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*The Ohio State University*  
Ph.D. 1983

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NONLINEAR ANALYSIS OF SILO/MATERIAL INTERACTION UNDER LATERAL LOAD

DISSERTATION

Presented in Partial Fulfilment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

By

Shih-Zong Jang, B.S., M.S.

The Ohio State University

1982

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ACKNOWLEDGEMENTS

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The author would also like to thank his reading committee, especially Professor R. S. Sandhu, for their valuable suggestions.

Sincere thanks are also extended to my family for their patience and encouragement. To them, this research work is gratefully dedicated.
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CHAPTER I.
INTRODUCTION

1.1 Problem Statement

The response of a filled silo to external loads involves both the silo structure and the stored material. There is usually a great difference between the mechanical properties of the silo wall and those of the stored material. There is also a possibility of slip of the stored material along its contact-surface with the silo wall. This renders the composite action of the silo wall and the stored material to external lateral load effects to be questionable.

Although a great deal of previous research has been done dealing with wind effects as well as seismic effects on liquid-filled storage tanks(8,16,24), very little information on silo/material interaction under lateral load exists in available literature. The information obtained from investigations on liquid-filled tanks cannot be directly applied to silos because of the basic differences in material properties between liquids and materials stored in silos.

According to the latest experimental/analytical investigation by Ibrahim and Bishara(20) on the effects of quasi-static external lateral loads on circular cylindrical silos storing either silica sand, which is a fine granular material, or corn silage, which can be characterized as a viscoelastic material, the following conclusion was drawn:

Hoop tensions of silo wall in regions perpendicular to the direction of external lateral loads increase while those in regions parallel to the direction of external lateral loads decrease as compared to linear super-
position of effects due to external lateral loads and internal radial pressure. The resulting stress condition in silo wall becomes non-axisymmetric as illustrated in Fig. 1-1.

Owing to the variation of the internal pressure caused by the deformations in the silo cross sections, simple superposition evidently cannot be expected to apply. It is conceivable that there is interaction between the silo wall and the stored material while undergoing lateral disturbance.

In contrast to liquid-filled tanks in which the hydrostatic liquid pressures do not change in response to boundary deformations, this topic has thus posed a problem with great complexity.

1.2 Objectives and Scope of Research

The objectives of this research are:

1. To investigate how the stored materials interfere with circular cylindrical silos in resisting quasi-static external lateral loads.

2. To develop a rationale for the static analysis of the external lateral load effects on circular cylindrical silos.

The scope of this research will be limited to the interactions between granular materials whose mechanical properties are time-independent, and isotropic, linearly elastic silos only. Once the objectives of this research have been achieved, the methodology employed hereinafter should be capable of being extended to analyze interactions between more complicated stored materials and wall materials.
Figure 1-1 Variation of Internal Pressure due to Boundary Deformation

(a) Before deformation
(Axisymmetric internal pressure)

(b) After deformation
(Non-axisymmetric internal pressure)
CHAPTER II.
RESEARCH APPROACH

2.1 Introductory Remarks

It is not the intention of this research to develop a sophisticated material model for granular materials nor to elaborate a better nonlinear solution technique. Rather, it is aimed to utilize the best analysis tool available, with necessary improvements, to accomplish the objectives of this research within the specified scope.

The base analysis tool adopted for this research is the nonlinear finite element program NONSAP - a structural analysis program for static and dynamic response of nonlinear systems, developed by Professor Bathe et al at University of California, Berkeley, California(1,2,3). This program was designed for a general incremental solution of nonlinear problems, but naturally can also be used for linear analysis. It is apparent that the stress condition in the stored material as well as in the silo wall will not remain axisymmetric after external load is applied. Thus three dimensional analysis need be employed. For three dimensional analysis the NONSAP program is very efficient in the optimum usage of computer hardware and software where, specifically, the appropriate allocation of high- and low-speed storage of problem information during solution process.

Nevertheless, improvements need be made to incorporate the feature of geometric nonlinear analysis into the NONSAP program to account for probable large strains in the granular media stored in silos, since in the present version of NONSAP only material nonlinearities have been included in 3/D continuum elements. Modifications also need be made to consider the effect
of variation of deviatoric strains in updating the instantaneous shear moduli for the curve description material model used in the present version of NONSAP.

It is important to note that the term 'granular materials' used in this research is not meant to embrace all kinds of granular materials but just including those demonstrating time-independent mechanical properties only. Further restrictions and assumptions, and procedures for the above-mentioned improvements to the NONSAP program are to be discussed in the following sections.

2.2 Granular Materials

In analogy to cohesionless soils in soil mechanics, a granular material in this research is defined as an assemblage of particles which are not fibrous nor cohesive and whose mechanical properties are time-independent. It is further assumed that only two phases viz solid and air existing in the volume of a granular material, i.e., there is perfect drainage and effects of moisture content are negligible. Thus a granular material will be taken just as a 'particulate system' hereinafter.

One of the most distinct characteristics of particulate systems is that they can not have definite forms without containments. A particulate system can attain its equilibrium as long as any part of the system consisting an angle with the horizontal plane smaller than the angle of repose of the granular material. This is because no tension can be tolerated in these systems. As a result, granular materials exert lateral pressure to the silo wall in addition to vertical pressure to the silo base.

Evidence collected from confined compression tests, triaxial tests and
direct shear tests (10, 11, 14, 15, 19, 22, 23, 25, 26) have shown that the stress-strain behavior of particulate systems are far from being linear. Therefore, in the development to follow it is presumed that granular materials demonstrate nonlinear mechanical properties and may experience large strains.

2.3 Geometric Stiffness Matrix

The discretized matrix equation of motion of a typical finite element can be symbolically written as Eq. (2.1) using Updated Lagrange formulation:

\[
(t^K_L + t^K_{NL}) \delta u = t^\Delta t_Q - t^F
\]  

(2.1)

in which

\[
t^K_{NL} = \int t^{R^T} \cdot t^L \cdot t^B_NL \cdot t^d_v \quad t^v
\]

(2.2)

is the geometric stiffness matrix; and \(t^K_L\), \(t^\Delta t_Q\) and \(t^F\) are the linear element stiffness matrix, the externally applied nodal load vector and the unbalanced load vector, respectively.

For analysis includes geometric nonlinearities \(t^K_{NL}\) must also be evaluated at the Gauss integration points and added to the linear stiffness matrix \(t^K_L\).

In Eq. (2.2) \(t^B_NL\) is sometimes referred as the nonlinear strain-displacement transformation matrix (21) although it does not relate explicitly the nonlinear strains \(t^\gamma_{ij}\) with the nodal displacements \(u_i\). \(t^B_{NL}\) has been derived to be as follows:

* Refer to Appendix A for Derivations.
\[ t_B^{NL} = \begin{bmatrix}
   t_{h1,1} & 0 & 0 & t_{h2,1} & 0 & 0 & \cdots & 0 \\
   t_{h1,2} & 0 & 0 & t_{h2,2} & 0 & 0 & \cdots & 0 \\
   t_{h1,3} & 0 & 0 & t_{h2,3} & 0 & 0 & \cdots & 0 \\
   0 & t_{h1,1} & 0 & 0 & t_{h2,1} & 0 & \cdots & 0 \\
   0 & t_{h1,2} & 0 & 0 & t_{h2,2} & 0 & \cdots & 0 \\
   0 & t_{h1,3} & 0 & 0 & t_{h2,3} & 0 & \cdots & 0 \\
   0 & 0 & t_{h1,1} & 0 & 0 & t_{h2,1} & \cdots & t_{hN,1} \\
   0 & 0 & t_{h1,2} & 0 & 0 & t_{h2,2} & \cdots & t_{hN,2} \\
   0 & 0 & t_{h1,3} & 0 & 0 & t_{h2,3} & \cdots & t_{hN,3}
\end{bmatrix} \]

t in Eq. (2.2) has also been derived to be as follows:

\[ t_L = \begin{bmatrix}
   t_{r11} & t_{r12} & t_{r13} & 0 & 0 & 0 & 0 & 0 & 0 \\
   t_{r21} & t_{r22} & t_{r23} & 0 & 0 & 0 & 0 & 0 & 0 \\
   t_{r31} & t_{r32} & t_{r33} & 0 & 0 & 0 & 0 & 0 & 0 \\
   0 & 0 & 0 & t_{r11} & t_{r12} & t_{r13} & 0 & 0 & 0 \\
   0 & 0 & 0 & t_{r21} & t_{r22} & t_{r23} & 0 & 0 & 0 \\
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   0 & 0 & 0 & 0 & 0 & 0 & t_{r11} & t_{r12} & t_{r13} \\
   0 & 0 & 0 & 0 & 0 & 0 & t_{r21} & t_{r22} & t_{r23} \\
   0 & 0 & 0 & 0 & 0 & 0 & t_{r31} & t_{r32} & t_{r33}
\end{bmatrix} \]
in which \( t_{ij}^{r} (i,j=1,2,3) \) is the Cauchy stress tensor.

The geometric stiffness matrix \( t_{NL}^{K} \) can now be programmed for numerical evaluation at Gauss integration points and added to \( t_{NL}^{L} \).

2.4 Curve Description Model

There are two common procedures for incorporating a nonlinear stress-strain law into a finite element formulation for digital computation, i.e., the tabular and the functional representation\(^{(12,13)}\). In tabular representation, the stress-strain law derived from a laboratory test can be used directly in a tabular or digital form. Representative points on the constitutive curves like those in Fig. 2-1 are selected and are put in the form of number pairs denoting stress-strain relations at those points. The instantaneous moduli \( t_{K}^{L} \) and \( t_{G}^{L} \) are obtained from such curves by suitable interpolation. In the alternative procedure, the laboratory stress-strain curves are expressed in the form of a consistent mathematical function.

The curve description model used in the NONSAP program is a tabular representation of mechanical properties for geological materials\(^{(1,2,29)}\). It may not be the most sophisticated method, however, curve description model has the merit of direct representation of laboratory experimental data and high numerical efficiency, especially in three dimensional analysis in which usually very large storage region size of the computer CPU is required. No additional subprogram need be coded to calculate the instantaneous moduli and thus computational cost is reduced.

However, it is noted that merely using volumetric change \( t_{v}^{e} \) as the only parameter in updating the shear modulus \( t_{G}^{L} \) is far from being sufficient although for the bulk modulus it is. By definition the shear modulus
NOTE: \( t_{G_{UN}} = \frac{t_{K_{UN}}}{t_{K_{LD}}} t_{G_{LD}} \)

COMPRESSIVE VOLUMETRIC STRAIN IS POSITIVE

Figure 2-1 Curve Description Model
G is expressed as

\[ G = \frac{d S_d}{d \varepsilon_d} = \lim_{\Delta \varepsilon_d \to 0} \frac{\Delta S_d}{\Delta \varepsilon_d} \quad (2.3) \]

in which \( \varepsilon_d \) is the deviatoric strain and \( S_d \) is the associated deviatoric stress. The shear modulus for granular materials is also dependent on the mean normal stress \( \sigma_m \) and the initial relative density \( D_r \), namely

\[ G = f (\sigma_m, S_d, D_r) \quad (2.4) \]

Eq. (2.4) is best manifested in the experimental results of Domaschuk and Wade(14), Fig. 2-2, on Chattahoochee sand.

For a specific initial density \( D_r \), Eq. (2.4) can be written as

\[ G = f (\sigma_m, S_d) \quad (2.5) \]

Instead of using \( \varepsilon_v \) as the only parameter to update the shear modulus \( G \), it is proposed that \( \varepsilon_v \) be used to update the 'initial' tangent shear modulus \( G_i \) for each load step only. Relations between \( \varepsilon_v \) and \( G_i \) can be experimentally determined as those curves shown in Fig. 2-3(14). The deviatoric stress \( S_d \) will be calculated using the hyperbolic relation, Eq. (2.6), as suggested by Konder and Zelasko(23).

\[ t_S = \frac{t_{\varepsilon_d} G_i}{1 + t_b t_{\varepsilon_d} G_i} \quad (2.6) \]

in which \( t_b \) is the reciprocal of the ultimate value of \( S_d \) as illustrated in Fig. 2-4(23). Relations among \( t_{\sigma_m} \) and \( t_b \) and initial relative density \( D_r \) can also be experimentally determined as those curves shown in Fig. 2-5(14).
Figure 2-2 Effects of Mean Normal Stress and Initial Relative Density on Stress-Strain Relations during Shear
Figure 2-3  Relationship Between Initial Tangent Shear Modulus and Mean Normal Stress for Various Relative Densities

Figure 2-4  Notation for Idealized Hyperbolic Stress-Strain Response
Figure 2-5 Relationship between Parameter 'b' and Mean Normal Stress for Different Densities
Once \( t^*_S \) is calculated, the relation suggested by Domaschuk and Wade (14), shown as Eq. (2.7) will be used for the calculation of \( t^*_G \):

\[
\begin{align*}
t^*_G &= t^*_G \left(1 - t^*_b \ t^*_S \right)^2 \\
\end{align*}
\tag{2.7}
\]

The following steps summarize the proposed procedure for updating the shear modulus \( G \):

1. At load step \( t \), determine \( t^*_G \) and \( t^*_b \) according to \( t^*_e \).
2. Calculate \( t^*_d \) using the relation

\[
\begin{align*}
t^*_d &= \sqrt{(t^*_e - \frac{1}{3} t^*_d )^2 + (t^*_e - \frac{1}{3} t^*_d )^2 + (t^*_e - \frac{1}{3} t^*_d )^2} \\
\end{align*}
\]

3. Calculate \( t^*_S \) using Eq. (2.6).
4. Calculate \( t^*_G \) using Eq. (2.7).

This procedure will correct the discrepancy of the present curve description model for the shear modulus used in NONSAF in situations where an element is primarily subjected to shear distortion and having negligible volumetric change as schematically shown in Fig. 2-6.

The incremental stress-strain relations considered in the solution using the curve description model are assumed to be (1, 2, 3)

\[
\begin{align*}
s_{ij} &= 2 \ t^*_G \ g_{ij} \ \tag{2.8} \\
\sigma_m &= 3 \ t^*_K \ e_m \ \tag{2.9} \\
\end{align*}
\]

where \( s_{ij} \) and \( g_{ij} \) are the incremental deviatoric stresses and strains, and \( \sigma_m \) and \( e_m \) are the incremental mean stress and strain. For the numerical solution, Eqs. (2.8) and (2.9) are approximated at time \( t \) as
Figure 2-6 Deformation with Negligible Volumetric Change
follows

\[ t+\Delta t \sigma_m = 3 t_K (t+\Delta t \varepsilon_m - t \varepsilon_m) + t \sigma_m \]  
(2.10)

\[ t+\Delta t \varepsilon_{ij} = 2 t_G (t+\Delta t \varepsilon_{ij} - t \varepsilon_{ij}) + t \varepsilon_{ij} \]  
(2.11)

It is worth noting that the stresses at time \( t+\Delta t \) are calculated using the material moduli pertaining to time \( t \), which were also used in the calculation of the stiffness matrix at time \( t \).

The unloading criteria used in the NONSAP program are kept unchanged for this research, i.e.,

\[ t_K = \begin{cases} 
 t_{K_{LD}} & \text{when } t \varepsilon_m \leq t \varepsilon_{\text{min}} \\
 t_{K_{UN}} & \text{when } t \varepsilon_m > t \varepsilon_{\text{min}} 
\end{cases} \]  
(2.12)

and

\[ t_G = \begin{cases} 
 t_{G_{LD}} & \text{when } t \varepsilon_m \leq t \varepsilon_{\text{min}} \\
 t_{G_{UN}} & \text{when } t \varepsilon_m > t \varepsilon_{\text{min}} 
\end{cases} \]  
(2.13)

in which \( t \varepsilon_{\text{min}} \) was defined as the minimum mean strain ever reached during the solution. The values of \( t_{K_{LD}}, t_{K_{UN}}, t_{G_{UN}} \) and \( t_{G_{LD}} \) are obtained using the curves provided, and the modulus \( t_{G_{UN}} \) is approximated as

\[ t_{G_{UN}} = \frac{t_{G_{LD}}}{t_{K_{UN}} / t_{K_{LD}}} \]  
(2.14)

However, \( t_G \) in Eq. (2.13) will be calculated according to the foregoing proposed procedure.
2.5 Frame Work of Research

The frame work of this research is summarized as follows:

1. Computer Programming

   (i) Incorporation of new feature of geometric nonlinear analysis for 3/D continuum elements into the present version of NONSAP to take into consideration probable large strains which may occur in the granular media stored in silos.

   (ii) Modification of the current curve description material model used in the NONSAP program for 3/D continuum elements according to the procedure proposed in Section 2.4. The shear modulus \( G \) can be more truthfully updated during the solution process after this improvement.

2. Finite Element Discretization of the Problem Domain

   Given that with usual solution techniques the solution accuracy is largely depend on the following three factors: (i) appropriate discretization scheme for the problem domain, (ii) fine enough finite element mesh (representation) for the problem domain, and (iii) small enough load steps for the incremental loading process. Thus, the next chapter is dedicated to the determination of a proper finite element representation for this research.

3. Application of the Modified NONSAP Program on Filled Silos

   Investigation of interaction between the stored material and the silo wall under external lateral load will be performed. Only quasi-static loading will be considered for external disturbance.

4. Interpretation of Results of Investigation

   Conclusions and recommendations for future work are to be drawn based adherently on consistent interpretation of investigation results.
CHAPTER III.
FINITE ELEMENT DISCRETIZATION OF PROBLEM DOMAIN

3.1 Introductory Remarks

It is standard practice(2,3,4,12,30,32) to assign only translational degrees of freedom, for example $u_x$, $u_y$, $u_z$ in rectangular Cartesian coordinate system, to each nodal point of a 3/D continuum element. As a result, only continuity of translational displacements between elements is assured.

It is immediately obvious that a large number of elements, especially linear elements, would have to be used to achieve a given degree of accuracy of stresses in 3/D continuum since stresses are functions of strains, and strains are first derivatives of displacements. This will result in a very large number of solution equations in practical problems, which may place a severe limit on the use of the finite element method for three dimensional analysis in practice. Given that with usual solution techniques the computational effort is roughly proportional to the number of equations(32). It is not surprising therefore that efforts to improve accuracy by usage of complex or higher order elements have been strongest in the area of three-dimensional analysis(32).

One of the main purposes of this chapter is the choice of proper elements to use for the silo wall and the stored material so that sufficient solution accuracy can be obtained and the computational cost be reasonably reduced.

In principle, the geometry and displacements of a 3/D continuum element can be interpolated with polynomials of arbitrary high order or consistent mathematical functions. However, in practice, linear and quadratic elements are most often used. A general three-dimensional isoparametric
element routine is available in the NONSAP program. Fig. 3-1 shows a typical 8 to 21 variable-nodes 3/D continuum element. Subparametric elements are also allowed for use to save element stiffness formation time and to promote numerical efficiency in the solution process (1, 2, 3, 4, 5).

Hence, it is considered good practice to interpolate the geometry of all straight edges of the 3/D continuum elements used linearly, and curved edges parabolically. It is natural choice to interpolate the displacements on curved edges parabolically since superparametric elements are usually not recommended (1, 2, 3, 4, 5). The most crucial decision need be made is on the order of interpolation functions for displacements along the height (longitudinal direction) of a silo. In the next section, preliminary numerical experiments will be performed to find out the better choice.

It is important to note that a special type of elements would have to be used at the silo/material interface to account for probable slipage of the stored material along its contact-surface with the silo wall. This type of elements will be further discussed later in this chapter.

3.2 Discretization of the Silo Structure

It is also a natural choice to discretize a silo as schematically shown in Fig. 3-2. There are two important tasks to be fulfilled:

(i) To determine a fine enough finite element mesh to achieve a given degree of solution accuracy,

(ii) To determine the better choice between a 12-node and a 16-node thick shell element, shown as Fig. 3-3(a) and (b), for the analysis of stresses in the silo wall based on the resulting solution accuracy and computational cost.
Figure 3-1 A General 8 to 21 Variable-Number-Nodes 3/D Continuum Element
Figure 3-2 Finite Element Discretization of a Silo
Figure 3-3  Thick Shell Elements
A 60 ft. x 15 ft. concrete silo, wall thickness 4 inches, has been chosen for all numerical investigation purposes of this research. It is important to note that the final total number of 3D continuum elements will be several times as many as that of thick shell elements after the silo is filled. Thus, to avoid excessive large number of solution equations in the latter analyses, the mesh shown in Fig. 3-4 is considered adequate and will be used in the following numerical experiments.

Figs. 3-4(a) and (b) represent finite element models of the full size silo using 12-node and 16-node thick shell elements, respectively. It was intended to make the numbers of nodal points, NUMNP, of the two meshes approximately equal so that the required computational costs (CPU time and DISKIO) can be compared on a relatively equal basis for the same loading condition. It is worth noting that the total number of elements, NUMEL, of the 12-node mesh is about 1.6 times (54 elements) more than that of the 16-node mesh; however, the effort needed in the preparation of input data (nodal points data and elements data) for the 12-node mesh is more straightforward than that for the 16-node mesh because the 12-node mesh is more suitable for automatic generation of input data.

It should be noted that all body or surface loading must be transformed to nodal point loading prior to using the NONSAP program, i.e., the loading in the analysis can consist of only concentrated nodal point loading. This, in fact, leaves the user a choice to transform distributed loading to nodal point loading more realistically. It is considered more appropriate to transform gravity loads and surface tractions which are not alternating in their distribution, to nodal point loading using linear
Figure 3-4(a) Finite Element Discretization Using 12-Node Thick Shell Elements

NUMNP = 676
NUMEL = 150
E = 3,000 ksi
ν = 0.25
Figure 3-4(b) Finite Element Discretisation Using 16-Node Thick Shell Elements
interpolation functions. Quadratic interpolation functions, as shown in Fig. 3-5 for two-dimensional cases, usually generate relatively very small or reverse nodal point loads at corner nodes and, as a result, generate relatively very large nodal point loads at mid-side nodes. This is especially undesirable in cases where materials of low strength or no tension resistance, such as granular materials, are involved.

Nevertheless, correction need be made to linearly interpolated nodal point loads for elements having curved edges. For example, referring to Fig. 3-6, the correction factors for elements used in Figs. 3-4(a) and 3-4(b) were calculated as follows:

**Gravity Loads**

Actual Element Cross-Sectional Area: \( R(2R-\Delta R)/N \)  
Linearly Interpolated Cross-Sectional Area: \( R(2R-\Delta R)\sin\theta/2 \)  
Modification Factor: 1.0471976

**Surface Tractions**

Actual Element Arc Length: \( 2\pi R/N \)  
Linearly Interpolated Arc Length: \( 2R\sin^2\theta/2 \)  
Modification Factor: 1.0115152

Linearly interpolated nodal point loads must be multiplied the corresponding modification factors prior to being used as applied loads input data for the analysis.

Two computer runs were performed* for the two finite element models subjected to own weight of the silo structure. Wall material properties were assumed to be isotropic and linearly elastic. Results of analyses

---

* Refer to Appendix C for computer implementation of transformation of body or surface loading to nodal point loading.

* Refer to Reference 1 for use of the NONSAP program.
Figure 3-5 Interpolation Functions for Two-Dimensional Elements

(a) Interpolation Function for Node 1 (Corner)

(b) Interpolation Function for Node 7 (Mid-Side)
Figure 3-6 Calculation of Correction Factors
are compared as follows:

A. Stresses

Since the wall thickness is just four inches, small comparing to radius and height of the silo, the analytical solution can be obtained using general theory of cylindrical shells(33). The general solution for this case is

\[ w(x) = e^{-\beta x} (C_1 \cos \beta x + C_2 \sin \beta x) \] (3-1)

At \( x=0 \), \( w(x) = 0 \) and \( x = 0 \) can be easily calculated using the following relations:

\[ w(x) = -a \epsilon \phi (x) = -a \nu \epsilon \chi (x) \]

\[ \epsilon \chi (x) = \rho x/2E \] (3-2)

Refering to Fig. 3-7, the radial deformation at \( x=0 \) can be written in terms of \( M_0 \) and \( Q_0 \).

\[ (w)_{x=0} = -(M_0 + Q_0)/2\beta^2 D \] (3-3)

Similarly, the slope at \( x=0 \) can be written as

\[ \left( \frac{dw}{dx} \right)_{x=0} = (2M_0 + Q_0)/2\beta^2 D \] (3-4)

At the bottom of the silo, both radial deformation and slope vanish. Thus by solving the following two equations simultaneously, \( M_0 \) and \( Q_0 \) are obtained.

\[ -(M_0 + Q_0)/2\beta^2 D = C_1 \] (3-5)

\[ 2M_0 + Q_0 = 0 \] (3-6)
Figure 3-7  A Cylindrical Shell Loaded at the End
Once \( M_0 \) and \( Q_0 \) are solved, the effects due to the built-in end can be evaluated using the following equation and Eq. (3-2).

\[
\begin{align*}
\frac{w(x)}{w_0} &= e^{\beta x} \\
&= \frac{e^{\beta x}}{2 \lambda D} \left[ M_0 (\sin \beta x - \cos \beta x) - Q_0 \cos \beta x \right], 
\end{align*}
\]

Eq. (3-7)

The analytical solution was plotted in Fig. 3-8 together with the results of analyses for the two finite element models.

It is seen in Fig. 3-8 that both meshes yielded very good solution accuracy. The 12-node mesh seems to give slightly better accuracy at all locations until five feet from the bottom. This is because there is no bending in the shell due to own weight except in a short distance from the bottom. The 16-node mesh continued to give good solution until one foot from the bottom. Since in the latter numerical investigations the silo structure will be subjected to lateral loading which tends to induce bending in the silo wall, given the present two meshes the 16-node model should be the better choice.

B. Displacements

As can be expected, the 12-node mesh yielded slightly better accuracy until five feet from the bottom end of the silo. The 16-node mesh gave good solution at all nodal points. Since there will be bending in the silo wall other than near the bottom in the latter numerical investigations, the 16-node mesh is more favorable.

C. Computational Efficiency

Refering to Table 3-1, it is rather striking to see that the 12-node mesh resulted in better computational efficiency — less CPU time and DISKIO than the 16-node mesh despite the fact that the former mesh has 10 more nodal points (10.4%) and 54 more elements (56.3%) than the 16-node mesh. That the CPU time ratio and the charges ratio are not
Figure 3-8 Comparison of Solution Accuracy between a 12-Node and a 16-Node Thick Shell Element
<table>
<thead>
<tr>
<th>Job</th>
<th>CPU Time (sec.)</th>
<th>DISKIO</th>
<th>Charges* ($)</th>
<th>NUMNP</th>
<th>NUMEL</th>
</tr>
</thead>
<tbody>
<tr>
<td>12-Node</td>
<td>36.63</td>
<td>531</td>
<td>6.90</td>
<td>676</td>
<td>150</td>
</tr>
<tr>
<td>Mesh</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>16-Node</td>
<td>46.52</td>
<td>603</td>
<td>7.82</td>
<td>666</td>
<td>96</td>
</tr>
<tr>
<td>Mesh</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ratio</td>
<td>0.787</td>
<td>0.881</td>
<td>0.882</td>
<td>1.015</td>
<td>1.563</td>
</tr>
</tbody>
</table>

* This is based on deferred rate.

Table 3-1 Comparison of Computational Cost
the same is because more output records were required for the 12-node mesh (7785 lines vs 5347 lines).

D. Remarks

Although the 12-node mesh provides the advantages of cheaper computational cost and more straightforward effort in the preparation of input data, the overwhelming merit of the 16-node mesh is that it can still yield good solution accuracy even when bending exists. The other advantages of the 16-node mesh are fewer elements and nodal points could be used (thus less labor). The 16-node thick shell is therefore considered better element for use in the latter numerical investigations.

3.3 Discretization of the Stored material.

In the discretization of a circular domain, special attention need be paid to the center portion. In spite of the fact that the isoparametric formulation allows the use of wedge elements (triangular prism elements) as shown in Fig. 3-9, this type of elements usually result in very unsatisfactory solution accuracy. For maximum solution accuracy, each face of a 3/D continuum element need be as close as possible to a rectangular shape(4).

The discretization scheme as shown in Fig. 3-10 was therefore adopted for this research. Refering to Fig. 3-10 the outer layer consists of 16-node thick shell elements. The first inner layer serves as an intermediate layer between two different materials, i.e., the silo wall material and the stored granular material. Very small thickness is assigned to 16-node elements in this layer in order to simulate the interface action between two materials of very different mechanical properties.
NOTE: NODES 9-20 ARE OPTIONAL.

Figure 3-9 Triangular Prism Element
Figure 3-10 Finite Element Discretization of the Silo Wall and the Stored Material
The second inner layer consists of another type of 12-node 3/D continuum elements as shown in Fig. 3-11. This type of elements is sometimes termed transitional elements (34) for that they have eight nodes on the face \( t=1 \) and four nodes on the face \( t=-1 \). The third and the fourth inner layers consist of 8-node linear 3/D continuum elements. It is clearly seen in Fig. 3-11 that no incompatible mode, as illustrated in Fig. 3-12, is resulted between any two layers.

The material properties of elements in the outer layer (silo wall) are assumed to be isotropic and linearly elastic. Nonlinear material properties are presumed for elements in all inner layers. The modified curve description material model already discussed in Chapter II will be used to characterize the material properties of the stored material based on results of laboratory testings.

3.4 Interface Element

A filled silo cannot be analyzed as a composite system without accounting for the possibility of slipage at the interface of the silo wall and the stored material, as schematically shown in Fig. 3-13.

Chandrangsu (21) assumed Coulomb friction at the silo/material interface such that the tangential (logitudinal) displacements of the stored material were identical to those of the wall surface it contacted as long as the magnitude of the shear stresses were less than a preassigned fraction of the normal stresses. The interface condition for \( q \) (tangential displacements of the stored material) equal to zero is:

\[
\left| \sigma_{nt} \right| < \mu_{\text{static}} \cdot \sigma_{nn}
\]

in which \( \sigma_{nt} \) = the shear stress at the interface
\( \sigma_{nn} \) = the normal stress
Figure 3-11 Types of Elements in Layers
Coordinates and Displacements Vary Parabolically along Both Element Edges

(a) Compatible Elements

Three-Node Edge
Coordinates Vary Linearly but
Displacements Vary Parabolically

(b) Incompatible Elements

Figure 3-12 Compatible and Incompatible Elements
A. Before Loading

B. After Loading

Figure 3-13 Slipage at Silo/Material Interface
\[ \mu_{\text{static}} = \text{static coefficient of friction between silo wall and the stored material} \]

When the static regime ends, the stored material slides relative to the silo wall. He further postulated the motion was impeded by a shear stress equal to some fraction of the normal stress usually smaller than the fraction assigned to static friction. The interface condition for \( q_t \) unequal to zero is:

\[ |\sigma_{nt}| = \mu_{\text{dynamic}} \cdot \sigma_{nn} \]

in which \( \mu_{\text{dynamic}} < \mu_{\text{static}} \)

The above slip-stick process is suitable in axisymmetric cases, however, is inconvenient to apply in non-axisymmetric cases because slippage of stored material element nodes may occur in directions other than the tangential (longitudinal) direction.

Two-dimensional and one-dimensional joint elements, shown as Figs. 3-14 (a) and (b), proposed by Duncan et al (34, 37, 38) and Goodman et al (35, 36, 37, 38) may perform satisfactorily in two-dimensional cases but are not applicable in three-dimensional cases. In three-dimensional analysis, each nodal point has three degrees of freedom and stresses are usually evaluated at integration points within the element. Evidently, in order to simulate the interface action between the silo wall and the stored material a three-dimensional 'joint' element need be employed.

The interface element used in this research is essentially a 16-node solid element having the same configuration of a 3-D thick shell element except very small thickness is assigned.
Figure 3-14 One- and Two-Dimensional Joint Elements
It is assumed that the bulk modulus for this type of element is as high as 'incompressible' comparing to that of the stored material so that the radial pressure can be directly transmitted to the silo wall and no penetration would occur within the thickness of the interface element.

As for the shear modulus for this type of element, it need be determined experimentally through direct shear test on the stored material against the wall material.

A representative stress-strain curve of sand testing on constant strain rate direct shear apparatus is as schematically shown in Fig. 3-15. An idealized elasto-plastic model can be utilized to characterize the stress-strain relationship. It has also been reported(20) that, curves of the same pattern could also be obtained by shearing granular materials like dry sands, dry wheat grains against concrete discs or some other types of wall materials. This implicates that at the interface the constitutive relationship for shear is basically similar to that of the stored material.

Using the curve description model in the present version of NONSAP, the bulk modulus for interface elements can be specified as a 'large' multiple of that of the stored material. A series of direct shear tests need be performed testing on the stored material against the wall material to characterize the relationship between the initial tangent or secant modulus (if elasto-plastic model is selected) and the normal pressure \( \sigma_1 \). That is

\[
G_I = f(\sigma_1, D_x) \tag{3-8}
\]
Figure 3-15 Stress-Strain Curve from a Direct Shear Test
Since the direct shear sample is always confined in thick metal rings, very little radial deformation would occur, thus \( \epsilon = \epsilon_1 \). With known initial relative density \( D_r \), utilizing the relationship between \( \epsilon \) and \( \sigma_1 \), Eq. (3-8) can also be expressed in terms of \( \epsilon \) and \( D_r \), i.e.,

\[
G_I = f(\epsilon, D_r)
\]  

Moreover, it can be readily seen that Fig. 3-15 is essentially equivalent to Fig. 2-2. Relationships as shown in Figs. 2-3 and 2-5 can also be established for direct shear tests by varying \( \sigma_1 \). Thus the procedure proposed in Chapter II can also be applied to this type of elements.

3.5 Concluding Remarks

Comparing Fig. 3-4(b) and Fig. 3-10, it is readily seen that the computational cost for the 60'x15' silo filled with material discretized as shown in Fig. 3-10 for one load step (or one iteration) will cost at least four times more than the foregoing elastic analysis (one load step) for the same silo when it was empty. In terms of dollars, it will cost at least $31.28 for one load step or iteration. Thus, it is important to adopt a nonlinear solution technique which can yield satisfactory solution accuracy with reasonable computational cost.

There are basically two reasons that iterative methods are not considered suitable for the numerical investigations of this research. Firstly, there is, in general, no way to predict or guarantee the num-

* This is based on the deferred rate.
ber of iterations needed for convergence for each load step. Secondly, the actual loading process of silos would not permit very large load steps since it is not realistic. Consequently, piecewise-linear nonlinear solution technique is adopted for latter numerical investigations. Further discussion pertaining to simulation of loading process of silos and the piecewise linear method is given in the next chapter.
CHAPTER IV.
NUMERICAL INVESTIGATIONS

4.1 Introductory Remarks

The NONSAP program, after modification, is ready for use in the 3/D nonlinear analysis investigation of this research. The modified subroutines are listed in Appendix E.

It is important to note that there are many different possible combinations of external lateral loading and internal gravity loading resulting from own weight of the material stored in the silo. In the following numerical investigations it is considered more realistic to impose external lateral loading on the silo after the stored material has reached its equilibrium. It is further assumed that the silo was filled sequentially, i.e., layer by layer, in order to simulate the loading process. Therefore, each analysis will be proceeded in two steps:

Step(1) - The filled silo is subjected to gravity load only.
Step(2) - The filled silo is subjected to both gravity load and external lateral loading.

The execution of Step(2) can be achieved either by using the RESTART feature of the NONSAP program or by specifying proper time (loading) functions for gravity load and external lateral loading. Two separate computer jobs need be submitted using the RESTART option and thus cause higher computational cost. Therefore, the other option, i.e., time functions, is preferred in this research.

* Refer to Reference 1 for applications of the RESTART option.
4.2 Simulation of the Loading Process

Due to limited computer fund, the stored material in the 15'x60' silo will be divided into sixteen horizontal layers as already shown in Fig. 3-4(b). Since the computational cost for three-dimensional finite element analysis of any practical engineering problem is with no exception very expensive, no equilibrium iteration will be performed in each loading step. The loading functions of gravity loads at nodal points at different elevations are as shown in Fig. 4-1. It is seen that the gravity loads at nodal points at each elevation are actually subdivided into two increments and the ratio of the two increments depends on the heights of the two adjacent layers. The external lateral loading is not active until at the end of the 15th load step.

It should be pointed out that although more sophisticated nonlinear solution techniques are available, they will lead to much higher computational cost than the piecewise-linear method adopted since in reality silos are always filled sequentially which implicates the smaller the division of layers the better the solution. It is, in practice, not economical to allow for iterations after a large number of layers have been divided.

The convergence of solution for nonlinear problems is as symbolically shown in Fig. 4-2 for one-dimensional cases. In principle, the more load steps taken the smaller the error. It is assumed, for engineering purposes, that the loading functions shown in Fig. 4-1 which essentially divide the total weight of the stored material into 32 increments and the external loading into 4 increments, are sufficient. Stresses output at the end of

@ It was due to the consideration of (1) limited computer fund, and (2) particle arrays would be more densely interlocked after all layers were loaded.
Figure 4-2 Symbolic Load-Displacement Curve for Piecewise Linear Incremental Solution
load steps 15, 18 and 20 are to be looked into and compared.

4.3 **Numerical Investigations**

In the following numerical investigations, the own weight of the silo structure will not be included in the analyses since it is always existing. The main objective is to investigate how the combined effects of System A and System B differ from that of System C, Fig. 4-3. Cross sectional deformation at the top of the silo, lateral movements along the height of the silo and the two most significant membrane stresses $\sigma_x$ and $\sigma_\phi$ are items to be compared.

4.3.1 **Empty Silo Subjected to Lateral Loading (System A)**

A computer run was performed for the 60"x15' full size silo discretized as shown in Fig. 3-4(b) subjected to a lateral loading uniform in the circumferential direction but varying linearly in the axial direction of the silo, as illustrated in Fig. 4-4. It should be noted that lateral loading (surface traction) of any form of distribution can be transformed to equivalent nodal loads in the $X,Y,Z$ directions according to the interpolation principle, Fig. 4-5.

$$P_i(r,s,t) = \sum_{k=1}^{N} p_i^k h_k$$

(i = X,Y or Z)

in which

- $P_i(r,s,t)$ = Distribution of surface traction in the $i$ direction over the element
- $p_i^k$ = Intensity of $P_i$ at the $k^{th}$ node of the element
- $h_k$ = Interpolation function at the $k^{th}$ node of an isoparametric element
Figure 4-3 Comparison of Effects of Lateral Loading

(Filled with Granular Material)

(Filled with Granular Material)
Figure 4-4  Lateral Loading for Numerical Investigations
Figure 4-5 Interpolation Principle of Surface Tensions
There are infinitely many forms of distribution of lateral loadings, the one chosen as shown in Fig. 4-4 was just a convenient example used to produce lateral movements, and thus cross sectional deformations, to a silo. However, it is worth noting that the pressure density at the top of the silo is 100 lb./ft\(^2\) which is, in magnitude, equivalent to extremely strong wind. Higher pressure density than this magnitude would not be practical in engineering sense.

Fig. 4-6 shows vertical and horizontal deformations at the top of the silo from results of analysis. It is seen that the silo wall elongates on the side facing the assigned lateral loading and shortens on the other side. The transition is gradual and the elongation at \(\theta=180^\circ\) is larger than the shortening at \(\theta=0^\circ\). Almost no change in length at \(\theta=90^\circ\) which is close to conventional beam theory since this is a long silo (height/diameter ratio = 4). The lateral movement in the \(y\)-direction at \(\theta=180^\circ\) is also larger than that at \(\theta=0^\circ\) which indicates shortening of diameter and will cause compression to the stored material if loaded. The maximum lateral displacement in the \(z\)-direction at \(\theta=90^\circ\) is of the same order of magnitude of the vertical displacements at \(\theta=0^\circ\) and \(180^\circ\) and is small comparing to lateral displacements in the \(y\)-direction. It is worth noting that all the displacements are very small comparing to the diameter and height of the silo although the silo is subjected to a lateral loading equivalent to very strong wind. As a result, the assumption that the silo wall remains linearly elastic is proven to be permissible.

Fig. 4-7 shows lateral displacements of the silo in the \(y\)-direction along the height at various locations. Being a long cylinder,
Figure 4-6 Vertical and Horizontal Deformations at the Top of the Silo (System A)
Figure 4-7 Horizontal Displacements ($\delta_y$) of the Silo (ft x 10^{-2})
the silo deflects like a cantilever.

Fig. 4-8 shows the resulting longitudinal stresses in the silo wall. It is seen that their distribution is very similar to that obtained from conventional beam theory. However, it should be noted that conventional beam theory can not yield satisfactory solution accuracy for this case especially at locations near the built-in end. The effects of the end constraints are manifested in the solutions for locations below 10 ft. It is important that finer mesh be used near the constrained end for similar cases.

Fig. 4-9 shows circumferential (hoop) stresses in the silo wall. The oscillating curve shapes indicate the existence of longitudinal-bending as can be perceived from the following equations (34).

\[ M_x = -D \left( \frac{d^2 \nu}{dx^2} \right) \]  
\[ N_\phi = -E h w/ r \]

4.3.2 Silo Filled with Chattahoochee Sand (System B)

As an example, the granular material chosen to fill the 60'x15' silo was Chattahoochee sand. The physical properties of this sand are as listed in Table 4-1 (14). From Table 4-1 the weight density of this sand for initial relative density \( D = 60\% \) was calculated to be 47.4 lb/ft\(^3\).

The shear and bulk moduli for primary loading and for unloading-reloading which was derived and used in the latter analysis are mainly based on the previous works of Domaschuk and Wade (14), and * Refer to Appendix D for converting lab testing results into curve description material model.
Figure 4-8 Longitudinal Stresses ($\sigma_x$)
Figure 4-9  Circumferential Stresses ($\sigma_\phi$)
<table>
<thead>
<tr>
<th>Data</th>
<th>Quantities</th>
</tr>
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<tbody>
<tr>
<td>Specific Gravity</td>
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</tr>
<tr>
<td>Maximum Void Ratio</td>
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</tr>
<tr>
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<tr>
<td>$D_{10}$ Size</td>
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<tr>
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</tr>
<tr>
<td>Textural Classification</td>
<td>Medium-Fine Sand</td>
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<tr>
<td>Unified Soil Classification</td>
<td>SP</td>
</tr>
<tr>
<td>Mica Content</td>
<td>&lt; 0.5%</td>
</tr>
</tbody>
</table>

Table 4-1 Physical Properties of Chattahoochee Sand
Fig. 4-10 shows cross sectional deformations and longitudinal as well as hoop stresses resulting from the stored material. It is seen that cross sectional deformations along the height of the silo are very small and negligible comparing to those caused by lateraling. Thus, they will be ignored in the latter analysis for displacements. The lower portions of $\sigma_x$ and $\sigma_\phi$ show the interaction between these two membrane stresses. Since there is no lateral loading acting, System B is an axisymmetric case.

Fig. 4-11 compares the internal radial pressure generated by the stored material with the solutions calculated by conventional method by Janssen(31). In Appendix G it was already pointed out that same mechanical properties of the stored material were assumed for the interface elements in this numerical analysis. Refering to Fig. 2-2 it can be seen that if the direct shear tests were performed within the vertical stress level of 10 psi (125 lb. of weight loaded on a 4" diameter sample), the engineering internal friction angle $\phi$ for Chattahoochee sand with initial density $D_e=60\%$ is around 16.5°. As a result the wall friction coefficient $\mu' = \tan(\phi) = 0.3$.

It is readily seen that Janssen's solution over estimated the radial pressure in the top one third portion of the silo since in this portion the vertical stresses in the sand are lower than assumed. However, Janssen's solutions are consistent to the finite element solutions in the middle one third portion of the silo because in this portion the vertical stresses in the sand are closer to assumed. In the lower one third portion of the silo, the effect that the sand was compacted and resulting higher internal friction angle makes Janssen's
Figure 4-10  Cross Sectional Deformations and Stresses in the Silo Wall
Figure 4-11 Comparison of Radial Pressures
solutions again higher than the finite element solutions.

4.3.3 Filled Silo Subjected to Lateral Loading (System C)

Fig. 4-12 shows vertical and horizontal deformations at the top of the silo from results of analysis. It is rather surprising to see that the lateral movements in the y-direction increased in stead of being decreased. The axial deformations are no longer close to being nearly symmetric as shown in Fig. 4-6 for System A. The axial shortening at $\theta=0^\circ$ was increased about twice as much and the elongation at $\theta=180^\circ$ was reduced to half. The reason for this phenomenon to occur was speculated to be as follows: As the portion of the silo wall in $\theta=90^\circ$ to $\theta=180^\circ$ elongated in response to lateral loading, the friction (shear) at the wall/material interface somewhat caused unloading to the contagious material. In the mean while, the portion of the silo wall in $\theta=0^\circ$ to $\theta=90^\circ$ shortened and the interface friction caused further loading to the contagious material on this side. As a consequence bending moments were created and the silo deflected more to the positive y-direction.

The variation of internal radial pressures also would induce additional bending moments. The final results are the silo sideswayed more and caused more shortening to the whole silo structure as compared to Fig. 4-6(a).

Fig. 4-13 shows comparison between the longitudinal stresses generated in System C and the superposition of those generated in System A and B. As can be expected, due to more bending, the longitudinal stresses for System C were increased in both tension and compression. It is worth noting that the trend of variation in longitudinal stresses in Fig. 4-13 is consistent to Fig. 4-12(a) for axial deformation of the silo.
(a) Axial Deformation

(b) Cross Sectional Deformation
(Middle Plane of the Silo Wall)

Figure 4-12 Vertical and Horizontal Deformations at
the Top of the Silo (System C)
Compression Increased

Figure 4-13   Longitudinal Stresses, System (A+B) vs System C
Figure 4-13 Logitudinal Stresses, System(A+B) vs System C (Cont.)
It can be seen from Table 4-2 that internal radial pressures increased in regions perpendicular to the direction of the external lateral loading ($\theta = 0^\circ \sim 60^\circ$ and $\theta = 120^\circ \sim 180^\circ$) and decreased in a small region parallel to the direction of the external lateral loading ($\theta = 60^\circ \sim 120^\circ$). Although their magnitudes seem insignificant in engineering sense, they may not be negligible in cases of large diameter silos.

Comparing Figs. 4-10 and 4-11, it is seen that the curve shape of hoop stresses in the silo wall is similar to that of the internal radial pressures. It can thus be readily perceived that the hoop stresses generated in System C are higher than the superposition of those generated in System A and System B in regions perpendicular to the direction of external lateral loading, and are lower in regions parallel to the direction of the external lateral loading.
<table>
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<th>Ht. (ft)</th>
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<th>System C (psf)</th>
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</thead>
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</tr>
<tr>
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</tr>
<tr>
<td>7</td>
<td>-383.50</td>
<td>-284.38</td>
</tr>
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</table>

* Heights chosen are around centers of each one fourth of the total height.

Table 4-2 Comparison of Radial Pressures
CHAPTER V.

CONCLUSIONS AND RECOMMENDATIONS FOR FUTURE RESEARCH

5.1 Conclusions

1. The silo filled with granular material with mechanical properties as assumed in this research tends to deflect more in the direction of external lateral loading comparing to the same silo subjected to the same lateral loading when it is empty. This is equivalent to that the stored material contributes 'negative' stiffness to the silo structure.

2. As a result of larger bending, the longitudinal and circumferential stresses in the silo wall increase in regions perpendicular to the direction of external lateral loading and decrease in regions parallel to the direction of external lateral loading as compared to linear superposition of effects due to external lateral loading and internal radial pressure resulting from the stored material when there is no external lateral loading.

3. Owing to cross sectional deformations, the internal radial pressures resulting from the stored granular material also increase in regions perpendicular to the direction of external lateral loading and decrease in regions parallel to the external lateral loading. The increase is larger than the decrease.

4. It is thus concluded that a filled silo is more vulnerable to external lateral loading than the same silo subjected to the same lateral loading when it is empty.

5. Conclusions of this research may not be applicable to short silos (bunkers).
5.2 Recommendations for Future Research

1. Since three-dimensional finite element analysis of any practical engineering problem is no exception very expensive, the technique to follow a foregoing two-dimensional analysis by a unavoidable three-dimensional may be worth developing. For example, the first step of the finite element analysis in this research could have been performed using two-dimensional elements, however for the continuation of the latter analysis (lateral loading) three-dimensional analysis was performed.

2. Short silos with large diameters may be worth the same investigation.
REFERENCES


25. Lambe, T. W. and Whitman R. V.; Soil Mechanics, John Wiley & Sons Ltd.


28. Mondkar, D. P. and Powell, G. H.; "Static and Dynamic Analysis of Nonlinear Structures", SESM Report 75-10, Department of Civil
Engineering, University of California, Berkeley, 1975.


APPENDIX A
THREE-DIMENSIONAL ISOPARAMETRIC FINITE ELEMENT MATRICES

A.1 Introductory Remarks

In this research, isoparametric finite element discretisation is used. The finite element matrices for 8 to 21 variable-number-nodes 3/D continuum element, as shown in Fig. A-1, accounting for effects due to large displacements, large strains and nonlinear material properties will be derived. Referring to the standard procedures for assembling the structure matrices, in the derivation of the required finite element matrices, attention need only be given to the calculation of matrices corresponding to a single element(30,32).

A.2 Interpolation Functions

In the isoparametric finite element solution the coordinates and displacements of an element are interpolated using the same interpolation functions at any time point or load step, namely

\[ t_{x_j} = \sum_{k=1}^{N_x} h_k t_{x_{jk}} \quad (A.1) \]

\[ t_{u_j} = \sum_{k=1}^{N_u} h_k t_{u_{jk}} \quad (A.2) \]

\[ u_j = \sum_{k=1}^{N_u} h_k u_{jk} \quad (A.3) \]
where \( j = 1, 2, 3 \), and \( h_k \) is the interpolation function corresponding to element nodal point \( k \), and \( N \) is the total number of nodal points of the elements \((4, 6, 9, 12, 32)\).

In three-dimensional finite element analysis the element formation time is, in general, significant and it has been found more efficient in the solution process to interpolate the coordinates to a lower degree than the displacements, namely, \( N_x < N_u \) in Eqs. (A.1) to (A.3). Typically, for the straight-sided elements, the displacements \( u \) and \( v \) are interpolated parabolically, whereas the coordinates \( x \) and \( y \) of the element need be interpolated linearly because the element edges are straight. Elements with a lower degree of interpolation on coordinates than the displacements are called subparametric elements \((4, 6, 9, 12, 32)\). Such elements are complete and satisfy the monotonic convergence requirements (13).

The option of using subparametric elements has been included in the NONSAP program and would be used in this research to increase the efficiency of the solution process.

For a general 8 to 21 variable-number-nodes three-dimensional element the interpolation functions written in terms of the natural coordinates \((r, s, t)\), as indicated in Fig. A-2, are as tabulated in Table A-1.

The evaluation of strains requires the following derivatives:

\[
\frac{\partial u_i}{\partial x_j} = \sum_{k=1}^{N} \left( \frac{\partial h_k}{\partial x_j} \right) u_i^k
\]

\[
= \sum_{k=1}^{N} (\partial h_{k,j}) u_i^k, \quad i=1, 2, 3 \quad (A.4)
\]

and

\[
\frac{\partial u_i}{\partial t_x_j} = \sum_{k=1}^{N} \left( \frac{\partial h_k}{\partial t_x_j} \right) u_i^k
\]
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<td>$-0.5h_9$ $-0.5h_{12}$ $-0.5h_{17}$</td>
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<tr>
<td>$h_{20} = \frac{1}{4} RST$</td>
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</tr>
</tbody>
</table>

Table A-1 Interpolation Functions of Three-Dimensional Element

where

- $R = 1+x$
- $S = 1+s$
- $T = 1+t$
- $\bar{R} = 1-x$
- $\bar{S} = 1-s$
- $\bar{T} = 1-t$
- $R^* = 1-x^2$
- $S^* = 1-s^2$
- $T^* = 1-t^2$
\[ N = \sum_{k=1}^{N} (t^h_{k,i}) u^k_1, \quad i=1,2,3 \] (A.5)

The derivatives are calculated using a Jacobian transformation.

The chain rule relating \( t^{x_1}_i \) \((i=1,2,3)\) to \( r,s,t \) derivatives is written as:

\[
\begin{bmatrix}
\frac{\partial}{\partial r} \\
\frac{\partial}{\partial s} \\
\frac{\partial}{\partial t}
\end{bmatrix}
= J
\begin{bmatrix}
\frac{\partial}{\partial t^{x_1}} \\
\frac{\partial}{\partial t^{x_2}} \\
\frac{\partial}{\partial t^{x_3}}
\end{bmatrix}
\] (A.6)

in which

\[
J =
\begin{bmatrix}
\frac{\partial^{x_1}}{\partial r} & \frac{\partial^{x_2}}{\partial r} & \frac{\partial^{x_3}}{\partial r} \\
\frac{\partial^{x_1}}{\partial s} & \frac{\partial^{x_2}}{\partial s} & \frac{\partial^{x_3}}{\partial s} \\
\frac{\partial^{x_1}}{\partial t} & \frac{\partial^{x_2}}{\partial t} & \frac{\partial^{x_3}}{\partial t}
\end{bmatrix}
\] (A.7)

Inverting the Jacobian operator \( J \), it is obtained that

\[
\begin{bmatrix}
\frac{\partial}{\partial t^{x_1}} \\
\frac{\partial}{\partial t^{x_2}} \\
\frac{\partial}{\partial t^{x_3}}
\end{bmatrix}
= J^{-1}
\begin{bmatrix}
\frac{\partial}{\partial r} \\
\frac{\partial}{\partial s} \\
\frac{\partial}{\partial t}
\end{bmatrix}
\] (A.8)
Figure A-1 Typical 3/D Continuum Elements Derived from the General 21-Node Element
Figure A-2  Hexahedral Element in Natural Coordinates
To evaluate the Jacobian determinant and Eqs. (A.4) and (A.5), the derivatives of the interpolation functions with respect to r, s, and t are needed and they are tabulated in Table A-2.

With all required derivatives being defined it is now possible to establish the strain-displacement transformation matrices for the elements.

A.3 Strain-Displacement Transformation

Decomposition of the incremental Almansi's strain tensor into a linear and nonlinear components can be written in the following vector form:

\[
\begin{bmatrix}
\varepsilon_{11} \\
\varepsilon_{22} \\
\varepsilon_{33} \\
2\varepsilon_{12} \\
2\varepsilon_{23} \\
2\varepsilon_{31}
\end{bmatrix}
= \begin{bmatrix}
\varepsilon_{11} \\
\varepsilon_{22} \\
\varepsilon_{33} \\
2\varepsilon_{12} \\
2\varepsilon_{23} \\
2\varepsilon_{31}
\end{bmatrix}
+ \begin{bmatrix}
\eta_{11} \\
\eta_{22} \\
\eta_{33} \\
2\eta_{12} \\
2\eta_{23} \\
2\eta_{31}
\end{bmatrix}
\] (A.9)

In the Updated Lagrangian formulation,

\[ t^e_{ij} = \frac{1}{2} (t^{u}_{i,j} + t^{u}_{j,i}) \] (A.10)

Eq. (A.10) written in matrix form gives

\[ t^e = t^B u \] (A.11)

in which

\[ u^T = \{ u_1^1 u_2^1 u_3^1 u_1^2 u_2^2 u_3^2 \ldots u_1^N u_2^N u_3^N \} \]

* Refer to Appendix B for definition.
<table>
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<th>Interpolation Function</th>
<th>9</th>
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Table A-2 Derivatives of Interpolation Functions w.r.t. s

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where

\[ R = 1 + r \]
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\[ T = 1 + t \]
\[ \bar{R} = 1 - r \]
\[ \bar{S} = 1 - s \]
\[ \bar{T} = 1 - t \]
\[ R = 1 - r^2 \]
\[ S = 1 - s^2 \]
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where

$R = 1 + r$

$S = 1 + s$

$T = 1 + t$

$R^* = 1 - r$

$S^* = 1 - s$

$T^* = 1 - t$

Table A-2 Derivatives of Interpolation Functions w.r.t. $r$
Table A-2 Derivatives of Interpolation Functions w.r.t. t

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where

- $R = 1+t$
- $S = 1+s$
- $T = 1+t$
- $R^* = 1-t$
- $S^* = 1-s$
- $T^* = 1-t$

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A.4.4 Externally Applied Element Nodal Load Vector

The first and second term on the right-hand side of Eq. (A.13) are effects due to surface tractions and body forces, respectively. They can be written as

\[ \delta u^T \left[ \int_{0_A} t_{Q}^e \, 0 \, da + \int_{0_V} \nabla t_{Q}^e \, 0 \, dv \right] = \delta u^T \left[ t + \Delta t_{Q} \right] \]  

(A.28)

in which

\[ t + \Delta t_{Q} = \int_{0_A} \nabla t_{Q}^e \, 0 \, da + \int_{0_V} \nabla t_{Q}^e \, 0 \, dv \]  

(A.29)

is termed the consistent or the externally applied element nodal load vector and is evaluated at the Gauss integration points in numerical integration.

Thus the discretized matrix equation of motion of a typical element can be symbolically written as

\[ (\, t_{K} + t_{K_{NL}} ) \, \dot{u} = t + \Delta t_{Q} - t_{P} \]  

(A.30)

A.5 Remarks

The use of Eq. (A.30) depends on the specific material property matrix considered. The evaluation of the material property matrix depends on the material model used.
The strain-displacement transformation matrix $T_{BL}$ for U.L. formulation can be readily derived according to Eq. (A.5).

It is seen that $T_{BL}$ is a rather sparse matrix. This allows relative ease in programming and less calculation time for the matrix product $T_{BL}^T C T_{BL}$ in the U.L. formulation.

An explicit relation between the nonlinear strains $\eta_{ij}$ and nodal displacements $u$ is not needed as will be evident when the evaluation of the geometric stiffness matrix is considered in the next section.

A.4 Evaluation of Element Matrices

The equations of motion for the U.L. formulation can be written as follows(1,2,3,4,5)

$$\int t^{C_{ijrs}} t^{ers} \delta_{t} t^{e_{ij}} t^{dv} + \int t^{T_{ij}} t^{\eta_{ij}} t^{dv}$$
APPENDIX B

DEFINITIONS OF STRESS AND STRAIN TENSORS

B.1 The Strain Tensors

Referring to Fig. B-1, consider an infinitesimal line element connecting the point \( P(a_1, a_2, a_3) \) to a neighboring point \( P'(a_1 + da_1, a_2 + da_2, a_3 + da_3) \). The square of the length \( ds_0 \) in the original configuration is given by

\[
ds_0^2 = da_1^2 + da_2^2 + da_3^2\]

When \( P \) and \( P' \) are deformed to the points \( Q(x_1, x_2, x_3) \) and \( Q'(x_1 + dx_1, x_2 + dx_2, x_3 + dx_3) \), respectively, the square of the length \( ds \) of the new line element \( QQ' \) is

\[
ds^2 = dx_1^2 + dx_2^2 + dx_3^2\]

The deformation of the body is known if \( x_1, x_2, x_3 \) are known functions of \( a_1, a_2, a_3 \):

\[
x_i = x_i(a_1, a_2, a_3) \quad (B.1)
\]

This is a mapping from \( a_1, a_2, a_3 \) to \( x_1, x_2, x_3 \). In continuum mechanics it is assumed\( (16) \) that deformation is continuous. A neighborhood is transformed into a neighborhood, i.e., the functions of Eq. (B.1) are single-valued, continuous, and have unique inverse

\[
a_i = a_i(x_1, x_2, x_3) \quad (B.2)
\]

for every point in the body.

By Eqs. (B.1) and (B.2) we have

\[
dx_i = \frac{x_i}{a_j} da_j, \quad da_i = \frac{a_i}{x_j} dx_j \quad (B.3)
\]
Figure B-1  Deformation of a Body
Hence, on introducing the Kronecker delta, we may write

\[ ds_0^2 = \delta_{ij} \, da_i \, da_j = \delta_{ij} \, \frac{\partial a_i}{\partial x_1} \, \frac{\partial a_j}{\partial x_m} \, dx_1 \, dx_m \]  

\[ ds^2 = \delta_{ij} \, dx_i \, dx_j = \delta_{ij} \, \frac{\partial x_i}{\partial a_l} \, \frac{\partial x_j}{\partial a_m} \, da_l \, da_m \]  

(B.4)

(B.5)

The difference between the squares of the length elements may be written, either as

\[ ds^2 - ds_0^2 = (\delta_{\alpha\beta} \, \frac{\partial x_\alpha}{\partial a_i} \, \frac{\partial x_\beta}{\partial a_j} - \delta_{ij}) \, da_i \, da_j \]  

(B.6)

or

\[ ds^2 - ds_0^2 = (\delta_{ij} - \delta_{\alpha\beta} \, \frac{\partial x_\alpha}{\partial x_i} \, \frac{\partial x_\beta}{\partial x_j}) \, dx_i \, dx_j \]  

(B.7)

The strain tensors are defined as:

\[ E_{ij} = \frac{1}{2} \left( \delta_{\alpha\beta} \, \frac{\partial x_\alpha}{\partial a_i} \, \frac{\partial x_\beta}{\partial a_j} - \delta_{ij} \right) \]  

(B.8)

\[ e_{ij} = \frac{1}{2} \left( \delta_{ij} - \delta_{\alpha\beta} \, \frac{\partial x_\alpha}{\partial x_i} \, \frac{\partial x_\beta}{\partial x_j} \right) \]  

(B.9)

so that

\[ ds^2 - ds_0^2 = 2 \, E_{ij} \, da_i \, da_j \]  

(B.10)

\[ ds^2 - ds_0^2 = 2 \, e_{ij} \, dx_i \, dx_j \]  

(B.11)

The strain tensor \( E_{ij} \) was introduced by Green and St.-Venant and is called Green's strain tensor. The strain tensor \( e_{ij} \) was introduced by Cauchy for infinitesimal strains and by Almansi and Hamel for finite strains and is known as Almansi's strain tensor. In analogy with terminology in hydrodynamics, \( E_{ij} \) is often referred as Lagrangian and \( e_{ij} \) as Eulerian.

In rectangular Cartesian coordinates, if we introduce the displace-
moment vector \( \mathbf{u} \) with components

\[
\mathbf{u}_\alpha = x_\alpha - a_\alpha \quad (\alpha = 1, 2, 3)
\]

then

\[
\frac{\partial x_\alpha}{\partial a_i} = \frac{\partial u_\alpha}{\partial a_i} + \delta_\alpha^i \quad \frac{\partial a_\alpha}{\partial x_i} = \delta_\alpha^i - \frac{\partial u_\alpha}{\partial x_i}
\]  \( \text{(B.12)} \)

and the strain tensors reduce to the simple form:

\[
E_{ij} = \frac{1}{2} \left\{ \delta_{ij} \left( \frac{\partial u_\alpha}{\partial a_i} + \delta_{\alpha i} \right) \left( \frac{\partial u_\beta}{\partial a_j} + \delta_{\beta j} \right) - \delta_{ij} \right\}
\]

\[
= \frac{1}{2} \left\{ \frac{\partial u_i}{\partial x_i} + \frac{\partial u_i}{\partial a_j} + \frac{\partial u_\alpha}{\partial a_i} \frac{\partial u_\alpha}{\partial a_j} \right\} \quad \text{(Green)} \quad \text{(B.13)}
\]

and

\[
e_{ij} = \frac{1}{2} \left\{ \delta_{ij} - \delta_{i\beta} \left( \frac{\partial u_\alpha}{\partial x_i} + \delta_{\alpha i} \right) \left( \frac{\partial u_\beta}{\partial x_j} + \delta_{\beta j} \right) \right\}
\]

\[
= \frac{1}{2} \left\{ \frac{\partial u_i}{\partial x_i} + \frac{\partial u_i}{\partial x_j} - \frac{\partial u_\alpha}{\partial x_i} \frac{\partial u_\alpha}{\partial x_j} \right\} \quad \text{(Almansi)} \quad \text{(B.14)}
\]

Eq. (B.14) is equal to Eq. (3.9) in the U.L. formulation of this research.

If the components of displacement \( u_i \) are such that their first
derivatives are so small that the squares and products of the partial
derivatives of \( u_i \) are negligible, then \( e_{ij} \) reduces to Cauchy's infinitesimal strain tensor,

\[
e_{ij} = \frac{1}{2} \left\{ \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right\}
\]  \( \text{(B.15)} \)

In the infinitesimal displacement case, the distinction between the
Lagrangian and Eulerian strain tensor disappears, since then it is
immaterial whether the derivatives of the displacements are calcu-
lated at the position of a point before or after deformation.
B.2 The Stress Tensors

A stress tensor referred to the strained state is a natural physical choice. However, in the course of analysis we must relate stresses to strains. Hence, if strains were referred to the original position of particles in a continuum, it would be convenient to define stresses similarly with respect to the original configuration.

Consider an element of a strained solid as shown on the right-hand side of Fig. B-2. Assume that in the original (undeformed) state this element has the configuration as shown on the left-hand side of Fig. B-2. A force vector $dT$ acts on the surface $FQRS$. A corresponding force vector $dT_0$ acts on the surface $P_0Q_0R_0S_0$. If we assign a rule of correspondence between $dT$ and $dT_0$ for any pair of corresponding surfaces, and define stress vectors in each case as the limiting ratios $dT/dS$, $dT_0/dS_0$, where $dS$ and $dS_0$ are the areas of $FQRS$, $P_0Q_0R_0S_0$, respectively, then we can define stress tensors in both configurations. The assignment of a corresponding rule between $dT$ and $dT_0$ is arbitrary, but must be mathematically consistent.

The following rule is known as Kirchhoff rule:

$$
\frac{dT_0^{\nu}}{dS_0^{\nu}} = \frac{\partial a_i^{\nu}}{\partial x_j^{\nu}} \frac{dT_j}{dS_j}
$$

(B.16)

i.e., $dT_0$ and $dT$ are related by the same rule as the transformation

$$
da_i = \frac{\partial a_i}{\partial x_j} \, dx_j
$$

(B.3)

If $T_j$ is the stress tensor referred to the strained state, we have Cauchy's relation

$$
dT_j = T_{ij} \, \nu_j \, dS
$$

(B.17)
Figure B-2 The Corresponding Tensions in the Original and Deformed State of a Body
If Eq. (B.16) is used, we write

\[ dT_{ij} = S_{ij} \nu_{ij} dS_o = \frac{\partial a_i}{\partial x_\alpha} dT_\alpha \]  

(B.16)

\( r_{ij} \) and \( S_{ij} \) are called the Cauchy (Eulerian) and the Kirchhoff stress tensors, respectively.

The relationships between \( r_{ij} \) and \( S_{ij} \) can be derived as follows:

\[ S_{ij} = \frac{\rho_o}{\rho} \frac{\partial a_i}{\partial x_\alpha} \frac{\partial a_j}{\partial x_\beta} r_{\beta \alpha} \]  

(B.19)

\[ r_{ij} = \frac{\rho_o}{\rho} \frac{\partial a_i}{\partial a_\alpha} \frac{\partial a_j}{\partial a_\beta} S_{\beta \alpha} \]  

(B.20)

From Cauchy's formula(18), it is clear that the Cauchy Stress tensor is symmetric. From Eq. (B.19), it is seen that the Kirchhoff stress tensor is also symmetric.

Expressed in terms of displacement components \( u_1, u_2, u_3 \), Eqs. (B.19) and (B.20) may be written as

\[ S_{ij} = \frac{\rho_o}{\rho} \left\{ r_{ij} - (\delta_{j\beta} \frac{\partial u_i}{\partial x_\alpha} + \delta_{i\alpha} \frac{\partial u_j}{\partial x_\beta} - \frac{\partial u_i}{\partial x_\alpha} \frac{\partial u_j}{\partial x_\beta} ) r_{\beta \alpha} \right\} \]  

(B.21)

\[ r_{ij} = \frac{\rho_o}{\rho} \left\{ S_{ij} + (\delta_{j\beta} \frac{\partial u_i}{\partial a_\alpha} + \delta_{i\alpha} \frac{\partial u_j}{\partial a_\beta} + \frac{\partial u_i}{\partial a_\alpha} \frac{\partial u_j}{\partial a_\beta} ) S_{\beta \alpha} \right\} \]  

(B.22)
APPENDIX C

COMPUTER IMPLEMENTATION OF EXTERNALLY APPLIED NODAL LOAD VECTOR

C.1 Introductory Remarks

In the present version of NONSAP (1, 2), the loading in the analysis can consist only of concentrated nodal point loading, i.e., all distributed body or surface loading must be transformed to nodal point loading prior to using the program. To meet this requirement, it is necessary to formulate a subprogram to take upon the task since distributed body forces and surface traction both play essential roles in the loading condition of this research.

C.2 Surface Traction

It has been derived in Eq. (A.29) that for quasi-static condition

\[ t+\Delta t Q = \int_{T_1}^{T_2} d^O a + \int_{F_1}^{F_2} d^O v \]

The first term on the right-hand side of Eq. (C.1) is the externally applied nodal load vector due to surface traction and it can be further expressed as

\[ Q_s = \int_{O_A}^{O_A} H_s T_s d^O a \]

in which

\[ T_s^T = \begin{bmatrix} T_s^x & T_s^y & T_s^z \end{bmatrix} \]

and \( H_s \) is the displacement interpolation matrix.

In this research, the silo wall is simulated by 16-node subparametric thick shell elements as shown in Fig. C-1. Assuming that surface traction
acting on the surface $t=1$, the displacement interpolation matrix $H_s$
then consists only two local coordinates $r$ and $s$, that is

$$H_s(r,s) = \begin{bmatrix}
    h_1 & 0 & 0 & h_2 & 0 & 0 & \cdots & h_{16} & 0 & 0 \\
    0 & h_1 & 0 & 0 & h_2 & 0 & \cdots & 0 & h_{16} & 0 \\
    0 & 0 & h_1 & 0 & 0 & h_2 & \cdots & 0 & 0 & h_{16}
\end{bmatrix} \quad (t=1) \quad (C.4)$$

Writing Eq. (C.2) in terms of local natural coordinates yields

$$\int H^T_s(r,s) T^s(r,s) \det J \, dr \, ds = \sum_{i,j} a_{ij} f(r_i,s_j) + R_n \quad (C.5)$$
in which

$$f(r,s) = H^T_s(r,s) T^s(r,s) \det J \quad (C.6)$$

The right-hand side of Eq. (C.5) is the numerical integration form of
the left-hand side. The summations extend over all integration points
$i$ and $j$ specified, the $a_{ij}$ are weighting factors, and $f(r_i,s_j)$ are the
matrices $f(r,s)$ evaluated at the integration points specified in the
in the argument. The matrices $R_n$ are error matrices, which in practice,
are usually not evaluated. The integration over directions $r$ and $s$ need
not of the same order. Therefore, it is used that

$$f(r,s) \, dr \, ds = \sum_{i,j} a_{ij} f(r_i,s_j) \quad (C.7)$$

To evaluate the Jacobian determinant at specified Gauss integration
points, the derivatives of the interpolation functions with respect to
$r$ and $t$ are needed and they are tabulated in Table C-1.

With $T_s$, Eq. (C.3), given in local coordinates $r$ and $s$, the exter-
ally applied nodal load vector $Q_s$ is ready to be evaluated numerically.

C.5 Body Forces
Figure C-1  3/D Thick Shell Element
Table C-1 Derivatives of Interpolation Functions of Thick Shell Element w.r.t. r and s (t=1)

\[
\begin{align*}
\text{h}_1, r &= (1/4)(1+s) + (1/2)r(1+s) - (1/4)(1-s^2) \\
\text{h}_2, r &= -(1/4)(1+s) + (1/2)r(1+s) + (1/4)(1-s^2) \\
\text{h}_3, r &= -(1/4)(1-s) + (1/4)(1-s^2) + (1/2)r(1-s) \\
\text{h}_4, r &= (1/4)(1-s) + (1/2)r(1-s) - (1/4)(1-s^2) \\
\text{h}_5, r &= 0 \\
\text{h}_6, r &= 0 \\
\text{h}_7, r &= 0 \\
\text{h}_8, r &= 0 \\
\text{h}_9, r &= -r(1+s) \\
\text{h}_{10}, r &= -(1/2)(1-s^2) \\
\text{h}_{11}, r &= -r(1-s) \\
\text{h}_{12}, r &= (1/2)(1-s^2) \\
\text{h}_{13}, r &= 0 \\
\text{h}_{14}, r &= 0 \\
\text{h}_{15}, r &= 0 \\
\text{h}_{16}, r &= 0 \\
\text{h}_1, s &= (1/4)(1+r) - (1/4)(1-r^2) + (1/2)(1+r)s \\
\text{h}_2, s &= (1/4)(1+r) - (1/4)(1-r^2) + (1/2)(1-r)s \\
\text{h}_3, s &= -(1/4)(1-r) + (1/2)(1-r)s + (1/4)(1-r^2) \\
\text{h}_4, s &= -(1/4)(1+r) + (1/4)(1-r^2) + (1/2)(1+r)s \\
\text{h}_5, s &= 0 \\
\text{h}_6, s &= 0 \\
\text{h}_7, s &= 0 \\
\text{h}_8, s &= 0 \\
\text{h}_9, s &= (1/2)(1-r^2) \\
\text{h}_{10}, s &= -(1-r)s \\
\text{h}_{11}, s &= -(1/2)(1-r^2) \\
\text{h}_{12}, s &= -(1+r)s \\
\text{h}_{13}, s &= 0 \\
\text{h}_{14}, s &= 0 \\
\text{h}_{15}, s &= 0 \\
\text{h}_{16}, s &= 0
\end{align*}
\]
The second term on the right-hand side of Eq. (C.1) is the externally applied nodal load vector due to body forces and it can be further expressed as (4)

\[ Q_B = \int_{O_V} H^B_T F_B dO_V \]  \hspace{1cm} (C.8)

in which

\[ F^B_T = \begin{pmatrix} F_x^B & F_y^B & F_z^B \end{pmatrix} \]  \hspace{1cm} (C.9)

and \( H^B \) is the displacement interpolation matrix.

Since gravity is the only body force needed to be considered in this research, it is sufficient to distribute gravity loads to nodal points using linear interpolation functions. For a general 8-point 3/D isoparametric continuum element, Fig. C-2, \( H^B \) can be symbolically written as follows

\[
H^B(x,s,t) = \begin{bmatrix}
  h_1 & 0 & 0 & h_2 & 0 & 0 & \ldots & h_8 & 0 & 0 \\
  0 & h_1 & 0 & 0 & h_2 & 0 & \ldots & 0 & h_8 & 0 \\
  0 & 0 & h_1 & 0 & 0 & h_2 & \ldots & 0 & 0 & h_8 \\
\end{bmatrix}
\]  \hspace{1cm} (C.10)

Similar to the procedure in Eqs. (C.5) to (C.7), it can be derived that

\[
\int f(x,s,t) \, dr \, ds \, dt = \sum_{i,j,k} a_{ijk} f(x_i,s_j,t_k)
\]  \hspace{1cm} (C.11)

in which

\[ f(x,s,t) = H^B_T(x,s,t) F(x,s,t) \, \text{det} J \]  \hspace{1cm} (C.12)
Figure C-2  A General 8-Node 3/D Continuum Element
<table>
<thead>
<tr>
<th></th>
<th>( h_s )</th>
<th>( h_t )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>((1/8)(1+x)(1+t))</td>
<td>((1/8)(1+x)(1+s))</td>
</tr>
<tr>
<td>2</td>
<td>(-(1/8)(1+x)(1+t))</td>
<td>((1/8)(1-x)(1+s))</td>
</tr>
<tr>
<td>3</td>
<td>(-(1/8)(1-s)(1+t))</td>
<td>((1/8)(1-x)(1-s))</td>
</tr>
<tr>
<td>4</td>
<td>((1/8)(1-s)(1+t))</td>
<td>((1/8)(1+x)(1-s))</td>
</tr>
<tr>
<td>5</td>
<td>((1/8)(1+x)(1-t))</td>
<td>(-(1/8)(1-x)(1-s))</td>
</tr>
<tr>
<td>6</td>
<td>(-(1/8)(1-x)(1-t))</td>
<td>((1/8)(1+x)(1-s))</td>
</tr>
<tr>
<td>7</td>
<td>((1/8)(1-x)(1-t))</td>
<td>((1/8)(1-x)(1-s))</td>
</tr>
<tr>
<td>8</td>
<td>((1/8)(1-s)(1-t))</td>
<td>((1/8)(1+x)(1-s))</td>
</tr>
</tbody>
</table>

Table C-2 Derivatives of Interpolation Functions of
a 8-Node Isoparametric 3/D Continuum Element
To evaluate the Jacobian determinant at specified Gauss integration points, the derivatives of the interpolation functions with respect to \( r, s \) and \( t \) are needed and they are explicitly tabulated in Table C-2.

It should be noted that there is only one nonzero constituent in the vector \( F^B \), Eq. (C.9), depending on the coordinate system adopted. According to the Cartesian coordinate system shown in Fig. C-1, Eq. (C.9) can be rewritten as

\[
F^B = \begin{pmatrix} 0 & 0 & F^R_z \end{pmatrix}
\]  

(C.13)

C.4 Flow Chart and Program Listing

The flow chart for the computer implementation of the above formulation is shown in Fig. C-3. The computer program is listed hereinafter.
START

READ Nodal Point Coordinates, NUMNP, NUMEL, and Weight Densities of Element Groups

READ Pressure Intensities at Loaded Nodal Points

Initialize Externally Applied Loads Vector

READ 3/D Continuum Elements Data

Calculate RB(NP) and RS(NP)

IF N.EQ.NUMEL

YES

IF RB(NP) or RS(NP),EQ.0

YES

NP = NP + 1

NO

PRINT RB(NP) or RS(NP)

STOP

Figure C-3 Program Flow Chart
THIS PROGRAM TRANSFORMS DISTRIBUTED LOADS INTO NODAL LOADS

IMPLICIT REAL*8(A-H,O-Z)
DIMENSION A(3)
COMMUN/CUG/M(3),WS(3),NT(3),R(3),S(3),T(3)
COMMUN/GN/10=1,NDT,IOD,IOS,ITD,IT,IS,IT
COMMUN/AO/300=(300),RD(3500),RS(3500),X(3500),Y(3500),
Z(3500),SUMS,SUM2,SUM3,SUM4,SUM5,SUM6,SUM7,SUM8,SUM9
COMM/GST/(433),ST(16),DT(16),UT(16),ICODE,
DETJ,NUM,PP,

READ(5,10)NUGN1,NUGN2,TS,ICODE,MTYPE,IPR,IOS,IT,NEL,
*NUMNP,ISUR
10 FORMAT(2F10.9,f0,15)
DEGS=3.1416/180.

ICODE=1 BODY FORCES
MTYPE=NUMBER OF DIFFERENT TYPES ON MATERIALS
IPR=INTEGRATION ORDER IN THE R DIRECTION
IOS=INTEGRATION ORDER IN THE S DIRECTION
ITD=INTEGRATION ORDER IN THE T DIRECTION
ISUR=NUMBER OF ELEMENTS ON WHICH SURFACE TRACTION APPLIED
TS=SURFACE COORDINATE ON WHICH SURFACE TRACTION APPLIED

DATA R/0.77459669241483,-0.77459669241483,0.000/
DATA S/0.77459669241483,-0.77459669241483,0.000/
DATA T/0.77459669241483,-0.77459669241483,0.000/
DATA WR/0.55555555555556,0.55555555555556,0.55555555555556,
DATA WS/0.55555555555556,0.55555555555556,0.55555555555556,
DATA WI/0.55555555555556,0.55555555555556,0.55555555555556,

INITIALIZE BODY FORCE AND SURFACE TRACTION LOAD VECTORS

DO 41 NP=1,NUMNP
41 RS(NP)=0.

INITIALIZE ELEMENT NODAL LOAD VECTORS

DO 12 NUM=1,NEL
DO 12 LD=1,3
12 P(NUM,LD)=0.

SURFACE TRACTION INPUT
IF(ISUR.EQ.0) GO TO 13
DO 16 JK=1,ISUR
READ(5,14) NUML,(A(JL),JL=1,8)
14 FORMAT(14,DF=5.0)
DO 16 JM=1,8
16 PINUML,JM)=A(JM)
NOEL=ELEMENT NUMBER ON WHICH SURFACE TRACTION APPLIED
A(JL)=INTENSITY OF SURFACE TRACTION AT LOCAL NODES
READ NODAL POINTS AND ELEMENTS DATA
DO 30 NP=1,NUMP
READ(5,24) LT, NM, PSF, ID1, ID2, ID3, ID4, ID5, ID6, ZC, RC, TC
24 FORMAT(A1,14,A1,14,515,3F10.0)
X(NP)=ZC
Y(NP)=RC*COS(CT*DEGS)
Z(NP)=RC*SIN(CT*DEGS)
30 CONTINUE
DO 40 M=1,MTYPE
DO 40 NUM=1,NEL
READ(5,26) N, IELU, IELX, IPS, MTYP, IST, KG, (NUD(NUM,K),K=1,16)
26 FORMAT(715/615/815)
CALL SOLID
40 CONTINUE
DO 66 NP=1,NUMNP
IF (ICODE.EQ.0) GO TO 22
NGUR=1
IF (FAC.EQ.0.1) GO TO 66
IDIRN=1
GO TO 55
22 FAC=RS(NP)
IF (FAC.EQ.0.) GO TO 66
IDIRN=2
55 WRITE(*,77) NP,IDIRN,NGUR,FAC
77 FORMAT(16,15,F10.4)
66 CONTINUE
STOP
END

SUBROUTINE SOLID*
IMPLICIT REAL*A-H,0-Z
COMMON/CUE/WR(J),WS(3),WT(3),R(3),S(3),T(3)
COMMON/ULD/WD=NU1,WE=N2,INUD,IK,R1,IK,IT
COMMON/SAD/NUG(+400,16),RB(+3500),RS(+3500),X(+3500),Y(+3500)
*Z(+3500),SUM1,SUM2,SUM3,SUM4,SUM5,SUM6,SUM7,SUM8
COMMON /SUL/N(+490,3),SU(16),UR(16),US(16),UT(16)
COMMON/NUM,PP,M

INTERPOLATION FUNCTIONS OF 3-D SOLID ELEMENTS

DO 290 IR=1,IK
DO 290 IS=1,10
DO 280 IT=1,10
IF (IGOVE=70,2) T(1T)=TS
SO(1)=C.125*(1.+K(IR))*(1.+S(IS))*T(IT))
SO(4)=C.125*(1.+K(IR))*(1.+S(IS))*T(IT))
SU(1)=C.125*(1.+K(IR))*(1.-S(IS))*T(IT))
SU(4)=C.125*(1.+K(IR))*(1.-S(IS))*T(IT))
SU(5)=C.125*(1.+K(IR))*(1.+S(IS))*T(IT))
SU(6)=C.125*(1.+K(IR))*(1.+S(IS))*T(IT))
SU(7)=C.125*(1.+K(IR))*(1.-S(IS))*T(IT))
SU(8)=C.125*(1.+K(IR))*(1.-S(IS))*T(IT))

<<<<<<PRESSURE_VARIATION_OVER_ELEMENT_SURFACE>>>>>
PP=P(NUM,4)*SU(4)+P(NUM,5)*SU(5)+P(NUM,6)*SU(6)+*P(NUM,7)*SU(7)+P(NUM,8)*SU(8)

DERIVATIVES OF INTERPOLATION FUNCTIONS OF 3-D ELEMENTS

DR(1)=C.125*(1.+S(IS))*(1.+T(IT))
DR(2)=C.125*(1.+S(IS))*(1.+T(IT))
DR(3)=C.125*(1.+S(IS))*(1.+T(IT))
DR(4)=C.125*(1.+S(IS))*(1.+T(IT))
DR(5)=C.125*(1.+S(IS))*(1.+T(IT))
DR(6)=C.125*(1.+S(IS))*(1.+T(IT))
DR(7)=C.125*(1.+S(IS))*(1.+T(IT))
DR(8)=C.125*(1.+S(IS))*(1.+T(IT))

DS(1)=C.125*(1.+K(IR))*(1.+T(IT))
DS(2)=C.125*(1.+K(IR))*(1.+T(IT))
DS(3)=C.125*(1.+K(IR))*(1.+T(IT))
DS(4)=C.125*(1.+K(IR))*(1.+T(IT))
DS(5)=C.125*(1.+K(IR))*(1.+T(IT))
DS(6)=C.125*(1.+K(IR))*(1.+T(IT))
DS(7)=C.125*(1.+K(IR))*(1.+T(IT))
DS(8)=C.125*(1.+K(IR))*(1.+T(IT))

DT(1)=C.125*(1.+K(IR))*(1.+S(IS))
DT(2)=C.125*(1.+K(IR))*(1.+S(IS))
DT(3)=C.125*(1.+K(IR))*(1.+S(IS))
DT(4)=C.125*(1.+K(IR))*(1.+S(IS))
DT(5)=C.125*(1.+K(IR))*(1.+S(IS))
DT(6)=C.125*(1.+K(IR))*(1.+S(IS))
DT(7)=C.125*(1.+K(IR))*(1.+S(IS))
DT(8)=C.125*(1.+K(IR))*(1.+S(IS))

CALL GAUSS
280 CONTINUE
RETURN

SUBROUTINE GAUSS
IMPLICIT REAL*4(A-H,O-Z)
COMMON/C0,7*K(3),WS(3),WT(3),R(3),S(3),T(3)

EVALUATION OF JACOBIAN DETERMINANT AT GAUSSIAN POINTS

IF (ICODE .EQ. 1) GO TO 380
IF (PP .EQ. 9) GO TO 360

DO 111 K = 1, M
  SUM1 = SUM1 + YR(K) * X(NOUL(NUM, K))
  SUM2 = SUM2 + UR(K) * XY(NOUL NUM, K))
  SUM3 = SUM3 + UR(K) * X(NOD NUM, K))
  SUM4 = SUM4 + US(K) * X(NOD NUM, K))
  SUM5 = SUM5 + US(K) * Y(NOD NUM, K))
  SUM6 = SUM6 + JS(K) * X(NOD NUM, K))
  SUM7 = SUM7 + DT(K) * X(NOD NUM, K))
  SUM8 = SUM8 + JT(K) * Y(NOD NUM, K))
  SUM9 = SUM9 + JT(K) * Z(NOD NUM, K))

DETFJ = SUM1 * SUM5 + SUM2 * SUM3 + SUM6 * SUM7 + SUM8
       - SUM3 * SUM5 + SUM2 * SUM6 * SUM7 * SUM8
111 SUM1 = SUM1 + DI(R(K) * X(NOD NUM, K))
SUM2 = SUM2 + DI(R(K) * Y(NOD NUM, K))
SUM3 = SUM3 + DI(R(K) * Z(NOD NUM, K))

*** SURFACE TRACTION ***

IF (ICODE .EQ. 17) GO TO 310
DO 290 LOC = 1, R
  KS(NOUL, LOC) = R(NOUL, LOC) * WR(IR) * WS(IS) *
       (SU(LOC) * PP * DETJ)
290 GO TO 350

*** BUDY FORCES ***

310 DO 320 LOC = 1, 8
  IF (M .EQ. 2) GO TO 321
  KB(NOUL NUM, LOC) = B(NOD NUM, LOC) * WR(IR) * WS(IS) * WT(IT) *
       (SC(LOC) * DEN1 * DETJ)
320 CONTINUE
321 KB(NOD NUM, LOC) = B(NOD NUM, LOC) * WR(IR) * WS(IS) * WT(IT) *
       (SC(LOC) * DENZ * DETJ)
360 RETURN
ENTRY
APPENDIX D
CONVERSION OF EXPERIMENTAL DATA INTO CURVE DESCRIPTION MODELS

D.1 Bulk Modulus

Since the stress-strain relationship is nonlinear for sands, the bulk modulus defined in Eq. (4.10) need be rewritten as

\[ K = \frac{\frac{d\sigma_m}{d\epsilon_v}}{\epsilon_v^0} = \text{Limit} \left( \frac{\Delta \sigma_m}{\Delta \epsilon_v} \right) \]  \hspace{1cm} (D.1)

in which \[ \epsilon_v = \epsilon_{kk} + \epsilon_{11} + \epsilon_{22} + \epsilon_{33} \] is the volumetric strain because there is no volumetric change due to shear.

The relationships among volumetric strain \( \epsilon_v \), mean normal stress \( \sigma_m \) and initial relative density \( D_r \) of the sand investigated by Domaschuk and Wade (14) have been shown in Fig. 4-18. Relative densities of 40%, 60% and 80% are chosen in the following elaboration for demonstrative purpose as to represent the sand in loose, medium and dense state, respectively.

D.1.1 Loading Modulus \( K_{LD} \)

Reading directly from Fig. 4-8 values of \( \sigma_m \) and \( \epsilon_v \) are tabulated as follows:

<table>
<thead>
<tr>
<th>( D_r = 40% )</th>
<th>( \sigma_m ) (psi)</th>
<th>1</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>25</th>
<th>30</th>
<th>40</th>
<th>50</th>
<th>60</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Delta \sigma_m )</td>
<td>4</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>10</td>
<td>10</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \epsilon_v ) (%)</td>
<td>0</td>
<td>0.6</td>
<td>0.95</td>
<td>1.30</td>
<td>1.55</td>
<td>1.80</td>
<td>1.95</td>
<td>2.25</td>
<td>2.50</td>
<td>2.70</td>
<td></td>
</tr>
<tr>
<td>( \Delta \epsilon_v )</td>
<td>0.6</td>
<td>0.35</td>
<td>0.35</td>
<td>0.25</td>
<td>0.25</td>
<td>0.15</td>
<td>0.30</td>
<td>0.25</td>
<td>0.20</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( K_{LD} ) (psi)</td>
<td>667</td>
<td>1429</td>
<td>1429</td>
<td>2000</td>
<td>2000</td>
<td>3333</td>
<td>3333</td>
<td>4000</td>
<td>5000</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

109
$D_r = 60\%$

<table>
<thead>
<tr>
<th>$\epsilon_v (%)$</th>
<th>0</th>
<th>0.4</th>
<th>0.6</th>
<th>0.8</th>
<th>1.0</th>
<th>1.2</th>
<th>1.35</th>
<th>1.55</th>
<th>1.75</th>
<th>1.95</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta \epsilon_v$</td>
<td>0.4</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
<td>0.15</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
</tr>
<tr>
<td>$K_{ID}$ (psi)</td>
<td>1000</td>
<td>2500</td>
<td>2500</td>
<td>2500</td>
<td>2500</td>
<td>3333</td>
<td>5000</td>
<td>5000</td>
<td>5000</td>
<td>5000</td>
</tr>
</tbody>
</table>

$D_r = 80\%$

<table>
<thead>
<tr>
<th>$\epsilon_v (%)$</th>
<th>0</th>
<th>0.25</th>
<th>0.45</th>
<th>0.6</th>
<th>0.75</th>
<th>0.85</th>
<th>0.95</th>
<th>1.15</th>
<th>1.30</th>
<th>1.40</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta \epsilon_v$</td>
<td>0.25</td>
<td>0.20</td>
<td>0.15</td>
<td>0.15</td>
<td>0.1</td>
<td>0.1</td>
<td>0.2</td>
<td>0.15</td>
<td>1.1</td>
<td></td>
</tr>
<tr>
<td>$K_{ID}$ (psi)</td>
<td>1600</td>
<td>2500</td>
<td>3333</td>
<td>3333</td>
<td>5000</td>
<td>5000</td>
<td>5000</td>
<td>5000</td>
<td>6667</td>
<td>10000</td>
</tr>
</tbody>
</table>

It should be noted that for numerical solution $t_K$ is used for the load step between $t$ and $t + \Delta t$, as manifested in the following equation.

$$t + \Delta t \sigma_m = 3 t_K (t + \Delta t e_m - e_m) + t \sigma_m$$  \hspace{1cm} (4.26)

Small load step need be taken to obtain good approximation in Eq. (D.1).

D.1.2 Unloading-Reloading Modulus $K_{UN}$

The only unloading record kept by Domaschuk and Wade(14) during their experimental research on Chattahoochee sand was Fig. 4-13. Since it has been assumed that the unloading-reloading stress-strain relationship could be closely approximated as linear, Fig. 4-13 can still be used to calculated $K_{UN}$.

$D_r = 40\%$

$\Delta \epsilon_v = (1-0.58) \times 3.5\% = 1.47\%$

$\Delta \sigma_m = 100$ psi

Thus

$K_{UN} = \Delta \sigma_m / \Delta \epsilon_v = 6803$ psi
$D_x = 60\%$

\[ \Delta \varepsilon_v = (1-0.465) \times 2.55\% = 1.36\% \]

\[ \Delta \sigma_m = 100\ \text{psi} \]

Thus

\[ X_{UN} = 7330\ \text{psi} \]

$D_x = 80\%$

\[ \Delta \varepsilon_v = (1-0.345) \times 1.75\% = 1.15\% \]

\[ \Delta \sigma_m = 100\ \text{psi} \]

Thus

\[ X_{UN} = 8724\ \text{psi} \]

D.2 Shear Modulus

Similar to the foregoing manipulation, the shear modulus is expressed as

\[ G = \frac{d S_d}{d \varepsilon_d} = \text{Limit} \left( \frac{\Delta S_d}{\Delta \varepsilon_d} \right)_{\varepsilon_d \rightarrow 0} \]

It should be noted that $G$ is not only dependent on the mean normal stress $\sigma_m$ and initial relative density $D_x$ but also on the resultant deviatoric stress $S_d$, namely

\[ G = f(\sigma_m, S_d, D_x) \]  \hspace{1cm} (4.18)

D.2.1 Loading Modulus $G_{LD}$

Referring to Eq. (4.20), the parameter $b$ can be directly read

\[ G = G_i \left( 1 - b S_d \right)^2 \]  \hspace{1cm} (4.20)

from Fig. 4-23 for each total mean normal stress $\sigma_m$ as tabulated below.
The initial shear modulus $G_i$ can also be read directly from Fig. 4-21 for each total mean normal stress $\sigma_m$ as tabulated below.

<table>
<thead>
<tr>
<th>$\sigma_m$ (psi)</th>
<th>$\times (%)$</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>25</th>
<th>30</th>
<th>40</th>
<th>50</th>
<th>60</th>
</tr>
</thead>
<tbody>
<tr>
<td>40</td>
<td>0.084</td>
<td>0.066</td>
<td>0.053</td>
<td>0.044</td>
<td>0.036</td>
<td>0.031</td>
<td>0.024</td>
<td>0.019</td>
<td>0.017</td>
<td></td>
</tr>
<tr>
<td>60</td>
<td>0.077</td>
<td>0.061</td>
<td>0.048</td>
<td>0.039</td>
<td>0.033</td>
<td>0.027</td>
<td>0.021</td>
<td>0.017</td>
<td>0.014</td>
<td></td>
</tr>
<tr>
<td>80</td>
<td>0.069</td>
<td>0.055</td>
<td>0.042</td>
<td>0.034</td>
<td>0.028</td>
<td>0.023</td>
<td>0.017</td>
<td>0.014</td>
<td>0.011</td>
<td></td>
</tr>
</tbody>
</table>

**D.2.2 Unloading-Reloading Modulus $G_{UN}$**

The approximation formula, Eq. (4.21), proposed by Bathe and Wilson et al (1,2), and Nelson et al (29) was also adopted in this research. The NONSAP program automatically calculates $t_{G_{UN}}$ based on the current $t_{K_{LD}}$, $t_{K_{UN}}$ and $t_{G_{LD}}$. 
APPENDIX E

LISTING OF THE MODIFIED SUBROUTINES
SUBROUTINE QUAD5 (NOD, NP, XYZ, PROPE, EDIS, WA, NOD9)

HEXAHEDRAL CURVILINEAR THREE-DIMENSIONAL ELEMENTS

ISOPARAMETRIC OR SUBPARAMETRIC

COMMON /SEL/ INU, NCOUNT, NPAR(20), NUMEC, NEGL, NEUBL, IMASS, IDAMP, ISTAT
1
COMMON /VAR/ INU, KP, MODCA, KSIEP, IFE, IMAX, IREF, IEQREF, INDCMD
1
COMMON /IMAG/ DT, ICLO, IMEL, IEL, NT, ON, NND9
1
COMMON /MIND/ DI, STRESS(6), STRAIN(6), IPT, NEL

DIMENSION B(I), S(I), XYZ(I), PROP(I), RE(I), EDIS(I)
1

DIMENSION B(I), S(I), XYZ(I), PROP(I), RE(I), EDIS(I)
1

EQUIVALENCE (NPAR(15), MODEL), (NPAR(IU), NINT), (NPART(0), NINT)
1

DATA XG, YG, ZG / 0., 0., 0. ;
1

DATA XG, YG, ZG / 0., 0., 0. ;
1

DATA WGT(4,4), XG(14), YG(14), ZG(14), IAU(14)
1

IF (IND.EQ.4) GO TO 100

FIND STIFFNESS OF LINEAR ELEMENT

INTEGRATE H(TRANSPOSED) * B

DU 10 LX=1, NINT

E1 = XG(LX, NINT)

DO 10 LY=1, NINT

E2 = YG(LY, NINT)

DO 10 LZ=1, NINT

W1 = WGT(LX, NINT) * WGT(LY, NINT) * WGT(LZ, NINT)

EVALUATE STRAIN-DISPLACEMENT MATRIX B AND JACOBIAN DETERMINANT

AT THIS INTEGRATION POINT
CALL DERIOD (NEL,XYZ,B,DET,E1,E2,E3,NOD9)

FAC = W**DET
FAC = DSQRRT (FAC)

40 H(I) = FAC * S(I)
KL = 0
UU = 45 I = 1,NU
UU = 45 I = 1,NU
KL = KL + 1
45 S < KL > = S( KL ) + B(I) * B(J)
CONTINUE

MULTIPLY D BY THE INTEGRATED B (TRANSPOSED) * B

CALL SYST3C (NEL,XYZ,PROP,D)

ISOTROPIC CASE

IF (MOD(ELEG,1)) # 1) GOTO 70
KL = 0
D1 = D(1,1)
D2 = D(1,2)
D3 = D(4,4)
UU = 60 Ix = 1,VEL
UU = 60 Ij = 1,IEL
UU = 60 Ix = 1,IEL
UU = 60 J0 = 3*(J-J-1)

KS = KL
IC = 0
IU = 62 I = 0,3
IC = IC + 1
62 KS = KS + 1
63 KS = KS + NU - 10 - I - 3

KS1 = KS
KS2 = KS + ND - 10 - 2
KS3 = KS + 2*NU - 10 - 2
KS4 = KS + ND - 10 - 2
KS5 = KS + 2*NU - 10 - 2
S(KS1) = D1 + (BB(5) + BB(9)) * D3
S(KS1 + 1) = D2 + (BB(1) + BB(9)) * D3
S(KS1 + 2) = D3 + (BB(1) + BB(5)) * D3
IF (KL = 1) IC = 0
S(KS1 + 3) = H(I,J) * D2 + .44(4) * D3
S(KS2) = BB(4) * D2 + BB(3) * D3
S(KS2 + 1) = BB(9) * D2 + BB(8) * D3
S(KS2 + 2) = BB(5) * D2 + BB(6) * D3
GO TO 61

64 S(KS1 + 1) = BB(3) * D2 + D3
64 S(KS1 + 2) = BB(3) * D2 + D3
65 S(KS2 + 1) = BB(9) * D2 + D3

61 KL = KL + 2*(ND-10) - 3
60 KL = KL + 2*(ND-10) - 3
RETURN
ANISOTROPIC CASE

70 CONTINUE
    RETURN

FIND NONLINEAR ELEMENT MATRICES

UPDATE ELEMENT COORDINATES

100 IF (INONL .LE. 2) GO TO 122
    DO 120 J = 1, NO
120  AX(J,J) = XYZ(J) + EDIS(J)

EVALUATE STRESS-STRAIN LAW IF LINEAR MATERIAL MODEL USED IN THIS ELEMENT

122 IF (MODEL .GT. 2) GO TO 125
    IF (INONL .LE. 2) GO TO 124
    CALL SIS13L (INEL, XX, PROP, D)
125 CALL SIS13L (INEL, XYZ, PROP, D)

INTEGRATE STIFFNESS MATRIX AND ELEMENT NODAL FORCE EXPRESSION

125 IPT = 0
130  IF (LX = I, NINT)
    E1 = XG(LX, NINT)
140  IF (LY = I, NINT)
    E2 = XG(LY, NINT)
150  IF (LZ = I, NINT)
    E3 = XG(LZ, NINT)
    WF = WGT(LX, NINT) * WGT(LY, NINT) * WGT(LZ, NINT)
    IPT = IPT + 1

UPDATED LAGRANGIAN FORMULATION (BOTH MATERIAL AND KINEMATIC NONLINEARITIES ACCOUNTED FOR)

CALL DERIV3 (INEL, XX, B, DET, E1, E2, E3, NOU9)

CALCULATE DISPLACEMENT DERIVATIVES

130 DJ = 1, 9
140 UDISU(J) = 0, 0
150 UDISU(J) = UDISU(J) + B(K) * EDIS(K)
    DISU = UDISU(2) + B(1) * EDIS(1)
EVALUATE STRESS-STRAIN LAW AND CURRENT STRESSES
CALL SIST3N (XX, PROP, DISD, IOW, WA)

ADD STRESS CONTRIBUTION TO ELEMENT FORCE VECTOR
FAC = T * DET
DU, Z15 = E*1.0
215 TAU(I) = STRES(S(I)) * FAC
JEK = 2
JEK = -1
KE(I) = KE(I) + H(I) * TAU(I) + B(J) * TAU(4) + B(K) * TAU(5)
KE(J) = KE(J) + B(I) * TAU(2) + B(J) * TAU(4) + B(K) * TAU(6)
KE(K) = KE(K) + B(I) * TAU(3) + B(J) * TAU(5) + B(J) * TAU(6)

ADD_LINEAR_CONTRIBUTION_TO_ELEMENT_STIFFNESS_MATRIX
ISOTROPIC VERSION OF B(TRANSPOSED) * D * B
GO TO 320
232 D01 = D(I+1) * FAC
D02 = D(I+2) * FAC
D03 = D(I+3) * FAC
D04 = D(I+4) * FAC
D05 = D(I+5) * FAC
D06 = D(I+6) * FAC
KL = KL + 1
S(KL) = S(KL) + B(I+1) * DB1 + B(I+1) * DB4 + B(I+2) * DB5
S(KL) = S(KL) + B(I+1) * DB2 + B(I) * DB4
S(KL) = S(KL) + B(I+1) * DB3 + B(I) * DB5
KL = KL + 1
250 KL = KL + 2 * (ND - J) - 1

C
KL = ND + 1
DU, Z24 = 2,4, ND, 3
D01 = D(I+2) * FAC
D02 = D(I+3) * FAC
D03 = D(I+4) * FAC
D04 = D(I+5) * FAC
D05 = D(I+6) * FAC
S(KL) = S(KL) + n(J) * DB1 + B(J-1) * DB4 + B(J+1) * DB6
KL = KL + 1
S(KL) = S(KL) + B(J+1)*DB3 + B(J)*DB6
KL = KL + 1
JM = JM + 2

253 DO 252 I = JM, ND + 1
S(KL) = S(KL) + B(I+1)*DB1 + B(I+1)*DB4
KL = KL + 1
S(KL) = S(KL) + B(I+1)*DB2 + B(I)*DB4 + B(I+2)*DB6
KL = KL + 1

254 KL = KL + 1
KL = KL + 2*(J-M-J) + 1
C
KL = 2*ND
D1 = U12*B(J)
D2 = U12*B(J)
D3 = U11*B(J)
D5 = U14*B(J-2)
D6 = U14*B(J-1)
S(KL) = S(KL) + B(J)*DB3 + B(J-2)*DB5 + B(J-1)*DB6
KL = KL + 1
JM = J + 1

257 IF (ND-JM) 258, 257, 257
S(KL) = S(KL) + B(I)*DB1 + B(I+2)*DB5
KL = KL + 1
S(KL) = S(KL) + B(I+1)*DB2 + B(I+2)*DB5
KL = KL + 1
S(KL) = S(KL) + B(I+2)*DB3 + B(I)*DB5 + B(I+1)*DB6
KL = KL + 1

256 KL = KL + 2*(ND-J) - 1

C
ADD NONLINEAR CONTRIBUTION TO STIFFNESS MATRIX
C
DIA(1,1) = 3(1)
DIA(2,1) = 3(2)
DIA(3,1) = 3(3)
DIA(4,1) = 3(4)
DIA(5,1) = 3(5)
DIA(6,1) = 3(6)
DIA(7,1) = 3(7)
DIA(3,2) = 3(2)
DIA(4,2) = 3(3)
DIA(5,2) = 3(4)
DIA(6,2) = 3(5)
DIA(7,2) = 3(6)
DIA(1,3) = 3(1)
DIA(2,3) = 3(2)
DIA(3,3) = 3(3)
DIA(4,3) = 3(4)
DIA(5,3) = 3(5)
DIA(6,3) = 3(6)
DIA(7,3) = 3(7)
DIA(3,4) = 3(2)
DIA(4,4) = 3(3)
DIA(5,4) = 3(4)
DIA(6,4) = 3(5)
DIA(7,4) = 3(6)
DIA(1,5) = 3(1)
DIA(2,5) = 3(2)
DIA(3,5) = 3(3)
DIA(4,5) = 3(4)
DIA(5,5) = 3(5)
DIA(6,5) = 3(6)
DIA(7,5) = 3(7)
DIA(3,6) = 3(2)
DIA(4,6) = 3(3)
DIA(5,6) = 3(4)
DIA(6,6) = 3(5)
DIA(7,6) = 3(6)
DIA(1,7) = 3(1)
DIA(2,7) = 3(2)
DIA(3,7) = 3(3)
DIA(4,7) = 3(4)
DIA(5,7) = 3(5)
DIA(6,7) = 3(6)
DIA(7,7) = 3(7)
SUBROUTINE STSS?N (XX,PROP,DISD,IDW,WA)
IMPLICIT REAL*8(A-H,O-Z)

SUBROUTINE TO FIND STRESSES FOR ALL MATERIAL MODELS AND STRESS-STRAIN LAW FOR NONLINEAR MATERIAL MODELS

COMMON /EL/ IND,ICOUNT, NPAR(20),NUMEG,NEG1,NEGNL, IMASS,IDAMP,ISTAT
COMMON /MTMP3P/ DT(6),STRESS(4),STRAIN(6),IPT, Nel

DIMENSION DISO(1), DMTMP3P(2),STRESS(4),STRAIN(6),IPT, Nel

EQUIVALENCE (NPAR(3),INDNL), (NPAR(15),MOEL)

DEFINITION OF STRAIN

LINEAR STRAIN TERMS

STRAIN(1) = DISO(1)
STRAIN(2) = DISO(2)
STRAIN(3) = DISO(3)
STRAIN(4) = DISO(4) + DISO(6)
STRAIN(5) = DISO(5) + DISO(8)
STRAIN(6) = DISO(7) + DISO(9)

IF (INDNL.EQ.1) GO TO 80

NONLINEAR STRAIN TERMS

DN(1) = 0.5*(DISO(1)*DISO(1) + DISO(6)*DISO(6) + DISO(8)*DISO(8))
DN(2) = 0.5*(DISO(1)*DISO(4) + DISO(4)*DISO(4) + DISO(7)*DISO(7) + DISO(9)*DISO(9))
DN(3) = 0.5*(DISO(5)*DISO(5) + DISO(8)*DISO(8) + DISO(3)*DISO(3))
DN(4) = 0.5*(DISO(1)*DISO(1) + DISO(4)*DISO(4) + DISO(7)*DISO(7) + DISO(9)*DISO(9))
DN(5) = 0.5*(DISO(5)*DISO(5) + DISO(8)*DISO(8) + DISO(3)*DISO(3))
DN(6) = 0.5*(DISO(5)*DISO(5) + DISO(8)*DISO(8) + DISO(3)*DISO(3))

IF (INDNL.EQ.3) GO TO 60

CALCULATE GREEN-LAGRANGE STRAINS (TOTAL LAGRANGIAN FORMULATION)

DO 20 I = 1,6
20 STRAIN(I) = STRAIN(I) + DN(I)
GO TO 80

CALCULATE ALMANSI STRAINS (UPDATED LAGRANGIAN FORMULATION)

DO 40 I = 1,6
40 STRAIN(I) = STRAIN(I) - DN(I)
CALCULATION OF STRESS-STRAIN MATRIX AND STRESSES

80 GO TO (1, 2, 3, 4, 5, 6), MODEL

***** MODEL = 1 LINEAR ISOTROPIC
1 DO 100 I=1,3
   STRESS(I)=D(I,1)*STRAIN(1)+D(I,2)*STRAIN(2)+D(I,3)*STRAIN(3)
100 STRESS(I+3)=D(I+3,4)*STRAIN(I+3)
   RETURN

***** MODEL = 2 LINEAR ORTHOTROPIC
2 RETURN

***** MODEL = 3 NONLINEAR CURVE DESCRIPTION MODEL
3 CALL ELT3D3
   RETURN

***** MODEL = 4 USER SUPPLIED MODELS
4 CALL ELT3D4
   RETURN
5 CALL ELT3D5
   RETURN
6 CALL ELT3D6
   RETURN
END
SUBROUTINE CMOD30 (NEL, EKK, KLD, KUN, GLD, SIG, EPS, EVMAX, 
STRESS, STRAIN, C, IPT, EU, SD, B) 
IMPLICIT REAL*8(A-H, O-Z) 

EKK = STRAIN ABSCISSAE 
KLD = LOADING BULKS MODULUS 
KUN = UNLOADING BULKS MODULUS 
GLD = LOADING SHEAR MODULUS 
SIG = STRESSES FROM THE PREVIOUS TIME STEP 
EPS = STRAINS FROM THE PREVIOUS TIME STEP 
EVMAX = MAXIMUM VOLUMETRIC STRAIN EVER REACHED (COMP *) 
STRESS = CURRENT STRESSES 
STRAIN = CURRENT STRAINS 
C = CURRENT ELASTICITY MODULUS MATRIX 
IPT = INTEGRATION POINT INDICATOR 

COMMON /EL/ INDICUANT, NPAR(20), NUMEG, NEUL, NEGNL, IMASS, IDAMP, ISTAT 
COMMON /VAR/ NG, KPRI, KMUDX, KSTEP, ITE, ITMAX, IREF, IEQREF, INOCMD 
DIMENSION EKK(1), GLD(1), SIG(1), EPS(1), STRESS(1), STRAIN(1), C(6,1) 
REAL KLD(1), KUN(1), K,KUNLO 
REAL*8 KLD(1), KUN(1), K,KUNLO 
EQUIVALENCE (NPAR(10), NINT), (NPAR(11), NINTZ), (NPAR(17), NCON) 

NPT=NINT*NINTZ*NINTZ 
IPOINT=NCUN/4 

1. CALCULATE STRESS AND STRAIN DEVIATORS 
OF THE PREVIOUS TIME STEP 

IMM=(SIG(1)+SIG(2)+SIG(3))/3. 
T11=SIG(1) - IMM 
T22=SIG(2) - IMM 
T33=SIG(3) - IMM 
T12=SIG(4) 
T13=SIG(5) 
T23=SIG(6) 

USER SUPPLIED STATEMENT 
SD=USQRT(T11**2+T22**2+T33**2) 

EVV=(EPS(1)+EPS(2)+EPS(3)) 
EMM=EVV/3. 
E11=EPS(1) - EMM
E22 = EPS(2) - EMM
E33 = EPS(3) - EMM
E12 = EPS(4)
E23 = EPS(5)

2. CALCULATE PARAMETERS BASED ON STRESSES AND STRAINS OF THE PREVIOUS TIME STEP

USER SUPPLIED STATEMENT
ED = DSORT (E11**2 + E22**2 + E33**2)

28 I = J - 1

DELEV = EKK(J) - EKK(I)
DELE1 = EVTOT - EKK(I)
RATIO = DELE1 / DELEV

IF (IKAS) 30,35,35
30 KUNL0 = KUN(I) + RATIO * (KUN(J) - KUN(I))
35 K = KUNL0 + RATIO * (KLD(J) - KLD(I))

MODIFY G FOR EFFECTS OF DEVIATORIC STRESS AND STRAIN
G = G * (I*1 - I**5) * 2
B IS FUNCTION OF EVTOT. USER SHOULD PROVIDE THE FUNCTION ACCORDING LABORATORY TESTING RESULTS.

3. USING PARAMETERS CALCULATED IN (2.), FIND CURRENT STRESSES BASED ON GIVEN STRAINS

EVI = -(STRAIN(1) + STRAIN(2) + STRAIN(3))
EPSMM = EVI/3.
EPS11 = STRAIN(1) - EPSMM
EPS22 = STRAIN(2) - EPSMM
EPS33 = STRAIN(3) - EPSMM
EPS12 = STRAIN(4)
EPS13 = STRAIN(5)
EPS23 = STRAIN(6)

C PEE = 3. * K * (EPSMM-EMM) + TMM
C
S11 = 2. * G * (EPS11 - E11) + T11
S22 = 2. * G * (EPS22 - E22) + T22
S33 = 2. * G * (EPS33 - E33) + T33
S12 = G * (EPS12 - E12) + T12
S13 = G * (EPS13 - E13) + T13
S23 = G * (EPS23 - E23) + T23

C STRESS(1) = S11 + PEE
STRESS(2) = S22 + PEE
STRESS(3) = S33 + PEE
STRESS(4) = S12
STRESS(5) = S13
STRESS(6) = S23

C PRINT STRESSES

IF (KPRI.NE.0) GO TO 50
IF (IPT.NE.1) GO TO 55
WRITE (6,2030)
WRITE (6,2035) NEL
55 WRITE (6,2040) IPT, (STRESS(I), I=1,6)

C 50 IF (ITE.GE.1 .OR. KPRI.EQ.0) RETURN

C 4. UPDATE STRESSES AND STRAINS

SIG(1) = STRESS(1)
SIG(2) = STRESS(2)
SIG(3) = STRESS(3)
SIG(4) = STRESS(4)
SIG(5) = STRESS(5)
SIG(6) = STRESS(6)

C EPS(1) = STRAIN(1)
EPS(2) = STRAIN(2)
EPS(3) = STRAIN(3)
EPS(4) = STRAIN(4)
EPS(5) = STRAIN(5)
EPS(6) = STRAIN(6)

C IF (EVI.GT.EVMAX) EVMAX = EVI
C
C IF (IPT.EQ.NPT) RETURN

C 5. FORM THE STRESS-STRAIN RELATIONSHIP
IF (EV1.LT.EVMAX) GO TO 75

LOADING

EVTOT=EVMAX
IK=J
IKAS=1
ITEM=2
GO TO 10

75 IF (IKAS) 100, 85, 85

WAS LOADING, BUT NOW UNLOADING.

85 KUNLO=KUN(I) + RATIO * ( KUN(J) - KUN(I) )

G = G * (KUNLO/K)

WAS UNLOADING, AND NOW UNLOADING

100 CONTINUE

DUM=0.6666667 * G

A1 = K + 2. * DUM

B1 = K - DUM

DO 110 I = 1, 6

DO 110 J = 1, 6

110 C(I, J) = 0.0

C(I, 1) = A1

C(I, 2) = B1

C(I, 3) = B1

C(1, 2) = A1

C(1, 3) = B1

C(2, 3) = A1

C(1, 2) = B1

C(1, 3) = B1

C(2, 3) = B1

DO 120 I = 4, 6

120 C(I, I) = G

RETURN

END

2015 FORMAT (///36H ***ERRX: CURRENT VOLUMETRIC STRAIN, E14.6, 2X, 1

2030 FORMAT (/)

2035 FORMAT (13X, I5, 6F15.4)