteristics and assigned the out-going characteristics as the "inner wake," which extended
less than 2% of the wake thickness from the centerline. In this inner wake region, the length scale was taken as the product of the local friction velocity and the reciprocal of the velocity gradient, while it varied with the $5/2$nd power of the lateral coordinate outside the inner layer. Comparison of the shear stress profile with the data of Ref. 45 at 9.6 initial momentum thicknesses downstream showed excellent agreement in the inner wake, though the agreement became progressively poorer towards the edge. Calculated and measured displacement thicknesses showed fair agreement.

Using an "interaction hypothesis" Morel, Torda and Bradshaw (60) and Morel and Torda (61) applied the hyperbolic shear stress equation of Ref. 55 to analyze free shear flows. The interaction hypothesis was partly formulated to eliminate the two problems associated with Eq. (2.29), as discussed above. For shear flows with velocity extremum, the flow is considered to be made up of two adjoining "simple" shear layers which interact with each other only through the mean velocity field. Each simple layer has its own shear stress field and the shear stress in the overlap region about the extremum is the algebraic sum of the two simple layers in the region. The momentum equation was therefore given by
where the superscripts, + and −, denote the simple layer of positive and negative shear stresses, respectively.

Their calculations revealed that interaction did affect the structure of the simple layer, but only changed the magnitudes of the constants and influenced the diffusion function, $g$ only mildly. Thus, constant values of $a_4$ and $1/\delta$ were used in the calculations, while $g$ was slightly modified. For compressible free shear flows they invoked the postulate of Morkovin (62) of passive turbulence convection for Mach numbers less than 5 and thus assumed the compressible shear layer structure to be Mach number independent. Consequently, they used incompressible parameters in the compressible flow calculations. Comparisons of the theoretical solutions were, in general, fair. Reference 61 also considered an additional auxiliary transport equation for the product $1/k$ from which the dissipation length scale was obtained. The solutions for the two length scales were practically the same.

Huffman and Ng (63) applied the interaction hypothesis to symmetric and asymmetric, two-dimensional and axisymmetric near wake flows. Applying Burggraf's step size criterion to account for Goldstein's
singularity, they integrated the equations of motion using an implicit difference scheme. Initial conditions, obtained either experimentally or from the calculation of the upstream boundary layer, were used. Their solutions showed acceptable agreement with measurements.

Leuchter(64) conducted both theoretical and experimental investigations of several incompressible free shear layers, with and without pressure gradients, using the turbulent kinetic energy transport equation. Like Refs.50 to 54, he used both gradient diffusion and isotropic turbulence models for the diffusion and dissipation terms. The diffusion exchange coefficient was taken to be proportional to the eddy viscosity while the dissipation length scale was obtained from an additional conservation equation for the product $k^2$. He employed the Prandtl-Kolomogrov eddy viscosity model

$$
\nu = \frac{1}{k^2} \frac{1}{k}
$$

and solved the system of equations numerically. In general, the solutions agreed with his measurements and those of others. However, the measurements of Chevray and Kovasney(45) showed that his shear stress profiles underpredicted the data for the flat plate zero pressure gradient near wake.
2.2 Experimental Studies

Experimental investigation of turbulent free shear flows has continued for the past four decades. In recent years, advances in instrumentation, diagnostics and data acquisition have aided in the measurements of important characteristic parameters with improved accuracy and reliability. Such information has helped to provide a better understanding of free shear layer turbulence.

In 1949, Liepmann and Laufer (13) performed extensive hot-wire measurements of mean velocities, shear stress, turbulence intensities and scales in two-dimensional incompressible half jet--still air mixing layer. The measurements revealed that exchange coefficient and constant mixing length eddy viscosity models do not adequately describe turbulence transport. Correlations with the theoretical solutions of Tollmien and Goertler gave spreading parameters of 12 and 11 respectively, though the shear stress profiles showed marked disagreement with the data for these values. Both the micro-scale and the integral scale were found to be constant at a given cross-section, the integral scale varying linearly with the streamwise coordinate. Their data showed that the turbulence field was non-isotropic.
Laurence (65) measured mean velocity, intensity, scale and spectra in a subsonic free jet of a 3.5 inch diameter nozzle with hot-wires. From his correlation measurements, he deduced the integral scales to be linear with the streamwise coordinate and that they were nearly Mach number and/or Reynolds number independent. The spectra exhibited a shift of energy to low frequency components as the downstream distance increased. The turbulence intensities became self-preserving at about 8 nozzle diameters downstream.

Davies, Fisher and Barrett (66) and Davies (67) investigated the turbulence structure and noise production of a 1.0 inch diameter nozzle free jet at Mach numbers of 0.20 to 0.55. Hot-wires were used to measure velocity, shear stress, intensity, scales, correlations and the integral time scale of the maximum energy convection reference frame. They also found the integral length scales to vary linearly with the streamwise coordinate, whereas the time scale was inversely proportional to the local shear. The convection velocity of the large scale motion, defined as the velocity for a given time delay at which a wire-separation curve is tangent to streamwise correlation envelop, was determined to be a little more than half the jet efflux velocity. The turbulent intensities showed Mach number dependence, decreasing with
increasing Mach number. The mixing layer was inferred to consist of a system of disk-shaped eddies which originate at the jet exit.

Kocin (68) also investigated the structure and noise generation mechanism in low to high subsonic turbulent free jets with hot-wire measurements and optical observations. The observations revealed a double structure, as was proposed by Townsend (39), with large scale events occuring very close to the jet nozzle exit. The temperature and Mach number profiles for hot and cooled jets showed self-similarity about 4 diameters downstream. Other results were consistent with those of Refs. 66 and 67.

Bradshaw, Ferriss and Johnson (69) studied the structure of turbulence in the noise-producing region of a 2 inch diameter round jet at a Mach number of 0.3. They measured mean velocity, shear stress, intensity and spectra with hot-wires and inferred that the noise-producing region was dominated by a group of large eddies which contain about 25 % of the shear stress. They observed that the large eddies were inclined at 45 with the lateral coordinate and appeared to diffuse early in the development of the asymptotic region. The finer structure was found to be non-isotropic.

Using the same facility Bradshaw and Ferriss (70) analyzed the longitudinal wavenumber spectrum 2
diameters downstream. From spectral energy balance, they discovered that close to the line of symmetry energy was transferred from the high wavenumber components to the low wavenumbers to compensate for low wavenumber energy lost through diffusion, exhibiting reverse energy cascade. This negative energy production was speculated to be due to a rapid increase in intensity, saturating the low wavenumbers. Bradshaw and Ferriss also deduced the turbulence field to consist of large eddies superimposed on a background of nearly isotropic fine structure.

Wygnanski and Fiedler (71) made extensive hot-wire measurements of conventional and conditioned flow properties in an incompressible single stream shear layer formed downstream of a backward facing step with trip. They measured velocity, Reynolds stress, intensity, scales and spectra as well as higher order correlations and intermittency. Their mean velocity data agreed with those of Liepmann and Laufer (13) for a spreading parameter of 9. The intensity profiles also compared well with those of Ref. 13. The zone-averaged intensity data showed the turbulence field to be non-isotropic. They determined a convection velocity of approximately 85% of the free-stream velocity, which is higher than that of Refs. 66 and 67. The integral scales were also found to vary linearly with the
streamwise coordinate while the micro-scale increased only slowly with downstream distance. From their data Wygnanski and Fiedler inferred that the eddy viscosity and gradient diffusion concepts appeared to be adequate, and the eddy viscosity was found to be approximately constant across the entire flow. The turbulent-potential flow intermittency had the double error function (bell shape) distribution.

Spencer(72) employed both hot-wire and hot-film bleed pressure probes to measure mean velocity, shear stress, intensity, scale, turbulence kinetic energy, fluctuating pressure and higher order moments in a two-stream plane mixing layer at various free-stream velocity ratios. Self-similarity was achieved by the mean velocity and intensity profiles sooner than the fluctuation pressure field. The longitudinal integral scale increased linearly with downstream distance whereas the micro-scale was found to be reasonably uniform across the layer. Spencer observed that the turbulence structure comprised large, energetic eddies which emanated in the central region of the layer. The mixing layer with a velocity ratio of 0.3 was inferred to be isotropic while the 0.6 layer was not. The inertial subrange, in which the wavenumber spectrum grows by the \(-5/3\)rd power, was notable.
Antonić and Silger (1973) measured mean velocity, Reynolds stress, intensity, scale and spectra in incompressible shear layers of an axisymmetric jet and two-dimensional co-flowing streams. Hot-wires and Pitot probes were used to acquire the data. The streamwise intensity, dissipation length scale and spectra did not attain asymptotic forms by the last measurement station though the mean velocity profile quickly became self-similar. Applying Taylor's hypothesis to the centerline auto-correlation, they found the integral scale to decrease with downstream distance, while the micro-scale, deduced from isotropic dissipation, showed only a slight increase.

Ikawa (1974) investigated the turbulence structure of a two-dimensional shear layer of a 2.47 Mach number free-stream and still air, with mass injection, behind a rearward facing step. Pitot, static and hot-wire probes were used to measure pressure, mean velocity, temperature, shear stress, intensity, scale and correlations. The data showed the spectral density profiles to attain self-similarity farther downstream than the mean velocity field. Ikawa also found that the effect of compressibility is to reduce the spreading rate of the shear layer. The large scale eddies convected downstream at the mean velocity of the dividing streamline. The streamwise and lateral
integral scales increased linearly with the streamwise coordinate and were approximately equal.

The work of Jones, Planchon and Hammersley (75) involved the measurement of mean velocity, intensity, scale, correlations and intermittency in an incompressible shear layer of two isothermal parallel streams with hot-wires in the self-similar region. The streamwise auto-correlations decayed exponentially in lag time and the integrals scales showed a linear increase with downstream distance. The convection velocity of the large scale motion was approximately equal to the layer centerline value and approached the local mean velocity with increasing wavenumber.

Demetriades (76, 77) made measurements of mean velocity, mean temperature, intensities, spatial scales and spectra in a supersonic flat plate wake at a Mach number of 3.0. Hot-wires, Pitot and static pressure probes were used for the measurements. Using a similarity variable, Demetriades fitted the mean values to Gaussian distributions. The intensities as well as the fluctuating density field relaxed to asymptotic trends much farther downstream than the mean flow field. From the spectra the integral scales were found to vary with the $-1/2$nd power of the downstream distance.
Bailey and Kuethe(73) employed a fine Pitot probe to measure mean velocity in the mixing layer of two supersonic free-streams at various Mach number, pressure and initial boundary layer thickness ratios. The velocity traverses showed that Reynolds number effect was negligible. The different downstream distributions were not compared for similarity. Baker and Weinstein(79) measured mean velocity and intensity in a heterogeneous shear layer of parallel free-streams at various density and pressure ratios while Hill and Page(80) made mean velocity measurements in the initial stage of development of two-dimensional supersonic mixing layers behind a rearward facing step. Weidner and Trexler(81) investigated the mixing of two supersonic streams in the presence of a network of oblique shocks and Eggers and Torrence(82) and Eggers(83) measured mean velocity and species concentration profiles in an air-air and air-hydrogen coaxial mixing layers, respectively, with hot-wires.

Patel(84) made mean velocity, shear stress and intensity measurements in the two-dimensional shear layer of a jet exhausting in quiescent air at three streamwise locations in the asymptotic region. His mean velocity, shear stress and intensity data were persistently larger than those of Liepmann and Laufer and smaller than the measurements of Wygnanski and
Fiedler(71).

Morris(35) made two-component measurements of mean velocities and intensities, as well as skewness and kurtosis (flatness factor) in the mixing layer of a compressible jet and a parallel stream with a two-color laser Doppler velocimeter. The mean velocity and intensity distributions exhibited self-similarity. Morris inferred from the skewness and kurtosis that the turbulence structure was independent of the ratio of the jet and free-stream velocities.

Fiedler(86) studied the turbulence structure of the free shear of a heated jet exhausting into stagnant air from temperature measurements, using a resistor probe. The data showed that the asymptotic mean temperature had three inflection points and the intensity had two corresponding maxima and a minimum. From these observations, Fiedler inferred that the turbulence structure was dominated by large-scale coherent motion or vortices, across each of which the temperature variation was approximately linear.

Weir and Bradshaw(37), Sguier, Fulachier and Keffer(85) and Andreopolus and Bradshaw(39) employed conditional sampling techniques to analyze the structure of turbulent free shear layers. In these investigations the shear layer was temperature-conditioned by slightly heating the layer such that the
heat was transported only passively. Weir and Bradshaw treated the shear layer beyond the potential core of a jet, one half of which was slightly heated. They measured zone-averaged shear stresses, triple velocity products and the intermittency of the warm fluid. From their results they concluded that the interaction of the warm and cold shear layers, initially separated by the potential core, strongly altered their turbulence structure.

Céguier, Fulachier and Keffer measured mean velocity, mean temperature, velocity and temperature intensities and intermittencies in the shear layer of slightly heated free jet with temperature asymmetry. The jet was bounded on one side by stagnant air while the other side had a free-stream of equal velocity. The data showed that the mean profiles and the intermittency factor became self-similar about 20 diameters downstream, though the mean temperature appeared to evolve more slowly to the asymptotic state. These profiles also revealed a region of negative temperature fluctuation, resulting from the transport of turbulent energy up the wavenumber gradient to the low-frequency spectral components.

Andreopoulos and Bradshaw made conventional and zone-averaged velocity, shear stress, intensity and triple velocity product and temperature intermittency
measurements in symmetric and asymmetric flat plate wake flows, with one of the upstream boundary layer slightly heated. They observed that in the near wake the mixing mechanism involved only the fine-scale structure. Their results also revealed a three-layer structure, consisting of central region of mixed fluid and bounded on either side by layers in which mixed and unmixed fluids continuously interact to change the turbulence structure.

Rajagopalan and Antonia(90, 91) also made conventional and conditional averaged measurements of the turbulence structure in a two-dimensional shear layer of a slightly heated jet issuing into quiescent atmosphere with hot- and cold-wire probes. Also, intensities, auto- and cross-correlation and spectra were measured. The mean data of Ref. 90 showed similarity just after 1.5 diameters downstream, though the velocity and temperature intensities became self-preserving farther downstream. Like Fiedler(36), Ref. 90 also reported that the temperature intensity distributions exhibited two maxima, corresponding to three inflexion points in the mean profile. The conditional-averaged measurements of Ref. 91 revealed signatures of vortex-like large structures. The large structures were found to be inclined at 40 to the jet axis. The convection velocity was determined from
auto-correlation as well as the cube of the temporal derivative of the temperature. Both methods showed the same variation of the convection velocity across the layer.

Most recently, Ramaprian, Patel and Sastry \(^{(92)}\) measured mean velocities and Reynolds stresses in the developing symmetric wake of a flat plate with hot-wire, Pitot and static pressure probes. Near the trailing edge the shear stress and lateral normal stress distributions were found to overshoot their asymptotic values. The measurements showed self-similarity after about 25 initial momentum thicknesses downstream.

Some previously cited work (Refs. 22 to 31, etc.) also included measurements of flow properties in various mixing layers with hot-wires and/or pressure probes. References 45 to 48 give data for mean velocities, shear stress and/or intensities. Chevray Kovasnavy \(^{(45)}\) observed that the maximum intensities generally occurred near the zone of maximum shear, away from the line of symmetry.

2.3 Effect of Initial Conditions

It has long been recognized that the upstream boundary layer does influence the development of the free shear layer, though the upstream memory would
eventually be lost to asymptotic trends in the far, self-preserving region. However, the structure of the initial stages of the shear layer and the relaxation distance to self-preservation directly depend on the upstream history. Thus it is appropriate to account for initial effects in theoretical analysis of free shear layers. Some theoretical treatments that included the effects of the initial boundary layer were discussed in Section 2.1. In this section the experimental investigation of this effect is discussed.

Bradshaw(92) studied the effect of the upstream boundary layer on the shear layer of a 2 inch diameter jet exhausting into quiescent atmosphere. He found that the shear layer was sensitive to upstream history up to about 1000 initial momentum thicknesses, for both laminar and turbulent boundary layers. For both initial boundary layers, the shear layer relaxed rather slowly to the asymptotic state because of initial overshoot in turbulence parameters. With increasing Reynolds number, the virtual origin changed from a negative to a positive value for the initially laminar boundary layer but remained positive for turbulent boundary layers.

For both tripped and untripped initially laminar boundary layers Batt(94) determined the relaxation distance to self-similarity of the mixing layer of a
half jet and still air to be 1500 initial momentum thicknesses. The tripped case also resulted in larger mass entrainment and consequently thicker layer. In the near region the fluctuations were more intense for the triped case, while in the self-preserving region the intensities decayed approximately to the same asymptotic values, for both initial conditions.

Champagne, Poa and Wygnanski (35) arrived at conclusions generally consistent with those of Bradshaw and Satt, for the two-dimensional shear layer of a half jet and still air behind a step. The boundary layers on the step were both laminar, with one case triped just upstream to hasten transition. However, unlike Bradshaw's observations, the virtual origin remained non-negative for both triped and untriped cases. For the initially excited two-stream shear layer, Öster et al (36) observed a dramatic increase in the spreading rate, which reached a peak with increasing forcing amplitude and then decayed to that of the unexcited layer in the asymptotic region.

Similar observations were made by Fiedler and Thies (37) in their high Reynolds number incompressible mixing layer of a two-dimensional jet exhausting into still air. The initial laminar boundary layer was triped with various wires for one case. The shear layer structure was reported to be sensitive to the
upstream history and other disturbances of the test facility. Also, Oster and Wygnanski (98) reported that excitation of the flow about the trailing edge of the splitter plate induced a non-linear growth of the shear layer.

Using Bradshaw's 1000 initial momentum thickness criterion and assuming the large structure spacing to be ten times the spacing of the initial oscillations, Dimotakis and Brown (99) estimated the relaxation distance to self-preservation to be about 40 centimeters. Their measurements, however, showed the spacing to be self-preserving just after 15 centimeters downstream. Also, Oster, Wygnanski and Fieoler (100) found that the history of a tripod laminar boundary was manifest beyond 1000 initial momentum thicknesses downstream in a two-stream mixing layer, at various free-stream velocity ratios. They also found that the spreading rate and fluctuation levels were consistently smaller for the tripod cases at all velocity ratios except for the zero velocity ratio flow. The virtual origin was downstream of the splitter plate trailing edge (positive) only for the tripod flow with zero velocity ratio.

Foss (101) investigated the effects of the upstream laminar and turbulent boundary layers on the development of a two-dimensional incompressible
turbulent mixing layer. He employed favorable and adverse pressure gradients to induce the laminar and turbulent boundary layers, respectively. His results were generally consistent with those of Latt(94). The virtual origin was downstream of the start of mixing for the initially turbulent boundary layer and upstream for the initially laminar case. The experiments of Birch(102) of two-stream mixing layers at Reynolds numbers of about 4 million, revealed that the spreading rate was insensitive to upstream memory at high Reynolds numbers. At Reynolds numbers of about 2 million, Birch observed that the spreading rate became independent of the initial conditions. A similar inference was made by Bailey and Kuethe(78) for supersonic shear layers at high Reynolds numbers.

Hussain(103), Hussain and Zedan(104,105) and Hussain and Hussain(106) carried out extensive analyses of the effect of the initial boundary layer conditions on free shear layers. Hussain(103) treated both plane and axisymmetric mixing layers of jets exhausting into stagnant environment, the works of Refs.104 and 105 dealt with axisymmetric shear layers only, while that of Ref.106 concerned circular jet mixing layers. Hussain observed that in the turbulent region of the shear layer, the intensity became self-similar much sooner for the initially laminar boundary layer than
that of the initially turbulent flow. His data also showed that an increase in the initial intensity caused a concomitant increase in the spreading rate, a decrease in the spreading (Goertler's) parameter, an increase in the asymptotic intensity and a decrease in the relaxation distance, with a lower limit of about 250 initial momentum thicknesses downstream. Self-similarity was also found to be reached sooner with increasing Reynolds number for both initial laminar and turbulent boundary layers. The findings of Refs. 104 and 105 are essentially the same as those of Hussain.

The investigation of Husain and Hussain(106) involved moderate Reynolds number (about 1 million) mixing layers of circular air jets. The upstream laminar boundary layer was triped to give a fully developed initial turbulent boundary layer on an extension plate attached to the lip of the jet nozzle. Shorter extension plates were employed for the shear layer flows with untripped initial laminar boundary layers. The virtual origin was found to be upstream of the tip of the extension plate, where mixing started, for the initially laminar shear layer but downstream for the initially turbulent case. The fluctuation intensities achieved self-similarity sooner for the initially laminar case than for the shear layer with an upstream turbulent boundary layer, which became
self-similar at about 450 initial momentum thicknesses downstream. Husain and Hussain also pointed out that vortex interaction (pairing) appeared to influence the momentum thickness spreading rate for both initial conditions. The initially laminar shear layer had a large spreading rate near the plate tip while a smaller rise in the spreading rate further downstream was seen for the initially turbulent shear layer.

Brown and Latigo(107) examined the effect of the initial boundary layer on a two-stream incompressible shear layer. For the initially turbulent case, the laminar boundary layer on the high speed side of the splitter plate was tripped to give turbulent flow at a Reynolds number of about 1 million. An approximately linear growth of a representative "integral thickness" was found for both initial conditions, though the growth rate was greater for the initially turbulent case. The initially turbulent shear layer thickness grew more slowly but relaxed to the initially laminar case in the asymptotic region. The intensities achieved self-similarity at about 800 initial momentum thicknesses downstream while the mean field was self-similar in about half that distance, for both initial conditions. These relaxation distances appeared to be independent of the initial boundary layers.
2.4 The Spreading Parameter

Various formulations have been given to relate Goertler's similarity or spreading parameter to free-stream velocities. This parameter is important since it is a measure of the relaxation distance necessary for a given turbulent free shear layer to achieve self-preservation. It also serves as a test parameter for consistency of (or agreement between) various experimental results obtained under similar initial and/or reference conditions. Most often, such results have yielded widely different values of the spreading parameter, thus raising concern about different experimentation procedures. However, these discrepancies are often said to be due to initial conditions.

By fitting the theoretical solutions of Tollmien and Goertler to their mean velocity data, Liepmann and Laufer(13) determined the zero free-stream velocity ratio spreading parameter, $\delta_0$, to be 12 and 11, respectively. Reichardt obtained a value of 13.5 with Goertler's solution. Also Wygnanski and Fiedler(71) matched their mean velocity data to that of Liepmann and Laufer with a $\delta_0$ of 9, while Ref.90 used a value of 10.2 to fit a similarity profile to their mean velocity. Batt(94), on the other hand, used values of 12 and 9 to correlate the initially untripped and tripped
shear layer data respectively, with similarity profiles.

In his incompressible two-stream mixing layer analysis, Sabin (23) derived the relation

\[ \frac{\sigma_0}{\sigma} = \frac{1 - r}{1 + r} = \Lambda \]  \hspace{1cm} (2.32)

where \( r = \frac{u_2}{u_1} \).

Equation (2.32) was also derived by Abromovich (9) and Korst and Chow (32). Birch and Eggers (108) used Prandtl's exchange coefficient relation to derive this equation also.

Assuming an error function velocity profile, Lilienthal (34) gave the expression

\[ \sigma = \frac{\partial U(u)}{\partial y} \bigg|_{\text{max}} \frac{-\pi x}{(1 - r)} \]  \hspace{1cm} (2.33)

Miles and Shih (109), on the other hand, correlated the incompressible relation

\[ \frac{\sigma_0}{\sigma} = \frac{1 + 5r^2}{(1 + r)}^{-1} \]  \hspace{1cm} (2.34)

and Yule (110) proposed the variation of the spreading parameter with the free-stream velocities to be

\[ \frac{\sigma_0}{\sigma} = \frac{(1 - r)}{(1 + r)^{\frac{1}{2}}} \]  \hspace{1cm} (2.35)

The three relations, Eqs. (2.32), (2.34) and (2.35) generally show fair agreement with experimental data.
(see Ref.108), while Eq.(2.33) has not been similarly tested.

Harsha(54) defines the spreading parameter as

\[
\theta = 1.855 \frac{x_2 - x_1}{y_2 - y_1} \tag{2.36}
\]

where \( y_1 \) and \( y_2 \) are the lateral points at which

\[
\frac{\bar{u} - u_2}{u_1 - u_2} = \begin{cases} 
0.1 & \text{at } x_1, \\
0.9 & \text{at } x_2.
\end{cases}
\]

Equation (2.37) compared favorably with Mach number and density ratio variations. Browr and Roshko(111) used the definition

\[
\theta = 1/\Delta \eta \tag{2.37}
\]

where \( \Delta \eta \) is an angular displacement, between the rays \( \eta_1 \) and \( \eta_2 \), which were defined by

\[
\frac{\bar{u} - u_2}{u_1 - u_2} = \begin{cases} 
(C*9)^{1/2} & \text{at } \eta_1, \\
(C*1)^{1/2} & \text{at } \eta_2.
\end{cases}
\]

\( \eta \) is Tollmien's similarity variable. For different velocity ratios, their velocity measurements gave values of the spreading parameter (Eq.(2.37)) that showed reasonable agreement with those of Miles and Shih, with the supersonic mean velocity data of Sirieix and Solignac(31) and Maydew and Reed(112). Brown and Roshko compared the spreading parameters to
those determined from their heterogeneous shear layer profiles. From this comparison they noted that density ratio was not the only factor responsible for the reduction in the spreading of supersonic shear layers.

Fiedler and Thies evaluated several single- and two-stream incompressible free shear layer experiments to determine the single-stream spreading parameter, $\sigma_0$, using Eq.(2,3,2). The two-stream mixing layer spreading parameter was calculated from the relation

$$\sigma = \frac{2}{\eta_1 \eta_{0.95}}$$

where

$$\eta = \frac{(y - y_o)}{(x - x_o)}$$

$x_o$ and $y_o$ are the coordinates of the virtual origin, $\eta_1$ and $\eta_{0.95}$ represent the values of $\eta$ where the $U/u_1$ are 0.1 and 0.95 respectively. From the data of Refs.13,71,94,95,100,101 and those of Brown and Weidman(113), Fiedler and Thies calculated $\sigma_0$ for each case. The comparison showed that $\sigma_0$ was smaller for shear layers with tripped initial laminar boundary layers and converged to an asymptotic value of 9 at Reynolds number greater than 5 million, for all initial conditions.

For supersonic shear layers, the only correlation for the spreading parameter was given by Korst and Tripp(114). They gave the expression
\[ \Theta = 12.0 + 2.758 M_\infty \]  

(2.38)

for a single-stream mixing layer, where \( M_\infty \) is the free-stream Mach number. Maydew and Reed estimated a spreading parameter of approximately 11 at subsonic speeds and stated that at supersonic speeds, the spreading parameter decreased with increasing Mach number. These estimates of the spreading parameter were made by fitting the theoretical solutions of Crane(21) to their data.

For compressible jet flows, Channacragada(115) derived the semi-empirical relation

\[ \frac{\Theta_\infty}{\Theta} = R \left[ \frac{1 + (T_{\infty}/T)}{1 + \frac{\gamma - 1}{2} M_{\infty}^2} \right] \]  

(2.39)

where \( R \) was defined as a compressibility factor, determined graphically from a R-C curve. \( C \) is the Crocco number, \( \gamma \), the jet Mach number and \( T_{\infty} \) and \( T \) are respectively the stagnation temperatures of the jet and still air. Here \( \Theta_\infty \) is the incompressible or reference value. The data of Maydew and Reed gave good agreement with Eq.(2.39). However, with other data it did not compare as well.

The variation of the spreading parameter with density ratio in heterogeneous shear layers was investigated by Brown and Roshko(111). Using Eq.(2.37), their mean velocity data gave values of the
spreading parameter that compared favorably with the empirical relation of Miles and Shih. Miles and Johnson(116) also measured mean velocity in homogeneous and heterogeneous two-stream mixing layers with laser Doppler velocimeter and deduced the spreading parameter from the relation

$$\sigma = x \Delta \eta / \Delta y$$  \hspace{1cm} (2.40)

$\Delta \eta$ is the width of the shear layer in the $\eta$ plane, where $\sigma$ is Goertler's similarity variable. Likewise, $\Delta y$ is the shear layer width in the physical plane. In either plane, the width of the shear layer is the distance between the points where

$$\frac{(\bar{u} - u_1)/(u_1 - u_2)}{(u - u_2)/(u_1 - u_2)} = 0.2$$

and

$$\frac{(\bar{u} - u_1)/(u_1 - u_2)}{(u - u_2)/(u_1 - u_2)} = 0.5.$$  \hspace{1cm} (2.41)

With Eq.(2.40) their measured velocity profiles gave values of $\sigma$ which showed fair agreement with those of Brown and Roshko and Eq.(2.34). For heterogeneous mixing layers, Birch and Eggers(106) suggested the relation

$$\frac{\sigma_e}{\sigma} = \left| \frac{1 - (P/\rho) r}{1 + r} \right|.$$
2.5 The "Third" Boundary Condition

The lack of a simple third boundary condition in free shear layer flows was discussed in the opening paragraphs of Section 2.1. It thus becomes necessary to prescribe a third condition in order to obtain a unique solution of the free shear layer flow. Generally the specification of the third condition has been done rather arbitrarily.

For the single semi-infinite stream mixing layer, Tollmien imposed the condition

$$ v(\eta_i) = 0 $$

which forces the lateral velocity component to vanish on the streaming edge. Goertler used only the two asymptotic conditions to solve the third order differential equation, while Kuethal applied Eq. (2.3). Pai used the lateral origin of the shear layer to be the point where \( \bar{u} \) equals \( u_\infty \). This arbitrary origin is often employed in graphical presentation of experimental data. Crane also applied this point as the third boundary, though its lateral position was left arbitrary. He even pointed out that because of this arbitrariness, an infinite number of equally valid solutions therefore exist.

Lessen defined the third boundary of the shear layer to be the dividing streamline, on which the
stream function vanishes. That is

\[ \eta = \eta_d, \quad f = 0 \quad (2.42) \]

where \( \eta \) and \( f \) are the dimensionless Blasius' similarity variable and stream function (see Ref. 11, for example), respectively. The subscript \( d \) denotes the dividing streamline. Also, Lessen gave the condition

\[ \int_{-\infty}^{y_d} \rho \bar{u}(u_1 - \bar{u})dy + \int_{y_d}^{\infty} \rho \bar{u}(u_2 - \bar{u})dy = 0 \quad (2.43) \]

which he showed to satisfy the Blasius' laminar flow ordinary differential equation and hence the momentum equation. Lock (117) also assigned zero stream function to the dividing streamline while Pai (113) assumed this condition at some arbitrary location across the layer.

Ting (119) gave a formal analysis to resolve the indeterminacy problem of the third boundary condition. He applied matched asymptotic expansion method to obtain higher order solution of the Navier-Stokes equations for a two-dimensional flow. In this method, the stream function, the velocity components and the static pressure were expanded in asymptotic series, with a gauge factor of \( 1/\sqrt{\eta} \). First order matching was then performed on the inner (shear layer) and the outer (potential) flow solutions of the Navier-Stokes equation. By balancing first order pressure terms
across the interface of the lower and upper semi-infinite streams, Ting derived a compatibility condition. Then applying linearized Bernoulli's equation to the compatibility condition he arrived at Eq. (2.8) for the incompressible mixing of semi-infinite streams. For the mixing of two supersonic streams, Ting's result is

\[ u_1 v_1 / B_1 + u_2 v_2 / B_2 = 0 \quad (2.44) \]

where \[ B_1 = M_1^2 - 1 \] and \[ B_2 = M_2^2 - 1 \]

while for the mixing of supersonic and subsonic streams, he gave the third boundary condition as

\[ v_1 = 0 \quad (2.45) \]

Ting and Ruger (120) performed second order matching for the mixing of two semi-infinite subsonic streams. Klemp and Acrivos (121) repeated the analysis of Ting and pointed out that Ting's solution was only exact to first order expansion of the dividing streamline trajectory.

Yen (122) also carried out an analysis to determine the third boundary condition for the mixing of two semi-infinite streams. He introduced the similarity variable

\[ \eta = \eta - \eta_q \]
where the similarity variable $\eta$ was defined like that of Blasius, with $u_q$ as the reference velocity. He gave the asymptotic values

$$
\eta^* \to \infty \quad f(\eta^*) \to (1 + \lambda)\eta^* - \alpha
$$

$$
\eta^* \to -\infty \quad f(\eta^*) \to (1 - \lambda)\eta^* - \beta
$$

where $\alpha$ and $\beta$ are empirical constants. Using the boundary conditions

$$
\eta^* = \infty \quad f'(\eta^*) = 1 + \lambda
$$

$$
\eta^* = -\infty \quad f'(\eta^*) = 1 - \lambda
$$

$$
\eta = 0 \quad f(\eta^*) = 0
$$

he performed a momentum balance on a control volume of the shear layer which gave the result

$$
\frac{1 + \lambda}{1 - \lambda} = \frac{\beta + \eta_q (1 - \lambda)}{\alpha + \eta_q (1 + \lambda)} \quad (2.46)
$$

From Eq. (2.46), the lateral position of the dividing streamline can be calculated for known values of $\alpha$ and $\beta$. For values of $r$ of 0 and 0.5, Yen calculated the dividing streamline location and found it to be shifted towards the faster stream side for the two-stream mixing layer while for the single-stream ($r$ equal to 0) layer, his calculation showed the dividing streamline line to be on the zero velocity side.
These solutions are inconsistent since the dividing streamline is generally known to shift towards the slower stream side of the shear layer. Repeating Yen's analysis, Baker and Weinstein (19) showed that Eq. (2.46) is indeed in error for non-zero velocity ratios, thus giving the incorrect solution for the lateral location of the dividing streamline.

Most recently, Small (123) applied variational calculus to calculate the dividing streamline. He used the similarity variable

$$\tilde{\eta} = \frac{1}{2} \left[ y + a(x) \right] \left( \frac{u_x}{\partial x} \right)^{\frac{1}{2}}$$

and assuming the error function velocity profile

$$\tilde{u} = u^a \left[ 1 + \lambda \operatorname{erf}(\tilde{\eta}) \right]$$

the "local potential,"

$$E = \int \int \left\{ \tilde{u} \left( \frac{\partial u_0}{\partial x} \right)^2 - u_0 v_0 \frac{\partial \tilde{u}}{\partial y} + v_0 u_0 \frac{\partial \tilde{u}}{\partial y} \right\} dy dx + \int u_0 v_0 \tilde{u} dx$$

was then extremized for a control volume of the shear layer. He defined \(-a(x)\) as the lateral shift of the error function profile or the location of the dividing streamline. The superscript 0 refers to some stationary state. From the analysis Small obtained the result
thus giving an explicit expression of the dividing streamline with the streamwise coordinate.

Leuchter(64) and Brown and Roshko(111) defined the dividing streamline to be the position of maximum shear stress, \( \tilde{\tau}_{\text{max}} \). Ref.111 also gave the condition

\[
\tilde{\tau}_{\text{max}} = \frac{\int_{0}^{\infty} \rho \vec{u} (\vec{u} - \vec{u}_L) d\eta}{\int_{0}^{\infty} \rho \vec{u} (u - \vec{u}) d\eta}
\]

which is similar to Eq.(2.43), given previously by Lessen(20), though Lessen did not assume it to satisfy the maximum shear stress position. Birch and Eggers(108) also stated that the "plane of symmetry" where the velocity gradient vanishes at a given cross-section is often applied as a third boundary.

From the above review a few remarks can be made. The review reveals that experimental efforts in the study of free shear layers have been directed to low speed flows. Consequently, this has created a paucity of supersonic mixing layer data, particularly for the two-stream mixing layers. Also, it is seen that initial or upstream boundary layer history strongly affects the development and growth of the free shear layer. Therefore, theoretical solutions should account for this effect. This has, generally, not been the
case. The review also shows that the "third" boundary condition, which is fundamental to the flow physics of free shear layers, has been prescribed rather arbitrarily in theoretical analyses.

In this study, both the theoretical and experimental analyses will be primarily concerned with the supersonic mixing of two free-streams, in light of the limited information available in this area. In the theoretical solutions, the initial condition will be included and Tien's third boundary conditions will be employed.
III. THEORETICAL ANALYSIS

3.1 The Theory

Within the restrictions of Prandtl's assumptions for thin shear layer flows, the governing equations of motion for free shear layers are the same as the conservation laws of boundary layer flows. Therefore, using the conventional Reynolds decomposition and averaging scheme, the conservation equations for two-dimensional mixing layer flows are generally given by

continuity: \[ \frac{\partial \rho \hat{u}}{\partial x} + \frac{\partial \rho \hat{v}}{\partial y} = 0 \] (3.1)

x-momentum: \[ \rho \hat{u} \frac{\partial \hat{u}}{\partial x} + \rho \hat{v} \frac{\partial \hat{u}}{\partial y} = \rho \hat{u} \frac{\partial \hat{u}}{\partial x} + \frac{\partial \hat{p}}{\partial y} \] (3.2)

ergy: \[ \rho \hat{u} \frac{\partial \hat{u}}{\partial x} + \rho \hat{v} \frac{\partial \hat{u}}{\partial y} = \frac{\partial}{\partial y} \left[ \frac{\hat{u} \hat{u}}{\rho} \right] - \hat{p} \frac{\partial \hat{v}^2}{\partial y} + \rho \hat{u} \hat{v} \hat{v} \]
\[ + \left( 1 - \frac{1}{Pr} \right) \mu \frac{\partial \hat{u}}{\partial y} \] (3.3)

x and y are the streamwise (axial) and lateral coordinates. \( \hat{u} \) and \( \hat{v} \) are the temporal mean velocity components in the x and y directions respectively and
$u'$ and $v'$ are the corresponding fluctuating components. $ar{p}$ is the local mean density, $\mu$ is the dynamic (laminar) viscosity and $Pr$, the laminar Prandtl number. $h'$ is the fluctuating static enthalpy. The subscript $e$ denotes layer edge values.

The product $\bar{p}v$ in Eqs. (3.1) to (3.3) is given by

$$\bar{p}v = (\bar{p} \bar{v} + \rho' v') .$$

$\bar{H}$, the mean total enthalpy relates to the mean static enthalpy, $\bar{h}$ by

$$\bar{H} = \frac{1}{2} \bar{u}^2 + \bar{h} .$$

The total shear stress, $\bar{\tau}$ is the sum of the shear stress due to molecular (laminar) diffusion and that due to turbulence transfer. That is

$$\bar{\tau} = \mu \frac{\partial \bar{u}}{\partial y} - \rho \bar{u} \bar{v}' .$$

where $-\rho \bar{u} \bar{v}'$ is the apparent or Reynolds stress.

Employing Boussinesq's eddy viscosity formulation, the total shear stress becomes (Ref. 6)

$$\bar{\tau} = \mu (1 + \bar{\varepsilon}) \frac{\partial \bar{u}}{\partial y}$$

where $\bar{\varepsilon}$ is the dimensionless eddy viscosity. With the turbulent Prandtl number, $Pr_t$, commonly defined as
the momentum and energy equations become

\[ \tilde{\rho} \frac{\partial \tilde{u}}{\partial x} + \tilde{\rho} \frac{\partial \tilde{u}}{\partial y} = \tilde{p} \frac{\partial \tilde{u}}{\partial x} + \frac{\partial}{\partial y} \left( \mu(1 + \tilde{\epsilon}) \frac{\partial \tilde{u}}{\partial y} \right) \]  

(3.2a)

\[ \tilde{\rho} \frac{\partial \tilde{u} \tilde{H}}{\partial x} + \tilde{\rho} \frac{\partial \tilde{u} \tilde{H}}{\partial y} = \frac{\partial}{\partial y} \left( \frac{\mu}{\tilde{R}_f} \frac{\partial (1 + \tilde{\epsilon})}{\partial y} \right) + \frac{\mu}{\tilde{R}_e} \left(1 + \tilde{\epsilon} \right) \]  

(3.3a)

For laminar flow solution, \( \tilde{\epsilon} \), as well as all the fluctuating quantities, become identically zero and all the dependent flow variables assume the corresponding laminar values (without the overbar).

3.1.1 The Closure Problem - Turbulence Models

The momentum and energy equations, Eqs. (3.2a) and (3.3a), contain the eddy viscosity which does not relate constitutively to other flow variables. Therefore, in order to solve the equations of motion, it becomes necessary to prescribe or model the eddy viscosity in terms of the flow variables which can be directly solved for. In this section, the various turbulence models employed in the solution of both the upstream boundary layer and the free shear layer flows.
are discussed.

For the upstream or initial boundary layer flows, two turbulence models are used. One involves an auxiliary transport equation for the turbulent kinetic energy, commonly classified as a one-equation model, while the other is a zero-equation or algebraic model. For the free shear flows, two one-equation and two zero-equation models are employed. In general, for two-dimensional compressible thin shear flows, the conservation equation for the transport of turbulent kinetic energy is given by

\[
\frac{\partial \bar{\rho}}{\partial x} \frac{\partial \bar{k}}{\partial x} + \frac{\partial \bar{\rho}}{\partial y} \frac{\partial \bar{k}}{\partial y} = -\frac{\partial}{\partial y} \left( \bar{\rho} \bar{v} \bar{v}_x + \bar{k} \bar{v}_y \right) - \bar{\rho} \bar{u} \bar{v}_x \frac{\partial \bar{u}}{\partial y} - \bar{\rho} \bar{k} (3.4)
\]

where the left hand side of the equation represent the advection of turbulent kinetic energy, and on the right hand side are, from left to right, the diffusion, the production and the dissipation of turbulent kinetic energy respectively. The turbulent kinetic energy per unit mass, \( \bar{k} \) is defined by

\[
\bar{k} = \frac{1}{2} \left( \bar{u}^2 + \bar{v}^2 + \bar{w}^2 \right)
\]

where \( \bar{w} \) is the fluctuation in the spanwise direction.
3.1.2 Initial Boundary Layer Models

For this case, the one-equation model of Glushko (124) is used. Glushko employed a gradient diffusion model for the diffusion term and modelled the dissipation after isotropic turbulence. Glushko's model is therefore given by

\[
\frac{\partial}{\partial x} \left( \rho \frac{\partial k}{\partial x} \right) + \frac{\partial}{\partial y} \left( \rho u \frac{\partial k}{\partial y} \right) = \frac{\partial}{\partial y} \left[ \mu \left( 1 + \frac{\varepsilon}{k} \right) \frac{\partial k}{\partial y} \right] + \frac{\varepsilon}{k} \left( k \frac{\partial u}{\partial y} \right)^2 - c \frac{\mu}{k} \left( 1 + \frac{\varepsilon}{k} \right) \frac{k}{\Lambda^2} \quad (3.5)
\]

where \( \varepsilon \) is the dimensionless turbulent kinetic exchange coefficient, \( \Lambda \) is the dissipation length scale, given algebraically from experimental correlation and \( c \) is an empirical constant.

Glushko gave the eddy viscosity by the relation

\[
\tilde{\nu}(r) = \alpha_\varepsilon \rho H(r) \quad (3.6)
\]

where \( \alpha_\varepsilon \) is an empirical constant. \( r \), the Reynolds number of turbulence was defined by

\[
r = (\kappa)^{\frac{1}{2}} \frac{\Lambda}{\nu}
\]

\( \nu \), being the local kinematic viscosity. The function \( H(r) \) and the dissipation length scale were correlated from flat plate boundary layer flow data and were given by the relations.
with $\delta$, the boundary layer thickness defined as the lateral distance from the wall where the normalized streamwise velocity is 0.99. Also, for $y/\delta$ greater than 0.595, $H(r)$ was set to unity regardless of the local turbulence Reynolds number, $r$. Glushko related the turbulent kinetic energy exchange coefficient or diffusivity to the eddy viscosity by

$$ \frac{\varepsilon}{\kappa} = \tilde{\varepsilon}(Kr) $$

$K$, being an empirical constant. For the incompressible flat plate boundary layer calculation, he used the numerical values of 3.93, 0.2, 0.4 for $c, \alpha, \gamma$, and $K$, respectively.

This model was originally formulated for incompressible flat plate boundary layers. Burggraf(6) extended it to calculate adiabatic compressible boundary layers as well as symmetric wake flows (discussed later), while Bodonyi(125) included
non-adiabatic effects.

The other model employed in the solution of the upstream boundary layer flow is the segmented zero-equation model of Cebeci, Smith and Monsinski(44). The two parts, an inner and outer eddy viscosities, reflect the turbulence production rates in the regions close to and away from the wall in the boundary layer. That is

\[
\bar{e}_i = \frac{K_1}{D} y \left[ 1 - \exp \left( - \frac{y}{26.4} \left( \frac{\tau_w}{\rho} - \rho \beta \frac{dy}{dx} \right)^{1/2} \right) \right] \left| \frac{du}{dy} \right|^2
\]

where the subscript w denotes wall conditions. The outer eddy viscosity was expressed by

\[
\bar{e}_o = \frac{K_0}{D} \int_0^\infty (u_w - \bar{u}) dy
\]

where \( K_1 \) and \( K_0 \) are empirical constants, for which Ref.44 used 0.4 and 0.0168 respectively. Switching from the inner to the outer eddy viscosity was accomplished by determining the point where the two values are equal.

Since zero-equation models cannot generate turbulent kinetic energy profiles required to start mixing layer solutions, for which these profiles may be desired, Eq.(2.29) is therefore employed to furnish the initial turbulent energy profiles. As pointed out in Section 2.1, for incompressible boundary layers, the proportionality constant \( a \) was determined to be 0.15
by Ref. 55 while Ref. 56 obtained a value of 0.3. These values were also used in free shear layer solutions, as was discussed there. For compressible boundary layers, the measurements of Rose and Murphy (126) showed a variation in \( \alpha \) across the layer, with an average value of 0.14. In the present study, the constant value of 0.3 is used for both compressible and incompressible boundary layers.

3.1.3 Free Shear Layer Flow Models

The first of the one-equation models is the wake flow version (Burggraf (6)) of Glushko's model. For wake flow calculations the original model apparently requires the dissipation length scale to vanish when \( y \) is zero, consequently forcing the turbulent kinetic energy to vanish there also, which is physically not plausible. To circumvent this oddity, Burggraf imposed a relaxation law to the length scale in the neighborhood of the plane of symmetry:

\[
\bar{\alpha}(x,0) \frac{d\alpha}{dx} = (\Lambda_1 - \Lambda_0)/\tau_r
\]

for symmetric wakes. \( \tau_r \) is a relaxation time constant and \( \Lambda_1 \), was defined as an equilibrium wake length. Equation (3.11) can readily be integrated to give
\[ \Lambda_0 = \Lambda_1 \left[ 1 - \exp \left( -\xi / \tau \right) \right] \]  
\[ \xi = \int_0^x \frac{dx}{u(x, \theta)} \]  

Burggraf obtained 0.6 and 0.355 from the best fit to the symmetric wake data of Chevray and Kovasny (45) for \( \tau \) and \( \Lambda_1 \) respectively.

As just mentioned, Burggraf's length scale modification is applicable to symmetric wakes only. Therefore if the modified model has to be used to calculate asymmetric shear layers, which is central to the present study, the asymmetry must be built into the relaxation law as well as the length scale distribution across the layer. With careful formulation, the redefinitions should reduce to the symmetric case in the limit.

For the asymmetric free shear layer, the dissipation length scale distribution (Eq. (3.8)) across the layer is simply modified to

\[ \Lambda / \delta = \begin{cases} 
\bar{y} / \delta, & 0 \leq \bar{y} / \delta < 0.235 \\
(0.47 + \bar{y} / \delta) / 3, & 0.235 \leq \bar{y} / \delta < 0.595 \\
0.6049 - \bar{y} / \delta, & 0.595 \leq \bar{y} / \delta 
\end{cases} \]  

with \( \bar{y} = |y - y_d| \) and \( \delta = |y_e - y_d| \)

where \( y_d \) is conveniently chosen to be the dividing
streamline on which the stream function vanishes at any cross-section. \( \delta \) is defined as the lateral distance from \( y_d \) to either edge, \( y_e \) of the shear layer.

Burggraf's relaxation law is similarly restated as

\[
u(x, y_d) \frac{dA_d}{dx} = \left( A_d - A_{d, 0} \right) \frac{b_d}{t}
\]

(3.15)

which again gives the solution

\[
A_d = A_d \left[ 1 - \exp(-\xi/b_d) \right]
\]

(3.16)

with

\[
\xi = \int_0^x \frac{dx}{u(x, y_d)}
\]

(3.17)

It can be seen that Eqs. (3.14) to (3.17) readily reduce to those of the symmetric wake, Eqs. (3.8) and (3.11) to (3.13), as \( y_d \) approaches zero. \( H(r) \) is also taken to be unity for \( \tilde{y}/\delta \) greater than 0.595.

The other one-equation model is also based on Eq. (3.4). Here the diffusion term is modelled after gradient diffusion process, as well. That is

\[
- \left( \rho \delta \nu^2 + \rho \kappa \nu^2 \right) = \xi_k \frac{\partial \delta}{\partial y}
\]

(3.18)

where \( \xi_k \), the turbulent kinetic energy diffusivity is expressed in terms of the eddy viscosity by

\[
\xi_k = \mu(1 + \bar{e})
\]

(3.19)
For the eddy viscosity, $\bar{\varepsilon}$, the Prandtl-Kolomogorov model, with a proportionality constant of unity, is employed. That is

$$\bar{\varepsilon} = \left(k^{3/2}\right) l_k$$

where $l_k$ is the dissipation length scale.

Equation (3.19) evidently presupposes the same momentum and turbulent energy transfer rates. Though Townsends(41) points out that $\bar{\varepsilon}_k$ is rather greater than $\bar{\varepsilon}$, Mellor and Herring(127) have found this relation to be quite satisfactory, for boundary layer flows, after a good many numerical experiments.

Following common practice, the dissipation term is modelled after isotropic turbulence. Thus,

$$\bar{\rho} \bar{\varepsilon} = c_2 \bar{\rho} k^{3/2}/l_k$$

where $c_2$ is a proportionality constant, determined from measurements. Therefore the second one-equation model becomes

$$\bar{\rho} \frac{\partial \bar{u}}{\partial x} + \bar{\rho} v \frac{\partial \bar{u}}{\partial y} = \frac{\partial}{\partial y} \left[ \mu (1 + \bar{\varepsilon}) \frac{\partial \bar{u}}{\partial y} \right] + \mu \bar{\varepsilon} \left( \frac{\partial \bar{u}}{\partial y} \right)^2 - c_2 \bar{\rho} k^{3/2}/l_k. \tag{3.22}$$

The dissipation length scale, $l_k$, which is the characteristic scale of the energy containing large eddies (Refs.64,128) is often assumed to vary linearly with the shear layer width. Likewise it is given as
1_b = c_b \quad (3.23)

where \( b \) is the shear layer thickness. Constraints on the lateral extent of \( b \) are given later. This linear variation of the dissipation length scale is supported by various experiments as discussed in Section 2.2. Also several theoretical investigations (see, for example, Refs. 49 to 54, 61, 62) have used Eq. (3.23).

Also, the streamwise turbulence intensity is calculated from the semi-empirical relation (Csanyi(129), Gasperas(130))

\[
\overline{u'^2} = A_k \quad (3.24)
\]

Similarly

\[
\overline{v'^2} = B_k \quad (3.25)
\]

\( A \) and \( B \) are proportionality constants.

The first of the two zero-equation models for the free shear layer flow solution is an extension of the outer eddy viscosity model of Cebeci, Smith and Monsinski(44). That is

\[
\overline{\varepsilon_0} = \int_{-\infty}^{\infty} (u_i - \bar{u})dy \quad (3.26)
\]

The second is Prandtl's exchange coefficient model, given by

\[
\overline{\varepsilon} = \frac{K_b}{\rho} b (u_1 - u_2) \quad (3.27)
\]
Both of these algebraic expressions imply constant eddy viscosities at a given cross-section of the layer.

3.1.4 Boundary Conditions

In order to solve the conservation equations, it is necessary to specify appropriate boundary conditions. For the upstream boundary layers, the no-slip wall conditions and the asymptotic boundary condition:

\[ y = 0 \, , \, \bar{u} = \bar{v} = 0 \]

\[ y \to \infty \, , \, \bar{u} \to u_\infty \]

apply. And for symmetric wakes, the boundary conditions are given by

\[ y \to \pm \infty \, , \, \bar{u} \to u_\infty \]

\[ y = 0 \, , \, \frac{\partial \bar{u}}{\partial y} = \bar{v} = 0 \, , \]

the last condition being the symmetry condition. For asymmetric free shear layers, the two asymptotic boundary conditions can readily be imposed. That is:

\[ y \to +\infty \, , \, \bar{u} \to u_1 \]

\[ y \to -\infty \, , \, \bar{u} \to u_2 \, . \]

For the third boundary condition, Ting's compatibility conditions, as given by Eqs. (2.8) and (2.44) are employed. Thus:
for the mixing of two incompressible free-streams, while for the case of two supersonic streams, the boundary condition is

\[ y = y_d, \quad u_{12}^{1/5} + u_{22}^{1/5} = 0 .\]

The boundary layer thickness follows conventional definition and is taken to be the distance from the wall where the streamwise velocity, \( u \), reaches the value of \( 0.995u_0 \). For symmetric wakes the layer thickness is twice this value, while the asymmetric free shear layer thickness is defined by

\[ b = y_1 - y_2 \]

where at

\[
y = y_1, \quad (\bar{u} - u_2)/(u_1 - u_2) = 0.99
\]

\[
y = y_2, \quad (\bar{u} - u_2)/(u_1 - u_2) = \pm 0.01
\]

The negative sign applies to the recovery region. Also

\( y_e = y_1 \) and \( y_e = y_2 \)

for the faster and slower streams respectively. However, for comparisons with experimental data, the layer thicknesses as defined in the report of the data, if different from the above definitions, will be used.
3.2 The Non-dimensionalization Scheme

The normalization procedure follows that of Burggraf(6). The different flow variables are normalized by reference values. These reference values are arbitrary and can be chosen to be the free-stream values for boundary layer flows or those of the faster stream for free shear layers. The reference values and edge conditions are assigned the subscripts $\ast$ and $e$, respectively.

The non-dimensional variables are defined by

$$
X = x/x_\ast, \quad U = \bar{u}/u_\ast, \quad U_e = u_e/u_\ast, \quad G = \bar{H}/H_e, \quad K = \bar{k}/u_\ast^2.
$$

The normalized lateral coordinate is

$$
Y = \frac{R_\ast}{X_\ast} \int_0^y \frac{\rho}{\rho_\ast} dy
$$

where $R_\ast$ is the reference Reynolds number, given by

$$
R_\ast = \frac{\rho_\ast u_\ast x_\ast}{\mu_\ast}
$$

and $\Delta(x)$ is a local layer growth rate scale. The non-dimensional stream function, $F$ is given by

$$
F = \int_0^Y \frac{U}{U_e} dY.
$$

Finally, the dynamic viscosity, $\mu$ is assumed to vary
with the mean static temperature, $\bar{T}$ in the form

$$\frac{\mu}{\mu_*} = \left(\frac{\bar{T}}{T_*}\right)^\omega$$

where the exponent, $\omega$, is an empirical constant, taken to be 0.75 in this study. This value was used by Bodonyi(171) and Gasperas(176).

After eliminating the term $\bar{p}v$ in the momentum equation, Eq.(2.2a), by means of the continuity equation, Eq.(3.1), and using the new variables, the momentum equation reduces to

$$\left[C \left(1 + \frac{\varepsilon}{R}\right) F_{YY}\right]_Y + \beta_1(F - F_0) F_{YY} + \beta_2(G - F_Y^2)$$

$$= \beta_3 \left[F_Y F_{XY} - (F - F_0) Y F_{YY}\right] \quad (3.28)$$

where the subscripts $X$ and $Y$ denote partial differentiations with respect to $X$ and $Y$ respectively. $C$ is the Chapman-Rubesin ratio, given by

$$C = \frac{\rho U}{\rho_* U_*}$$

$F_0$ is the value of the stream function where $Y$ equals 0 and is, by definition, identically zero for boundary layer and symmetric wake flows. The coefficients $\beta_1$, $\beta_2$ and $\beta_3$ are:
\[ \beta_4 = \Delta(x) \frac{d}{dx} [u_e(x) \Delta(x)] \]

\[ \beta_2 = \frac{d u_e(x)}{d x} \Delta^2(x) \left[ 1 + \frac{\gamma - 1}{2} \frac{M_e^2}{u_e} \right] \]

\[ \beta_3 = u_e(x) \Delta^2(x) \]

\( \gamma \) is the ratio of specific heats.

Similarly the energy equation, Eq. (3.3a), becomes

\[ \left[ c (1 + \frac{P_e}{P_e^2}) \right] G_Y + \left[ \beta_4 (F - F_0) + \beta_3 (F - F_0) X \right] \]

\[ + \left[ c (1 + \frac{P_e}{P_e^2}) \right] G_Y + \beta_4 F_Y G = \beta_3 \frac{P_e}{P_e^2} G_X \]

\[ + \beta_3 \left[ c (1 + \frac{P_e}{P_e^2}) \right] G_Y + \beta_4 F_Y G \]

\[ \text{(3.29)} \]

where

\[ \beta_4 = \frac{P_e d H_e}{H_e} \]

\[ \beta_5 = \frac{u_e(x)}{H_e} \left\{ \frac{2(\gamma - 1) M_e^2}{2 + (\gamma - 1) M_e^2} \right\} \]

The non-dimensional form of the Glushko's model is

\[ \left[ c (1 + \epsilon_e) \right] G_Y + \beta_4 (F - F_0) G - \beta_6 (1 + \epsilon_e) X \]

\[ = \beta_3 \left[ F_Y X - (F - F_0) X K \right] - C \epsilon U_e^2(x) F_Y^2 \]  

\[ \text{(3.30)} \]

where

\[ \beta_6 = c C (P_e/x)^2/\beta_4 \]

\[ \beta_4 = \frac{\Lambda M_e^2}{x_A X} \]
In dimensionless form, the model of Cebeci, Smith and Monsinsky becomes:

\[
\bar{E}_i = \frac{k_i^2}{C} \left\{ \frac{P_e}{\rho} \int_0^Y \frac{P_e}{\rho} \, dY \right\}^{2/3} \left\{ 1 - \exp \left[ -\frac{Q^2}{26C} \left( \frac{F_{YY}}{Y} \right)^4 \right] \right\}
\]

\[-\Delta^2(x) \frac{dU^e(x)}{dx} \int_0^Y \frac{P_e}{\rho} \, dY \left( \int_0^Y \frac{P_e}{\rho} \, dY \right)^{1/2} \left( \int_0^Y \frac{P_e}{\rho} \, dY \right)^{1/2} \right\}^2 (3.31a)

\[
\bar{E}_0 = \frac{k_0^2}{C} \frac{U_e(x)}{\rho_e} \Delta(x) \left( \frac{\rho}{\rho_e} \right) \left( \frac{\rho}{\rho_0} \right) \int_0^\infty \left( 1 - F \left( \frac{\rho}{\rho_e} \right) \right) \, dY \quad (3.31b)
\]

where \( G = U_e(x) \Delta(x) \left( \frac{\rho}{\rho_e} \right)^3 \left( \frac{\rho}{\rho_0} \right) \)

and \( \frac{\rho_e}{\rho} = \left( 1 + \frac{x-1}{2} \rho_e^2 \right) G + \frac{x-1}{2} \rho_e^2 \Phi_e \).

Also Eq. (3.22) reduces to

\[
\left[ C(1 + \bar{E} \cdot \kappa) \right]_Y + \beta_1 (F - F_0) \kappa_Y - \beta_3 \kappa_Y^{3/2}
\]

\[= \beta_3 \left[ F \cdot \kappa_Y - \left( F - F_0 \right) \kappa_Y \right] - C\bar{E}_0 U_e^2(x) \kappa_Y \quad (3.32)
\]

where \( \beta_3 = c_2^{-1} \Delta^2(x) / \kappa_Y \)

while the two zero-equation models for the free shear layer calculations become
The edges $Y_1$ and $Y_2$ in the transformed plane correspond to $y_1$ and $y_2$ in the physical plane. $r$ is the velocity ratio.

To arrive at Eqs. (3.28) to (3.30) and Eq. (3.32), it was first necessary to eliminate $\bar{p}\bar{v}$ by means of the continuity equation, as pointed out before. This was done by integrating Eq. (3.1), giving

$$\bar{p}\bar{v} = - \int_0^y \frac{\partial}{\partial x} (\bar{p}\bar{u}) \, dy + \bar{v}(x,0).$$

Following common practice, the constant of integration, $\bar{p}\bar{v}(x,0)$ was neglected.

3.3 The Numerical Computational Scheme

The above non-dimensional equations were solved numerically using two finite difference schemes. In one scheme, a backward difference was used to represent the partial derivatives with respect to $X$ and a centered difference representation for the derivatives with respect to $Y$. This scheme is...
hereafter called the Backward Difference Scheme. The other scheme was essentially the Crank-Nicholson method (Burggraf(6)). With these two differencing schemes, the non-dimensional equations were transformed into implicit difference equations, which were further reduced to tridiagonal systems. The resulting tridiagonal difference equations were then solved simultaneously and iteratively using the fast Thomas(131) algorithm. Details of the two finite difference schemes are given in the Appendix.

3.3.1 The Computational Grid

Following Burggraf, a non-uniform grid or mesh, which reflects the different scales of turbulence, was employed. The non-uniform mesh was generated by a hyperbolic-tangent coordinate transformation of the normalized lateral coordinate. The required transformation is

$$S'(Y) = \frac{1}{2} \left\{ \left[ S'_{N} + S'_{1} \right] + \left[ S'_{N} - S'_{1} \right] \tanh 12(S - 1/2) \right\}$$

where the prime denotes differentiation with respect to $Y$. $N$ is the number of mesh points and the subscript 1 refers to the wall boundary. In the free shear layer, the $Y$ equal to zero position is assigned the subscript 125. $S'_{1}$ and $S'_{N}$ are given by
where $\tilde{Y}_N$ is an approximation to the desired $Y_N$. $S$ varies from zero to 1 (or any other desired upper limit). The derivatives of the transformation are related by

$$\frac{d}{d\tilde{Y}} = S \frac{d}{dS} \quad \text{and} \quad \frac{d^2}{d\tilde{Y}^2} = (S^2) \frac{d^2}{dS^2} + S \frac{d}{dS} .$$

In the transformed coordinate, $S$, a uniform mesh is used to achieve higher numerical accuracy.

### 2.3.2 The Computer Program

The original version of the current computer program was written by Burggraf(6). He developed it for incompressible and adiabatic compressible boundary layer and symmetric wake flows and embodied the model of Glushko. Bodonyi(125) extended the program to allow for heat transfer. Gasoeras(130) added the model of Cebeci, Smith and Monsinski and other models in his investigation of the temperature field of supersonic symmetric wakes. For the present investigation, the program is further extended to include the new turbulence models and to handle asymmetric free shear flows. In addition to the Crank-Nicholson type difference scheme already present (Burggraf(6)), it is also made to accommodate the direct centered difference
scheme. The solution algorithm is the same for both difference schemes.

3.3.3 Numerical Stability

The application of difference equations to represent differential equations introduces truncation, discretization and machine-related round-off errors in numerical solutions. Depending on the step size(s) employed, these errors may compound and grow, and may lead to instability and convergence problems which can render the solution worthless. Therefore, when difference schemes are used to solve differential equations it is necessary to establish appropriate stability and convergence criteria.

For linear differential equations, ordinary or partial, formal treatment of stability and convergence is available in the literature. However, a similar treatment for non-linear differential equations, such as the conservation equations of fluid motion, is lacking. Therefore one must resort to heuristic methods to determine necessary stability and convergence criteria for non-linear differential equations. Since numerically stable difference equations are also convergent if the difference equations are consistent (Ames(132)), only the stability criterion needs be addressed.
Ames(8,132) applied von Mises transformation to convert the laminar flat plate boundary layer equation to an analogous non-linear heat condution equation. The resulting equation was then solved numerically in the von Mises plane, using an explicit difference scheme. Based on heuristic arguments, Ames applied the linear heat conduction equation stability criterion to the non-linear case. That is

\[
\frac{\partial \psi}{\partial t} \frac{A_{x,i}}{A} < \frac{1}{2}
\]  

(3.36)

where \(\psi\) is the von Mises stream function and \(A\), here, indicates step size. The indices (or subscripts) \(i\) and \(j\) denote the \(i\)th and the \(j\)th position on the mesh. Anderson(29) also applied the inequality (3.36) for turbulent free shear layer calculations.

Another heuristic, but more general approach is the electrical network analogy of Karplus(133). Karplus argued that a computational grid may be likened to a rectangular resistance network. This network, as in a mesh, is composed of fundamental units or resistance blocks, with each resistance block having a central node and four boundary nodes which connect it to the four adjacent units. He then imposed Kirchoff's voltage law on a given resistance block or the current law on an equivalent current loop to obtain the result
\[
(\bar{u}_{i,j} - \bar{u}_{i,j}^n) + l(\bar{u}_{i,j} - \bar{u}_{i,j}^n) + m(\bar{u}_{i,j} - \bar{u}_{i,j}^n) + n(\bar{u}_{i,j} - \bar{u}_{i,j}^n) = 0
\]

(3.37)

where \(i, m\) and \(n\) are simply coefficients. Karplus showed that, if all the coefficients are positive, Eq. (3.37) is stable and that any electrical disturbance or noise will decay and die out after the source is removed. However, if any of the coefficients is negative, a sufficient condition to guarantee stability is that the algebraic sum of the coefficients be negative.

For an analogous resistance block or nodal unit of a computational grid for the solution of non-linear differential equations, the constraints on Eq. (3.37) can be applied to establish necessary stability criterion. Karplus applied this analogy to the difference equations of various linear partial differential equations. For one-dimensional equations, he derived stability criteria which were exactly the same as those given by the classical methods. However, for the two-dimensional wave equation, his stability criterion was somewhat more conservative than the classical conditions.

Lee (30) also applied Karplus' method to the numerical solution of turbulent free shear flows (ignoring the pressure gradient term) to establish the
stability criterion

\[
\frac{1}{\bar{U}_{i,j}} \frac{\Delta X \bar{U} \bar{E}_{i,j}}{\left(\Delta Y\right)^2} < \frac{1}{2} \quad \text{(3.36)}
\]

For the present study, Karplus' heuristic approach is employed to establish a stability criterion. Thus using Eq. (3.37) and the constraints on it leads to

\[
\frac{1}{\psi_{i}(F_{i,j})} \frac{\Delta X \bar{C}_{i} \left(1 + E_{i,j}\right)}{\left[E^{(x)} S^{(y)} \Delta S\right]^2} < \frac{1}{2} \quad \text{(3.39)}
\]

where \(\Delta X\) and \(\Delta S\) are local step sizes in the streamwise and lateral coordinates respectively in the transformed plane, while \(\Delta(X)\) is the growth rate scale function, as previously defined. \(C_{i}, E_{i},\) and \((F_{i,j})\) are the Chapman-Rubesin ratio, eddy viscosity and dimensionless streamwise velocity, respectively, at the lateral node \(j\). The inequality (3.39) will therefore be used as a guide to ensure numerically stable solutions. Zero streamwise pressure gradient was assumed in the derivation of the inequality (3.39).
IV. EXPERIMENTAL APPARATUS AND PROCEDURE

The experimental analysis involved the measurement of two-component velocities and turbulence intensities. The diagnostic tool was the laser Doppler velocimeter (LDV). The wind tunnel used in the measurements, the diagnostics and related equipment are described below.

4.1 Test Facility

The test facility is a new asymmetric supersonic blow-down wind tunnel, designed and built at the Aeronautical and Astronautical Research Laboratory (AARL), The Ohio State University. The wind tunnel consisted of two differently contoured supersonic nozzle-walls. The different contours provided the desired asymmetry of the shear layer flow field. The faster stream nozzle had a design Mach number of 3.5 while the nominal Mach number of the other was 1.5. The Mach 3.5 nozzle was newly designed. The Mach 1.5 nozzle was a little too long and was shortened about an inch and half from its fully expanded design length to match the Mach 3.5 nozzle. Boundary layer displacement effects were accounted for in the tunnel design. The
full scale test facility is shown in Fig.1.

The two nozzles, and their respective settling chambers, were separated from each other by a centrally placed thin steel plate, which ran along the streamwise axis from the end wall of the settling chambers and terminated at the beginning of the rectangular test section. The splitter plate had a nominal thickness of 0.125 inch and a flow length of 30 inches. The last 3 inches of the flat plate was tapered on the faster stream side to within 0.03 inch at the trailing edge.

The length of the test section measured 24 inches, while its cross-section was 6 X 6 inches. The floor of the test section had an 3° slope, beginning 5 inches from the trailing edge of the splitter plate and extended for 3.5 inches. This slope was made to accommodate test models. A schematic of the wind tunnel is shown in Fig.2. The contours in this figure are only illustrative.

The tunnel flows were supplied from the laboratory (AARL) compressed air storage and exhausted into the outside atmosphere. Typical values of the stagnation pressures of the Mach 3.5 flow and the Mach 1.5 flow are 123 psia and 11 psia, respectively. Run times were short, typically about a minute or less, but long enough to establish equilibrium conditions in the settling chambers before data were taken. The facility
was fully instrumented to allow accurate determination of tunnel operating conditions.

The facility was a high Reynolds number wind tunnel. Unit Reynolds numbers of the order of 6 million per foot could readily be attained. In order to minimize free-stream turbulence levels, highly dense porous plates were installed in both settling chambers to damp out initial disturbances. In addition, a piece of octagonal-celled honeycomb, 6 inches long and of 1/4 inch diameter cells, was also installed in the settling chamber of the slower stream to further control free-stream turbulence, since the turbulence level was higher in this flow. Preliminary measurements indicated a maximum streamwise intensity level of about 2.0% in the Mach 1.5 free-stream, while it was as high as 20.0% in the Mach 3.5 free-stream.

4.2 Flow Field Diagnostics

Measurements of the time-resolved velocities in the supersonic mixing layer as made with a laser Doppler velocimeter (LDV). Laser velocimetry is a non-intrusive measurement technique and has become very valuable in fluid mechanics research. The principle of laser anemometry is highly developed and its application and reliability are extensively documented in the literature (see, for example, Goldstein and
Kraico(134), Durst, Melling and Whitelaw(135), Currani and Grested(136) and Ref. 137). The method allows the
direct measurement of flow velocities even in complex
flow fields where conventional probes simply cannot be
used. A rather simple account of the working principle
of the method is subsequently given.

In laser velocimetry, the velocity component of a
light scattering particle being convected in a flow can
independently and unambiguously be determined from the
Doppler frequency of the scattered radiation. If the
particle or scattering center is small enough to be
transported at the local fluid velocity, the local
velocity component can be measured by measuring the
Doppler frequency.

In this investigation, the dual beam or fringe
mode of operation of the LDV was employed to determine
the Doppler frequency. In this mode, two laser beams
of equal intensity, wavelength and length optically
interfere to give a pattern of dark and bright fringes.
A scattering particle, crossing the fringes, gives a
burst of scattered light whose amplitude and frequency
are proportional to the component of the particle
velocity normal to the fringes. The resulting relation
is

\[ f = \frac{2v \sin(\theta/2)}{\lambda} \] (4.1)
where \( f_D \) is the Doppler frequency, \( \lambda \), the wavelength of laser beam and \( \phi \) is the scattering angle (see Fig. 3). Thus the local fluid velocity component is seen to vary linearly with the Doppler frequency.

The LDV employed in this investigation consisted of a Spectra Physics 164 5-Watt argon ion laser, transmitting and receiving optics arrangements (see Figs. 4 and 5 respectively) and a single Thermo-Systems Inc. (TSI) 1980 burst counter and accompanying electronics, for the two-component signal processing. The argon ion laser operates in the TEM\(_{00}\) multimode, with the blue and green lines being the most intense of the excited axial modes. High optical quality mirrors, lenses and prisms as well as path compensated beam splitters and polarization rotators were used to minimize beam attenuation.

The polarization rotators were used to compensate for changes in the polarization caused by the passage of the beams through the optical components. Proper adjustment of the polarization vectors is required to maximize the intensity of the scattered light.

Each receiving channel was made up of a collecting compound lens, an aperture at its focus, a collimating lens, line filter and focusing lens (in that sequence), followed by an EMI 9812B high gain photomultiplier, see Fig. 5. The collimating and focusing lenses were
identical plano-convex lenses with a focal length of 5 cm. The photomultipliers were rated at a response time of 2.7 nanosecond and a frequency sensitivity of 60 MHz.

The velocimeter was operated in the conventional two-color, fringe mode in the forward scatter, off-axis configuration. The multimode laser beam was first passed through a polarization rotator and then collimated. The collimated beam was next dispersed with a triangular prism and the dispersed lines were divided into two groups with a triangular plane reflector. The less intense lines of each group were then masked out, leaving the brightest blue or green. After passing through polarization rotator, each line was split into a pair of equally intense monochromatic blue or green beams with a path compensated, constant beam separation beam splitter. One of the blue beams was down-shifted with a TSI Model 9180 Bragg Cell to 10 MHz to eliminate directional ambiguity of the measured velocities. The pair of green lines were used to measure the streamwise component while the blue lines measured the lateral component. The two pairs of beams were then focused with a single 150 cm. focal length lens onto the probe volume in the mixing layer.

A Bragg cell is an acousto-optic cell, which is used to shift the frequency of a given laser beam, in
order to increase the range of application of the LDV. The Model 9160 shifts the light beam to a fixed 40 MHz. In this experiment, the Bragg cell was electronically down-shifted from the 40 MHz to 10 MHz. Thus scattering particles moving in the same direction as the moving (shifted) fringes give negative velocities, while particles moving in the opposite direction give positive velocities. Therefore, frequency shifting gives the orientation or direction of the local velocity vector, thus eliminating directional ambiguity.

At a given downstream measurement position, the lateral translation of the probe volume to acquire data at various lateral points was accomplished by an arrangement of four plane mirrors (not shown in Fig. 4). These mirrors were oriented such that the first and second mirrors (positioned behind the focusing lens) were inclined at -45° and 45° respectively to the vertical while the third and fourth, placed above the focusing lens, were at 135° and 45° to the horizontal, respectively. These inclination angles gave the proper path length between the focusing mirror and the probe volume.

With this multiple mirror arrangement, the refracted beams from the focusing lens were incident on the first mirror whose reflections were again incident
on the second mirror. The reflections from this mirror, traveling in a direction opposite to their original direction, were incident on the third mirror which reflected them onto the fourth. The fourth mirror redirected the beams back to their original direction and to the focus at the probe volume. The focusing lens and the first mirror were coupled to a one degree of freedom manual drive which translated in the initial direction of the beams. Also the third and fourth mirrors were coupled to another single degree of freedom manual drive, translating perpendicular to the initial beam direction. This mechanism enabled the probe volume to be translated at the same focal length. For each lateral or streamwise translation of the probe volume, the receiving optics arrangements were manually aligned.

Since proper seeding of the flow with scattering particles was essential to obtaining satisfactory data rate, the shear layer was seeded with polystyrene spheres of about 0.5 micron in diameter. These particles were believed to be aerodynamically small enough to be convected at the local fluid velocity with minimum slippage. Also, because of the their small sizes, inertial effects of the particles on the flow dynamics were considered negligible. Each settling chamber was individually seeded. The particles were
generated and injected into the supply line with a particle generator which was designed and built at the AARL. This generator had been successfully used in previous LDV measurements (Emmer(138)). A schematic of the particle seeding system is given in Fig.6.

Optical access to the flow in the test section was obtained by means of high optical quality plexiglass windows. Using a good window is critical to LDV measurements, particularly on the receiving side since the photodetector signal is a function of the intensity of the transmitted wavelengths. Since only a single signal processor (burst counter) was available in the course of this investigation, the photomultiplier signals were acquired one after the other, with the normal velocity component more frequently processed first.

As mentioned above, the signal processor used in the experiment was a TSI 1980 burst counter. It consisted of an input conditioner module with low and high pass filter banks, an electronic clock, a zero crossing detector and counter units, a discriminating timer/analog module and an added digital interface module. The processor accurately and quickly measures photodetector signals. As a burst counter, it is particularly suitable for individual realization signals or frequency bursts (Doppler signals) of single
scattering centers, typical of LCVs. For an input Doppler signal, the processor first removes the pedestal and low frequency noise with the high pass filter, while the high frequency noise is filtered out with the low pass filter. After filtering, the signal is amplified to the set gain and then digitized for digital operation. In this investigation, the processor was used in the digital mode, which means purely digital signals were sent to the computer. That is, no analog to digital conversion was required.

In addition to the Doppler velocimeter, a pitot probe was used to obtain pressure and Mach number profiles across the shear layer. Other equipment used in the investigation include a pair of pressure transducers. These were used to monitor stagnation conditions. A Validyne DP15TL 0-250 psia transducer and a 0-10 psid Statham 950 transducer were used to measure stagnation pressures. The Validyne transducer was used to measure the Mach 3.5 stagnation pressure while the Statham transducer measured the pressure of the Mach 1.5 reservoir. A Carrier CD 15 demodulator was used in conjunction with the Validyne transducer. The Statham transducer was used with an amplifier, at a gain of 50, and had a potentiometer bridge excitation of 5.075 volts.
4.3 Tunnel Run Procedure

A typical tunnel run to acquire data was as follows: The compressed air was introduced into each settling chamber through a manifold and a set of distributor lines. Each manifold was controlled by a high pressure manual valve. To make a run, the valves were opened until the tunnel had reached condition, that is, when the stagnation pressures as well as the static pressures had balanced.

When the tunnel was on condition, the data acquisition software operator was then notified to proceed with the data acquisition by first opening the electrical valve that controlled the particle flow line. The particle flow as driven by a 1000 psi pressure from a nitrogen bottle and the flow was fed directly into each manifold. The particle flow rate into each manifold was regulated by a needle valve. When the electrical valve was opened, the tunnel flow was seeded and data was acquired until a preset number of data samples had been collected. When the required number of samples had been acquired, the electrical valve was automatically shut off to terminate the particle seeding and the tunnel shut down.
4.4 Data Acquisition and Reduction

Real-time data were acquired and reduced with the AARL digital computer system. This system includes a Harris Slash 6 central processor, an 80 megabyte removable disk pack and drive, a 75 ips vacuum 9-track magnetic tape drive, analog-to-digital (A/D) converters, a patch panel, a 36-inch drum plotter, several interactive terminals (CRTs) and several other peripherals interfaced with the Slash 6. The system allows totally interactive on-line real-time data acquisition and reduction. Data can also be acquired and stored on magnetic tape for off-line processing or future reference.

The signal processor was operated in the digital mode and the LUV data was acquired through a locally built digital controller interface. The controller interface consisted of signal drivers and receivers. A schematic of the main features of the controller interface is given in Fig. 7.

During a tunnel run, the signal processor would give a data ready signal when data were available. The data were then latched (or locked) in the processor. In this mode, the processor was disabled to process and update any incoming photomultiplier signals. Through the controller interface, the computer would then read the locked in data by means of a 16 bit digital input.
card. After the data had been read, the computer would release the processor from the locked in mode to process incoming photomultiplier signals. When data were available in the processor again, the data collection cycle was repeated until the preset number of samples was acquired, after which the particle seeding was shut off as explained in Section 4.3.

All other signals from the test facility were introduced to the system via the patch panel interface. These analog signals were routed to the different signal conditioners and/or A/D converters. One of the A/D systems was used to acquire data from the test facility. This is a medium speed low-level multichannel which accepts up to 16 differential low-level inputs, with a full scale voltage of plus or minus 1 volt and has a throughput rate of 8 KHz and a resolution of 12 bits. These data were the tunnel operating conditions and were displayed (as long as desired) on the interactive CRT for a consistency check from run to run.
V. RESULTS

Before embarking on the discussion of the theoretical and the experimental results, it was deemed necessary to make a comparative study of the theoretical predictions of the two numerical methods employed in the computer program. This is to ascertain if one scheme has any particular advantage over the other or if either may not be suitable for the algorithm, solution procedure or for any given flow configuration.

5.1 Comparison of the Numerical Methods

A rather extensive comparison of the predictions of the Crank-Nicholson and the Backward Difference schemes was undertaken for various reference or free-stream Reynolds numbers. Only a few selected results of the comparison are reported here. Most of the results presented are for a Reynolds number of 5 million. For each comparison, the empirical constants in the turbulence models, as well as other input parameters, are the same. All the presented results are for incompressible flat plate flows. The plotted
Reynolds stresses are the absolute values. In all the figures presented in this chapter, the lateral coordinate $y$, was normalized by the sum of the calculated trailing edge momentum thicknesses of the upstream boundary layers, unless otherwise specified in any figure.

The numerical grid or mesh is the same for both schemes, with 125 $Y$(or $S$) mesh points and 134 $X$-points along the wall for boundary layers. For wakes or free shear layers, the mesh consisted of 250 $S$-points while the $X$-points continued from 134 at the trailing edge to 250 at the last $X$-station. These mesh configurations were also used for the subsequent theoretical predictions. These grids were fine enough for the calculations to converge to a preset tolerance after a few iterations, the number depending on the turbulence model being used.

A typical flow solution was usually started by solving the upstream boundary layers first. The trailing edge solutions were then used as initial conditions for the wake or free shear layer calculations.

Figure 8 shows the mean velocity profiles for the lower and upper boundary layers at the trailing edge, calculated with the model of Cebeci, Smith and Monsinski(44), while Fig. 9 gives the corresponding
Reynolds stress distributions. The sharp points in these distributions (Fig. 9) indicate the change from the inner to the outer eddy viscosity. For this model the two numerical schemes give practically the same results. On the other hand, the predictions of the model of Glushko (124) show quite different results, with the Crank-Nicholson scheme giving higher mean velocities but lower Reynolds stresses as shown in Figs. 10 and 11 respectively. The sharp points in Fig. 11 are due to the segmented length scale distribution of the model. For all computations, the Crank-Nicholson scheme is about 30% faster in terms of computing time.

The rest of the figures in this section are for the symmetric wake of the upstream flat plate. Figures 12 through 17 are the comparisons at about 12\% downstream while those of Figs. 18 to 21 are for the station about 23\% downstream. Figures 12 and 13 show the predictions of the modified model of Cebeci, Smith and Monsinski, Figs. 14 to 17, for the new one-equation model (2nd One-Equation model) while the predictions of the modified Glushko model are given in Figs. 18 to 21.

In addition to the observable differences in the profiles in these figures, a characteristic spike r jitter about the centerline is readily seen in the mean velocity and Reynolds stress predictions of the
Crank-Nicholson scheme. Figures 16, 17 and 20 are presented to show clearly this peculiar feature for cases where it may be obscured in the figures by identification symbols. Figure 21 shows the comparison for which the initial boundary layer is laminar. Similar observations of this feature of the Crank-Nicholson scheme was also made by Gasperas(139).

This feature of the Crank-Nicholson scheme does seem to indicate that the scheme may not be suitable for solving wake or free shear layer flows when solved in the range $-\infty < y < \infty$ with the Thomas algorithm as in these computations; or that the mesh may not have been fine enough for this scheme. Efforts made to verify the latter were unsuccessful due to memory limitations of the AARL Harris Slash 6 computer. Consequently, subsequent theoretical predictions were made with the Backward Difference scheme only. It may be added that if the symmetric wake flow was solved only in the half plane, $0 \leq y < \infty$, with the Crank-Nicholson scheme, the observed spike or jitter would not result in the mean velocity and Reynolds stress profiles (see Burggraf(6) and Gasperas(130)).

5.2 Trailing Edge Step Size Effects

Due to the trailing edge singularity, trailing edge numerical solutions of wake and free shear layer
flows are particularly sensitive to step sizes in the neighborhood of the trailing edge. Eurggraf(6) recommends a step size of \( \Re^{\frac{3}{5}} \) in order to obtain accurate numerical solution about the trailing edge. The effect of the step size near the trailing edge is demonstrated in Fig. 22 for a flat plate incompressible symmetric laminar wake. A laminar wake was chosen since no empiricism is involved in the flow physics.

Figure 22 shows the comparison between the experimental data of Fage and Faulkner (see Ref. 2) and the theoretical predictions for different constant step sizes, 2 % downstream. It is clearly seen that for the step sizes much greater than \( \Re^{\frac{3}{5}} \), the laminar theory under predicts the experimental data, the difference being more pronounced about the centerline. The agreement between the theory and the experiment is most satisfactory for the step size of 0.00145 which is equal to \( \Re^{\frac{3}{5}} \) for the data of Ref. 2.

It is therefore necessary that to obtain accurate numerical solutions near the trailing edge, the step size must not be greater than \( \Re^{\frac{3}{5}} \), as given by Eurggraf. In the subsequent theoretical calculations, the step size at the trailing edge was made as small as \( 2 \times 10^{-7} \) and was gradually and unevenly increased downstream.
5.3 Comparisons: Theory and Experiments

In this section various comparisons are made between the theoretical predictions of the turbulence models and the data of four previous experiments.

Figures 23 through 30 show the comparisons with the data of Chevray and Kovasney (45) for incompressible flat plate boundary layers as well as the ensuing symmetric wake. Figures 23 and 24 are trailing edge profiles. Figures 25 and 26 are the comparisons at about 2% downstream, Figs. 27 and 28, being those at 20% downstream while Figs. 29 and 30 are the profiles at a full plate length into the wake.

At the trailing edge, the model of Cebeci, Smith and Monsinski and that of Glushko give mean velocity profiles which compare favorably with the data, though both Reynolds stress distributions underpredict the data in the inner layer region (Fig. 24). Outside this region, the Cebeci-Smith-Monsinski model shows better agreement with the data. The Glushko model fails to exhibit the asymptotic trend towards the edges. Also a characteristic kink is noticable at the outer boundary in the profiles predicted with the model of Glushko. These kinks are also present in the wake profiles discussed below.

The reported trailing edge momentum and boundary layer thicknesses were 0.58 cm. and 5.50 cm.,
respectively. The corresponding values calculated with the model of Cebeci, Smith and Monsinski were 0.57 cm. and 5.01 cm. while the Glushko model ave 0.58 cm. and 4.19 cm. The agreement of these numbers is generally good, though the Glushko model predicted a somewhat smaller boundary layer thickness.

In the near wake, at 2% downstream Figs. 25 and 26), the modified Glushko model gives a much better agreement with the data, with respect to the other two models. This is more evident in the Reynolds stress distributions (Figs. 26). At 20% downstream (Figs. 27 and 28), the agreement between the data and the predictions of the modified Glushko model have become poorer while the 2nd One-Equation and the modified Cebeci-Smith-Monsinski models show improved agreement, particularly in the Reynolds stress profiles.

Comparatively the 2nd One-Equation model appears to give better agreement than the other two models. At a plate length downstream (Figs. 29 and 30), the 2nd One-Equation and the modified Cebeci-Smith-Monsinski models compare quite well with the mean velocity data while the agreement with the modified Glushko model has progressively worsened. The Reynolds stress profiles do not compare as well. Again the 2nd One-Equation model does appear to give the best agreement.
The fact that the modified Glushko model gives good agreement with the data in the near wake does suggest that models with varying length scales are more appropriate for near wake solutions, while in the far wake or asymptotic region the comparisons do indicate that, at a given cross section, a constant length scale or eddy viscosity be used. Indeed, in the near region, the history of the upstream boundary layer, and hence the distributed length scales, still influence the near wake development. In the asymptotic region, this effect has been lost in the recovery process.

Figures 31 to 36 show the comparisons of the mean velocity and mean static temperature profiles with the asymptotic correlations of Demetriades (76) for a Mach 3.0 supersonic flat plate wake. The comparisons are made only with the predictions of the modified Cebeci-Smith-Monsinski model since the modified Glushko and the 2nd One-Equation models did not give convergent solutions due to the rather large step size ($\Delta x = 0.75$) required for $X > 2.0$.

In these figures, the comparisons show marked disagreement between the theory and experimental correlations, for the experimental "virtual body thickness" ($VBT$) of 0.00909, reported by Demetriades. By definition, the virtual body thickness is two times the momentum thickness. Similar discrepancies were
reported by Gasperas(130). Though the differences are rather obvious in these figures, the maximum difference which is on the centerline is less than 2 % for both the mean velocity and static temperature for all stations. However when the calculated virtual body thickness of 0.00577 is used with the asymptotic correlations, a much better agreement is found between the correlations and the theoretical predictions as shown in the figures. This tends to suggest that the reported units Reynolds number of 66500 per cm. might be somewhat low.

The upstream boundary layer was assumed laminar because of the rather low Reynolds number of 19551 of the flow. The calculated trailing edge momentum thickness was 0.0013 cm.

Theoretical mean velocity profiles are compared with the data of Bailey and Kuethe(78) in Figs.37 to 41. The experimental data is for the mixing of two supersonic streams at Mach numbers of 2.5 and 1.9, downstream of a double-corner nozzle lip. These data were taken with a fine Pitot probe. Bailey and Kuethe fitted the trailing edge data to the 1/nth power law, with an n of 5 and 7 for the faster and slower streams, respectively.

Since the pressure distributions in the double-corner nozzle were not given in Ref.78, fully turbulent
"equivalent flat plate" solutions were sought to predict the initial profiles. Figure 36 shows this comparison. The comparison with the slower stream profile \( (n = 7) \) shows acceptable agreement while the faster stream profile \( (n = 5) \) does not agree with the model predictions as well. Consequently the experimental initial velocity distributions were used as the initial profiles for the downstream free shear layer calculations. Other initial conditions such as turbulent kinetic energy, mean static temperature, energy and eddy viscosity were obtained from the equivalent flat plate solution.

The reported trailing edge boundary layer thicknesses were 0.89 cm. for the faster stream and 0.54 cm. for the slower stream. The Cebeci-Smith-Monsinski model gave 0.888 cm. and 0.551 cm., respectively, while those of the Glushko model were 0.844 cm. and 0.535 cm. These experimental and theoretical values do agree quite reasonably. No initial momentum thicknesses were reported in Ref. 78. The calculated values were 0.064 cm. and 0.044 cm. for the faster and slower streams, respectively, with the model of Cebeci, Smith and Monsinski. Those obtained with Glushko's model were 0.067 cm. and 0.049 cm.

Figures 38 to 41 show poor agreement for all the four turbulent models, though the agreement does
improve with downstream distance. The models generally underpredict the experimental data. Even though no one model appears to give a very good agreement with the data, the zero-equation models seem to fair better in this case, particularly further downstream. Similar disagreement was also reported by Bailey and Kuethe with their theory (see Ref. 32).

The incompressible two-stream mixing layer mean velocity data of Spencer (72) and the theoretical predictions of the 2nd One-Equation model, the modified model of Cebeci, Smith and Monsinski and Prandtl's exchange coefficient model are given in Figs. 42 to 46. The Glushko model solutions did not converge because of the initial conditions. These figures show that at the first measurement station, about 12% downstream of the splitter plate, the free shear layer has recovered from the upstream momentum deficit.

Although the trend is very well predicted by all the three models, the agreement is only fair in each case. The 2nd One-Equation and the exchange coefficient models appear to give better agreement towards the edge of the faster stream while the modified model of Cebeci, Smith and Monsinski gives better comparison on the slower stream side, as shown in Figs. 42 to 44. Further downstream (Figs. 45 and 46), the reverse situation appears to be true. This general
lack of agreement with Spencer's data has also been reported by Morel, Torca and Bradshaw(60) as well as Spencer who compared his data with Goertler's error function asymptotic solution.

For this flow the upstream boundary layer was laminar. The calculated trailing edge momentum thicknesses were 0.0044 inch and 0.0057 inch for the faster and slower streams, respectively.

The empirical constants used in the various turbulence models are given in Tables 1 and 2. The tables show different values for some of the constants. These constants were chosen to give the best agreement between experimental data and model predictions, and were dictated by both the numerical scheme employed and the experimental input parameters. All other constants of both the original and modified versions of the Glushko model were used as given by Glushko(124) and Eurggraf(6).

5.4 Results of the Present Experiment

5.4.1 The Near Region Flow Field

The first part of the experiment involved flow visualization. Schlieren photography was used to investigate the flow field in the neighborhood of the splitter plate trailing edge. Figure 47 is a Schlieren photograph of the flow about the trailing edge. In
this figure, the wave system is clearly displayed. On the Mach 3.5 stream side of the splitter plate, a Prandtl-Meyer fan can be seen projecting out from the turn while an oblique shock is distinct at the lip. In the Mach 1.5 stream, an expansion fan is also situated at the lip. As can be seen, the strengths of the waves are such that the pressure across the shear layer is balanced, though a slight turning of the flow towards the Mach 3.5 stream is discernable, which would be expected due to the presence of the wave system. The tunnel was run with this wave system, that is, with an expansion wave in the Mach 1.5 flow, because even a very weak compression turn would cause shock detachment, thus forcing the tunnel to go out of flow.

5.4.2 Pitot Probe Surveys

The next step in the experiment involved pitot probe traverses across the free shear layer at different downstream stations. The pitot cans gave pressure and Mach number profiles across the mixing layer. Typical pressure and Mach number profiles are given in Figs. 48 and 49, respectively. In these figures, the mixing layer is marked by the region of large pitot pressure or Mach number jump. In this region isentropic relations do not hold because of viscous dissipation.
Also, other regions of non-uniform flow on either side of the shear layer are noticeable in these figures. This is due to the interaction of the wave system with the isentropic free-streams. Figure 48 shows a decrease in pitot total pressure across the Prandtl-Meyer fan on the Mach 3.5 free-stream while the oblique shock results in an increase in pressure, as would be expected. A slight drop in pressure is also seen across the expansion fan in the Mach 1.5 flow. Outside these regions, the free-streams appear reasonably uniform.

The Mach number distribution gives a free-stream value of 3.2 for the faster stream and 1.4 for the slower stream. These are less than the corresponding design Mach numbers of 3.5 and 1.5. The reason for the lesser free-stream Mach numbers is twofold: the first is evidently due to the presence of the wave system about the lip or trailing edge of the splitter plate (see Fig. 47), while the second may be associated with the shortening of the Mach 1.5 nozzle as pointed out in Section 4.1.

5.4.3 The LDV Measurements

As a first step in ensuring accurate measurements of the time-resolved velocity components, the equipment, such as the photomultipliers, preamplifiers
and signal processor, were checked thoroughly to make sure that they functioned properly.

A confidence test was run on the photomultipliers by simulating or generating Doppler signals and observing the photomultiplier output on a high resolution oscilloscope. The Doppler signals were generated by spinning a fine steel wire of approximately $7 \times 10^{-4}$ inch diameter on a small disc through the probe volume of either the green or blue line in the center of the test section. The disc was driven by a miniature motor, rated at 800 rpm. Doppler bursts were clearly seen on the oscilloscope, indicating that the photomultipliers were in working order. A typical unfiltered Doppler signal from the spinning wire for the green line is given in Fig. 50.

The fringe spacing in the probe volume was determined experimentally. This was done by positioning a very small steel wire, of the same diameter as the one used on the spinning disc, in the probe volume which was targeted onto an eyepiece in the center of the test section. The lens resolved the probe volume, showing clearly the fringe pattern on a background. The number of fringes occupying the wire diameter could therefore be directly counted on the background or on a photograph of it. The fringe spacing was then easily determined by dividing the wire
diameter by the number of fringes occupying this distance. In this manner, the fringe spacing in the probe volume was found to be 17.3 microns for the green line. By simple proportion, that of the blue line was 16.4 microns.

This method of determining the fringe spacing was used because of the relatively thick plexiglass windows of the test section. These windows were about 1 inch thick and could considerably change the fringe spacing if it were calculated from the geometry of the transmitting optics, as is conventionally done. A photograph of the fringe pattern and the wire is given in Fig. 51.

The signal processor was repeatedly tested by supplying it with a signal of known frequency from a wavetek signal generator and comparing the velocity obtained from the data acquisition program to the value found by multiplying the said frequency and the fringe spacing. This also served as a test for the data acquisition program. Consistent agreement was found between the two values for the various input frequencies.

The problem of noise in LDV measurements is of major concern. In laser velocimetry, several sources of noise arise. These include photomultiplier shot noise, electronic noise from any of the
electronic/electrical equipment being used, scattered radiation and even room lights. Thus considerable effort was put into optimizing equipment operational conditions in order to obtain reasonable signal to noise ratios. To ensure this, several preliminary wind tunnel runs were made to establish optimum laser power, photomultiplier input voltages and the settings of the low and high pass filters as well as the gain on the signal processor.

The following conditions were determined to give the best signal to noise ratio: the laser power was 0.8 Watt, the voltage on the photomultipliers was 1700 ± 60 volts, the low and high pass filters were set at 30 MHz and 1 MHz respectively for the v-component measurements while the high pass filter was set at 3 MHz and the low pass set wide open (equivalent to 100 MHz) for the u-component measurements. The signal processor gain was varied from one component measurement to the other and from one measurement point to another. Though the gain was always set at 1 or greater, it was set as low as possible to minimize processor inherent noise, at the expense of high data rate.

To establish some degree of confidence in the measured velocity data, several test runs were made in order to demonstrate repeatability of the data at the same measurement position. This measurements were made
in the free-stream of the slower stream. This in effect served as a complete check for the whole system. The repetitions showed only a very slight variation, less than 0.1%, in the mean streamwise velocity data, and in most cases the repeated data were practically the same.

In order to remove persistent noise, such as signal processor noise from the acquired data, data editing was used to throw out recognizable noise in cases where this was necessary. Three edit options were available. These were 2-sigma, 3-sigma and the manual option. The 2- and 3-sigma edit options removed all data outside the range of data greater or lesser than two and three times the root mean square value of the raw data respectively, while in manual edit, the number of data points removed was simply a matter of choice and the editing could be repeated as required. In general, 3-sigma edit resulted only in an insignificant change in the mean velocity data, while significant difference could result in the root mean square values. This would naturally be expected since noise would contribute more to the root mean square values which represent fluctuations. Table 3 shows the effect of 3-sigma editing. All reported data were 3-sigma edited.
In the course of this experiment, two preamplifiers were used. The first was an Avantek, UAA-11355, with a bandwidth of 5 KHz to 400 MHz and a gain of 40. This preamplifier was used for all the preliminary tests as well as part of the actual velocity measurements at 6 inches downstream of the splitter plate trailing edge, where it failed. The second preamplifier was a pair of Anzac AM-102. Each had a bandwidth of 5 - 300 MHz and a gain of 10. The two were used in series to give a total gain of 100. This higher gain introduced otherwise unamplified noise in the measured data. Much of this noise was eliminated using the experimental method described earlier.

Measurements were made at two stations, at 2 and 6 inches downstream of the splitter plate trailing edge. Measurements were first made at the 6-inch station. In individual realization LDV, there is a higher probability of measuring velocities greater than the temporal mean (McLaughlin and Tiederman(140)). That is, signals of faster particles are sampled more often than slower ones, thus giving a velocity bias towards the higher velocities. Therefore, the streamwise mean velocity and its root mean square values reported here have been corrected for velocity bias using the method of McLaughlin and Tiederman. The lateral components
were not corrected since a correction scheme for frequency shifted data is lacking.

The data are presented in Figs. 52 to 59. Figures 52 to 55 are the profiles 2 inches downstream of the trailing edge of the splitter plate while Figs. 56 to 59 are the data for the 6-inch station.

Figures 52 and 53 show the mean velocity data of the streamwise and lateral components, respectively, while the corresponding root mean square data are given in Figs. 54 and 55. Figure 52 exhibits the recovery trend of the mixing layer from the momentum deficit of the upstream boundary layers. A slight scatter is noticable in the data. Figure 53 is the plot of the lateral mean component. No particular trend is readily discernible here, though it does appear to show the recovery process as well. The velocity is in general negative, indicating the slight turning of the shear layer towards the faster stream side, as discussed in Section 5.4.1.

The root mean square velocities 2 inches downstream are given in Figs. 54 and 55 for the streamwise and lateral components, respectively. These data do show reasonable scatter, which may be due to noise or free-stream turbulence. Although some trend in the data is apparent in either figure, it is not well defined primarily because of the scatter about the
edges of the shear layer. But for two data points, Fig. 55 shows a fairly constant value of about 0.06 of $v_{rms}/u_1$ across the layer.

The mean velocity data of the streamwise and lateral components 6 inches downstream are given in Figs. 56 and 57, respectively. The corresponding root mean square data are given in Figs. 58 and 59. The mean streamwise velocity (Fig. 56) clearly shows that the mixing layer is still recovering from the upstream momentum deficit. Scatter in this data is minimal and the trend is typical of recovering free shear layers. The data of Fig. 57 show more scatter. However, the expected profile across the layer is apparent. A similar pattern is noticeable in the streamwise root mean square velocity distribution (Fig. 58). The lateral root mean square data, Fig. 59, show considerable scatter such that it is difficult to make out any trend in the data.

It is seen in Figs. 57 and 58 that the data increase rapidly towards the edge of the faster stream. Besides noise and free-stream turbulence, this may also be due to the interaction of the test section boundary layer with the free shear layer. Such an interaction would evidently increase the lateral mean velocity and its root mean square values.
A rather unanticipated observation is that the streamwise mean velocity data in the Mach 1.5 side of the mixing layer, 6 inches downstream (Fig. 57), are smaller than the corresponding data at the 2-inch station. Ordinarily, the reverse would be the case for a recovering free shear layer. This decrease in the mean velocity could be caused by the complex interacting wave system in the downstream flow field.

5.5 Comparisons: Theory and Present Data

A comparison of the mean streamwise data with theoretical predictions of the four free shear layer models is given in Fig. 60 for the 2-inch station. Those for the 6-inch station are given in Fig. 61. Though the trends are well predicted, the agreement is only fair for all the models in each case. The 2nd One-Equation model appears to give the best agreement, comparatively. However, at the 6-inch station (Fig. 61), the predictions of the models show very little difference, except that of the modified Glushko model which deviates from the others somewhat.

Comparisons for the turbulence intensities are given in Figs. 62, 63, 64 and 65. Figures 62 and 63 show the comparisons for the streamwise and lateral intensities, respectively, at the 2-inch station while Figs. 64 and 65 show the corresponding data at the
In Fig. 64, both turbulence models appear to predict the trend in the data. Agreement between the data and the predictions is again, at best, fair, though the 2nd One-Equation model shows a slightly better agreement. In the Mach 3.5 free-stream, the comparison show rather poor agreement. This may be due to the high free-stream intensity in this flow, as pointed out previously. Due to the considerable scatter in the lateral intensity data at the 6-inch station (Fig. 65), no agreement with model predictions is seen. However, some of the data do seem to follow the predicted trends.

The model constants used in the calculations are given in Tables 1 and 2. Also, the proportionality constants of the correlation equations, Eqs. (3.24) and (3.25), used to predict the intensities are given in Table 4.
VI. CONCLUSIONS AND RECOMMENDATIONS

A theoretical and experimental study of free shear flows had been undertaken. The theoretical treatment covered both incompressible and supersonic mixing layers while the experiment involved asymmetric supersonic mixing only.

The main aspect of the theoretical analysis involved turbulence modelling of free shear flows. Four turbulence models, the modified Glushko model, the 2nd One-Equation model, the modified outer eddy viscosity model of Cebeci, Smith and Monsinski and Prandtl's exchange coefficient model were used in the theoretical predictions. The modified Cebeci-Smith-Monsinski model was simply an extension of the outer eddy viscosity model, while for the free shear layer calculations the 2nd One-Equation model was newly applied. The upstream boundary layer calculations were made with the models of Glushko and Cebeci, Smith and Monsinski.

Comparison with the symmetric wake data of Chevray and Kosvasnay shows the modified Glusko model to be more suited for near wake predictions. In the far wake
the 2nd One-Equation and the modified Cebeci-Smith-Monsinski models fared much better. Comparisons were also made with asymmetric mixing layer data. The four models generally gave similar predictions, though the 2nd One-Equation model did appear to show somewhat better agreement with experimental data, except with those of Ref.78. In general, the agreement between the theoretical predictions and the asymmetric data was fair.

Another point considered in the theory was the fundamental problem of step size effects on trailing edge numerical solutions. Solutions of incompressible symmetric laminar flat plate wake, 2 % downstream, for different trailing edge step sizes, were compared with the laminar data of Fage and Faulkner. The comparison showed that step sizes in the neighborhood of the trailing edge must not be greater than $Re^{-3/5}$, as had been given by Burggraf, to obtain accurate numerical solutions in this region.

The experiment involved the measurement of real-time velocity components in a new asymmetric blow down wind tunnel. The free shear layer was formed by two supersonic streams with nominal Mach numbers of 3.5 and 1.5. Laser Doppler anemometry was used to acquire the data. The anemometer was operated in the dual beam or fringe mode and data were collected in the forward
scatter, off-axis configuration. Both $u$ and $v$ velocities were measured.

Measurements were made at 2 and 6 inches downstream of the splitter plate. In this region, the $u$ profiles reveal the flow to be recovering from the upstream momentum losses. Comparison with the model predictions is, at best, fair, in general. Also the data do show some scatter, particularly in the lateral mean velocity and the root mean square distributions. The scatter may be due to noise and/or free-stream turbulence.

A rather surprising observation from the data was that the streamwise mean velocities in the slower stream, 6 inches downstream were lesser than the values at the 2-inch station. It is speculated that this trend may be due to wave interactions.

Limitations of time and resources prevented extensive measurements of the asymmetric free shear layer to be made. This has made it very difficult to draw solid conclusions as to the structure of the mixing layer. However, the mean streamwise distributions do show that by the last measurement station, the shear layer is yet to reach asymptotic conditions.

It is therefore suggested that a more thorough investigation of this flow field be made. As a first
step, the "wedgeo" floor of the test section should be replaced to eliminate the wave system generated by this wedge. Another aspect of the experiment that requires further attention is the relatively high level of free-stream turbulence in both supersonic streams. The scatter in the data, mainly in the lateral mean velocity distributions, as well as the root mean square data, may be largely due to the high free-stream turbulence levels. Therefore more effort should be made to significantly reduce the free-stream turbulence levels. Finally, simultaneous u and v measurements should be made to give Reynolds stress data since Reynolds stress measurements were not made due to lack of a coincidence sensing equipment as well as a second signal processor.
FIGURE 1. THE ASYMMETRIC SUPERSONIC WIND TUNNEL
FIGURE 2. THE ASYMMETRIC SUPERSONIC WIND TUNNEL SCHEMATIC
Interference Fringes in Scattering Volume

FIGURE 3. DUAL BEAM OR FRINGE LDV OPERATION MODE
FIGURE 4. LDV TRANSMITTING OPTICS
(For Legend see Page 144)
LEGEND
(Figure 4)

1 LASER HEAD
2 POLARIZATION ROTATOR
3 BEAM COLLIMATOR
4 MIRROR
5 DISPERSION PRISM
6 MASK
7 BEAM SPLITTER
8 BRAGG CELL
9 BEAM STEERING MODULE
10 FOCUSING LENS
FIGURE 5. LDV RECEIVING OPTICS
(For Legend see Page 146)
LEGEND
(Figure 5)

1  300 mm Focal Length Lens
2  130 mm Focal Length Lens
3  50 mm Focal Length Lens
4  Retaining Ring
5  Spacing Ring
6  Lens Tube
7  Existing Aperture Holder
8  Lens Tube
9  Lens Tube Adaptor Ring
10  Lens Holder
11  Retaining Ring
12  Photomultiplier Tube
13  Filter
14  Support Point
15  Connecting Rod (not shown)
16  Vertical Tube Support Assembly (not shown)
17  Filter Retainer
18  Aperture (Pin-Hole)
19  Plane of Photomultiplier Tube
FIGURE 6. PARTICLE SEEDING SYSTEM

A 2500 psi Nitrogen Bottle
B 3-Way Valve
C High Pressure Bleed
D High Pressure Lime
E Particle Reservoir
F Electrical Valve
G Needle Valve
H Manifold
FIGURE 7. DIGITAL CONTROLLER INTERFACE
$X/X^* = 1.0000$

Reynolds No. = 5000000

Mach No. = 0.00

$U_2/U_1 = 1.000$

**CSM Model:**

- Backward differencing
- Crank-Nicholson scheme

**Figure 8. Mean Velocity Distribution**
\( X/X^* = 1.0000 \)

\( \text{REYNOLDS NO.} = 5000000 \)

\( \text{MACH NO.} = .000 \)

\( U_2/U_1 = 1.000 \)

**CSM MODEL:**
- \( \times \) BACKWARD DIFFERENCING
- \( + \) CRANK-NICHOLSON SCHEME

**FIGURE 9. REYNOLDS STRESS DISTRIBUTION**
$X/X* = 1.0000$
REYNOLDS NO. = 5000000
MACH NO. = 0.000
$U_2/U_1 = 1.000$

GLUSHKO MODEL:
$\times$ BACKWARD DIFFERENCING
$+$ CRANK-NICHOLSON SCHEME

FIGURE 10. MEAN VELOCITY DISTRIBUTION
$X/X_*= 1.000$  
$\text{REYNOLDS NO.} = 5000000$  
$\text{MACH NO.} = .000$  
$U_2/U_1 = 1.000$

**GLUSHKO MODEL:**  
$\times$ BACKWARD DIFFERENCING  
$+$ CRANK-NICHOLSON SCHEME

**FIGURE 11. REYNOLDS STRESS DISTRIBUTION**
$X/X^* = 1.1167$
REYNOLDS NO. = 5000000
MACH NO. = 0.000
$U_2/U_1 = 1.000$

FIGURE 12. MEAN VELOCITY DISTRIBUTION
MODIFIED CSM MODEL:
\[ \times \text{ BACKWARD DIFFERENCING} \]
\[ + \text{ CRANK-NICHOLSON SCHEME} \]

FIGURE 13. REYNOLDS STRESS DISTRIBUTION
$X/X_*=1.1167$

REYNOLDS NO. = 5000000

MACH NO. = .000

$U_2/U_1 = 1.000$

**2ND ONE-EQUATION MODEL:**

× BACKWARD DIFFERENCING

+ CRANK-NICHOLSON SCHEME

**FIGURE 14. MEAN VELOCITY DISTRIBUTION**
$X/X_\ast = 1.1167$

REYNOLDS NO. = 5000000
MACH NO. = .000
$U_2/U_1 = 1.000$

2ND ONE-EQUATION MODEL:
X BACKWARD DIFFERENCING
+ CRANK-NICHOLSON SCHEME

FIGURE 15. REYNOLDS STRESS DISTRIBUTION
$X/X^* = 1.1167$
Reynolds No. = 5000000
Mach No. = 0.000
$U_2/U_1 = 1.000$

2nd One-Equation Model:
Crank-Nicholson Scheme

Figure 16: Mean Velocity Distribution
2nd one-equation model:
CRANK-NICHOLSON SCHEME

FIGURE 17. REYNOLDS STRESS DISTRIBUTION
MODIFIED GLUSHKO MODEL:
X/BACKWARD DIFFERENCING
+ CRANK-NICHOLSON SCHEME

FIGURE 18. MEAN VELOCITY DISTRIBUTION
$X/X_* = 1.8333$

Reynolds No. = 5000000

Mach No. = 0.000

$U_2/U_1 = 1.000$

**FIGURE 19. REYNOLDS STRESS DISTRIBUTION**
$X/X_*= 1.8333$

REYNOLDS NO. = 5000000

MACH NO. = .000

$u_2/U_1 = 1.000$

FIGURE 20. REYNOLDS STRESS DISTRIBUTION
X/X* = 1.8333
REYNOLDS NO. = 1000000
MACH NO. = .000
U_2/U_1 = 1.000

MODIFIED GLUSHKO MODEL:
+ BACKWARD Differencing
+ CRANK-NICHOLSON SCHEME

FIGURE 21. REYNOLDS STRESS DISTRIBUTION
$X/X_*= 1.0200$

REYNOLDS NO. = 53600

MACH NO. = .000

$U_2/U_1 = 1.000$

FIGURE 22. VELOCITY DISTRIBUTION
$X/X_\ast = 1.0000$
Reynolds No. = 654500
Mach No. = 0.000
$u_2/u_1 = 1.000$

**Figure 23. Mean Velocity Distribution**
\[ X/X_* = 1.0000 \]

\[ \text{REYNOLDS NO.} = 654500 \]

\[ \text{MACH NO.} = 0.000 \]

\[ u_2/u_1 = 1.000 \]

**FIGURE 24. REYNOLDS STRESS DISTRIBUTION**
\[ \frac{X}{X} = 1.0208 \]

REYNOLDS NO. = 654500

MACH NO. = .000

\[ \frac{U_2}{U_1} = 1.000 \]

**FIGURE 25. MEAN VELOCITY DISTRIBUTION**
\[ X/X_* = 1.0208 \]
\[ \text{REYNOLDS NO.} = 654500 \]
\[ \text{MACH NO.} = 0.000 \]
\[ U_2/U_1 = 1.000 \]
$X/X_* = 1.2083$

Reynolds No. = 654500

Mach No. = .000

$u_2/u_1 = 1.000$

**Figure 27. Mean Velocity Distribution**
$X/X_* = 1.2083$

REYNOLDS NO. = 654500

MACH NO. = .000

$U_2/U_1 = 1.000$

**FIGURE 28. REYNOLDS STRESS DISTRIBUTION**
\[ X/X_0 = 2.0000 \]
\[ \text{REYNOLDS NO.} = 654500 \]
\[ \text{MACH NO.} = 0.000 \]
\[ U_2/U_1 = 1.000 \]

**FIGURE 29. MEAN VELOCITY DISTRIBUTION**
$X/X_*= 2.0000$
REYNOLDS NO. = 654500
MACH NO. = 0.000
$U_2/U_1 = 1.000$

**Figure 30. Reynolds Stress Distribution**

- DATA OF REF. 45
- MODIFIED GLUSHKO MODEL
- 2ND ONE-EQUATION MODEL
- MODIFIED CSM MODEL
$X/X_o = 34.2500$

REYNOLDS NO. = 19550

MACH NO. = 3.000

$u_2/u_1 = 1.000$

ASYMPTOTIC SOLUTION OF REF. 76

- $VBT = 0.00909$ (REF. 76)
- $VBT = 0.00557$ (CSM MODEL)
- MODIFIED CSM MODEL

FIGURE 31. MEAN VELOCITY DISTRIBUTION
\[ \frac{X}{X_*} = 34.2500 \]

REYNOLDS NO. = 19550

MACH NO. = 3.000

\[ \frac{u_2}{u_1} = 1.000 \]

ASYMPTOTIC SOLUTION OF REF. 76

- □ VBT = 0.00909 (REF. 76)
- ○ VBT = 0.00557 (CSM MODEL)
- △ MODIFIED CSM MODEL

FIGURE 32. MEAN TEMPERATURE DISTRIBUTION
$X/X_\ast = 41.0000$

Reynolds No. = 19550

Mach No. = 3.000

$u_2/u_1 = 1.000$

**Figure 33. Mean Velocity Distribution**
$X/X_\star = 41.0000$
REYNOLDS NO. = 19550
MACH NO. = 3.000
$U_2/U_1 = 1.000$

ASYMPTOTIC SOLUTION OF REF. 76

\[ VBT = 0.00909 \text{ (REF. 76)} \]
\[ VBT = 0.00557 \text{ (CSM MODEL)} \]
\[ \text{MODIFIED CSM MODEL} \]

FIGURE 34. MEAN TEMPERATURE DISTRIBUTION
$X/X_* = 51.5000$

Reynolds No. = 19550

Mach No. = 3.000

$U_2/U_1 = 1.000$

Figure 35. Mean velocity distribution
$X/X^* = 51.5000$

REYNOLDS NO. = 19550

MACH NO. = 3.000

$U_2/U_1 = 1.000$

FIGURE 36. MEAN TEMPERATURE DISTRIBUTION
\[ x/x_0 = 1.000 \]
REYNOLDS NO. = 5115000
MACH NO. = 2.500
\[ u_2/u_1 = 0.873 \]

**FIGURE 37. MEAN VELOCITY DISTRIBUTION**
$X/X^* = 1.0253$

REYNOLDS NO. = 5115000

MACH NO. = 2.500

$U_2/U_1 = 0.873$

**DATA OF REF. 78**

MODIFIED GLUSHKO MODEL

2ND ONE-EQUATION MODEL

MODIFIED CSM MODEL

EXCHANGE COEF. MODEL

**FIGURE 38. MEAN VELOCITY DISTRIBUTION**
$X/X_* = 1.1522$

REYNOLDS NO. = 5115000
MACH NO. = 2.500
$U_2/U_1 = .873$

**Figure 39. Mean Velocity Distribution**
$X/X_\star = 1.4530$

REYNOLDS NO. = 5115000
MACH NO. = 2.500
$U_2/U_1 = .873$

DATA OF REF. 78
MODIFIED GLUSHKO MODEL
2ND ONE-EQUATION MODEL
MODIFIED CSM MODEL
EXCHANGE COEF. MODEL

FIGURE 40. MEAN VELOCITY DISTRIBUTION
\begin{align*}
X/X_*= 1.5415 \\
\text{REYNOLDS NO.} = 5115000 \\
\text{MACH NO.} = 2.500 \\
U_2/U_1 = .873
\end{align*}

\textbf{FIGURE 41. MEAN VELOCITY DISTRIBUTION}
$X/X_* = 1.1167$
REYNOLDS NO. = 3180000
MACH NO. = .000
$u_2/u_1 = .610$

FIGURE 42. MEAN VELOCITY DISTRIBUTION
$X/X_\ast = 1.1833$

REYNOLDS NO. = 3180000

MACH NO. = .000

$U_2/U_1 = .610$

**FIGURE 43. MEAN VELOCITY DISTRIBUTION**
$X/X_\ast = 1.3667$
REYNOLDS NO. = 3180000
MACH NO. = .000
$u_2/u_1 = .610$

**FIGURE 44. MEAN VELOCITY DISTRIBUTION**
Figure 45. Mean Velocity Distribution
$X/X_\ast = 1.8333$
REYNOLDS NO. = 3160000
MACH NO. = .000
$U_2/U_1 = .610$

DATA OF REF. 72
× 2ND ONE-EQUATION MODEL
△ MODIFIED CSM MODEL
◇ EXCHANGE COEF. MODEL

FIGURE 46. MEAN VELOCITY DISTRIBUTION
FIGURE 47. TUNNEL WAVE SYSTEM
FIGURE 48. PITOT PRESSURE DISTRIBUTION
FIGURE 49. TUNNEL MACH NUMBER DISTRIBUTION
FIGURE 50. DOPPLER SIGNALS FROM SPINNING WIRE
FIGURE 51. INTERFERENCE FRINGES AND WIRE
$X = 2.0$ INCHES
REYNOLDS NO. = 17103000
MACH NO. = 3.200
$U_2/U_1 = .991$

FIGURE 52. MEAN VELOCITY DISTRIBUTION
Figure 53. Mean Velocity Distribution

X = 2.0 INCHES
REYNOLDS NO. = 17103000
MACH NO. = 3.200
U₂/U₁ = 0.991

PRESENT DATA
$X = 2.0$ INCHES
REYNOLDS NO. = 17103000
MACH NO. = 3.200
$U_2/U_1 = 0.991$

PRESENT DATA

FIGURE 54. RMS VELOCITY DISTRIBUTION
FIGURE 55. RMS VELOCITY DISTRIBUTION

X = 2.0 INCHES
REYNOLDS NO. = 17103000
MACH NO. = 3.200
$U_2/U_1 = .991$

PRESENT DATA
$X = 6.0$ INCHES
REYNOLDS NO. $= 17103000$
MACH NO. $= 3.200$
$U_2/U_1 = .795$

FIGURE 56. MEAN VELOCITY DISTRIBUTION
Figure 57. Mean velocity distribution

$x = 6.0$ inches
Reynolds No. = 17103000
Mach No. = 3.200
$U_2/U_1 = .795$
X = 6.0 INCHES
REYNOLDS NO. = 17103000
MACH NO. = 3.200
$U_2/U_1 = .795$

FIGURE 58. RMS VELOCITY DISTRIBUTION
$X = 6.0$ INCHES
REYNOLDS NO. = 17103000
MACH NO. = 3.200
$U_2/U_1 = 0.795$

![Graph showing RMS velocity distribution](image)

**FIGURE 59. RMS VELOCITY DISTRIBUTION**
X/X* = 1.0667
REYNOLDS NO. = 17100000
MACH NO. = 3.200
U_2/U_1 = .991

FIGURE 60. MEAN VELOCITY DISTRIBUTION
$X/X_* = 1.2000$
REYNOLDS NO. = 17100000
MACH NO. = 3.200
$u_2/u_1 = .795$

FIGURE 61. MEAN VELOCITY DISTRIBUTION
$X/X_* = 1.0667$

REYNOLDS NO. = 17100000

MACH NO. = 3.20C

$U_2/U_1 = .991$

□ PRESENT DATA
○ MODIFIED GLUSHKO MODEL
× 2ND ONE-EQUATION MODEL

**FIGURE 62. TURBULENCE INTENSITY.**
$X/X^* = 1.0667$
REYNOLDS NO. = 17100000
MACH NO. = 3.200
$U_2/U_1 = .991$

\[ \text{PRESENT DATA} \]
\[ \odot \text{MODIFIED GLUSHKO MODEL} \]
\[ \times \text{2ND ONE-EQUATION MODEL} \]

\text{FIGURE 63 : TURBULENCE INTENSITY.}
FIGURE 64. TURBULENCE INTENSITY.
$X/X_* = 1.2000$
REYNOLDS NO. = 1710000
MACH NO. = 3.200
$u_2/u_1 = 0.795$

- Present Data
- Modified GLUSHKO Model
- 2nd One-Equation Model

FIGURE 65. TURBULENCE INTENSITY.
### TABLE 1.

**EMPIRICAL CONSTANTS IN THE UPSTREAM BOUNDARY LAYER CALCULATIONS**

<table>
<thead>
<tr>
<th>Experiment</th>
<th>Glushko Model</th>
<th>CSM Model</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$C_k$</td>
<td>$\alpha_E$</td>
</tr>
<tr>
<td>Ref. 45</td>
<td>3.93</td>
<td>0.100</td>
</tr>
<tr>
<td>Ref. 78</td>
<td>3.93</td>
<td>0.165</td>
</tr>
<tr>
<td>Present</td>
<td>3.93</td>
<td>0.165</td>
</tr>
</tbody>
</table>
TABLE 2.
EMPIRICAL CONSTANTS IN THE FREE SHEAR LAYER CALCULATIONS.

EMPIRICAL CONSTANTS

<table>
<thead>
<tr>
<th>Experiments</th>
<th>Modified Glushko Model</th>
<th>Modified CSM Model</th>
<th>2nd One-Equation Model</th>
<th>Exchange Coef. Model</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$C_k$</td>
<td>$\alpha_E$</td>
<td>$K$</td>
<td>$\kappa_0$</td>
</tr>
<tr>
<td>Ref. 45</td>
<td>3.93</td>
<td>0.100</td>
<td>0.40</td>
<td>0.0168</td>
</tr>
<tr>
<td>Ref. 72</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>0.005</td>
</tr>
<tr>
<td>Ref. 76</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>0.0168</td>
</tr>
<tr>
<td>Ref. 78</td>
<td>3.93</td>
<td>0.200</td>
<td>0.40</td>
<td>0.0100</td>
</tr>
<tr>
<td>Present (2-inch Station)</td>
<td>3.93</td>
<td>0.125</td>
<td>0.40</td>
<td>0.0125</td>
</tr>
<tr>
<td>Present (6-inch Station)</td>
<td>3.93</td>
<td>0.100</td>
<td>0.40</td>
<td>0.0050</td>
</tr>
</tbody>
</table>
TABLE 3
EFFECT OF 3-SIGMA DATA EDITING

RUN TITLE: FREE SHEAR LAYER MEASUREMENTS

- - - - - REDUCED RESULTS FOR LVY STATION NO. 1 - - - - -

DATA FOR THIS STATION COLLECTED ON 11-09-81 AT 14:30:31
WIND POINT EDITING USED: NONE
LVY VELOCITY COMPONENT PROCESSED: "U"
NUMBER OF SAMPLES ACQUIRED ON STATION: 2000
NUMBER OF POINTS EDITED FROM DATA: 0
UNCORRECTED STATISTICS (FT/SEC):

<table>
<thead>
<tr>
<th>STATISTIC</th>
<th>VALUE</th>
</tr>
</thead>
<tbody>
<tr>
<td>MEAN VELOCITY</td>
<td>1271.29</td>
</tr>
<tr>
<td>RMS</td>
<td>0.5284E+02</td>
</tr>
<tr>
<td>SKEW</td>
<td>0.3645E+01</td>
</tr>
<tr>
<td>EXCESS</td>
<td>0.5917E+02</td>
</tr>
</tbody>
</table>

STATISTICS CORRECTED FOR VELOCITY BIAS:

<table>
<thead>
<tr>
<th>STATISTIC</th>
<th>VALUE</th>
</tr>
</thead>
<tbody>
<tr>
<td>MEAN VELOCITY</td>
<td>1269.287</td>
</tr>
<tr>
<td>RMS</td>
<td>0.5433E+02</td>
</tr>
<tr>
<td>SKEW</td>
<td>0.2391E+01</td>
</tr>
<tr>
<td>EXCESS</td>
<td>0.4493E+02</td>
</tr>
</tbody>
</table>

APPROX. "X" LOCATION OF OPTICAL SUBSYSTEM: 6.000 INCHES
APPROX. "Y" LOCATION OF OPTICAL SUBSYSTEM: -0.900 INCHES
LVY PROCESSOR BEING USED: 1990
LVY PROCESSOR INTERFACE USED: "DIGITAL"
LVY PROCESSOR "N" CYCLES SETTING: 9
LVY PROCESSOR DATA EXPONENT SETTING: AUTO-RANGE
FRINGE SPACING: 17.300 MICRONS
Bragg CELL DOWNMIX FREQUENCY: 0 MHZ
**TABLE 3 (continued)**

**EFFECT OF 3-SIGMA DATA EDITING**

**RUN TITLE:** FREE SHEAR LAYER MEASUREMENTS

---

**REDUCED RESULTS FOR LDV STATION NO. 1**

DATA FOR THIS STATION COLLECTED ON 11-09-83 AT 14:30:31

**WILD POINT EDITING USED:** 3 SIGM

**LDV VELOCITY COMPONENT PROCESSED:** U

**NUMBER OF SAMPLES ACQUIRED ON STATION:** 2000

**NUMBER OF POINTS EDITED FROM DATA:** 10

**UNCORRECTED STATISTICS (FT/SEC):**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>MEAN VELOCITY</td>
<td>1270.030</td>
</tr>
<tr>
<td>RMS</td>
<td>0.4744E+02</td>
</tr>
<tr>
<td>SKEW</td>
<td>-0.1020E+00</td>
</tr>
<tr>
<td>EXCESS</td>
<td>-0.6633E+00</td>
</tr>
</tbody>
</table>

**CORRECTED FOR VELOCITY DIAD:**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>MEAN VELOCITY</td>
<td>1268.600</td>
</tr>
<tr>
<td>RMS</td>
<td>0.4225E+02</td>
</tr>
<tr>
<td>SKEW</td>
<td>-0.1767E+00</td>
</tr>
<tr>
<td>EXCESS</td>
<td>-0.7038E+00</td>
</tr>
</tbody>
</table>

**APPROX. "X" LOCATION OF OPTICAL SUBSYSTEM:** 6.000 INCHES

**APPROX. "Y" LOCATION OF OPTICAL SUBSYSTEM:** -0.500 INCHES

**LDV PROCESSOR BEING USED:** 1980

**LDV PROCESSOR INTERFACE USED:** DIGITAL

**LDV PROCESSOR "NM CYCLES SETTING: o**

**LDV PROCESSOR DATA EXPONENT SETTING:** AUTO-RANGE

**FRINGE SPACING:** 17.300 MICRONS

**SHAGE CELL DOWNMIX FREQUENCY:** 0 MHZ
TABLE 4.

INTENSITY CORRELATION CONSTANTS

<table>
<thead>
<tr>
<th>Measurement Station</th>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Modified Glushko Model</td>
<td>2nd One-Equation Model</td>
</tr>
<tr>
<td>2-Inch</td>
<td>1.750</td>
<td>1.500</td>
</tr>
<tr>
<td>6-Inch</td>
<td>0.100</td>
<td>0.0750</td>
</tr>
</tbody>
</table>
As stated in Section 3.3, the dimensionless equations of motion were solved numerically, using a Backward and the Crank-Nicholson Difference Schemes. The basic difference between the two schemes, as used in this study, lies in the finite difference expressions for the derivatives with respect to independent variable S (or Y). Therefore, this discussion will focus on the difference in the difference expressions for the S-derivatives only.

In the Backward Difference Scheme, a backward difference was used to represent the derivatives with respect to X, hence the name Backward Difference Scheme, while centered difference representation was used for the derivatives with respect to S. Using a single index or subscript notation, at a node along the S-axis, the partial derivatives with respect to S were approximated by the difference expressions

\[ (P_S)_j = \frac{1}{2H} \left[ p_{j+1} - p_{j-1} \right] \]  \hspace{1cm} (A.1)

and

\[ (P_{SS})_j = \frac{1}{H^2} \left[ p_{j+1} - 2p_j + p_{j-1} \right] \]  \hspace{1cm} (A.2)
where \( P \) is any flow property.

In the Crank-Nicholson Scheme, averages and differences of flow properties were defined as follows:

\[
\bar{P}_j = \frac{1}{2} (P_j + \hat{P}_j) \tag{A.3}
\]

\[
\Delta P_j = P_j - \hat{P}_j. \tag{A.4}
\]

\( \bar{P}_j \) is the average value, \( \hat{P}_j \), the old solution and \( P_j \), the new solution at the node \( j \). Therefore, in the Crank-Nicholson Scheme, the first and second partial derivatives with respect to \( S \) were given by

\[
(P_S)_j = \frac{1}{2H} \left[ p_j \right]_{j+1} - p_j - \frac{1}{2} \left[ \Delta P_j \right]_{j+1} - \Delta P_j \right] \tag{A.5}
\]

and

\[
(P_{SS})_j = \frac{1}{H^2} \left[ p_j \right]_{j+1} - 2p_j + p_j \right]_{j-1} - \frac{1}{2} \left[ \Delta P_j \right]_{j+1} - 2\Delta P_j + \Delta P_j \right]_{j+1} \right. \tag{A.6}
\]

The difference equations for the \( X \)-derivatives were centered about \( j - 1/2 \). Therefore, the basic difference in the two schemes is seen by examining Eqs. (A.1), (A.2), (A.5) and (A.6).
LIST OF REFERENCES


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