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CENTROID ANGLE OF ARRIVAL TEMPORAL POWER SPECTRUM FOR SPHERICAL WAVE PROPAGATION THROUGH THE TURBULENT ATMOSPHERE BETWEEN TWO MOVING VEHICLES

The Ohio State University

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CENTROID ANGLE OF ARRIVAL TEMPORAL POWER SPECTRUM FOR SPHERICAL WAVE PROPAGATION THROUGH THE TURBULENT ATMOSPHERE BETWEEN TWO MOVING VEHICLES

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy in the Graduate School of the Ohio State University

By

Yu-Jih Liu, B.S., M.S.

The Ohio State University

1983

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# TABLE OF CONTENTS

| DEDICATION | i |
| ACKNOWLEDGEMENT | ii |
| VITA | iii |
| TABLE OF CONTENTS | iv |
| LIST OF FIGURES | vi |

## Chapter

### I. INTRODUCTION

1. General Consideration ........................ 1
2. Previous Work ............................... 4
3. Nontracking and Tracking System ............. 7
4. Content Outline ............................. 10

### II. CENTROID ANGLE OF ARRIVAL POWER SPECTRUM FOR A NONTRACKING RECEIVER

1. Introduction .................................. 12
2. Definition and Simplification of Centroid Angle of Arrival ............ 15
3. Derivation of Random Phase for Atmospherically Perturbed Wave from a Moving Source . . 26
4. Computation of Phase Correlation Function ..................... 44
5. Computation of Angle of Arrival Covariance Function ................ 48
6. Computation of Angle of Arrival Temporal Power Spectrum ............ 50
7. Derivation for a Special Case ........................ 52
8. Summary ..................................... 60

### III. ANGLE OF ARRIVAL POWER SPECTRUM FOR A TRACKING RECEIVER

1. Introduction .................................. 61
2. Angle of Arrival Equations ................... 64
3. Moving Coordinate Systems ................... 65
4. Wave Equation in the Rotating Coordinate System .................... 69
5. Solution of the Wave Equation in a Moving Coordinate System ......... 74
6. Discussion of the Approximations .................. 78
7. Evaluation of the Deterministic Portion of the Complex Log Wave Amplitude ..................... 79
8. Evaluation of the Random Portion of the Complex Log Amplitude ........ 86
9. Phase Correlation Function .................... 96
10. Angle of Arrival Covariance ........................ 103
11. Angle of Arrival Power Spectrum ............. 108
TABLE OF CONTENTS (Continued)

<table>
<thead>
<tr>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>12   Numerical Integration Results</td>
</tr>
<tr>
<td>13   Physical Discussion</td>
</tr>
<tr>
<td>14   Summary</td>
</tr>
<tr>
<td>IV. PROPOSED EXPERIMENTAL VERIFICATION</td>
</tr>
<tr>
<td>1    Introduction</td>
</tr>
<tr>
<td>2    Transmitter Optics</td>
</tr>
<tr>
<td>3    Receiver Optics</td>
</tr>
<tr>
<td>4    Spot Motion</td>
</tr>
<tr>
<td>5    Signal Processing</td>
</tr>
<tr>
<td>6    C² Measurement</td>
</tr>
<tr>
<td>7    Summary</td>
</tr>
<tr>
<td>V. DISCUSSION AND SUMMARY</td>
</tr>
<tr>
<td>1    Summary and Conclusion</td>
</tr>
<tr>
<td>2    Discussion</td>
</tr>
<tr>
<td>APPENDIXES</td>
</tr>
<tr>
<td>A. Investigation of the Effect of Log Amplitude Fluctuation</td>
</tr>
<tr>
<td>B. Derivation of Orthogonality Relationship</td>
</tr>
<tr>
<td>C. Some Remarks on Taylor's Hypothesis</td>
</tr>
<tr>
<td>D. Some Physical Explanation on the Deterministic Phase Shift</td>
</tr>
<tr>
<td>E. Estimated Wave Equation in the Rotating Coordinate System</td>
</tr>
<tr>
<td>F. Simplification of Wave Equation in the Rotating Coordinate System</td>
</tr>
<tr>
<td>G. Justification of Constant Propagation Range Within Correlation Time</td>
</tr>
<tr>
<td>H. Detector Signal-to-Noise Ratio</td>
</tr>
<tr>
<td>I. Computer Program Listings</td>
</tr>
<tr>
<td>BIBLIOGRAPHY</td>
</tr>
<tr>
<td>Figure</td>
</tr>
<tr>
<td>--------</td>
</tr>
<tr>
<td>1.</td>
</tr>
<tr>
<td>2.</td>
</tr>
<tr>
<td>3.</td>
</tr>
<tr>
<td>4.</td>
</tr>
<tr>
<td>5.</td>
</tr>
<tr>
<td>6.</td>
</tr>
<tr>
<td>7.</td>
</tr>
<tr>
<td>8.</td>
</tr>
<tr>
<td>10.</td>
</tr>
<tr>
<td>11.</td>
</tr>
<tr>
<td>12.</td>
</tr>
<tr>
<td>13.</td>
</tr>
<tr>
<td>14.</td>
</tr>
<tr>
<td>15.</td>
</tr>
<tr>
<td>16.</td>
</tr>
<tr>
<td>17.</td>
</tr>
<tr>
<td>18.</td>
</tr>
<tr>
<td>Figure</td>
</tr>
<tr>
<td>--------</td>
</tr>
<tr>
<td>19.</td>
</tr>
<tr>
<td>20.</td>
</tr>
<tr>
<td>21.</td>
</tr>
<tr>
<td>22.</td>
</tr>
<tr>
<td>23.</td>
</tr>
<tr>
<td>24.</td>
</tr>
<tr>
<td>25.</td>
</tr>
<tr>
<td>26.</td>
</tr>
<tr>
<td>27.</td>
</tr>
<tr>
<td>28.</td>
</tr>
<tr>
<td>29.</td>
</tr>
<tr>
<td>30.</td>
</tr>
<tr>
<td>31.</td>
</tr>
<tr>
<td>32.</td>
</tr>
<tr>
<td>33.</td>
</tr>
</tbody>
</table>
# LIST OF FIGURES (Continued)

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>34.</td>
<td>Transmitter Optics</td>
<td>150</td>
</tr>
<tr>
<td>35.</td>
<td>Receiver Optics</td>
<td>155</td>
</tr>
<tr>
<td>36.</td>
<td>Centroid Angle of Arrival Data Digitizer</td>
<td>163</td>
</tr>
<tr>
<td>37.</td>
<td>Centroid Angle of Arrival Correlation Function Versus Time Delay on a Log-Linear Scale</td>
<td>166</td>
</tr>
<tr>
<td>38.</td>
<td>Centroid Angle of Arrival Correlation Function Versus Time Delay on a Linear-Linear Scale</td>
<td>167</td>
</tr>
<tr>
<td>39.</td>
<td>Typical Airborne Measured Microthermal Power Spectrum from Flight 3</td>
<td>169</td>
</tr>
<tr>
<td>40.</td>
<td>Explanation of Doppler Shift</td>
<td>197</td>
</tr>
</tbody>
</table>
CHAPTER I

INTRODUCTION

1 GENERAL CONSIDERATION

As the light ray propagates through the atmosphere, many quantities such as amplitude, phase, and angle of arrival fluctuate in a random manner. The reason is that the refractive index of the atmosphere changes from one time to another and from one place to another because of the variations in temperature, humidity, etc. A simple illustration of this effect can be given by considering the atmosphere as composed of many volumes of different sizes and refractive index values. As the light ray propagates from one volume to another or equivalently from one medium to another, it will be deflected according to Snell's law of refraction. If the refractive index varies in a random manner, the received focused light spot will dance around and the beam will not stay focused. Stated differently, the angle of arrival of the incoming beam will fluctuate in a random manner. The angle of arrival studies are the central issue in this dissertation.

Intuitively, angle of arrival is the quantity which gives the direction of the incoming wave. Without random refractive index fluctuation, the wave propagation direction is well defined. With random
refractive index fluctuation, there is no fixed instantaneous propagation direction and the effect such as spot dancing and defocusing described before will result. In this dissertation, the angle of arrival fluctuation is analyzed in detail.

The angle of arrival is very important. This is because angle of arrival is the quantity to indicate the source direction. It is applied to astronomical imaging and tracking of stationary or moving vehicles (satellites and aircraft for example).

Technically, angle of arrival can be defined in many different ways depending on the particular problem. For example, it can be defined as the received wavefront normal for a small aperture where the wavefront is flat over the aperture. The angle of arrival in this case is then defined in terms of the angle between the axis and the ray from the center of the lens to the point. It can also be defined as the direction of maximum average energy transfer for a large aperture. The definition we are going to use can also be applied to a large aperture and is called the "centroid of angle of arrival", where the term centroid defined here refers to the intensity centroid in the image plane. This can be explained more clearly as follows. Suppose the incoming wave from the point source is focused by the lens. When the incoming wavefront is not spherical, the focused spot is not a point and the centroid of the intensity distribution in the image plane is a good indication of angle of arrival.
Experimentally, centroid angle of arrival can be measured using a detector which generates an output $V_y$ proportional to the integrated product of its $y$-direction displacement and signal intensity. $V_y$ is the signal given by the following integral

$$V_y = \iint y I(y, z) \, dy\,dz \quad (1.1)$$

where the integration is performed over the image plane $(y,z)$. To normalize, another signal $V$ given by

$$V = \iint I(y, z) \, dy\,dz \quad (1.2)$$

and is generated to give the average intensity in the image plane. The centroid angle of arrival $\alpha$ is then the ratio given below.

$$\alpha = \frac{V_y}{V} \quad (1.3)$$

Usually there are two angles of arrival corresponding to the two directions $y$ and $z$ in the image plane. The $z$ component of the angle of arrival is given by $\beta = \frac{V_z}{V}$ where $V_z$ is similarly defined.

The atmospheric refractive index varies not only spatially but also temporally. Thus, the measured angle of arrival will also fluctuate temporally. What we are interested in is the temporal variation of the angle of arrival. A natural source of this temporal variation is the wind motion. If both the transmitter and receiver move, they can be considered as extra sources of the so called "superficial wind velocity". A good quantity to show the rapidity of the temporal variation is the temporal power spectrum.
In this dissertation, the problem taken is thus to carry through a derivation of the centroid angle of arrival temporal power spectrum with both the transmitter and receiver moving and the wind blowing along the path. Because the transmitter moves, the situation is a little more complicated. If the transmitter moves very far, the receiver must somehow track it in order that the receiver may keep in view. In this dissertation, the cases for both a tracking and a nontracking receiver are analyzed. Finally, the design of an experiment for measuring the centroid angle of arrival temporal spectrum is presented.

2 PREVIOUS WORK

For wave propagation in the turbulent atmosphere, much work has been done since the pioneer work by Tatarskii [1, 2], Chernov [3], Protheroe [4], and Fried [5, 6, 7, 8, 9]. Statistical quantities such as spatial and temporal variance, covariance, structure function, and spectrum are all of interest. They have been used to analyze the fluctuation of log amplitude [2, 3, 10], phase [2, 5, 11, 12], intensity [13, 14, 15, 16], depolarization [17, 18], angle of arrival [2, 19, 20], etc. Much work has also been done on the saturation of scintillation intensity variance [21, 22, 23, 24, 25, 26]. The atmospheric wave propagation theory was also applied in detection and estimation for optical communication [27, 28, 29, 30], refractive index probing [31, 32, 33, 34], and wind velocity sensing [35, 36]. Recently, some methods such as speckle interferometry [37, 38], adaptive optics [39, 40, 41, 42, 43, 44, 45], and some other techniques [46, 47, 48, 49], have been successfully applied to compensate for the effect of the atmospheric turbulence.
Another topic of interest is the temporal power spectrum. The temporal power spectrum has been presented for different kinds of wave propagation [2, 57, 58, 59] as will be described.

The angle of arrival fluctuations which are the center issue in this dissertation have a close relation with the phase fluctuation. This is because the normal to the constant phase front in general gives the angle of arrival. The problem of determining the phase fluctuation statistics has been solved for the simple cases of the plane wave [2, 5], spherical wave [2, 50], to the more complicated laser beam wave [51, 52], pulse wave [53], finite laser beam [54], and speckle wave [55, 56].

The angle of arrival can be related to the phase difference measured at two points separated by finite distance [2, 58, 60]. In this area, some work has also been done to analyze spatial and temporal behavior [2, 57, 58, 61].

Work has also been done using other definitions about angle of arrival. Tatarskii and Strobehn and Clifford define the angle of arrival to be the normal to the phase front [2, 62]. This is useful for a small aperture. For a large aperture, Heidbreder associated the angle of arrival with the direction of maximum instantaneous power received [63]. deWolf calculated the difference angle of arrival power spectrum of two initially parallel rays using the small aperture definition [64]. Lutomirsky and Buser relate the angle of arrival to the phase difference between two points as their separation distance approaches zero [65].
The definition we are going to use is based on the centroid angle of arrival. This is because a position proportional detector which can be used in the image plane of a lens aperture is consistent with the centroid definition. The earliest work with this definition is by Tatarskii [2]. However, in Tatarskii's original work, only the variance of the angle of arrival is calculated. Recently, Hogge and Butts [66] calculated the centroid angle of arrival power spectrum by using geometrical representation to the distortion of the wavefront.

All the previous work mentioned so far except the one by Lutomirsky and Buser has considered only wave propagation between a stationary source and a stationary receiver. There are also some references on wave propagation between moving vehicles. Lutomirsky and Buser calculated the angle of arrival fluctuation for light propagating from a moving source to a stationary receiver. This is valid only for a small aperture [65]. Stroehn developed the phase and log amplitude statistics for laser beam propagation between two moving vehicles [67]. S. A. Collins and Y. J. Liu evaluated the centroid angle of arrival temporal power spectrum between a moving source and a stationary receiver [68]. This was for the simple case of a nontracking receiver.

All the previous work on angle of arrival has considered wave propagation either between two stationary vehicles or between a moving source and a stationary receiver. We derive a more general formula for both source and receiver moving and the wind blowing. There are further two other factors which have not yet been considered. The first is the
Doppler effect as the source moves. The second is that the receiver must track the source in order to assure long term operation. Both of these factors are included in this dissertation. We first derive the centroid angle of arrival power spectrum for a nontracking system using the fixed coordinate system and then the centroid angle of arrival power spectrum for a tracking system using the rotating coordinate system. In the tracking system, the receiver has to follow the source motion, while in the nontracking system, the receiver is stationary. These two systems are briefly discussed in the next section.

3 NONTRACKING AND TRACKING SYSTEM

The typical nontracking system is shown in Figure 1. The source sends out a spherical wave which propagates downward through the turbulent atmosphere. This spherical wave is collected by the lens and imaged to form a spot in the image plane. A position proportional detector is used in the image plane to measure the spot motion. This system can be analyzed by using a fixed coordinate system. The problem with the nontracking system is that the spot will move too far from the optical axis as the source moves. Thus, the receiver will have difficulty in collecting the incoming beam spot and thus cannot function properly.

The tracking system is shown in Figure 2. Here the receiver rotates to face the source all the time. The advantage of this system is that the beam spot will fall around the optical axis to enable proper operation of the receiver. This system is analyzed by using a rotating coordinate system.
Figure 1. Wave Propagation Geometry for a Nontracking System
Figure 2. Wave Propagation Geometry for a Tracking System
In Chapter 2, the development of centroid angle of arrival formulation is briefly reviewed. This formulation is then used to compute the angle of arrival temporal power spectrum for a nontracking system. We solve the time dependent wave equation by using the well known method of smooth perturbations. A general formula is then derived for both the source moving and the wind blowing. At the end, a special numerical example is presented.

In Chapter 3, we evaluate the angle of arrival temporal power spectrum for a tracking system. Because of the rotation of the receiver, the wave equation is first transformed from the fixed coordinate system to the rotating coordinate system. This transformed equation is then solved by using Tatarskii's smooth perturbation method. More numerical examples with physical interpretations are given at the end.

In Chapter 4, an experimental design is presented for measuring centroid angle of arrival temporal power spectrum for a moving source and a tracking receiver.

Several appendices are listed at the end. In Appendix A, we show that log amplitude gives only a second order correction to the angle of arrival power spectrum evaluations. In Appendix B, we derive the statistical orthogonality relationship. In Appendix C, we analyze the Taylor hypothesis from mathematical grounds. In Appendix D, we relate the deterministic phase shift coming from the solution of wave equation of a nontracking system to the Doppler shift. In Appendix E, we discuss
the wave equation in rotating coordinate system. In Appendix F, we simplify the wave equation in the rotating coordinate system by neglecting all those terms which are \((\frac{V}{c})^2\) smaller than those terms which are left. In Appendix G, we justify the approximation that the range can be considered as constant within the correlation time of interest. In Appendix H, we discuss the detector signal-to-noise ratio requirements. In Appendix I, we list the computer program.
CHAPTER II
CENTROID ANGLE OF ARRIVAL
POWER SPECTRUM FOR A NONTRACKING RECEIVER

1 INTRODUCTION

In this chapter we present the centroid angle of arrival formula-
tion [69] and apply it to the case of a nontracking receiver. The math-
ematical derivation is given here and previous results are reviewed
[68]. This formulation will be used later to derive the centroid angle
of arrival temporal correlation function and then the centroid angle of
arrival temporal power spectrum in a nontracking system.

In this system, we consider spherical wave propagation between a
moving source and a stationary receiver with wind blowing across the
propagation path. With source and receiver stationary, only time inde-
dendent wave equation needs to be solved. Because the source may move,
we have to solve time dependent wave equation. The solution from this
time dependent wave equation is then used to compute the angle of arri-
val temporal power spectrum. Even though the received spot may be away
from the optical axis as was discussed in Chapter 1, there is no re-
quirement in the nontracking system that the receiver follows the source
motion.
The optical system for a nontracking system considered is shown in Figure 3. In this figure, the wave sent out from the source propagates downward through the turbulent atmosphere. This wave is then collected by a lens aperture and focused to an image plane. A position proportional detector is used to give a voltage proportional to one component of the centroid of the intensity pattern.

Section 2 deals with the initial angle of arrival formulation. The final result is an expression for angle of arrival temporal spectrum in terms of phase correlation function for an atmospherically degraded wave. In section 2, we first define the centroid angle of arrival and simplify the expression by writing it in terms of the random phase. The angle of arrival covariance function is thus formulated in terms of the phase correlation function. The angle of arrival temporal power spectrum is then formulated as the Fourier transform of the angle of arrival temporal covariance function.

Section 3 and section 4 deal with atmospheric propagation. The result is an expression for phase correlation function to be inserted in the angle of arrival covariance expression. In section 3, we derive an expression for the atmospherically perturbed field. We solve the time dependent wave equation using the method of smooth perturbations to obtain an expression for phase. The phase is found to consist of a random component and a deterministic component. The deterministic component is related to the Doppler shift. The random phase derived in section 3 is used in section 4 in the derivation of the phase correlation function.
Figure 3. Fixed Coordinate System in a Nontracking System
Section 5 and section 6 deal with the final calculation. The result of this is an integral expression to be calculated later for a special case. In section 5, we derive the centroid angle of arrival covariance function using the phase correlation function derived in section 4. The centroid angle of arrival power spectrum, the Fourier transform of the centroid angle of arrival covariance function, is derived in section 6. In section 7, we present a special example. In that example, the source and the wind are all assumed to move in the y direction. Finally in section 8, we summarize the major results.

In the next section, we start by defining centroid angle of arrival.

2 DEFINITION AND SIMPLIFICATION OF CENTROID ANGLE OF ARRIVAL

Figure 4. Optical System in Defining Angle of Arrival
In this section, we first define centroid angle of arrival and then modify it. Consider Figure 4 for a thin lens placed at a distance $d_1$ in front of the image plane. The source moves in a plane a distance $L$ in front of the lens. Let the intensity distribution in the image plane be given by $I(y,z,t)$. $I(y,z,t)$ can also be written in terms of $y$ and $z$ components of arrival angles $\alpha$ and $\beta$ defined below.

\[
\alpha = \frac{y}{d_1} \quad (2.1a)
\]

\[
\beta = \frac{z}{d_1} \quad (2.1b)
\]

The angles $\alpha$ and $\beta$ give the direction the light coming to point $y, z$. They can also be considered as angles between normal to the aperture and the direction from the center of the aperture to the observation point $(d_1, y, z)$ as shown in Figure 4.

The $y$ component of centroid angle of arrival is then defined from the formula

\[
\alpha_0(t) = \frac{\iiint \alpha I(\alpha, \beta, t) d\alpha d\beta}{\iiint I(\alpha, \beta, t) d\alpha d\beta} \quad (2.2)
\]

where $\alpha_0(t)$ is the centroid angle of arrival definition in the $y$ direction. It gives the average direction of the source.

The intensity distribution $I(y,z,t)$ can be derived by first finding an expression for the field $E(y,z,t)$ in the image plane in terms of $E_0''(y',z',t)$ immediately before the lens. The field $E(y,z,t)$ in the image plane is related to the field $E'(y',z',t)$ immediately behind the
lens through the Fresnel diffraction formula [70]. This well-known Fresnel diffraction formula is given below.

\[ E(y,z,t) = \frac{1}{\lambda d} \iint E'(z',y',t) e^{-\frac{i k}{2 d} \left[ (z-z')^2 + (y-y')^2 \right]} \, dz' \, dy' \]  

(2.3)

where \( \lambda \) is the wavelength, \( k \) is the wave number and \((y',z')\) denote the position in the plane of the lens.

The field immediately behind the lens is related to the field immediately in front of the lens by a phase factor \( \Delta(z',y') \). Through matrix optics [70] or thickness function of a thin lens [71], this phase factor \( \Delta(z',y') \) can be shown to be given by

\[ \Delta(z',y') = e^{\frac{-i k}{2 f} (z'^2 + y'^2)} \]  

(2.4a)

where \( f \) is the focal length of the lens. In order for Eq.(2.4a) to be true, two assumptions should be made. One is the assumption of a thin lens, and the other is the following paraxial ray condition [71] for a symmetrical lens

\[ \frac{k}{8R^3} [(z'^2+y'^2)^2] \text{max} \ll 1 \]  

(2.4b)

where \( R \) is the radius of curvature on both sides of the lens. Using Eq. (2.4a), \( E'(y',z',t) \) can be written as

\[ E'(y',z',t) = E_0''(y',z',t) e^{\frac{-i k}{2 f} (z'^2 + y'^2)} \]  

(2.5)

Assuming that a point source is located at \((0, y_s, z_s)\), the spheroid wave incident on the lens is then given by

\[ E_0''(y',z',t) = \frac{E_0'(y',z',t)}{j \lambda L} \exp\left\{ \frac{ik}{2L} [(y'-y_s)^2 + (z'-z_s)^2] + jkL \right\} \]  

(2.6)
where $E_0'$ is a factor to account for atmospheric turbulence. $E_0'$ will be discussed in detail later. Combining Eq. (2.5), Eq. (2.6) and Eq. (2.3), the field $E(y,z;t)$ in the image plane then becomes

$$E(y,z,t) = \frac{k^2}{4\pi^2 L d_i} \exp\left[j\frac{k}{2d_i}(y^2 + z^2)\right] \exp[jkL(y_s^2 + z_s^2)]$$

$$\exp(jkL) \iint E_0'(y',z',t) \exp\left[j\frac{k}{L}(\frac{1}{L} + \frac{1}{d_i} - \frac{1}{f})(y'^2 + z'^2)\right]$$

$$\exp\left[-jk\left[\frac{y_s}{L}y' + \frac{z_s}{d_i}z'\right]\right]dy'dz'$$

(2.7)

We take the detector in the image plane, so from geometric optics, the following lens law is satisfied.

$$\frac{1}{L} + \frac{1}{d_i} = \frac{1}{f}$$

(2.8a)

Two more parameters, $\alpha_s$ and $\beta_s$, are defined. They give the direction of the source and are given by

$$\alpha_s = \frac{y_s}{L}$$

(2.8b)

$$\beta_s = \frac{z_s}{L}$$

(2.8c)

Substituting $\alpha_s$, $\beta_s$, $\alpha$, and $\beta$ from Eq. (2.8b), Eq. (2.8c), Eq. (2.1a), and Eq. (2.1b) into Eq. (2.7) and using the lens law given in Eq. (2.8a), the field $E(y,z,t)$ then becomes

$$E(y,z,t) = \frac{k^2}{4\pi^2 L d_i} \exp\left[j\frac{k}{2d_i}(y^2 + z^2)\right] \exp[jkL(y_s^2 + z_s^2)] \exp(jkL)$$

$$\iint E_0'(y',z',t) \exp\left[-jk[(\alpha_s + \alpha) y' + (\beta_s + \beta) z']\right] dy'dz'$$

(2.9)

The intensity distribution $I(y,z,t)$ which is equal to $E(y,z,t)E^*(y,z,t)$ is then given by
\[ I(y,z,t) = E(y,z,t)E^*(y,z,t) \]

\[ I(y,z,t) \] is renamed as \( I(\alpha,\beta,t) \) and we have

\[ I(\alpha,\beta,t) = \frac{k^4}{16\pi^2 L^2 d^2} \iiint E_0'(y',z',t) E_0'^*(y'',z'',t) \]

\[ -jk[(\alpha+\alpha_s)(y'-y'')+(\beta+\beta_s)(z'-z'')] \]

\[ e^{-jk[(\alpha+\alpha_s)(y'-y'')+(\beta+\beta_s)(z'-z'')] d\alpha d\beta} dy'dz'dy''dz'' \]

The expression in Eq.(2.10) for the image plane intensity written in terms of aperture plane integration will be used to simplify the expression in Eq.(2.2) for angle of arrival.

The expression for the angle of arrival \( \alpha_0(t) \) defined by Eq.(2.2) is now going to be simplified. We first evaluate the denominator of Eq.(2.2)

\[ \iiint I(\alpha,\beta,t)d\alpha d\beta = \frac{k^4}{16\pi^2 L^2 d^2} \iiint E_0'(y',z',t) E_0'^*(y'',z'',t) \]

\[ -jk[(\alpha+\alpha_s)(y'-y'')+(\beta+\beta_s)(z'-z'')] \]

\[ \left[ \int e^{-jk[(\alpha+\alpha_s)(y'-y'')+(\beta+\beta_s)(z'-z'')] d\alpha} dy'dz'dy''dz'' \right] \]

\[ = \frac{k^2}{4\pi^2 L^2 d^2} \iint |E_0'(y',z',t)|^2 dy'dz' \] (2.11)

where we have used the following relationships.

\[ \int e^{-jk(\alpha+\alpha_s)(y'-y'')} d\alpha = 2\pi \delta[k(y'-y'')] \] (2.12a)

\[ \delta[k(y'-y'')] = \frac{1}{k} \delta(y'-y'') \] (2.12b)

\[ \int f(x,y) \delta(x-x_0) \delta(y-y_0) dx dy = f(x_0, y_0) \] (2.12c)
The integration procedure was to perform the image plane, i.e., $\alpha$ and $\beta$ integrations first using Eq.(2.12a) and then the aperture plane, i.e., $y''$ and $z''$ integration using Eq.(2.12b) and Eq.(2.12c).

We now turn to the evaluation of the numerator of Eq.(2.2)

$$\iiint aI(\alpha, \beta, t)d\alpha d\beta = \frac{k^4}{16\pi^4 L^2 d_i^2} \iiint E_0'(y', z', t) \left( -jk(\alpha + \alpha_s)(y' - y'') \right) \frac{\partial E_0'^*(y', z', t)}{\partial y'} dy' dz'$$

$$- \frac{k^2 \alpha_s}{4\pi^2 L^2 d_i^2} \iint |E_0'(y', z', t)|^2 dy' dz'$$

where we have used the following relationships

$$\int_{-\infty}^{\infty} \alpha e^{-jk(\alpha + \alpha_s)(y' - y'')} d\alpha = 2\pi \delta'[k(y' - y'')] - 2\pi \alpha_s \delta[k(y' - y'')]$$

$$= \frac{2\pi j}{k^2} \frac{\partial}{\partial y'} \delta(y' - y'') - \frac{2\pi \alpha_s}{k} \delta(y' - y'')$$

$$\delta'[k(y' - y'')] = \frac{1}{k^2} \delta'(y' - y'')$$

$$\int f(x) \delta'(x - \chi_0) dx = -f'(%\chi_0)$$

Substituting Eq.(2.11) and Eq.(2.13) back into Eq.(2.2), we obtain

$$\alpha_0(t) = \frac{1}{j k} \frac{\int E_0'(y', z', t) \frac{\partial E_0'^*(y', z', t)}{\partial y'} dy' dz'}{\iint |E_0'(y', z', t)|^2 dy' dz'} - \alpha_s$$
To reduce to the standard working expression, we integrate by parts. The expression for \( \alpha_0(t) \) can then be written as

\[
\alpha_0(t) = -\frac{1}{jk} \iint \frac{\partial E_0(y',z',t)}{\partial y'} \frac{E_0^*(y',z',t)}{\sum |E_0(y',z',t)|^2} \, dy' \, dz' - \alpha_s \quad (2.16)
\]

Taking the half sum of Eq.(2.15) and Eq.(2.16) we obtain

\[
\alpha_0(t) = \frac{1}{k} \iint \frac{1}{2j} \left[ E_0(y',z',t) \frac{\partial E_0^*(y',z',t)}{\partial y'} - \frac{\partial E_0^*(y',z',t)}{\partial y'} \right] dy' \, dz' - \alpha_s
\]

\[
= \frac{1}{k} \iint I_M \frac{\partial E_0^* (y,z,t)}{\partial y} \, dy dz - \alpha_s \quad (2.17)
\]

where \( I_M \) stands for imaginary part.

Equation (2.17) is the desired expression for the centroid angle of arrival now written in terms of integration of field \( E_0 \) over the plane immediately before the lens.

We now give a simple example of a computation of \( \alpha_0(t) \) using the expression given in Eq.(2.17). Consider a point source located at \((0,y_s,0)\). Assuming no atmospheric turbulence, \( E_0' = 1 \), \( \alpha_0(t) \) in Eq.(2.17) then becomes

\[
\alpha_0(t) = -\alpha_s = \frac{-y_s}{L} \quad (2.18)
\]

The angle \( y_s/L \) gives precisely the direction of the source.
Equation (2.17) will be made useful by expressions used for wave propagation in the turbulent atmosphere. There, the formulas for optical fields are usually written in terms of log amplitude and phase. The reason for interest in log amplitude is because the experiment has shown that amplitude fluctuations are log-normally distributed and, hence, the log amplitude is normally distributed. All the statistical quantities can be derived from log amplitude and phase.

Let the field received without turbulence be given by

$$E_0 = A_0 e^{jS_0} e^{-j\omega_0 t}$$

(2.19)

where $A_0$ is the amplitude and $S_0$ is the phase. If there is turbulence, the actual received field can be written as

$$E_0'' = A_0' e^{jS_0'} e^{-j\omega_0 t}$$

(2.20)

where $A_0'$ and $S_0'$ are the perturbed amplitude and phase respectively.

By using Eq.(2.19), Eq.(2.20) can also be written in the following form:

$$E_0'' = A_0 e^{jS_0} e^{\ln\left(\frac{A_0'}{A_0}\right)} e^{j(S_0' - S_0)} e^{-j\omega_0 t}$$

$$= E_0 e^{\ln\left(\frac{A_0'}{A_0}\right)} e^{j(S_0' - S_0)}$$

(2.21)

The log amplitude fluctuation $\chi$ and phase fluctuation $S$ are then defined as

$$\chi = \ln\left(\frac{A_0'}{A_0}\right)$$

(2.22a)
\[ S_0 = S_0' - S_0 \]  \hspace{1cm} (2.22b)

Using Eq.(2.22a) and Eq.(2.22b), the perturbed field \( E_0' \) can then be written as

\[ E_0'' = E_0 e^{\chi + jS_0} = E_0 E_0' \]  \hspace{1cm} (2.22c)

Where \( E_0' \) is given by

\[ E_0' = \exp(\chi + jS_0) \]  \hspace{1cm} (2.22d)

The symbols \( \chi \) and \( S_0 \) are frequently used in later discussions.

Inserting \( E_0' \) in Eq.(2.22d) into Eq.(2.17), we have

\[
\alpha_0(t) = \frac{1}{k} \left( \iint_M e^{\chi(y,z,t)} [e^{jS(y,z,t)} e^{\chi(y,z,t)} - jS(y,z,t) \left[ \frac{\partial \chi(y,z,t)}{\partial y} - j \frac{\partial S(y,z,t)}{\partial y} \right]] \, dy \, dz \right) - \alpha_s
\]

\[
- \int e^{2\chi(y,z,t)} \, dy \, dz
\]

\[
= \frac{1}{k} \left( \iint_M [e^{2\chi(y,z,t)} \left[ \frac{\partial \chi(y,z,t)}{\partial y} - j \frac{\partial S(y,z,t)}{\partial y} \right]] \, dy \, dz \right) - \alpha_s
\]

\[
= \frac{1}{k} \left( \iint [e^{2\chi(y,z,t)} \frac{\partial S(y,z,t)}{\partial y}] \, dy \, dz \right) - \alpha_s
\]  \hspace{1cm} (2.23)

Some simplification can be obtained if we neglect log amplitude \( \chi \). This will be justified in Appendix A. In that appendix, the log amplitude \( \chi \) is shown only to give a second order correction to \( \alpha_0(t) \). Neglecting \( \chi \), the expression for \( \alpha_0(t) \) in Eq.(2.23) can be written as

\[
\alpha_0(t) = \frac{1}{kA} \sum \frac{\partial S(y,z,t)}{\partial y} \, dy \, dz - \alpha_s
\]  \hspace{1cm} (2.24)
where $A$ is the area of the aperture. The derivative of the phase can be explained as relating to the normal of the wave front. With the nonuniformity of the wavefront distortion, the integration over the aperture gives the average value of the angle of arrival. However, the phase front normal is a variable over the aperture. That is different from the small aperture definition in which the phase front normal is considered as constant.

The centroid angle of arrival covariance function is defined by

$$R_{\alpha}(t_1,t_2) = \langle(\alpha_0(t_1) - \langle\alpha_0(t_1)\rangle)(\alpha_0(t_2) - \langle\alpha_0(t_2)\rangle)\rangle$$  \hspace{1cm} (2.25)

where $\langle\alpha_0(t)\rangle$ is the ensemble average.

For the cases to be considered in this dissertation, the phase $S$ in Eq.(2.24) can be written in general as the sum of the deterministic phase $S_d$ and the random phase $S_r$ with zero mean. It will be shown in section 3 that the deterministic part comes from a Doppler shift due to moving source. The angle of arrival $\alpha_0(t)$ in Eq.(2.24) then becomes

$$\alpha_0(t) = \frac{1}{kA} \int\int_{\Sigma} \frac{\partial S_r(y,z,t)}{\partial y} dydz + \frac{1}{kA} \int\int_{\Sigma} \frac{\partial S_d(y,z,t)}{\partial y} dydz - \alpha_s$$  \hspace{1cm} (2.26a)

The expected value of $\alpha_0(t)$ is then given by the expression

$$\langle\alpha_0(t)\rangle = \frac{1}{kA} \int\int_{\Sigma} \frac{\partial S_d(y,z,t)}{\partial y} dydz - \alpha_s$$  \hspace{1cm} (2.26b)

Substituting $\alpha_0(t)$ from Eq.(2.26a) and $\langle\alpha_0(t)\rangle$ from Eq.(2.26b) into Eq.(2.25), cancelling terms and combining the integrals gives for $R_{\alpha}(t_1,t_2)$.

$$R_{\alpha}(t_1,t_2) = \langle \frac{1}{k^2A} \int\int\int\int \frac{\partial S_r(y_1,z_1,t_1)}{\partial y_1} \frac{\partial S_r(y_2,z_2,t_2)}{\partial y_2} dy_1dz_1dy_2dz_2 \rangle$$  \hspace{1cm} (2.27a)
The expected value sign shown in Eq. (2.27a) can be moved inside the integral and the expression for \( R_\alpha(t_1, t_2) \) then becomes

\[
R_\alpha(t_1, t_2) = \frac{1}{k^2\pi^2} \iiint \frac{\partial^2}{\partial y_1 \partial y_2} \langle s_r(y_1, z_1, t_1) s_r(y_2, z_2, t_2) \rangle \, dy_1 dz_1 dy_2 dz_2
\]

\[
= \frac{1}{k^2\pi^2} \iiint \frac{\partial^2}{\partial y_1 \partial y_2} B_s(y_1, z_1, y_2, z_2, t_1, t_2) \, dy_1 dy_2 dz_1 dz_2
\]

(2.27b)

where the expression \( B_s(y_1, z_1, y_2, z_2, t_1, t_2) = \langle s_r(y_1, z_1, t_1) s_r(y_2, z_2, t_2) \rangle \) is defined as the phase correlation function.

\( R_\alpha(t_1, t_2) \) can also be written in terms of the sum variable \( \delta \) and difference variable \( \tau \) as \( R_\alpha(\tau, \delta) \). The \( \tau \) and \( \delta \) are defined below.

\[
\tau = t_1 - t_2
\]

(2.28a)

\[
\delta = \frac{1}{2}(t_1 + t_2)
\]

(2.28b)

For a stationary process, \( R_\alpha(\tau, \delta) \) depends only on \( \tau \) and Eq. (2.27b) then reduces to the following expression.

\[
R_\alpha(\tau) = \frac{1}{k^2} \int \int \int \frac{\partial^2 B_s(\overline{r}_1, \overline{r}_2, \tau)}{\partial y_1 \partial y_2} \, d\overline{r}_1 d\overline{r}_2
\]

(2.29)

where

\[
\overline{r}_1 = (y_1, z_1)
\]

\[
\overline{r}_2 = (y_2, z_2)
\]

The power spectrum defined by the Fourier transform of \( R_\alpha(\tau) \) is then given by

\[
W_\alpha(w, \delta) = \int R_\alpha(\tau, \delta) e^{-j\omega \tau} \, d\tau
\]

(2.30)

\( \alpha_0(t), R_\alpha(\tau, \delta), \) and \( W_\alpha(w, \delta) \) are the general formulation of centroid angle of arrival, centroid angle of arrival covariance function and the
centroid angle of arrival power spectrum. Equation (2.29) and Eq.(2.30) are the final results of this section, expressions for the angle of arrival temporal power spectrum written in terms of phase correlation function, $B_s$, of an atmospherically degraded wave.

In order to proceed further, we must calculate the expression for $B_s$. The next sections are devoted to that task.

3 DERIVATION OF RANDOM PHASE FOR ATMOSPHERICALLY PERTURBED WAVE FROM A MOVING SOURCE

In section 3, we derive the expression for the random phase of an atmospherically degraded wave from a moving source. We start with the wave equation, postulate an expression for the field from the moving source, incorporate random atmospheric effects, and finally determine the random phase.

3.a The Time Dependent Wave Equation For a Moving Source

The time dependent wave equation has the following form:

$$\nabla^2 \vec{E} - \mu \frac{\partial^2 \vec{E}}{\partial t^2} = 0$$  \hspace{1cm} (2.31)

where $\vec{E}$ in general is a vector. There is no loss of generality if we consider $\vec{E}$ in one direction only. We thus drop the vector notation on $\vec{E}$ by considering the following scalar equation for a linearly polarized wave.

$$\nabla^2 E - \mu \frac{\partial^2 E}{\partial t^2} = 0$$  \hspace{1cm} (2.32)
It will be convenient to decompose $\varepsilon$ as

$$\varepsilon = \varepsilon_0(1 + \varepsilon_1) \quad (2.33a)$$

where $\varepsilon_1$ is a small random quantity. This is because $\varepsilon_1$ has a magnitude much less than 1. Typical values of $\varepsilon_1$ range from $10^{-6} \sim 10^{-5}$ [52]. The refractive index fluctuation $n_1$ is defined by the equation $1 + n_1 = \sqrt{1 + \varepsilon_1}$. For very small $\varepsilon_1$, $1 + n_1 \approx 1 + (1/2)\varepsilon_1$, $\varepsilon_1$ is thus approximately equal to $2n_1$. In terms of $n_1$, $\varepsilon$ is Eq.(2.33a) is given by the following expression

$$\varepsilon = \varepsilon_0(1 + 2n_1) \quad (2.33b)$$

For the time dependent field, we write $E$ as

$$E(\vec{r},t) = \psi(\vec{r},t)e^{-j\omega_0t} \quad (2.34)$$

where $\psi(\vec{r},t)$ is the complex wave amplitude. Substituting $E(\vec{r},t)$ from Eq.(2.34) into Eq.(2.32), we obtain

$$\nabla^2 \psi = \mu \varepsilon_0 \frac{\partial^2 \psi}{\partial t^2} - 2j\omega_0 \frac{\partial \psi}{\partial t} - \omega_0^2 \psi \quad (2.35)$$

where we have used the following relations

$$\frac{\partial E}{\partial t} = \frac{\partial \psi}{\partial t}e^{-j\omega_0t} - j\omega_0 \psi e^{-j\omega_0t} \quad (2.36a)$$

$$\frac{\partial^2 E}{\partial t^2} = \frac{\partial^2 \psi}{\partial t^2}e^{-j\omega_0t} - 2j\omega_0 \frac{\partial \psi}{\partial t}e^{-j\omega_0t} - \omega_0^2 \psi e^{-j\omega_0t} \quad (2.36b)$$

Substituting $\varepsilon$ from Eq.(2.33b) into Eq.(2.35) and rearranging, we obtain

$$\nabla^2 \psi = \mu \varepsilon_0 \frac{\partial^2 \psi}{\partial t^2} - \mu \varepsilon_0(2j\omega_0 \frac{\partial \psi}{\partial t} - \omega_0^2 \psi

+ 2\mu \varepsilon_0 \frac{\partial^2 \psi}{\partial t^2} - 2\mu \varepsilon_0(2j\omega_0 \frac{\partial \psi}{\partial t} - \omega_0^2 \psi

= 2\mu \varepsilon_0 \frac{\partial^2 \psi}{\partial t^2} - 2\mu \varepsilon_0(2j\omega_0 \frac{\partial \psi}{\partial t} - \omega_0^2 \psi

(2.37)$$

If the source is stationary at origin, the spherical wave from the source is given by $\psi = \exp[jk|\vec{r}|]/|\vec{r}|$. If the source is at a fixed distance $\vec{r}_s$ from the origin, this spherical wave becomes $\psi_0 = \exp[jk|\vec{r}-\vec{r}_s|]/|\vec{r}-\vec{r}_s|$. If the source moves with velocity $\vec{v}$, we write
\[ \vec{r}_s = \vec{v}t \text{ and } \psi \text{ becomes} \]

\[ \psi_0 = \frac{e^{ik|\vec{r} - \vec{v}t|}}{|\vec{r} - \vec{v}t|} \quad (2.38) \]

where \( k^2 = \omega_0^2 \mu \varepsilon_0 \). However, Eq.(2.38) is not the exact solution to the time dependent wave equation in Eq.(2.37). As will be shown next, the error induced is only of the order of \( \left( \frac{V}{c} \right)^2 \) smallness.

Let \( I_0, I_1, I_2, I_3, I_4, \) and \( I_5 \) represent the third, the second, the first, the fourth, the fifth, and the sixth term on the right hand side of Eq.(2.37). Substituting \( \psi \) from Eq.(2.38) into Eq.(2.37) and differentiating, we obtain roughly

\[ I_0 = -\mu \varepsilon_0 \omega_0^2 \psi \quad (2.39a) \]

\[ I_1 = -\mu \varepsilon_0 (2 j \omega_0) \frac{\partial \psi}{\partial t} \]
\[ = -\mu \varepsilon_0 (2 \omega_0 k |\vec{v}| \psi) \quad (2.39b) \]

\[ I_2 = -\mu \varepsilon_0 \frac{\partial^2 \psi}{\partial t^2} \]
\[ = +\mu \varepsilon_0 k^2 u^2 \psi \quad (2.39c) \]

\[ I_3 = 2 \mu \varepsilon_0 n_1 \frac{\partial^2 \psi}{\partial t^2} = 2n_1 I_2 \quad (2.39d) \]

\[ I_4 = -2 \mu \varepsilon_0 n_1 (2 j \omega_0) \frac{\partial \psi}{\partial t} = 2n_1 I_1 \quad (2.39e) \]

\[ I_5 = -2 \mu \varepsilon_0 n_1 \omega_0^2 \psi = 2n_1 I_0 \quad (2.39f) \]

In Eq.(2.39), \( I_2, I_3, \) and \( I_4 \) are all small compared with \( I_0, I_1, \) and \( I_5 \). This may be demonstrated by forming the following ratio:
Because \( \frac{V}{c} \ll 1 \) and \( n_1 \) is of order 10^{-6}, we see that \( I_0 \) is the most important and hence the zero order term, \( I_1 \) and \( I_5 \) are the next important and hence the first order terms, \( I_2 \) and \( I_4 \) are the second order terms and \( I_3 \) is the third order term. Equation (2.36) can be regrouped to give the following equation.

\[
\nabla^2 \psi + k^2 \psi = -k^2 \left( 2n_1 \psi + \frac{2i}{w_0} \frac{\partial \psi}{\partial t} \right) + k^2 \left( \frac{1}{w_0^2} \frac{\partial^2 \psi}{\partial t^2} - \frac{4in_1}{w_0} \frac{\partial \psi}{\partial t} \right) \\
+ k^2 \left( \frac{2n_1}{w_0^2} \frac{\partial^2 \psi}{\partial t^2} \right)
\]

(2.41)

In the next section, we are going to solve Eq.(2.41) using the method of smooth perturbations.
3.b Derivation of Complex Log Amplitude

It is easiest to solve Eq.(2.41) through successive approximations. We first rewrite Eq.(2.41) in terms of the complex phase \( \phi \) which is related to \( \psi \) by

\[
\phi = 2\pi \psi
\]  

(2.42)

Inserting \( \phi \) from Eq.(2.42) into Eq.(2.41) and using simple differentiations then gives

\[
\nabla^2 \phi + (\nabla \phi \cdot \nabla \phi) + k^2 = -k^2 \left( 2n_1 + \frac{\partial \phi}{\partial t} \right) + k^2 \left\{ \frac{1}{w_0^2} \left[ \frac{\partial^2 \phi}{\partial t^2} + (\frac{\partial \phi}{\partial t})^2 \right] - \frac{2in_1}{w_0} \frac{\partial \phi}{\partial t} \right\} 
\]

\[
+ k^2 \left( \frac{2n_1}{w_0^2} \left[ \frac{\partial^2 \phi}{\partial t^2} + (\frac{\partial \phi}{\partial t})^2 \right] \right)
\]  

(2.43)

where we have used the following relations

\[
\nabla \psi = (\nabla \phi)e^\phi
\]  

(2.44a)

\[
\nabla^2 \psi = (\nabla^2 \phi + \nabla \phi \cdot \nabla \phi)e^\phi
\]  

(2.44b)

\[
\frac{\partial \psi}{\partial t} = e^\phi \frac{\partial \phi}{\partial t}
\]  

(2.44c)

\[
\frac{\partial^2 \psi}{\partial t^2} = e^\phi \left[ \frac{\partial^2 \phi}{\partial t^2} + (\frac{\partial \phi}{\partial t})^2 \right]
\]  

(2.44d)

Introducing the small parameter \( \mu \), we may make a series expansion on \( \phi \) to give

\[
\phi = \phi_0 + \mu \phi_1 + \mu^2 \phi_2 + \ldots
\]  

(2.45)

We choose \( \phi_0 = 2\pi \psi_0 = jk |\vec{r} - \vec{v}t| - 2\pi |\vec{r} - \vec{v}t| \). With \( v = 300 \) m/sec for an aircraft velocity or \( v = 6 \times 10^3 \) m/sec for a satellite velocity, Eq.(2.40) thus shows that \( 2n_1 + \frac{\partial \psi}{\partial t} \phi \) ranges from \( 10^{-6} \) to \( 2 \times 10^{-5} \), \( \frac{1}{w_0^2} \left[ \frac{\partial^2 \phi}{\partial t^2} + (\frac{\partial \phi}{\partial t})^2 \right] \)
ranges from $10^{-12}$ to $4 \times 10^{-10}$ and $\frac{2n_1}{w_0^2} \frac{\partial^2 \phi}{\partial t^2} + (\frac{\partial \phi}{\partial t})^2$ ranges from $10^{-18}$ to $8 \times 10^{-15}$. So the terms in successive angular brackets of Eq.(2.44) are smaller by a factor of at least $4 \times 10^{-4}$. With the choice of $u$ at most $4 \times 10^{-4}$, we write $2n_1 + \frac{\partial j}{w_0} \frac{\partial \phi}{\partial t}$ as $u (2n_1 + \frac{\partial j}{w_0} \frac{\partial \phi}{\partial t}), \frac{1}{w_0} \frac{\partial^2 \phi}{\partial t^2} + (\frac{\partial \phi}{\partial t})^2$ as $-\frac{4jn_1}{w_0} \frac{\partial \phi}{\partial t}$ as $u^2 \left\{ \frac{1}{w_0} \left[ \frac{\partial^2 \phi}{\partial t^2} + (\frac{\partial \phi}{\partial t})^2 \right] - \frac{4jn_1}{w_0} \frac{\partial \phi}{\partial t} \right\}$ and $\frac{2n_1}{w_0^2} \frac{\partial^2 \phi}{\partial t^2} + (\frac{\partial \phi}{\partial t})^2$ as $\frac{2n_1}{w_0^2} \left[ \frac{\partial^2 \phi}{\partial t^2} + (\frac{\partial \phi}{\partial t})^2 \right]$. Substituting the expression for $\Phi$ from Eq.(2.45) into Eq.(2.43), we obtain

$$\nabla^2 (\phi_0 + u \phi_1 + u^2 \phi_2 + \ldots)$$

$$+ \nabla (\phi_0 + u \phi_1 + \ldots)$$

$$\cdot \nabla (\phi_0 + u \phi_1 + \ldots)$$

$$+ k^2 + k^2 u \epsilon_1 + k^2 u \frac{2j}{w_0} \frac{\partial \phi}{\partial t} [\phi_0 + u \phi_1 + \ldots]$$

$$- \frac{k^2 u^2}{w_0^2} \{ \frac{\partial^2 \phi}{\partial t^2} (\phi_0 + u \phi_1 + \ldots) + \left[ \frac{\partial}{\partial t} (\phi_0 + u \phi_1 + \ldots) \right]^2 \}$$

$$+ \frac{4jk^2 u^2 n_1}{w_0} \frac{\partial}{\partial t} (\phi_0 + u \phi_1 + \ldots)$$

$$- \frac{2k^2 n_1}{w_0^2} u^3 \{ \frac{\partial^2 \phi}{\partial t^2} (\phi_0 + u \phi_1 + \ldots) +$$

$$\left[ \frac{\partial}{\partial t} (\phi_0 + u \phi_1 + \ldots) \right]^2 \} = 0 \quad (2.46)$$

Equating both sides of Eq.(2.46) those terms with the same power of $u$, we obtain

$$\nabla^2 \phi_0 + (\nabla \phi_0)^2 + k^2 = 0 \quad (2.47)$$
\[ \nabla^2 \phi_1 + 2 \nabla \phi_0 \cdot \nabla \phi_1 = -k^2 \left( 2n_1 + \frac{2j}{\omega_0} \frac{\partial \phi_0}{\partial t} \right) \] \hspace{1cm} (2.48)

Equation (2.48) for \( \phi_1 \), the first order correction for \( \phi_0 \), is the one that will be used in the derivations to come. It is interesting to note that to first order the source is moving sufficiently slow so as to be regarded as stationary. Using Eq.(2.44b) and Eq.(2.42), we obtain the following relationship.

\[ \nabla^2 \psi_0 + k^2 \psi_0 = \left( \nabla^2 \phi_0 + \nabla \phi_0 \cdot \nabla \phi_0 \right) \psi_0 + k^2 e^{\phi_0} \]
\[ = \left( \nabla^2 \phi_0 + \nabla \phi_0 \cdot \nabla \phi_0 + k^2 \right) e^{\phi_0} \] \hspace{1cm} (2.49a)

Thus, Eq.(2.47) is the same as the zero order time independent wave equation

\[ \nabla^2 \psi_0 + k^2 \psi_0 = 0 \] \hspace{1cm} (2.49b)

We solve Eq.(2.48) by means of the substitution

\[ \phi_1 = e^{-\phi_0} \sigma \] \hspace{1cm} (2.50a)

Inserting the expression for \( \phi_1 \) in Eq.(2.50a) into Eq.(2.48), we obtain

\[ \nabla^2 \sigma + k^2 \sigma = -e^{\phi_0} f_1 \] \hspace{1cm} (2.50b)

where we have used

\[ \nabla \phi_1 = e^{-\phi_0} \nabla \sigma - \sigma \nabla \phi_0 e^{-\phi_0} \] \hspace{1cm} (2.51a)

\[ \nabla^2 \phi_1 = \nabla^2 \sigma e^{-\phi_0} - 2(\nabla \sigma \cdot \nabla \phi_0) e^{-\phi_0} \]
\[ - \sigma \nabla^2 \phi_0 e^{-\phi_0} + (\nabla \phi_0 \cdot \nabla \phi_0) \sigma e^{-\phi_0} \] \hspace{1cm} (2.51b)

and

\[ f_1 = 2k^2 \left( n_1 + \frac{2j}{\omega_0} \frac{\partial \phi_0}{\partial t} \right) \] \hspace{1cm} (2.51c)

The solution to Eq.(2.50b) is the well known Green's function solution given by [88]

\[ \sigma = \sigma_{1r} + \sigma_{1d} \]
where

\[ W_{1r} = \frac{k^2}{2\pi} \iint \frac{\phi_0(\vec{r}', t)}{\vec{r} - \vec{r}'} d^3 \vec{r}' \]  

(2.52)

\[ W_{1d} = \frac{jk^2}{2\pi \omega_0} \iint \frac{\phi_0(\vec{r}', t)}{\vec{r} - \vec{r}'} \frac{\partial \phi_0(\vec{r}', t)}{\partial t} d^3 \vec{r}' \]  

(2.53)

By using Eq. (2.50a), we then obtain

\[ \phi_1 = \phi_{1r} + \phi_{1d} \]  

(2.54)

where

\[ \phi_{1r}(\vec{r}, t) = \frac{k^2}{2\pi} \iint n_1(\vec{r}', t) e^{jk|\vec{r} - \vec{r}'|} \frac{\psi_0(\vec{r}', t)}{|\vec{r} - \vec{r}'|} \frac{\partial \psi_0(\vec{r}', t)}{\partial t} d^3 \vec{r}' \]  

(2.55)

\[ \phi_{1d}(\vec{r}, t) = \frac{jk^2}{2\pi \omega_0} \iint \frac{\psi_0(\vec{r}', t)}{|\vec{r} - \vec{r}'|} \frac{\partial \psi_0(\vec{r}', t)}{\partial t} \frac{1}{\psi_0(\vec{r}, t)} d^3 \vec{r}' \]  

(2.56)

The complete field in terms of \( \phi_1 \) is written to first order in the following form:

\[ E(\vec{r}, t) = \psi_0(\vec{r}, t) e^{\Phi_1(\vec{r}, t)} e^{-j\omega_0 t} \]  

(2.57)

\( \Phi_{1r}(\vec{r}, t) \) is a random quantity because of the appearance of \( n_1(\vec{r}', t) \) while \( \Phi_{1d} \) is a deterministic quantity. In the next section we are going to evaluate both terms, assuming a spherical wave propagation from a moving transmitter to a stationary receiver.
3.c Evaluation of the Random Quantity

We now proceed to evaluate the integral for the random quantity, $\Phi_{1R}$, in Eq.(2.55). The coordinate system is fixed in space at the starting point of the source motion as shown in Figure 3. The x axis is defined to be along the initial propagation direction while the yz plane is perpendicular to the x axis. We take the source to be moving with velocity $\mathbf{v}_t = v_{2t} \hat{y} + v_{3t} \hat{z}$. Then at any time $t$, the position of the source relative to source reference origin is $\mathbf{r}_t = y_t \hat{y} + z_t \hat{z} = v_{2t} t \hat{y} + v_{3t} t \hat{z}$. If a spherical wave is emitted from the source at time $t$ and collected by a receiver located at $\mathbf{r}_r = (L, y, z)$, then the zero order solution at $\mathbf{r}_r$ given by Eq.(2.38) is

$$
\psi_0(\mathbf{r}_r, t) = \frac{e^{jk|\mathbf{r}_r - \mathbf{r}_t|}}{|\mathbf{r}_r - \mathbf{r}_t|} \quad (2.58)
$$

Utilizing the small angle approximation, we have

$$
|\mathbf{r}_r - \mathbf{r}_t| \approx L + \frac{(y-y_t)^2 + (z-z_t)^2}{2L} \quad (2.59a)
$$

where

$$
[(y-y_t)^2 + (z-z_t)^2]/L^2 \ll |
$$

Similarly,

$$
|\mathbf{r}_r - \mathbf{r}'| \approx (L-x') + \frac{(y-y')^2 + (z-z')^2}{2(L-x')} \quad (2.59b)
$$
Substituting the expressions from Eq.(2.59a) and Eq.(2.59b) into Eq.(2.55), we obtain

\[
\Phi_{1r} = \frac{k^2}{2\pi} \iint n_1(\vec{r}',t) \frac{\psi_0(\vec{r}',t)}{|\vec{r}' - \vec{r}|} \psi_0(\vec{r},t) d^3\vec{r}'
\]

\[
= \frac{k^2}{2\pi} \exp\left\{ \frac{-jk[(y-y_t)^2+(z-z_t)^2]}{2L} \right\}
\]

\[
\iint \frac{n_1(x',y',z')}{x'(L-x')} \exp\left\{ \frac{jk[(y'-y_t)^2+(z'-z_t)^2]}{2x'} \right\} \exp\left\{ \frac{jk[(y'-y_t)^2+(z'-z_t)^2]}{2(L-x')} \right\} dx'dy'dz'
\]

(2.60)

where the integration of \(y'\) and \(z'\) is from \(-\infty\) to \(\infty\) and the integration of \(x'\) is from 0 to the propagation distance \(L\).

The exponent in the integrand of Eq.(2.60) can be simplified. It is

\[
J = \frac{jk}{2x'}[(y'-y_t)^2 + (z'-z_t)^2] + \frac{jk}{2(L-x')}[(y-y_t')^2 + (z-z'_t)^2]
\]

(2.61)

Equation (2.61) can be rearranged by grouping terms containing \(y'^2\), \(z'^2\), \(y'\), and \(z'\) together to give

\[
J = \frac{jk}{2} \frac{L}{x'(L-x')} \{y'^2 - 2y'\frac{y_t L + x'(y-y_t)}{L} + J_1 + z'^2 - 2z' \frac{z_t L + x'(z-z_t)}{L} + J_2 \}
\]

(2.62)
where

\[ J_1 = \frac{y_t^2(L-x')}{L} + \frac{y'^2x'}{L} \]  
\[ J_2 = \frac{z_t^2(L-x')}{L} + \frac{z'^2x'}{L} \]  

One next completes the square in \( y' \) and \( z' \)

\[ J = \frac{ik}{2} \frac{L}{x'(L-x')} \left[ (y' - \frac{y_{tL+x'y'-x'y_t}}{L})^2 + J_3 + (z' - \frac{z_{tL+x'z'-x'z_t}}{L})^2 + J_4 \right] \]  

where

\[ J_3 = \left[ \frac{y_{tL+x'}(y-y_t)^2}{L} + \frac{y_{t}^2(L-x')}{L} \right] + \frac{y'^2x'}{L} \]  
\[ J_4 = \left[ \frac{z_{tL+x'}(z-z_t)^2}{L} + \frac{z_{t}^2(L-x')}{L} \right] + \frac{z'^2x'}{L} \]  

\( J_3 \) and \( J_4 \) can be simplified to give

\[ J_3 = \frac{x'(L-x')}{L^2}(y-y_t)^2 \]  
\[ J_4 = \frac{x'(L-x')}{L^2}(z-z_t)^2 \]  

Substituting \( J_3 \) and \( J_4 \) in Eq.(2.65a) and Eq.(2.65b) into Eq.(2.64a), we have

\[ J = \frac{ik}{2} \frac{L}{x'(L-x')} \left[ (y'+y_a)^2 + (z'+z_a)^2 \right] \]  
\[ + \frac{ik}{2} \left[ \frac{(y-y_t)^2+(z-z_t)^2}{L} \right] \]
where

\[ y_a = \frac{-y_t L - x'(y - y_t)}{L} \]  

\[ z_a = \frac{-z_t L - x'(z - z_t)}{L} \]  

Substituting \( J \) in Eq. (2.66) into Eq. (2.60), gives the following expression:

\[
\phi_{1r}(\vec{r}, t) = \frac{k^2 \Delta}{2\pi} \iint_{S'} \frac{n_1(x', y', z', t)}{x'(L-x')} \exp\left\{ \frac{ik}{2} \frac{L}{x'(L-x')} \right\} \left\{ (y'+y_a)^2 + (z'+z_a)^2 \right\} dx'dy'dz' 
\]  

If there is wind blowing when the wave propagates from the source to the receiver, Taylor's frozen turbulence hypothesis may be used. A detailed discussion on the validity of the Taylor hypothesis is given in Appendix C. In this situation, the random variable \( n_1(\vec{r}', t) \) can be written as

\[ n_1(\vec{r}', t) = n_1(\vec{r}' - \vec{r}_w(t), 0) \]  

where \( \vec{r}_w(t) = y_w \hat{y} + z_w \hat{z} = v_{w1} t \hat{y} + v_{w2} t \hat{z} \) is the distance the wind travels.

Substituting \( n_1 \) in Eq. (2.69) into Eq. (2.68) and defining new integration variables \( x'' = x', \ y'' = y' - y_w, \ z'' = z' - z_w, \) \( \phi_{1r} \) becomes

\[
\phi_{1r}(x, y, z, t) = \frac{k^2 \Delta}{2\pi} L \iint_{S'} \frac{n_1(x'', y'', z'')}{x''(L-x'')} \exp\left\{ \frac{ik}{2} \frac{L}{x''(L-x'')} \right\} \left\{ (y''+y_c)^2 + (z''+z_c)^2 \right\} dx''dy''dz'' 
\]
where

\[ y_c = y_a + y_w = y_w + \frac{-y_{L} L - (y-y_{L}) x''}{L} \]  \hspace{1cm} (2.71a)  

\[ z_c = z_a + z_w = z_w + \frac{-z_{L} L - (z-z_{L}) x''}{L} \]  \hspace{1cm} (2.71b)  

\((x'',y'',z'')\) is just the dummy integration variable in Eq.(2.70). It makes no difference if we change \((x'',y'',z'')\) back to \((x',y',z')\). Thus, we have

\[
\phi_1(x,y,z,t) = \frac{k^2}{2\pi} \int \int \int \frac{n_1(x',y',z')}{{x'}(L-{x'})} \exp \left\{ \frac{j}{2} \frac{L}{x'} (y'y_c + z'z_c)^2 \right\} dx' dy' dz' 
\]  \hspace{1cm} (2.72)  

where

\[ y_c = y_w + \frac{-y_{L} L - (y-y_{L}) x'}{L} \]  \hspace{1cm} (2.73a)  

\[ z_c = z_w + \frac{-z_{L} L - (z-z_{L})}{L} \]  \hspace{1cm} (2.73b)  

\(\phi_1\) in Eq.(2.72) can be simplified by writing \(n_1(x',y',z')\) in terms of its two dimensional Fourier transform \(\mu_1(x',K_y,K_z)\)

\[
n_1(x',y',z') = \int \mu_1(x',K_y,K_z) e^{jK_y y' + jK_z z'} dK_y dK_z \]  \hspace{1cm} (2.74)  

Substituting \(n_1\) in Eq.(2.74) into Eq.(2.72), we obtain

\[
\phi_1(x,y,z,t) = \frac{k^2}{2\pi} \int \int \int \frac{\mu_1(x',K_y,K_z)}{{x'}(L-{x'})} \exp \left\{ \frac{j}{2} \frac{L}{x'} [(y'y_c + z'z_c)^2] \right\} dx' dy' dz' + jK_y y' + jK_z z' \]  \hspace{1cm} (2.75)
The exponent inside the bracket of Eq. (2.75), calling it M, can be rewritten by grouping terms containing $y'^2$, $y'$, $z'^2$, $z'$ together to give

$$M = \frac{jk}{2} L_x \left[ y'^2 + 2y' \left( y_c + \frac{y}{kL_x} \right) + y_c^2 + z'^2 + 2z' \right. \left. \left( z_c + \frac{z}{kL_x} \right) + z_c^2 \right]$$

(2.76a)

where

$$L_x = \frac{L}{x'(L-x')}$$

(2.76b)

M in Eq. (2.76a) can be put in a square form as

$$M = \frac{jk}{2} L_x \left[ (y'+y_c^2 + \frac{y}{kL_x})^2 - 2y_c \frac{1}{L_x} - \frac{y^2}{k^2} \frac{1}{L_x^2} \right.$$

$$+ \left. (z'+z_c^2 + \frac{z}{kL_x})^2 - 2z_c \frac{1}{L_x} - \frac{z^2}{k^2} \frac{1}{L_x^2} \right]$$

(2.77)

Substituting M in Eq. (2.77) into Eq. (2.75) and integrating the $y'$ and $z'$ from $-\infty$ to $\infty$, we obtain

$$\Phi(x,y,z,t) = \int \int \mu_1(x',K_y,K_z) e^{jK_y y_c - jK_z z_c}$$

$$H(x',K_y,K_z) dx' dK_y dK_z$$

(2.78)

where

$$H = \frac{jk}{2} \exp \left\{ \frac{-j}{2k} \left[ K_y^2 + K_z^2 \right] \frac{x'(L-x')}{L} \right\}$$

(2.79)

and we have used the following Gaussian integral:

$$\frac{1}{\sqrt{2\pi\sigma}} \int_{-\infty}^{\infty} e^{-\frac{(y-\mu)^2}{2\sigma^2}} dy = 1$$

(2.80)

Next we split up $\Phi(x,y,z,t)$ to obtain the phase component. $\Phi(x,y,z,t)$ is the complex log amplitude; it can be written as the sum of the log amplitude $\chi$ and the random phase $S_r$. 
\[ \phi_{1r}(x,y,z,t) = \chi(x,y,z,t) + j\Sigma_r(x,y,z,t) \] (2.81)

\( \Sigma_r \) is the imaginary part of \( \phi_{1r}(x,y,z,t) \), and can be derived simply as follows:

\[
\Sigma_r = \frac{1}{2i} [\phi_{1r} - \phi_{1r}^*] \\
= \frac{1}{2i} \left[ \iiint \mu_1(x', K_y, K_z) e^{-jK_y y_c - jK_z z_c} H dx'dK_ydK_z \\
- \iiint \mu_1^*(x', K_y, K_z) e^{jK_y y_c + jK_z z_c} H^* dx'dK_ydK_z \right] \\
= \iiint \mu_1(x', K_y, K_z) e^{-jK_y y_c - jK_z z_c} \\
\left[ \frac{1}{2i}(H-H^*) \right] dx'dK_ydK_z \] (2.82)

where we have used \( \mu_1^*(x', -K_y, -K_z) = \mu_1(x', K_y, K_z) \), which is possible because \( \mu_1 \) is the Fourier transform of \( n_1 \), a real quantity. Substituting \( H \) in Eq.(2.79) into Eq.(2.82), we then obtain

\[
\Sigma_r(L,y,z) = k\iiint \mu_1(x', K_y, K_z) e^{-jK_y y_c - jK_z z_c} \\
\cos \frac{(K_y^2+K_z^2)x'(L-x')}{2KL} dx'dK_ydK_z \] (2.83)

where

\[
y_c = v_{2w} t - \frac{1}{L} [v_{2t} tL + yx' - v_{2t}tx'] \] (2.84a)

\[
z_c = v_{3w} t - \frac{1}{L} [v_{3t} tL + zx' - v_{3t}tx'] \] (2.84b)

Equation (2.83) is the final expression for the phase. This expression will be used in section 4 when we derive the phase correlation function. In the next section we are going to derive an expression for the deterministic quantity \( \Phi_{1d}(\vec{r}, t) \).
3.4 Evaluation of the Deterministic Quantity

The deterministic quantity $\Phi_{1d}(\vec{r},t)$ given by Eq. (2.56) will now be evaluated. The time derivative of $\psi_0(\vec{r}',t)$ in the integrand is given approximately by

$$\frac{\partial \psi_0(\vec{r}',t)}{\partial t} \approx -\frac{jk}{|\vec{r}'-\vec{r}_t|^2} \left( \vec{\nabla}_t \cdot (\vec{r}'-\vec{r}_t) \right)$$

(2.85)

where the effects of changes in the denominator have been neglected, since $k \gg |\vec{r}'-\vec{r}_t|^{-1}$. Substituting the expressions in Eq. (2.85) and the approximate forms in Eq. (2.59) into Eq. (2.56), we obtain

$$\Phi_{1d}(\vec{r},t) = \frac{k^3}{\omega_0 n} \exp \left\{ \frac{-jk}{2L} [(y-y_t)^2+(z-z_t)^2] \right\}$$

$$\int \int \int \frac{e^{\frac{jk}{2x'}[(y'-y_t)^2+(z'-z_t)^2]}}{(L-x')x'^2}$$

$$\{e^{\frac{jk}{2x'}[(y'-y_t)^2+(z'-z_t)^2]} \} \left[ \vec{\nabla}_t \cdot (\vec{r}'-\vec{r}_t) \right] d^3r'$$

(2.86)

Rearranging the exponent in the integrand of Eq. (2.86), we obtain the following expression:

$$\Phi_{1d}(\vec{r},t) = \frac{k^3}{\omega_0 n} \int_0^L \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{v_2(t)(y'-y_t)^2+v_3(t)(z'-z_t)}{(L-x')x'^2}$$

$$\exp \left\{ \frac{jkL}{2x'} \left[ (y' - \frac{y_t L + x'(y-y_t)^2}{L}) + (z' - \frac{z_t L + x'(z-z_t)^2}{L}) \right] \right\} d^3r'$$

(2.87)

where the integration of $y'$ and $z'$ are from $-\infty$ to $\infty$ and the integration of $x'$ is from 0 to $L$. 


The integrals in Eq.(2.87) can be performed quite easily by using the following two formulas:

\[ \int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \sqrt{2\pi} \sigma \]  
\[ \int_{-\infty}^{\infty} x e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \mu \sqrt{2\pi} \sigma \]  

For \( y' \) integration, \( x \) is set to \( y' \), \( \mu \) is set to \( [y'T + x'(y-y_T)]/L \) and \( \sigma \) is set to \( \sqrt{-x'(L-x')/jkL} \) in Eq.(2.88). The \( z' \) integration can be done similarly. Using Eq.(2.88), we obtain the following result

\[ \Phi_{1d}(\bar{r}_r, t) = \frac{jk^2}{w_0} [\bar{v}_t \cdot (\bar{r}_r - \bar{r}_t)] \]  

The deterministic phase \( S_d \) which is the imaginary part of \( \Phi_{1d} \) is then given by

\[ S_d = \text{Im} [\Phi_{1d}] = \frac{k^2}{w_0} [\bar{v}_t \cdot (\bar{r}_r - \bar{r}_t)] \]  

Equation (2.89a) shows that \( \Phi_{1d}(\bar{r}_r, t) \) is a purely imaginary quantity. Thus, the source motion will induce an extra deterministic phase shift only, but no amplitude change. This is understandable, since the orientation of the wavefront received by the detector aperture changes as the source moves. If the source remains stationary, i.e., \( \bar{v}_t = 0 \), then \( \Phi_{1d}(r, t) \) is zero as expected.

Another interesting question is the direction of motion under which there should be no extra deterministic phase tilt. Setting Eq.(2.89a) to zero, we obtain the following equation:

\[ \bar{v}_t \cdot (\bar{r}_r - \bar{r}_t) = 0 \]  

(2.90)
The physical meaning of Eq. (2.90) is shown in Figure 5. The vector \( \vec{r}_r - \vec{r}_t \) is directed from the source toward any point on the receiver aperture. Since the aperture size is much smaller than the propagation distance, the vector \( \vec{r}_r - \vec{r}_t \) is actually very nearly parallel to the vector \( \vec{r}_0 \) from the source toward the origin of the receiver aperture (to be considered parallel, the angle between \( \vec{r}_0 \) and \( \vec{r}_r - \vec{r}_t \) should be close to zero). Equation (2.90) thus says that the source velocity should be perpendicular to the vector \( \vec{r}_0 \) in order for the deterministic phase shift to be zero.

Qualitatively, this is the same as the classic Doppler shift. Appendix D reviews the classic Doppler shift from physical grounds and shows quantitatively that Eq. (2.89) is indeed equal to the Doppler shift. The approach used in Appendix D is to try to find the frequency drift after the source has moved a time \( T \) seconds corresponding to \( u_0 / 2\pi \). The frequency drift given by \( S_d / T \) is then equal to the classic Doppler shift.

Figure 5. The Geometry Used to Explain the Deterministic Phase Shift
Equation (2.89b) gives the desired expression for \( S_d \) which will be used later. This finishes the discussion on the derivation of \( S_d \), the deterministic part of the phase. In the next section, we are going to derive an expression for phase correlation function.

4 COMPUTATION OF PHASE CORRELATION FUNCTION

Equation (2.83) gives the expression for \( S_r(L,y,z) \) the random component of the phase at any point on the receiver aperture. The phase correlation function between any two points on the aperture at two different times will be derived now.

Using Eq.(2.83), the phase correlation function between point \((L,y_1,z_1)\) at time \(t_1\) and \((L,y_2,z_2)\) at time \(t_2\) is given by

\[
B_s(y_1,y_2,z_1,z_2,t_1,t_2) = \langle S_r(y_1,z_1,L,t_1)S_r(L,y_2,z_2,t_2) \rangle
\]

\[
= k^2 \iint < \mu_1(x',K_y,K_z)\mu_1(x'',K_y'',K_z'') e^{-jK_y y_c -jK_z z_c -jK_y'' y_c'' -jK_z'' z_c''} 
\]

\[
\cos\left[\frac{(K_y^2+K_z^2)x'(L-x')}{2kL}\right] \cos\left[\frac{(K_y''^2+K_z''^2)x''(L-x'')}{2kL}\right] dx'dx''dK_y dK_z dK_y'' dK_z'' \quad (2.91a)
\]

\[
= k^2 \iint < \mu_1(x',K_y,K_z)\mu_1(x'',-K_y,-K_z') e^{-jK_y y_c -jK_z z_c +jK_y'' y_c'' +jK_z'' z_c''} 
\]

\[
\cos\left[\frac{(K_y^2+K_z^2)x'(L-x')}{2kL}\right] \cos\left[\frac{(K_y''^2+K_z''^2)x''(L-x'')}{2kL}\right] dx'dx''dK_y dK_z dK_y'' dK_z'' \quad (2.91b)
\]

where \( y_c \) and \( z_c \) are defined in Eq.(2.84a) and Eq.(2.84b).
The average value given by the brackets inside the integral in Eq. (2.91) can be simplified using the following important relation valid for homogeneous refractive index fluctuations [89].

\[ \mu_1(x', K_y, K_z) \mu_1(x'', -K_y', -K_z') = \delta(K_y - K_y') \delta(K_z - K_z') F_n(K_y', K_z', x' - x'') \]  (2.92)

This relationship is also derived in Appendix B. Substituting the expression in Eq. (2.92) into Eq. (2.91b) and performing the \( K_y' \) and \( K_z' \) integrations, the phase correlation function \( B_s \) then becomes

\[ B_s(y_1, y_2, z_1, z_2, \tau_1, \tau_2) = k^2 \iint F_n(K_y, K_z, x' - x'') e^{-jk_y(y_C - y'_C)} - jK_z(z_C - z'_C) \cos\left(\frac{K_y^2 + K_z^2}{2kL} x'(L - x')\right) \cos\left(\frac{K_y^2 + K_z^2}{2kL} x'(L - x')\right) \right] dx'dx'dK_y dK_z \]  (2.93)

We now consider the \( x' \) and \( x'' \) integrations. The rectangular integration in the \( x', x'' \) domain can be transformed to a rhombus in the \( \eta, \xi \) domain with \( \eta \) and \( \xi \) defined below

\[ \eta = \frac{x' + x''}{2} \]  (2.94a)
\[ \xi = x' - x'' \]  (2.94b)

The rectangle in the \( x', x'' \) domain and the rhombus in the \( \eta, \xi \) domain are also shown in Figure 6 and Figure 7. The \( x', x'' \) integration given in Eq. (2.91) then becomes

\[ \int_0^L \int_0^L dx'dx'' = \int_{-L}^0 d\xi \int_{-\xi/2}^{\xi/2} d\eta + \int_0^L d\xi \int_{\xi/2}^{-\xi/2} d\eta \]  (2.95a)

where we have used the fact that the main contribution to the integration of \( F_n(K_y, K_z, \xi) \) with \( \xi \) is for \( K\xi < 1 \). But \( L \geq L_0 \geq \frac{1}{K} \), where \( L_0 \) is the outer scale and \( K = (K_y^2 + K_z^2)^{1/2} \), we thus have \( L \geq L_0 \geq \frac{1}{K} > \xi \). Thus, the \( \xi \) integration limit can be changed to infinity without changing its value, and the \( \xi \) in the \( \eta \) integration limit can be set to zero [90].
Figure 6. Rectangular Integration Region in $x', x''$ Domain

Figure 7. Rhombus Integration Region in $\eta, \xi$ Domain
Using the property $F_n$ is symmetric with respect to $\xi$ and changing $\xi$ to $-\xi$ in the $\xi$ integration, Eq. (2.95a) then becomes

$$\int_0^L \int_0^L dx' dx'' = 2 \int_0^L d\eta \int_0^\infty d\xi$$  \hspace{1cm} \text{(2.95b)}$$

With the use of Eq. (2.95b), $B_\delta$ given in Eq. (2.93) then becomes

$$B_\delta(y_1, y_2, z_1, z_2, t_1, t_2) = 2\pi k^2 \int \int \int \phi_n(K_y, K_z) e^{-jK_y y_c - jK_z z_c}$$

$$\cos^2 \frac{(K_y^2+K_z^2)\eta(L-\eta)}{2kL} \frac{d\eta}{dK_y} dK_z$$ \hspace{1cm} \text{(2.96)}$$

where $y_c$ and $z_c$ introduced in Eq. (2.73) have become

$$y_c = v_{3w}(t_1-t_2) + [(\frac{n}{L} - 1)v_{2t}](t_1-t_2) - \frac{n}{L} (y_1-y_2)$$ \hspace{1cm} \text{(2.97)}$$

$$z_c = v_{3w}(t_1-t_2) + [(\frac{n}{L} - 1)v_{3t}](t_1-t_2) - \frac{n}{L} (z_1-z_2)$$ \hspace{1cm} \text{(2.98)}$$

and we have used the following relation: \cite{90}

$$\int_0^\infty F_n(\xi, K_y, K_z) d\xi = \pi \phi_n(0, K_y, K_z)$$ \hspace{1cm} \text{(2.99)}$$

Equation (2.96) can be written in a more compact form by defining

$$\tilde{\rho} = (y_1-y_2, z_1-z_2)$$ \hspace{1cm} \text{(2.100a)}$$

$$\tau = (t_1-t_2)$$ \hspace{1cm} \text{(2.100b)}$$

$$\tilde{\kappa} = (K_y, K_z)$$ \hspace{1cm} \text{(2.100c)}$$

$$\tilde{\rho}_c = (y_c, z_c)$$ \hspace{1cm} \text{(2.100d)}$$

$$K^2 = K_y^2 + K_z^2$$ \hspace{1cm} \text{(2.100e)}$$

Substituting $y_c$ and $z_c$ given in Eq. (2.97) and Eq. (2.98) into Eq. (2.100d), $\tilde{\rho}_c$ becomes

$$\tilde{\rho}_c = \tilde{v}_{3w} \tau + [(\frac{n}{L} - 1) \tilde{v}_t] \tau - \frac{n}{L} \rho$$ \hspace{1cm} \text{(2.101)}$$
Using Eq.(2.100) and Eq.(2.101), the phase correlation function $B_s$ is rewritten as

$$ B_s(\bar{\rho}_c, t) = 2\pi k^2 \iiint \phi_n(K_y, K_z)e^{-jK\cdot\bar{\rho}_c} \cos^2 \left( \frac{K^2 \eta(L-n)}{2KL} \right) d\eta dK_y dK_z \quad (2.102) $$

Writing $B_s$ in terms of $\bar{\rho}_c$ will simplify the derivation of angle of arrival power spectrum later on.

This finishes the discussion on the spherical wave phase correlation function. The resultant expression, Eq.(2.102) will be used later. In the next section we are going to derive the centroid angle of arrival covariance function.

5 COMPUTATION OF ANGLE OF ARRIVAL COVARIANCE FUNCTION

In this section we derive an expression for the centroid angle of arrival covariance function. Starting with Eq.(2.25), we need to compute $<\alpha_0(t)>$. Substituting the expression for $S_d$ in Eq.(2.89b) into Eq.(2.24), and performing the differentiation and integration, we have

$$ <\alpha_0(t)> = \frac{-v_2t}{C} - \frac{-v_2t^2}{L} \quad (2.103) $$

Where we have used $\eta = v_2t$ in Eq.(2.8b).

Substituting the expression for $B_s(\bar{\rho}, \tau)$ from Eq.(2.101) and Eq.(2.102) into Eq.(2.27b), we obtain
\[ R_\alpha(t) = \frac{2\pi k^2}{k^2/\Lambda^2} \iiint \phi_n(K_y, K_z) \left[ \cos^2 \frac{n(L-n)K^2}{2kL} \right] \]

\[ = \iiint \frac{\partial^2}{\partial y_1 \partial y_2} e^{jk \cdot (\vec{r}_1 - \vec{r}_2) \frac{n}{L}} d\vec{r}_1 d\vec{r}_2 d\phi  \\
\[ = \iiint \frac{\partial^2}{\partial y_1 \partial y_2} e^{jk \cdot (\vec{r}_1 - \vec{r}_2) \frac{n}{L}} d\vec{r}_1 d\vec{r}_2  \\
\[ = (\frac{n}{L})^2 K_y^2 \left[ \int_{-a/2}^{a/2} e^{jy_1 \frac{nK}{L} y dy_1} \right] \left[ \int_{-a/2}^{a/2} e^{jy_2 \frac{nK}{L} y dy_2} \right] \]

Each individual integration in each bracket of Eq.(2.105b) can be performed easily using the following formula:

\[ \int_{-a/2}^{a/2} e^{jax} dx = \text{sinc}(\frac{ax}{2})  \\
\[ = 4a^2 \left[ \text{sinc}(\frac{2\pi a \frac{n}{L}}{2}) \right]^2 \sin^2(K_y \frac{a \frac{n}{L}}{2}) \]
Substituting the integral given in Eq.(2.106) into Eq.(2.104) then gives the following expression for $R_{\alpha}(\tau)$

$$
R_{\alpha}(\tau) = \frac{8\pi L a^2}{A^2} \iiint \Phi_n(K_y, K_z) \cos^2\left(\frac{L\sigma(1-\sigma)K_z^2}{2k}\right) \sin^2\left[\frac{a}{2} K_y \sigma\right] \frac{jK_y(1-\sigma) - \vec{v}_w \tau}{\sin\left(\frac{\theta}{2} K_z \sigma\right)} e^{-j\omega t} \, d\omega \, dK_y \, dK_z \, d\tau
$$

(2.107a)

where $\sigma$ is the normalized range variable

$$
\sigma = \eta/L
$$

(2.107b)

and $A = a^2$.

Equation (2.107a) is the desired expression for angle of arrival covariance function. In the next section, we derive the angle of arrival temporal power spectrum.

### 6 COMPUTATION OF ANGLE OF ARRIVAL TEMPORAL POWER SPECTRUM

In this section we derive an expression for the angle of arrival temporal power spectrum. Substituting $R_{\alpha}(\tau)$ in Eq.(2.107a) into the definition of power spectrum given in Eq.(2.30), we then have.

$$
W_{\alpha}(\omega) = \int_{-\infty}^{\infty} R_{\alpha}(\tau) e^{-j\omega t} \, d\tau
$$

$$
= \frac{8\pi L a^2}{A^2} \iiint \Phi_n(K_y, K_z) \cos^2\left[\frac{L\sigma(1-\sigma)K_z^2}{2k}\right] \sin^2\left[\frac{a}{2} K_y \sigma\right] \frac{jK_y(1-\sigma) - \vec{v}_w \tau}{\sin\left(\frac{\theta}{2} K_z \sigma\right)} e^{-j\omega t} \, d\omega \, dK_y \, dK_z \, d\tau
$$

(2.108)

The $t$ integration in Eq.(2.108) can be done using the following integral.

$$
\int \exp(j\omega t) \, dt = 2\pi \delta(\omega)
$$

(2.109)
Substituting the above integral into Eq.(2.108), the following expression for $W_\omega(\omega)$ results

$$W_\omega(\omega) = \frac{16n^2L}{a^2} \iint \phi_n(K_y, K_z) \cos^2\left[\frac{L\sigma(1-\sigma)K^2}{2k}\right]$$

$$\sin^2\left[\frac{a}{2} K_y \sigma\right] [\text{sinc}(\frac{a}{2} K_z \sigma)]^2$$

$$\delta[K_y(\tilde{v}_t(1-\sigma) - \tilde{v}_s(1-\sigma)) - \omega]d\tilde{v}_ydK_y$$

(2.110)

To simplify the writing in Eq.(2.110), we define two more variables.

$$f_1(\sigma) = v_{2t}(1-\sigma) - v_{2w}$$

$$f_2(\sigma) = v_{3t}(1-\sigma) - v_{3w}$$

(2.111a)

(2.111b)

Substituting the expressions for $f_1(\sigma)$ and $f_2(\sigma)$ in Eq.(2.111a) and Eq.(2.111b) into Eq.(2.110), we have

$$W_\omega(\omega) = \frac{16n^2L}{a^2} \iint \phi_n(K_y, K_z) \cos^2\left[\frac{L\sigma(1-\sigma)K^2}{2k}\right] \sin^2\left[\frac{a}{2} K_y \sigma\right]$$

$$[\text{sinc}(\frac{a}{2} K_z \sigma)]^2 \delta[K_y f_1(\sigma) + K_z f_2(\sigma) - \omega]d\tilde{v}_ydK_y$$

(2.112)

The $K_y$ integration in Eq.(2.112) can be performed using the following formula:

$$\int g(K_y, K_z, \sigma) \delta(K_y f_1(\sigma) + K_z f_2(\sigma) - \omega)dK_y$$

$$= \frac{1}{|f_1(\sigma)|} g(f_3(K_z, w, \sigma), K_z, \sigma)$$

(2.113a)

where the $\delta$ function has required that we replace $K_y$ by $f_3$ where

$$f_3(K_z, w, \sigma) = \frac{-K_z f_2(\sigma) + w}{f_1(\sigma)} = \frac{-K_z [v_{3t}(1-\sigma) - v_{3w}] + w}{v_{2t}(1-\sigma) - v_{2w}}$$

(2.113b)
Using the above integral, we obtain for $W_\alpha(\omega)$ the following expression:

$$W_\alpha(\omega) = \frac{16\pi^2 L}{a^2} \int_0^L \int_{-\infty}^\infty \frac{1}{|f_\perp(\sigma)|} \phi_n [ f_3(K_z, w, \sigma), K_z ]$$

$$\cos^2\left[ \frac{\sigma(1-\sigma) L}{2k} (K_z^2 + f_3^2(K_z, w, \sigma)) \right]$$

$$\sin^2\left[ \frac{\sigma K_z}{2} \right] \left[ \text{sinc}\left( \frac{\sigma K_z}{2} \right) \right]^2 \frac{d\omega dK_z}{2 \pi}$$

Substituting $f_1$ in Eq.(2.111a) and $f_3$ in Eq.(2.113b) into Eq.(2.114a), we have

$$W_\alpha(\omega) = \frac{16\pi^2 L}{a^2} \int_0^L \int_{-\infty}^\infty \frac{1}{|v_{2t}(1-\sigma) - v_{2w}|} \phi_n \left\{ \frac{w-K_z[v_{3t}(1-\sigma) - v_{3w}]}{v_{2t}(1-\sigma) - v_{2w}} \right\}, K_z$$

$$\cos^2\left[ \frac{\sigma(1-\sigma) L}{2k} \left( K_z^2 + \left( \frac{w-K_z v_{3t}(1-\sigma) - K_z v_{3w}}{v_{2t}(1-\sigma) - v_{2w}} \right)^2 \right) \right]$$

$$\sin^2\left[ \frac{\sigma}{2} \frac{w-K_z v_{3t}(1-\sigma) - K_z v_{3w}}{v_{2t}(1-\sigma) - v_{2w}} \right] \left[ \text{sinc}\left( \frac{\sigma K_z}{2} \right) \right]^2 \frac{d\omega dK_z}{2 \pi}$$

Equation (2.114b) contains the general formulation of the angle of arrival power spectrum. In the next section we will give a numerical example.

7 DERIVATION FOR A SPECIAL CASE

We consider first the case where the source moves in the y direction. Thus, the source and wind velocity components in the z direction $v_{3t}$ and $v_{3w}$ are all zero. The source and wind velocity in the y direction $v_{2t}$ and $v_{2w}$ are not zero. We shall use von Kármán power spectrum which is given by
\[ \phi_n(K_y, K_z) = 0.033 C_n^2 \left( K_y^2 + K_z^2 + L_0^{-2} \right)^{-11/6} \]  

We also introduce new variables \( \gamma, K'_z, W_0, \) and \( \Omega \) which are defined below

\[
\begin{align*}
\Omega &= \frac{aw}{v_{2t}} \quad (2.116a) \\
K'_z &= aK_z \quad (2.116b) \\
\gamma &= \frac{v_{2w}}{v_{2t}} \quad (2.116c) \\
W_0 &= \frac{16n^2L_o2/3(0.033)}{v_{2t}} \quad (2.116d)
\end{align*}
\]

Substituting \( v_{3t} = v_{3w} = 0, \) \( \phi_n \) in Eq.(2.115) and using variables \( \gamma, K'_z, W_0, \) and \( \Omega \) defined in Eq.(2.116), \( W_\alpha(w) \) in Eq.(2.114b) then becomes

\[
W'_\alpha = \frac{W_\alpha}{W_0}
\]

\[
= \int_0^1 \int_{-\infty}^{\infty} \left[ K'_z^2 + \frac{\Omega^2}{(1-\sigma-\gamma)^2} + a^2L_0^{-2} \right]^{-11/6} \\
\left[ \frac{C_n^2(\sigma)}{1-\sigma-\gamma} \cos^2\left\{ \frac{L_0(1-\sigma)}{2ka^2} [K'_z^2 + \frac{\Omega^2}{(1-\sigma-\gamma)}] \right\} \right] \\
\sin^2\left[ \frac{\sigma\Omega}{2(1-\sigma-\gamma)} \right] \left[ \text{sinc}\left( \frac{\sigma K'_z}{2} \right) \right]^2 \text{d}\sigma \text{d}K'_z 
\]

\[
(2.117)
\]

The refractive index structure constant \( C_n^2(\sigma) \) and the outer scale \( L_0(\sigma) \) are both functions of the height parameter \( \sigma \). The profiles for \( C_n^2(\sigma) \) and \( L_0(\sigma) \) are given by the following equations [53].

\[
C_n^2(\sigma) = C_n^2(H_0) \left[ 1 + \left( 1-\sigma \right) \frac{(H_L-H_0)}{H_0} \right]^{-4/3} \quad (2.118a)
\]
\[ L_0(\sigma) = \frac{L_0(H_0)}{H_0} \left[ H_0 + (1-\sigma)(H_L-H_0) \right] \]  

Equation (2.118a) and Eq.(2.118b) will be used in Eq.(2.117) for computer calculations.

Besides power spectrum, another quantity of interest is the differential path contribution. The normalized differential path contribution \( D_q(\sigma,\Omega) \) is given by the following.

\[
D_q(\sigma,\Omega) = \int_{-\infty}^{\infty} \left[ K_z' \right]^{2} + \frac{\Omega^2}{(1-\sigma-\gamma)^2} \right]^{1/6} \sigma_0^2(\sigma) \\
\frac{1}{|1-\sigma-\gamma|} \cos^2\left\{ \frac{\lambda(1-\sigma)}{2k a^2} \left[ K_z' \right]^{2} + \frac{\Omega^2}{(1-\sigma-\gamma)^2} \right\} \\
sin^2\left\{ \frac{\sigma \Omega}{2(1-\sigma-\gamma)} \right\} \left\{ \text{sinc} \left( \frac{\sigma \Omega}{2 K_z} \right) \right\}^2 dK_z'
\]

Figure 8 is a plot of differential path contributions \( D_q(\sigma,\Omega) \) versus \( \sigma \) for \( \Omega = 0.01 \). \( \Omega = 0.01 \) means that the source moves one hundredth the aperture size in time \( T \) corresponding to \( \omega/2\pi \). Figure 8 shows that \( D_q(\sigma,\Omega) \) increases with \( \sigma \) measured from the transmitter. For other \( \Omega \) it behaves in the same way except for possible oscillation at very large \( \Omega \) due to the sine function in the integrand. Figure 9 is a plot of \( W_q(\Omega) \) versus \( \Omega \). That curve shows that the power spectrum will increase with \( \Omega \), reaching a maximum, and then decrease again.
Figure 8. Differential Path Contribution for the Moving Source and Stationary Receiver in a Nontracking System
Figure 9. Normalized Power Spectrum for the Moving Source and Stationary Receiver in a Nontracking System

RL = 1000.0
HO = 10.0

NORMALIZED FREQUENCY (Log ω)

NORMALIZED SPECTRUM (Log Nω)

-6 -5 -4 -3 -2 -1 0

-3 -2 -1 0

-5 -4 -3 -2 -1 0
According to Eq.(2.117), the spatial frequency $K_y$ is chosen according to $K_y = \Omega/a(1-\sigma)$. The wind velocity factor $\gamma$ is small compared with the source or receiver velocity and has been neglected without affecting the discussion. That simple relation between $K_y$ and $\Omega, \sigma$ can also be determined from the geometry shown in Figure 10. For a time period $T$ with corresponding detector frequency $\nu_2$ and normalized frequency $\Omega$, the source has moved the distances $v_2tT$. The equivalent eddy size in the same period is just the distance $AC$ in Figure 10 that can be determined from the triangle $RS_1S_2$. Using the triangular relations, we then obtain

$$\lambda_y = \frac{AC}{(1-\sigma)v_2tT} \quad (2.120)$$

Figure 10. Geometrical Determination of Eddy Size at Range $\sigma$ From the Source with the Source Moving and Receiver Stationary
From Eq. (2.120), the normalized spatial frequency $K_y$ is then given by

$$K_y = \frac{2\pi}{\lambda_y} = \frac{\omega}{(1-\sigma)v_2t}$$

$$= \frac{\Omega}{a(1-\sigma)} \quad \text{(2.121)}$$

where

$$\Omega = \frac{aw}{v_2t} \quad \text{(2.122)}$$

This $K_y$ value is just the result we expect.

The differential path contribution shown in Figure 8 increases with the range parameter $\sigma$. This is because at the receiver for the parameter chosen, the effective eddy size, taking into account the spherical nature of the wave, never gets a chance to reach a half cycle variation within the aperture. Mathematically, this can simply be determined from the argument of the sine function in Eq. (2.117). That argument is written again.

$$\text{ARG} = \frac{\sigma}{2} \frac{\Omega}{(1-\sigma)} \quad \text{(2.123)}$$

By setting ARG equal to $\pi$ corresponding to half a cycle variation, we may solve for $\sigma$ to see whether it is in the propagation range. For $\Omega = 0.01$, we find $\sigma \approx 1$. This explains why the differential path contribution $D'_\sigma(\sigma)$ has a maximum at the receiver.

The angle of arrival power spectrum $W'_\sigma(\Omega)$ can be separated into three parts. In the left part, $\Omega$ is so small that Eq. (2.117) can be approximated by

$$W'_\sigma(\Omega) \propto \Omega^2 \int_0^1 \int_{-\infty}^{\infty} \left[ K_z^2 + a^2k_y^{-2} \right]^{-11/6} \frac{c_o^2(\sigma)}{|1-\sigma-\gamma|} \left[ \frac{\sigma}{2(1-\sigma-\gamma)} \right]^2 \left[ \sin c \left( \frac{\sigma}{2} K'_z \right) \right]^2 \sin \gamma \cos K'_z \quad \text{(2.124)}$$
where we have used $\sin \chi \approx \chi$ for very small $\chi$ and $\Omega$ is neglected in the von Kármán spectrum. Thus, the left part is dominated by aperture diffraction effect inherent in the $\sin^2$ function and $W'_\alpha(\Omega)$ increases with $\Omega$ at a slope about equal to 2.

In the middle part, $\Omega$ is not so small as to be completely neglected from von Kármán spectrum, and thus Eq.(2.117) becomes

$$W'_\alpha(\Omega) \propto \int_{0}^{1} \int_{-\infty}^{\infty} \left[ K'_z^{2} + \frac{\Omega^2}{1-\sigma-\gamma} \right] \frac{\sigma^2 \Omega^2}{2(1-\sigma-\gamma)^2} dK'_z d\sigma$$

where we have approximated sinc function by 1 and $\sin \chi$ by $\chi$. Equation (2.125a) can be simplified using the following substitution.

$$K'_z = \left[ \frac{\Omega}{(1-\sigma-\gamma)} \right] \tan \theta$$

We thus have

$$W'_\alpha(\Omega) \propto \int_{0}^{1} \int_{0}^{\pi/2} \left( \frac{1}{1-\sigma-\gamma} \right)^{-2/3} \Omega^{-2/3} \sec^{-5/3} \theta \frac{\sigma^2 \Omega^2}{2(1-\sigma-\gamma)^2} d\theta d\sigma$$

Thus the middle is dominated by both turbulence and aperture diffraction effect and $W'_\alpha(\Omega)$ decreases at a slope roughly equal to 2/3.

In the right part, $W'_\alpha(\omega)$ decreases much faster with $\Omega$ and the sinc function is approximately a constant. Making the same substitution given in Eq.(2.125b), $W'_\alpha$ is found to decrease with $\Omega$ following a $-8/3$ slope. This is because this region is completely dominated by turbulence effect.
This finishes the discussion on the power spectrum for a nontracking receiver using the fixed coordinate system. In the next section, we summarize the major results.

8 SUMMARY

In this chapter, the centroid angle of arrival formulation is reviewed. This centroid angle of arrival can be measured using a centroid detector whose output is proportional to the integrated product of intensity and position. Using this formulation, we then derive the centroid angle of arrival power spectrum for a nontracking receiver. In this system, the receiver does not face the source all the times.

The wave equation is first derived in the fixed coordinate system. All those terms which are of order \((\frac{V}{c})^2\) smaller are neglected. The complex wave amplitude is then solved using the method of smooth perturbations. The complex wave amplitude is found to consist of a deterministic component and a random component. The deterministic component is just the conventional Doppler shift. The random component is then used to derive the power spectrum with both source and receiver moving in the transverse direction.

Finally, a numerical example is given. In that example, the source moves horizontally. The temporal power spectrum has various frequency ranges which are explained on physical grounds. The differential path contribution has a peak at the receiver which is similarly explained.
CHAPTER III

ANGLE OF ARRIVAL POWER SPECTRUM FOR A TRACKING RECEIVER

1 INTRODUCTION

In this chapter, we compute the angle of arrival temporal power spectrum for a tracking receiver. We have already considered the non-tracking receiver case in chapter II. If the source moves but the receiver remains stationary, the light spot will fall farther and farther away from the optical axis. In this case, the receiving system cannot work for very long and something must be done to alleviate the situation.

The way chosen to solve this problem is to have the receiver tracking the source. That means the receiver aperture must be rotated in the correct manner to face the source all the time. The source, receiver input aperture, and the center of the output plane are always kept on the same line. Thus the light spot should always be confined to the small region around the optical axis.

A rotating coordinate system should be used in the analysis due to the fact that the receiver aperture has to rotate. Hence, we must first define this rotating coordinate system and then solve the wave equation in this rotating coordinate system. We expect that the analysis to be
presented for the tracking receiver will have the same approach as that for the nontracking receiver and that there will be additional complexity due to the rotating coordinate system.

In section 2, we review the centroid angle of arrival formalism stated in chapter II. The result is an expression for the angle of arrival temporal spectrum in terms of the phase correlation function of atmospherically degraded input wave. Sections 3 through 9 deal with deriving the necessary expression for the phase correlation function. The derivations are more complex because of the use of the rotating coordinate system. In section 3, we derive the coordinate transformation equations which relate the fixed coordinates \( x', y', \) and \( z' \) to the rotating coordinates \( x, y, \) and \( z \). Using these equations in section 4, we express the wave equation in terms of the variables of the rotating coordinate system. During the transformation, all those terms which are in order of \( \frac{v^2}{C^2} \) in smallness are neglected. In section 5, we first derive the wave equation in terms of complex wave amplitude. Again during the derivation, all those terms which are in order of \( \frac{v^2}{C^2} \) in smallness are neglected. This wave equation is then solved for complex log amplitude. The complex log amplitude is found to contain both random and deterministic components. All the approximations made are summarized in section 6. In section 7, we evaluate the deterministic part of the complex log amplitude. This deterministic component is found to have two imaginary parts. Both parts are due to Doppler shift. The first part is radial and the second part is due to coordinate rotations. In section 8, we evaluate the random part of the complex log amplitude first, and then use it to derive the expression for the phase of the
atmospherically degraded wave. During the derivation, the extension is made for the Taylor frozen turbulence hypothesis in the rotating coordinate system. Also during the derivation, the justification for neglecting longitudinal velocity is given. The phase computed in section 8 is used in the derivation of the phase correlation function in section 9.

Sections 10 and 11 lead up to an integral expression for the angle of arrival temporal spectrum. In section 10, we derive the centroid angle of arrival covariance function. The centroid angle of arrival power spectrum which is the Fourier transform of the covariance function is derived in section 11. The expression for power spectrum is simplified using the argument that the propagation range remains nearly constant within the correlation time of interest. This approximation is justified both physically and mathematically in Appendix G.

Sections 12 through 14 deal with numerical results and their interpretation. In section 12, we present numerical integration results for both the power spectrum and its differential path contributions. These numerical results are interpreted physically in section 13. Finally, in section 14, we summarize the major results.

In the next section we start by reviewing the centroid angle of arrival formulations.
In this section, we review the formulation of centroid angle of arrival, centroid angle of arrival covariance function and centroid angle of arrival power spectrum which were discussed in chapter II. These same formulations will also be used in this chapter where the receiver has to track the source.

The x-direction centroid angle of arrival \( \alpha_0(t) \) which was derived in Eq.(2.24) is given by

\[
\alpha_0(t) = \frac{1}{kA} \int \frac{3S(y,z,t)}{y} dydz - \alpha_s
\]  

(3.1)

where the integration is over the aperture plane \((y,z)\) and the phase \(S\) is in general the sum of the deterministic quantity and the random quantity.

The centroid angle of arrival covariance function \( R_\alpha(t_1,t_2) \) which is derived in Eq.(2.27b) is given by

\[
R_\alpha(t_1,t_2) = \frac{1}{k^2A} \iiint \frac{\partial^2}{\partial y_1 \partial y_2} B_s(y_1,y_2,z_1,z_2,t_1,t_2) dy_1 dz_1 dy_2 dz_2
\]

(3.2)

where \(B_s\) is the phase correlation function.

The centroid angle of arrival power spectrum, which is defined as the Fourier transform of the covariance function, is then given by
\[ W_\alpha(\omega, \delta) = \int R_\alpha(t, \delta) e^{-j\omega t} dt \]  

(3.3)

where \( t \) and \( \delta \) are the usual time lag and mean time.

\[ t = t_1 - t_2 \]  

(3.4a)

\[ \delta = \frac{1}{2}(t_1 + t_2) \]  

(3.4b)

Equation (3.2) and Eq.(3.3) are the general formulas used later in sections 10 and 11 to compute the angle of arrival covariance function and the angle of arrival power spectrum. In the next section, we start the derivation of the phase of the atmospherically disturbed wave described in the rotating coordinate system by first talking about the moving coordinate system.

3 MOVING COORDINATE SYSTEM

Now we begin consideration of wave propagating in the atmosphere. As shown in Eq.(3.1), we have to solve for the phase, \( S \), in order to compute the angle of arrival, \( \alpha_0(t) \). The phase, \( S \), is the imaginary part of the complex log amplitude, \( \Phi \). \( \Phi \) is related to the field, \( E \), through the equation \( E = e^{\Phi} e^{-j\omega t} \). To obtain \( E \), we have to solve the wave equation. Because both the source and receiver may move and the receiver tracks the source, we solve the wave equation in the moving coordinate system. In order to do that, we must derive the wave equation in the moving coordinate system.
The general configuration we consider is shown in Figure II. In that figure, the source sends out spherical waves which propagate downward through the atmosphere toward the receiver. The source and receiver move in an arbitrary direction with constant velocities. The receiver not only translates, but also rotates so as to track the source. As will be described in the next paragraph, the reference coordinate system $x'', y'', z''$ is centered on the receiver aperture and maintains constant orientation. The moving coordinate system $x, y, z$ is also centered on the receiver aperture, but rotates to face the source. The receiver is fixed in this coordinate system. Thus, from the point of view of the receiver, the moving coordinate system does not translate, but always faces the source.

The directions of the three axes in the reference coordinate system can be uniquely defined. Assume that the source, receiver, and wind move with constant velocities $\vec{v}_T$, $\vec{v}_R$, and $\vec{v}_W$, respectively. The reference coordinate system designated by $x'', y'', z''$ is centered at the center of the receiver aperture. Thus, from the point of view of the receiver, it sees a source moving with velocity $\vec{v}_T - \vec{v}_R$. The reference $y''$ axis is by definition aligned parallel to the $\vec{v}_T - \vec{v}_R$ direction as shown in Figure II. The shortest distance between the source trajectory and the receiver is in a direction perpendicular to the $y''$ direction. The reference $x''$ direction is thus defined to be aligned along this direction and the shortest distance is called $L$. The reference $z''$ axis is then perpendicular to the $x''$, $y''$ plane in the direction to form a right handed coordinate system. From the point of view of the receiver, the wind blows with velocity $\vec{v}_W - \vec{v}_R$. We assume for simplicity that $\vec{v}_W - \vec{v}_R$ has components only in the $x''$ and $y''$ directions.
Figure 11. Rotating Coordinate System in a Tracking System
The angle of rotation, \( \theta(t) \), shown in Figure 11 is related to the source velocity, \( \vec{v}_T \), the receiver velocity, \( \vec{v}_R \), and the path separation, \( L \), by the expression.

\[
\cos \theta = \frac{L}{\sqrt{L^2 + v_T^2 t^2}}
\]  

(3.5a)

where

\[
v = |\vec{v}_T - \vec{v}_R|
\]  

(3.5b)

The rotating coordinates \( x, y, \) and \( z \) can be related to the reference coordinates \( x'', y'', \) and \( z'' \) by considering Figure 11. For an arbitrary point \( P \) in space with coordinate \( (x,y,z) \) in the rotating coordinate system and with coordinate \( (x'',y'',z'') \) in the reference coordinate system, the following expression can be easily derived.

\[
x'' = x \cos \theta - y \sin \theta
\]  

(3.6a)

\[
y'' = x \sin \theta + y \cos \theta
\]  

(3.6b)

\[
z'' = z
\]  

(3.6c)

Equation (3.6) can also be solved to give \( x \) and \( y \) in terms of \( x'' \) and \( y'' \)

\[
x = y'' \sin \theta + x'' \cos \theta
\]  

(3.7a)

\[
y = -x'' \sin \theta + y'' \cos \theta
\]  

(3.7b)

\[
z = z''
\]  

(3.7c)

We are now going to transform the wave equation in the fixed coordinate system to the moving coordinate system in the next section.
In this section, we transform the wave equation from the reference coordinate system to the rotating coordinate system. The wave equation in the reference coordinate system is given by

$$\frac{\partial^2 E}{\partial x'^2} + \frac{\partial^2 E}{\partial y'^2} + \frac{\partial^2 E}{\partial z'^2} - \mu \frac{\partial^2 E}{\partial t'^2} = 0$$  \hspace{1cm} (3.8)

where $t''$ in the reference coordinate system is related to the $t$ in the rotating coordinate system by

$$t = t''$$  \hspace{1cm} (3.9)

Using the chain rule and Eq.(3.7) and Eq.(3.9), we have

$$\frac{\partial E}{\partial x''} = \frac{\partial E}{\partial x'} \frac{\partial x'}{\partial x''} + \frac{\partial E}{\partial y'} \frac{\partial y'}{\partial x''} + \frac{\partial E}{\partial t'} \frac{\partial t'}{\partial x''}$$

$$= \frac{\partial E}{\partial x}(\cos\theta) + \frac{\partial E}{\partial y}(-\sin\theta)$$  \hspace{1cm} (3.10)

where from Eq.(3.9)

$$\frac{\partial t}{\partial x''} = 0$$  \hspace{1cm} (3.11)

The second derivative of $E$ with respect to $x''$ then becomes

$$\frac{\partial^2 E}{\partial x''^2} = \left[ \frac{\partial}{\partial x'} \left( \frac{\partial E}{\partial x''} \right) \right] \frac{\partial x'}{\partial x''} + \left[ \frac{\partial}{\partial y'} \left( \frac{\partial E}{\partial x''} \right) \right] \frac{\partial y'}{\partial x''}$$

$$+ \left[ \frac{\partial}{\partial t'} \left( \frac{\partial E}{\partial x''} \right) \right] \frac{\partial t'}{\partial x''}$$

$$= \frac{\partial^2 E}{\partial x^2}\cos^2\theta - 2 \frac{\partial^2 E}{\partial x\partial y}\sin\theta\cos\theta + \frac{\partial^2 E}{\partial y^2}\sin^2\theta$$  \hspace{1cm} (3.12)

The second derivative of $E$ with respect to $y''$ can be derived in a similar fashion and we obtain

$$\frac{\partial^2 E}{\partial y''^2} = \frac{\partial^2 E}{\partial x^2}\sin^2\theta + 2 \frac{\partial^2 E}{\partial x\partial y}\sin\theta\cos\theta + \frac{\partial^2 E}{\partial y^2}\cos^2\theta$$  \hspace{1cm} (3.13)
The second derivative of $E$ with respect to $z''$ can be transformed quite easily to give

$$\frac{\partial^2 E}{\partial z''^2} = \frac{\partial}{\partial z'} \left[ \frac{\partial E}{\partial z''} \right] = \frac{\partial^2 E}{\partial z'^2}$$  \hfill (3.14)

We now turn to the derivation of the time derivative. Again, using the chain rule and the coordinate and time transformations given by Eq.(3.7) and Eq.(3.9), we have

$$\frac{\partial E}{\partial t''} = \frac{\partial E}{\partial x} \frac{\partial x}{\partial t''} + \frac{\partial E}{\partial y} \frac{\partial y}{\partial t''} + \frac{\partial E}{\partial t} \frac{\partial t}{\partial t''}$$

$$= \frac{\partial E}{\partial x} \frac{\partial x}{\partial t} y - x \frac{\partial E}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial E}{\partial t} \frac{\partial t}{\partial t}$$  \hfill (3.15)

where we have used the following expression

$$\frac{\partial x}{\partial t} = y'' \cos \theta \frac{\partial \theta}{\partial t''} - x'' \sin \theta \frac{\partial \theta}{\partial t''}$$

$$= y \frac{\partial \theta}{\partial t}$$ \hfill (3.16)

$$\frac{\partial y}{\partial t} = -x'' \cos \theta \frac{\partial \theta}{\partial t''} - y'' \sin \theta \frac{\partial \theta}{\partial t''}$$

$$= -x \frac{\partial \theta}{\partial t}$$ \hfill (3.17)

From Eq.(3.15), we can derive the second derivative of $\psi$ with respect to $t''$ as follows:

$$\frac{\partial^2 E}{\partial t''^2} = \frac{\partial}{\partial x} \left[ \frac{\partial E}{\partial t} \right] \frac{\partial x}{\partial t} y - x \frac{\partial E}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial E}{\partial t} \frac{\partial t}{\partial t}$$

$$+ \frac{\partial}{\partial y} \left[ \frac{\partial E}{\partial t} \right] \frac{\partial y}{\partial t} y - x \frac{\partial E}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial E}{\partial t} \frac{\partial t}{\partial t}$$

$$+ \frac{\partial}{\partial t} \left[ \frac{\partial E}{\partial t} \right] \frac{\partial t}{\partial t} y - x \frac{\partial E}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial E}{\partial t} \frac{\partial t}{\partial t}$$

$$= \left[ \frac{\partial^2 E}{\partial x^2} \frac{\partial x}{\partial t} y - \frac{\partial E}{\partial y} \frac{\partial y}{\partial t} - x \frac{\partial^2 E}{\partial x \partial y} \frac{\partial x}{\partial x} \frac{\partial x}{\partial t} \right] \left[ y \frac{\partial \theta}{\partial t} \right]$$

$$+ \left[ \frac{\partial^2 E}{\partial x \partial y} \frac{\partial x}{\partial t} y + \frac{\partial E}{\partial y} \frac{\partial y}{\partial t} - x \frac{\partial^2 E}{\partial y^2} \frac{\partial y}{\partial y} \frac{\partial y}{\partial t} \right] \left[ -x \frac{\partial \theta}{\partial t} \right]$$

$$+ \left[ \frac{\partial^2 E}{\partial x \partial t} \frac{\partial x}{\partial t} y + \frac{\partial E}{\partial y} \frac{\partial y}{\partial t} - x \frac{\partial^2 E}{\partial y \partial x} \frac{\partial y}{\partial y \partial t} - \frac{\partial E}{\partial y} \frac{\partial^2 E}{\partial y \partial x} \frac{\partial x}{\partial y} \frac{\partial x}{\partial t} \right] \left[ -x \frac{\partial \theta}{\partial t} \right]$$
Substituting Eq.(3.12), Eq.(3.13), Eq.(3.14), and Eq.(3.18) into Eq.(3.4), we finally obtain the wave equation in the rotating coordinate system in the following form.

\[
\frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} + \frac{\partial^2 E}{\partial z^2} - \mu \varepsilon \left[ \frac{\partial^2 E}{\partial t^2} + \gamma^2 \left( \frac{\partial}{\partial t} \right)^2 \frac{\partial^2 E}{\partial x^2} - 2xy \left( \frac{\partial}{\partial t} \right) \frac{\partial}{\partial y} \frac{\partial^2 E}{\partial x \partial y} + 2y \frac{\partial}{\partial t} \frac{\partial}{\partial x} \frac{\partial^2 E}{\partial x \partial t} \right]
- 2x \frac{\partial}{\partial t} \frac{\partial^2 E}{\partial y \partial t} - x \frac{\partial^2 E}{\partial x \partial t} + \frac{\partial E}{\partial y} \frac{\partial^2 E}{\partial y^2} + \gamma^2 \frac{\partial}{\partial t} \frac{\partial^2 E}{\partial y^2} + y \frac{\partial}{\partial x} \frac{\partial^2 E}{\partial x \partial t} + \gamma \frac{\partial}{\partial y} \frac{\partial^2 E}{\partial x \partial y} = 0
\]  

Equation (3.19) looks complicated, but can be simplified because some terms can be neglected due to their smallness. We now roughly estimate the magnitude of all the terms containing the time derivative in the bracket of Eq.(3.19). To do this, we assume the source radiates monochromatic spherical wave with \( \exp(-j\omega t + jkr) \) dependence. Thus, the following estimates are obtained.

\[
\frac{\partial E}{\partial x} \approx kE \quad \text{(3.20a)}
\]
\[
\frac{\partial^2 E}{\partial x^2} \approx k^2 E \quad \text{(3.20b)}
\]
\[
\frac{\partial E}{\partial y} \approx k \left( \frac{\gamma}{x} \right) E \quad \text{(3.20c)}
\]
\[
\frac{\partial^2 E}{\partial y^2} \approx k^2 \left( \frac{\gamma}{x} \right)^2 E \quad \text{(3.20d)}
\]
The first and second time derivative of \( \theta \), with respect to \( t \), can be obtained from Eq. (3.5a) and are given by

\[
\frac{\partial \theta}{\partial t} = \frac{v}{L} \cos^2 \theta \quad (3.21a)
\]

\[
\frac{\partial^2 \theta}{\partial t^2} = -\frac{v^2}{L^2} (\sin \theta)(\cos^2 \theta) \quad (3.21b)
\]

Using Eq. (3.20) and Eq. (3.21) the magnitude estimates for all the terms in the bracket of Eq. (3.19) are hence given by

\[
f_1 = \frac{\partial^2 E}{\partial t^2} \approx \omega^2 E \quad (3.22a)
\]

\[
f_2 = y^2 \left( \frac{\partial \theta}{\partial t} \right)^2 \frac{\partial^2 E}{\partial x \partial t} \approx y^2 \left( \frac{\partial \theta}{\partial t} \right)^2 k^2 E = \frac{v^2}{L^2} y^2 k^2 \cos^4 \theta E \quad (3.22b)
\]

\[
f_3 = 2 xy \left( \frac{\partial \theta}{\partial t} \right)^2 \frac{\partial^2 E}{\partial x \partial y} \approx 2 xy \left( \frac{\partial \theta}{\partial t} \right)^2 k^2 \left( \frac{v}{x} \right)^2 E = 2 k^2 y^2 \left( \frac{v^2}{L^2} \right) \cos^4 \theta E \quad (3.22c)
\]

\[
f_4 = 2 y \left( \frac{\partial \theta}{\partial t} \right) \frac{\partial^2 E}{\partial x \partial t} \approx 2 y \left( \frac{\partial \theta}{\partial t} \right) k \omega E = 2 y k \omega \left( \frac{v}{L} \right) \cos \theta E \quad (3.22d)
\]

\[
f_5 = 2 x \left( \frac{\partial \theta}{\partial t} \right) \frac{\partial^2 E}{\partial y \partial t} \approx 2 x \left( \frac{\partial \theta}{\partial t} \right) k \omega E = 2 x k \omega \left( \frac{v}{L} \right) \cos \theta E \quad (3.22e)
\]

\[
f_6 = x \left( \frac{\partial E}{\partial x} \right) \left( \frac{\partial \theta}{\partial t} \right)^2 \approx k x \left( \frac{\partial \theta}{\partial t} \right)^2 = x k \frac{v^2}{L^2} \cos^4 \theta E \quad (3.22f)
\]

\[
f_7 = y \left( \frac{\partial E}{\partial y} \right) \left( \frac{\partial \theta}{\partial t} \right)^2 \approx k \left( \frac{v}{x} \right)^2 y \left( \frac{\partial \theta}{\partial t} \right)^2 = \left( \frac{v^2}{x} \right) k \left( \frac{v^2}{L^2} \right) \cos^4 \theta E \quad (3.22g)
\]

\[
f_8 = x^2 \left( \frac{\partial \theta}{\partial t} \right)^2 \frac{\partial^2 E}{\partial y^2} \approx x^2 \left( \frac{v^2}{L^2} \cos \theta \right) k^2 \frac{y^2}{x^2} E = y^2 k^2 \frac{v^2}{L^2} \cos^4 \theta E \quad (3.22h)
\]

\[
f_9 = y \left( \frac{\partial E}{\partial x} \right) \frac{\partial^2 \theta}{\partial t^2} \approx ky \left( \frac{\partial^2 \theta}{\partial t^2} \right) \approx ky \left( \frac{v}{L} \right)^2 \cos \theta \cos^2 \theta E \quad (3.22i)
\]

\[
f_{10} = x \left( \frac{\partial E}{\partial y} \right) \frac{\partial^2 \theta}{\partial t^2} \approx k \left( \frac{v}{x} \right) x \left( \frac{\partial^2 \theta}{\partial t^2} \right) \approx ky \left( \frac{v}{L} \right)^2 \sin \theta \cos^2 \theta E \quad (3.22j)
\]
By forming the ratio $\frac{f_i}{f_1}$, $i = 2-10$, we obtain the following set of equations.

\begin{align*}
\frac{f_2}{f_1} & \propto \frac{v^2}{c^2} \left( \frac{Y}{L} \right)^2 \quad (3.23a) \\
\frac{f_3}{f_1} & \propto \frac{v^2}{c^2} \left( \frac{Y}{L} \right)^2 \quad (3.23b) \\
\frac{f_4}{f_1} & \propto \left( \frac{Y}{c} \right) \left( \frac{Y}{L} \right) \quad (3.23c) \\
\frac{f_5}{f_1} & \propto \left( \frac{Y}{c} \right) \left( \frac{L}{Y} \right) \quad (3.23d) \\
\frac{f_6}{f_1} & \propto \frac{v^2}{L w_0 c} \quad (3.23e) \\
\frac{f_7}{f_1} & \propto \frac{v^2}{L w_0 c} \left( \frac{Y}{L} \right)^2 \quad (3.23f) \\
\frac{f_8}{f_1} & \propto \frac{v^2}{c^2} \left( \frac{Y}{L} \right)^2 \quad (3.23g) \\
\frac{f_9}{f_1} & \propto \frac{v^2}{L w_0 c} \left( \frac{Y}{L} \right) \quad (3.23h) \\
\frac{f_{10}}{f_1} & \propto \frac{v^2}{L w_0 c} \left( \frac{Y}{L} \right) \quad (3.23i)
\end{align*}

where we have assumed that $\theta$ is not near $\frac{\pi}{2}$ so that we could put $x \approx L$. 
In general we assume that optical fields are of significant magnitude only for values of \( \frac{V}{L} \leq 1 \) due to the assumed paraxial nature of the waves. Thus the ratio of \( f_4 \) and \( f_5 \) to \( f_1 \) are less than \( \frac{V}{C} \). However, they are still the most important terms compared with the others as shown in Eq.(3.23). Thus, we only retain \( f_1, f_4, \) and \( f_5 \), and neglect all those terms which are less than \( \left( \frac{V}{C} \right)^2 \) compared with \( f_1 \). Retaining \( f_1, f_4, \) and \( f_5 \), Eq.(3.19) can be rewritten in the following form.

\[
\frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} + \frac{\partial^2 E}{\partial z^2} = \mu \varepsilon \left( \frac{\partial^2 E}{\partial t^2} + 2y \frac{\partial \theta}{\partial x} \frac{\partial^2 E}{\partial x \partial t} - 2x \frac{\partial \theta}{\partial y} \frac{\partial^2 E}{\partial y \partial t} \right)
\]

Equation (3.24) is the wave equation in the rotating coordinate system. This wave equation is also discussed in Appendix E. In the next section, we are going to solve this equation including refractive index fluctuation.

5 SOLUTION OF WAVE EQUATION IN A ROTATING COORDINATE SYSTEM

In this section, we derive the expression for the complex log amplitude in the rotating coordinate system by using the method of smooth perturbations.

Considering again monochromatic wave propagation with \( e^{-j\omega_0 t} \) dependence, we can write

\[
E(\vec{r},t) = \psi(\vec{r},t)e^{-j\omega_0 t}
\]

Substituting \( E(\vec{r},t) \) in Eq.(3.18a) into Eq.(3.17), the following equation results
\[ \nabla^2 \psi = \mu e \left[ -\omega_0^2 \psi - 2j\omega_0 \frac{\partial \psi}{\partial t} - 2j\omega_0y \frac{\partial \psi}{\partial y} + 2j\omega_0x \frac{\partial \psi}{\partial x} \right] + \frac{\partial^2 \psi}{\partial t^2} + 2y \frac{\partial \psi}{\partial t} \frac{\partial^2 \psi}{\partial x \partial t} - 2x \frac{\partial \psi}{\partial t} \frac{\partial^2 \psi}{\partial y \partial t} \] (3.26)

where we have used

\[ \frac{\partial^2 E}{\partial t^2} = \left[ \frac{\partial^2 \psi}{\partial t^2} - 2j\omega_0 \frac{\partial \psi}{\partial t} - \omega_0^2 \psi \right] e^{-j\omega_0t} \] (3.27a)

\[ \frac{\partial^2 E}{\partial x \partial t} = \left[ \frac{\partial^2 \psi}{\partial x \partial t} - j\omega_0 \frac{\partial \psi}{\partial x} \right] e^{-j\omega_0t} \] (3.27b)

\[ \frac{\partial^2 E}{\partial y \partial t} = \left[ \frac{\partial^2 \psi}{\partial y \partial t} - j\omega_0 \frac{\partial \psi}{\partial y} \right] e^{-j\omega_0t} \] (3.27c)

In Eq.(3.18b), the last three terms are less than \((\frac{V}{c})^2\) compared with the first term and can be neglected. The details are shown in Appendix F.

We next split the permittivity \(\varepsilon\) up into its vacuum value and a small perturbation as was given in Eq.(2.33b). That expression is given by

\[ \varepsilon = \varepsilon_0 (1 + 2n_1) \] (3.28)

The quantity \(n_1 (x,y,z)\) is as before, a small random function of position. Furthermore, the square of the free space wave number is also given by

\[ k^2 = \omega_0^2 \mu_0 \varepsilon_0 \] (3.29)

Using the above two equations and neglecting the last three terms of Eq.(3.26), we then have

\[ \nabla^2 \psi + k^2 \psi = \left( 2k^2 n_1 \psi - \frac{2j\omega_0}{c^2} \frac{\partial \psi}{\partial t} - \frac{2j\omega_0 y}{c^2} \frac{\partial \psi}{\partial y} + \frac{2j\omega_0 x}{c^2} \frac{\partial \psi}{\partial x} \right) \] (3.30)

As in chapter II, we use the method of smooth perturbations to solve Eq.(3.30) (Tatarskii, p. 71).
We first introduce the complex log amplitude, \( \phi \), given by

\[ \phi = \ln \psi \]  

(3.31)

Using Eq. (3.31), Eq. (3.30) can be rewritten as

\[ \nabla^2 \phi + \nabla \phi \cdot \nabla \phi + k^2 = f_1(r,t) \]  

(3.32)

where

\[ f_1 = -2k^2 n_1 - \frac{2j\omega_y \partial \phi}{c^2} \frac{\partial}{\partial t} - \frac{2j\omega_y \theta \partial \phi}{c^2} \frac{\partial}{\partial x} + \frac{2j\omega_x \partial \phi}{c^2} \frac{\partial}{\partial y} \]  

(3.33)

and we have used the following relations:

\[ \nabla \psi = (\nabla \phi)e^\phi \]  

(3.34a)

\[ \nabla^2 \psi = (\nabla^2 \phi + \nabla \phi \cdot \nabla \phi)e^\phi \]  

(3.34b)

We next expand \( \phi(r,t) \) by writing it in a series of terms with each term smaller than the preceding one by a factor of \( \omega \) where \( \omega \) is the order of \( n_1 \) or \( \frac{\omega}{c} \).

\[ \phi(r,t) = \phi_0(r,t) + \omega \phi_1(r,t) + \omega^2 \phi_2(r,t) + \ldots \]  

(3.35)

Substituting \( \phi \) in Eq. (3.35) into Eq. (3.32) and by writing \( f_1(r,t) \) as \( \omega f_1(r,t) \), we obtain

\[ \nabla^2 (\phi_0 + \omega \phi_1 + \omega^2 \phi_2 + \ldots) + \nabla (\phi_0 + \omega \phi_1 + \ldots) \cdot \nabla (\phi_0 + \omega \phi_1 + \ldots) \]

\[ + k^2 = \omega f_1(r,t) \]  

(3.36)

All the terms in \( f_1 \) are about the same order of magnitude as is demonstrated in Appendix F.

Equating the terms of the same power \( \omega \) to zero in Eq. (3.22), the following expression then results up to the first order \( \omega \). (The second and higher order terms are proportional to \( n_1^i \leq (\frac{\omega}{c})^i, i \geq 2 \) and are too small to be included as a solution.)

\[ \nabla^2 \phi_0 + (\nabla \phi_0 \cdot \nabla \phi_0) + k^2 = 0 \]  

(3.37)

\[ \nabla^2 \phi_1 + 2(\nabla \phi_0 \cdot \nabla \phi_1) = f_1(r,t) \]  

(3.38)
The zero order equation given by Eq. (3.37) comes from the wave equation
\[ \nabla^2 \psi_0 + k^2 \psi_0 = 0. \]

The first order equation given by Eq. (3.38) can be solved exactly by letting
\[ \phi_1 = e^{-\Phi_0 W}. \]

Substituting \( \phi_1 \) in Eq. (3.39) into Eq. (3.38), we obtain
\[ \nabla^2 \psi_1 + k^2 \psi_1 = -e^{\Phi_0 W} \]
where we have used
\[ \nabla \phi_1 = e^{-\Phi_0 W} \nabla \psi_1 - \psi_1 \nabla \Phi_0 e^{-\Phi_0} \]  
\[ \nabla^2 \phi_1 = \nabla^2 \psi_1 - 2(\nabla \psi_1 \cdot \nabla \Phi_0) e^{-\Phi_0} \]
\[ -\nabla \psi_1 \Phi_0 e^{-\Phi_0} + (\nabla \Phi_0 \cdot \nabla \psi_1) e^{-\Phi_0} \]

The solution to Eq. (3.40) is the well known Green's function solution
\[ W(\bar{r},t) = \frac{k^2}{4\pi} \int \int \int \frac{e^{jk|\bar{r}-\bar{r}'|}}{|\bar{r}-\bar{r}'|} \psi_0(\bar{r}',t) d^3 \bar{r}'. \]

Substituting the expression for \( W \) from Eq. (3.42) into Eq. (3.39), a formula for \( \phi_1(\bar{r},t) \) then becomes
\[ \phi_1(\bar{r},t) = \frac{1}{4\pi} \int \int \int \frac{e^{jk|\bar{r}-\bar{r}'|}}{|\bar{r}-\bar{r}'|} \psi_0(\bar{r}',t) \psi_0(\bar{r},t) d^3 \bar{r}'. \]

where we have used
\[ \psi_0(\bar{r},t) = e^{\Phi_0(\bar{r},t)}. \]

Substituting \( f_1 \) in Eq. (3.33) into Eq. (3.43), we obtain the following set of equations.
\[ \phi_1 = \phi_{1a} + \phi_{1b} + \phi_{1c} + \phi_{1d} \]
where
\[ \phi_{1a} = \frac{k^2}{2\pi} \int \int \int \psi_0(\bar{r}',t) \frac{e^{jk|\bar{r}-\bar{r}'|}}{|\bar{r}-\bar{r}'|} d^3 \bar{r}'. \]
We note that $\Phi_{1a}$ depends on the random refractive index fluctuation $n_1$ and is therefore a random quantity. $\Phi_{1b}$, $\Phi_{1c}$, and $\Phi_{1d}$ do not depend on $n_1$ and are deterministic. To restate our result, the complete solution, accurate up to the first order, written in terms of $\Phi_1$ is from Eq.(3.25).

$$E = \psi_0 e^{\Phi_1 (\mathbf{r}, \mathbf{t})} e^{-j\omega_0 t}$$  \hspace{1cm} (3.50)$$

$\Phi_{1a}$ is a random quantity and can be treated only through statistical techniques. $\Phi_{1b}$, $\Phi_{1c}$, and $\Phi_{1d}$ are deterministic quantities and can be evaluated once the zero order solution $\psi_0(r,t)$ is known. In the next section, we are going to evaluate $\Phi_{1b}$, $\Phi_{1c}$, and $\Phi_{1d}$.

6 DISCUSSION ON THE APPROXIMATIONS

In getting Eq.(3.45), approximations have been used in several places. In this section, we summarize all the approximations used.

We first derive the wave equation in the rotating coordinate system. Some terms are dropped. The ratio of the terms neglected to the most significant term are smaller than $(\frac{Y}{c})^2$. 

$$\Phi_{1b} = \frac{j\omega_0}{2\pi c^2} \iint_{\mathbb{R}^3} \frac{\partial \psi_0(\mathbf{r}', t)}{\partial t} \frac{1}{\psi_0(\mathbf{r})} e^{jk|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}' \hspace{1cm} (3.47)$$

$$\Phi_{1c} = \frac{j\omega_0}{2\pi c^2} \iint_{\mathbb{R}^3} \frac{\partial \psi_0(\mathbf{r}', t)}{\partial x'} \frac{1}{\psi_0(\mathbf{r})} e^{jk|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}' \hspace{1cm} (3.48)$$

$$\Phi_{1d} = \frac{-j\omega_0}{2\pi c^2} \iint_{\mathbb{R}^3} \frac{\partial \psi_0(\mathbf{r}', t)}{\partial y'} \frac{1}{\psi_0(\mathbf{r})} e^{jk|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}' \hspace{1cm} (3.49)$$
Next we solve the complex log amplitude by using the method of smooth perturbations. The terms dropped compared with those kept are an order of magnitude \( n_1 \left( \frac{\nu}{c} \right) \) or \( \left( \frac{\nu}{c} \right)^2 \) smaller. However, \( n_1 \) is smaller than or equal to \( \frac{\nu}{c} \). Thus, the approximation used is consistent with what we discussed above.

In summary, we have neglected all those terms which are less than \( \left( \frac{\nu}{c} \right)^2 \).

7 EVALUATION OF DETERMINISTIC PORTION OF THE COMPLEX LOG AMPLITUDE

In this section, we compute \( \Phi_{ib} \), \( \Phi_{ic} \), and \( \Phi_{id} \), the integrals in Eq.(3.47), Eq.(3.48), and Eq.(3.49) for the deterministic terms in the solution for the complex log amplitude. The configuration we consider is shown in Figure 12. A spherical wave sent out from a moving source \( S \) propagates downward through the turbulent atmosphere and is accepted at the receiver aperture. In order to track the source, we have defined a rotating coordinate system which is always centered at the receiver. Because of the coordinate rotation, the receiver always sees the source along the \( x \) axis. Suppose the distance between the source and receiver is \( x_t \) at time \( t \), then the received spherical wave at position \( \vec{r} \) in the receiver aperture and time \( t \) can be written as

\[
\psi_0(\vec{r},t) = \frac{jk|\vec{r}-\vec{r}_t|}{|\vec{r}-\vec{r}_t|} e^{jk|\vec{r}-\vec{r}_t|} \tag{3.51a}
\]
Figure 12. A Plot to Explain the Deterministic Phase Shift for a Tracking System with Both Transmitter and Receiver Moving
where

\[ r_t = (x_t,0,0) \]  \hspace{1cm} (3.51b)

\[ r = (0,y,z) \]  \hspace{1cm} (3.51c)

\[ x_t = \sqrt{l^2+v^2t^2} \]  \hspace{1cm} (3.51d)

We note that \( \psi_0 \) satisfies Eq.(3.26) up to the first order.

To evaluate \( \Phi_{1b}, \Phi_{1c}, \) and \( \Phi_{1d}, \) we need derivatives of \( \psi_0. \) The first derivative of \( \psi_0(\bar{r}',t), \) where \( \bar{r}'=(x',y',z'), \) with respect to \( t, \)
\( x', \) and \( y' \) is easily obtained as

\[ \frac{\partial \psi_0(\bar{r}',t)}{\partial t} = \frac{jk|\bar{r}_t-\bar{r}'|}{|\bar{r}_t-\bar{r}'|^2} \]  \hspace{1cm} (3.52a)

\[ \frac{\partial \psi_0(\bar{r}',t)}{\partial x'} = -\frac{jk(x_t-x')e^{-jkr'-r'}|\bar{r}_t-\bar{r}'|}{|\bar{r}_t-\bar{r}'|^2} \]  \hspace{1cm} (3.52b)

\[ \frac{\partial \psi_0(\bar{r}',t)}{\partial y'} = jky'e^{-jkr'-r'}\frac{jk|\bar{r}_t-\bar{r}'|}{|\bar{r}_t-\bar{r}'|^2} \]  \hspace{1cm} (3.52c)

Substituting the expressions from Eq.(3.50) and Eq.(3.52) into
Eq.(3.47), Eq.(3.48) and Eq.(3.49), we then have

\[ \Phi_{1b} = -\frac{k^3}{2\pi\omega_0}\int\int\int (x_t-x')e^{-jkr'-r'}\frac{jk|\bar{r}_t-\bar{r}'|}{|\bar{r}_t-\bar{r}'|^2} e^{jk|\bar{r}_t-\bar{r}'|} \frac{1}{|\bar{r}_t-\bar{r}'|} d^3\bar{r}' \]  \hspace{1cm} (3.53)

\[ \Phi_{1c} = \frac{k^3}{2\pi\omega_0}\int\int\int (x_t-x')y'e^{-jkr'-r'}\frac{jk|\bar{r}_t-\bar{r}'|}{|\bar{r}_t-\bar{r}'|^2} e^{jk|\bar{r}_t-\bar{r}'|} \frac{1}{|\bar{r}_t-\bar{r}'|} d^3\bar{r}' \]  \hspace{1cm} (3.54)
\[ \phi_{1d} = \frac{k^3}{2\pi \omega_0} \frac{\partial}{\partial t} \iint \frac{y' x' e^{jk|\mathbf{r}_t - \mathbf{r}'|}}{|\mathbf{r}_t - \mathbf{r}'|^2} \frac{|\mathbf{r}_t - \mathbf{r}'| e^{jk|\mathbf{r}_r - \mathbf{r}'|}}{\mathbf{d}^3 \mathbf{r}'}, \]

(3.55)

Where \( \mathbf{r}' = (x', y', z') \) denotes a point between the source and the receiver. Restricting the derivation to the usual paraxial ray approximations

\[ x'^2 \gg (y-y')^2 + (z-z')^2, \quad (x_t - x')^2 \gg y'^2 + z'^2 \quad \text{and} \quad x_t^2 \gg y^2 + z^2, \]

\(|\mathbf{r}_t - \mathbf{r}'|, |\mathbf{r}_r - \mathbf{r}'|, \text{and} |\mathbf{r}_r - \mathbf{r}_t| \text{can be written as} \]

\[ |\mathbf{r}_r - \mathbf{r}'| = x' + \frac{(y-y')^2 + (z-z')^2}{2x'} \quad (3.56a) \]

\[ |\mathbf{r}_r - \mathbf{r}_t| = (x_t - x') + \frac{y'^2 + z'^2}{2(x_t - x')} \quad (3.56b) \]

\[ |\mathbf{r}_r - \mathbf{r}_t| = x_t + \frac{y^2 + z^2}{2x_t} \quad (3.56c) \]

Substituting the expressions in Eq.(3.56) into Eq.(3.53), Eq.(3.54) and Eq.(3.55), we have

\[ \phi_{1b} = \frac{-k^3}{2\pi \omega_0} \frac{\partial}{\partial t} \iint \frac{x_t}{x'(x_t - x')} \frac{dx_t}{dt} \exp\left\{ -jk(y^2 + z^2) \frac{2x_t}{2(x_t - x')} \right\} + jk \left\{ \frac{(y-y')^2 + (z-z')^2}{2x'} \right\} d^3 \mathbf{r}'. \]

(3.57)

\[ \phi_{1c} = \frac{k^3}{2\pi \omega_0} \frac{\partial}{\partial t} \iint \frac{x_t}{x'(x_t - x')} (y') \exp\left\{ -jk(y^2 + z^2) \frac{2x_t}{2(x_t - x')} \right\} + jk \left\{ \frac{(y-y')^2 + (z-z')^2}{2x'} \right\} d^3 \mathbf{r}'. \]

(3.58)

\[ \phi_{1d} = \frac{k^3}{2\pi \omega_0} \frac{\partial}{\partial t} \iint \frac{x_t}{(x_t - x')^2} \frac{y'}{(x_t - x')^2} \exp\left\{ -jk(y^2 + z^2) \frac{2x_t}{2(x_t - x')} \right\} + jk \left\{ \frac{(y-y')^2 + (z-z')^2}{2x'} \right\} d^3 \mathbf{r}'. \]

(3.59)
The exponent in the integral of Eq.(3.57), Eq.(3.58) and Eq.(3.59) is rewritten by first grouping the terms containing $y_2'$ and $z_2'$ together.

\[
\begin{align*}
\frac{-jk(y^2+z^2)}{2\xi_t} + \frac{jk y_2'^2 + z_2'^2}{2(\xi_t - \xi_t')} + \frac{jk}{2\xi_t} \frac{(y-y_1')^2 + (z-z_1')^2}{2\xi_t'} &= jk \frac{y_1'^2}{2\xi_t' (\xi_t - \xi_t')} \\
+ jkz_2'^2 \left[ \frac{\xi_t}{2\xi_t' (\xi_t - \xi_t')} \right] - 2jk \frac{yy_1'}{2\xi_t'} - 2jk \frac{zz_1'}{2\xi_t'} + jk y_1'^2 \frac{\xi_t - \xi_t'}{2\xi_t' \xi_t} + jkz_1'^2 \frac{\xi_t - \xi_t'}{2\xi_t' \xi_t} \\
= jk \frac{\xi_t}{2\xi_t' (\xi_t - \xi_t')} \left[ y_2'^2 - 2yy_1' \frac{\xi_t - \xi_t'}{\xi_t} + y_1'^2 \frac{\xi_t - \xi_t'}{\xi_t} \right] + jk \frac{\xi_t}{2\xi_t' (\xi_t - \xi_t')} \left[ (y_1' - z_1')^2 + (z_1' - z_1')^2 \right]
\end{align*}
\]

where

\[
\begin{align*}
y_1' &= \frac{(\xi_t - \xi_t')y}{\xi_t} \quad (3.61a) \\
z_1' &= \frac{(\xi_t - \xi_t')z}{\xi_t} \quad (3.61b)
\end{align*}
\]

Using the expression in Eq.(3.60), and rewriting, Eq.(3.57), Eq.(3.58), and Eq.(3.59) then become

\[
\begin{align*}
\Phi_{1b} &= \frac{-k^3}{2\pi m_0} \frac{\pi}{\xi_t'} \int \int \frac{\xi_t}{x'(\xi_t - \xi_t')} \frac{dx_t}{dt} \exp\left\{ \frac{jkx_t}{2(\xi_t - \xi_t')x_t'} \left[ (y_1' - y_1')^2 + (z_1' - z_1')^2 \right] \right\} d^3 x_t' \\
\Phi_{1c} &= \frac{k^3}{2\pi m_0} \frac{\partial}{\partial t} \int \int \frac{x_t y_1'}{x'(\xi_t - \xi_t')} \exp\left\{ \frac{jkx_t}{2(\xi_t - \xi_t')x_t'} \left[ (y_1' - y_1')^2 + (z_1' - z_1')^2 \right] \right\} d^3 x_t' \\
\Phi_{1d} &= \frac{k^3}{2\pi m_0} \frac{\partial}{\partial t} \int \int \frac{x_t y_1'}{(\xi_t - \xi_t')^2} \exp\left\{ \frac{jkx_t}{2(\xi_t - \xi_t')x_t'} \left[ (y_1' - y_1')^2 + (z_1' - z_1')^2 \right] \right\} d^3 x_t'
\end{align*}
\]
Equation (3.62), Eq.(3.63), and Eq.(3.64) can be integrated by using the following two well known integrals.

\[
\int_{-\infty}^{\infty} \exp\left\{ \frac{-(x-\mu)^2}{2\sigma^2} \right\} \, dx = \sqrt{2\pi}\sigma \tag{3.65a}
\]

\[
\int_{-\infty}^{\infty} x \exp\left\{ \frac{-(x-\mu)^2}{2\sigma^2} \right\} \, dx = \mu\sqrt{2\pi}\sigma \tag{3.65b}
\]

Using (3.65), we finally get the following results.

\[
\phi_{1_b} = \frac{jk^2}{w_0} \frac{dx_t}{dt} \tag{3.66}
\]

\[
\phi_{1_c} = \frac{-jk^2}{w_0} \frac{\partial \theta}{\partial t} \frac{x_t}{2} y \tag{3.67}
\]

\[
\phi_{1_d} = \frac{-jk^2}{w_0} \frac{\partial \theta}{\partial t} \frac{x_t}{2} y \tag{3.68}
\]

\(\phi_{1_b}, \phi_{1_c},\) and \(\phi_{1_d}\) are all imaginary. That means we have obtained extra deterministic phase shifts. These can all be interpreted physically. The total deterministic phase shift, \(\phi_{1_t}\), is the sum of \(\phi_{1_b}\), \(\phi_{1_c}\), and \(\phi_{1_d}\).

\[
\phi_{1_t} = \phi_{1_b} + \phi_{1_c} + \phi_{1_d}
\]

\[
= \frac{jk^2}{w_0} \frac{dx_t}{dt} \left[ \frac{\partial \theta}{\partial t} - y \frac{\partial \theta}{\partial t} \right] \tag{3.69}
\]

The first term in the bracket of Eq.(3.69) is the regular Doppler shift that we obtained earlier; while the second term is associated with the rotation of the coordinate system. A simple model can be built to explain Eq.(3.69).
Consider Figure 12. If we have a point receiver, then \( y = 0 \) and \( \phi_{1t} \) is the same as what we obtained earlier given in (2.89b). However, the receiver aperture is finite, the radial velocity should be calculated based upon \( x'_t \) instead of \( x_t \). Using the law of cosines in the triangle SOA, we have

\[
x'_t = \frac{x_t^2 + y'^2 - 2x_ty' \cos(90^\circ - \theta)}{x_t^2 - y^2}
\]

for \( y' \ll x_t \) (3.70)

Hence

\[
\frac{dx'_t}{dt} = \frac{dx_t}{dt} - y' \cos \theta \frac{\partial \theta}{\partial t} = \frac{dx_t}{dt} + y \frac{d\theta}{dt}
\]

(3.71)

where we have used

\[
|x'_t| = x_t
\]

(3.72)

The deterministic phase shift calculated from the time rate change of \( x'_t \) is then

\[
\phi_{1t} = \frac{jk^2}{w_0} x'_t \frac{dx'_t}{dt} = \frac{jk^2}{w_0} x_t \left( \frac{dx_t}{dt} + y \frac{d\theta}{dt} \right)
\]

(3.73)

Equation (3.73) is the same as Eq.(3.69). Thus, this simple model explains why there is an extra phase shift. This extra phase shift is zero for a point aperture or a stationary coordinate system.

This concludes the discussion on the evaluations of the deterministic phase shift. In the next section, we are going to evaluate the random phase \( \phi_{1a} \).
In this section, the random phase, \( S \), is computed for the tracking receiver and the spherical wave source. The complex wave perturbation \( \phi_1 a \) from Eq.(3.46) is rewritten below for convenience.

\[
\phi_1 a = \frac{k^2}{2\pi} \iiint \text{n}_1(\bar{r}',t) e^{jk|\bar{r}-\bar{r}'|} \frac{\psi_0(\bar{r}',t)}{|\bar{r}-\bar{r}'|} \psi_0(\bar{r},t) \, d^3\bar{r}' \tag{3.74}
\]

Substituting \( \psi_0(\bar{r},t) \) in Eq.(3.50) into Eq.(3.74), we obtain

\[
\phi_1 a = \frac{k^2}{2\pi} \iiint \text{n}_1(\bar{r}',t) \frac{\text{x}_t}{\text{x}'}(\frac{\text{x}_t-x'}{2\text{x}_t}) \text{exp}\{\frac{-j\text{k}(y'^2+z'^2)}{2\text{x}_t} \}
\]

\[
+ \frac{jk}{2\pi} [(y-y')^2 + (z-z')^2] + \frac{jk}{2} \frac{y'^2+z'^2}{x_t-x'} \} \, dx'dy'dz' \tag{3.75}
\]

For the later development of random phase derivation, the Taylor hypothesis has to be used. However, this important relation is well defined only in the reference coordinate system but not in the rotating coordinate system. There is an uncertainty about whether the same relationship can be applied in both the reference and the rotating coordinate systems. Later results show that the Taylor hypothesis in the reference coordinate system is indeed different from that in the rotating coordinate system. Thus, the Taylor hypothesis in the reference coordinate system will not be applied directly to the rotating coordinate system. What will be done is to apply the Taylor hypothesis to the reference coordinate system first. The coordinate variables are then transformed back from the reference coordinate system to the rotating coordinate system to obtain the Taylor hypothesis in the rotating coordinate system.
Assuming \( \ddot{v}_w - \ddot{v}_R \) has components in both \( y'' \) and \( x'' \) directions, we thus have

\[
\ddot{v}_w - \ddot{v}_R = v_{1w} \hat{x}'' + v_{2w} \hat{y}''
\]

(3.76)

where \( \hat{x}'' \) and \( \hat{y}'' \) are unit vectors. Applying Taylor hypothesis to the reference coordinate system, the following equation is obtained:

\[
n_1(x'', t) = n_1(x'' - v_{1w} t \hat{x}'' - v_{2w} t \hat{y}'')
\]

\[
= n_1(x'' - v_{1w} t, y'' - v_{2w} t, z'')
\]

(3.77)

\( x'', y'', z'' \) in Eq.(3.77) are related to \( x', y', z' \) through Eq.(3.6a), Eq.(3.6b), and Eq.(3.6c), and are given by

\[
x'' = x' \cos \theta - y' \sin \theta
\]

(3.78a)

\[
y'' = x' \sin \theta + y' \cos \theta
\]

(3.78b)

\[
z'' = z'
\]

(3.78c)

\( \cos \theta \) and \( \sin \theta \) are defined by

\[
\cos \theta = \frac{L}{x_t}
\]

(3.79a)

\[
\sin \theta = \frac{v_t}{x_t}
\]

(3.79b)

Substituting \( \cos \theta \) and \( \sin \theta \) in Eq.(3.79a) and Eq.(3.79b) into Eq.(3.78a), Eq.(3.78b), and Eq.(3.78c), we obtain

\[
x'' = x' \frac{L}{x_t} - y' \frac{v_t}{x_t}
\]

(3.80a)

\[
y'' = x' \frac{v_t}{x_t} + y' \frac{L}{x_t}
\]

(3.80b)

\[
z'' = z'
\]

(3.80c)
Substituting \( x'', y'', z'' \) in Eq.(3.80a), Eq.(3.80b), and Eq.(3.80c) into Eq.(3.77), we obtain

\[
n_1(\tilde{r}'', \tilde{r}'', t) = n_1(x''(\tilde{r}'') - v_{1w} t, y''(\tilde{r}'') - v_{2w} t, z'')
\]

\[
= n_1(x' \frac{L}{x_t} - y' \frac{v_t}{x_t}, x' \frac{v_t}{x_t} + y' \frac{L}{x_t} - v_{2w} t, z')
\]  (3.81)

Equation (3.82) is also the Taylor hypothesis in the rotating coordinate system

\[ n_1(\tilde{r}'', \tilde{r}'', t) \text{ is now a function of } x', y', \text{ and } z'. \] We thus rename \( n_1(\tilde{r}'', \tilde{r}'', t) \) as \( n_1(\tilde{r}'', t) \), i.e.,

\[
n_1(\tilde{r}'', t) = n_1(\tilde{r}'', \tilde{r}'', t)
\]  (3.83)

Substituting Eq.(3.83) and Eq.(3.81) into Eq.(3.75), we have

\[
\phi_{1a} = \frac{k^2}{2\pi} \iint n_1(x''(\tilde{r}'') - v_{1w} t, y''(\tilde{r}'') - v_{2w} t, z'') \frac{x_t}{x''(\tilde{r}' - \tilde{r}'')}
\]

\[
\exp\{-\frac{jk(y^2+z'^2)}{2x_t} + \frac{jk}{2x_t}[(y-y'')^2 + (z-z'')^2]
\]

\[
+ \frac{jk}{2} \frac{y'^2+z'^2}{x_t-x''} \}dx'dy'dz'
\]  (3.84)

By grouping terms in Eq.(3.84) containing \( y'^2, z'^2, y', \) and \( z' \) first, and completing the squares, the following expression results:

\[
\phi_{1a} = \frac{k^2}{2\pi} \iint n_1(x''(\tilde{r}'') - v_{1w} t, y''(\tilde{r}'') - v_{2w} t, z'') \frac{x_t}{x'(x_t-x'')}
\]

\[
\exp\{\frac{jk}{2} \frac{x_t}{x'(x_t-x'')}(y' - \frac{x_t-x'}{x_t} y)^2
\]

\[
+ (z' - \frac{x_t-x'}{x_t} z)^2 \}dx'dy'dz'
\]  (3.85)
\( n_1(x'', y'', z'') \) can be represented in the spectral domain by using the following Fourier transform integral.

\[
n_1(x'', y'', z'') = \iiint \mu_1(K_x'', K_y'', K_z'') e^{j(K''x'' + K''y'' + K''z'')} dK''_x dK''_y dK''_z
\]

Using Eq. (3.86), Eq. (3.81) can be written as

\[
n_1(x''(r') - v_1w, y''(r') - v_2t, z'') \]

\[
= \iiint \mu_1(K_x'', K_y'', K_z'') e^{jK''x'\left(\frac{r}{x_t} - \frac{v_1w}{x_t}\right) + jK''y'\left(\frac{r}{y_t} - \frac{v_2t}{y_t}\right) + jK''z'} dK''_x dK''_y dK''_z
\]

Substituting the expression in Eq. (3.87) into Eq. (3.85), we have

\[
\phi_{1a} = \frac{k^2}{2\pi} \iiint \mu_1(K_x'', K_y'', K_z'') \exp\left\{ jK''x'\left(\frac{r}{x_t} - \frac{v_1w}{x_t}\right) + jK''y'\left(\frac{r}{y_t} - \frac{v_2t}{y_t}\right) + jK''z'\left(\frac{r}{z_t} - \frac{v_1w}{z_t}\right) \right\}
\exp\left\{ \frac{jk}{2} \left(\frac{z_t}{x_t} y' - \frac{z_t}{x_t} z'\right)^2 \right\} dx'dy'dz'dK''_x dK''_y dK''_z
\]

By grouping terms containing \( y'^2, z'^2, y', z' \), together and then completing the squares in the exponent of Eq. (3.88), the following expression results.
\[ \phi_{1a} = \frac{k^2}{2\pi} \int \int \int \frac{x_t}{x_t'(x_t - x_t')} \mu_1(K_x', K_y', K_z') \]

\[ \exp\left\{ j \frac{x_t - x_t'}{x_t} (K_x y x_t - K_z z) + \frac{jk}{2} \frac{x_t}{x_t'(x_t - x_t')} \right\} \]

\[ [y' + \frac{K_y}{k} \frac{x_t'(x_t - x_t')}{x_t} - \frac{x_t - x_t'}{x_t} y - \frac{K_x}{k} \frac{x_t'(x_t - x_t')}{x_t} \frac{v_t}{x_t}]^2 \]

\[ + \frac{jk}{2} \frac{x_t}{x_t'(x_t - x_t')}(z' - \frac{K_x}{k} \frac{x_t'(x_t - x_t')}{x_t} - \frac{x_t - x_t'}{x_t} z)^2 \]

\[ - \frac{i}{2k} \frac{x_t'(x_t - x_t')}{x_t} (K_y^2 \frac{x_t}{x_t} + K_z^2) \exp[jK_y y v_t] \]

\[ \exp\left\{ jK_x' \frac{L}{x_t} x_t' + v_{1w} x_t + K_y \frac{L}{x_t} \frac{1}{k} \frac{x_t'(x_t - x_t')}{x_t} \frac{v_t}{x_t} \right\} \]

\[ - v \frac{x_t - x_t'}{x_t} \frac{v_t}{x_t} + \frac{K_x}{2k} \frac{x_t'(x_t - x_t')}{x_t} \frac{v^2 t^2}{x_t} \] 

dx'dy'dz'dK'dK'dK''dK''

(3.89)

where

\[ V_{ey} = \frac{v x_t'}{x_t} - v_{2w} \]  

(3.90)

We are going to show next the magnitude for each of the last three time dependent terms in Eq.(3.89) is much smaller than \( 2\pi \) and can be neglected. \( V_{ey} \) in Eq.(3.89) can be considered as equivalent transverse velocity. Even though \( V_{2w} \) may have magnitude comparable to \( v x_t' / x_t \), it won't affect the following analysis by dropping \( V_{2w} \) from \( V_{ey} \). \( V_{ey} \) is thus approximated by

\[ V_{ey} \approx \frac{v x_t'}{x_t} \]  

(3.91)
The transverse velocity \( V_{ey} \) is in the direction in which diffraction pattern is formed. The characteristic dimension in the transverse direction is given by \( \sqrt{L\lambda} \) which is the first Fresnel zone size [52].

The characteristic time is then given by

\[
    t = \frac{\sqrt{L\lambda}}{V_{ey}} = \frac{\sqrt{L\lambda} x_t}{v y}
\]

We now substitute \( t \) from Eq.(3.92) into the last three terms of Eq.(3.89).

The estimate for the third term in the last exponential term of Eq.(3.89) is given by

\[
    K'' K''_{xy} \int \frac{1}{x^k} \frac{(x_t-x')}{x_t} \frac{v}{x_t} t\]

\[
    = K'' K''_{xy} (\cos \theta) \frac{\lambda}{2\pi} x' (1-x') \frac{v}{x_t} \frac{\sqrt{L\lambda x_t}}{y x'}
\]

\[
    \leq K'' K''_{xy} \frac{\lambda}{2\pi} \sqrt{L\lambda} \leq \frac{\sqrt{L\lambda}}{e_0} \ll 2\pi
\]

The estimate for the fourth term in the last exponential term of Eq.(3.89) is given by

\[
    K'' K''_{xy} \int \frac{x_t-x'}{x_t} \frac{v}{x_t} t\]

\[
    = K'' K''_{xy} (1-x') \frac{v}{x_t} \frac{\sqrt{L\lambda x_t}}{y x'}
\]

\[
    \leq K'' \frac{\sqrt{L\lambda}}{x'} \leq K'' \frac{\lambda}{x} \sqrt{\frac{L}{e_0}} \leq \sqrt{\frac{\lambda}{L e_0}} \leq 2\pi
\]
The estimate for the fifth term in the last exponential term of Eq. (3.89) is given by

\[
\frac{K''}{2k} \frac{x'}{x_t} \left(1 - \frac{x'}{x_t}\right) \frac{v^2 t^2}{x_t^2}
\]

\[
= \frac{K''}{2k} \frac{v^2 L \lambda x_t^2}{x_t}
\]

\[
= \frac{K''}{2k} \frac{L \lambda \lambda^2}{x_t} \approx \frac{K''}{2k} \frac{\lambda^2}{2\delta} \leq 2\pi
\]

(3.95)

where \(x'\) is approximated by \(L\), and \(\delta_0\) is the inner scale of turbulence. Typical value of \(\delta_0\) is from 1mm to 1cm, \(L\) is 1km for an aircraft and \(10^5\)m for a satellite and \(\lambda\) is \(6328 \times 10^{-10}\)m for He-Ne laser. With substitution of these values, the last three terms of Eq. (3.89) are all much less than \(2\pi\) and hence can be neglected.

The relative longitudinal wind velocity \(V_{1w}\) can be neglected also. This is allowed if the ratio of \(V_{1w}\) versus \(V_{ey}\) is less than \(\sqrt{\frac{L}{\lambda}}\) [91]. That condition is easily satisfied unless \(V_{ey}\) approaches zero. \(V_{ey}\) can approach zero only if both the relative transverse source velocity \(v\) and the relative transverse wind velocity \(V_{2w}\) are zero. That is highly unlikely because we consider wave propagation between moving vehicles. The relative longitudinal velocity \(V_{1w}\) is thus dropped from Eq. (3.89).
There is one special case which is worth mentioning. As $\theta \to 90^\circ$, $x_t \to \infty$ and $V_{ey}$ in (3.89) becomes $v_{2w}$. In other words, the relative transverse wind velocity is the dominant transverse velocity as $\theta \to 90^\circ$. A similar analysis will show Eq.(3.93), Eq.(3.94) and Eq.(3.95) still hold.

Neglecting the last three terms of Eq. (3.89) we then have

$$\phi_{1a} = \frac{k^2}{2\pi} \int \int \int \int \mu_1(K_x'', K_y'', K_z'') \frac{x_t}{x_t'} \exp[jK_y'' V_{ey} t]$$

$$\exp\left[\frac{x_t - x_t'}{x_t} (K_y' y - K_z' z)\right] \exp[jK_y'' \frac{L}{x_t} x_t']$$

$$\exp\left\{\frac{jk}{2} \frac{x_t}{x_t'} \left[\frac{K''}{y} \frac{x}{x_t} + \frac{K''}{z} \frac{x}{x_t} \right] - \frac{x_t - x_t'}{x_t} \right\}^2$$

$$+ \frac{jk}{2} \frac{x_t}{x_t'} \left[\frac{K''}{z} \frac{x}{x_t} + \frac{K''}{y} \frac{x}{x_t} \right] \left[za' - \frac{x_t - x_t'}{x_t} \right]$$

$$- \frac{k}{2k} \frac{x_t}{x_t'} \left(\frac{K''}{y} \frac{L^2}{x_t} + K''^2\right) dx'dy'dz' \frac{dK''}{x} \frac{dK''}{y} \frac{dK''}{z}$$

(3.96)

The $y'$ and $z'$ integrals can be performed using Eq.(3.65). The results are

$$\phi_{1a} = -jk \int \int \int \mu_1(K_x'', K_y'', K_z'') \frac{L}{x_t} \frac{x_t'}{x_t} \exp[jK_y'' \frac{L}{x_t} x_t']$$

$$\exp[j(K_y'' \frac{L}{x_t} - K_z'' \frac{x_t'}{x_t})]$$

$$\exp[-\frac{j}{2kx_t} (K_y'' \frac{L^2}{x_t} + K_z'')] dx'dy'dz' \frac{dK''}{x} \frac{dK''}{y} \frac{dK''}{z}$$

(3.97)
The exponent containing $K''_y$ in Eq. (3.97) can be regrouped to give

$$\phi_{1a} = -jk\int \int \int \mu(x, y, z)e^{jK''_x \frac{L}{x_t} x^t} \exp[jK''_y y_b(r, t) + jK''_z z_b(r, t)]$$

$$= \exp[-j(K''_y \frac{L^2}{y x_t^2} + K''_z \frac{x'(x_t - x')} k x_t^2)]dx'dK''_x dK''_y dK''_z$$

(3.98)

where $\bar{r}$ is a function of $y$ and $z$ in the receiver aperture

and

$$y_b(\bar{r}, t) = \frac{v x^t}{x_t} - V_2 w + y \frac{L}{x_t} \frac{x_t - x'}{x_t^2}$$

(3.99)

$$z_b(\bar{r}, t) = -z \frac{x_t - x'}{x_t}$$

(3.100)

$\phi_{1a}$ given in Eq. (3.98) is the final expression for the complex log amplitude.

The phase $S(\bar{r}, t)$ is obtained by finding the imaginary part of $\phi_{1a}(\bar{r}, t)$, thus

$$S(\bar{r}, t) = \frac{1}{2j}[\phi_{1a}(\bar{r}, t) - \phi_{1a}^*(\bar{r}, t)]$$
\[ S(\vec{r}, t) = \frac{1}{2j} \left\{ -jk \iiint \mu_1(K''_x, K''_y, K''_z) \exp[jK''_x \frac{L}{x_t} x'] \right. \\
\left. \exp[jK''_y y_b(\vec{r}, t) + jK''_z z_b(\vec{r}, t)] \exp[-j(K''_y \frac{L^2}{x_t^2} + K''_z^2) \frac{x'(x_t - x')} {2kx_t}] \ dx'dK''_y dK''_z - (-j)k \iiint \mu_1(K''_x, K''_y, K''_z) \exp[-jK''_x \frac{L}{x_t} x'] \right. \\
\left. \exp[-jK''_y y_b(\vec{r}, t) - jK''_z z_b(\vec{r}, t)] \exp[j(K''_y \frac{L^2}{x_t^2} + K''_z^2) \frac{x'(x_t - x')} {2kx_t}]dx'dK''_y dK''_z \right\} \\
\right. \\
\] (3.101)

Because \( n_1 \) is real, \( \mu_1(K''_x, K''_y, K''_z) \) which is the Fourier transform of \( n_1 \) satisfies the following relationship:

\[ \mu_1(K''_x, K''_y, K''_z) = \mu^*_1(-K''_x, -K''_y, -K''_z) \] (3.102)

Substituting Eq. (3.101) and then changing \( K''_x, K''_y, \) and \( K''_z \) to \( -K''_x, -K''_y, \) and \( -K''_z \) in the second integral of Eq. (3.101), Eq. (3.101) becomes

\[ S(\vec{r}, t) = -k \iiint \mu_1(K''_x, K''_y, K''_z) \exp[jK''_x \frac{L}{x_t} x'] \right. \\
\left. \exp[jy_b(\vec{r}, t)K''_y + jK''_z z_b(\vec{r}, t)] \right. \\
\left. \frac{x'(x_t - x')} {2kx_t} (K''_y^2 \frac{L^2}{x_t^2} + K''_z^2) \right. \\
\left. \right\} dx'dK''_y dK''_z \\
\right. \\
\]
\[
-k \int \int \int \mu_1(K''_x, K''_y, K''_z) \exp\left[jK''_y \frac{L}{x_1} x''\right] \\
\exp\left[jy_b(\bar{r}_1, t)K''_y + jK''_z z_b(\bar{r}_1, t)\right] \\
\frac{x'(x_t - x'')}{\cos\left[-\frac{2kx_t}{2} (K''_y \frac{L}{x_t} + K''_z)\right]} \, dx' \, dx'' \, dK''_y \, dK''_z
\]
(3.103)

Where we have used the following relationship
\[
\cos\theta = \frac{1}{2} [e^{i\theta} + e^{-i\theta}]
\]
(3.104)

Having obtained the phase \( S(\bar{r}, t) \), we go ahead and find the phase correlation function.

9 PHASE CORRELATION FUNCTION

In this section, we derive an expression for the phase correlation function. Using the expression for the phase, \( S \), derived in Eq.(3.103), the expression for the phase correlation function \( B_s(\bar{r}_1, \bar{r}_2, t_1, t_2) \) is then given by

\[
B_s(\bar{r}_1, \bar{r}_2, t_1, t_2) = \langle S(\bar{r}_1, t_1)S(\bar{r}_2, t_2) \rangle \\
= \frac{k^2}{2} \int_{t_1}^{t_2} \int_{t_1}^{t_2} \int \int \int \int < \mu_1(K''_x, K''_y, K''_z)u_1(K''_x, K''_y, K''_z)> \\
\exp[jy_b(\bar{r}_1, t_1) + jy_b(\bar{r}_2, t_2) + jK''_z z_b(\bar{r}_1, t_1) + jK''_z z_b(\bar{r}_2, t_2)] \\
\exp[jK''_y \frac{L}{x_1} x'' + jK''_y \frac{L}{x_2} x'''] \cos[-\frac{2kx_t}{2} (K''_y \frac{L}{x_2} + K''_z) x'''] (x_t - x''') \\
\cos[-\frac{2kx_t}{2} (K''_y \frac{L}{x_2} + K''_z) x'''] \, dx' \, dx''' \, dK''_y \, dK''_z \, dK''_z
\]
(3.105)
Next, we perform the $K''_x$, $K''_y$, and $K''_z$ integrations. In order to do that, we introduce the orthogonality relationship [92].

\[
\langle \mu_1(K''_x, K''_y, K''_z) \mu_1(K''_x, K''_y, K''_z) \rangle = \phi_n(K''_x, K''_y, K''_z) \delta(K''_x+K''_y) \delta(K''_y+K''_z) \delta(K''_z+K''_x)
\]

(3.106)

Substituting the expression in Eq.(3.106) into Eq.(3.105), we obtain

\[
B_s(\vec{r}_1, \vec{r}_2, t_1, t_2) = k^2 \int_{t_1}^{t_2} \int_{t_1}^{t_2} \int_{0}^{\infty} \int_{0}^{\infty} \phi_n(K''_x, K''_y, K''_z) \delta(K''_x+K''_y) \delta(K''_y+K''_z) \delta(K''_z+K''_x)
\]

\[
\exp[jK''_y y_b(\vec{r}_1, t_1) + jK''_y y_b(\vec{r}_2, t_2) + jK''_z z_b(\vec{r}_1, t_1) + jK''_z z_b(\vec{r}_2, t_2)]
\]

\[
\exp[jK''_x \left( \frac{L}{x_{t_1}} - x' \right) + jK''_x \left( \frac{L}{x_{t_2}} - x'' \right)]
\]

\[
\cos[\frac{2k}{L} \left( K''_x \frac{L}{x_{t_1}} + K''_y \right)]
\]

\[
\cos[\frac{2k}{L} \left( K''_x \frac{L}{x_{t_2}} + K''_y \right)] dx' dx'' dx'' dx'' dx'' dx'' dx'' dx'' dx'' dx'' dx'' dx'' dx'' dx'' dx'' dx'' dx'' dx'' dx'' dx''
\]

(3.107)

Performing the $K''_x$, $K''_y$, and $K''_z$ integrations, the following equation is obtained

\[
B_s(\vec{r}_1, \vec{r}_2, t_1, t_2) = k^2 \int_{t_1}^{t_2} \int_{t_1}^{t_2} \int_{0}^{\infty} \int_{0}^{\infty} \phi_n(K''_x, K''_y, K''_z)
\]

\[
\exp[jK''_y y_c(t_1, t_2) + jK''_z z_c(t_1, t_2)]
\]

\[
\exp[jK''_x \left( \frac{L}{x_{t_1}} - x' \right)] \cos[\left( \frac{K''_y L}{x_{t_1}} + K''_x \right)] dx' dx'' dx'' dx'' dx'' dx'' dx'' dx'' dx'' dx'' dx'' dx'' dx'' dx'' dx'' dx'' dx'' dx'' dx'' dx'' dx''
\]

(3.108)
where \( y_c \) and \( z_c \) are given by

\[
y_c = y_b(\bar{r}_1, x', t_1) - y_b(\bar{r}_2, x'', t_2)
\]

\[
= \nu \left( \frac{x'}{x_{t_1}} t_1 - \frac{x'''}{x_{t_2}} t_2 \right) - v_2 w(t_1 - t_2)
\]

\[
+ (y_1 \frac{L}{x_{t_1}} - y_2 \frac{L}{x_{t_2}}) - (y_1 \frac{Lx'}{x_{t_1}^2} - y_2 \frac{Lx'''}{x_{t_2}^2})
\]  

(3.109)

\[
z_c = z_b(\bar{r}_1, x', t_1) - z_b(\bar{r}_2, x'', t_2)
\]

\[
= -(x_1 - x_2) + \left( \frac{z_1 x'}{x_{t_1}} - \frac{z_2 x'''}{x_{t_2}} \right)
\]  

(3.110)

Next, we transform the rectangle integration region in the \( x' \), and \( x'' \) domain to a rhombus in the \( \eta \), and \( \zeta \) domain. The required transformation is given by the following equation:

\[
\zeta = \frac{x_{t_2}}{x_{t_1}} x' - x'''
\]  

(3.111)

\[
\eta = \frac{x_{t_2}}{2} \frac{x'}{x_{t_1}} + \frac{x'''}{2}
\]  

(3.112)

Solving Eq.(3.111) and Eq.(3.112) for \( x' \) and \( x''' \), we have

\[
x' = \frac{x_{t_1}}{x_{t_2}} \left( \eta + \frac{\zeta}{2} \right)
\]  

(3.113)

\[
x''' = \eta - \frac{\zeta}{2}
\]  

(3.114)
Using Eq. (3.113) and Eq. (3.114), the lines $x' = 0$, $x'' = 0$, $x' = x_{t_1}$, and $x''' = x_{t_2}$ are transformed respectively to $\eta + \frac{\zeta}{2} = 0$, $\eta - \frac{\zeta}{2} = 0$, $\eta + \frac{\zeta}{2} = x_{t_2}$, and $\eta - \frac{\zeta}{2} = x_{t_2}$. The two integration regions are also shown in Figures 13 and 14.

The Jacobian of transformation is given by

$$
\begin{vmatrix}
\frac{dx'}{d\eta} & \frac{dx'}{d\zeta} \\
\frac{dx''}{d\eta} & \frac{dx''}{d\zeta}
\end{vmatrix}
= \begin{vmatrix}
x_{t_1} & x_{t_1} \\
x_{t_2} & \frac{1}{2} x_{t_2}
\end{vmatrix}
= \frac{x_{t_1}}{x_{t_2}}.
$$

(3.115)

The differential area $dx'$ and $dx''$ in the $x'$ and $x''$ domain is then transformed to $d\eta$ and $d\zeta$ in the $\eta$ and $\zeta$ domain given by the following equation

$$
dx' dx'' = \frac{x_{t_1}}{x_{t_2}} d\eta \ d\zeta
$$

(3.116)
Figure 13. Rectangular Integration Region in $x'$ and $x'''$ Domain

Figure 14. Rhombus Integration Region in $\xi$ and $\eta$ Domain
Referring to Figure 14, the double integration with respect to \(x'\) and \(x''\) can then be written as

\[
\int \int dx' dx'' = \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dx'' \frac{x_{t_2}}{x_{t_1}} d\eta d\zeta
\]

\[
+ \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dx'' \frac{x_{t_2}}{x_{t_1}} d\eta d\zeta
\]

(3.117)

Using the same arguments given in Section 4 of Chapter 2, Eq.(3.117) is approximately given by the expression [90]

\[
\int \int dx' dx'' = \int_{-\infty}^{\infty} d\zeta \int_{-\infty}^{\infty} dx' \frac{x_{t_2}}{x_{t_1}} d\eta
\]

(3.118)

Substituting the integral in Eq.(3.118) and \(x', x''\) in Eq.(3.113) and Eq.(3.114) into Eq.(3.108), we then have

\[
B_s(\bar{r}_1, \bar{r}_2, t_1, t_2) = k^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi_n(K''', K'''', K''''') \exp[jK''''y_c(t_1, t_2) + jK''''z_c(t_1, t_2)] \exp[jK''''\frac{L}{x_{t_2}}] \]

\[
\cos[(\frac{y}{x_{t_1}} + K''')^2 + (\frac{z}{x_{t_2}} + K''')^2] \frac{x_{t_1}^2}{2k x_{t_2}}
\]

\[
\cos[(\frac{y}{x_{t_2}} + K''')^2 + (\frac{z}{x_{t_2}} + K''')^2] \frac{x_{t_2}^2}{2k x_{t_2}}
\]

(3.119)
where $y_c$ and $z_c$ are given by

$$y_c(t_1, t_2) = v\left(\frac{n}{x_{t_2}} - \frac{n}{x_{t_1}}\right) - v_{2w}(t_1-t_2) + (y_1 \frac{L}{x_{t_1}} - y_2 \frac{L}{x_{t_2}}) - (y_1 \frac{Ln}{x_{t_1}x_{t_2}} - y_2 \frac{Ln}{x_{t_2}})$$

(3.120)

$$z_c(t_1, t_2) = -(z_1-z_2) + (z_1 \frac{n}{x_{t_2}} - \frac{z_2n}{x_{t_2}})$$

(3.121)

Now, we perform the $K_x''$ and $\zeta$ integration in Eq.(3.119) using the following Fourier transform pair

$$F_n(K_y, K_z', \zeta) = \int_{-\infty}^{\infty} e^{jK_x'x} \phi_n(K_x, K_y, K_z') dK_x$$

(3.122)

$$\phi_n(K_x, K_y, K_z') = \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-jK_x'x} F_n(K_y, K_z', \zeta) d\zeta$$

(3.123)

Using Eq.(3.122) and Eq.(3.123), the following formula is obtained

$$\int jK_x'\left(\frac{L}{x_{t_2}}\zeta\right) d\zeta dK_x''$$

$$= \int F_n(K_y', K_z', \frac{L}{x_{t_2}}\zeta) d\zeta$$

$$= 2\pi \phi_n(0, K_y', K_z') \frac{x_{t_2}}{L}$$

(3.124)
Substituting the integral in Eq.(3.124) into Eq.(3.119), we then have

\[
B_s(\mathbf{r}_1, \mathbf{r}_2, t_1, t_2) = \frac{2\pi k^2}{L} \int_0^{X_t} \int_x \Phi_n(0, K_y', K_z') \exp[jK'_{y_c} + jK''z_c] \\
\cos[(\frac{v}{x_{t_1}} + K'_{z_1})\eta(x_{t_1} - \frac{\eta x_{t_1}}{x_{t_2}})/2kx_{t_2}] \\
\cos[(\frac{v}{x_{t_2}} + K''_{z_2})\eta(x_{t_2} - \eta)/2kx_{t_2}] \, d\eta \, dK''_{y} \, dK''_{z} (3.125)
\]

Eq.(3.125) is the desired expression for the phase correlation function. In the next section, we are going to derive the centroid angle of arrival covariance function.

10. Angle of Arrival Covariance Function

In this section, we derive an expression for the centroid angle of arrival covariance function defined as

\[
R_{\alpha}(t_1, t_2) = \langle \alpha_0(t_1) - \langle \alpha_0(t_1) \rangle \rangle \langle \alpha_0(t_2) - \langle \alpha_0(t_2) \rangle \rangle (3.126)
\]

To evaluate the expression in Eq.(3.126), we need the expected value of angle of arrival \( \langle \alpha_0(t) \rangle \). The expression for \( \langle \alpha_0(t) \rangle \) given in Eq.(2.26b) is rewritten here

\[
\langle \alpha_0(t) \rangle = \frac{1}{ka} \int \int \frac{\partial S_d(y, z, t)}{\partial y} \, dy \, dz (3.127)
\]

Where \( S_d \) is the total deterministic phase shift and \( \alpha_s = 0 \) for a tracking receiver. The total deterministic phase shift which is the imaginary part of \( \Phi_{1t} \) given in Eq.(3.73) is given by

\[
S_d = \frac{k^2}{\omega_0 t} \left( \frac{dx_t}{dt} - y \frac{d\theta}{dt} \right) (3.128)
\]
Substituting $S_d$ in Eq.(3.128) into Eq.(3.127), we have

\[
\langle \alpha_0(t) \rangle = \int \frac{k}{w_0 A} x_t \frac{\partial \theta}{\partial t} \; dydz
\]

Substituting $\langle \alpha_0(t) \rangle$ in Eq.(3.129a) into Eq.(3.126) and following the analysis similar to that in Chapter 2, we obtain the expression

\[
R_o(t_1, t_2) = \frac{1}{k^2 A} \frac{\partial^2 B_s(r_1, r_2, t_1, t_2)}{\partial y_1 \partial y_2} \; dr_1 dr_2
\]

Substituting $B_s$ from Eq.(3.125) into Eq.(3.129b) and performing the differentiations with respect to $y_1$ and $y_2$ we have

\[
R_o(t_1, t_2) = -\frac{2\pi}{L A^2} \int_0^t \int \int \int \int x_1 \phi(0, K_y', K_z')
\]

\[
\exp[j K''_y' y_c + j K''_z' z_c] K''_y (\frac{\partial y_c}{\partial y_1}) (\frac{\partial y_c}{\partial y_2})
\]

\[
\cos[(\frac{x_2}{x_1} + K''_z)\eta(x_1 - \frac{\eta x_1}{x_2})] / 2k x_2
\]

\[
\cos[(\frac{x_2}{x_1} + K''_z)\eta(x_2 - \eta) / 2k x_2]
\]

\[
d\eta dK''_y dK''_z dr_1 dr_2
\]

\[
\frac{\partial y_c}{\partial y_1} \text{ and } \frac{\partial y_c}{\partial y_2}
\]

can be computed using Eq.(3.120) to give

\[
\frac{\partial y_c}{\partial y_1} = \frac{L}{x_t} (1 - \frac{\eta}{x_t})
\]
\[
\frac{\partial y_c}{\partial y_2} = \frac{-L}{x_{t_2}}(1 - \frac{\eta}{x_{t_2}}) \quad (3.130c)
\]

Substituting \(\frac{\partial y_c}{\partial y_1}\) in Eq. (3.130b) and \(\frac{\partial y_c}{\partial y_2}\) in Eq. (3.130c) into Eq. (3.130a), we then have

\[
R_\Omega(t_1,t_2) = \frac{2\pi L}{A^2} \iint \Phi_n(K''_{y},K''_{z})(1 - \frac{\eta}{x_{t_2}})(1 - \frac{\eta}{x_{t_2}})K''_{y}^2 \exp[jK''_{y}y_c + jK''_{z}z_c] \cos[(\frac{y}{x_{t_1}} + K''_{z})\eta(x_{t_1} - \frac{\eta x_{t_1}}{x_{t_2}})/2kx_{t_2}]
\]

\[
\cos[(\frac{y}{x_{t_2}} + k''_{z})\eta(x_{t_2} - \eta)/2kx_{t_2}]
\]

\[
d\eta dK''_y dK''_z d\bar{r}_1 d\bar{r}_2 \quad (3.131)
\]

Next we perform the \(\bar{r}_1\) and \(\bar{r}_2\) integrations. This aperture integration can be performed as follows:

\[
\iint e^{jK''_{y}y_c + jK''_{z}z_c} d\bar{r}_1 d\bar{r}_2
\]

\[
\frac{1}{x_{t_1}} - \frac{\eta}{x_{t_2}})
\]

\[
\mathcal{J}_{y_1} \left( 1 - \frac{\eta}{x_{t_1}} \right) K''_{y} - jy_2 \frac{L}{x_{t_2}} (1 - \frac{\eta}{x_{t_2}}) K''_{y}
\]

\[
- jK''_{z_1} (1 - \frac{\eta}{x_{t_2}}) + jK''_{z_2} (1 - \frac{\eta}{x_{t_2}})
\]

\[
dy_1 dy_2 dz_1 dz_2
\]
\[ jK''y \left( \frac{n}{x_{t_2}} - \frac{n}{x_{t_2}} - jK''y_2 \omega(t_1 - t_2) \right) = e^{jK''y \left( \frac{n}{x_{t_2}} - \frac{n}{x_{t_2}} \right)} \]

\[
\left[ \int_{-a}^{a} \int_{-a}^{a} e^{jK''y \left( \frac{n}{x_{t_2}} - \frac{n}{x_{t_2}} \right)} y_{1} - jK''y \left( \frac{n}{x_{t_2}} - \frac{n}{x_{t_2}} \right) y_{2} \right] \text{dy}_{1} \text{dy}_{2} \]

\[
\left[ \int_{-a}^{a} \int_{-a}^{a} e^{jK''y \left( \frac{n}{x_{t_2}} - \frac{n}{x_{t_2}} \right)} - jK''z_{1} \left( \frac{n}{x_{t_2}} - \frac{n}{x_{t_2}} \right) + jK''z_{2} \left( \frac{n}{x_{t_2}} - \frac{n}{x_{t_2}} \right) \right] \text{dz}_{1} \text{dz}_{2} \]

Equation (3.132) can be evaluated rather easily using the following formula:

\[
\int_{-a}^{a} e^{j\alpha x} \text{dx} = \text{asinc}(\alpha a) \quad (3.133)
\]

Using Eq.(3.133) the following two integrals are obtained.

\[
\left[ \int_{-a}^{a} \int_{-a}^{a} e^{jK''y \left( \frac{n}{x_{t_2}} - \frac{n}{x_{t_2}} \right)} y_{1} - jK''y \left( \frac{n}{x_{t_2}} - \frac{n}{x_{t_2}} \right) y_{2} \right] \text{dy}_{1} \text{dy}_{2} \]

\[
\left[ \int_{-a}^{a} \int_{-a}^{a} e^{jK''y \left( \frac{n}{x_{t_2}} - \frac{n}{x_{t_2}} \right)} - jK''z_{1} \left( \frac{n}{x_{t_2}} - \frac{n}{x_{t_2}} \right) + jK''z_{2} \left( \frac{n}{x_{t_2}} - \frac{n}{x_{t_2}} \right) \right] \text{dz}_{1} \text{dz}_{2} \]

\[
= a^2 \text{sinc} \left[ \frac{aK''}{y} \left( \frac{n}{x_{t_2}} - \frac{n}{x_{t_2}} \right) \right] \text{sinc} \left[ \frac{aK''}{y} \left( \frac{n}{x_{t_2}} - \frac{n}{x_{t_2}} \right) \right] \quad (3.134)
\]

\[
= a^2 \text{sinc} \left[ \frac{aK''}{x_{t_2}} \left( \frac{n}{x_{t_2}} - \frac{n}{x_{t_2}} \right) \right] \text{sinc} \left[ \frac{aK''}{x_{t_2}} \left( \frac{n}{x_{t_2}} - \frac{n}{x_{t_2}} \right) \right] \quad (3.135)
\]
Substituting the results from Eq. (3.134) and Eq. (3.135) into Eq. (3.132), we then have

\[
\int_{-a}^{a} \int_{-a}^{a} \exp[jK''y_c + jK''z_c] \, dr_1 \, dr_2 = a^4 \exp[jK''y\left(\frac{\eta}{x_{t_1}} - \frac{\eta}{x_{t_2}}\right) - jK''y z_w(t_1-t_2)]
\]

\[
\text{sinc}\left[\frac{a y}{x_{t_1}} \left(1 - \frac{\eta}{x_{t_2}}\right)\right] \text{sinc}\left[\frac{a y}{x_{t_2}} \left(1 - \frac{\eta}{x_{t_1}}\right)\right] \text{sinc}\left[\frac{a z}{x_{t_2}} \left(1 - \frac{\eta}{x_{t_1}}\right)\right] \text{sinc}\left[\frac{a z}{x_{t_1}} \left(1 - \frac{\eta}{x_{t_2}}\right)\right]
\]

(3.136)

Further substituting the integral in Eq. (3.136) into Eq. (3.131), we then have

\[
R_q(t_1, t_2) = \frac{2\pi L a^4}{A^2} \iint \frac{1}{x_{t_1}} K''^2(1 - \frac{\eta}{x_{t_2}}) (1 - \frac{\eta}{x_{t_1}}) \Phi_n(K''y, K''z)
\]

\[
\exp[jK''y\left(\frac{\eta}{x_{t_1}} - \frac{\eta}{x_{t_2}}\right) - jK''y z_w(t_1-t_2)]
\]

\[
\text{sinc}\left[\frac{a y}{x_{t_1}} \left(1 - \frac{\eta}{x_{t_2}}\right)\right] \text{sinc}\left[\frac{a y}{x_{t_2}} \left(1 - \frac{\eta}{x_{t_1}}\right)\right] \text{sinc}\left[\frac{a z}{x_{t_2}} \left(1 - \frac{\eta}{x_{t_1}}\right)\right] \text{sinc}\left[\frac{a z}{x_{t_1}} \left(1 - \frac{\eta}{x_{t_2}}\right)\right]
\]

\[
\text{sinc}\left[\frac{a y}{x_{t_1}} \left(1 - \frac{\eta}{x_{t_2}}\right)\right] \text{sinc}\left[\frac{a z}{x_{t_2}} \left(1 - \frac{\eta}{x_{t_1}}\right)\right]
\]

\[
K''^2 e^{\frac{\eta x_{t_1}}{2k x_{t_2}}} \cos\left[\frac{K''^2}{x_{t_1}} + K''^2 \eta(x_{t_1} - \frac{\eta}{x_{t_2}})/2k x_{t_2}\right] \, d\eta \, dK''y \, dK''z
\]

(3.137)
Equation (3.137) is the desired expression for the angle of arrival covariance function. In the next section, we are going to compute angle of arrival power spectrum.

11 ANGLE OF ARRIVAL TEMPORAL POWER SPECTRUM

The angle of arrival temporal power spectrum is defined to be the Fourier transform of the covariance function given in Eq.(3.137).

\[
W_\phi(\omega) = \int_{-\infty}^{\infty} R_\phi(t_1,t_2) e^{-j\omega \tau} d\tau
\]

where \( \tau = t_1 - t_2 \) is the time difference variable.

Substituting \( R_\phi(t_1,t_2) \) in Eq.(3.137) into Eq.(3.138) we then have

\[
W_\phi(\omega) = \frac{2\pi\text{La}^4}{A^2} \int \int \int \frac{1}{x_{t_2}} k''^2 (1-\frac{\eta}{x_{t_2}})(1-\frac{\eta}{x_{t_2}})^2 \phi_{n}(K''y,K'')
\]

\[
\times \exp[jk''y\left(\frac{\eta}{x_{t_2}} - \frac{\eta}{x_{t_2}}\right) - jK''y_2\omega t - j\omega \tau]
\]

\[
sinc[\frac{a}{z}k''y\left(\frac{x_{t_1}}{x_{t_2}} - \frac{x_{t_1}}{x_{t_2}}\right)]
\]

\[
sinc[\frac{a}{z}k''y\left(\frac{x_{t_2}}{x_{t_2}} - \frac{x_{t_2}}{x_{t_2}}\right)]sinc[\frac{a}{z}k''(1-\frac{\eta}{x_{t_2}})]
\]

\[
sinc[\frac{a}{z}k''(1-\frac{\eta}{x_{t_2}})]
\]

\[
\cos\left[\left(\frac{y}{x_{t_1}} + \frac{K''^2z}{x_{t_2}}\right)\eta(x_{t_2} - \frac{\eta}{x_{t_2}})/2kx_{t_2}\right]
\]

\[
\cos\left[\left(\frac{y}{x_{t_2}} + \frac{K''^2z}{x_{t_2}}\right)\eta(x_{t_2} - \eta)/2kx_{t_2}\right]\right]d\eta d\omega dK''
\]

(3.139)
Equation (3.139) is the expression for the angle of arrival power spectrum. \( x_{t_1} \) and \( x_{t_2} \) are the ranges at time \( t_1 \) and time \( t_2 \). They can also be written in terms of the time sum variable \( \delta \) and time difference variable \( \tau \). \( \tau \) and \( \delta \) are defined by

\[
\tau = t_1 - t_2 \quad \text{(3.140)}
\]
\[
\delta = \frac{1}{2}(t_1 + t_2) \quad \text{(3.141)}
\]

In terms of \( \tau \) and \( \delta \), \( x_{t_1} \) and \( x_{t_2} \) are written as

\[
x_{t_1} = \sqrt{L^2 + v^2 \delta^2}
\]
\[
= \sqrt{L^2 + v^2 (\delta - \frac{1}{2} \tau)^2} \quad \text{(3.142)}
\]
\[
x_{t_2} = \sqrt{L^2 + v^2 \tau^2}
\]
\[
= \sqrt{L^2 + v^2 (\delta + \frac{1}{2} \tau)^2} \quad \text{(3.143)}
\]

The angle of arrival power spectrum thus depends in a quite complicated manner on \( \tau \) which renders any further simplifications quite difficult. However, the propagation range maintains nearly constant value within the correlation time of interest. We thus make a Taylor series expansion of \( x_{t_1} \) and \( x_{t_2} \) versus \( \tau \) and get

\[
x_{t_1} = \sqrt{(L^2 + v^2 \delta^2) + v^2 (\delta \tau + \frac{1}{4} \tau^2)}
\]
\[
= \sqrt{L^2 + v^2 \delta^2} \left[ 1 + \frac{1}{2} \frac{v^2 (\delta \tau + \frac{1}{4} \tau^2)}{L^2 + v^2 \delta^2} + \cdots \right] \quad \text{(3.144)}
\]
\[
x_{t_2} = \sqrt{(L^2 + v^2 \delta^2) + v^2 (\delta \tau - \frac{1}{4} \tau^2)}
\]
\[
= \sqrt{L^2 + v^2 \delta^2} \left[ 1 - \frac{1}{2} \frac{v^2 (\delta \tau - \frac{1}{4} \tau^2)}{L^2 + v^2 \delta^2} + \cdots \right] \quad \text{(3.145)}
\]
The zero order term of \( x_{t_1} \) and \( x_{t_2} \) is called \( x_0 \) and is given by

\[
x_0 = \sqrt{L^2 + \nu^2 \delta^2} \tag{3.146}
\]

If \( x_{t_1} \) and \( x_{t_2} \) can be approximated by \( x_0 \), the integral can be significantly simplified.

That \( x_{t_1} \) and \( x_{t_2} \) can be approximated by \( x_0 \) is justified in Appendix G. There we first show that within the correlation time of interest, the following condition is satisfied.

\[
\frac{x_{t_1} - x_{t_2}}{x_0} \ll 1 \tag{3.147}
\]

Next, we make a Taylor series expansion of the power spectrum versus \( \Delta = x_{t_1} - x_{t_2} \). The same condition given by Eq. (3.147) is obtained in order for the second order term to be neglected.

Making the approximation of \( x_{t_1} \) and \( x_{t_2} \) by \( x_0 \) and replacing the first two sinc(X) functions of Eq. (3.139) by \( \sin(X)/X \), we obtain

\[
W_q(\omega, \delta) = \frac{8\pi x_0 a^2}{L A^2} \int \int \int \phi_n(K''', K'') \exp[j(K'' y x_0 - K'' y_2w - \omega t)]
\]

\[
\{\sin[\frac{aK''}{2y} x_0 (1 - \eta_0)]\}^2 \{\sin[\frac{aK''}{2z} (1 - \eta_0)]\}^2
\]

\[
\cos^2[(-\frac{\nu}{x_0} + K''^2 y) \eta(x_0 - \eta)/2kx_0]
\]

\[
d\eta d\delta dK'' dK'''
\]

(3.148)
The $\tau$ integration in Eq.(3.148) is next performed giving

$$
\int_{-\infty}^{\infty} e^{j(K''\frac{\eta}{y} x_0 - K'' v_z - w)\tau} d\tau
$$

$$
= 2\pi\delta[K''\frac{\eta}{y} x_0 - K'' v_z - w]
$$

$$
= 2\pi\delta[K''[v \frac{\eta}{y} x_0 - v_z] - w]
$$

$$
= 2\pi\delta[[v \frac{\eta}{y} x_0 - v_z][K'' - \frac{w}{v \frac{\eta}{y} x_0 - v_z}]]
$$

$$
= \frac{2\pi}{|v \frac{\eta}{y} x_0 - v_z|} \delta[K'' - \frac{w}{v \frac{\eta}{y} x_0 - v_z}]
$$

(3.149)

Substituting the integral in Eq.(3.149) into Eq.(3.148), we then have

$$
W_A(w,\delta) = \frac{16\pi^2 x_0 a^2}{L A^2} \iiint \frac{\phi(K'' y, K'')}{|v \frac{\eta}{y} x_0 - v_z|} \delta[K'' - \frac{w}{v \frac{\eta}{y} x_0 - v_z}]
$$

$$
\{\sin[\frac{a}{2} K' y x_0 (1 - \frac{\eta}{x_0})]\}^2 \{\text{sinc}[\frac{a}{2} K'' z (1 - \frac{\eta}{x_0})]\}^2
$$

$$
\cos^2[(\frac{K'' y}{x_0 z} + K'' z)\eta(x_0 - \eta)/2k x_0]
$$

$$
d\eta dK'' dK'' (3.150)
$$
Next, the \( K''_y \) integration in Eq. (3.150) is computed using the following formula

\[
\int f(x) \delta(x-x_0) \, dx = f(x_0) \quad (3.151)
\]

Performing the \( K''_y \) integration in Eq. (3.150), we then have

\[
W_0(w, \delta) = \frac{16\pi^2}{L A^2} \int_0^\infty \phi_n \left( -\frac{w}{\sqrt{\frac{\eta}{x_0} - v_{2w}}} \right) \left[ \frac{\eta}{x_0} - v_{2w} \right]^{-1} \left[ \frac{\eta}{x_0} - v_{2w} \right]^2 \sin \left( \frac{a}{2} \right) \left( \frac{w}{x_0} - v_{2w} \right) \left[ \frac{\eta}{x_0} - v_{2w} \right]^2 \cos^2 \left( \frac{L^2}{x_0} \right) \left[ \frac{\eta}{x_0} - v_{2w} \right]^2 \left( \frac{\eta}{x_0} - v_{2w} \right)^2 \right] \frac{\eta(x_0 - \eta)}{2kx_0} \, d\eta dK''_z \quad (3.152)
\]

Equation (3.152) will be evaluated numerically. Before doing that, we first define the refractive index power spectrum \( \Phi_n \) and related parameters and then simplify the expression by transforming to reduced variables. We shall use the von Kármán power spectrum for \( \Phi_n \) which is given by [87]

\[
\Phi_n = \frac{0.033C_n^2(h)}{[K''_y + K''_z + L_0^2(h)]^{11/6}} \quad (3.153)
\]

where \( C_n^2(h) \) is the structure constant and \( L_0(h) \) is the outer scale of turbulence. Both \( C_n^2(h) \) and \( L_0(h) \) are functions of vertical height and are modeled by the following equations [68, 75]
\[ C_n^2(h) = C_n^2(H_o) \left( \frac{h}{H_o} \right)^{-4/3} e^{-h/5940} \]  
\[ L_o(h) = L_o(H_o) \left( \frac{h}{H_o} \right) \]  

where \( H_o \) is the vertical height of the receiver. The exponential decay is necessary in Eq.(3.154) in order to explain the negligible atmospheric turbulence at height close to satellite range. Actually at height greater than 5940m, the intensity of refractive index fluctuations significantly decreases.

Next, we define several normalized variables. The normalized variable \( \sigma \) is defined so that range is measured from the transmitter. The \( \sigma \) thus defined has value 0 at the transmitter and value 1 at the receiver, \( \sigma \) is given by

\[ \sigma = 1 - \frac{h}{x_0} \]  

Defining \( H_L \) to be the vertical height of the source, the vertical range \( \text{I}_h \) is then given by

\[ \text{I}_h = H_L - H_o \]  

\( K_m \) is defined to be the normalized spatial frequency and is given by

\[ K_m = K_m'' \text{L}_h \]
is defined to be the normalized temporal frequency and is given by

\[ \Omega_n = \frac{\omega L_h}{v} \]  

is also the ratio of \(2\pi\) times the vertical range \(L_h\) versus the distance the source moves in one period corresponding to the temporal frequency \(\Omega_n\). \(R_a\) is defined to be the ratio of vertical propagation range versus vertical receiver height and is given by

\[ R_a = \frac{H_L - H_o}{H_o} = \frac{L_h}{H_o} \]  

\(v_n\) is defined to be the ratio of relative wind velocity versus relative source velocity and is given by

\[ v_n = \frac{v_{2w}}{v} \]  

\(\theta_o\) is defined to be the tracking angle and is given by

\[ \cos \theta_o = \frac{L}{x_o} \]  

\(W_n\) is defined to be the normalization constant for the power spectrum and is given by

\[ W_n = \frac{\omega \cos^2 \theta_o |v| a^2}{16\pi^2 (0.033) C_n^2 (H_o) H_o^{4/3} L_h^{4/3} L} \]  

Now, we compute \(C_n^2(h)\), \(I_o(h)\) and \(\Phi_n\) using the previously defined parameters. In terms of \(\sigma\), \(h\) is given by

\[ h = H_o + (1 - \sigma)(H_L - H_o) \]  

\[ = H_o + (1 - \sigma)L_h \]
Using the $h$ just defined, $C^2_n(h)$ and $L_o(h)$ are functions only of $\sigma$ and are renamed as $C^2_n(\sigma)$ and $L_o(\sigma)$. They are given by

\[
C^2_n(\sigma) = C^2_n(H_0) R-a^{-4/3} (1 - \sigma + \frac{1}{R-a})^{-4/3} e^{-L_h(1 - \sigma + \frac{1}{R_a})/5940} \tag{3.165}
\]

\[
L_o(\sigma) = L_o(H_0) R-a (1 - \sigma + \frac{1}{R-a}) \tag{3.166}
\]

We shall use the model that outer scale is equal to the vertical propagation height. In that case, $L_o(H_0) = H_0$ and $L_o(\sigma)$ in Eq.(3.165) is given by

\[
L_o(\sigma) = L_h(1 - \sigma + \frac{1}{R_a}) \tag{3.167}
\]

Using the normalized variables defined before, the von Kármán power spectrum $\Phi_n$ in Eq.(3.152) then becomes

\[
\Phi_n \left[ \frac{\omega}{v_n} , K_m \right]_{x}^{v_n} = \Phi_n \left[ \frac{\omega}{v(1 - \sigma - v_n)} , K_m \right]_{L_h}^{L_h}
\]

\[
= 0.033 C^2_n(\sigma) \left[ \frac{\omega^2}{v^2(1 - \sigma - v_n)^z} + \frac{K_m^2}{L_h^2} + L_o^{-2}(\sigma) \right]^{-11/6}
\]

\[
= 0.033 C^2_n(\sigma)L_h^{11/3} \left[ \frac{\Omega_n^2}{(1 - \sigma - v_n)^z} + K_m^2 + (1 - \sigma + \frac{1}{R_a})^{-2} \right]^{-11/6}
\]  

\[(3.168)\]
Substituting $\Phi_n$ in Eq.(3.168) into Eq.(3.152) and using previously defined normalized variables $K_m, \sigma, \Omega_n, \mathcal{W}_n$, we then have

$$\mathcal{W}_n = \int_0^\infty \int_0^1 \frac{1}{\sqrt{|1-\sigma v_n|}} (1-\sigma + \frac{1}{R_a})^{-4/3}$$

$$\{K_m^2 + \frac{\Omega_n^2}{(1-\sigma v_n)^2} + (1-\sigma + \frac{1}{R_a})^{-2}\}^{-11/6}$$

$$\Omega_n \cos \theta_0 \sigma \frac{a}{2(1-\sigma v_n) L_h} \left[ \text{sinc}\left(\frac{K_m a}{2L_h}\right) \right]^2$$

$$\cos^2 \Omega_n \frac{L_m (1-\sigma) \sec \theta_0}{2k L_h^2}$$

Equation (3.169) is the final normalized angle of arrival power spectrum.

Another quantity of interest is the differential path contribution. The differential path contribution gives the contribution to the power spectrum as a function of range along the propagation path. Let $D_n$ represent the differential path contribution, then $D_n$ is defined by

$$\mathcal{W}_n = \int D_n(\Omega_n, \sigma) d\sigma$$

where

$$D_n(\sigma, \Omega_n) = \int_0^\infty \frac{1}{\sqrt{|1-\sigma v_n|}} \{K_m^2 + \frac{\Omega_n^2}{(1-\sigma v_n)^2} + (1-\sigma + \frac{1}{R_a})^{-2}\}^{-11/6}$$

$$\Omega_n \cos \theta_0 \sigma \frac{a}{2(1-\sigma v_n) L_h} \left[ \text{sinc}\left(\frac{K_m a}{2L_h}\right) \right]^2$$

$$\cos^2 \Omega_n (1-\sigma v_n)^2 + \frac{L_m (1-\sigma) \sec \theta_0}{2k L_h^2} \} dK_m$$

In the next section, the numerical integration results are given.
12. NUMERICAL INTEGRATION RESULTS

In this section, we present the numerical integration results. There are two quantities of interest; one is the differential path contribution, the other is power spectrum itself. We consider wave propagation from moving source to ground. We assume also $L$ is along vertical direction and $L_h = L$. For differential path contribution, we consider ranges at 1000m and $10^5$m and $\Omega_n$ from 1 to 1000. For the power spectrum, we consider ranges from 1km to $10^5$m and tracking angles at 45°, 60° and 75°. Further a discussion of the numerical integration procedure is given and explanation of the curve is presented.

Gaussian quadrature algorithm is used to compute both $D_n(\sigma,\Omega_n)$ and $W_n(\Omega_n)$[86]. The integration is done so that each integration division length is much less than the possible oscillation periods of the sine, cosine and sinc function of Eq.(3.169). The oscillation period of cosine function is much greater than that of the sinc function so only the latter will be considered. The sinc function has a period greater than 0.1L. From Eq.(3.169), the von Karman spectrum drops at $K_m$ greater than

$$
[\Omega_n^2/(1-\sigma-v_n)^2 + (1-\sigma + 1/R_a)^{-2}]^{0.5}
$$

The smaller of 0.1L and

$$
0.1[\Omega_n^2/(1-\sigma-v_n)^2 + (1-\sigma + 1/R_a)^{-2}]^{0.5}
$$
is chosen initially as the integration division length. Later, this length is compared to \( \pi \) and the minimum integration length is set equal to \( \pi \) which is always smaller than the possible oscillation period (0.1L). Each length is then integrated using 16 point Gaussian quadrature algorithm. The choice of 16 points is because integration length is guaranteed to be smaller than 1/4 of the possible oscillation period and the previous experience justifies its applicability. The integration then proceeds one length after another until the one division length integration result is less than \( 10^{-5} \) of previous sum. Because of the use of normalized variables defined in the previous section, the oscillation period is approximately \( K_m > 2L \) for sinc function and \( K_m > \sqrt{2kL} \) for cosine function. The oscillation period is so large that the numerical integration of Eq. (3.169) can be done quite easily.

Figure 15 to Figure 18 show log-log plots of differential path contribution versus normalized range \( \sigma \) for \( \Omega_n \) from 1 to 1000 at \( L_h = 1000 \text{m}, \theta_o = 45^\circ, v_n = 0.001 \) and \( H_o = 10 \text{m} \). Figure 19 to Figure 22 show corresponding plots of \( D_n(\sigma) \) using log-linear scale. Figure 23 shows the log-log plot of \( D_n(\sigma) \) versus \( \sigma \) at \( L_h = 10^5 \text{m} \). The source is toward the left in these plots and the receiver is at the right. As will be shown next, the differential path contribution normally increases monotonically with \( \sigma \) for small \( \Omega_n \) and starts oscillating near the receiver at large \( \Omega_n \). At large \( L_h, (=10^5 \text{m}) \), the differential path contribution significantly decreases near the transmitter due to the exponential decrease in \( C_n^2 \) with height.
Figure 15. Differential Path Contribution on a Log-Log Scale with $\Omega_n = 1.0$ and $L = 1000.0\text{m}$ in a Tracking System
Figure 16. Differential Path Contribution on a Log-Log Scale with $\Omega_n = 10.0$ and $L = 1000.0\text{m}$ in a Tracking System
Figure 17. Differential Path Contribution on a Log-Log Scale with $\Omega_n = 100.0$ and $L = 1000.0m$ in a Tracking System
Figure 18. Differential Path Contribution on a Log-Log Scale
with $\Omega_n = 1000.0$ and $L = 1000.0m$ in a Tracking System
Figure 19. Differential Path Contribution on a Log-Linear Scale with $\Omega_n = 1.0$ and $L = 1000.0 \text{m}$ in a Tracking System
Figure 20. Differential Path Contribution on a Log-Linear Scale with $\Omega_n = 10.0$ and $L = 1000.0$ in a Tracking System
Figure 21. Differential Path Contribution on a Log-Linear Scale with $\Omega_n = 100.0$ and $L = 1000.0\,\text{m}$ in a Tracking System
Figure 22. Differential Path Contribution on a Log-Linear Scale with $\Omega_n = 1000.0$ and $L = 1000.0m$ in a Tracking System
Figure 23. Differential Path Contribution on a Log-Log Scale with $\Omega_n = 1.0$ and $L = 10^5$m in a Tracking System
Now we are going to examine the plots of differential path contribution, \( D_n(\sigma) \). We first examine \( D_n(\sigma) \) versus \( \sigma \) for small \( \Omega_n \). The log-log plot of \( D_n(\sigma) \) shows that \( D_n(\sigma) \) increases with \( \sigma \) initially with a slope 2. This can be explained. To explain this we note that for very small \( \sigma \), \( 1-\sigma \approx 1 \), Eq.(3.171) then becomes

\[
D_n(\sigma) \approx \int_0^\infty \frac{(1+\frac{1}{R_a})^{-4/3}}{|1-\nu_n|} \left\{ K_m^2 + \frac{\Omega_n^2}{(1-\nu_n)^2} + (\frac{1}{R_a})^{-2} \right\}^{-11/6} \sin[\frac{\Omega_n \cos \theta_0 \sigma}{2(1-\nu_n)L_h}] \left[ \text{sinc}(\frac{K_m \sigma}{2L_h}) \right]^2 \cos^2[\Omega_n \cos \theta_0 \sigma (1-\nu_n)^2 + K_m^2] \sigma \sec \theta_0 \cdot \text{d}K_m \quad (3.172)
\]

Also, for small \( \sigma \) such that the argument of sine and sinc function is smaller than \( \pi \), the following approximations hold.

\[
\sin[\frac{\Omega_n \cos \theta_0 \sigma}{2(1-\nu_n)L_h}] \approx \frac{\Omega_n \cos \theta_0 \sigma}{2(1-\nu_n)L_h} \quad (3.173)
\]

\[
\left[ \text{sinc}(\frac{K_m \sigma}{2L_h}) \right]^2 \approx 1 \quad (3.174)
\]

Substituting the expressions in Eq.(3.173) and Eq.(3.174) into Eq.(3.172), the differential path contribution \( D_n(\sigma) \) becomes

\[
D_n(\sigma) \approx \frac{1}{|1-\nu_n|} \left[ \frac{\Omega_n^2 \cos \theta_0 \sigma}{2(1-\nu_n)L_h} \right]^2 \sigma \int_0^\infty \left\{ K_m^2 + \frac{\Omega_n^2}{(1-\nu_n)^2} + (\frac{1}{R_a})^{-2} \right\}^{-11/6} \text{d}K_m \quad (3.175)
\]

This explains \( D_n(\sigma) \propto \sigma^2 \) for very small \( \sigma \). Also, for small \( \Omega_n \), the plot of \( D_n(\sigma) \) shows that \( D_n(\sigma) \) increases with \( \sigma \) along the whole propagation path.
Now we examine $D_n(\sigma)$ versus $\sigma$ for large $\Omega_n$. For large $\Omega_n$ such as $\Omega_n = 1000$, the plot of $D_n(\sigma)$ shows $D_n(\sigma)$ still increases with $\sigma$ initially and then starts oscillating beyond certain range of $\sigma$. The source of oscillation is the sine function in Eq.(3.171). The oscillation point can be predicted from that equation by setting the argument of sine function to be equal to $\frac{\pi}{2}$. The results gives

$$\sigma = \frac{\pi L_h}{\pi L_h + a n \Omega_n \cos \theta_o} \quad (3.176)$$

For $L_h = 1000 m$, $\Omega_n = 1000$, $\theta_o = 45$, $a = 1m$ the above equation gives $\sigma = 0.81$. For $\Omega_n = 100$, the above equation gives $\sigma = 0.97$. These $\sigma$ values check with the curve shown in Figure 17 and Figure 18. For $\Omega_n$ smaller than 10, the above equation gives $\sigma = 1$ and hence no oscillation occurs.

Now we examine the differential path contribution $D_n(\sigma)$ versus $\sigma$ for very long propagation distance such as $L_h = 10^5 m$. The differential path contribution peaks at the receiver ($\sigma = 1$) and gradually decreases toward the transmitter. This indicates that the propagation near the transmitter or the upper atmosphere contributes negligibly to the power spectrum. Physically there are two pictures which all contribute to these phenomena. First, as the point along propagation range becomes higher and higher, the air density becomes thinner and thinner and the effect of atmospheric turbulence is small. The exponential decay of the $C_n^2$ model has implicitly taken this into account. Second, the equivalent velocity is maximum at the transmitter and linearly decreases toward the receiver. This can also be explained from the sine function given in Eq.(3.171). That sine function says the following:
\[
\sin \left( \frac{\Omega_n \sigma}{(1-\sigma)L_h^2} \right) \alpha \sin \frac{\omega \sigma a}{2v(1-\sigma)}
\]

The differential path contribution is plotted using normalized frequency \( \Omega_n \). For \( L_h = 10^5 \text{m} \), the argument of sine is negligibly small for almost all the range and \( \sin(\Omega_n \sigma)/2(1-\sigma)L_h \alpha \alpha \frac{\Omega_n \sigma}{(1-\sigma)L_h^2} a \). This explains that \( D_n(\sigma) \) increases toward the receiver also. Hence, the contribution to the power spectrum increases toward the receiver for a fixed \( \Omega_n \).

Now we examine the power spectrum \( \tilde{W}_n \) as a function of range \( L_h \) and tracking angle \( \theta_o \). Figures 24 to Figures 28 show \( \tilde{W}_n \) for vertical propagation height ranging from 1000m to 8000m. Figure 29 shows \( \tilde{W}_n \) for \( \theta_o = 60^\circ \) and Figure 30 shows \( \tilde{W}_n \) for \( \theta_o = 75^\circ \) at range equal to 1000m. Figure 31 shows \( \tilde{W}_n \) at range \( 10^5 \text{m} \). All these curves follow a similar trend with an initial increasing slope and a later decreasing slope with the peak at \( \Omega_n \) somewhere between 0.1 and 1. This suggests that the peak normalized frequency is not a strong function of propagation height.

This is because the peak is determined by outer scale and outer scale varies with height according to \( 1-\sigma + \frac{1}{R_a} \). The propagation range normally is greater than 1km and \( R_a \) is greater than 100 for \( H_o = 10\text{m} \). Thus, \( \frac{1}{R_a} \) is negligibly small for almost all ranges under consideration. This explains the peak normalized frequency is quite insensitive to range variation.
Figure 24. Normalized Power Spectrum \( W_n \) Versus \( \Omega \) at \( L = 1000.0m \) and \( \theta_0 = 45^\circ \) in a Tracking System
Figure 25. Normalized Power Spectrum ($\tilde{W}_n$) Versus $\Omega_n$ at $L = 2000.0$ m and $\theta_0 = 45^\circ$ in a Tracking System
Figure 26. Normalized Power Spectrum ($W_n$) Versus $\Omega$ at $L = 4000.0$ m and $\theta_0 = 45^\circ$ in a Tracking System.
Figure 27. Normalized Power Spectrum ($W_n$) Versus $\Omega_n$ at $L = 6000.0$ m and $\theta_0 = 45^\circ$ in a Tracking System
Figure 28. Normalized Power Spectrum ($W_n$) Versus $\Omega$ at $L = 8000.0$ m and $\theta_0 = 45^\circ$ in a Tracking System
Figure 29. Normalized Power Spectrum \( W_n \) Versus \( \Omega_n \) at \( L = 1000.0 \text{m} \) and \( \theta_0 = 60^\circ \) in a Tracking System
Figure 30. Normalized Power Spectrum ($W_n$) Versus $\Omega_n$ at $L = 1000.0$ m and $\theta_0 = 75^\circ$ in a Tracking System
Figure 31. Normalized Power Spectrum \( (W_n) \) Versus \( \Omega \) at \( L = 10^5 m \) and \( \theta_0 = 45^\circ \) in a Tracking System.
Now, we examine the asymptotic behavior of the power spectrum $W_n$ versus $\Omega_n$. The Log-Log plot of the power spectrum increases with $\Omega_n$ initially with a slope 2 until it reaches a maximum. Because for small $\Omega_n$, the following approximations are true.

\[ [K_m^2 + \frac{\Omega_n^2}{(1-\sigma-v_n)^2} + (1-\sigma+\frac{1}{R_a})^{-2}]^{-11/6} \approx [K_m^2 + (1-\sigma+\frac{1}{R_a})^{-2}]^{-11/6} \] (3.177)

\[ \sin^2 \frac{\Omega_n \cos \theta_0 a}{2(1-\sigma-v_n) L_h} \approx \frac{\Omega_n^2 \cos^2 \theta_0 \sigma^2 a^2}{2(1-\sigma-v_n)^2 L_h^2} \] (3.178)

Equation (3.177) is true for $K_m < 1$ and (3.178) is true for the argument of sine function smaller than $\pi$. Substituting Eq.(3.177) and Eq.(3.178) into Eq.(3.169) we have $W_n \propto \Omega_n^2$.

When the power spectrum reaches a maximum, it drops initially with a -2/3 slope. This behavior can also be predicted again using the following approximations:

\[ \{K_m^2 + \frac{\Omega_n^2}{(1-\sigma-v_n)^2} + (1-\sigma+\frac{1}{R_a})^{-2}\}^{-11/6} \approx \{K_m^2 + \frac{\Omega_n^2}{(1-\sigma-v_n)^2}\}^{-11/6} \] (3.179)

\[ \sin^2 \left[ \frac{\Omega_n \cos \theta_0 a}{2(1-\sigma-v_n) L_h} \right] \approx \frac{\Omega_n^2 \cos^2 \theta_0 \sigma^2 a^2}{4(1-\sigma-v_n)^2 L_h^2} \] (3.180)

Setting $K_m = \frac{\Omega_n}{1-\sigma-v_n} \tan \theta$, Eq.(3.179) then becomes

\[ \{K_m^2 + \frac{\Omega_n^2}{(1-\sigma-v_n)^2} + (1-\sigma+\frac{1}{R_a})^{-2}\}^{-11/6} \approx \frac{\Omega_n^{-11/3}}{(1-\sigma-v_n)^{-11/3}} (\sec \theta)^{-11/3} (3.181) \]
Substituting the expressions in Eq. (3.180) and (3.181) into Eq. (3.169) we have

\[
W_n \propto \left( \frac{1}{\sigma^2} \right)^{4/3} \int_0^{\pi/2} \int_0^{\pi/2} \frac{(1-\sigma R)^{-4/3}}{|1-\sigma v_n|} \frac{(\sec \theta)^{-11/3}}{(1-\sigma v_n)^{-11/3}} \Omega_n^{-11/3} \Omega_n^2
\]

\[
\cos^2 \theta_0 \sigma^2 a^2 s \left[ \sin \left( \frac{2}{2L_h} \right) \right]
\]

\[
\Omega_n^{-11/3} \Omega_n^2 \Omega_n = \Omega_n^{-2/3} \quad (3.182)
\]

As \( \Omega_n \) continues increasing, the sine function can no longer be approximated by its argument and the power spectrum starts dropping with a gradually increasing slope. This is because refractive index fluctuation plays a dominant role in this region. The refractive index fluctuation spectrum given by the von Kármán power spectrum decreases for spatial frequency greater than the inverse of outer scale. But spatial frequency \( K_y \) is linearly proportional to temporal frequency \( \omega \) as will be shown in Eq. (3.187). Thus, once the temporal frequency is greater than what corresponds to the outer scale, the power spectrum starts decreasing.

Finally, we examine the dependence of power spectrum \( W_\alpha \) on aperture size \( a \), tracking angle \( \theta_0 \) and wavelength \( \lambda \). Rewriting Eq. (3.163), we have the expression
For very small $\Omega_n$, the aperture diffraction dominates as will also be explained physically in the next section. From Eq. (3.169), $\sin x \approx x$ for very small $\Omega_n$ and we have the proportionality:

$$W_\alpha \propto \frac{W_n (\sec^2 \theta_0) L_h^{7/3}}{a^2} \quad (3.183)$$

Combining Eq. (3.183) and Eq. (3.184), we see $W_\alpha$ has no dependence on tracking angle $\theta_0$ and aperture size $a$. This is understandable, because atmospheric turbulence does not play a role in this region. For large $\Omega_n$, atmospheric turbulence dominates and Eq. (3.184) is no longer valid. In that case $W_\alpha$ increases with $\theta_0$ and $L_h$. This is because there are more atmospheric turbulences as propagation range increases. $W_\alpha$ also decreases with $a$, due to the aperture averaging. Both $W_\alpha$ and $W_n$ have no strong dependence on wavelength $\lambda$. This is because angle of arrival is proportional to be normal of the wavefront.

In the next section, we give the physical interpretation of the power spectrum curve.
In this section, we explain on physical grounds the various slopes of the power spectrum. Consider first the region where the power spectrum increases with $\Omega_n$. Mathematically, for small $\Omega_n$, the refractive index power spectrum remains constant and the $\sin^2$ term in Eq. (3.169) dominates. We shall use the phase grating model to explain this behavior. The phase grating is a thin slab which contains sinusoidal spatial variations of the refractivity field. It imposes phase perturbations on the incoming wave. Figure 32 is a plot of the geometry used to explain the increase of the power spectrum with $\Omega_n$. The source is at point 0 moving with velocity $v$. The receiver is at the bottom of the plot. As the source moves, the line of sight traverses the phase grating variations. We assume a sinusoidal phase grating of spatial wavelength $\lambda_y$ and spatial frequency $K_y$ located at range $\sigma$ from the source. At the receiver, assume that the spatial wavelength to be $\lambda_r$ and the spatial frequency to be $K_r$. Because of the spherical nature of the wave, $K_r$ and $K_y$ can be related by examining Figure 32.

Figure 32. Triangular Relationship Explaining Spherical Wave Nature
We thus obtain

\[
\frac{\lambda_y}{\lambda_r} = \frac{\overline{AB}}{\overline{CD}} = \frac{OE}{OF} = \frac{\sigma}{1 - \sigma + \sigma} = \frac{K_r}{K_y}
\]

or

\[K_r = K_y \sigma \quad (3.185)\]

Hence, at the receiver, the phase is perturbed by

\[S_r = e^{jK_y y} = e^{jK_y \sigma y} \quad (3.186)\]

But \(K_y\) is related to temporal frequency, \(\omega\), by the equation:

\[K_y = \frac{\omega}{v(1 - \sigma)} \quad (3.187)\]

Equation (3.185) is also explained on physical grounds in section 7 of chapter 2. Substituting \(K_y\) in Eq. (3.187) into Eq. (3.186), the expression for \(S_r\) then becomes

\[S_r = e^{j\omega \frac{\sigma}{v(1 - \sigma)} y} \quad (3.188)\]

Using the normalized temporal frequency \(\Omega_n\) given in Eq. (3.159a), Eq. (3.186) then becomes

\[S_r = e^{j\frac{\Omega_n}{\Omega_n} \frac{\sigma}{1 - \sigma} y} \quad (3.189)\]

After passing through the aperture, the centroid angle of arrival defined in Eq. (2.24) then becomes

\[\alpha_r \propto \int_{-a}^{a} \frac{\partial S_r}{\partial y} dy\]
Hence, for small $\Omega_n$, $\alpha_r$ is proportional to $\Omega_n$ and the power is proportional to $\Omega_n^2$. This explains the power spectrum increases on the log-log plot for small $\Omega_n$ with a slope 2.

For very large $\Omega_n$, the power spectrum starts decreasing after reaching a maximum corresponding to outer scale. This was also discussed in the previous section. The physical discussion is thus finished. In the next section, we summarize the main results.
The centroid angle of arrival power spectrum for a tracking receiver is derived in this chapter. The wave equation is first derived in the rotating coordinate system and then solved by using the method of smooth perturbations. The final solution is obtained by neglecting all those terms which are order of magnitude $(\frac{V}{c})^2$ smaller than the dominant term. The phase is found to consist of a deterministic component and a random component. The deterministic component is related to Doppler shift and coordinate rotation. The random component is used to derive power spectrum. During the derivation, two approximations have been used. One is to neglect longitudinal velocity compared with transverse velocity. The other is to neglect range variation within angle of arrival correlation time. The Taylor hypothesis in the rotating coordinate system is also derived.

The differential path contribution shows that the region near the receiver has dominant contribution to the power spectrum. Using normalized variables, the power spectrum is found to have a peak without strong dependence on propagation ranges. The power spectrum initially increases with a slope 2 and reaches a maximum around the outer scale. It then starts decreasing following an initial -2/3 power low and subsequently decreases with a gradually increasing slope.

This finishes chapter 3. In the next chapter, we shall discuss a suggested experimental configuration.
CHAPTER IV
PROPOSED EXPERIMENT VERIFICATION

1 INTRODUCTION

In the past chapters, we have computed theoretical expressions for the centroid angle of arrival power spectrum for laser beam propagation between two moving vehicles. In this chapter, we discuss how an actual experiment might be designed.

In the experiment, a spherical wave source on an aircraft or a satellite is considered. Figure 33 shows the experimental design. In that figure, the source on an aircraft emits a spherical wave which is collected by the telescope aperture and focused to a position proportional detector, the output of which is proportional to centroid angle of arrival. This centroid angle of arrival signal is then recorded on the analog tape. The data recorded on the analog tape are then digitized and transferred to magnetic tape. The magnetic tape is then accessed by the computer and power spectrum computed via FFT.

In section 2, the transmitter optics are discussed. We derive an expression for both the source power requirement and the received S/N requirement. In section 3, the receiver optics are discussed. We derive a value for the required receiver aperture size and discuss the narrow band color filter bandwidth. The limitation due to recorder and
A/D converter are also discussed. In section 4, the spot motion is discussed. The spot motion distance is computed as the product of image distance and the centroid angle of arrival standard deviation. The S/N of this spot is also discussed. In section 5, we discuss the sampling rate and the use of FFT to compute power spectrum. We also compute averaging time by deriving an expression for the centroid angle of arrival correlation time first. In section 6, the associated measurement of $C_n^2$ is briefly considered. Finally, in section 7, we summarize the major results.
Figure 33. Experimental Setup
The transmitter optics are shown in Figure 34. The source is an argon laser with wavelength 0.45 μ. The output from the source is spatially filtered to generate a clean spherical wave. This spherical wave then propagates downward to the receiver.

The pinhole diameter $D_t$ of the spatial filter can be adjusted to control the divergence of the spherical wave. Calling the half angle of divergence $\theta$, then $\theta$ is given by

$$\theta = 1.22 \frac{\lambda}{D_t} \quad (4.1)$$

At range $r$ from the source, the area illuminated by the source is then given by

$$A = \pi \left( \frac{1.22 \lambda r}{D_t} \right)^2 \quad (4.2)$$

The received power density $P_{rd}$ at distance $r$ from the source with power $P_s$ is then given below.

$$P_{rd} = \frac{P_s}{\pi \left( \frac{1.22 \lambda r}{D_t} \right)^2} = \frac{P_s}{\pi} \left( \frac{D_t}{1.22 \lambda} \right)^2 \frac{1}{r^2} \quad (4.3)$$

Suppose the receiver diameter is $D_r$, the total power $P_r$ collected by the receiver is then given by the expression

$$P_r = P_{rd} \cdot \pi \left( \frac{D_r}{2} \right)^2 = \frac{P_s}{5.95} \frac{D_t^2 D_r^2}{\lambda^2 r^2} \quad (4.4)$$
Figure 34. Transmitter Optics

Argon laser

Spatial filter
The power collected by the receiver aperture is focused to a photodiode. The photodiode is a position sensing detector which measures the centroid of light intensity. The internal noise of the detector is indicated by the noise equivalent power (N.E.P.). The noise equivalent power is defined as the light power required to generate a signal equal to the internal noise of the photodiode. The best N.E.P. is obtained when the photodiode is shot-noise limited and is given by the expression [77].

\[
P_n = \text{N.E.P.} = \frac{(2e_i d B)^{\frac{1}{2}}}{R'}
\]

where

- \( e \) = electronic charge
- \( i_d \) = dark current of photodiode
- \( R' \) = detector responsitivity (Amp/Watt)
- \( B \) = bandwidth

Dividing Eq.(4.4) by Eq.(4.5), we then have

\[
\frac{P_r}{P_n} = \frac{P_s}{5.95 \frac{D_t^2 D_r^2}{\lambda^2 r^2}} \frac{R'}{(2e_i d B)^{\frac{1}{2}}}
\]

From Eq.(4.6), the source power \( P_s \) is then given by

\[
P_s = 5.95 \frac{P_r r^2 \lambda^2}{P_n D_t^2 D_r^2} \frac{(2e_i d B)^{\frac{1}{2}}}{R'}
\]

The incoming signal is random but the average incoming signal is predictable. Appendix H shows the desirable average \( P_r/P_n \) to achieve 99% detection accuracy given as \( P_r/P_n = 21 \). Because the spectral amplitude at high frequency is much smaller, \( P_r/P_n \) should be multiplied by a frequency dependent factor, \( \alpha \), in order to achieve the comparable accuracy in high frequency. Thus \( P_r/P_n \) is then given by
\[
\frac{P_r}{P_n} = 21\alpha \tag{4.8}
\]

Substituting \(P_r/P_n\) in Eq. (4.8) into Eq. (4.7) we then have for the required source power

\[
P_s = 125\alpha - \frac{r^2\lambda^2(2e/d_B)^2}{D_t^2D_r^2R} \tag{4.9}
\]

Now two examples are given to show the numeric value of \(P_s\) for propagation distance 1km and 10^5 m. For \(r = 1\text{km} = 10^5\text{cm} \text{ and } \theta = 1^\circ\), Eq. (4.1) gives \(D_t = 31.45 \mu\text{m}\). Also, using a United Detector Technology PIN-SC/10D position sensing detector at a temperature of \(T = 25^\circ\text{C}\), we have \(i_d = 3 \times 10^{-7}\text{ Amp}\), and \(R = 0.35\text{ Amp/Watt}\). We also choose \(\lambda = 0.45\mu\text{m}\) for an argon laser and \(D_r = 38\text{cm}\) from the discussion given in the next section. The bandwidth \(B\) can be determined from the theoretical plot of power spectrum against normalized frequency in Figure 23. From Eq. (3.159), \(\Omega_n\) is given by

\[
\Omega_n = \frac{\omega\lambda h}{v}\tag{4.10}
\]

At \(\Omega_n = 6 \times 10^3\), the power spectrum is about \(10^5\) down from the peak. With \(v = 100\text{m/sec}\), \(\Omega_n = 6 \times 10^3\), and \(L_h = 1000\text{m}\), Eq. (4.10) gives \(f = \frac{\omega}{2\pi} = 100\). To be conservative, let \(B = 100\text{ Hz}\), then \(P_s\) in Eq. (4.9) becomes

\[
P_s \bigg|_{r=1\text{km}} = \alpha(1.568 \times 10^{-3})\text{ mW}\tag{4.11}
\]

At \(L_h = 10^5\text{m}\) and velocity equal to \(10^4\text{m}\), the results are the same.

Ideally, we would like the accuracy of power spectral measurement at frequency corresponding to inner scale to be comparable to that at frequency corresponding to peak power. The frequency corresponding to
inner scale is \( w = v/\ell_0 \). Thus the normalized frequency corresponding to \( \ell_0 \) is given by

\[
\Omega_n|\ell_0 = \frac{L_h}{\ell_0}
\] (4.12)

For \( L_h = 10^3 \text{m} \) and \( \ell_0 = 1\text{mm} \), \( \Omega_n \) becomes \( 10^6 \). At \( \Omega_n = 6 \times 10^3 \) the power spectrum is already \( 10^5 \) down from the peak, thus \( \Omega_n = 10^6 \) tends to be a difficult requirement. Requiring the power spectral measurement at \( \Omega_n = 6 \times 10^3 \) to have comparable accuracy to that at peak power, gives \( \alpha = 10^5 \). This corresponds to \( P_{r}/P_n = 2.1 \times 10^5 = 2.1 \times 10^6 \). Substituting \( \alpha = 10^5 \) into Eq.(4.9), we have

\[
P_s|_{r=1\text{km}} = 156.8 \text{ mW}
\] (4.13)

At \( r = 10^5 \text{ m} \), with \( D_r = 94 \text{ cm} \) and all the other parameters the same, \( P_s \) becomes

\[
P_s = 2.563\alpha \text{ mW}
\] (4.14)

If \( \alpha = 10^5 \), then \( P_s = 256.3 \text{W} \). Commercial available source power at 4500 A° is less than 20W [93]. Several ways can be used to reduce the power requirement. The first is to increase \( D_t \) from 31.45\( \mu \) to 331.5\( \mu \) but keep \( T = 25^\circ \), \( D_r = 94\text{cm} \), and then \( P_s = 2.56\text{W} \). The second is to keep \( D_r = 94\text{cm} \) and \( D_t = 331.5\mu \), but reduce temperature to \(-25^\circ \) or \( i_d = 4 \times 10^{-9} \text{Amp} \), and then \( P_s = 0.295\text{W} \). Depending on the application, several other combinations are possible to get the desirable source power.

In the next section we discuss the receiver optics.
3 RECIIVER OPTICS

The essential elements of the receiver optics are shown in Figure 35. The telescope aperture will collect the incoming light and converge it to the image plane. The purpose of the optical filter is to reject all sunlight frequency components except those around 4500 Å. Ideally, the bandwidth of this filter should be as small as possible to pass only one wavelength at 4500 Å. The best available optical filter bandwidth appears to be 0.4 Å [93]. On a clear sunny day at sea level, the irradiance at 4500 Å is 0.12 W/m² Å [78]. With 0.4 Å bandwidth and 50% transmittance, the power transmitted is $0.12 \times 0.4 \times 1/2 = 2.4 \times 10^{-2} \text{ W/m}^2 = 2.4 \times 10^{-3} \text{ mW/cm}^2$. The total sun power irradiance is 1390 W/m² [79]. This means only $2.4 \times 10^{-2}/1390 = 1.72 \times 10^{-5} = 1.72 \times 10^{-3}\%$ is transmitted and 99.998% is rejected.

The diaphragm is inserted on the image plane to prevent any undesirable background light from entering into the detector. The sensitive area of United Detector Technology PIN-SC/10D is 1 cm. This also means the size of the diaphragm is at most 1 cm. Another way is to enclose detector and aperture in a black box so that no interfering background sunlight may enter.

Normally, the aperture size should be as large as possible so as to focus the incoming light to a fine spot and also to collect as much incoming signal power as possible. But, because of the turbulence, the beam spreading becomes more severe. A compromise is to choose the aperture size equal to the beam coherence size $\rho_o$. $\rho_o$ is defined to be
Figure 35. Receiver Optics
the transverse separation such that the atmospheric modulation mutual coherence function (MCF) is reduced to $e^{-1}$. For a spherical wave [80] the $\rho_0$ is given by

$$\rho_0 = [1.5 k^2 \int_0^z C_n^2(s)(s^3/z^5) ds]^{-3/5}$$

(4.15)

where $z$ is the propagation height and $s$ is the range variable. Defining the dimensionless variable $\sigma$ such that $\sigma = 0$ corresponds to $s = z$ and $\sigma = 1$ corresponds to $s = 0$, we have

$$s = (1 - \sigma)z$$

(4.16)

Substituting $s$ in Eq.(4.16) into Eq.(4.15), $\rho_0$ becomes

$$\rho_0 = [1.5 k^2 z \int_0^1 C_n^2(\sigma)(1 - \sigma)^{5/3} d\sigma]^{-3/5}$$

(4.17)

The $C_n^2$ model which is given in Eq.(3.154) is rewritten here

$$C_n^2(\sigma) = C_n^2(H_0)[1+(1 - \sigma)\frac{H_L - H_0}{H_0}]^{-4/3}$$

(4.18)

where we have used $z = H_L - H_0$. Substituting $C_n^2(\sigma)$ in Eq.(4.18) into Eq.(4.17), we then have

$$\rho_0 = [1.5 k^2 z C_n^2(H_0)\int_0^1 (1 - \sigma)^{5/3}(1 + (1 - \sigma)\frac{z}{H_0})^{-4/3} d\sigma]^{-3/5}$$

(4.19)

For $H_L = 1000$ m, $H_0 = 10$ m, and medium $C_n^2(10$m) = $10^{-15}$, Eq.(4.19) is evaluated to give $\rho_0 = 38$ cm. For $H_L = 10^5$ m, $H_0 = 10$ m and the same $C_n^2(10$m) = $10^{-15}$, Eq.(4.19) is evaluated to give $\rho_0 = 94$ cm. Thus, at 1 km range, the aperture size is chosen to be 38 cm while at 10 m range, the aperture size is chosen to be 94 cm.
The focused light in the image plane is collected by a position proportional detector. The output of this detector is proportional to the product of the signal intensity and the detector position. The average reading is an indication of centroid angle of arrival. This centroid angle of arrival signal can be recorded continuously on an analog tape using a tape recorder. One area of concern is the recorder noise that may be generated. For a Marantz Superscope recorder and TDK D-C 60 analog tape, the recorded S/N is at most 60 dB. This means signal voltage is at most $10^3$ times the recorder noise voltage.

The output from the recorder is digitized and stored on the magnetic tape for later power spectral computation. One area of concern is the quantization noise of the A/D converter. The normal commercial A/D converter is 12 bits. For a 2's complement number, the maximum recorded integer is $\pm 2^{11}$. The quantization error is within 1 bit. Hence, the signal to quantization noise ratio in voltage is $2^{11} = 2048$. This figure is about the same as that previously discussed for recorder. The recommended procedure is to digitize the detector signal and record it on computer magnetic tape.

In the next section, we discuss the spot motion in the image plane.
4 SPOT MOTION

Another quantity which is of interest is how far the spot may move in the image plane. Of course, it is desirable to have the spot motion limited in the sensitive area of the detector. The spot motion can be derived from the centroid angle of arrival variance $\sigma_\alpha^2$. If the distance from the receiver aperture to the image plane is $d_i$, then the RMS distance $d$ the spot moves away from the center is given by

$$d = \sigma_\alpha d_i$$

(4.20)

The variance $\sigma_\alpha^2$ can be derived from the power spectrum $W_\alpha$ using the following Fourier transform relationship.

$$\sigma_\alpha^2 = R_\alpha(0) = \frac{1}{2\pi} \int W_\alpha(w)dw$$

(4.21)

$w$ is relative to the normalized frequency $\Omega_n$ through Eq.(3.159) which is rewritten here.

$$w = \frac{\Omega_n v}{L_h}$$

(4.22)

Substituting $w$ in Eq.(4.22) into Eq.(4.21), we have

$$\sigma_\alpha^2 = \frac{v}{2\pi L_h} \int W_\alpha(\Omega_n) d\Omega_n$$

(4.23)

$W_\alpha$ is related to the normalized power spectrum $W_n$ through Eq.(3.163) which is rewritten here.

$$W_\alpha = \frac{16\pi^2(0.033)c_n^2(H_o)[L_o(H_o)]^{4/3} L_h^{4/3} a^{-2} x_0}{\cos \theta_0 |v|} W_n$$

(4.24)
Substituting $\tilde{W}_\alpha$ in Eq.(4.24) into Eq.(4.23) we have

$$\sigma_\alpha^2 = \frac{8\pi (0.033) C_n^2 (H_o) [L_o (H_o)]^{4/3} L_h^{1/3} a^{-2} x_0}{\cos \theta_0} \int \tilde{W}_n (\Omega_n) d\Omega_n (4.25)$$

where

$$C_n^2 (H_o) = C_n^2 \text{ at height } H_o \quad (4.26)$$

$$L_o (H_o) = \text{ outer scale at height } H_o \quad (4.27)$$

$$x_0 = \frac{L}{\cos \theta_0} \quad (4.28)$$

$L_o (H_o)$ is equal to $H_o$, hence Eq.(4.25) becomes

$$\sigma_\alpha = \left( \frac{8\pi (0.033) C_n^2 (H_o) H_o^{4/3} L_h^{1/3} a^{-2} L^{1/2}}{\cos^2 \theta_0} \right) \left[ \int \tilde{W}_n (\Omega_n) d\Omega_n \right]^{1/3} (4.29)$$

Equation (4.29) shows $\sigma_\alpha$ is a function of $C_n^2$. An increase of $C_n^2$ also increases $\sigma_\alpha$.

The numerical value of $\sigma_\alpha$ can be computed quite easily by first computing the area under the normalized power spectrum curve. For $H_o = 10m$, $2a = 38cm$, $\theta_0 = 45^\circ$, $L = 10^3m$, and consider normal $C_n^2 (1km) = 10^{-16}$ and extremely large $C_n^2 (1km) = 10^{-13}$ we have $\sigma_\alpha (C_n^2 = 10^{-16}) = 4.19 \times 10^{-5}$ and $\sigma_\alpha (C_n^2 = 10^{-13}) = 1.32 \times 10^{-13}$. For image distance $d_i = 5m$, we have from Eq.(4.20) $d(C_n^2 = 10^{-16}) = 0.021cm$, $d(C_n^2 = 10^{-13}) = 0.66cm$.

For the United Detector Technology PIN-SC/10D photodetector, the sensitive area is 1cm. This also means under most turbulence levels, the spot motion is large enough to be sensed by the detector.

The two spot distances computed above are greater than the resolution range 0.000254 cm quoted for the PIN-SC/10D detector and so S/N is no problem. However, the actual S/N can be roughly estimated. For the PIN-SC/10D photodiode, the position sensitivity is $s = 0.5 \text{ Amp/Watt/cm}$,
the dark current is \( i_d = 0.3 \, \mu A = 3 \times 10^{-7} \, \text{Amp} \), and the detector responsivity is \( R' = 0.35 \, \text{Amp/Watt} \). The shot noise current is given by \( i_n = \sqrt{2ei_d B} \). For \( B = 100 \, \text{Hz} \), we obtain \( i_n = 3.09 \times 10^{-12} \, \text{Amp} \). The noise equivalent power \( P_n \) is then computed as \( P_n = i_n / R' = 8.85 \times 10^{-12} \, \text{Watts} \). Equation (4.9) gives the required input power \( P_r \) to be computed as \( P_r = 21 \alpha P_n \). For \( \alpha = 10^5 \), \( P_r \) is then computed as \( P_r = 21 \times 10^5 \times 8.85 \times 10^{-12} = 1.86 \times 10^{-5} \, \text{W} \). The position sensitivity \( i_p \) in unit \( \text{Amp/cm} \) is given by \( i_p = s \times P_r = 0.5 \, \text{Amp/Watt/cm} \times 1.86 \times 10^{-5} \, \text{W} = 0.93 \times 10^{-6} \, \text{Amp/cm} \). Assuming the output is linearly proportional to the distance from center of detector, then \( i(d = 0.021 \, \text{cm}) = 0.93 \times 10^{-5} \times 0.021 = 1.95 \times 10^{-7} \, \text{Amp} \), and \( i(d = 0.66 \, \text{cm}) = 0.93 \times 10^{-5} \times 0.66 = 6.13 \times 10^{-6} \, \text{Amp} \). The ratio of output current versus shot noise current is then \( i_p / i_n (d = 0.021 \, \text{cm}) = 1.95 \times 10^{-7} / 3.09 \times 10^{-12} = 6.31 \times 10^4 \) and \( i_p / i_n (d = 0.66 \, \text{cm}) = 6.13 \times 10^{-6} / 3.09 \times 10^{-12} = 1.98 \times 10^6 \).

In the next section, we discuss the required signal processing to compute power spectrum.

5 SIGNAL PROCESSING

As was discussed, the output from the centroid detector is recorded on the analog tape and then subsequently digitized. An important question is what the sampling rate should be. According to the Nyquist theorem, the sampling rate should be at least twice the signal bandwidth. In the discussion on signal-to-noise ratio, the bandwidth is found to be 100 Hz. One hundred Hz is obtained based on the assumption that outer scale is equal to vertical propagation height. If the outer
scale is 0.1 times the vertical propagation height, the bandwidth then becomes 1 kHz. Due to the uncertainty of the correct outer scale model, a safe choice is to assume that the bandwidth is equal to 2 kHz. Then the sampling rate should be 4 kHz. One possible product of the angle of arrival power spectrum is the determination of the correct outer scale model after the bandwidth is determined from the measurement. If necessary, the sampling rate can be changed according to the correctly measured bandwidth. Figure 36 shows the signal processing procedure.

The digitized data can be processed by the computer off line to compute the power spectrum using fast Fourier transform. In order to reduce the variance of power spectrum estimate, the data should be multiplied by a window function first before applying FFT. A typical window function is Hamming window which is given below [81].

\[ W_i = 0.54 - 0.46 \cos \left(\frac{2\pi i}{N-1}\right) \quad 0 \leq i \leq N-1 \]  

(4.30)

where \( N \) is the total number of digitized samples. Assuming the \( N \) digitized samples are given by \( \alpha_i, i = 0, N-1 \), then the FFT output \( \alpha_w(k) \), \( k = 0, N-1 \) is computed by the well-known formula.

\[ \alpha_w(k) = \sum_{i=0}^{N-1} \alpha_i W_i e^{-j \frac{2\pi}{N} ik} \quad k = 0, N-1 \]  

(4.31)

where \( i \) corresponds to time \( iT_s \), \( k \) corresponds to frequency \( \frac{k}{T_s N} \) and \( T_s \) is the sampling period. \( T_s \alpha_w(k) \) also gives the corresponding Fourier transform integral. The power spectrum \( \tilde{W}_d(k) \) is then computed by the square of \( \alpha_w(k) \).
\[ W_α(k) = |α_w(k)|^2 \] (4.32)

One question is what the number of samples, \( N \), or the averaging time, \( T \), should be. Suppose the sampling rate is \( f_c \) Hz, then \( T \) and \( N \) are related by the following expression.

\[ T = \frac{N}{f_c} \] (4.33)

The averaging time \( T \) depends on the accuracy of measurement required. It is given by the expression [82]

\[ T = \frac{4\tau}{\varepsilon^2} \] (4.34)

where

- \( \tau \) is the correlation time
- \( \varepsilon \) is percentage error from the ensemble average

The above expression is derived by trying to find how long it takes for a time average to represent an ensemble average. For any random signal, one first derives the variance between the time average and ensemble average and the averaging time \( T \) is then subsequently obtained. During the derivation, the only assumption required is that the correlation time to be much less than the averaging time. The autocorrelation function of the random signal should drop to zero in a time much less than the averaging time.

The correlation time \( \tau \) is then computed using the power spectrum derived in the main text. The procedure is to find the correlation function by taking the one dimensional inverse FFT of the computed power spectrum. Suppose \( W_α(k) \) is the power spectrum, the correlation function \( R_α(\tau) \) is then given by
Figure 36. Centroid Angle of Arrival Data Digitizer
\[ R_x(i) = \frac{1}{M} \sum_{k=0}^{M-1} \mathbb{W}_k(i) e^{\frac{j2\pi ik}{M}} i = 0, M-1 \quad (4.35) \]

where \( i \) and \( k \) are explained following Eq.(4.31). \( R_x(i)/T \) also gives the corresponding inverse Fourier transform integral where \( M \) is the total number of points. The correlation time \( \tau \) is then estimated by using the following formula

\[ \tau = \frac{A}{R_x(0)} \quad (4.36) \]

where \( A \) is the area under the autocorrelation curve.

In computing \( R_x(i) \) using Eq.(4.35), \( M \) is chosen to be 32768. Any point which is not computed is interpolated logarithmically. Suppose the normalized frequency between two consecutive points is \( \Delta \Omega_n \), the maximum normalized frequency \( \Omega_M \) is then

\[ \Omega_M = (\Delta \Omega_n) M \quad (4.37) \]

\( \Omega_M \) can be converted to the actual frequency \( f_M \) using the following relation

\[ \Omega_M = \frac{2\pi f_M L_h}{v} \quad (4.38) \]

Combining Eq.(4.37) and Eq.(4.38), we then have

\[ f_M = \frac{\Omega_M v}{2\pi L_h} = \frac{\Delta \Omega_n M v}{2\pi L_h} \quad (4.39) \]

In the time domain, the duration between two consecutive points is then

\[ \Delta t = \frac{1}{f_M} = \frac{2\pi L_h}{\Delta \Omega_n M v} \quad (4.40) \]

With \( \Delta \Omega_n = 6.5 \times 10^{-3} \), \( L_h = 10^3 m \), \( v = 100 \text{ m/sec} \), and \( M = 32768 \), then
\[ \Delta t = 0.295 \text{ sec} \]

After computing the total area after the correlation curve, Eq. (4.36) gives

\[ \tau = 2.5 \text{ sec} \]  \hspace{1cm} \text{(4.41)}

In Appendix G, correlation time is computed as \( \tau = L_0/v \). Assuming \( L_0 = L_h = 10^3 \text{ m} \), and \( v = 100 \text{ m/sec} \), \( \tau \) is found to be 10. sec. This value is close to what is actually computed in Eq.(4.41). The difference is because outer scale is a function of height and effective outer scale value should be used.

Figure 37 is a plot of the normalized correlation function \( R_a/R_a(0) \), versus time delay on a log-linear scale and Figure 38 is a similar plot on a linear-linear scale. With \( \epsilon = 1\% = 0.01 \), Eq.(4.34) gives averaging time as follows:

\[ T = \frac{4\tau}{\epsilon^2} = \frac{4(2.5)}{(10^{-2})^2} = 10^5 \text{ sec} = 27.7 \text{ (hr)} \]  \hspace{1cm} \text{(4.42)}

27.7 hours is too large for any practical measurements. It also suggests 1% accuracy cannot be achieved. Suppose \( \epsilon \) is 10\% or \( \epsilon = 0.1 \), then (4.34) gives the averaging time as follows:

\[ T = \frac{4(2.5)}{(10^{-1})^2} = 10^3 \text{ sec} = 16 \text{ (min)} \]  \hspace{1cm} \text{(4.43)}

This value is reasonable for practical measurements and also suggests 10% precision can be achieved. However, this conclusion is also based on the model that the outer scale is numerically equal to propagation height.

In the next section, we discuss the measurement of \( C_n^2 \) which is the key parameter in computing centroid angle of arrival power spectrum.
Figure 37. Centroid Angle of Arrival Correlation Function Versus Time Delay on a Log-Linear Scale
Figure 38. Centroid Angle of Arrival Correlation Function Versus Time Delay on a Linear-Linear Scale
\( C_n^2 \) is the key parameter that determines the fluctuation of angle of arrival. The \( C_n^2 \) profile should be on the same day the experiment is performed.

\( C_n^2 \) profile can be determine once \( C_T^2 \) is measured [74]. The conversion from \( C_T^2 \) to \( C_n^2 \) is given by the well-known formula

\[
C_n^2 = C_T^2 \left( \frac{79 \times 10^{-6} P}{T} \right)^2
\]

(4.44)

where \( P \) is the pressure in millibars and \( T \) is the temperature in °K. To determine \( C_T^2 \), microthermal fluctuations are recorded on an analog tape. The thermal probe is mounted on the nose of the aircraft flying at various altitudes. \( C_n^2 \) in general is a function of height. Thus each recording at a given altitude determines \( C_n^2 \) at that altitude. The data recorded on the analog tape can be digitized and power spectrum computed via fast Fourier transform. The \( C_n^2 \) is then determined for each height by finding the best match of one dimensional von Kármán power spectrum with the measured power spectrum [75]. This one dimensional power spectrum is given by the following formula [75]

\[
V(k) = \frac{A}{(k^2 + B)^{5/6}}
\]

(4.45)

where \( A \) is the parameter associated with \( C_n^2 \) and \( B \) is the parameter associated with outer scale. One such typical power spectrum is shown in Figure 39.
Figure 39. Typical Airborne Measured Microthermal Power Spectrum from Flight 3.
SUMMARY

The experimental setup to measure centroid angle of arrival power spectrum is discussed in this chapter. The moving source sends out a spherical wave whose divergence angle can be controlled. The received wave is collected by a tracking receiver aperture and then focused to a position sensing detector. The output from the detector is recorded on the analog tape first and then subsequently digitized. The power spectrum is then computed from the digitized data using FFT.

The transmitter source power requirements are analyzed in detail. Depending on the specific application, the desired source power can be obtained by adjusting the pinhole size, detector temperature, S/N requirements, and telescope aperture size. Requiring the received S/N to be $2.1 \times 10^6$, the source power was found to be 156.8 mw for 1 km propagation range at $D_r = 38 \text{ cm}$, $T = 25^\circ$, $D_t = 331.5 \mu$ and 0.295 w for 10 km propagation range at $D_r = 94 \text{ cm}$, $T = -25^\circ$ and $D_t = 331.5 \mu$.

The receiver telescope aperture size is set equal to the coherence size. Typical coherence size is computed. The maximum recorder S/N in voltage is $10^3$. That sets the limit on the optical power versus noise power to be $10^3$. The A/D converter can achieve a S/N of about $10^3$ in voltage using a 12 bit quantizer. Higher S/N can be obtained using more quantization bits.

Concerning signal processing, the sampling rate is set equal to 4 kHz. The averaging time is found to be 27.7 hours for 1% measurement precision and 16 minutes for 10% measurement precision.
The spot motion in the image plane of the detector is also analyzed. It is found under all turbulence levels, the spot range is always within 1 cm. Thus, the spot can always be collected using a United Detector Technology PIN-SC/10D detector. The approximate signal-to-noise ratio is also computed.

This finishes Chapter 4. In the next chapter we summarize and discuss all major results.
CHAPTER V
SUMMARY AND DISCUSSIONS

1  SUMMARY AND CONCLUSION

The center issue in this dissertation is the centroid angle of ar­rival temporal power spectrum. Particular emphasis is given to spherical wave propagations between two moving vehicles with wind blowing across the propagation path. Both nontracking and tracking systems are considered. In the nontracking system, the receiver does not have to face the source all the time and eventually will not function properly. In the tracking system, the receiver has to face the source all the time in order to insure proper detection. For both systems, we derive an expression for the power spectrum and the differential path contributions with numerical examples given. An experimental apparatus is also designed in order to verify the theory.

In the nontracking system, a quite general formula is derived for the optical phase using the method of smooth perturbations. The phase is found to consist of a random component and a deterministic component. The deterministic component is found to be the Doppler shift due to the moving source. The random component is used to compute the centroid angle of arrival power spectrum. One specific example is considered and numerical integration results are given. A simple geometric model is used to explain the physical phenomena.
In the tracking system, the wave equation is derived in the rotating coordinate system. The complex wave amplitude is then solved using the method of smooth perturbations. In obtaining the final formula, all terms which have order of magnitude \( \left( \frac{V}{c} \right)^2 \) smaller than the most significant term are neglected. The phase derived from the complex wave amplitude is again found to be composed of a deterministic component and a random component. The deterministic component consists of two parts. The first part is due to conventional Doppler shift. The second part is associated with the rotation of coordinate system. A simple geometric model is shown to explain the deterministic phase shift. The random phase component is then used to derive the phase correlation function, the centroid angle of arrival correlation function, and finally, the centroid angle of arrival temporal power spectrum.

In the process of deriving centroid angle of arrival power spectrum, two approximations are used. In the first approximation, longitudinal velocity is neglected compared with the transverse velocity. In the second approximation, the propagation range within the correlation time of centroid angle of arrival is found to have little variation. This second approximation is also justified from both physical and mathematical grounds. Taylor's hypothesis in the rotating coordinate system is also derived. It has a form different from that in the stationary coordinate system.
Both differential path contribution and power spectrum are evaluated numerically. The differential path contribution is found to increase toward the receiver. The power spectrum increases initially with a slope of 2. It reaches a maximum at a frequency determined by outer scale and then subsequently decreases with a gradually increasing slope. The peak of the normalized frequency, \( \omega L_n/v \), is found to be between 0.1 and 1 with no strong propagation range dependence.

An experimental apparatus is also designed to measure the centroid angle of arrival temporal power spectrum. The moving source sends out a spherical wave whose divergence angle can be controlled. The received wave is collected by a tracking receiver aperture and then focused to a position sensing detector. The output from the detector is recorded on the analog tape first and then subsequently digitized. The power spectrum is then computed from the digitized data using FFT.

The transmitter source power requirements are analyzed in detail. Depending on specific application, the desired source power can be obtained by adjusting the pinhole size, detector temperature, S/N requirements, and telescope aperture size. Requiring the power spectrum measurement at \( \Omega_n = 6 \times 10^3 \) to have comparable accuracy to that at peak power, the received S/N is set equal to \( 2.1 \times 10^6 \). The source power was then found to be 156.8 mW for 1 Km propagation range at receiver diameter, \( D_r = 38 \) cm, temperature, \( T = 25^o \), transmitter diameter, \( D_t = 33.15 \mu \) and 0.295 W for \( 10^5 \) propagation range at \( D_r = 94 \) cm, \( T = -25^o \) and \( D_t = 331.5 \mu \).
The receiver telescope aperture size is set equal to the coherence size. Typical coherence size is computed. The maximum recorder S/N in voltage is $10^3$. That sets a limit on the optical power versus the noise power to be $10^3$. Using a 12 bit A/D converter, the signal to quantization noise is found to be about 60 dB also. But higher S/N can be obtained by using more quantization bits.

Concerning signal processing, the sampling rate is set equal to 4 kHz. The averaging time is found to be 27.7 hours for 1% measurement precision and 16 minutes for 10% measurement precision.

The spot motion in the image plane of the detector is also analyzed. It is found under all turbulence levels that the spot range is expected to be within 1 cm. Thus, the spot can always be collected using a United Detector Technology PIN-SC/10D detector. The approximate signal-to-noise ratio is also computed.

2 DISCUSSION

The formulas we derived in this dissertation have several new features. For both the nontracking and tracking system, we solve the time dependent wave equation using the method of smooth perturbations. We thus consider not only the fluctuations due to refractive index but also the Doppler shift due to the source motions. Further, in the tracking system, the receiver is required to face the source all the time. We thus solve the time dependent wave equation in the rotating coordinate system. The Taylor hypothesis which is valid in the rotating
coordinate system is also derived. Also, in deriving the power spectrum in the tracking system, the source and receiver are allowed to move in the arbitrary direction with constant velocity. The formulas thus derived are quite general and can be applied in many different situations.

An experimental apparatus to verify the theory is also designed. There are several interesting aspects. First, we found the recorder is the limiting factor in achieving higher signal-to-noise ratio. Second, the bandwidth of the power spectrum tends to be small due to the choice that outer scale is equal to propagation height. Third, the correlation time is determined by outer scale and is verified numerically. Fourth, the averaging time tends to be too large to achieve 1% measurement accuracy but reasonable for 10% measurement accuracy.

In the analysis for both the nontracking and tracking system, the wind is considered to move with constant velocity. We could also consider the wind moving with random velocity. A good model is to assume the wind velocity follows a Gaussian distribution. In that case we computed the expected power spectrum by averaging through all possible wind velocity fluctuations.

Another limitation of this dissertation is we consider only weak atmospheric turbulence. The theory can be extended to the strong turbulence regime. One approach is to apply Tatarskii's quantum field theory [94]. Another approach is to consider the angle of arrival temporal spectrum under the condition when intensity scintillation is saturated [21].
About experimental apparatus, we use analog tape to record our data. Another approach is to replace the recorder by a microprocessor. In that case, the output from the detector is directly digitized and sent to the microprocessor. In this case, the power spectrum can be computed in real time.
APPENDIX A
INVESTIGATION OF THE EFFECT OF LOG AMPLITUDE FLUCTUATION

A.1 INTRODUCTION

We have neglected the log amplitude in deriving the centroid angle of arrival power spectrum for both the tracking and nontracking system. In this appendix, we show that log amplitude contributes only a second-order effort to the angle of arrival.

The approach is to make a Taylor series expansion of angle of arrival in terms of log amplitude. The angle of arrival autocorrelation function is then subsequently derived. In order to compute the statistics, we further show that the derivative of a Gaussian distributed variable is Gaussian. The first-order term of the autocorrelation function is then shown to be zero independent of whether the log amplitude mean is zero or not.

A.2 POWER SERIES EXPANSION OF ANGLE OF ARRIVAL

From Eq.(2.23), the angle of arrival \( \alpha_0(t) \) is given below.

\[
\alpha_0(t) = - \frac{1}{k} \sum e \frac{2\chi(\vec{r}_1, t) \partial S(\vec{r}_1, t)}{\partial y_1} \, d\vec{r}_1
\]

\[
- \frac{2}{\Sigma} \int e 2\chi(\vec{r}_2, t) \, d\vec{r}_2
\]

(A.1)
By making a Taylor series expansion of \(e^{2\chi}\), \(\alpha_0(t)\) then becomes,

\[
\alpha_0(t) = -\frac{1}{k} \sum \frac{\operatorname{AS}(\vec{r}_1, t)}{\partial y_1} \, d\vec{r}_1 \tag{A.2}
\]

where \(A\) is the area of the aperture.

\(\alpha_0(t)\) given in Eq. (A.2) is manipulated further to give the following expression:

\[
\alpha_0(t) = -\frac{1}{kA} \left[ \int_{\Sigma_1} \int [(1+2\chi(\vec{r}_1, t) + 2\chi^2(\vec{r}_1, t) + \cdots) \frac{\partial \operatorname{AS}(\vec{r}_1, t)}{\partial y_1} \, d\vec{r}_1] \right.

\[\left.\quad \left[1 - \frac{1}{A} \int_{\Sigma_2} (2\chi(\vec{r}_2, t) + 2\chi^2(\vec{r}_2, t) + \cdots) \, d\vec{r}_2 \right] \right]

\[= -\frac{1}{kA} \left[ \int_{\Sigma_1} \int \frac{\partial \operatorname{AS}(\vec{r}_1, t)}{\partial y_1} \, d\vec{r}_1 + 2 \int_{\Sigma_1} \chi(\vec{r}_1, t) \frac{\partial \operatorname{AS}(\vec{r}_1, t)}{\partial y_1} \, d\vec{r}_1 \right.

\[\left.\quad - \frac{2}{A} \int_{\Sigma_2} \int \frac{\partial \operatorname{AS}(\vec{r}_1, t)}{\partial y_1} \chi(\vec{r}_2, t) \, d\vec{r}_1 \, d\vec{r}_2 \right]

\[= -\frac{4}{A} \int_{\Sigma_1} \int \int \chi(\vec{r}_2, t) \frac{\partial \operatorname{AS}(\vec{r}_1, t)}{\partial y_1} \, d\vec{r}_1 \, d\vec{r}_2 \]

\[+ \int_{\Sigma_1} \int \chi^2(\vec{r}_1, t) \frac{\partial \operatorname{AS}(\vec{r}_1, t)}{\partial y_1} \, d\vec{r}_1 \]

\[= -\frac{2}{A} \int_{\Sigma_1} \int \int \chi^2(\vec{r}_2, t) \frac{\partial \operatorname{AS}(\vec{r}_1, t)}{\partial y_1} \, d\vec{r}_1 \, d\vec{r}_2 + \cdots \right] \tag{A.3}
\]
Equation (A.3) is the power series expression of $a_0(t)$ in terms of log amplitude $\chi$. In the next section, we derive an expression for the angle of arrival autocorrelation function.

**A.3 POWER SERIES EXPANSION OF ANGLE OF ARRIVAL AUTOCORRELATION FUNCTION.**

The angle of arrival autocorrelation functions $R_{\alpha}(t_1,t_2) = \langle a_0(t_1)a_0(t_2) \rangle$ is obtained by multiplying Eq.(A.3) with itself and then taking the expected value. We obtain

$$R_{\alpha}(t_1,t_2) = \langle a_0(t_1)a_0(t_2) \rangle = R_0 + R_1 + R_2 + \cdots$$  

where the zero-order term $R_0$ and the first-order term $R_1$ are given by the following expression:

$$R_0 = \frac{1}{k A} \iiint \frac{\partial S(r_1,t_1)}{\partial y_1} \frac{\partial S(r_3,t_2)}{\partial y_3} \, dr_1 dr_3$$  

$$R_1 = \frac{2}{k A} \iiint \frac{\partial S(r_1,t_1)}{\partial y_1} \frac{\partial S(r_3,t_2)}{\partial y_3} \chi(r_3,t_2) \, dr_1 dr_3$$

$$+ \frac{2}{k A} \iiint \chi(r_1,t_1) \frac{\partial S(r_1,t_1)}{\partial y_1} \frac{\partial S(r_3,t_2)}{\partial y_3} \, dr_1 dr_3$$

$$- \frac{2}{k A} \iiint \frac{\partial S(r_1,t_1)}{\partial y_1} \frac{\partial S(r_3,t_2)}{\partial y_3} \chi(r_4,t_2) \, dr_1 dr_3 dr_4$$

$$- \frac{2}{k A} \iiint \frac{\partial S(r_1,t_1)}{\partial y_1} \frac{\partial S(r_3,t_2)}{\partial y_3} \chi(r_2,t_1) dr_1 dr_2 dr_3$$  

(A.6)
In order to compute $R_1$, we need to find the expected value of the product of three random variables. $\chi$ and $S$ are in general Gaussian. There is an uncertainty whether the derivative of $S$ is also Gaussian. We thus digress here and show that the derivative of a Gaussian variable is also Gaussian.

A.4 PROBABILITY DENSITY FUNCTION OF $\frac{\partial S}{\partial y}$

From the basic definition of a derivative in calculus, we write

$$\frac{\partial S(x,y)}{\partial y} = \lim_{\varepsilon \to 0} \frac{S(x,y+\varepsilon)-S(x,y)}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{S_1-S_2}{\varepsilon}$$

where $S_1 = S(x,y+\varepsilon)$, $S_2 = S(x,y)$ and $\varepsilon$ is any number greater than zero.

Physically, $S_1$ and $S_2$ correspond to the phase of two close points on the aperture. In general, they must be dependent with a correlation coefficient $\rho$. However, $S_1$ and $S_2$ are both Gaussian distributed with the mean value zero and the same variance $\sigma^2$. The joint statistics of $S_1$ and $S_2$, then, follow the following bivariate Gaussian distribution $f(S_1,S_2)$:

$$f(S_1,S_2) = \frac{1}{2\pi\sigma^2\sqrt{1-\rho^2}} \exp\left[\frac{-1}{2(1-\rho^2)} \left(\frac{S_1^2}{\sigma^2} - 2\rho \frac{S_1S_2}{\sigma^2} + \frac{S_2^2}{\sigma^2}\right)\right]$$

where $\rho$, the correlation coefficient, is defined by

$$\rho = \frac{\mathbb{E}[S_1S_2]}{\sigma^2} = \frac{R_S(0,\varepsilon)}{\sigma^2}$$

The next step is to derive the probability density function for $S_1-S_2$. Thus, we make the following change of variables:
Solving \( S_1 \) and \( S_2 \) in terms of \( u \) and \( v \), we have

\[
S_1 = \frac{1}{2} (u+v) \quad \text{(A.11a)}
\]

\[
S_2 = \frac{1}{2} (v-u) \quad \text{(A.11b)}
\]

The Jacobian of the transformation is then

\[
\begin{vmatrix}
\frac{\partial S_1}{\partial u} & \frac{\partial S_1}{\partial v} \\
\frac{\partial S_2}{\partial u} & \frac{\partial S_2}{\partial v}
\end{vmatrix} = \frac{1}{2} \quad \text{(A.12)}
\]

Using Eq. (A.11a), Eq. (A.11b) and Eq. (A.12), the probability distribution function of \( f(u,v) \) is thus given by

\[
f(u,v) = (J) \frac{1}{2\pi\sigma^2 \sqrt{1-\rho^2}} \exp\left\{ -\frac{1}{2(1-\rho^2)} \left[ \frac{(u+v)^2}{4\sigma^2} - 2\rho \frac{(u+v)(u-v)}{\sigma^2} + \frac{(u-v)^2}{4\sigma^2} \right]\right\}
\]

\[
= \frac{1}{4\pi\sigma^2 \sqrt{1-\rho^2}} \exp\left\{ -\frac{u^2}{4\sigma^2(1-\rho)} - \frac{v^2}{4\sigma^2(1+\rho)} \right\} \quad \text{(A.13)}
\]

The probability density function \( f(u) \) is obtained after integrating Eq. (A.13) with respect to all possible \( v \) values. Hence,

\[
f(u) = \int_{-\infty}^{\infty} f(u,v) dv
\]

\[
= \frac{1}{\sigma \sqrt{4\pi(1-\rho)}} \exp\left\{ -\frac{u^2}{4\sigma^2(1-\rho)} \right\} \quad \text{(A.14)}
\]

Actually, we are interested in the probability density function \( f(v) \) where \( v = u/\varepsilon \). Using the new variable \( v \), we obtain the following expression for \( f(v) \):
\[ f(v) = \frac{\varepsilon}{\alpha \sqrt{4\pi(1-\rho)}} \exp\left[ -\frac{v^2\varepsilon^2}{4\sigma^2(1-\rho)} \right] \quad (A.15) \]

As Eq. (A.7) shows, the probability density function of \( \frac{\partial S(x,y)}{\partial y} \) is obtained after taking the limit \( \varepsilon \to 0 \) in Eq. (A.16). Thus,

\[ f \left[ \frac{\partial S(x,y)}{\partial y} \right] = \lim_{\varepsilon \to 0} \frac{\varepsilon}{\sqrt{(1-\rho)4\pi \sigma}} \exp\left[ -\frac{v^2\varepsilon^2}{4\sigma^2(1-\rho)} \right] \quad (A.16) \]

Equation (A.16) is Gaussian distributed with variance \( \sqrt{2(1-\rho)} \sigma/\varepsilon \).

The next question is whether the limit exists in Eq. (A.16). We first recognize that \( \rho \) defined by Eq. (A.9) is an even function of \( \varepsilon \) because of the even property of the phase correlation function. Let us make a Taylor Series expansion of \( \rho \) and note that \( \rho(\varepsilon=0) = 1 \). We then obtain

\[ \rho(\varepsilon) = 1 + \frac{\rho''(\varepsilon=0)}{2} \varepsilon^2 + \frac{\rho'''(\varepsilon=0)}{4!} \varepsilon^4 + \cdots \quad (A.17) \]

Equation (A.17) can be used to show that the variable \( w = \frac{\varepsilon}{\sqrt{1-\rho}} \) is a finite number independent of \( \varepsilon \). That is,

\[ w = \lim_{\varepsilon \to 0} \frac{\varepsilon}{\sqrt{1-\rho}} = \frac{2}{\sqrt{p''(\varepsilon=0)}} \quad (A.18) \]

Using the new variable \( w \) which is a constant in (A.16), we finally obtain the following expression:

\[ f(\frac{\partial S(x,y)}{\partial y}) = \frac{w}{\sqrt{4\pi\sigma}} \exp\left[ -\frac{(\frac{\partial S(x,y)}{\partial y})^2}{4\sigma^2} \right] \quad (A.19) \]

This completes the proof that \( \frac{\partial S(x,y)}{\partial y} \) is Gaussian distributed if \( S(x,y) \) is Gaussian distributed.
A.5 COMPUTATION OF $R_0$ AND $R_1$

The zero-order term $R_0$ is the easiest one to evaluate. By interchanging the integration and expectation in Eq.(A.5), we have the following expression:

$$R_0 = \frac{1}{k^2 A^2} \iiint < \frac{\partial S(\vec{r}_1, t_1)}{\partial y_1} \frac{\partial S(\vec{r}_3, t_2)}{\partial y_3} > d\vec{r}_1 d\vec{r}_3$$

$$= \frac{1}{k^2 A^2} \iiint \frac{\partial^2}{\partial y_1 \partial y_3} < S(\vec{r}_1, t_1) S(\vec{r}_3, t_2) > d\vec{r}_1 d\vec{r}_3$$

$$= \frac{1}{k^2 A^2} \iiint \frac{\partial^2}{\partial y_1 \partial y_3} B_S(\vec{r}_1 - \vec{r}_3, \tau) d\vec{r}_1 d\vec{r}_3 \quad (A.20)$$

where $\tau = t_1 - t_2$ and we have used the homogeneity of the phase spatial correlation functions.

In order to evaluate $R_1$, we need a formula for the expected value of the product of three Gaussian distributed variables. Suppose $x_1$, $x_2$, and $x_3$ are jointly Gaussian distributed with mean $\mu_1$, $\mu_2$, and $\mu_3$. The following well known formula results [85].

$$<x_1 x_2 x_3> = \sigma_{23} \mu_1 + \sigma_{13} \mu_2 + \sigma_{12} \mu_3 + \mu_1 \mu_2 \mu_3 \quad (A.21a)$$

where

$$\sigma_{ij} = <(x_i - \mu_i)(x_j - \mu_j)> \quad i=1,3, \ j=1,3 \quad (A.21b)$$

Using Eq.(A.21a) and $<\frac{\partial S}{\partial y}> = 0$, Eq.(A.6) then becomes after interchanging integration and expectation.
\[ k^2 A^2 R_1 = 2 \iint < \frac{\partial S(\mathbf{r}_1, t_1)}{\partial y_1} \frac{\partial S(\mathbf{r}_3, t_2)}{\partial y_3} > \chi(\mathbf{r}_3, t_2) \, d\mathbf{r}_1 d\mathbf{r}_3 \]

\[ + 2 \iint < \frac{\partial S(\mathbf{r}_1, t_1)}{\partial y_1} \frac{\partial S(\mathbf{r}_3, t_2)}{\partial y_3} > \chi(\mathbf{r}_1, t_1) \, d\mathbf{r}_1 d\mathbf{r}_3 \]

\[ - 2 \iint \iint \frac{\partial S(\mathbf{r}_1, t_1)}{\partial y_1} \frac{\partial S(\mathbf{r}_3, t_2)}{\partial y_3} > \chi(\mathbf{r}_4, t_2) \, d\mathbf{r}_1 d\mathbf{r}_3 d\mathbf{r}_4 \]

\[ - \frac{2}{A} \iint \iint \frac{\partial S(\mathbf{r}_1, t_1)}{\partial y_1} \frac{\partial S(\mathbf{r}_3, t_2)}{\partial y_3} > \chi(\mathbf{r}_2, t_1) \, d\mathbf{r}_1 d\mathbf{r}_2 d\mathbf{r}_3 \]

\[ = 2 \chi_0 \iint \frac{\partial^2 B_S(\mathbf{r}_1 - \mathbf{r}_3, t)}{\partial y_1 \partial y_3} \, d\mathbf{r}_1 d\mathbf{r}_3 + 2 \chi_0 \iint \frac{\partial^2 B_S(\mathbf{r}_1 - \mathbf{r}_3, t)}{\partial y_1 \partial y_3} \, d\mathbf{r}_3 d\mathbf{r}_1 \]

\[ - \frac{2 \chi_0}{A} \iint \frac{\partial^2 B_S(\mathbf{r}_1 - \mathbf{r}_3, t)}{\partial y_1 \partial y_3} \, d\mathbf{r}_1 d\mathbf{r}_3 \] [\int d\mathbf{r}_4]

\[ - \frac{2 \chi_0}{A} \iint \frac{\partial^2 B_S(\mathbf{r}_1 - \mathbf{r}_3, t)}{\partial y_1 \partial y_3} \, d\mathbf{r}_3 d\mathbf{r}_1 \] [\int d\mathbf{r}_2]

\[ = 0 \]

That \( R_1 \) is zero shows that angle of arrival contributes only a second-order effect in the angle of arrival autocorrelation function.
APPENDIX B

DERIVATION OF ORTHOGONALITY RELATIONSHIP

In this appendix, we derive the orthogonality relationship which is used in Eq. (2.92). This orthogonality relationship is given by

\[ \langle \mu_1(x', K_y, K_z) \mu_1(x'', -K_y'', -K_z'') \rangle \]

\[ = \delta(K_y - K_y'') \delta(K_z - K_z'') F_n(K_y', K_z', x' - x'') \] (B.1)

The refractive index \( n_1 \) and its spectral representation \( \mu_1 \) are related by the equation

\[ \mu_1(x', K_y, K_z) = \left( \frac{1}{2\pi} \right) \iint n_1(x', y', z') \right] e^{-jK_y'y' - jK_z'z'} \, dy' \, dz' \] (B.2)

Using \( \mu_1(x', K_y, K_z) \) given in Eq. (B.2), the product of \( \mu_1(x', K_y, K_z) \) and \( \mu_1(x'', +K_y'', +K_z'') \) is then given by the expression

\[ \mu_1(x', K_y, K_z) \mu_1(x'', +K_y'', +K_z'') \]

\[ = \left( \frac{1}{2\pi} \right) \iiint n_1(x', y', z') \, n_1(x'', y'', z'') \right] e^{-jK_y'y' - jK_z'z' - jK_y''y'' - jK_z''z''} \, dy' \, dz' \, dy'' \, dz'' \] (B.3)
Taking the expected value on both sides of Eq. (B.3) and changing 
$K_y''$ to $-K_y''$ and $K_z''$ to $-K_z''$, we have 
\[
\langle \mu_1(x', K_y, K_z) \mu_1(x'', -K_y'', -K_z'') \rangle
\]
\[
= \left( \frac{1}{2\pi} \right)^4 \iiint <n_1(x', y', z') \ n_1(x'', y'', z'')>
\]
\[
e^{-j K_y' y' + j K_y'' y'' - j K_z' z' + j K_z'' z''} dz' dz'' dy'' dy''
\]
\[ \text{(B.4)} \]

Assuming refractive index fluctuation is homogeneous, then the expected 
value inside the integral of Eq. (B.4) is given by the expression.
\[
\langle n_1(x', y', z') \ n_1(x'', y'', z'') \rangle
\]
\[
= B_{n_1} (x' - x'', y' - y'', z' - z'') \ \text{(B.5)}
\]

where $B_{n_1}$ is the refractive index correlation function. Substituting 
the expression in Eq. (B.5) into Eq. (B.4), we then have 
\[
\langle \mu_1(x', K_y, K_z) \mu_1(x'', -K_y'', -K_z'') \rangle
\]
\[
= \left( \frac{1}{2\pi} \right)^4 \iiint B_{n_1} (x' - x'', y' - y'', z' - z'')
\]
\[
e^{-j K_y' y' + j K_y'' y'' - j K_z' z' + j K_z'' z''} dz' dz'' dy'' dy''
\]
\[ \text{(B.6)} \]

We now introduce new integration variables $y$ and $z$ which are de-

fined by
\[
y' - y'' = y \ \text{(B.7a)}
\]
\[
z' - z'' = z \ \text{(B.7b)}
\]

Using the expressions in Eq. (B.7a) and Eq. (B.7b), Eq. (B.6) then becomes 
\[
\langle \mu_1(x', K_y, K_z) \mu_1(x'', -K_y'', -K_z'') \rangle
\]
\[
= \left( \frac{1}{2\pi} \right)^4 \iiint B_{n_1} (x' - x'', y, z) e^{-j K_y y + j K_z z}
\]
\[
e^{-j y''(K_y' - K_y'')} e^{-j z''(K_z' - K_z'')} dy dz dy'' dz''
\]
\[ \text{(B.6)} \]
\[ (\frac{-1}{2\pi})^4 \int \bigg( \int B_{n_1}(x' - x'', y, z) e^{-jK_y y - jK_z z} \, dy \, dz \bigg) \]

\[ \times \left[ \int e^{-jy''(K_y - K_y''')} \, dy'' \right] \left[ \int e^{-jz''(K_z - K_z''')} \, dz'' \right] \]  

(B.8)

The first integral of Eq.(B.8) is the definition of \( F_n(x' - x'', K_y, K_z) \) and is given below.

\[
\int B_{n_1}(x' - x'', y, z) e^{-jK_y y - jK_z z} \, dy \, dz
\]

\[ = (2\pi)^2 F_n(x' - x'', K_y, K_z) \]  

(B.9)

The second and the third integral can be integrated using the following formula:

\[
\int e^{jKx} \, dx = 2\pi \delta(K) \]  

(B.10)

Using the integral Eq.(B.10) and substituting the expression in Eq.(B.9) into Eq.(B.8), we have

\[
<\mu_1(x', K_y, K_z) \mu_1(x''', -K_y'', -K_z'')>
\]

\[ = F_n(x' - x'', K_y, K_z) \delta(K_y - K_y'') \delta(K_z - K_z'') \]  

(B.11)
APPENDIX C
SOME REMARKS ON TAYLOR'S HYPOTHESIS

C.1 INTRODUCTION

In Chapter 2, we talk in words about Taylor's hypothesis. In this appendix we want to investigate more deeply into its origin and validity. We first present a mathematical derivation of Taylor's hypothesis following the work by Hinze (1959) in free air flow. Then we discuss the Taylor hypothesis in shear flow according to the work by Lin (1953) and Lumley (1965). At last, we discuss the applicability of Taylor's hypothesis into wave propagation problems.

C.2 DERIVATION OF TAYLOR'S HYPOTHESIS

The Taylor hypothesis states that the turbulent eddies move as a whole with constant velocity. For mathematical derivation of this hypothesis, we start with the Navier-Stokes equation which governs the flow of the turbulent eddies. This equation is given by

\[ \rho \frac{dV_i}{dt} = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 V_i}{\partial x_j \partial x_j}, \quad i = 1, 2, 3 \]  

(C.1)

where
\( \rho \) is the fluid density
\( P \) is the pressure
\( \mu \) is the viscosity
\( V_i \) is the flow velocity along the \( x_i \) directions

There is no loss of generality if we concentrate on the flow field in the \( x \) direction only. In that case, Eq.(C.1) becomes

\[
\rho \frac{dV_1}{dt} = -\frac{\partial p}{\partial x_1} + \mu \frac{\partial^2 V_1}{\partial x_j \partial x_j} \tag{C.2}
\]

In the turbulent flow, the velocity \( v_1 \) and pressure \( p \) are not constants. They will fluctuate in a random manner with time and space. However, it is convenient to decompose the random field into the sum of a constant quantity and a fluctuating quantity. We thus have

\[
V_1 = \bar{v} + v_1 \tag{C.3a}
\]

\[
P = \bar{p} + p \tag{C.3b}
\]

where \( \bar{v} \) and \( \bar{p} \) are constant fields but \( v_1 \) and \( p \) are fluctuating fields.

Substituting \( V_1 \) in Eq.(C.3a) and \( P \) in Eq.(C.3b) into Eq.(C.2) we obtain

\[
\rho \frac{dV_1}{dt} = -\frac{\partial \bar{p}}{\partial x_1} + \mu \frac{\partial^2 \bar{v}}{\partial x_j \partial x_j} \tag{C.4}
\]

The total derivative \( \frac{dV_1}{dt} \) can be decomposed as follows:
\[
\frac{dV_i}{dt} = \frac{\partial V_i}{\partial t} + \frac{\partial V_i}{\partial x_i} v_i \\
= \frac{\partial V_i}{\partial t} + (v_i + \bar{v_i}) \frac{\partial V_i}{\partial x_i} \\
= \frac{\partial V_i}{\partial t} + (v_i + \bar{v}) \frac{\partial V_i}{\partial x_i}
\]

Where we have assumed that \( \bar{v_i} = \bar{v} \) for all \( i = 1,2,3 \).

Substituting \( \frac{dV_i}{dt} \) in Eq. (C.5) into Eq. (C.4) we obtain

\[
\rho \frac{\partial V_i}{\partial t} + \rho \bar{v} \frac{\partial V_i}{\partial x_i} = -\rho v_i \frac{\partial V_i}{\partial x_i} - \frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 V_i}{\partial x_i \partial x_j}
\]

The right hand side of Eq. (C.6) can be neglected in comparison with the left hand side. This is because \( v_i << \bar{v}, p \propto v_i^2 << \bar{v} \) and \( \mu \) is very small. Under this condition, Eq. (C.6) then becomes

\[
\frac{\partial}{\partial t} = -\bar{v} \frac{\partial}{\partial x_1} \quad (C.7a)
\]

or

\[
\frac{\partial x_1}{\partial t} = -\bar{v} \frac{\partial}{\partial t} \quad (C.7b)
\]

Integrating both sides of Eq. (C.7b), we obtain the equation

\[
x_i = x_0 = \bar{v}(t_i - t_0)
\]

where \( x_0 \) and \( t_0 \) have an arbitrary time and space origin. Equation (C.8) is another way of saying that the whole turbulent eddies are considered to move as a whole with constant velocity \( \bar{v} \) without worrying about the small random fluctuating component \( v_i \).
C.3 TAYLOR HYPOTHESIS IN SHEAR FLOW

The foregoing analysis of Taylor's hypothesis is based on the fact that the main flow is uniform. In the turbulent boundary layer, however, the flow is not uniform. The question is then whether Taylor's hypothesis can still be applied without any limit.

Lin (1953) is the first one to study the applicability of Taylor's hypothesis to shear flow. From some intuitive thought, he set up the following condition:

\[ \frac{\partial \bar{v}}{\partial y} \ll \bar{v} \theta \quad \text{(C.9)} \]

where \( K \) is the eddy spatial frequency and \( y \) is in the direction normal to the boundary. The meaning of Eq.(C.9) can be understood if we write Eq.(C.9) again in the following form:

\[ \frac{\partial \bar{v}}{\partial y} (\lambda) \ll \bar{v} \quad \text{(C.10)} \]

where \( \lambda \) is the eddy size corresponding to the spatial frequency \( K \). Thus, if the velocity change within the distance corresponding to the eddy size \( \lambda \) is smaller than the convection velocity, then the Taylor hypothesis still applies for the shear flow.

Lumley (1965) derived from physical grounds more or less similar constraints but phrased differently. This constraint is related to the eddy energy density \( E(K) \). An eddy of size \( K \) has an energy per unit mass roughly \( KE(K) \). The characteristic velocity is thus about \( (KE)^{\frac{1}{2}} \). The characteristic time scale is then \( 2\pi/K(KE)^{\frac{1}{2}} = 2\pi(K^3E)^{-\frac{1}{2}} \). The time
required for the eddy moving with velocity \( \bar{v} \) to pass through a distance of about \( \frac{2\pi}{k} \) is then \( \frac{2\pi}{k\bar{v}} \). The frozen turbulence condition is satisfied then, whenever

\[
\frac{2\pi}{k\bar{v}} \ll \frac{2\pi}{k^{3/2}E^{1/2}}
\]

or

\[
E(K) \ll \frac{\bar{v}^2}{K}
\]

Another criterion derived by Lumley (1965) is to say that the velocity change within the scale \( \frac{2\pi}{k} \) should be much smaller than the characteristic velocity \((KE)^{1/3}\). This condition will guarantee the isotropy of the turbulent field. We thus have

\[
\frac{\partial \bar{v}}{\partial y} \frac{2\pi}{k} \ll (KE)^{1/3}
\]

or

\[
E(K) \gg \frac{4\pi^2}{(\frac{\partial \bar{v}}{\partial y})^2 K^{-3}}
\]

Equation (C.11) and Eq.(C.12) can be combined to give

\[
4\pi^2 (\frac{\partial \bar{v}}{\partial y})^2 K^{-3} \ll E(K) \ll \frac{\bar{v}^2}{K}
\]

If both Eq.(C.10) and Eq.(C.13) are satisfied, we may then say that the turbulence is isotropic, frozen, and that the eddy moves with essentially the same speed \( \bar{v} \).

C.4 APPLICABILITY OF TAYLOR HYPOTHESIS TO PROPAGATION PROBLEM

In the propagation problem we often want to find the temporal spectrum. Taylor hypothesis provides a means to equate the spatial correlation with time correlation. Suppose \( f \) is the physical quantity of interest. The spatial correlation function \( R_f(a) \) is defined by
\( R_f(a) = <f(x)f(x-a)> \)

The time correlation function \( R_f(t) \) is defined by \( R_f(t) = <f(t)f(t-T)> \)

If the Taylor hypothesis is true, i.e., \( a = \nu t \), we can then set \( R_f(a) = R_f(t) \).

From the above discussion we know that the Taylor hypothesis may not be quite perfect for shear flow close to the ground boundary layer which is about one meter thick at most. We consider in this dissertation propagation between two moving vehicles which should be far above the ground. Thus, the possible shear flow near the ground should not cause a problem in our propagation study.

Another question is whether the outer scale should impose any limitation on applying the Taylor hypothesis to the free space propagation problem. Tatarskii (1961) derived a frozen turbulence condition concerning outer scale as follows:

\[ \sqrt{L\lambda} \ll L_0 \]

where

\( L_0 \) is the outer scale

\( L \) is the propagation distance

\( \lambda \) is the wavelength
The outer scale $L_0$ increases with height above ground. Thus, Eq.(C.14) can always be satisfied and is even not a constraint.

We may conclude in this appendix that as long as the turbulence is low and the flow is uniform, the Taylor hypothesis is well justified.
APPENDIX D

SOME PHYSICAL EXPLANATION ON THE DETERMINISTIC PHASE SHIFT

D.1 INTRODUCTION

In Chapter 2, we discussed the origin of the deterministic phase shift $S_d$. It is shown there that the phase shift is zero if the source moves in a direction perpendicular to the radial vector connecting the source and the receiver. That is the same as the well-known Doppler shift in the direction sense. We shall show here that quantitatively $S_d$ is also in consistence with the Doppler shift. In addition, we are going to explain this Doppler shift from the fundamental physics for the case that the source moves both with a transverse and longitudinal velocity with respect to the receiver.

D.2 PHYSICAL EXPLANATION OF DOPPLER SHIFT

Consider first that the source and the receiver are stationary as shown in Figure 40(a). At $t = 0$ the source starts sending out waves with speed $c$ and frequency $f_0$. After $t = T$ seconds, the observer sees a total of $N = f_0T$ waves in $L = CT$ meters. The wavelength $\lambda_0$ is then given by

$$\lambda_0 = \frac{L}{N} = \frac{CT}{f_0T} = \frac{c}{f_0}$$  \hspace{1cm} (D.1)
Figure 40. (a) The Source and Receiver are Both Stationary
(b) The Case Associated with the Longitudinal Source Velocity $v_L$
(c) The Case Associated with the Transverse Source Velocity $v_t$

Consider now that the source starts to move with respect to the receiver. We may resolve at any time the source velocity into two components, one longitudinal velocity $v_L$, and one transverse velocity $v_t$. We then investigate their effect separately below.
Figure 40(b) shows the case associated with the longitudinal velocity \( v_L \). After \( t = T \) seconds, the source has moved along the longitudinal direction a distance \( v_L T \) and the observer at \( R \) sees a total \( N = f_0 T \) waves in \( L - v_L T \) meters. The new wavelength the observer sees is then

\[
\lambda' = \frac{L - v_L T}{f_0 T} = \frac{1}{f_0} \left( \frac{L}{T} - v_L \right)
\]

\[
= \frac{C - v_L}{f_0}
\]

where \( C = \frac{L}{T} \) is the wave propagation velocity. The new frequency \( f' \) is hence given by

\[
f' = \frac{C}{\lambda'} = \frac{C}{C - v_L} f_0
\]

(F.2)

Figure 40(c) shows the case associated with the transverse velocity component \( v_t \). Because \( v_t \) is perpendicular to \( SR \) the observer still sees a total of \( N = f_0 T \) waves in \( L \) meters. Hence the frequency shift is zero.

In general, the velocity \( v_L \) is much smaller than \( c \). Hence we may approximate Eq. (F.3) by the binomial series expansion:

\[
f' = \left( \frac{1}{1 - \frac{v_L}{c}} \right) f_0 = (1 + \frac{v_L}{c} + \frac{v_L^2}{c^2} + \cdots) f_0
\]

\[
= (1 + \frac{v_L}{c}) f_0 = f_0 + \frac{v_L}{c} f_0
\]

where we have neglected all the second order terms. The observer thus sees a frequency shift given by

\[
\Delta f = f' - f_0 = \frac{v_L}{c} f_0
\]

(F.4)
D.3 CONNECTION OF $S_d$ TO DOPPLER SHIFT

The deterministic phase shift from Eq.(2.89b) is rewritten here for convenience.

$$S_d(r,t) = \frac{\hbar k}{w_0} [\bar{v}_t \cdot (\bar{r}_r - \bar{r}_t)]$$  \hspace{1cm} (D.5)

The transverse velocity component which is perpendicular to $\bar{r}_r - \bar{r}_t$ produces zero phase shift. We thus only have to consider the radial velocity component $v_\parallel$.

From Figure 5, we see that $\bar{r}_r - \bar{r}_t$ is a vector from the source to the receiver and is along the longitudinal direction. We thus rename $\bar{r}_r - \bar{r}_t$ as $\bar{r}_G$ and Eq.(D.5) becomes

$$S_d = \frac{k^2}{w_0} \bar{v}_t \cdot \bar{r}_G$$  \hspace{1cm} (D.6)

Defining the velocity $\bar{v}_t$ along the longitudinal direction to be $v_\parallel$, $S_d$ in Eq.(D.6) then becomes

$$S_d = \frac{k^2}{w_0} [\bar{v}_t \cdot \frac{\bar{r}_G}{|\bar{r}_G|}]$$

$$= \frac{k^2}{w_0} v_\parallel |\bar{r}_G|$$  \hspace{1cm} (D.7)

The average frequency shift within $T$ seconds is then

$$\Delta \omega = \frac{S_d}{T} = \frac{w_0}{c^2} \frac{v_\parallel}{T} |\bar{r}_G|$$

$$= \frac{w_0}{c^2} \frac{v_\parallel}{T}$$

$$= \frac{v_\parallel}{c} w_0$$  \hspace{1cm} (D.8)

Equation (D.8) thus agrees with the result obtained from the classic Doppler shift described earlier.
APPENDIX E
ESTIMATED WAVE EQUATION IN
THE ROTATING COORDINATE SYSTEM

In the turbulent atmosphere, \( \varepsilon \) is in general a function of space variable and time. The wave equation thus contains the derivative of \( \varepsilon \) with respect to time. In the stationary coordinate system, the term containing time derivative of \( \varepsilon \) is in general too small to be considered. In this appendix, we want to show the same thing is true in the rotating coordinate system.

In the stationary coordinate system, the equation is given by

\[
\nabla \varepsilon = -\mu \frac{\partial \varepsilon}{\partial t^2} \quad (E.1)
\]

\[
\nabla \varepsilon = \frac{\partial \varepsilon}{\partial t^2} = \frac{\partial \varepsilon}{\partial t^2} \quad (E.2)
\]

Equation (E.1) and Eq.(E.2) can be combined to give the equation

\[
\nabla^2 \varepsilon - \mu \varepsilon \frac{\partial^2 \varepsilon}{\partial t^2} - 2 \mu \varepsilon_0 \frac{\partial \varepsilon}{\partial t} \frac{\partial \varepsilon}{\partial t} - \mu \varepsilon_0 \frac{\partial^2 \varepsilon}{\partial t^2} = 0 \quad (E.3)
\]

where \( \varepsilon = \varepsilon_0 (1+\varepsilon_1) \)

Assume an eddy with size \( l_0 \) and velocity \( v_0 \), then the fourth term is proportional to \( \mu \varepsilon_1 \varepsilon_0 v_0^2 / l_0^2 \) and the third term is proportional to \( \mu \varepsilon_0 \varepsilon_1 v_0 \omega l_0 / l_0 \). Clearly, the fourth term can be neglected and Eq.(E.3) becomes

200
To transform Eq.(E.4) in the rotating coordinate system, we may use the transformation law of Eq.(3.15) and the estimation result of Eq.(3.24) to give the equation:

\[
\begin{align*}
\nabla^2 E - \mu_0 \frac{\partial^2 E}{\partial t^2} - 2 \mu_0 \epsilon_1 \frac{\partial E}{\partial t} \epsilon_1 \frac{\partial E}{\partial t} = 0 \quad (E.4)
\end{align*}
\]

Using the approximation from Eq.(3.20a) to Eq.(3.21b) and the estimation

\[
\begin{align*}
\frac{\partial \epsilon_1}{\partial x} & \approx \frac{\partial \epsilon_1}{\partial y} = \frac{\epsilon_1}{\lambda_0} \quad (E.7) \\
\frac{\partial \epsilon_1}{\partial t} & \approx \frac{\epsilon_1 \nu}{\lambda_0} \quad (E.8)
\end{align*}
\]

It can be found quite easily that the ratio of each term of Eq.(E.6) to \(2y \frac{\partial \epsilon}{\partial t} \frac{\partial^2 E}{\partial x \partial t} \) is proportional to \(\frac{\epsilon_1 \nu}{\lambda_0 \omega_0 \nu} \) or \(\frac{\epsilon_1 \nu}{\lambda_0 \omega_0 \nu} \). With \(\epsilon_1 \sim 10^{-6} \), \(\omega_0 \) equal to optical frequency and \(k \) equal to optical wavelength, it is quite obvious that Eq.(E.6) can indeed be neglected.
APPENDIX F

SIMPLIFICATION OF WAVE EQUATION IN
THE ROTATING COORDINATE SYSTEM

In this appendix, we show the details why the last three terms of Eq.(3.26) can be neglected. Each term of Eq.(3.26) can be roughly estimated as

\[
S_1 = w_0^2 \psi \\
|S_2| = 2w_0 \frac{\partial \psi}{\partial t} \approx w_0 k \nu \psi \\
|S_3| = 2y w_0 \frac{\partial \theta}{\partial t} \frac{\partial \psi}{\partial y} \approx (w_0 k \nu) \left( \frac{\nu}{L} \right) \psi \\
|S_4| = 2w_0x \frac{\partial \theta}{\partial t} \frac{\partial \psi}{\partial y} \approx (w_0 k \nu) \left( \frac{\nu}{L} \right) \psi \\
|S_5| = \frac{\partial^2 \psi}{\partial t^2} = k^2 \nu^2 \psi \\
|S_6| = 2y \frac{\partial \theta}{\partial t} \frac{\partial^2 \psi}{\partial x \partial t} \approx y \left( \frac{\nu}{L} \right) \left( k^2 \nu \right) \psi \\
|S_7| = 2x \frac{\partial \theta}{\partial t} \frac{\partial^2 \psi}{\partial y \partial t} \approx x \left( \frac{\nu}{L} \right) \left( \frac{\nu}{x} \right) k^2 \nu = y \left( \frac{\nu}{L} \right) \left( k^2 \nu \right) \psi
\]

where we have used

\[
\frac{\partial \theta}{\partial t} = \left( \frac{\nu}{L} \right) \cos^2 \theta \\
\frac{\partial \psi}{\partial x} \approx k \psi \\
\frac{\partial \psi}{\partial y} \approx k \left( \frac{\nu}{x} \right) \psi \\
\frac{\partial \psi}{\partial t} \approx k \nu \psi
\]
The ratio of $S_i/S_1$, $i = 2,7$ is then given by the following set of equations:

\[
\frac{S_2}{S_1} = \frac{v}{c} \quad (F.9a)
\]

\[
\frac{S_3}{S_1} = \frac{S_4}{S_1} = \left(\frac{v}{c}\right)\left(\frac{v}{L}\right) \quad (F.9b)
\]

\[
\frac{S_5}{S_1} = \frac{v^2}{c^2} \quad (F.9c)
\]

\[
\frac{S_6}{S_1} = \frac{S_7}{S_1} = \left(\frac{v^2}{c^2}\right)\left(\frac{v}{L}\right) \quad (F.9d)
\]

Equation (F.9) shows $S_5$, $S_6$, $S_7$ are an order of magnitude $v^2/c^2$ smaller than $S_1$ and can be neglected. Neglecting $S_5$, $S_6$, and $S_7$, Eq.(3.26) is written as

\[
\nabla^2 \psi = \mu_0 \left(-\omega_0^2 \psi - \omega_0^2 \epsilon_1 \psi - 2jw_0 \frac{\partial \psi}{\partial t} - 2jw_0 \frac{\partial \theta}{\partial t} \frac{\partial \psi}{\partial x} + 2jw_0 \frac{\partial \theta}{\partial t} \frac{\partial \psi}{\partial y} \right)
\]

\[(F.10)\]

where we have used Eq.(3.28) and the multiplication of $\epsilon_1$ with the last three terms of Eq.(F.10) are neglected because they are an order of magnitude $\epsilon_1 (\frac{v}{c})$ compared with the first term.

Equation (F.10) is the same as Eq.(3.30) in the text.
APPENDIX G
JUSTIFICATION OF CONSTANT PROPAGATION
RANGE WITHIN THE CORRELATION TIME

In this appendix, we justify the approximation that $x_{t_1}$ and $x_{t_2}$ can be approximated by $x_0$. The exact power spectrum is given by Eq. (3.139). It depends in a quite complicated manner on $t_1$ and $t_2$ and prevents any further simplifications. With the approximation of $x_{t_1}$ and $x_{t_2}$ by $x_0$, more physical insights and simplifications may be obtained.

There are two approaches used in justifying the approximations. One approach is to show that the propagation range maintains nearly constant value within the correlation time of interest. The other approach is to show the same thing by making a Taylor series expansion of the exact power spectrum versus $\Delta = x_{t_1} - x_{t_2}$.

In order for the propagation distance to maintain nearly constant value, the following equation must be true:

$$\frac{x_{t_1} - x_{t_2}}{x_0} < 1 \quad (G.1)$$

where

$$x_0 = \sqrt{L^2 + v^2} \delta^2 \quad (G.2)$$

and the expressions for $x_{t_1}$ and $x_{t_2}$ given in Eq. (3.142) and Eq. (3.143) are rewritten here

$$x_{t_1} = \sqrt{L^2 + v^2} \left(\delta + 1/2 \tau\right)^2 \quad (G.3)$$
Using binomial expansion, \( x_{t_1} \) and \( x_{t_2} \) can be approximated by

\[
x_{t_1} \approx \sqrt{L^2 + v^2 \delta^2} + \frac{1}{2} \frac{v^2 \delta t}{\sqrt{L^2 + v^2 \delta^2}}
\]

\[
x_{t_2} \approx \sqrt{L^2 + v^2 \delta^2} - \frac{1}{2} \frac{v^2 \delta t}{\sqrt{L^2 + v^2 \delta^2}}
\]

Substituting the expressions in Eq.(G.2), Eq.(G.5), and Eq.(G.6) into Eq.(G.1), the requirement for the approximation becomes

\[
\frac{v^2 \delta t}{(\sqrt{L^2 + v^2 \delta^2})^2} < 1
\]

or

\[
\tau < \frac{L^2 + v^2 \delta^2}{v^2 \delta}
\]

Equation (G.7) defines another time scale \( \tau_2 \) which is equal to the quantity on the right-hand side of Eq.(G.7). \( \tau_2 \) is given by

\[
\tau_2 = \frac{L^2 + v^2 \delta^2}{v^2 \delta}
\]

\( \tau_2 \) actually is the maximum time period that can be allowed in order to have constant propagation distance.

In order for our approximation to hold, the propagation distance should have nearly constant value within the correlation time of interest. The correlation time can be predicted from the outer scale, \( L_0 \), of the refractive index inhomogeneities. Based on \( L_0 \), the correlation time \( \tau_1 \) is given by

\[
\tau_1 = \frac{L_0}{v}
\]
\( \tau_1 \) must be smaller than \( \tau_2 \) in order for conditions given by Eq.(G.1) to hold. Similarly, the time deviation \( \tau \) must be smaller than \( \tau_1 \). In summary, we require the following condition

\[
\tau < \tau_1 < \tau_2 \tag{G.10}
\]

Now we prove that \( \tau_1 \) is always smaller than \( \tau_2 \). Assuming outer scale \( L_0 \) to be equal to vertical propagation distance \( L \), then Eq.(G.8) becomes

\[
\tau_2 = \frac{L_0^2 + v^2 \delta^2}{v^2 \delta} \tag{G.11}
\]

for \( L_0 < v \delta \), \( \tau_2 \) is approximately given by

\[
\tau_2 = \frac{v^2 \delta^2}{v^2 \delta} = \delta > \frac{L_0}{v} > \tau_1 \tag{G.12}
\]

For \( L_0 > v \delta \), \( \tau_2 \) is approximately given by

\[
\tau_2 = \frac{L_0^2}{v^2 \delta} = \frac{L_0}{v} \frac{L_0}{v \delta} = \tau_2 \frac{L_0}{v \delta} > \tau_1 \tag{G.13}
\]

where we have used the factor \( L_0/v \delta \) is greater than 1. Equation (G.12) and Eq.(G.13) thus show that \( \tau_2 \) is greater than \( \tau_1 \). Combining Eq.(G.12) and Eq.(G.13), we thus have

\[
\tau_1 < \tau_2 \tag{G.14}
\]

Equation (G.14) shows that the correlation time based on outer scale \( L_0 \) is smaller than \( \tau_2 \).

The angle of arrival power spectrum is computed from angle of arrival correlation function. The contribution to angle of arrival power spectrum is mainly from the region where angle of arrival functions are highly correlated or during the correlation time period. In order not to affect the computation of power spectrum, the approximation of \( x_{\tau_1} \) and \( x_{\tau_2} \) by \( x_0 \) has to be true at least within the correlation
time. According to our earlier discussion, the propagation distance can maintain nearly constant value within time period as large as $\tau_2$. Because the correlation time $\tau_1$ is smaller than $\tau_2$, the propagation distance can maintain constant value within $\tau_1$. Hence, the approximation of $x_{t_1}$ and $x_{t_2}$ by $x_0$ is justified.

Now we start using the second approach to justify quantitatively the approximation. We first make an expansion of the exact power spectrum about the point $\Delta = x_{t_1} - x_{t_2}$. By expanding the first two sinc(x) functions of Eq.(3.139) by $\sin x/x$, we have

$$W_0(w) = \frac{8\pi}{LA^2} \int \int \int x_{t_1} x_{t_2} \phi_n \exp \left[ jK'' y (\tau) \sigma \right]$$

$$- jK'' y \omega \tau - j\omega \tau] \sin \left[ \frac{\alpha}{2} K'' \frac{L}{y x_{t_1}} (1-\sigma) \right]$$

$$\sin \left[ \frac{\alpha}{2} K'' \frac{L}{y x_{t_2}} (1-\sigma) \right] \sin \left[ \frac{\alpha}{2} K'' \frac{L}{y x_{t_2}} (1-\sigma) \right]$$

$$\cos \left( K'' \frac{L}{y x_{t_1}} + K'' \frac{L}{x_{t_1}} \right) \frac{x_{t_0} (1-\sigma) \sigma}{2k} \cos \left[ \frac{\sqrt{K''^2 L^2}}{x_{t_2}} + \frac{K''^2}{K''} \right]$$

$$\frac{\sigma (1-\sigma) x_{t_2}}{2k} \int dK'' dK'' d\omega d\tau \quad \text{(G.15)}$$

Introducing the variables $\Delta$ and $x_0$ which are given by

$$\Delta = x_{t_1} - x_{t_2} \quad \text{(G.16)}$$

$$x_0 = \frac{1}{2} (x_{t_1} + x_{t_2}) \quad \text{(G.17)}$$

Equation (G.15) then becomes
\[ W(\Delta) = \frac{8\pi}{L \Delta^2} \iint \int (x_0 + \frac{1}{2}\Delta)(x_0 - \frac{1}{2}\Delta) \Phi_n \exp\left[jK''y \tau \sigma - jK''v_2 \omega t - j\omega t\right] \sin[a_{K''} y \frac{L(1-\sigma)}{x_0 + \frac{1}{2}\Delta}] \sin[a_{K''} \frac{L(1-\sigma)}{x_0 - \frac{1}{2}\Delta}] \{\text{sinc}\left[\frac{a_{K''}}{2} K''(1-\sigma)\right]\}^2 \]

\[
\cos[(K''^2 \frac{L^2}{y^2} + K''^2 \frac{L^2}{z^2}) \frac{(x_0 + \frac{1}{2}\Delta) \sigma(1-\sigma)}{2k}] \] 

\[
\cos[(K''^2 \frac{L^2}{y^2} + K''^2 \frac{L^2}{z^2}) \frac{\sigma(1-\sigma)}{2k}] \frac{dK''^2 dK'' \omega d\tau}{y^2 z^2} \]

\[ = \frac{8\pi}{L \Delta^2} \iint \int (x_0 + \frac{1}{2}\Delta)(x_0 - \frac{1}{2}\Delta) \sin C_1 \sin C_2 \cos C_3 \]

\[
\cos C_4 \cdot f \frac{dK''^2 dK'' \omega d\tau}{y^2 z^2} \quad (G.18) \]

where

\[ C_1 = \frac{a_{K''} L(1-\sigma)}{2 y (x_0 + \frac{1}{2}\Delta)} \quad (G.19) \]

\[ C_2 = \frac{a_{K''} L(1-\sigma)}{2 y (x_0 - \frac{1}{2}\Delta)} \quad (G.20) \]

\[ C_3 = \frac{(K''^2 \frac{L^2}{y^2} + K''^2 \frac{L^2}{z^2}) \frac{\sigma(1-\sigma)}{2k}(x_0 + \frac{1}{2}\Delta)}{(x_0 + \frac{1}{2}\Delta)^2} \quad (G.21) \]

\[ C_4 = \frac{(K''^2 \frac{L^2}{y^2} + K''^2 \frac{L^2}{z^2}) \frac{\sigma(1-\sigma)}{2k}(x_0 - \frac{1}{2}\Delta)}{(x_0 - \frac{1}{2}\Delta)^2} \quad (G.22) \]

\[ f = \Phi_n \exp\left[jK'' y \tau \sigma - jK'' v_2 \omega t - j\omega t\right] \{\text{sinc}\left[\frac{a_{K''}}{2} K''(1-\sigma)\right]\}^2 \quad (G.23) \]

The Taylor series expansion of \( W(\Delta) \) about \( \Delta = 0 \) is given by the following equation

\[ W(\Delta) = W(\Delta) \bigg|_{\Delta = 0} + \frac{W''(\Delta)}{2} \bigg|_{\Delta = 0} \Delta^2 + \ldots \quad (G.24) \]
The first term of Eq. (G.24) is easily written as

$$W(\Delta)|_{\Delta=0} = \frac{8\pi}{L\alpha} \iiint x_0^2 \sin^2D \cos^2E \int \frac{dK''}{y} \frac{dK''}{z} \sigma \, d\tau$$  \hspace{1cm} (G.25)

where

$$D = C_1|_{\Delta=0} = C_2|_{\Delta=0} = \frac{a}{2} \frac{K''}{y} \frac{L(1-\sigma)}{x_0}$$  \hspace{1cm} (G.26)

$$E = C_3|_{\Delta=0} = C_4|_{\Delta=0} = \left(\frac{K''^2}{y^2} + \frac{K'^2}{z^2}\right) \frac{\sigma(1-\sigma)}{2K} \, x_0$$  \hspace{1cm} (G.27)

Also at \(\Delta=0\), \(x_{t_1}\) is equal to \(x_{t_2}\). This implies that \(\tau=0\) and \(x_{t_1} = x_{t_2} = x_0\).

$$x_0 = \beta|_{\Delta=0}$$  \hspace{1cm} (G.28)

The second term of Eq. (G.24) contains the first derivative of \(W\) which is given below

$$\frac{\partial W}{\partial \Delta} = \frac{8\pi}{L\alpha} \iiint \left[\left(\frac{1}{2}(x_0 - \frac{1}{2}\Delta) - \frac{1}{2}(x_0 + \frac{1}{2}\Delta)\right) \sin C_1 \sin C_2 \cos C_3 \cos C_4 \int \frac{dK''}{y} \frac{dK''}{z} \sigma \, d\tau + \iiint (x_0 + \frac{1}{2}\Delta) \cdot (x_0 - \frac{1}{2}\Delta) \right] \left[\cos C_1 \sin C_2 \frac{\partial C_1}{\partial \Delta} + \sin C_1 \cos C_2 \frac{\partial C_2}{\partial \Delta} \right] \cdot$$

$$\cos C_3 \cos C_4 \int \frac{dK''}{y} \frac{dK''}{z} \sigma \, d\tau + \iiint (x_0 + \frac{1}{2}\Delta) \cdot (x_0 - \frac{1}{2}\Delta) \sin C_1 \sin C_2 \left[-\sin C_3 \cos C_4 \frac{\partial C_3}{\partial \Delta} - \right.$$  

$$\cos C_3 \sin C_4 \frac{\partial C_4}{\partial \Delta} \int \frac{dK''}{y} \frac{dK''}{z} \sigma \, d\tau \right]$$  \hspace{1cm} (G.29)

where

$$\frac{\partial C_1}{\partial \Delta} = \frac{a}{4} \left(\frac{K''}{y} \frac{L(1-\sigma)}{(x_0 + \frac{1}{2}\Delta)^2}\right)$$  \hspace{1cm} (G.30)

$$\frac{\partial C_2}{\partial \Delta} = \frac{a}{4} \left(\frac{K''}{y} \frac{L(1-\sigma)}{(x_0 + \frac{1}{2}\Delta)^2}\right)$$  \hspace{1cm} (G.31)
At $A=0$, Eq.(G.30), Eq.(G.31), Eq.(G.32), and Eq.(G.33) become

$$\frac{\partial C_3}{\partial \Delta} = \left( K''^2 \frac{L^2}{(x_0 + \frac{1}{2} \Delta)^2} + K''^2 \frac{\sigma(1-\sigma)}{4k} \right) - \frac{K''^2 L^2}{(x_0 + \frac{1}{2} \Delta)^2} \frac{\sigma(1-\sigma)}{2k} \quad (G.32)$$

$$\frac{\partial C_4}{\partial \Delta} = - \left( K''^2 \frac{L^2}{(x_0 - \frac{1}{2} \Delta)^2} + K''^2 \frac{\sigma(1-\sigma)}{4k} \right) - \frac{\sigma(1-\sigma)}{2k} \frac{K''^2 L^2}{(x_0 - \frac{1}{2} \Delta)^2} \quad (G.33)$$

At $\Delta=0$, Eq.(G.30), Eq.(G.31), Eq.(G.32), and Eq.(G.33) become

$$\frac{\partial C_1}{\partial \Delta} \bigg|_{\Delta=0} = - \frac{\partial C_2}{\partial \Delta} \bigg|_{\Delta=0} = - \frac{a_{n} K''^2}{4y} L(1-\sigma) \frac{1}{x_0^2} \quad (G.34)$$

$$\frac{\partial C_3}{\partial \Delta} \bigg|_{\Delta=0} = - \frac{\partial C_4}{\partial \Delta} \bigg|_{\Delta=0} = \left( K''^2 \frac{L^2}{x_0} + K''^2 \frac{\sigma(1-\sigma)}{4k} \right) - \frac{\sigma(1-\sigma)}{2k} \frac{K''^2 L^2}{x_0^2} \quad (G.35)$$

Substituting the expressions in Eq.(G.26), Eq.(G.27), Eq.(G.34), and Eq.(G.35) into Eq.(G.29), it is easily shown that

$$\frac{\partial W}{\partial \Delta} \bigg|_{\Delta=0} = 0 \quad (G.36)$$

This means that the errors due to the approximation $x_{t_1} = x_{t_2} = x_0$ is of second order smallness.

The second order derivative of $W$ is given in Eq.(G.37)
\[
\frac{\partial^2 W}{\partial \Delta^2} = \frac{8\pi}{LA^2} \left\{ \int \int \int - \frac{1}{2} \sin C_1 \sin C_2 \cos C_3 \cos C_4 
\right. \\
+ \int \int \int \frac{1}{2} \Delta [\cos C_1 \sin C_2 \frac{\partial C_1}{\partial \Delta} + \\
\sin C_1 \cos C_2 \frac{\partial C_2}{\partial \Delta}] \cos C_3 \cos C_4 f dK^1 dK^2 d\Omega \\
+ \int \int \int - \frac{1}{2} \Delta [\sin C_3 \frac{\partial C_3}{\partial \Delta} \cos C_4 - \sin C_4 \cos C_3 \frac{\partial C_4}{\partial \Delta}] \\
\sin C_1 \sin C_2 f dK^1 dK^2 d\Omega + \int \int \int (x_0 + \frac{1}{2} \Delta) (x_0 - \frac{1}{2} \Delta) \\
[\cos C_1 \sin C_2 \frac{\partial C_1}{\partial \Delta} + \sin C_1 \cos C_2 \frac{\partial C_2}{\partial \Delta}] \\
[- \sin C_3 \cos C_4 \frac{\partial C_3}{\partial \Delta} - \cos C_3 \sin C_4 \frac{\partial C_4}{\partial \Delta}] \\
\left. \ight. \\
+ \sin C_1 \cos C_2 \frac{\partial C_2}{\partial \Delta} \right\} \\
(G.37)
\]

In fashion similar to that discussed before, at \( \Delta = 0 \), all terms except the first term in Eq.(G.37) are zero. Hence,

\[
\frac{\partial^2 W}{\partial \Delta^2} \bigg|_{\Delta = 0} = \frac{8\pi}{LA^2} \int \int \int - \frac{1}{2} \sin^2 D \cos^2 E f dK^1 dK^2 d\Omega \\
(G.38)
\]

Combining Eq.(G.38) and Eq.(G.25), the Taylor series expansion of Eq.(G.24) then becomes

\[
W(\Delta) = \frac{8\pi}{LA^2} [\int \int \int \sin^2 D \cos^2 E f dK^1 dK^2 d\Omega] x_0^2 \\
- \frac{8\pi}{LA^2} [\int \int \int \sin^2 D \cos^2 E f dK^1 dK^2 d\Omega] (-\frac{1}{4}) \Delta^2 \\
+ \ldots \\
(G.39)
\]

In order for the second term to be smaller than the first term, we require

\[
-\frac{1}{4} \Delta^2 + x_0^2 < 0 \\
(G.40)
\]
\[ \Delta < 2 x_0 \]  

Equation (G.40) also suggests \( x_{t_1} - x_{t_2} < 2 x_0 \). But this is just the same condition we use in Eq.(G.1). Thus, the quantitative analysis further justifies what we have already discussed before. Therefore, we conclude that the approximation \( x_{t_1} = x_{t_2} \approx x_0 \) is justified.
APPENDIX H
DETECTOR SIGNAL-TO-NOISE RATIO

In the discussion of experimental setup in Chapter 4, one of the questions is what should be the desirable input signal-to-noise ratio of the photodiode. Naturally, the signal power should be stronger than the noise power. But it is not necessary to have the signal power stronger than required. In this appendix, statistical detection theory is used to determine the input signal-to-noise ratio.

At the receiver, noise \( n(t) \) is added to signal \( s(t) \) to give the output \( y(t) = s(t) + n(t) \). At any given time \( t \), a decision is made based on observation \( y(t) \) whether correct detection is made. Using the criterion of maximum a posteriori probability [86], correct decision is made whenever

\[
P[y/H_1] \geq P[y/H_0]
\]

(H.1)

where

\( H_1: \) hypothesis with signal present
\( H_0: \) hypothesis with no signal present

\( P[y/H_1] \): probability of \( y \) when \( H_1 \) is time

\( P[y/H_0] \): probability of \( y \) when \( H_0 \) is time
Assuming noise $n(t)$ is Gaussian distributed with zero mean and variance $\sigma$, then

$$P[y/H_1] = \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{(y-s)^2}{2\sigma^2}}$$

(H.2)

$$P[y/H_0] = \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{y^2}{2\sigma^2}}$$

(H.3)

Substituting Eq.(H.3) and Eq.(H.2) into Eq.(H.1), we have

$$\frac{-\frac{(y-s)^2}{2\sigma^2}}{e^{-\frac{y^2}{2\sigma^2}}} \geq \frac{-\frac{y^2}{2\sigma^2}}{e^{-\frac{(y-s)^2}{2\sigma^2}}}$$

(H.4)

Taking the natural logarithm on both sides of Eq.(H.4), we have

$$-(y-s)^2 \geq -y^2$$

or

$$y \geq \frac{s}{2}$$

(H.5)

The probability of correct decision with $H_1$ time is then

$$P[c/H_1] = \frac{1}{\sqrt{2\pi} \sigma} \int_{s/2}^{\infty} e^{-\frac{(y-s)^2}{2\sigma^2}} dy$$

(H.6)

Making new variable $z = (y-s)/\sqrt{2\sigma}$, Eq.(H.6) becomes

$$P[c/H_1] = \frac{1}{\sqrt{\pi}} \int_{s/2}^{\infty} e^{-\frac{z^2}{2}} dz$$

(H.7)

$s/\sigma$ is equal to the square root of signal power to noise power. Hence defining $s/\sigma = \sqrt{P_s/P_n}$, Eq.(H.7) then becomes

$$P[c/H_1] = \frac{1}{\sqrt{\pi}} \int_{s/2}^{\infty} e^{-\frac{z^2}{2}} dz$$

(H.8)

The criterion used is that choosing $P_s/P_n$ such that $P[c/H_1] = 0.99$. 

The criterion used is that choosing $P_s/P_n$ such that $P[c/H_1] = 0.99$. 

The equation to be solved is then
\[ \frac{1}{\sqrt{\pi}} \int_{-\gamma}^{\infty} e^{-z^2} \, dz = 0.99 \]  \hspace{1cm} (H.9)

where
\[ \gamma = \frac{1}{2\sqrt{2}} \sqrt{\frac{P_s}{P_n}} \]

However,
\[ \int_{-\gamma}^{\infty} e^{-z^2} \, dz = \int_{-\gamma}^{\gamma} e^{-z^2} \, dz + \int_{\gamma}^{\infty} e^{-z^2} \, dz \]
\[ = 2 \int_{0}^{\gamma} e^{-z^2} \, dz + \int_{\gamma}^{\infty} e^{-z^2} \, dz \]
\[ = 2 \sqrt{\pi} \operatorname{erf}\gamma + \sqrt{\pi} (1 - \operatorname{erf}\gamma) \]  \hspace{1cm} (H.10)
\[ = \sqrt{\pi} \left[ \frac{1}{2} + \frac{1}{2} \operatorname{erf}\gamma \right] \]

Substituting Eq.(H.10) into Eq.(H.9), we then require
\[ \operatorname{erf}\gamma = 0.98 \]
or
\[ \gamma = 1.64 \]  \hspace{1cm} (H.11)

Substituting the definition of \( \gamma \) from Eq.(H.9) to Eq.(H.11) then gives
\[ \frac{1}{2\sqrt{2}} \sqrt{\frac{P_s}{P_n}} = 1.64 \]
or
\[ \frac{P_s}{P_n} = 21 \]  \hspace{1cm} (H.12)

This means the desirable signal power should be 21 times the noise power in order to assume 99% detection accuracy.
In this appendix, we list the programs to compute the power spectrum and differential path contribution for both the nontracking and tracking systems.

Program 1 lists the program to compute the differential path contribution, \( D'_\alpha(\sigma, \Omega) \), and power spectrum, \( W'_\alpha(\Omega) \), for spherical wave propagation between a moving source and a stationary receiver in the nontracking system. Both the source and wind are assumed to move in the same direction. \( D'_\alpha(\sigma, \Omega) \) and \( W'_\alpha(\Omega) \) are defined in Eq.(2.119) and Eq.(2.117) and are also written here

\[
W'_\alpha(\Omega) = \int_0^1 D'_\alpha(\sigma, \Omega) \, d\sigma \quad (I.1)
\]

\[
D'_\alpha(\sigma, \Omega) = \int_{-\infty}^{\infty} \left[ \frac{\Omega^2}{(1-\sigma-\gamma)^2} + a^2 L_0^{-2}(\sigma) \right]^{-11/16} C_n^2(\sigma) \times \frac{1}{|1-\sigma-\gamma|} \cos^2 \left[ \frac{L_0(1-\sigma)}{2ka} \left( K_z' + \frac{\Omega^2}{(1-\sigma-\gamma)^2} \right) \right] \sin^2 \left[ \frac{\sigma \Omega}{2(1-\sigma-\gamma)} \right] \left[ \text{sinc} \left( \frac{\sigma K_z'}{2} \right) \right]^2 \, dK_z' \quad (I.2)
\]

where \( L \) is the vertical propagation range, \( a \) is the receiver radius, \( \Omega \) is the normalized temporal frequency, \( \gamma \) is the ratio of wind velocity versus source velocity, \( k \) is the wave number, \( \sigma \) is the normalized range, \( L_0(\sigma) \) is the outer scale, and \( C_n^2(\sigma) \) is the structure constant. Both \( C_n^2(\sigma) \) and \( L_0(\sigma) \) are functions of vertical height which are given in Eq.(2.118a) and Eq.(2.118b).
Program 2 lists the program to compute the differential path contribution \( D_n(\sigma, \Omega) \) and power spectrum \( W_n(\Omega) \) for both the source and receiver moving in the tracking system. The source and receiver move with constant velocity in an arbitrary direction. \( D_n(\sigma, \Omega) \) and \( W_n(\Omega) \) are defined in Eq.(3.170) and Eq.(3.171) and are also written here

\[
W_n(\Omega) = \int_0^1 D_n(\sigma, \Omega) \, d\sigma \quad (1.3)
\]

\[
D_n(\sigma, \Omega_n) = \int_0^\infty \frac{(1-\sigma+{\frac{1}{R_n}})^{-4/3}}{|1-\sigma-v_n|} \left[ K_m^2 + \frac{\Omega_n^2}{(1-\sigma-v_n)^2} + (1-\sigma+{\frac{1}{R_n}})^{-2}\right]^{-11/6}
\]

\[
= \frac{\sin^2 \left[ \frac{\Omega_n \cos \theta_0 \sigma}{2(1-\sigma-v_n)} \frac{a^2}{L_h} \right]}{\left[ \frac{\Omega_n^2}{|1-\sigma-v_n|^2} + \frac{K_m^2}{2kL_h^2} \right]} \left[ \sin \left( \frac{K_m \sigma a}{2L_h} \right) \right]^2
\]

\[
= \cos^2 \left[ \left( \frac{\Omega_n \cos \theta_0 \sigma}{(1-\sigma-v_n)^2} + \frac{K_m^2}{2kL_h^2} \right) \frac{L_0(1-\sigma) \sec \theta_0}{2L_h} \right] \, dK_m \quad (1.4)
\]

where \( R_n \) is the ratio of vertical propagation range versus receiver height, \( v_n \) is the ratio of relative wind velocity versus relative source velocity, \( L \) is the shortest distance between the source and receiver, \( L_h \) is the vertical propagation range between the source and receiver and \( \theta_0 \) is the tracking angle. The structure constant \( C_n^2(\sigma) \) and the outer scale \( L_0(\sigma) \) are given in Eq.(3.165) and Eq.(3.166). An exponential decay is included in the \( C_n^2(\sigma) \) model in order to explain the negligible atmospheric turbulence effect in outer space.
PROGRAM 1

C THIS PROGRAM COMPUTES CENTROID ANGLE OF ARRIVAL TEMPORAL POWER SPECTRUM FOR SPHERICAL WAVE PROPAGATION BETWEEN A MOVING SOURCE AND A STATIONARY RECEIVER
C
C EXTERNAL FSIG
C INCLUDE 'PROW.VCM'
C INTEGER*2 PRT
C LOGICAL*1 PHFLG
C
C PROP.DAT IS THE OUTFILE FOR POWER SPECTRA
C DDIFF.DAT IS THE OUTFILE FOR DIFFERENTIAL PATH CONTRIBUTION
C PROPG.RUN IS THE INPUT DATA FILE
C
CALL ASSIGN(6,"POWER.DAT")
CALL ASSIGN(8,"PDIFF.DAT")
CALL ASSIGN(7,"POWER.RUN")

C IF PHFLG IS FALSE COMPUTE POWER SPECTRA. IF PHFLG ID TRUE COMPUTE DIFFERENTIAL PATH CONTRIBUTION
C
PHFLG=.FALSE.
READ(7,35)PHFLG

35 FORMAT(1X)

IF(PHFLG)GO TO 16
PRT=6
GO TO 26

16 PRT=8

26 CONTINUE

CALL INITC(PRT)

IF(.NOT. PHFLG)GO TO 45
CALL DIFF
GO TO 55

45 CONTINUE

HERE WE COMPUTE POWER SPECTRA WITH FREQUENCY RANGING FROM 1.E-4 TO 10.0
BEG=1.E-7
DO 50 K=1,6
FRE=BEG*(10**K)
W=0.0
DO 60 L=1,5
W=W+2.0*FRE

50 CONTINUE

HERE WE INTEGRATE WITH RESPECT TO SIG FROM 0 TO 0.01
AN=0.0
BN=0.001
ANS=0.0
DO 10 I=1,2
CALL DRP16(AN,BN,FSIG,RS)
ANS=ANS+RS
WRITE(5,14)AN,BN,RS,ANS
C FORMAT(AN=F8.4,BN=F8.4,RS=F12.4,ANS=F12.4) 
10 CONTINUE
C HERE WE INTEGRATE WITH RESPECT TO SIG FROM 0.01 TO 0.1
C AN=0.01
BN=0.03
DO 30 I=1,4
CALL DRP16(AN,BN,FSIG,RS)
ANS=ANS+RS
WRITE(5,14)AN,BN,RS,ANS
C
AN=BN
BN=AN+0.02
30 CONTINUE
AN=0.09
BN=0.1
CALL DRP16(AN,BN,FSIG,RS)
ANS=ANS+RS
WRITE(5,14)AN,BN,RS,ANS
C HERE WE INTEGRATE WITH RESPECT TO SIG FROM 0.1 TO 1.0
C AN=0.1
BN=AN+0.1
DO 20 I=1,9
CALL DRP16(AN,BN,FSIG,RS)
ANS=ANS+RS
WRITE(5,14)AN,BN,RS,ANS
C
20 BN=AN+0.1
C WRITE(6,85)W,ANS
C 85 FORMAT(W=F12.4,ANS=F12.4) 
C 60 CONTINUE
C 50 CONTINUE
C 55 END SUBROUTINE INIT(PRT)
C ** THIS ROUTINE INITIALIZES ALL PARAMETERS
C INCLUDE "PROW.VCM"
C INTEGER#2 PRT
C PI=3.141592653589793
R2=0.001
DIA=1.0
HO=10.0
ML=1010.0
R=(ML-HO)
CDN=6328.0E-10/(2*PI)
C
READ(7,*)HO
WRITE(PRT,12)HO
12 FORMAT(' INITIAL PROPAGATION HEIGHT= ',F8.1)
READ(7,*)R
WRITE(PRT,15)R
15 FORMAT(' PROPAGATION RANGE= ',F11.1)
C
LO=HO
ML=HO+R
RETURN
END SUBROUTINE OIFF

C THIS ROUTINE COMPUTES THE DIFFERENTIAL PATH CONTRIBUTION
C
C INCLUDE 'PROW.VCM'
C
W=5.0E-4
DO 40 KL=1,5
W=W*10.0
WRITE(8,99)W
99 FORMAT(' W=',F8.2,\nSTA=1.0E-3
DO 10 K=1,2
AI=STA*(10.0**K)
DIN=AI
DO 10 IL=1,9
DPATH=FSIG(AI)
WRITE(8,21)AI,DPATH
21 FORMAT(' SIG=',E12.4, ' DPATH=',E12.4)
10 CONTINUE

C AI=0.91
DIN=0.01
DO 30 IL=1,10
DPATH=FSIG(AI)
WRITE(8,21)AI,DPATH
AI=AI+DIN
30 CONTINUE

RETURN
END FUNCTION FSIG(AI)

C THE FUNCTION COMPUTES THE DIFFERENTIAL PATH CONTRIBUTION
C AT FIXED TEMPORAL FREQUENCY
C
INCLUDE 'PROW.VCM'
C

\[ \text{CNH} = (1 - UA) \times \frac{HL}{HC + UA} \times (-4/3) \]
\[ \text{FA} = 1 - UA - R2 \]
\[ \text{ST} = \text{ABS}(\text{FA}) \]
\[ \text{CSI} = \text{SIN}(0.5 \times \text{W} \times UA / FA) \times 2 \]
\[ \text{FSIG} = \text{CNH} \times \text{CSI} \times \text{FKZ}(UA) / \text{ST} \]
\[ \text{RETURN} \]

FUNCTION FKZ(UA)

C THIS FUNCTION COMPUTES THE SPATIAL FREQUENCY INTEGRATION AT
C THE GIVEN NORMALIZED RANGE AND TEMPORAL FREQUENCY

EXTERNAL FCT

INCLUDE "PROW.VCM"

SIG=U
REN=PI/SIG
AI=0.
WA=W/(1-SIG-R2)
OL=LO/HD
CL=OL*HD*(((1-SIG)*HL/HD+SIG)
BI=SQR((DIA*DI)/(CL*OL)+WA*WA)
BI=BI/4.0
CALL DQG16(AI,BI,FCT,RE)
FKZ=RE
FIN=RE

20 IF(AI .GE. 1.0) GO TO 71
31 BI=31+PI GO TO 21
41 BI=31+REN GO TO 21

71 BI=31+PI
GO TO 21
41 BI=31+REN
GO TO 21
11 RETURN

FUNCTION FCT(U)

C THIS FUNCTION COMPUTES ALL SPECIAL FREQUENCY DEPENDENT TERMS
C AT FIXED TEMPORAL FREQUENCY AND NORMALIZED RANGE

INCLUDE "PROW.VCM"

WA=W/(1-SIG-R2)
OL=LO/HD
CL=OL*HD*(((1-SIG)*HL/HD+SIG)
ARG=0.5*(HL-HD)*SIG*(1-SIG)*\text{CON}*(\text{WA*WA}+\text{U*U})/(\text{DIA*DIA})
FTM=\text{COS}(\text{ARG}) \times 2
PROGRAM 2

C THIS PROGRAM COMPUTES CENTROID ANGLE OF ARRIVAL TEMPORAL
C POWER SPECTRUM FOR SPHERICAL WAVE PROPAGATION BETWEEN TWO
C MOVING VEHICLES WITH RECEIVER TRACKING THE TRANSMITTER ALL
C THE TIME
C
C EXTERNAL FSIG
C INCLUDE "PROP.VCM"
C INTEGER*2 PRT
C LOGICAL*1 PHFLG
C
C PROP.GDAT IS THE OUTFILE FILE OF POWER SPECTRA
C DIFF.DAT IS THE OUTFILE OF DIFFERENTIAL PATH CONTRIBUTION
C PROP.GRUN IS THE INPUT DATA FILE
C
C CALL ASSIGN(6,"PROP.GDAT")
C CALL ASSIGN(8,"DIFF.DAT")
C CALL ASSIGN(7,"PROP.GRUN")
C
C IF PHFLG IS FALSE COMPUTE POWER SPECTRA. IF PHFLG ID TRUE
C COMPUTE DIFFERENTIAL PATH CONTRIBUTION
C
C PHFLG=.FALSE.
C READ(7,35)PHFLG
C
C IF(PHFLG)GO TO 16
C PRT=6
C GO TO 26
C 16 PRT=8
C 26 CONTINUE
C CALL INIT(PRT)
C
C IF (.NOT. PHFLG)GO TO 45
C CALL DIFF
C GO TO 55
C 45 CONTINUE
C
C HERE WE COMPUTE POWER SPECTRA WITH FREQUENCY RANGING FROM
C 1.E-3 TO 1000.0
C BEG=1.0E-6
C DD=50 K=1.8
C FRE=BEG*(10**K)
C CMEG=0.0
C DO 50 L=1,5
C CMEG=CMEG+2.0*FRE
C 50 CONTINUE
C
C HERE WE INTEGRATE WITH RESPECT TO SIG FROM 0 TO 0.01
C AN=0.0
Here we integrate with respect to SIG from 0.01 to 0.1

AN = 0.01
BN = 0.03
DO 30 I = 1, 4
   CALL DRP16(AN, BN, FSIG, RS)
   ANS = ANS + RS
   WRITE(5, 14) AN, BN, RS, ANS
   AN = BN
   BN = AN + 0.02
30 CONTINUE
AN = 0.09
BN = 0.1
CALL DRP16(AN, BN, FSIG, RS)
ANS = ANS + RS
WRITE(5, 14) AN, BN, RS, ANS

Here we integrate with respect to SIG from 0.1 to 1.0

AN = 0.1
BN = AN + 0.1
DO 20 I = 1, 9
   CALL DRP16(AN, BN, FSIG, RS)
   ANS = ANS + RS
   WRITE(5, 14) AN, BN, RS, ANS
   AN = BN
20 BN = AN + 0.1

WRITE(6, 85) CMSEG, ANS
FORMAT('CMSEG=', E12.4, ' ANS=', E12.4)
60 CONTINUE
50 END

** This routine initializes all parameters

INCLUDE 'PROP.VCM'

INTEGER*2 PRT
PI = 3.141592653589793
DIA = 1.0
LG = 100.0
THIS ROUTINE COMPUTES THE DIFFERENTIAL PATH CONTRIBUTION

INCLUDE 'PROP.VCM'

OMEG=0.1

DO 40 KL=1,1
OMEG=OMEG*10.0
WRITE(99)OMEG
99 FORMAT('OMEG=',F8.2)

STA=1.0*E-3

DO 10 K=1,2
AI=STA*(10.0**K)
DIN=AI

DO 10 IL=1,9
DPATH=FSIG(AI)

WRITE(6,21)SIG,DPATH
21 FORMAT('SIG=',E12.4,'DPATH=',E12.4)

AI=AI+DIN
CONTINUE

AI=0.91
DIN=0.01

DO 30 I=1,10
DPATH=FSIG(AI)

WRITE(6,21)SIG,DPATH
AI=AI+DIN
CONTINUE

CONTINUE
RETURN

FUNCTION FSIG(A)

return
THE FUNCTION COMPUTES THE DIFFERENTIAL PATH CONTRIBUTION
AT FIXED TEMPORAL FREQUENCY

\[ \text{C INCLUDE "PROP.VCM"} \]

\[ \text{SIG=UA} \]
\[ \text{B1=(1.0\text{-SIG}+1.0/\text{RA})^{(-4.0/3.0)}}, \]
\[ \text{ARG=OMEG*COS(THEA)}*\text{SIG}^{0.5}\text{DIA/}(\text{LO*(1.0\text{-SIG-R})RA)}, \]
\[ \text{B2=(SIN(ARG))^{+2}}, \]
\[ \text{B3=AB5}*(1.0\text{-SIG-R}) \]
\[ \text{B4=RL*(1.0\text{-SIG}+1.0/\text{RA})}, \]
\[ \text{B5=EXP(-B4/5940.)}, \]
\[ \text{FSIG=B1*B2*B5*FKZ(SIG)/B3}, \]
\[ \text{RETURN} \]

FUNCTION FKZCUG)

THE FUNCTION COMPUTES THE SPATIAL FREQUENCY INTEGRATION AT
THE GIVEN NORMALIZED RANGE AND TEMPORAL FREQUENCY

\[ \text{C EXTERNAL FCT} \]

\[ \text{C INCLUDE "PROP.VCM"} \]

\[ \text{AI=0.0} \]
\[ \text{C1=OMEG*OMEG/(1.0-SIG-R)^2}, \]
\[ \text{C2=(1.0\text{-SIG}+1.0/\text{RA})^{(2)(-2)}}, \]
\[ \text{C3=LO*RA^0.1} \]
\[ \text{BI=SQRT(C1+C2)*0.1} \]
\[ \text{IF(BI .LT. C3)GO TO 15} \]
\[ \text{BI=C3} \]

15 \[ \text{CONTINUE} \]
\[ \text{IF(BI .GE. PI)GO TO 25} \]
\[ \text{SIN=PI} \]
\[ \text{GO TO 35} \]
\[ \text{SIN=BI} \]

35 \[ \text{CONTINUE} \]
\[ \text{FKZ=0.0} \]
\[ \text{CALL DCG16(AI,BI,FCT,RE)}, \]
\[ \text{WRITE(5,'( Ai,Bi,Re)')} \]
\[ \text{C12 FORMAT( 'Ai=',E12.4,' Bi=',E12.4,' Re=',E12.4) \]
\[ \text{FKZ=RE} \]
\[ \text{AI=BI} \]
\[ \text{BI=AI+SIN} \]
\[ \text{CALL DCG16(AI,BI,FCT,RE)}, \]
\[ \text{FKZ=FKZ+RE} \]
\[ \text{WRITE(5,'( Ai,Bi,Re)')} \]
\[ \text{C16 FORMAT( 'Ai=',E12.4,' Bi=',E12.4,' Re=',E12.4) \]
\[ \text{IF(CRATIO .GT. 1.0E-5)GO TO 10} \]
\[ \text{RETURN} \]

END FUNCTION FCT(UK)
THIS FUNCTION COMPUTES ALL SPECIAL FREQUENCY DEPENDENT TERMS
AT FIXED TEMPORAL FREQUENCY AND NORMALIZED RANGE

INCLUDE "PROP.VCM"

A1=OMEG*OMEG/((1.0-SIG-R)**2)
A2=UK*UK
A4=(1.0-SIG+1.0/RA)**(-2)
A5=SIG**(1.0-SIG)*RL*0.5/(KAP*DIA/DIA*COS(THEA))
A7=SIG**(-1.0/6.0)
ARG1 = CA1 + A2 + A4)**(-1.0/6.0)
A3=UK*SIG*0.5*IA/(LO*RA)
ARG2 = (SIG/(3.0**2))**(2)
ARG3 = (OMEG*OMEG/((1.0-SIG-R)**2))
ARG4 = A5*(A2)**(-1)
ARG5 = COS(ARG4)**2
FCT=ARG1*ARG2*ARG3

RETURN

END

SUBROUTINE DQG16(YL,YU,FCT,Y)

AA=0.5*(YU+YL)
BB=YU-YL
CC=0.49470046749582E0*BB
Y=0.13576229705877E-1*(FCT(AA+CC)+FCT(AA-CC))
CC=0.47228751193661E0*BB
Y=Y+0.31326761693235E-1*(FCT(AA+CC)+FCT(AA-CC))
CC=0.43261560119351E0*BB
Y=Y+0.47579255841246E-1*(FCT(AA+CC)+FCT(AA-CC))
CC=0.37770220411775E0*BB
Y=Y+0.62314485627766E-1*(FCT(AA+CC)+FCT(AA-CC))
CC=0.30683812220132E0*BB
Y=Y+0.74973420081969E-1*(FCT(AA+CC)+FCT(AA-CC))
CC=0.2290083836286150*BB
Y=Y+0.8457825969750E-1*(FCT(AA+CC)+FCT(AA-CC))
CC=0.14080177538962E0*BB
Y=Y+0.9130170752246E-1*(FCT(AA+CC)+FCT(AA-CC))
CC=0.47506254918818E-1*BB
Y=B3*(Y+0.9472353052753E-1*(FCT(AA+CC)+FCT(AA-CC))
RETURN

END

SUBROUTINE DRP16(YL,YU,FCT,Y)

AA=0.5*(YU+YL)
BB=YU-YL
CC=0.49470046749582E0*BB
Y=0.13576229705877E-1*(FCT(AA+CC)+FCT(AA-CC))
CC=0.47228751193661E0*BB
Y=Y+0.31326761693235E-1*(FCT(AA+CC)+FCT(AA-CC))
CC=0.43261560119351E0*BB
Y=Y+0.47579255841246E-1*(FCT(AA+CC)+FCT(AA-CC))
CC=0.37770220411775E0*BB
Y=Y+0.62314485627766E-1*(FCT(AA+CC)+FCT(AA-CC))
CC=0.30683812220132E0*BB
Y=Y+0.74973420081969E-1*(FCT(AA+CC)+FCT(AA-CC))
CC=0.2290083836286150*BB
Y=Y+0.8457825969750E-1*(FCT(AA+CC)+FCT(AA-CC))
CC=0.14080177538962E0*BB
Y=Y+0.9130170752246E-1*(FCT(AA+CC)+FCT(AA-CC))
CC=0.47506254918818E-1*BB
Y=B3*(Y+0.9472353052753E-1*(FCT(AA+CC)+FCT(AA-CC))
RETURN

END
\[ Y = Y + 0.84578259697505 \times 10^{-1} \times (FCT(AA+CC) + FCT(AA-CC)) \]
\[ CC = 0.14080177538962 \times 10^0 \times BB \]
\[ Y = Y + 0.9130170752246 \times 10^{-1} \times (FCT(AA+CC) + FCT(AA-CC)) \]
\[ CC = 0.47506254918818 \times 10^{-1} \times BB \]
\[ Y = BB \times (Y + 0.9472530522753 \times 10^{-1} \times (FCT(AA+CC) + FCT(AA-CC))) \]
RETURN
END


4. Protheroe.


230

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BIBLIOGRAPHY (Continued)


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