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INTERFERENCE EFFECTS IN LOW ENERGY RADIATIVE CAPTURE REACTIONS IN LIGHT NUCLEI

The Ohio State University

PH.D. 1983

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INTERFERENCE EFFECTS IN LOW ENERGY RADIATIVE CAPTURE REACTIONS IN LIGHT NUCLEI

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

By

James Charles Brown, B.A., M.S.

* * * * *

The Ohio State University

1983

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ACKNOWLEDGEMENTS

The author wishes to express his appreciation to Professor Richard G. Seyler for his encouragement, guidance and continued interest throughout the course of this investigation. The author would also like to thank Professor S. Leslie Blatt for greatly assisting in the initiation of this investigation.
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I. INTRODUCTION

There is a particular class of nuclear reactions in which the incident projectile is absorbed by the target nucleus followed by the emission of gamma radiation. These reactions are known as radiative capture reactions and are important for two main reasons. First, the radiative capture reaction is one of the most important processes for the formation of various species of elements in the universe\textsuperscript{63}. Second, it is an important class of nuclear reactions because of its great value in facilitating the determination of the properties of nuclear states\textsuperscript{28}.

The study of nuclear transmutation by electromagnetic processes is an especially important tool for probing nuclear structure, because this interaction is relatively simple and has well established properties, i.e., the description of the electromagnetic interaction based on Maxwell's equations for the propagation of the electromagnetic field and its coupling to the charge-current density has been found to have a range of validity extending to all the atomic and nuclear systems so far studied\textsuperscript{66}. The study of nuclear gamma-rays can, in principle, reveal the energies, spins, parities, and reduced matrix elements.
$<F|H(L)|I>$ for the initial (I) and final (F) states of a nucleus interacting via the electromagnetic operator $H(L)$. The first three are occasionally called the gross properties of a level and they reveal only the main outlines of nuclear structure. A determination by partial width measurement (or direct measurement of the lifetime) of a gamma ray transition matrix element provides a more exacting test of a model. A feature of the electromagnetic operators that in certain ways increases the sensitivity to nuclear structure is the operation of numerous selection rules. In addition to the requirements on angular momentum and parity, selection rules for the various multipole operators with regard to isospin, particle configurations and collective wavefunctions also come into play.

If a transition has only one multipole $L$, knowledge of its lifetime enables one to determine the absolute value of the matrix element $<F|H(L)|I>$. If it is a mixed transition, i.e., different multipoles contributing to a gamma transition between well defined states, then observation of the interference structure in angular correlation experiments provides information about the magnitude as well as the relative sign of the ratios of the reduced matrix elements (mixing ratios). Other quantities, such as internal conversion coefficients, combine with the lifetime to give only the absolute values of the matrix elements. If the static electromagnetic moments (diagonal matrix elements)
are known in addition to several transition (or off diagonal) matrix elements of different $L$ connecting the same state, a sensitive test of a nuclear structure model is possible.

Every radiative capture reaction must be considered in the problem of stellar evolution at various stages, if only to be discarded on the basis of insufficient number densities of the reacting particles and/or a temperature not right for a particular reaction to compete with other reactions. The stellar evolution problem, encompassing energy generation and nucleosynthesis in stars, for a given initial isotopic composition and total mass is complicated by poorly understood dynamic mechanisms such as the degree of convective mixing between shells of stellar matter, non-thermal particle acceleration processes, energy loss via neutrino emission, etc. However, the early stages of stellar evolution (hydrogen burning) are reasonably well understood and scenarios for the building of solar system isotopic abundances have been developed.

Element synthesis may begin with pure hydrogen gravitationally condensed to stars and undergoing the proton-proton chain of reactions by which hydrogen is converted into helium. In the pp chain, after $^3$He has been produced via the very slow sequence of reactions

$$^1\text{H}(p,\gamma)^2\text{D}(p,\gamma)^3\text{He},$$
the chain is normally terminated by converting the \(^3\)He into \(^4\)He through one of the two \(^3\)He burning reactions

\[ ^3\text{He}(^3\text{He},2p)^4\text{He}, \]

or

\[ ^3\text{He}(^4\text{He},\gamma)^7\text{Be}. \]

The latter reaction may branch into one of the following

\[ ^7\text{Be}(e^-,\gamma)^7\text{Li}(p,^4\text{He})^4\text{He}, \]

or

\[ ^7\text{Be}(p,\gamma)^8\text{B}(e^+\gamma)^9\text{Be}(\gamma)^7\text{Be}(*^4\text{He})^4\text{He}. \]

The terminations via the \(^3\)He(\(^4\)He, \(\gamma\))\(^7\)Be reaction produce one \(^4\)He for every \(^1\)H(p,e\(^+\gamma)^2\)D reaction, compared to the yield of only one \(^4\)He for every two \(^1\)H(p,e\(^+\gamma)^2\)D reactions through the \(^3\)He(\(^3\)He,2p)\(^4\)He termination. Therefore since the rate of the pp chain is determined by the positron decay of an unbound state of two protons, completion of the chain via the \(^3\)He(\(^4\)He, \(\gamma\))\(^7\)Be reaction instead of the \(^3\)He(\(^3\)He,2p)\(^4\)He reaction doubles the rate of \(^4\)He production and nearly doubles the rate of energy generation\(^2\). (In the hydrogen burning phase of stellar evolution the energy carried away by neutrinos is only a small part of that carried by electromagnetic photons; the diffusion time for radiation is, in contrast to neutrinos, very long and thus
the electromagnetic energy is nearly confined to the stellar volume.) The difference in the efficiencies of these two terminations to the pp chain makes a knowledge of the relative importance of these reactions essential to the study and generation of models of stellar interiors for stars operating on the proton-proton chain. Further, it has been suggested that the $^7\text{Be}(p,\gamma)^7\text{B}$ reaction prevails over the $^7\text{Be}(e^-,\nu)^7\text{Li}$ reaction only for $T > 20 \times 10^6$ K. However, above about $20 \times 10^6$ K, the carbon cycle will dominate over the pp chain in all stars other than those with no carbon, oxygen, or nitrogen, so that this method of termination of the pp chain can be neglected for such stars.

More complicated reactions arise in connection with the burning of hydrogen in the presence of elements produced in reaction sequences subsequent to the pp chain. When $^{12}\text{C}$ produced in helium burning [$^4\text{He}(^4\text{He})^8\text{Be}(^6\text{He},\gamma)^{12}\text{C}$] is mixed with hydrogen at high enough temperatures (e.g., in stars on the upper part of the main sequence in the Hertzprung-Russell diagram), the hydrogen is converted to helium via the carbon (CNO) cycle in addition to the pp chain. In this sequence of reactions, $^{12}\text{C}$ acts as a catalyst in the conversion of hydrogen to helium. The reactions are

$$^{12}\text{C}(p,\gamma)^{13}\text{N}(e^+,\nu)^{13}\text{C}(p,\gamma)^{14}\text{N}(p,\gamma)^{15}\text{O}(e^+,\nu)^{15}\text{N}(p,^3\text{He})^{12}\text{C}.$$  

The loss of the $^{12}\text{C}$ catalyst through the reaction
$^{12}\text{N}(p,\gamma)^{13}\text{O}$ is partially replaced by the sequence of reactions

$^{16}\text{O}(p,\gamma)^{17}\text{F}(e^-\gamma)^{17}\text{O}(p,\alpha)^{14}\text{N}.$

Further, it has been suggested that the reaction $^{17}\text{O}(p,\gamma)^{18}\text{F}$ is the start of a third branch within the carbon cycle acting to replace the loss of $^{12}\text{C}$ catalyst through the sequence

$^{17}\text{O}(p,\gamma)^{18}\text{F}(e^-\gamma)^{18}\text{O}(p,\alpha)^{14}\text{N},$

and thus reentering the main carbon cycle. It is thought that the addition of a third cycle to the carbon chain will not change significantly the energy release; rather the effect is to change the $^{17}\text{O}$ and $^{18}\text{O}$ abundances - the $^{17}\text{O}$ isotope survives hydrogen burning in greater abundances and thus can serve as a source of neutrons in helium burning via $^{17}\text{O}(\alpha,\gamma)^{20}\text{Ne}.$

At high stellar temperatures occurring just before the hydrogen is exhausted, additional hydrogen burning cycles involving heavier elements become more probable due to reduced penetrabilities. One of these cycles is the Ne-Na sequence of reactions

$^{20}\text{Ne}(p,\gamma)^{21}\text{Na}(e^-\gamma)^{21}\text{Ne}(p,\gamma)^{22}\text{Na}(e^-\gamma)^{22}\text{Ne}(p,\gamma)^{23}\text{Na}(p,\alpha)^{20}\text{Ne}.$

The Ne-Na cycle is not important as an additional energy source is stars (in comparison to the carbon cycle and/or
pp chain) but rather as a process for generating $^{21}$Ne. $^{21}$Ne owes its importance to the exoergic reaction $^{21}$Ne($^4$He,n)$^{24}$Mg which has been proposed as the main source of neutrons for building heavy elements through neutron capture processes. Many of the radiative capture cross sections of the nuclear processes involved in hydrostatic stellar burning are very small at the relevant stellar energies; they cannot be measured directly in the laboratory using any presently known techniques. In the usual procedure, the nuclear reactions are studied instead in experimentally accessible regions and then extrapolated down to thermal energies for use in astrophysical calculations. The extrapolative procedure is usually guided by theoretical considerations. For the radiative capture reactions, these considerations invariably include the extranuclear direct capture mechanism proposed by E.G. Thomas in 1951 to account for an abnormally high thermal neutron cross section in the reaction $^7$Li(n,$\gamma$)$^{7}$Li. The importance of the extranuclear contributions to capture gamma reactions was recognized much earlier. The beginning of theoretical investigations into the general problem of reaction mechanisms, including the radiative capture mechanisms, predated most of the rudimentary astrophysical cross section measurements (e.g., Refs. 13, 27, 53, 58, 74, 75, 76, 77).
Many of the astrophysically relevant capture cross section measurements were made some years ago. Only recently was there an experimental interest in combining the direct capture mechanism with the compound nucleus mechanism for the production of gamma rays to account for interference structure in gamma excitation curves\(^1\). The type of interference (whether constructive or destructive) is of some relevance for the astrophysical problem due to low lying resonances - just above or just below particle emission threshold - which can influence reaction rates. Examples of such complications are encountered in the reactions: \(^{12}\text{C}(p,\gamma)^{13}\text{N}\) (Ref. 22); \(^{15}\text{N}(p,\gamma)^{16}\text{O}\) (Ref. 73); \(^{12}\text{C}(^4\text{He},\gamma)^{16}\text{O}\) (Ref. s 79, 79, 80); \(^{20}\text{Ne}(p,\gamma)^{21}\text{Na}\) (Ref. s 52, 81); \(^{17}\text{O}(p,\gamma)^{18}\text{F}\) (Ref. s 82, 83); \(^{18}\text{O}(p,\gamma)^{19}\text{F}\) (Ref. 84). This particular problem motivated, in part, an experimental investigation of low energy radiative capture analysing powers at The Ohio State University in the mid seventies. In particular, an experimental study of the \(^{12}\text{C}(p,\gamma)^{13}\text{N}\) reaction analysing power was undertaken\(^4\) to determine whether the currently accepted view\(^2\) of the direct capture mechanism could account for interference terms that make up an analysing power. A large analysing power was found in the vicinity of the second excited state of \(^{13}\text{N}\) - much larger than could be accounted for on the basis of compound nucleus formation alone.
The direct capture process considered in the present work represents a transition from an initial continuum state to a final bound state via interaction with the electromagnetic field. These reactions are non-resonant in the sense that they do not follow the formation of a compound state. All else being equal, the probability of direct capture with concurrent emission of a gamma photon will depend on the degree to which the actual final state resembles a bound state formed of the projectile in the field of the target nucleus with no residual interactions. The two body bound state so formed is not intended to be an eigenstate of the nuclear Hamiltonian.

It is convenient in considering the direct capture process in combination with processes leading to compound nucleus formation to separate the spatial region of interaction into external and internal parts. The external region is defined by a radius outside of which the nuclear potential is negligible. In this region, the direct radiative capture process is essentially without competition. In the internal region, however, the direct radiative capture process is in competition with direct elastic scattering (also referred to as shape-elastic scattering), direct reactions and compound reactions (of which compound elastic scattering is an example). The internal direct capture process is described phenomenologically by the complex optical potential model. According to this model, an inci-
dent particle moves an appreciable distance through the target nucleus before being absorbed out of the entrance channel (or single particle state) through collisions. During the initial period of motion in the mean nuclear field, the particle may radiate energy in the form of a photon and drop into a bound particle orbit. In contrast to the compound state radiative capture process which is delayed or multi-step, the direct capture process is immediate or single-step.

No distinction is made between intermediate states and compound nucleus states. Rather, in the intermediate interaction model which incorporates internal direct capture processes, the entrance channel is coupled directly to the proper states of R-matrix theory. If the compound state (leading to a compound nucleus with few degrees of freedom excited) is formed immediately at low bombarding energies in light nuclei, i.e., the mean free path of the incident nucleon is small in comparison to nuclear dimensions, then the direct internal radiative capture contribution is negligible. Whether the projectile and target persist in a single particle configuration to the extent that a direct internal radiative contribution is necessary to account for the capture reactions is a question examined in the present work. The capture reactions considered are characterized by resonance bumps and background as determined by the nuclear system under scrutiny, i.e., the fine
structure as would be obtained with good energy resolution is considered.

A detailed accounting of the interference structure in the resonance region may require the inclusion of the direct capture contributions to the radiative capture reactions. The broad term interference structure is meant to include analysing powers, odd order differential cross section Legendre coefficients, etc. The purpose of the present work is to incorporate a direct reaction contribution into the currently accepted picture of low energy capture reactions in the resonance region. It will be shown that the single-step reaction mechanism model emerges in a natural way from a rigorous theory of nuclear reactions. In the approach presented, the compound nucleus and direct mechanisms arise in the capture reaction problem on an equal footing; as having the same origin. Consequently, no phenomenology is required in the formal description of these reactions. The direct capture radiative transitions are clearly separated into those that occur inside or outside the target nucleus. The extent to which any internal radiative transitions (either direct or compound state processes) may be inferred to occur, by way of comparison with capture data, is examined.

In Chapter II, expressions are developed relating the general observables in a radiative capture reaction to a set of well defined collision matrix elements. The derivati-
tion is from first principles whenever it is necessary to clearly delimit the range of applicability of the derived results and to establish a consistent and obvious set of phase choices. In electing to use the Lane and Thomas collision matrix, the normalization of the asymptotic outgoing particle (radiation field) is determined. An arbitrary phase is selected for convenience. The normalization and phase of the outgoing photon then determines the form of the photon creation operator. A photon reaction amplitude is then easily deduced. The interesting question of what approximation to a nuclear wavefunction, within the R-matrix framework, is appropriate to the problem at hand is addressed at this point.

In Chapter III, a thorough examination of the derived formalism is given in terms of applications to radiative capture data in the form of excitation functions and angular distributions of total and differential cross sections and analysing powers. Detailed formulae are presented for the observables considered. In Chapter IV, conclusions are presented.
II. THEORY OF REACTIONS INVOLVING PHOTONS

The qualitative role played by photons in nuclear reactions is in many respects similar to that of non-zero rest mass particles. This is made plausible by the Bohr picture in which the compound nucleus decays by competition through the various channels in a manner independent of the mode of formation. In this picture, photon emission is just one particular competing mode of decay so it is expected to damp the formation of the compound state just like any other mode. The treatment of photon processes is however different from the treatment of massive particles.

Processes involving the formation of photons may be described by an electromagnetic interaction which under certain conditions is completely specified, i.e., independent of the nucleon hamiltonian (Siegert conditions). More important, however, is the fact that, since the coupling of the nucleons to the electromagnetic field is relatively small, a perturbative approach to these electromagnetic phenomena may be followed with some confidence. In particular, since the coupling is small, the probability of two photons being present simultaneously is much less than that for one photon and so the one photon restriction is
imposed. This does not constitute a significant limitation in the derived results—it does, however, preclude analysis of certain anomalous cases such as, for example, spin zero to spin zero transitions, which require more than one photon. Also, concatenated processes such as Coulomb excitation followed by capture to the excited state and subsequent emission of a single photon will not be considered on similar grounds.

A. PHOTON EIGENSTATES

Before discussing the collision matrix for gamma ray production, a suitable representation for a photon wavefunction must be found. The criterion for suitability is determined by the context within which the representation will be used. In the density matrix formalism of Ref. (1), for example, a suitable representation consists of specifying the photon radiation parameters. The concern in this approach centers about determining the unitary transformation elements that take an ordinary vector plane wave into a representation in which a spin projection of zero does not exist. To resolve phase ambiguity, it is assumed that the photon wavefunction is the vector potential. There is more than adequate theoretical justification for this assumption; but a more pedestrian development (the inverse of that found in Ref. (2)) makes the identification of the vector potential as the photon wavefunction more compel-
ling, and comes up with the radiation parameters rather incidentally. This approach will be followed only as a matter of taste—both developments being equivalent.

The general solution to the Schrodinger equation may be expressed in the asymptotic region in terms of coefficients of the incoming waves (in an \( \mu \ell sJm \) scheme):

\[
\psi^{Jm} = \psi'^{Jm} + \sum_{c' \ell} \left( e^{i \omega_c s_c} \delta_{c, c'} - \omega_{c}\right) \mathcal{O}_{c}^{Jm} \gamma_{c}^{Jm}. \tag{2.1}
\]

In order that \( \psi^{Jm} \) represent an incoming plane wave in one channel, \( \gamma_{c}^{Jm} \) must be of a specific form. It is the outgoing wave that is of current interest, however, in this standard expression. The outgoing wave is generally specified by two spin quantum numbers (and projections), relative orbital angular momentum quantum number (and projection), and coordinates of relative motion. Unless indicated otherwise, the single axis on which projections are made is the axis defined by the momentum vector of the incident particle. As a matter of choice in the asymptotic region, the spin of one of the particles may be coupled to the relative orbital angular momentum to form the quantum number \( L \); and this quantum number coupled explicitly with the residual spin quantum number \( B \) to form total angular momentum \( J \). Thus,

\[
\mathcal{O}_{L \ell B}^{Jm} = \sum_{\lambda, m_{\lambda}} \chi_{\lambda \ell B}^{Jm} \mathcal{O}_{L \lambda}^{P} \chi_{m_{\lambda} B}^{m_{\lambda}}. \tag{2.2}
\]
It is useful to classify the states $O^\pm_{\lambda}$ according to their parity $p$. The order of coupling, $L+B=J$, represents a definite choice of phase. Conventions established in Ref. (4) will be followed. The function $O^\pm_{\lambda}$ is an outgoing wave composed of an orbital and spin part $(\lambda, J)$ and is subject to the restrictions of being transverse and an eigenfunction of the parity operator (eigenvalue $(-)^p$) to reflect the properties of the radiation field. In order to form this function, it is worthwhile to define first:

$$\phi_{\lambda L}^{(\ell m)} = \sum_{m', q} \frac{i^L Y_{\ell m'}^m}{\sqrt{q}} O_{\mathbf{q}} \chi_{m', q J} \phi_0.$$  \hspace{1cm} (2.3)

This construction is identical in form to those used in Ref. (3) for particle channels. The factor $i^L$ is a normalization constant to be determined. It is convenient to work with spherical harmonic functions with the additional factor $i^L$ because they satisfy a time reversal condition of the type

$$K + J m = (-)^{-m} \chi_{J - m},$$ \hspace{1cm} (2.4)

and because the $i^L$ factors which appear in plane wave expansions can be absorbed, thus simplifying the expressions. The $Y_{\ell m}^{m'}$'s are the Condon and Shortley normalized spherical harmonics. The function $O_{\mathbf{q}}(\mathbf{r})$ is a solution to the radial Schrödinger equation for positive energy chan-
nels corresponding to asymptotically outgoing waves. These are proportional to the spherical Hankel functions of the first kind:

\[ O_{\ell \ell}(r) = \rho h^{(+)}_{\ell \ell} \quad \rho = k r, \quad (2.5) \]

with asymptotic behavior:

\[ h^{(+)}_{\ell \ell} \to (-i)^{\ell+1} \frac{e^{ikr}}{r}. \quad (2.6) \]

The order of coupling, i.e., \( \ell + 1 = L \), follows Ref. (4). The spin functions \( \hat{e}_b \) are unit vectors in a spherical basis; these are the angular momentum eigenfunctions in the space of basis vectors and correspond to spin one. If the vector spherical harmonics of Ref. (6) are used, i.e.,

\[ \bar{\gamma}^{\ell \ell_1}_m = \sum_{m_1} \bar{\gamma}^{\ell m_1}_m \hat{e}_b, \quad (2.7) \]

the functions \( \bar{\phi}_{\ell \ell_1} \) may be written as

\[ \bar{\phi}^{\ell \ell_1}(\omega) = (-i)^{\ell+1} \frac{h^{(-)}_{\ell \ell_1}}{\sqrt{2\ell + 1}} h^{(+)}_{\ell \ell} \bar{\gamma}^{\ell \ell_1}. \quad (2.8) \]

The vector spherical harmonics are the angular momentum eigenfunctions in the compound space \( (\ell + 1) \) and are irreducible tensors of rank \( L \) and parity \((-)^L\). There are three linearly independent irreducible tensors of rank \( L \):
$\tilde{\phi}_{\ell m}$ and $\tilde{\phi}_{\ell m}^\wedge$. Since these tensors span a three-dimensional space, for given $L, \Lambda$, they form a complete set.

A sufficient condition in order that the radiation field be transverse is that the divergence of the field be zero: the divergence operator acting on linear combinations of the outgoing wave basis vectors $\tilde{\phi}_{\ell m}^\wedge$ is imposed to satisfy

$$\nabla \cdot \sum_{\ell} c_\ell \tilde{\phi}_{\ell m}^\wedge = 0 \quad (2.9)$$

One form of the divergence operator is

$$\nabla \cdot (f_\ell \tilde{\gamma}_{\ell m}^\wedge) = \sum_{\nu, \mu} c_{\nu, \mu} \tilde{\phi}_{\nu, \mu} \cdot \tilde{\phi}_{\mu} \quad (2.10)$$

Using the gradient formula of Ref. (6), this results in,

$$\nabla \cdot (f_\ell \tilde{\gamma}_{\ell m}^\wedge) = \sum \left\{ \frac{df_\ell}{d^2} + \frac{L+1}{r} f_\ell \right\} + \sqrt{\frac{L}{2\ell+1}} \delta_{L-1, \ell} \tilde{\gamma}_{\ell m}^\wedge \quad (2.11)$$

where

$$f_\ell(r) = (i)^{L+1} \frac{\hbar}{\sqrt{2r}} \cdot \ell^{(+)}. \quad (2.12)$$
Recursion relations for spherical Bessel functions, i.e.,
\[
\frac{d^{(n+1)} j_n(z)}{dz^{n+1}} = j_{n-1}(z) + j_{n+1}(z),
\]
(2.13)
and
\[
\frac{d^2 j_0}{dz^2} = \frac{1}{z^2} \left[ z^2 j_0(z) - j_{1}(z) \right],
\]
(2.14)
may be used to reduce Eq. (2.11) to the form
\[
\nabla \cdot (f \phi^L \phi^L) = \left[ -\sqrt{\frac{n+1}{2n+1}} \, \frac{d}{dz} - \frac{n}{z} \right] (\delta^{L+2} \frac{d}{dz} - \frac{n}{z}) h^{(4)}_L \phi^L.
\]
(2.15)
Application of this formula to the (-)\(L\) parity case, i.e.,
\[
\bar{O}^{(-)}_{L} = c_L \phi^L\phi^L \),
\]
(2.16)
shows that this function is automatically transverse. Coefficient normalization imposes
\[
c_L = e^{i \delta_m}.
\]
(2.17)
For the parity (-)\(L+1\):
\[
\bar{O}^{(-)}_{L+1} = c_{L+1} \phi^{L+1} + c_{L-1} \phi^{L-1},
\]
(2.18)
and
\[ \nabla \cdot \vec{\phi}_{\ell \hbar} = 0 = \left\{ c_{\ell+1} \sqrt{\frac{l+1}{2\lambda l}} - c_{\ell-1} \sqrt{\frac{\ell - 1}{2\lambda l}} \right\}. \tag{2.19} \]

This condition requires that:
\[ \frac{c_{\ell-1}}{c_{\ell+1}} = + \left\{ \frac{L+1}{L} \right\}^{\frac{1}{2}}. \tag{2.20} \]

This is the solenoidal gauge constant of Ref. (5). It differs in sign from that reference only because of the factor \((i)^{\frac{L}{2}}\) being present in these expressions. Normalization applied to the \(c\)'s gives:
\[ \vec{\phi}_{\ell \hbar} = e^{-i\xi_e} \left\{ \frac{\ell}{L} \vec{\phi}_{\ell \hbar}^{\text{(even)}} + \frac{\ell+1}{L} \vec{\phi}_{\ell \hbar}^{\text{(odd)}} \right\}. \tag{2.21} \]

where use is made of the definition
\[ \frac{\sqrt{L}}{L} \equiv \sqrt{2L+1}. \tag{2.22} \]

The phases \(\xi_e\) and \(\xi_m\) are completely arbitrary. If one chooses \(\xi_e = \pi\) and \(\xi_m = 0\), then
\[ \vec{\phi}_{\ell \hbar} = \frac{\ell}{L} \vec{\phi}_{\ell \hbar}^{\text{(even)}} + \frac{\ell+1}{L} \vec{\phi}_{\ell \hbar}^{\text{(odd)}} \quad \text{or} \quad \vec{\phi}_{\ell \hbar}^{\text{(even)}} = \frac{\ell}{L} \vec{\phi}_{\ell \hbar}, \tag{2.23} \]

where \(\vec{\phi}_{\ell \hbar}^{\text{(even)}}\) and \(\vec{\phi}_{\ell \hbar}^{\text{(odd)}}\) are defined by
\[ \tilde{A}_{\ell \lambda} = \frac{1}{2} \left( \tilde{A}_{\ell \lambda}^{(+)} + \tilde{A}_{\ell \lambda}^{(-)} \right), \]  
\hspace{1cm} (2.24)

using the relation

\[ J_{\ell} = \frac{1}{2} \left( J_{\ell}^{(+)} + J_{\ell}^{(-)} \right), \]  
\hspace{1cm} (2.25)

where \( J_{\ell} \) is the spherical Bessel function. The vector function \( \tilde{A}_{\ell \lambda} \) has the choice of phases of the vector potential of Ref. (7), although the normalization is of necessity different. The expression for \( \Omega_{\ell \lambda} \) may be written is a compact (and eventually useful) form:

\[ \Omega_{\ell \lambda} = -\sqrt{\frac{2}{\ell + 2}} \sum_{\ell} \mathcal{S}(\ell, \ell') \langle \ell - 1, 1 \mid 1 20 \rangle \mathcal{F}_{\ell \ell}^{(\ell')}. \]  
\hspace{1cm} (2.26)

This expression may be compared (indirectly) with Eq. (3) of Ref. (2). The \( \mathcal{S}(\ell, p) \) specify \( \ell \) as,

\[ p = 1 \quad (e) \quad \ell = \ell \]

\[ p = 2 \quad (c) \quad \ell = \ell + 1, \ell - 1 \]  
\hspace{1cm} (2.27)

A comparable development may be found in Ref. (8).

It will prove convenient to normalize the vector functions \( \Omega_{\ell \lambda} \) to unit flux crossing a sphere centered at the origin (henceforth referred to as unit flux area); recognizing that \( \Omega_{\ell \lambda} \) is the vector potential. To evaluate the particle flux, consider the outgoing wave fields in the radiation zone \((kr >> 1)\) where the net time averaged energy
flux crossing a sphere of radius $r$ is,

$$S_{\ell m}^{(+)} = \frac{c}{4\pi} \int d\Omega \hat{r} \cdot (\mathbf{E}_{\ell m}^{(+)} \times \mathbf{H}_{\ell m}^{(+)}^*)$$  \hspace{1cm} (2.28)

where the real electric field is given by

$$\mathbf{E}_{\ell m}^{(+)} = \mathbf{E}_{\ell m}^{(+)} + \mathbf{E}_{\ell m}^{(-)}$$  \hspace{1cm} (2.29)

and similarly for $\mathbf{H}$. It is sufficient to consider only the magnetic multipole field, since the Poynting flux, $S_{\ell m}^{(+)}$, is invariant under the dual transformation:

$$\mathbf{H}(\mathbf{e}) \rightarrow -i \mathbf{E}(\mathbf{m}) ,$$

$$\mathbf{E}(\mathbf{e}) \rightarrow +i \mathbf{H}(\mathbf{m}) .$$  \hspace{1cm} (2.30)

This split of an overall minus sign into a shift of 90° backward for one field and 90° forward for the other is by no means universal (see, for example, Ref.s (5) and (9)); it does produce the relations of Ref. (7), viz.,

$$\nabla \times \mathbf{O}_{\ell m}^{(e)} = +h \mathbf{O}_{\ell m}^{(h)} .$$  \hspace{1cm} (2.31)

This particular choice for the dual transformation gives convenient forms for the electric and magnetic vector potentials. With the outgoing solutions to the scalar wave equation written as,
\[ \ell_{lm}^{(\nu)} = N \ell \ell' \ell^2 \gamma_{lm}^{\nu}, \quad (2.32) \]

then

\[ \ell_{lm} = \frac{-i \ell \ell_{lm}^{(\nu)}}{\ell(\ell+1)}, \quad (2.33) \]

and

\[ \ell_{lm} = \frac{-i \ell \ell_{lm}^{(\nu)}}{h \ell(\ell+1)} \quad (2.34) \]

There are no additional i's between \( \ell(e) \) and \( \ell(m) \). The Poynting vector in terms of the \( \ell \) 's is

\[ \ell_{lm}(\mu) \times \ell_{lm}(\mu) = i h \ell_{lm} \times \ell_{lm}^{*} / i, \quad (2.35) \]

where

\[ \ell_{lm}(\mu) = -i h \ell_{lm} \ell_{lm}^{*} \quad (2.36) \]

So,

\[ \ell_{lm}(\mu) \times \ell_{lm}(\mu) = i h^2 (\ell_{lm} \times \ell_{lm}^{*}) \quad (2.37) \]

The net energy flux out of a large sphere is given by the perpendicular components of the fields \( \ell(e) \) and \( \ell(m) \), i.e.,
In order to discuss the Poynting flux, it is convenient to decompose the field vectors into radial and tangential components. The radial part of any vector is given by

\[ \vec{\mathbf{A}}_{\parallel} = \hat{r} \left( \hat{\mathbf{r}} \cdot \vec{\mathbf{A}} \right). \]  

(2.39)

The tangential part of any vector is given by

\[ \vec{\mathbf{A}}_{\perp} = -\hat{\mathbf{r}} \times \left( \hat{\mathbf{r}} \times \vec{\mathbf{A}} \right). \]  

(2.40)

From Eqs. (2.33) and (2.39), the magnetic vector potential \( \vec{\mathbf{A}}(\mathbf{m}) \) is perpendicular to \( \hat{\mathbf{r}} \) \( (\vec{\mathbf{E}}(\mathbf{m}) \cdot \hat{\mathbf{r}} = 0) \). The electric vector potential, i.e.,

\[ \vec{\mathbf{A}}_{\text{em}} = -\hat{\mathbf{r}} \times \left( \frac{\nabla \times \vec{\mathbf{A}}_{\text{em}}}{\omega \mathbf{r} (\mathbf{r})} \right), \]  

(2.41)

has a parallel component. The right hand side of Eq. (2.41) is

\[ \nabla \times \overline{\mathbf{A}}_{\text{em}} = \left[ \nabla \times \overline{\mathbf{A}}_{\text{em}} \right] + \mathbf{r} \left\{ \rho \overline{\mathbf{V}}^{\mu}_{\text{em}} + \omega \overline{\mathbf{V}}^{\mu}_{\text{em}} \right\}. \]  

(2.42)

The second term on the right hand side of Eq. (2.42) is,

\[ \nabla \times \overline{\mathbf{V}}^{\mu}_{\text{em}} = -i \nabla \times \hat{\mathbf{r}} \times \overline{\mathbf{V}}^{\mu}_{\text{em}}. \]  

(2.43)
Using the identity,
\[ \nabla \times \vec{r} \times \vec{\nabla} = \vec{r} \nabla^2 - \vec{\nabla} \left( 1 + r \frac{d}{dr} \right), \quad (2.44) \]

Eq. (2.43) may be written,
\[ \nabla \times \vec{\upsilon}^m = -i \left[ -r \frac{e^{(l+1)}}{r^2} \upsilon^m - \vec{\nabla} \upsilon^m \right], \quad (2.45) \]

So that Eq. (2.39) is,
\[ \bar{O}^{\uparrow}_{l,m} = \frac{N \langle 0 | h_e^{(l+1)} \upsilon^m \rangle}{\lambda (l+1)}, \quad (2.46) \]

and Eq. (2.40) is
\[ \bar{O}^{\downarrow}_{l,m} = \frac{N \langle 0 | h_e^{(l+1)} \upsilon^m \rangle}{\lambda (l+1)} \left[ \nabla h_e^{(l+1)} \times \vec{\upsilon}^m + i \nabla \upsilon^m \right], \quad (2.47) \]

The first term on the right hand side of Eq. (2.47) is
\[ \bar{O}^{\downarrow}_{l,m} = \frac{N \langle 0 | h_e^{(l+1)} \upsilon^m \rangle}{\lambda (l+1)} \left( -r \nabla \upsilon^m \right), \quad (2.48) \]

So, Eq. (2.47) becomes
\[ \bar{O}^{\downarrow}_{l,m} = \frac{(i)^{l+2} N}{\lambda (\ell+1)} G_{\ell} \nabla \upsilon^m, \quad (2.49) \]

where
The expression for the net energy flux, Eq. (2.28), becomes,

\[ S_{\ell m} = \frac{e^{i \pi} c^2 \pi \alpha \beta}{2 \pi \ell (\ell + 1)} \int \mathbf{J} \cdot (\mathbf{\hat{r}} \times \mathbf{\hat{r}} \times \mathbf{\hat{r}}) \, d\Omega \. \] (2.51)

Since,

\[ \mathbf{J} \cdot (\mathbf{\hat{r}} \times \mathbf{\hat{r}} \times \mathbf{\hat{r}}) = i \ell \ell + 1 \left| \nabla \mathbf{\hat{r}} \mathbf{\hat{r}} \mathbf{\hat{r}} \right|^2, \] (2.52)

one is left with evaluating the integral,

\[ \int d\Omega \left| \nabla \mathbf{\hat{r}} \mathbf{\hat{r}} \mathbf{\hat{r}} \right|^2 = \frac{\ell (\ell + 1)}{\ell !}. \] (2.53)

The Poynting flux, Eq. (2.26), is

\[ S_{\ell m}^{(+)} = -i c \frac{e^{i \pi} c^2 \pi \alpha \beta}{2 \pi} \mathbf{\hat{r}} \times \mathbf{\hat{r}} \times \mathbf{\hat{r}} G_{\ell}^* \] (2.54)

From the recursion relations for spherical Bessel functions, Eqs. (2.13) and (2.14), \( G_{\ell}^* \) may be written,

\[ G_{\ell}^* = -i \ell \mathbf{\hat{r}} \times \mathbf{\hat{r}} \times \mathbf{\hat{r}} + b \mathbf{\hat{r}} \times \mathbf{\hat{r}} \times \mathbf{\hat{r}}. \] (2.55)

The asymptotic behavior of this function, from Eq. (2.6), is
To terms of order $(1/r)$, the radial product in Eq. (2.52) is,

$$\imath \hbar \ell^{(\ell)} G_{\ell} = \imath \hbar \ell^{(\ell)} \frac{e^{i \kappa r}}{\kappa r} e^{-i \kappa r} \frac{1}{\hbar}.$$

So that the Poynting flux in this limit is,

$$S_{\ell m}^{(+)} = \frac{c}{2\pi} N^2 \hbar \omega.$$

In this limit the parallel field $0_{\ell m}^{(\ell)}(e)$ is zero.

The probability of emission of a light quantum $\hbar \omega$ of a given multipole type and mode (electric or magnetic) per unit time is obtained from the rate of emission of energy by dividing by the quantum energy $\hbar \omega$. Thus, the particle flux may be determined, i.e.,

$$P_{\ell m} = \frac{S_{\ell m}^{(+)}}{\hbar \omega} = \frac{c}{2\pi} \frac{N^2}{\omega} \hbar \omega.$$

In order that this particle flux area be 1,

$$N = \sqrt{\frac{2\pi}{\omega \hbar}}.$$

The expression for $\ell^{(\ell)}_{\ell m}$ eqn (2.32), may be written in the form of Ref. (3),
The nonalization factor, \( \nu \), of Eq. (2.3) is,

\[
\nu = \frac{k}{\pi h} \tag{2.62}
\]

This may be identified as the photon channel 'velocity' to bring this form into correspondence with a particle channel - this quantity does not have the dimensions of a velocity.

An incoming wave vector potential may be developed analogously. With,

\[
\Phi_{\pm \ell m}^{(-)} = \sqrt{\frac{m^*}{\hbar}} \frac{i \ell}{r} \hat{\ell} \gamma_m. \tag{2.63}
\]

where

\[
\hat{\ell} = -i \gamma \partial_{\ell}, \quad \ell = \hbar \varphi,
\]

and \( \hat{\ell} \) is the spherical Hankel function of the second kind. The vector functions have the form:

\[
\vec{\Phi}_{\pm \ell m} = \frac{-i \ell \Phi_{\pm \ell m}^{(-)}}{\sqrt{\ell (\ell + 1)}}, \tag{2.65}
\]

and

\[
\vec{\Phi}_{\pm \ell m} = \frac{-\nabla \times \frac{-i \ell \Phi_{\pm \ell m}^{(-)}}{\sqrt{\ell (\ell + 1)}}}{\hbar \sqrt{\ell (\ell + 1)}}, \tag{2.66}
\]
The connection between the outgoing waves $\tilde{\mathbf{E}}_{\text{out}}(r)$ and sources is given classically by Maxwell's equations. In particular, the time independent vector potential, satisfying the Lorentz condition, is a solution to the vector Helmholtz equation with a source term $4\pi \tilde{J}/c$, where

$$ \tilde{\mathbf{J}} = \mathbf{j} + c \tilde{\nabla} \times \tilde{\mathbf{m}}, \quad (2.67) $$

$\mathbf{j}$ is the current density and $\tilde{\mathbf{m}}$ is the magnetization. The solution, in terms of the Green's function appropriate to this equation in the infinite domain is,

$$ \tilde{\mathbf{A}}(\tilde{r}) = \frac{4\pi}{c} \int \frac{\tilde{\mathbf{J}} \cdot \tilde{r}}{r'} G(\tilde{r}, \tilde{r}') \, d^3 r', \quad (2.68) $$

where

$$ G(\tilde{r}, \tilde{r}') = i \hbar \sum_{\ell m} \mathcal{E}_\ell(k r') \mathcal{H}_\ell(k r) Y^\ell_m(\hat{r})^* \tilde{Y}^\ell_m(\hat{r}'), \quad (2.69) $$

The radial part of Eq. (2.69) satisfies the boundary conditions of finiteness at the origin and outgoing waves at infinity. The introduction of the unit dyadic, i.e.,

$$ \tilde{\mathbf{J}} \cdot \hat{r} = \tilde{\mathbf{J}} \cdot \tilde{r} = \tilde{\mathbf{J}}, \quad (2.70) $$

where
in a spherical basis permits an expansion of the Green's function in terms of the vector potentials. The result is,

\[
G(\vec{r},\vec{r}^\prime) \frac{\vec{1}}{\vec{r}} = i\hbar \sum_{\ell m} \bar{O}^p_{\ell m}(\vec{r}) \frac{\bar{A}^p_{\ell m}(\vec{r})}{\sqrt{\ell(\ell+1)}} , \tag{2.72}
\]

where \( p = e, m, l \). This expression differs only in normalization and implicit phases from the comparable expression in Ref. (5). \( \bar{O}^p_{\ell m}(\vec{r}) \) is the vector function defined previously as the outgoing wave solution to the source free vector Helmholtz equation. The coordinates are explicitly shown. The new vector function \( \bar{\mathbf{A}}^p_{\ell m}(\vec{r},r) \) is

\[
\bar{\mathbf{A}}^e_{\ell m} = i \vec{\nabla} \times \frac{\vec{1}}{\hbar} \vec{\phi}_s , \tag{2.73}
\]

\[
\bar{\mathbf{A}}^m_{\ell m} = \frac{\vec{1}}{\hbar} \vec{\phi}_s , \tag{2.74}
\]

\[
\bar{\mathbf{A}}^l_{\ell m} = \frac{\vec{\nabla} \vec{\phi}_s}{i\hbar} , \tag{2.75}
\]

where

\[
\vec{\phi}_s = i^{\ell+1} \int \vec{\ell} \cdot \gamma^m_{\ell} . \tag{2.76}
\]
Therefore the vector field outside the source, from Eq. (2.68), is

\[ \mathbf{A}(\mathbf{r}) = \frac{4\pi i \hbar}{\sqrt{2\pi c}} \sum_{\ell m} \mathbf{O}_{\ell m}^P \int d^3r' \mathbf{j}(\mathbf{r}') \cdot \mathbf{A}_\ell^P(\mathbf{r}') \]  

(2.77)

The transverse fields (e and m) are in the solenoidal gauge. Since the dyadic Green's function is not solenoidal, the longitudinal field (Q) is required in the general expansion.

A general transverse radiation field may be written,

\[ \mathbf{A}(\mathbf{r}) = \sum_{\ell m} (a_{\ell m}^e \mathbf{O}_{\ell m}^e + a_{\ell m}^m \mathbf{O}_{\ell m}^m) \]  

outside the source. On comparing Eqs. (2.77) and (2.78), the coefficients are found to be,

\[ a_{\ell m}(r) = i \sqrt{\frac{8\pi \hbar}{x c^2}} \int d^3r' \mathbf{j}(\mathbf{r}') \cdot \mathbf{A}_{\ell m}^P(\mathbf{r}') \]  

(2.79)

A parallel calculation for the particle flux crossing a large sphere in terms of this general vector field, i.e.,

\[ \mathbf{E}_{\ell m}^{(m)} \times \mathbf{H}_{\ell m}^{*(m)} = i\hbar^2 |a_{\ell m}^{(m)}|^2 (\mathbf{O}_{\ell m}^{(m)} \times \mathbf{O}_{\ell m}^{*(e)}) \]  

(2.80)

gives,

\[ \mathcal{P}_{\ell m} = |a_{\ell m}(r)|^2 \]  

(2.81)
since the \( \bar{O}_{\lambda_m} \) vector functions have been normalized to unit particle flux area. The function \( P_{\lambda_m} \) is the probability of the emission of a light quantum of a given multipole type and mode per unit time. This suggests the identification

\[
P_{\lambda_m} = \frac{\alpha \pi \rho}{\hbar} |\langle f | H(\omega) | i \rangle|^2, \tag{2.83}
\]

based on the Fermi golden rule II. \( H(\text{int}) \) is that part of the hamiltonian causing transitions from \( |i\rangle \) to \( |f\rangle \); \( \rho \) is the number of final states per unit energy interval. To within a phase,

\[
a_{\lambda m}(\rho) = \sqrt{\frac{\alpha \pi \rho}{\hbar}} \langle f | H(\omega) | i \rangle. \tag{2.84}
\]

The next section is directed toward making this identification explicit and incorporating these results into the general scattering formalism of Ref. (3).

B. COLLISION MATRIX FOR PHOTONS

Wave functions with and without photon particle coupling are denoted by \( \Phi \) and \( \Phi^\ast \), respectively, so that

\[
H \Phi = E \Phi, \quad H = H_{\text{pot}} + H_{\text{em}} + H(\text{int}), \tag{2.85}
\]

and
\[ \mathcal{H} \Phi = \varepsilon_\beta \Phi, \quad \mathcal{H} = \mathcal{H}_{\text{part}} + \mathcal{H}_{\text{em}}. \quad (2.86) \]

\( \mathcal{H}(\text{int}) \) may be decomposed into

\[ \mathcal{H}_{\text{int}} = \mathcal{H}' + \mathcal{H}'' \quad (2.87) \]

where \( \mathcal{H}' \) creates single photons and \( \mathcal{H}'' \) absorbs them. Consider the state \( \Phi_c \) corresponding to unit incoming flux in the channel \( c = e \). Allowing for at most one photon to be present, this state may be written as,

\[ \Phi_c = \Phi_c^{(0)} + \sum_\beta A_\beta \Phi_c^{(\beta)}. \quad (2.88) \]

Superscripts signify the number of photons present in the state. Thus,

\[ \Phi_c^{(0)} = \mid 0 \rangle \mid 0 \rangle, \quad \Phi_c^{(\beta)} = \mid 0 \rangle \mid \mathbf{1}_\beta \rangle, \quad (2.89) \]

where \( \mid 0 \rangle \) is the vector state of vacuum in the occupation number space of the photons and \( \mid 1 \rangle \) is the vector state of one photon which may be taken to be in the \((k;L\mathbf{M};p)\) representation (\( k \) = photon wave number; \( L \) = angular momentum; \( \mathbf{M} \) = projection on some axis; and \( p \) = parity). The nucleus is the state \( \alpha \) is represented by the eigenvector \( \mid \Psi_\alpha \rangle \):

\[ \mathcal{H}_{\text{part}} \mid \Psi_\alpha \rangle = \varepsilon_\alpha \mid \Psi_\alpha \rangle. \quad (2.90) \]
To further clarify the notation, one has

$$H_{	ext{e.m.}} | \psi > = E_\gamma | \psi > , \quad E_\gamma = \hbar \omega , \quad (2.91)$$

$$H | \Psi_\beta > = E_\beta | \Psi_\beta > , \quad E_\beta = E_\alpha + E_\gamma , \quad (2.92)$$

$$H | \Psi_\beta^{(o)} > = E_\beta^{(o)} | \Psi_\beta > , \quad E_\beta^{(o)} = E_\delta , \quad (2.93)$$

and

$$(H + H_{\text{int}}) \left[ \Phi_e^{(o)} + \sum_\beta A_\beta \Phi_\beta \right] =$$

$$= c \Phi_e^{(0)} + \sum_\beta A_\beta \Phi_\beta . \quad (2.94)$$

With these definitions,

$$\Phi_e^{(o)} = \sum_\beta \langle \Psi_\beta^{(o)} | \Phi_e > \langle \Psi_\beta > , \quad (2.95)$$

and

$$A_\alpha = \langle \Psi_\beta^{(1)} | \Phi_e > . \quad (2.96)$$

Substituting Eq. (2.88) into Eq. (2.85) and taking account of Eqs (2.86) and (2.93) yields,
Both members of Eq. (2.97), i.e., $\Phi^{(e)}$ and $\Phi^{(\omega)}$, may be projected on to the state $\Phi^{(b)}$. Taking into account the relations

$$<\Phi^{(b)}|\mathbf{H}_{\text{int}}|\Phi^{(e)}> = \epsilon_\alpha \alpha^*, <\Phi^{(\omega)}|\Phi^{(e)}> = \delta_{\epsilon_\alpha} \alpha^*, (2.98)$$

as well as Eq.s (2.94) and (2.95) gives,

$$<\Phi^{(\omega)}|\mathbf{H}_{\text{int}}|\Phi^{(e)}> = A_\beta (\epsilon - \epsilon_\beta), (2.99)$$

An equation of this form is examined in considerable detail in Ref. (11). The general solution is,

$$A_\beta = \frac{<\Phi^{(\omega)}|\mathbf{H}_{\text{int}}|\Phi^{(e)}>}{\epsilon - \epsilon_\beta} \lambda(\beta, \epsilon) \delta(\epsilon - \epsilon_\beta), (2.100)$$

where the integral over energy of the first term will consist of the principle value (cf. Ref. (12)). From the general solution, it is seen that Eq. (2.99) is inadequate to determine the representative $A_\beta$ completely, on account of the arbitrary function $\lambda(\beta, \epsilon)$ occurring in Eq. (2.100). $\lambda$ must be chosen such that $A_\beta$ represents only outward moving particles. The one photon part of Eq. (2.88), in an $x$-representation, is
\[<\tilde{r}|\tilde{\Phi}_e^{(u)}> = \sum_{\rho} A_{\rho} \sqrt{\frac{2\omega}{\pi \gamma}} \beta^\gamma <\tilde{r}|\tilde{1}|\tilde{r}>, \quad (2.101)\]

where

\[<\tilde{r}|\tilde{1}|\tilde{r}> = -\frac{i}{\hbar} \sqrt{\frac{\pi \omega}{2\gamma}} \sum_{\alpha} \frac{e^{i\alpha\Omega_{e\beta}}}{\Omega_{e\beta}} \delta_{\alpha\beta} \frac{\delta_{\alpha\beta}}{\hbar^2} \epsilon_{\alpha\beta} \epsilon_{\alpha\beta}(2.102)\]

where \(\epsilon_{(\pm)}\) is the photon spin projection on its direction of motion and \(D_{\alpha\beta}\) is a rotation matrix element as defined in Ref. (6). The rotation involved here consists of taking an arbitrary \(z\) axis into the direction of the photon wave vector \(k_y\). This expansion readily follows by the Rayleigh expansion of a vector plane wave in terms of functions defined previously and normalized to have unit particle flux (time average) crossing any 'cube' centered on the origin, i.e.,

\[\tilde{A}_\epsilon(k_y, \tilde{r}) = \left(\frac{e^{i\epsilon\tilde{y}}}{\sqrt{\pi \gamma}}\right)^\gamma \epsilon_{\epsilon\tilde{y}} e^{i\epsilon\tilde{y}}.\quad (2.103)\]

This is the conventional normalization for plane waves; corresponding to one photon per unit volume. Eq. (2.101) becomes

\[<\tilde{r}|\tilde{\Phi}_e^{(u)}> = -i \sqrt{\frac{\pi \omega}{2\gamma}} \sum_{\alpha} \frac{e^{i\alpha\Omega_{e\beta}}}{\Omega_{e\beta}} \frac{\delta_{\alpha\beta}}{\hbar^2} \epsilon_{\alpha\beta} \epsilon_{\alpha\beta} \epsilon_{\alpha\beta} \frac{dE_y}{\hbar^2} X(\Omega_{e\beta}) \left[<\tilde{\Phi}_{e\beta}^{(u)}|H_{\text{int}}|\tilde{\Phi}_{e\beta}^{(u)}> + \frac{\lambda}{\epsilon^2} (\epsilon^2 - E_y) \right] \quad (2.104)\]

\[\times \left(\delta_{\alpha\beta} - \Omega_{e\beta}\right) \frac{e^{i\epsilon\tilde{y}}}{\sqrt{\pi \gamma}}.\]
What was previously a sum over $\phi$ ($E_{\phi} = E_{\gamma} + E_{\nu}$) has been rearranged into a sum over $\alpha$ and an integral over the gamma ray energies $-\rho(E_{\nu})$ is the density of final photon states and $E'$ is $E - E_{\alpha}$; it is clear that a statement of energy conservation will result. The first term in brackets may be written in an asymptotic version (with temporary neglect of irrelevant terms):

$$\sum_{E_{\nu}} dE \frac{g(E)}{E' - E} \left[ e^{-\frac{E_{\nu}}{\hbar c}} - e^{\frac{E_{\nu}}{\hbar c}} \right] =$$

$$= g(E') \int_{0}^{E} dE \left[ e^{-\frac{E}{\hbar c}} - e^{\frac{E}{\hbar c}} \right] \left( \frac{E'}{E' - E} \right) \quad (2.105)$$

with neglect of terms involving $1/\epsilon$, for any continuous function $g(E)$, which formula holds since $\int K(E) e^{\frac{E_{\nu}}{\hbar c}} dE$ is of order $1/\epsilon$ for any continuous function $K(E)$ and since the difference

$$\frac{g(E)}{E' - E} - \frac{g(E')}{E' - E}$$

is continuous. The first term in the numerator of Eq. (2.105) may be rewritten,

$$g(E') \int_{0}^{E} dE \frac{E'}{E' - E} = g(E') \int_{0}^{E} dE \frac{E'}{E' - E} \quad (2.106)$$

Alternatively, one may view the matrix element and density function as reasonably constant over an energy interval in the neighborhood of $E'$. But, $e^{\frac{E_{\nu}}{\hbar c}}$ in Eq. (2.105) oscil-
lates rapidly as a function of \( E \) in this same interval for all \( r \) satisfying \( r \gg \hbar c/E \); and has a pronounced peak at \( E = E^* \). For this reason also, the limits of the integral may be replaced by \(-\infty \) to \(+\infty \) to arrive at the integral form derived in Ref. (11). From Eq. (2.106), one has the well known integrals,

\[
\int_{-\infty}^{\infty} e^{\pm i x} \frac{dx}{x} = \pm i \pi, \quad (2.107)
\]

So that Eq. (2.105) is

\[
\int_{-\infty}^{\infty} \frac{du}{E - u} \left[ e^{i E_0/\hbar c} - e^{-i E_0/\hbar c} \right] = \frac{g(E')}{E' - E} e^{-i E_0/\hbar c} ( + i \pi) - \frac{g(E)}{E - E'} e^{i E_0/\hbar c} ( - i \pi). \quad (2.108)
\]

Eq. (2.104) becomes,

\[
\langle \Phi | \Phi^{(1)} \rangle = -i \sum_{m=\pm \lambda} \sum \frac{\hbar}{\omega_{\lambda m}} \frac{g(E_m)}{E_m - E} \left[ \frac{1}{E_0} - \frac{1}{E'} \right] \lambda (\omega_{\lambda m} \Phi_{\lambda m} - \lambda m \Theta_{\lambda m}\Phi_{\lambda m}) + \frac{\lambda (\omega_{\lambda m} \Phi_{\lambda m} - \lambda m \Theta_{\lambda m}\Phi_{\lambda m})}{\hbar} \quad (2.109)
\]

The integral over the delta function was evaluated in forming Eq. (2.109) from Eq. (2.104).

\[
E_0 = E_\beta - E_\alpha = \frac{\hbar c}{\lambda} \quad (2.110)
\]

In order that this equation shall represent only outgoing
waves,

\[ \chi(\omega \mu \rho \varepsilon) = -i \pi \langle \Phi_{\rho}^{(\varepsilon)} | H_{\text{int}} | \Phi_{\varepsilon}^{(\omega)} \rangle. \]  (2.111)

Which is the result of Ref. (11). Eq. (2.109) becomes,

\[ \langle \tilde{\eta} | \Phi_{\varepsilon}^{(\omega)} \rangle = -i \sum_{\alpha} \sqrt{\frac{n_{\alpha}}{2\pi \omega_{\alpha}}} \sum_{L} \frac{\Delta_{\alpha L}}{2 \Omega_{\alpha L}} \left\{ \frac{2 \pi \pi}{\hbar \gamma} \right\}^{1/2} \phi(\varepsilon) \langle \Phi_{\rho}^{(\varepsilon)} | H_{\text{int}} | \Phi_{\varepsilon}^{(\omega)} \rangle O_{\text{int}} \int \frac{d\omega}{1 + \frac{\omega}{\omega_{\text{cut}}}}. \quad (2.112) \]

The number of photon modes for each polarization per unit energy interval is given by

\[ \rho(\varepsilon) = \frac{\sqrt{\varepsilon}}{\omega^2} \frac{E^2}{c^3} = \frac{\sqrt{\hbar^2}}{\omega c (\Theta^* \rho)^2}. \quad (2.113) \]

The volume may be eliminated in terms of the density function to give for Eq. (2.112),

\[ \langle \tilde{\eta} | \Phi_{\varepsilon}^{(\omega)} \rangle = \sum_{\alpha \rho} \sqrt{\frac{n_{\alpha \rho}}{2\pi \omega_{\alpha \rho}}} \sum_{L} \frac{\Delta_{\alpha L}}{2 \Omega_{\alpha L}} \left\{ \frac{2 \pi \pi}{\hbar \gamma} \right\}^{1/2} \phi(\varepsilon) \langle \Phi_{\rho}^{(\varepsilon)} | H_{\text{int}} | \Phi_{\varepsilon}^{(\omega)} \rangle. \quad (2.114) \]

The angular function is normalized properly in this asymptotic expression.

A completely general solution to the wave equation in the asymptotic region may be written in terms of incoming and outgoing solutions to the asymptotic Schrodinger equation (channel wave functions) as
The numbers $y_c$ are the amplitudes of the incoming waves $\phi_c$ in the various channels $c$, while the $x_c$ are the amplitudes of the outgoing waves $\phi_c$. For a given total system, when the $y_c$ are given, the numbers $x_c$ are determined by the nature of the system. The role of the collision matrix is just to give an expression for the $x_c$ in terms of the $y_c$ as follows:

$$x_c = - \sum_c U_{c'c} y_c. \quad (2.116)$$

If $\phi_c$ and $\phi_c$ are normalized to unit flux areas, then this collision matrix is the collision matrix of Ref. (3). Explicitly, consider the ratio of the outgoing flux area in channel $c'$ to the incoming flux area in channel $c$:

$$|U_{c'c}|^2 = \frac{|x_c|^2 v_c' \int |\phi_c'|^2 dA}{|y_c|^2 v_c \int |\phi_c|^2 dA} \quad (2.117)$$

or,

$$U_{c'c} = - x_c' / y_c. \quad (2.118)$$

where the minus sign is introduced for convenience; and it is assumed that there are incident waves in one channel only. The channels and/or collision matrix indices may be
defined so that there is always an incident wave in one channel only. In this case \( y = \delta_{c'} \) and Eq. (2.115) becomes

\[
\Phi_{c'}(\omega) = \Phi_{c'} - \sum_{c} U_{c'}^{0} a_{c} \cdot \Omega_{c} . \tag{2.119}
\]

Thus, the collision matrix corresponding to the emission of a photon along \( k \) with circular polarization \( \varepsilon \) is

\[
U_{L \omega \mu \alpha} e = -\sqrt{\frac{2\pi\mu e}{\hbar}} \langle \Phi_{\omega}^{(0)} | H_{\omega} | \Phi_{\omega}^{(0)} \rangle . \tag{2.120}
\]

Thus, Eq. (2.114) may be written

\[
\langle \tilde{r} | \Phi_{\omega}^{(0)} \rangle = -\sum_{L \omega \mu \alpha} U_{L \omega \mu \alpha} e \cdot \frac{\epsilon^b \hat{L}}{\sqrt{8\pi}} \frac{\Delta^L_{\omega \mu \alpha}}{\Omega_{\omega \mu \alpha}} . \tag{2.121}
\]

This is the reaction wavefunction, and it is in implicit agreement with the R.G. Thomas result\(^3\). The factor

\[
\frac{\epsilon^b \hat{L}}{\sqrt{8\pi}} \frac{\Delta^L_{\omega \mu \alpha}}{\Omega_{\omega \mu \alpha}}
\]

is the generalized expansion coefficient of the Racah radiation parameter; representing part of the eigenfunction of the radiation referred to an arbitrary coordinate axis\(^1\).

The discussion may now be specialized to the case of first order theory without damping by replacing \( \Phi_{\omega}^{(0)} \) in Eq. (2.120) by \( \Phi_{\omega}^{(0)} \), the nuclear wave function in the absence of photon coupling. It is shown in considerable detail in Ref. (3) that this wave function has a form in the internal
region for unit incoming flux area in any channel given by

$$\Phi_e^{(1)} = -i \sum_{\lambda \mu} A_{\lambda \mu} \Gamma_{\lambda \mu} \chi_{\lambda}^{(2)},$$

where $\Phi_e$ is a phase (for protons in the $e$ channel it has the form $e^{i(\omega - \omega_e)}$ where $\omega_e$ and $\omega$ are the Coulomb and hard sphere phases, respectively); $A_{\lambda \mu}$ is the level matrix of Ref. (3); $\Gamma_{\lambda \mu}$ is a partial width; and $\chi_{\lambda}$ is a nuclear eigenfunction. As this equation is fundamentally important, a digression will be taken to discuss R matrix theory and the origin of this equation. Only those points of immediate applicability will be discussed.

C. R MATRIX THEORY

First, a Green's theorem relation will be shown to facilitate the development. The wave equation for a system of $A$ nucleons is written for general solutions at two energies,

$$H \Phi_1 = \varepsilon_1 \Phi_1, \quad H \Phi_2 = \varepsilon_2 \Phi_2, \quad (2.122)$$

where $H$ is the Hamiltonian operator. The first of these is multiplied by $\Phi_2^*$ and the complex conjugate of the second by $\Phi_1^*$; the difference between the resulting equations is integrated over the internal region $\mathcal{C}$. (The internal region is defined as a volume of nuclear dimensions in physical space in which all nucleons are close together -
the potential energy, assumed Hermitian, is completely unknown in this region.): 

$$\left(\varepsilon_2^* - \varepsilon_1^*\right) \int T^*_2 \nabla T_1^* d\tau = \frac{\hbar^2}{\alpha M} \int \left(\nabla T^*_2 \cdot T_1^* - T^*_2 \cdot \nabla T_1^*\right) d\tau. \quad (2.123)$$

The Hamiltonian is a sum of kinetic energy terms and potential terms. If the potential is Hermitian (conservation of probability demands this), then the potential part of the right hand side of Eq. (2.123) is zero. The kinetic energy terms which remain are integrated by Green's theorem (in 3A dimensional space) to give,

$$\left(\varepsilon_2^* - \varepsilon_1^*\right) \int T^*_2 \nabla T_1^* d\tau = \frac{\hbar^2}{\alpha M} \int \left(\nabla T^*_2 \cdot T_1^* - T^*_2 \cdot \nabla T_1^*\right) d\tau. \quad (2.124)$$

The occurrence of the ordinary reduced mass ($\alpha$) is a consequence of the two body nature of the theory as given in Ref. (3). The right hand side of Eq. (2.124) may be reexpressed in terms of the value and derivative quantities of Ref. (3). In order to define these convenient quantities, it is first necessary to introduce the surface functions,

$$\Psi_{\alpha s u_0} = \frac{i e^0}{\alpha} \Psi_{\alpha s u}. \quad (2.125)$$

The subscript $\alpha$ refers to a division of the $A$ nucleons into two groups of, say, $A-n$ nucleons and $n$ nucleons. The channel spin wave functions, $\Psi_{\alpha s u}$, of the pair $\alpha$ is con-
structured by vector coupling the normalized internal wave functions of the individual fragments $\alpha_i$ (A-n) and $\alpha_j$ (n). The surface functions are mutually orthogonal and normalized on the totality of all surfaces: $S = \xi_S S$. An element of the surface $S_{\xi}$ is

$$dS_{\xi} = a_{\xi}^2 d\Omega_{\xi} d\delta_{\xi} \quad (2.126)$$

where $d\Omega_{\xi}$ is the element of solid angle of the relative separation between the pair $\xi$; $q_{\xi}$ represents the internal coordinates of the pair $\xi$; and $a_{\xi}$ is the minimum radial distance of separation of the pair at which neither nucleus experiences any polarizing force from the other.

In subsequent developments, which involve the matching of internal and external functions on the surface of the internal region, expressions are needed for the values and derivatives of the radial parts of the internal and external wave functions on the surface. Therefore, it is convenient to define value and derivative quantities. The value quantity is proportional to the radial part of a wave function evaluated at $r = a_{\xi}$. Thus,

$$V_{\xi} = \left\{ \frac{t_2}{2\mu_{A_n}} \right\}^{1/2} \sum \alpha_{\xi}^* \Psi dS \quad (2.127)$$

The derivative quantity is proportional to the derivative of that part of the wave function satisfying the radial
The radial Schrödinger equation for the internal region when treated as an eigenvalue problem will have solutions (eigenvectors) that depend on the boundary conditions imposed. The boundary condition at the origin is usually fixed at zero; that leaves one condition to be specified at, say, a channel radius. On specifying this boundary condition at \( r = a \), a set of eigenvectors may be found. As is occasionally the case with a second order eigenvalue equation, the boundary condition may be a relation involv-
ing the function and its first derivative. If the size of the solution is of no immediate concern, a condition may be imposed on the logarithmic derivative of the function. The derivative and value quantities of the radial eigenvectors in the internal region may be expressed in terms of the boundary condition (real number) at the channel surface on the radial eigenvectors,

$$\frac{\Delta x_c}{V_{x_c}} = \beta_c \quad (2.131)$$

As one boundary condition gives rise to a set of eigenvectors, it cannot depend on the index used to specify one eigenvector, $a$. The radial function is the solution to the radial Schrödinger equation divided by $r$, i.e., the full radial dependance. By applying the Green's function relation, Eq. (2.130), to any two eigenvectors $x_a, x_{a'}$ belonging to energy eigenvalues $E_a, E'_{a'}$

$$\left(E_{a'}^2 - E_a\right) \int x_{a'}^* x_a d\tau = \sum_c V_{a,c} V_{a,c} \left[\beta_c - \beta_{c}^{'}\right] \quad (2.132)$$

where the reality of $V_{a,c}$ has been used. The reality of the proper value quantities (and thus the proper derivative quantities) follows from the observation that these quantities, representing the proportion of some channel in the proper state $x_a$ cannot depend on the spatial orientation of the whole system, i.e., on the projection numbers. On
application of the anti-unitary time reversal operator, defined in Eq. (2.4), one obtains,

\[ \mathcal{V}_{\lambda c} = \int \mathcal{J}_m \phi \mathcal{J}_m^* ds = (\phi_{\lambda}, \chi_{\lambda}) \]

\[ = (\chi_{\lambda}, \mathcal{J}_m^*)^* = (\mathcal{K} \chi_{\lambda}, \mathcal{K} \mathcal{J}_m^*) = (2.133) \]

\[ = (-)^{\mathcal{J} - \mathcal{J}'} (\chi_{\lambda}, \mathcal{J}_m^*) = (\mathcal{K}_{\lambda}, \mathcal{J}_m^*)^* \]

Invariance of the internal Hamiltonian under time reversal is required. From Eq. (2.132), the eigenvalues are real and the eigenfunctions, for \( E_{\lambda} \neq E'_{\lambda} \), are mutually orthogonal. Linear combinations of degenerate eigenfunctions may be found which are orthogonal. The eigenvectors are taken to be normalized to 1 over the internal region only.

Since a physical operator (the internal Hamiltonian) has a sufficient number of eigenfunctions to represent an arbitrary state, any wave function may be expanded,

\[ \psi = \sum_{\lambda} A_{\lambda} \chi_{\lambda} \quad (2.134) \]

The energy dependent coefficients \( A_{\lambda} \) being given by

\[ A_{\lambda} = \int \chi_{\lambda}^* \psi \ d\tau \quad (2.135) \]

Using the Green's function relation, Eq. (2.130), these coefficients may be expressed in terms of surface quanti-
ties (including the actual boundary conditions),

$$A_\lambda = (E_\lambda - \mathcal{E})^{-1}_c \sum_c \left( \nu_{\lambda c} b_c - V_c \delta_{\lambda c} \right). \quad (2.136)$$

$$A_\lambda = (E_\lambda - \mathcal{E})^{-1}_c \sum_c \gamma_{\lambda c} (b_c - B_c V_c). \quad (2.136a)$$

Eq. (2.136a) is a defining equation for the reduced partial width amplitude. An alternate form for this amplitude is

$$\gamma_{\lambda c} \equiv \nu_{\lambda c} = \left\{ \frac{e^2}{\hbar m_c a_c} \right\} \nu_\lambda (\Omega_c, \chi_\lambda). \quad (2.137)$$

Expansion (2.133) becomes,

$$\Psi = \sum_c \left[ \sum_\lambda \frac{x_\lambda \gamma_{\lambda c}}{E_\lambda - \mathcal{E}} \right] \delta_c \quad (2.138)$$

where

$$\delta_c = b_c - B_c V_c \quad (2.139)$$

Eq. (2.138) relates the value of $\Psi$ at any point in the internal region to its derivative and value quantities on the surface; the quantity in the square bracket in Eq. (2.138) being considered the Green's function. By evaluating Eq. (2.138) at the surface, one obtains the fundamental $R$ matrix relation,
\[ V_c' = \sum_c R_{c'c} \delta_c \]  

(2.140)

where

\[ R_{c'c} = \sum_{\alpha} \frac{\tilde{\delta}_{c'c} \tilde{\delta}_{\alpha}}{\epsilon_{c} - \epsilon} \]  

(2.141)

Eq.s (2.140) and (2.141) give the connection between the derivative and the value of the radial wave function on the surface which must hold if the wave function is to be continuous into the internal region. Since the $R$ matrix specifies the form of the wave function on the surface of the internal region and the $U$ matrix specifies an asymptotic wave function (external region), the connection between the two is established by joining the two regions.

In the external region, any solution of the Schrodinger equation may be expressed as a sum of incoming and outgoing waves, the coefficients of which are $\gamma_c$ and $\chi_c$ respectively, i.e.,

\[ \Phi = \sum_c (\chi_c D_c + \gamma_c \Theta_c) \]  

(2.115)

The basis functions in this expansion are the particle channel wave functions. The channel wave functions are the solutions to the Schrodinger equation in the external region (where at most, a Coulomb potential is present). In an \textit{(also \checkmark)} scheme, these functions are,
\[
\mathcal{D}_{\alpha \ell m}^{(+)} = \mathcal{D}_{\alpha \ell m}^{(+)} \frac{I_{\alpha \ell}}{\sqrt{\nu_{\alpha}}}, \quad (2.142)
\]

and

\[
\mathcal{O}_{\alpha \ell m}^{(+)} = \mathcal{O}_{\alpha \ell m}^{(+)} \frac{O_{\alpha \ell}^{(+)}}{\sqrt{\nu_{\alpha}}}, \quad (2.143)
\]

normalized to unit flux area. The functions \( I_{\alpha \ell} \) and \( O_{\alpha \ell} \)
are, for a Coulomb potential and no potential,

\[
I_{\alpha \ell}^{(+)} = (G_{\ell} - iF_{\ell}) e^{i\omega_{\alpha \ell}} - i\rho h_{\ell}^{(-)} \quad (2.144)
\]

and

\[
O_{\alpha \ell}^{(+)} = (G_{\ell} + iF_{\ell}) e^{i\omega_{\alpha \ell}} + i\rho h_{\ell}^{(+)} \quad (2.145)
\]

where \( G_{\ell} \) (\( F_{\ell} \)) are the irregular (regular) Coulomb wave
functions \(^{14}\) and \( h_{\ell}^{(+)} \) and \( h_{\ell}^{(-)} \) are the spherical Hankel
functions of the first and second kind, respectively. The
phase is,

\[
\omega_{\alpha \ell} = \omega_{\alpha \ell}^{(+)} - \omega_{\alpha}^{(-)} = \sum_{s=1}^{2} \tan^{-1}(\nu_{\alpha} s) \quad (2.146)
\]

and the Coulomb field parameter is,

\[
\eta_{\alpha} = \frac{2e^{2}}{\alpha_{\ell}^{2} \sqrt{\nu_{\alpha}}} \quad (2.147)
\]
This particular choice of phase, especially the offset phase \( S_{\alpha} \), facilitates the development of the general scattering amplitude (cf. section VIII in Ref. (3) and section 21 of Ref. (15)) in a neat form, as well as making clear from the outset that, for numerical work, it is not necessary to calculate the individual phases \( S_{\alpha k} \), which is a more complicated expression than Eq. (2.146). The superscript + on the I and O functions indicates positive energy channels. For negative energy channels only the outgoing wave occurs in the usual applications. For a Coulomb potential, it is identical to the real Whittaker function, i.e., the solution to

\[
\left\{ \frac{d^2}{dr^2} - \frac{2(G + \nu)}{r} - \frac{2\nu}{r} - \hbar^2 \right\} \psi_{\nu}(r, \rho) = 0, \quad (2.148)
\]

with a boundary condition of zero at infinity. The value and derivative quantities of the external wave, Eq. (2.115), are from Eqs (2.143) and (2.144)

\[
V_c = \left( \frac{G}{2} \right)^2 \left[ \frac{O_c x_c}{p_c y_c} + \frac{T_c y_c}{p_c y_c} \right], \quad (2.149)
\]

and

\[
\delta_c = \left( \frac{G}{2} \right)^2 \left[ p_c y_c O_c x_c + p_c x_c T_c y_c \right]. \quad (2.150)
\]

The derivatives are with respect to \( \rho (= kr) \). Since the
external wave and its first derivative must be continuous at the surface, i.e., Eq. (2.140) must be satisfied; an equation relating \( x_c \) and \( y_c \) results,

\[
\left[ \frac{O_c x_c}{\rho_c v_x} + \frac{T_c y_c}{\rho_c v_x} \right] = \sum_c R_{cc'} \left\{ \left[ \frac{O_{c'} x_{c'}}{\rho_{c'} v_x} + \frac{T_{c'} y_{c'}}{\rho_{c'} v_x} \right] + \frac{\rho_{c'} v_x}{\rho_c v_x} T_{c'} y_{c'} \right\} - B_{c'} \left[ \frac{O_{c'} x_{c'}}{\rho_{c'} v_x} + \frac{T_{c'} y_{c'}}{\rho_{c'} v_x} \right].
\]  

(2.151)

Since the collision matrix is defined as that quantity which relates the outgoing wave coefficients to the incoming wave coefficients, i.e.,

\[
x_{c'} = - \sum_c U_{c'c} y_c .
\]  

(2.116)

Eq. (2.151) may be manipulated into the form of Eq. (2.116). The result, in matrix notation, is

\[
\mathbf{W}^T = - \mathbf{R} \mathbf{W}^T \Omega .
\]  

(2.152)

where an element of the diagonal \( \Omega \) matrix is

\[
\Omega_c = \left\{ \frac{T_c}{O_c} \right\}^{1/2} = e^{i(\omega_c - \omega_c)}. \]  

(2.153)

The radial functions \( I_c \) and \( O_c \) are defined in Eqs. (2.144) and (2.145) for positive energy channels.
for positive energy channels,

\[ W^J = 1 + 2i \mathbf{P}^J (1 - R^J L_0)^{-1} R^J \mathbf{P}^J, \quad (2.154) \]

where the logarithmic derivative matrix is,

\[ L^0 = L - B^J. \quad (2.155) \]

\( B^J \) is the real diagonal boundary value matrix. An element of the \( L \) matrix is

\[ L_e = \left( \begin{array}{c} \rho_e \Theta'_e \\ \Theta_e \end{array} \right) \right|_{e \in \mathcal{A}_e} = S_e + i P_e, \quad (2.156) \]

- in terms of elements of the real diagonal shift and penetrability matrices. Eq. (2.152) gives the collision matrix in terms of the real quantities \( \chi_e, \, \beta_e, \, \omega_e, \, S_e, \) and \( P_e. \) Although these quantities depend on the parameters \( \alpha_e \) and \( \beta_e, \) the \( U \) matrix, which is a property of the physical system, is independent of these parameters. The \( U \) matrix is symmetric and unitary - these results follow from time reversal properties and conservation of probability.

To relate the \( R \) matrix to reaction cross sections, which essentially depend on \( |U_e|^2, \) it is necessary to invert the channel matrix \( (1 - R L_0), \) of dimension equal to the number of channels. Useful general expressions for the
inverse cannot be given explicitly. However, it is possible to transform the matrix for \( U \) from its present form involving the inversion of a channel matrix, the components of which refer to channels \( c \), to one involving the inversion of a level matrix, the components of which refer to the proper levels, \( \lambda \), of the system.

Before adopting this procedure, we note that it may be possible to invert \((1-B^0L^0)\), where \( R^0 \) is a part of \( R \), i.e.,

\[
R = R^0 + R'.
\]  

(2.157)

The problem then reduces to that of inverting \((1-R'L')\). The superscript \( J \) has been dropped for economy of notation. So that the matrix product, \((1-BL^0)^{-1}\) of Eq. (2.154), becomes

\[
(1 - R^0 L^0)^{-1} R = (1 - R^0 L^0)^{-1} R^0 +
\]

\[
+ (1 - R^0 L^0)^{-1} (1 - R'L')^{-1} R' (1 - L' R')^{-1},
\]

(2.158)

where \( L' \) is defined

\[
L' = L^0 (1 - R^0 L^0)^{-1}.
\]

(2.159)

The inversion procedure is to assume an expansion

\[
(1 - R'L')^{-1} = 1 + \sum_{\mu\nu} (\xi_{\mu} \times \beta_{\nu}) A_{\mu\nu},
\]

where
\[ \beta_{\nu} = \gamma_{\nu} \]  
(2.161)

and \( A_{\mu \nu} \) is referred to as the level matrix (to be determined). The sum in Eq. (2.160) extends only over those levels contained in \( B' \). The cross denotes an outer product of channel column vectors. Equations for the elements of this level matrix may be found by multiplying both sides of Eq. (2.160) by the inverse of \((1-B'L')^{-1}\), viz.,  
\[ 1 - R_{\nu}' \xi = 1 - \sum_{\lambda} \frac{\sigma_{\lambda} \times \beta_{\lambda}}{E_{\lambda} - E} . \]  
(2.162)

Eq. (2.162) follows directly from the definition of \( R \) and \( R' \), Eqs. (2.141) and (2.157); the sum over levels is restricted to those contained in \( R' \). The equation to be satisfied by the level matrix is  
\[ (1 - R_{\nu}' \xi)(1 - R_{\nu}' \xi)^{-1} = 1 = 1 - \sum_{\lambda} \frac{\sigma_{\lambda} \times \beta_{\lambda}}{E_{\lambda} - E} + \sum_{\mu \nu} (\gamma_{\mu} \times \beta_{\nu}) A_{\mu \nu} - \sum_{\mu \nu \lambda} \frac{(\gamma_{\mu} \times \beta_{\lambda})}{(E_{\lambda} - E)} \cdot (\gamma_{\mu} \times \beta_{\nu}) A_{\mu \nu} . \]  
(2.163)

The matrix product in the last term may be simplified  
\[ (\sigma_{\lambda} \times \beta_{\lambda})(\gamma_{\mu} \times \beta_{\nu}) = \sigma_{\lambda} \beta_{\lambda}^{T} \gamma_{\mu} \beta_{\nu}^{T} = \]  
\[ = \sigma_{\lambda} \delta_{\lambda \mu} (\beta_{\nu}^{T} - \delta_{\lambda \nu} (\sigma_{\lambda} \times \beta_{\nu}) , \]  
(2.164)
where

\[ \xi_{\lambda \mu} = (\beta_{\lambda}^T \beta_{\mu} = (\beta_{\lambda}, \beta_{\mu}) \right. \]  
(2.165)

is a channel scalar which is symmetric in \( \lambda \mu \) because the matrix \( L' \) is symmetric (the inverse of a symmetric matrix is symmetric). One is left with,

\[- \sum_{\lambda} \frac{\gamma_{\lambda} \times \beta_{\lambda}}{E_{\lambda} - E} + \sum_{\lambda \nu} (\gamma_{\lambda} \times \beta_{\nu}) A_{\lambda \nu} - \]
(2.166)

\[- \sum_{\lambda \nu} \frac{\gamma_{\lambda} \times \beta_{\nu}}{E_{\lambda} - E} \sum_{\mu} \xi_{\lambda \mu} A_{\mu \nu} = 0.\]

The common term may be factored to give,

\[ \sum_{\lambda \nu} (\gamma_{\lambda} \times \beta_{\nu}) \left[ \frac{\delta_{\lambda \nu}}{E_{\lambda} - E} + A_{\lambda \nu} - \sum_{\mu} \xi_{\lambda \mu} A_{\mu \nu} \right] = 0. \]  
(2.167)

which is satisfied for all \( \lambda \nu \) if

\[ (E_{\lambda} - E) A_{\lambda \nu} - \sum_{\mu} \xi_{\lambda \mu} A_{\mu \nu} = \delta_{\lambda \nu}. \]  
(2.168)

Eq. (2.168) may be written as a matrix equation,

\[ A = (e - E - \xi)^{-1}, \]  
(2.169)

where the real diagonal matrix \( e \) has components \( e_{\lambda} \) and the matrix \( E \) is the energy times the unit matrix. \( A \) is symme-
The quantity in Eq. (2.158) to be evaluated is

\[(1 - R^\dagger R)^{-1} R^\dagger = \left[1 + \sum_{\mu \nu} (\gamma_\mu \times \beta_\nu) A_{\mu \nu} \right] \left[ \sum_{\lambda} \frac{\sigma_\lambda \times \sigma_\lambda}{E_\lambda - E} \right] \]

where Eq. (2.164) was used in the last step. By applying Eq. (2.168) to the sum over \( \nu \), Eq. (2.170) becomes,

\[(1 - R^\dagger R)^{-1} R^\dagger = \sum_{\lambda} \frac{\sigma_\lambda \times \sigma_\lambda}{E_\lambda - E} + \sum_{\mu \lambda} \frac{(\gamma_\mu \times \sigma_\lambda)}{E_\lambda - E} A_{\mu \lambda}

\[\left\{ (E_\lambda - E) A_{\lambda \mu} - \delta_{\lambda \mu} \right\}, \]

where the level symmetry of \( q \) and \( A \) has been used. Eq. (2.171) is

\[(1 - R^\dagger R)^{-1} R^\dagger = \sum_{\mu \lambda} (\gamma_\mu \times \sigma_\lambda) A_{\lambda \mu}. \]
\[ \alpha_\gamma = (1 - R^0 L^0)^{-1} \delta \gamma . \] (2.174)

The value and derivative quantities, Eqs (2.149) and (2.150), may be expressed in terms of the level matrix. When there is a unit flux area incoming wave in only one channel then,

\[ \gamma_c = \delta \gamma_c , \quad \chi_c = - U \gamma_c . \] (2.175)

By substituting these in Eq. (2.149), one obtains,

\[ V = \left( \frac{k}{2} \right) \gamma_c \left[ - \frac{\rho \gamma_c}{\rho \gamma_c} + \frac{\rho}{\rho \gamma_c} \right] . \] (2.176)

Using the original form of the collision matrix, i.e.,

\[ U = \rho \gamma_c \delta^{-1} (1 - R L^0)^{-1} (1 - R L^0*) I \rho^{-\gamma_c} , \] (2.177)

one finds on direct substitution of Eq. (2.177) in Eq. (2.176) and recalling that diagonal matrices commute that

\[ V = \left( \frac{k}{2} \right) \gamma_c \left[ - (1 - R L^0)^{-1} (1 - R L^0*) I \rho^{-\gamma_c} + \rho^{-\gamma_c} I \right] . \] (2.178)

Since,

\[ (1 - R L^0)^{-1} (1 - R L^0*) = (1 - R L^0)^{-1} (1 - R L^0 + 2i \kappa P) = 1 + 2i (1 - R L^0)^{-1} R P , \] (2.179)
Eq. (2.178) becomes,
\[ V = \left( \frac{k}{\pi} \right)^{1/2} \left[ - \pi \rho^{-\gamma} - \pi (1 - \mathbf{R}\mathbf{L}^0)^{-1} \mathbf{P} \mathbf{I} \rho^{-\gamma} + \pi \rho^{-\gamma} \right] , \]  
(2.180)

or,
\[ V = -i \sqrt{\pi \kappa} (1 - \mathbf{R}\mathbf{L}^0)^{-1} \mathbf{P} \rho^{\gamma} \mathbf{J} , \]  
(2.181)

where
\[ \mathbf{P} \rho^{\gamma} \mathbf{J} = \frac{\rho^{\gamma}}{\mathbf{I} \rho^{\gamma} \mathbf{O}^{\gamma}} \frac{\mathbf{I} \rho^{\gamma}}{\mathbf{O}^{\gamma}} = \mathbf{P} \mathbf{I} \rho^{-\gamma} . \]  
(2.182)

The expression for the derivative quantity is, from Eqs. (2.150), (2.175) and (2.177),
\[ \mathbf{D} = \left( \frac{k}{\pi} \right)^{1/2} \left[ - \rho^{\gamma} \mathbf{O} \rho^{\gamma} \mathbf{O}^{-1} (1 - \mathbf{R}\mathbf{L}^0)^{-1} (1 - \mathbf{R}\mathbf{L}^0^*) \right] \mathbf{J} \rho^{-\gamma} + \mathbf{I} \rho^{\gamma} \mathbf{J} , \]  
(2.183)

From Eq. (2.179) and the defining relation for the logarithmic derivative matrix, Eq. (2.156), one obtains,
\[ \mathbf{D} = \left( \frac{k}{\pi} \right)^{1/2} \left[ - \mathbf{L} (1 + \lambda i (1 - \mathbf{R}\mathbf{L}^0)^{-1} \mathbf{P} \mathbf{I} \rho^{-\gamma} + \mathbf{I} \rho^{\gamma} \mathbf{J} . \right] \]  
(2.184)

From Eqs. (2.135) and (2.181),
The term $[I'O'-O'I]$ in Eq. (2.185) is $-2i$ by virtue of the Wronskian relation for positive energy channels. Using Eq. (2.182), Eq. (2.185) may be written,

$$\Delta^0 = -i \frac{\delta}{\delta \alpha} \left[ (1 - R^0) \Gamma^0 \rho \right]^+ \quad (2.186)$$

The $V$ and $D^0$ quantities are now considered as matrices, rather than as column vectors, with each column referring to a different entrance channel. Upon substituting Eqs (2.158) and (2.172) into Eq. (2.186), the derivative matrix becomes,

$$\Delta^0 = -i \frac{\delta}{\delta \alpha} \left[ 1 + L^0 (1 - R^0) \Gamma^0 \right] P^0 Y - \Omega \quad (2.187)$$

The first two terms in brackets may be summed to give

$$1 + \frac{L^0 R^0}{1 - R^0 L^0} = \frac{1}{1 - R^0 L^0} \quad (2.188)$$

This follows from the symmetry of the $R$ matrix, i.e., $L^0 R^0 = R^0 L^0$. From definitions, Eqs (2.161) and (2.174),

$$L^0 \alpha^\mu = \beta^\mu \quad (2.189)$$
so that,

\[ D^0 = \frac{1}{2\pi} \left[ (1 - \mathbf{R}_0^0 \mathbf{L}_0^0) ^{-1} + \sum_{\mu\nu} (\mathbf{a}_\mu \times \mathbf{b}_\nu) A_{\mu\nu} \right] . \]  

From Eq. (2.137), the internal wave function may be expressed in terms of the channel scalar product of the vector \( \xi_\lambda \) and the derivative matrix \( D_0 \), i.e.,

\[ A_\lambda = \left( \mathcal{E}_\lambda - \mathcal{E} \right)^{-1} \langle \xi_\lambda, D_0 \rangle . \]  

For unit incoming flux area in any given channel \( e \) in the internal region, the derivative matrix becomes a column vector, i.e.,

\[ \langle \xi_\lambda, D_0^e \rangle = \frac{1}{2\pi} \left\{ \langle \xi_\lambda, (1 - \mathbf{R}_0^0 \mathbf{L}_0^0)^{-1}_e \rangle + \sum_{\mu\nu} \langle \xi_\lambda, (\mathbf{b}_\mu \times \mathbf{a}_\nu)_e \rangle A_{\mu\nu} \right\} P_e \Omega_e . \]  

Define a column vector \( \mathbf{f} \) with zeros everywhere except for 1 in the \( e \)th row. Also, define a matrix \( N \)

\[ N = (1 - \mathbf{R}_0^0 \mathbf{L}_0^0)^{-1} = \frac{1}{\mathbf{L}_0^0} , \]  

so that the quantity in braces in Eq. (2.192) may be written,
\[\left( \gamma_2, N \bar{x} \right) + \sum_{\mu \nu} \left( \gamma_2, (\beta_\mu \times \alpha_\nu) \bar{x} \right) A_{\mu \nu} \] (2.194)

The first term is

\[\left( \gamma_2, N \bar{x} \right) = \left( N^T \gamma, \bar{x} \right) = \left( N \gamma, \bar{x} \right). \] (2.195)

The second term of eqn (2.194) is

\[\left( \gamma_2, (\beta_\mu \times \alpha_\nu) \bar{x} \right) = \gamma_2^T \beta_\mu \alpha_\nu \bar{x} = \left( \beta_\mu \gamma_2 \right)^T \alpha_\nu \bar{x}. \] (2.196)

But, \( \beta_\mu \gamma_2 \) is the channel scalar \( \beta_\mu \gamma_2 \). So,

\[\left( \gamma_2, (\beta_\mu \times \alpha_\nu) \bar{x} \right) = \gamma_\mu \lambda \left( \alpha_\nu, \bar{x} \right), \] (2.197)

where \( \alpha_\nu = N \gamma_\nu \). The sum in Eq. (2.194) may be written,

\[\sum_{\nu} \left( \alpha_\nu, \bar{x} \right) \sum_{\mu} \gamma_\mu \lambda \ A_{\mu \nu} = \sum_{\nu} \left( \alpha_\nu, \bar{x} \right) \left\{ (\xi_\nu - \xi) \ A_{\nu \nu} - \delta_{\nu \nu} \right\}. \] (2.198)

from Eq. (2.168). So that Eq. (2.192) may be written,

\[\left( \gamma_2, \beta_\nu \right) = -i \sqrt{\epsilon} \left\{ \left( \alpha_\nu, \bar{x} \right) + \sum_{\nu} \left( \alpha_\nu, \bar{x} \right) \right\} \left( \xi - \xi \right) A_{\nu \nu} - \left( \alpha_\nu, \bar{x} \right) \right\} \beta_{\nu} \Omega_e \] (2.199)

or,
\begin{equation}
(\gamma, \delta) = -i \frac{\hbar}{2} \int \frac{d^3k}{(2\pi)^3} \sum_n \int \frac{d^3l}{(2\pi)^3} \left( \frac{\omega_{nl}}{\omega_{n'l'}} \right) (E_{nl} - E_{n'l'}) \cdot \mathbf{A}_{2n} \cdot \mathbf{P}_{e'}. \tag{2.200}
\end{equation}

From Eqs. (2.191) and (2.134), it follows that, for unit incoming flux area in channel \( e \),

\begin{equation}
\Phi_e = -i \frac{\hbar}{2} \sum \omega_{ne} A_{2n} A_{2n}^2 \mathbf{A} \mathbf{P}_{e} \mathbf{X}_e. \tag{2.201}
\end{equation}

If \( R^o = 0 \), so that \( \omega_{ne} = \gamma_{ne} \), then

\begin{equation}
\Phi_e = -i \frac{\hbar}{2} \sum \gamma_{ne} A_{2n} \mathbf{P}_{e} \mathbf{X}_e. \tag{2.202}
\end{equation}

where

\begin{equation}
\gamma_{ne} = 2 \mathbf{P}_{e} \mathbf{X}_e. \tag{2.202a}
\end{equation}

A particularly straightforward method of arriving at the collision matrix \( U \) in terms of the level matrix \( A \) is to evaluate the value quantity \( V_c \) for the internal wave function \( \Phi_e \) and equate it to the corresponding value quantity for an external wave function. From Eq. (2.202),

\begin{equation}
V_c = \left( \frac{\hbar}{2 \omega_{na}} \right) \gamma_{ne} (\omega_{na}) \Phi_e = -i \frac{\hbar}{2} \sum \gamma_{ne} A_{2n} \mathbf{P}_{e} \mathbf{X}_e. \tag{2.203}
\end{equation}

where the defining relation, Eq. (2.137), for the reduced level widths has been used. The value quantity for the external wave, Eq. (2.149), is given by
The second term on the right hand side of Eq. (2.204) is zero, so that,

\[ V_e = \left( \frac{k_e}{\pi} \right)^{1/2} \left\{ - \frac{U_{ee} \Omega_e}{\rho_e^{1/2}} + \frac{T_c}{\rho_e^{1/2}} \delta_{e,e} \right\} \]  

(2.204)

or,

\[ U_{ee} = \sum_{\lambda \nu} \Gamma_{\nu e} \Lambda_{\lambda \nu} \Omega_e \frac{\rho_e^{1/2} \delta_{\lambda \nu}}{\Omega_e}, \]  

(2.205)

or,

\[ U_{ee} = i \sum_{\lambda \nu} \Gamma_{\nu e} \Lambda_{\lambda \nu} \Omega_e \rho_e \Gamma_{\lambda \nu}. \]  

(2.206)

The background \( B \) matrix is zero in this expression.

The digression will be completed after developing a relation between the square integral of the wavefunction given by Eq. (2.201) over the internal region and the partial widths. The square integral of the internal region is a measure of the probability of all particles being together in a compound system of nuclear dimensions. This is a purely formal quantity in that nuclear interactions are not necessarily implied. The square integral, for unit incoming flux area in channel \( e \), in terms of the level matrix, is, from Eq. (2.201)

\[ \Sigma \chi_{e}^{2} \lambda \nu = \text{det} \left\{ \sum_{\lambda \nu} \alpha_{\lambda \nu} A_{\lambda \nu} \rho_{\lambda \nu}^{1/2} \chi_{\lambda \nu} \right\}^{2}. \]  

(2.207)

Due to the orthonormality of the eigenfunctions \( \chi_{\lambda} \), in the
internal region, Eq. (2.207) may be written,

\[ \int_0^\infty \left| \psi_e \right|^2 d\tau \, = \, 2 \pi \rho_e \sum_\lambda \overline{C}_{\lambda e} C_{\lambda e}^*, \quad (2.208) \]

where generalized radial limits on the integral are used to indicate that only the internal region is involved, and

\[ C_{\lambda e} = \sum_\nu A_{\lambda \nu} \alpha_{\nu e}. \quad (2.209) \]

The value quantity for the internal wave function, from Eq. (2.201) is,

\[ |\psi_e|^2 = 2 \pi \rho_e \sum_\lambda \overline{C}_{\lambda e} C_{\lambda e} \sum_{\lambda'} \overline{C}_{\lambda'e} C_{\lambda'e}. \quad (2.210) \]

Define an energy dependant quantity:

\[ \Theta_{\epsilon}^{-2} = \frac{\mu_e a_e^2}{\hbar^2} \sum_\lambda \overline{\gamma_{\lambda e}} C_{\lambda e} \sum_{\lambda'} \overline{\gamma_{\lambda'e}} C_{\lambda'e}, \quad (2.211) \]

so that Eq. (2.207) may be written,

\[ \int_0^\infty \left| \psi_e \right|^2 d\tau \, = \, \frac{\mu_e a_e^2}{\hbar^2} \frac{|\psi_e|^2}{\Theta_{\epsilon}^{-2}}. \quad (2.212) \]

As always, the expansion of the internal wave function on surfaces \( S_e \) in terms of the two body surface wave functions (complete in \( \Theta, \Psi \) and channel spin functions) assumes that a general n-cluster configuration may be
expanded in a series of two-cluster configurations.

It is of some interest to examine Eq. (2.211) for negative energy channels. For negative energy channels, the logarithmic derivative quantity \( L \) is real, i.e., for charged particles in some channel \( c \), it is

\[
L_c = \rho_c \frac{W_c'(x, p_c)}{W_c(x, p_c)}.
\]  

Consequently, one may choose the boundary condition \( B_c \) equal to \( L_c \), i.e., \( L^0=0 \). Since \( L^0 \) is the quantity which occurs as a multiplier of \( R^0 \) in the preceding equations, the presence of a background \( R \) matrix is eliminated with this choice. Therefore, the background \( R \) matrix may be chosen to contain all but one level; leaving only one level in \( R' \). With only one level, Eq. (2.211) becomes,

\[
\Theta_c^2 = \frac{\mu_c a_c^2}{\xi_c^2} \Theta_c^2,
\]  

an energy independent quantity. From any other perspective these are obvious conclusions, i.e., if one imposes the correct boundary conditions, one will get a correct solution. One further thing to be noted for negative energy channels is that, since \( W_c = V_c \), the value quantity for bound states is real. These are not general conclusions; it is easy to see that for multiple bound states of the same spin and parity, the channel quantity \( \Theta_c \) will be
shared over these multiple bound states. But for many light nuclei, there are few bound states for most channels. The specialization to light nuclei is made here.

D. REACTION WAVEFUNCTION

The initial compound system wave function in the internal region is, with no photons, from Eq. (2.202),

$$\Psi_{n}^{(w)} = -i \sum_{\lambda,\mu} \Lambda_{\lambda \mu} \sum_{\lambda} \chi_{\lambda}^{(0)}$$

where the sum over levels ($\lambda, \mu$) is complete in this expression, i.e., there are no levels in the background $R$ matrix ($R^0=0$).

If one defines the photon partial width amplitude and phase factor:

$$\Omega_{\mu, b} \Gamma_{\lambda, b} = \frac{2 \pi}{\hbar} \langle \Psi_{b}^{(w)} | H_{\text{int}} | \chi_{\lambda}^{(0)} \rangle$$

then the collision matrix, Eq. (2.120), may be written in analogy with particle channels as

$$U_{\mu b} = i \Omega_{\mu b} \sum_{\lambda \mu} \Lambda_{\lambda \mu} \Gamma_{\mu e} \Gamma_{\lambda b}$$

if the internal region is large enough to include all significant contributions to the matrix element, Eq. (2.120). If the operator $H(\text{int})$ in Eq. (2.216) transforms under time
reversal as,

\[ \mathbf{K} \mathbf{H}_{\text{int}} \mathbf{K}^{-1} = (-)^{l_{\text{int}}} \mathbf{H}_{l_{\text{int}}} \]  \hspace{1cm} (2.218)

and the wave functions in accordance with Eq. (2.4), then the matrix element in Eq. (2.216) is real. It is always possible to choose these phases so that Eqs. (2.4) and (2.218) are satisfied. Consequently, the phase, \( \varphi_{\text{b}} \), may be set equal to 1 without loss in generality.

If the particle channel surfaces are chosen in the normal way, however, the external region of configuration space may contribute to the electromagnetic transition matrix elements, i.e.,

\[ \mathbf{U}_{\text{be}} = \mathbf{U}_{\text{be}}^{(\text{ent})} + \mathbf{U}_{\text{be}}^{(\text{ext})} \]  \hspace{1cm} (2.219)

or,

\[ \mathbf{U}_{\text{be}} = -\mathbf{U}_{\text{be}}^{(\text{int})} \left\{ \sum_{a} (\mathbf{S}_{\text{be}}^{(u)} \mathbf{H}_{\text{int}} \mathbf{S}_{\text{be}}^{(v)} + \mathbf{S}_{\text{be}}^{(u)} \mathbf{H}_{\text{int}} \mathbf{S}_{\text{be}}^{(v)} + \sum_{a} \mathbf{S}_{\text{be}}^{(u)} \mathbf{H}_{\text{int}} \mathbf{S}_{\text{be}}^{(v)} \right\} \]  \hspace{1cm} (2.220)

Thus, quite generally, terms occur in the collision matrix element in addition to the usual dispersion sum. It will be shown that these terms, arising from integration in the entrance channel, are largest for those final states which have strong channel components such as single particle states.
At this point it is worthwhile to consider the normalization of the bound state, i.e., the nucleon part of $\Phi^b$. With a bound state, there is no difficulty in imposing the condition that there is unit probability of the system of $A$ nucleons being inside the internal region or outside the internal region. This condition is,

$$\int_0^\infty |\psi_b|^2 \, d\tau + \int_0^\infty |\psi_e|^2 \, d\tau = 1 \ . \ (2.221)$$

If the system extends outside the internal region, it may reside in a number of channels (say, $N$ channels). From Eqs. (2.212) and (2.214), the first term on the left hand side of Eq. (2.221) may be written

$$\frac{m_c a_c^2}{\tau_c^2} \frac{\nu_c^2}{\nu_c^2} + \sum_a \frac{m_c a_c^2}{\tau_c^2} \frac{\nu_c^2}{\nu_c^2} = 1 \ . \ (2.222)$$

There is one bound state of a given spin and parity; so that the value quantity is real. The external part of the bound wave function may be expanded in terms of negative energy outgoing channel wave functions, i.e.,

$$\psi_b^\text{ext} (\vec{r}) = \sum_c N_c \langle \Omega_c | \frac{\Omega_c^-}{\Omega_c^-} \rangle \ . \ (2.223)$$

For each channel, the value quantity of the interior will equal the value quantity of the exterior, viz.,
\[ (\mathbf{Q}_c, \mathbf{\Phi}_p^t) = V_c \left( \frac{\partial w_c \mathbf{a}_c}{\partial \mathbf{e}_c} \right) \mathbf{v}_c = \mathbf{N}_c \frac{O_c(w_c)}{\mathbf{v}_c} \quad (2.224) \]

So, \( \mathbf{N}_c \) may be eliminated in favor of \( V_c \), giving

\[ \mathbf{\Phi}_p^t = \sum_c \left( \frac{\partial w_c \mathbf{a}_c}{\partial \mathbf{e}_c} \right) V_c \frac{O_c(w_c)}{\mathbf{v}_c} \mathbf{e}_c \quad (2.225) \]

The normalization condition, Eq. (2.222), may be written

\[ \frac{\mathbf{m}_c a_c^2}{\mathbf{v}_c^2} + \sum_c \frac{\partial w_c \mathbf{a}_c}{\partial \mathbf{e}_c} V_c^2 \int_{a_c}^{\infty} \mathbf{E}_c^2 \mathbf{d}r = 1 \quad (2.226) \]

where orthogonality of the external functions has been used, and \( E_c (\mathbf{r}) = O_c (\mathbf{r}) / O_c (a_c) \). The first term on the left hand side of Eq. (2.226) is specified in terms of an arbitrary channel \( c \). The square integral over the interior region, Eq. (2.212), could equally well be written in terms of any of the other possible channels - giving an additional \( N-1 \) equations:

\[ \frac{\mathbf{m}_1 a_1^2}{\mathbf{v}_1^2} + \sum_c \frac{\partial w_1 a_1}{\partial \mathbf{e}_1} V_1^2 \int_{a_1}^{\infty} \mathbf{E}_1^2 \mathbf{d}r = 1 \quad (2.227) \]

These equations may be used to eliminate the value quantity in favor of \( \mathbf{v}_c \). The result is,
Eq. (2.225) may be reexpressed as

\[ \psi_b^{\text{int}} = \sum_{c} \left( \frac{2 \Theta_c^2}{\alpha_c} \right)^{1/2} \sqrt{E_c(r)} \sqrt{\psi_c^{\text{int}}} \left\{ 1 + \sum_c \frac{2 \Theta_c^2}{\alpha_c} E_c^2 \right\} \]

There is ambiguity here of only a plus or minus sign, which may be absorbed in \( \Theta_c \). With the aid of Eq. (2.228) the square integral for bound states becomes

\[ \int_0^a \psi_b^{\text{int}} \psi_b^{\text{int}} \ dx = \left\{ 1 + \sum_c \frac{2 \Theta_c^2}{\alpha_c} E_c^2 \right\}^{-1} \]

The internal collision matrix element \( U_{bc}^{\text{int}} \) of Eq. (2.219) is proportional to the width of the bound state in any channel, i.e.,

\[ \Theta_c^2 = \frac{\mu_c a_c}{k^2} \frac{V_c}{S_0} \]

If the internal region is sufficiently large, the integral in the denominator of Eq. (2.231) is 1. Even if this integral is less than 1, one may retain the same explicit dependence of the bound state partial width on the channel radius if the partial widths are renormalized as,

\[ \frac{\Theta_c^2}{\left\{ 1 + \sum_c \frac{2 \Theta_c^2}{\alpha_c} E_c^2 \right\}} = \frac{\mu_c a_c}{k^2} V_c \]
from Eq. (2.230). Thus, $U^{(\text{int})}$ should be renormalized with a factor $N_b^{y_2}$, where

$$N_b = \left\{ 1 + e \frac{3}{a_c} \int_0^\infty \delta^2 r d^3 \right\}^{-1}. \quad (2.233)$$

In an operational sense, there is little point in this renormalization, since the factor $N_b^{y_2}$ may be absorbed in the photon partial width. In fact, a renormalizing term is omitted in Ref. (13). Following the suggestion of R.G. Thomas, the internal collision matrix will include the renormalizing factor:

$$U_{b \ell}^{(\text{int})} = N_b^{y_2} i \Omega_e \sum_{\lambda \mu} A_{\lambda \mu} \Gamma_{\mu e}^{y_2} \Gamma_{\lambda b}^{y_2}. \quad (2.234)$$

All (implicit) spatial integrations in Eq. (2.234) are limited to the interior volume. It will be seen that the renormalizing term establishes an upper limit on the size of the external contribution, for a given channel radius.

The remaining term in the total collision matrix element, i.e.,

$$U_{b \ell}^{(\text{ext})} = - \sqrt{\frac{\pi^2 a_c}{\hbar}} \int_a^\infty \mathcal{I}_b^{(1)} h_i \mathcal{I}_c^{(0)} d\tau, \quad (2.235)$$

corresponds to transitions in the external region. The bound state wave function in the external region will be that part of Eq. (2.229) corresponding to the nucleon con-
figuration of the incident channel (e.g., proton and target). The initial wave function for a specified nucleon configuration in the external region will be the most general stationary free scattering state, specialized to no reactions on the reasonable assumption that external transitions are only significant in channels in which there are incident waves.

An initial scattering wave function (distorted plane wave) in the external (asymptotic) region of total angular momentum $J$ and projection $M$ may be written

$$
\Phi_{J^m} = \sum_{\epsilon_c'} \left( \delta_{\epsilon_c, \epsilon_c'} \Phi_{J^m}^{\epsilon_c'} - \epsilon_{c_c'} \Theta_{c_c'}^{J^m} \right) y_{\epsilon_c'}^{J^m} \quad (2.236)
$$

$\delta_{\epsilon_c, \epsilon_c'}$ and $\Theta_{c_c'}^{J^m}$ are the channel wave functions of Eqs. (2.142) and (2.143) vector coupled to form states of good $J$ and $M$, i.e.,

$$
\Phi_{J^m}^{\epsilon_c} = \sum_{J^m, \nu} \langle \epsilon_{\nu}, \epsilon_{\nu}' | J^m \rangle \frac{i \nu y_{\nu}^{J^m}}{\sqrt{W_{\nu}}},
$$

and similarly for $\Theta_{c_c'}^{J^m}$. The coupling order follows Ref. (4). The term $y_{\epsilon_c}^{J^m}$ is chosen so that when

$$
\epsilon_{c_c'} \rightarrow e^{i \omega_c} \delta_{c_c'},
$$

(2.237)

corresponding to point Coulomb scattering, the function $\Phi_{J^m}$ behaves as
which is the Coulomb scattered wave for unit incident particle flux along the z axis. $P_n(\theta)$ is the ordinary Legendre polynomial. This requires that

$$\psi_c = \frac{i}{\hbar} \sum_{J_m} \sum_{s' n'} \langle \ell^0, s' n' | J_m \rangle \psi_{J_m}.$$  

Eq. (2.236) may be written,

$$\sum_{J_m} \sum_{s' n'} \langle \ell^0, s' n' | J_m \rangle \psi_{J_m} = \frac{1}{\sqrt{n}} \sum_{s' n'} \sum_{J_m} \langle \ell, s' n' | J_m \rangle \psi_c \chi_{s' n'} \chi_{s' n'}.$$  

This is the most general scattering wave function. It reduces to the elastic scattering initial wave function of Ref. (16) almost immediately. In the case of Ref. (16), the scattering corresponds to spin 1/2 on spin 0, so that in cases such as this, where there is no possibility of channel spin change and parity conservation precludes a change in orbital angular momentum, there is no question that the initial channel will be the final channel, i.e., $\alpha ls = \alpha' l' s'$. In the general elastic scattering case, however, there will be channel spin changes and corresponding orbital angular momentum changes. These amplitudes will usually be small and may be neglected, as are the reaction...
channels. Thus, the initial wave function is taken to be,
\[
\psi^{J_m} = \frac{1}{\sqrt{2}} \sum_{\alpha \beta \sigma} \chi_{\alpha \beta \sigma}^{J_m} \otimes \phi_{\alpha \beta \sigma}^{J_m} i \hat{L} \phi_{\alpha \beta \sigma}^{J_m} \tag{2.241}
\]
\[x \left[ T_e - W_{O} e \right] \psi_{c}^{J_m}.
\]
Note that Eq. (2.241) may be easily generalized to include the outgoing waves corresponding to channel spin and orbital angular momentum changes in elastic scattering and all reaction channels. The inclusion of these additional outgoing waves would lead to direct external capture expressions involving bound states characterized by the channel specifications of these additional outgoing waves. For example, if a reaction channel were open [say, \(p, ^4\text{He}\)], an additional term to direct external capture of protons would be the direct external capture of alpha particles. In the case of spin 1/2 on spin 0, when the incident energy is low enough that reaction channels are not open (except for the usually very small damping of gamma emission), Eq. (2.241) is virtually exact.

A general reaction wavefunction, for channel \(c = \alpha ls\), is
\[
\psi_{\text{rec}}^{J_m} = \sum_{c'} \chi_{c'}^{J_m} O_{c'c}^{J_m}, \tag{2.242}
\]

where
\[
\chi_{c'}^{J_m} = -\left[ W_{c'c} \right]^{J_m} \psi_{c'}^{J_m}, \tag{2.243}
\]
and $\gamma_{c}^{J_{m}}$ is defined in Eq. (2.239). Eq. (2.242) becomes,

$$T_{\text{rec}} = - \sum_{c} U_{c^*}^{\dagger} y_{c}^{J_{-m}} \Theta_{c^*}^{J_{m}}, \quad (2.244)$$

where $c^*=LpB$ and $c=ls$. $\phi_{c}^{J_{m}}$ is the spatial representation of the photon wave function, Eq. (2.2).

Taking that part of the bound state wavefunction, Eq. (2.229), corresponding to the nucleon configuration of the incident channel ($\alpha = \alpha'$) - any one photon operator will not cause a shuffling among the available nucleons - and vector coupling the constituents to form a state of total angular momentum $B$ and projection $M$, one has,

$$\gamma_{c}^{J_{m}} = \sum_{\alpha s'\nu'} c_{\alpha}^{\nu'} \langle \alpha s'\nu' | \beta_{m} \rangle N_{\nu'}^{\nu}. \quad (2.245)$$

On substituting Eqs. (2.241) and (2.245) into the defining relation, Eq. (2.120), for the general coefficient of the outgoing wave ($0_{c}^{J_{m}}$) for any specified weighting of the incoming waves ($y_{c}^{J_{m}}$), one obtains,

$$U_{c^*c}^{\alpha s} \gamma_{c}^{J_{m}} = - \sqrt{\frac{2}{\pi}} \sum_{\alpha s'\nu'} \int_{0}^{\infty} d\tau \int_{0}^{\infty} \langle \alpha s'\nu' | \beta_{m} \rangle N_{\nu}^{\nu} \left( \frac{2\hat{Q}_{c}^{\nu}}{a_{s'}} \right)^{\frac{1}{2}} \frac{G_{c}^{\nu}}{\tau} \langle \nu' | \gamma_{c}^{J_{m}} \Phi_{c}^{s'} \rangle H_{\nu}^{\nu}, \quad (2.246)$$

$$\langle \nu' \nu | \gamma_{c}^{J_{m}} \Phi_{c}^{s'} \rangle = \sum_{\alpha s'\nu'} \langle \nu' \nu | \gamma_{c}^{J_{m}} \Phi_{c}^{s'} \rangle \gamma_{c}^{J_{m}} \Phi_{c}^{s'}, \quad (2.246)$$
The critical point in this development is in defining an operator $H_{LAP}(r')$ which creates precisely the outgoing photon wavefunction $\psi_{\chi}^J(r)$. The relation between the operator and wavefunction is displayed most succinctly by the Green's function relation, Eq. (2.72). Eq. (2.246) may be reduced and examined.

The collision matrix element, Eq. (2.246), has the wrong exit channel to be compatible with Eq. (2.244). The remedy is to transform to a coupled representation,

$$U_{k|\ell} = \sum_{\lambda, \mu} \langle \lambda | \psi_{\lambda} \rangle | J_{\mu} \rangle U_{\mu|\ell} \langle \mu | \psi_{\mu} \rangle$$  \hspace{1cm} (2.247)

If the operator, $H_{LAP}(r')$ does not involve spin operations (it will be shown, in another context, that only the electric dipole operator, which does not depend on spin, gives a significant contribution in the external region), then, orthogonality of the channel spin functions, i.e.,

$$\langle \chi s' | \chi s \rangle = \delta_{s,s'} \delta_{\nu,\nu'}$$  \hspace{1cm} (2.248)

may be used to obtain

$$U_{k|\ell} = \sum_{\lambda, \mu} \langle \lambda | \psi_{\lambda} \rangle \langle J_{\mu} | \psi_{\mu} \rangle N_{\mu} \langle \mu | \psi_{\mu} \rangle$$

\( \sum_{s, \nu} \langle \chi s' | J_{\mu} \rangle \langle \chi s | J_{\nu} \rangle \left( 2 \Theta^2 \right)^{\nu - \frac{\nu'}{2}} \langle \mu | \psi_{\mu} \rangle \langle \nu | \psi_{\nu} \rangle \)

\( \sum_{s, \nu} \langle \chi s' | J_{\mu} \rangle \langle \chi s | J_{\nu} \rangle \left( 2 \Theta^2 \right)^{\nu - \frac{\nu'}{2}} \langle \mu | \psi_{\mu} \rangle \langle \nu | \psi_{\nu} \rangle \)

\( \sum_{s, \nu} \langle \chi s' | J_{\mu} \rangle \langle \chi s | J_{\nu} \rangle \left( 2 \Theta^2 \right)^{\nu - \frac{\nu'}{2}} \langle \mu | \psi_{\mu} \rangle \langle \nu | \psi_{\nu} \rangle \)

\( \frac{1}{\Omega} \oint (\overrightarrow{r} - \overrightarrow{r}_e - \overrightarrow{r}_o) \overrightarrow{r} \)
Eq. (2.249) is in implicit agreement with the corresponding equation in Ref. (3).

The Wigner-Eckart theorem may be used to reduce Eq. (2.249) and to show that the matrix element \( U_{S_L,0s} \) is in fact independent of projection quantum numbers, viz.,

\[
U_{S_L,0s} = \sum_{\lambda, m} \langle \lambda, m | S_L | \lambda' \rangle N_\lambda \left( \frac{4\pi \rho \Theta^2}{\kappa N V a_0^2} \right)^{1/2}.
\]

(2.250)

\[ \sum_{\lambda, m} \langle \lambda, m | S_L | \lambda' \rangle = \langle \lambda, m | S_L | \lambda' \rangle \delta_{\lambda\lambda'} \delta_{m'm} \] 

While this may seem contrived, the phase and minus sign in front of \( \hat{\Lambda} \) appear quite naturally in actual calculations just as a result of

\[ \gamma_{m}^{*} = (-)^{m} \gamma_{-m}. \]

(2.251)

The sum over \( m, m' \) and \( v \) is,

\[
\sum_{\lambda, m} \langle \lambda, m | S_L | \lambda' \rangle = (-)^{3-\delta_{\lambda'\lambda}} \delta_{m'm} \langle \lambda, m | S_L | \lambda' \rangle W(Te Le' ; s L).
\]

(2.252)

The sum over \( \lambda \) and \( \Theta \) may be done immediately, leaving

\[
U_{S_L,0s} = N_\Lambda \left( \frac{4\pi \rho \Theta^2}{\kappa N V a_0^2} \right)^{1/2} (-)^{3-\delta_{\lambda'\lambda}} \delta_{m'm} \langle \lambda, m | S_L | \lambda' \rangle W(Te Le' ; s L) \] 

(2.253)
Thus the external collision matrix is independent of $M$ and diagonal in $J$ and $M$.

At this point, the reaction wavefunction is

$$
\uparrow_{\text{reac}} = - \sum_{J_{\text{int}}} \sum_{L_{\text{int}}, S_{\text{int}}} \left[ U_{L_{\text{int}} S_{\text{int}} J_{\text{int}}} + U_{L_{\text{int}} S_{\text{int}} J_{\text{int}}} \right] \times \Omega_{L_{\text{int}} S_{\text{int}} J_{\text{int}}}
$$

(2.254)

$U_{L_{\text{int}} S_{\text{int}} J_{\text{int}}}$ and $U_{L_{\text{int}} S_{\text{int}} J_{\text{int}}}$ have been defined except for the form of the operator $H_{\lambda \mu \nu}$. This operator is determined by the normalization and form of the outgoing photon wavefunction $\omega_{\lambda \mu \nu}$.

$$
\Gamma_{\lambda \mu \nu}(\omega_{\lambda \mu \nu}) = \sum \int_{\lambda \mu \nu} \omega_{\lambda \mu \nu} \chi_{\lambda \mu \nu} \cdot \mathbf{A}_{\lambda \mu \nu}
$$

(2.255)

where $b=(L \rho \lambda \mu \nu \omega)$. And from Eq. (2.79),

$$
\Gamma_{\lambda \mu \nu}(\omega_{\lambda \mu \nu}) = i \left\{ \frac{8 \pi \hbar}{c^2} \right\} \int d^3 \mathbf{r} \mathbf{F} \cdot \mathbf{A}_{\lambda \mu \nu}
$$

(2.256)

This expression agrees with the corresponding equation in Ref. (3). Explicitly,

$$
\mathbf{A}_{\lambda \mu \nu} = \mathbf{i} \frac{L+1}{L(L+1)} J_{\lambda \mu \nu} \chi_{\lambda \mu \nu} = \mathbf{i} \frac{L+1}{L(L+1)} J_{\lambda \mu \nu} \chi_{\lambda \mu \nu}
$$

(2.257)

using the definition of Ref. (10). Since,

$$
\mathbf{A}_{\lambda \mu \nu} = \mathbf{i} \frac{\mathbf{D} \times \mathbf{A}_{\lambda \mu \nu}}{\hbar}, \quad \mathbf{F} \cdot \mathbf{D} \times \mathbf{A}_{\lambda \mu \nu} = \mathbf{A}_{\lambda \mu \nu} \cdot \mathbf{D} \times \mathbf{F}
$$

(2.258)

Thus, Eq. (2.256) may be written as...
The source expressions in Eq. (2.259) are given by the quantum mechanical analogues:

\[
\overline{J}_{ab} = \sum_i \frac{e_i}{2m_i} \left( \hat{r}_i \cdot \hat{p}_i - \hat{p}_i \cdot \hat{r}_i \right) X_a + \left( \hat{p}_i \cdot \hat{p}_i \right) X_a,
\]

(2.260)

\[
\bar{M}_{ab} = \sum_i \frac{e_i}{\alpha mc} g_i \frac{1}{2} \bar{S}_i \cdot X_a,
\]

(2.261)

where the sum is to be taken over all nucleons. \( \hat{p}_i = -i \hbar \nabla_i \) is the linear momentum operator for the \( i \)th nucleon; \( \bar{S}_i \) is the Pauli spin operator and \( g_i \) is the magnetic moment expressed in units of the Bohr magneton.

These expressions may be developed, through a series of partial integrations, into forms that are more convenient for numerical evaluations. Eq. (2.259) may be written

\[
\eta_{2}^{(2)} = \left\{ \frac{2\pi \hbar}{\alpha mc (L+1)} \right\}^{1/2} \left( H_{um} + H_{\bar{u}m}^* \right) \left( E_{nm} + E_{\bar{u}m}^* \right),
\]

(2.262)

where

\[
H_{um} = -2i \frac{e \hbar}{\alpha mc} \int d\mathbf{r} \left[ j_{le} (i \mathbf{k} \mathbf{r} \mathbf{m}^*) \bar{S}_l \cdot \left( \hbar \mathbf{k}^* \right) \right]
\]

(2.263)
Summation over all nucleons is understood. The magnitudes of the various terms are suggested by the coefficients of these terms. On this basis it is clear that only the first part of the principle electric term, Eq. (2.265) will usually give significant contributions. This question will be examined in some detail later. The operator of this term summed over two particles and referred to the center of mass system is

$$ e^i \frac{\partial}{\partial \varphi_i} \left[ e \sum_{\rho} \frac{1}{\hbar c} \left\{ d \chi \left( \gamma_{\rho} \hat{\sigma} \cdot \hat{\tau} \right) \chi \right\} \right] $$

where
\[ G_{\ell}(\nu) = 2 \left[ k_{2} \, \nu \, J_{\ell-1}(k_{2} \, \nu) - k_{1} \, J_{\ell}(k_{1} \, \nu) \right] + (2.268) \]

\[ + (\nu) \, 2 \left[ k_{1} \, \nu \, J_{\ell-1}(k_{2} \, \nu) - k_{2} \, J_{\ell}(k_{1} \, \nu) \right], \]

and

\[ k_{\nu}^2 = k_{2} \frac{m_{1}^2}{m}, \quad m = m_{1} + m_{2}, \quad (2.269) \]

and \( Z \) is the charge of the particle in units of \( e \). With this expression, the reduced matrix element in Eq. (2.253) may be evaluated. For electric dipole transitions, the result is

\[ \mathcal{M}(\text{ee}) = (\nu) \, N_{\ell}^2 \, \sum_{\ell^1} \frac{6 \, k_{2}^3 \, m \, e^2}{k^2 \, a \, h_{\nu}} \int \, \ell \, L \, B. \quad (2.270) \]

\[ \sum_{\ell^1} \left( \ell - \ell^1 \right) \theta_{\ell^1 \ell} \, R_{\ell \ell^1} \, \langle \ell \xi_{1} 0 | 0 | \ell^1 \rangle \, W(\ell \xi_{1} \ell^1; \psi). \]

where

\[ R_{\ell \ell^1} = \int_{\ell}^{0} \, d\ell \, E_{\ell^1} \, \frac{G_{\ell}}{h_{\nu}} \left( \ell \xi_{1} - \ell \xi_{1}^J \right). \quad (2.271) \]

E. REACTION AMPLITUDE

A reaction amplitude for photons may be developed by considering the asymptotic form of the photon wavefunction. From Eq. (2.26),

\[ \text{(Expression for reaction amplitude)} \]
\[ \begin{align*}
\vec{\Delta}_{Lm} & = -\frac{\alpha e^{i \theta}}{r} \sum_{l_m} s(l, m) \left< L - l, 1l \mid \ell 0 \right> \tag{2.272} \\
\end{align*} \]

All projections are on a common z axis \((\vec{k}_z)\). Using the Clebsch-Gordan series*,

\[ \delta_{c_c}^c (e) \left< A a', B b' \mid c - c' \right> = \sum_{\alpha b} \alpha^c \Delta_{\alpha b} (e) \delta_{aa'} (e) \delta_{bb'} (e), \tag{2.273} \]

the last two terms of Eq. (2.272) may be written,

\[ \left< A-a', B b' \mid c - c' \right> \Delta_{aa'} (e) \Delta_{bb'} (e), \tag{2.274} \]

where \(e = (0, -\theta, -\phi)\) is the rotation which takes the \(z'\) axis \((k' = z)\) direction) to the \(z\) axis \((\vec{k})\). On substituting Eq. (2.274) in Eq. (2.272), one finds that the sum over \(l\) may be completed. Using Eq. (2.27) and evaluating the resulting Clebsch-Gordan coefficients results in

\[ \sum_\ell s(l, m) \left< L - l, 1l \mid \ell 0 \right> \left< L - l, 1l \mid \ell 0 \right> = \frac{(b)^P}{2}. \tag{2.275} \]

Eq. (2.272) becomes,

\[ \begin{align*}
\vec{\Delta}_{Lm} & = -\frac{\alpha e^{i \theta}}{r} \sum_{\ell} s(l, m) \frac{b^P \ell}{2} \left( - \right)^{1+b} \frac{1}{\sqrt{\pi}} \\
& \quad \times \delta_{\alpha b} (e) \Delta_{aa'} (e) \delta_{bb'} (e). \tag{2.276} \\
\end{align*} \]

It is convenient to project the photon's spin quantum number on its direction of propagation \((z'\ axis)\), i.e.,
Replacing \( \hat{e}_q \) in Eq. (2.276) with Eq. (2.277) and performing the sum over \( q \),

\[
\sum_q \delta_{bq}(q) \delta_{e(q)}(q) = \delta_{be}, \quad (2.278)
\]

results in,

\[
\bar{O}_{ba} \rightarrow -e^{i k_0 r} \frac{1}{r} \sum_q \frac{1}{L} \hat{e}_q(q) \hat{A}_e^*(q). \quad (2.279)
\]

Thus, from Eqs. (2.2), (2.239) and (2.254), the reaction wavefunction has the asymptotic form

\[
\Phi_{\text{rec}} = -e^{-ik_0 r} \sum \sum \frac{1}{L} \hat{e}_e(q) \hat{A}_e(q) \Phi_{asv} i \frac{r}{k_0} \hat{A}_e L_e^{-1}(q) \Phi_{asv}^* \Phi_{\text{rec}}. \quad (2.280)
\]

A scattering amplitude is the coefficient of the normalized outgoing wave, i.e.,

\[
A_{e \theta m_0; sv} = \frac{1}{\sqrt{2 \pi \rho}} e^{-i k_0 r} \hat{A}_{asv} \hat{e}_e \Phi_{\text{rec}}. \quad (2.281)
\]

On substituting Eq. (2.280) in Eq. (2.281), one obtains

\[
A_{e \theta m_0; sv} = i \frac{1}{\sqrt{2 \pi}} \sum \frac{1}{L} \hat{e}_e(q) \hat{A}_e(q) \Phi_{asv} i \frac{r}{k_0} \hat{A}_e L_e^{-1}(q) \Phi_{asv}^* \Phi_{\text{rec}} \Phi_{\text{rec}}. \quad (2.282)
\]
Since there are no interactions in the asymptotic region which would prevent a decomposition of the channel spin function into its constituents, Eq. (2.282) may be written in the more convenient incident spin uncoupled form, viz.,

\[
A_{\ell m a; m' a'} = \frac{i \ell!}{n a} \sum_{J m} \langle \ell a_m, a' m' | s v \rangle \hat{L}^\ell
\]

(2.283)

\[
\times U_{\ell, s, a, b} \sum_{J m} \langle \ell a, b m a | j m \rangle \langle \ell, s, v | j m \rangle \frac{\hat{L}}{\hat{R}^n} e^{i \hat{L}} D_{j a b \ell}(R).
\]

Not surprisingly, an identical reaction amplitude may be found by starting with an ordinary particle reaction amplitude (e.g., that given in Ref. [3]); recoupling angular momenta in the exit channel from \( \ell - s = \ell' + b \) to \( \ell' + b \); and changing the basis functions of the reaction wavefunction. The particle basis functions, with any spin projection permitted on its propagation direction, are the ordinary outgoing spherical waves. Whereas, the basis functions for photons are transverse spherical waves, i.e., the outgoing wave vector potentials. The elements of the unitary transformation from the one set of basis functions to the other set are related to the Racah radiation parameters of Ref. [1].

F. REACTION OBSERVABLES

It is possible to form a general density matrix from
the coefficients of the outgoing transverse spherical waves, viz.,

\[
\rho_{\mathbf{m} \mathbf{m}'}^{\mathbf{m} \mathbf{m}'} = A_{\mathbf{m} \mathbf{m}'} A_{\mathbf{m} \mathbf{m}'}^*.
\] (2.284)

By definition, this quantity is a positive definite (eigenvalues non negative) Hermitian matrix. The reaction wave function is not normalized, so the expression for an expectation value of some operator, e.g., \(\sigma_y\), is

\[
A_y = \frac{\text{Tr}(\rho \sigma_y)}{\text{Tr}(\rho)}.
\] (2.285)

The quantity in the denominator, i.e.,

\[
\text{Tr}(\rho) = \sum_{\mathbf{m} \mathbf{m}'} |A_{\mathbf{m} \mathbf{m}'}|^2,
\] (2.286)

is recognized as being proportional to the differential cross section for the emission of a photon in some direction \(\hat{\mathbf{k}}_\gamma\) and polarization of the photon and residual nucleus not observed:

\[
\text{Tr}(\rho) = A^2 A^2 \frac{d\sigma(\hat{\mathbf{k}}_\gamma)}{d\Omega}.
\] (2.287)

All polarization observables are completely defined by Eqs (2.283) and (2.284). For some purposes it is necessary to establish the connection between the properties of the
incident projectile beam and the general differential cross section.

The system of particles forming the initial configuration (beam and target) are generally uncorrelated. So the initial density matrix may be expressed as a product of the individual density matrices, i.e.,

\[ \rho_{\text{initial}} = \rho_{\text{beam}} \rho_{\text{target}}, \]

where the individual density matrices are understood to be ensemble averages. Note that since the intermediate nuclear states are well defined, no further ensemble averaging is necessary. The representation in terms of which the initial density matrix is formed is normalized, so one obtains,

\[ \text{Tr}(\rho) = 1. \]

Eq. (2.289), coupled with the two postulates of equal a priori probabilities and random a priori phases leads to the form of a density matrix element for an unpolarized system of spin \( A \),

\[ \rho_{m_1, m_1'} = \frac{\delta_{m_1, m_1'}}{2A + 1}. \]

It is convenient to introduce the statistical tensors in
terms of which a density matrix may be expanded, viz.,

\[ \rho_{\alpha \beta \gamma \delta} = \frac{1}{\hat{\alpha}} \sum_{\alpha'} \langle \alpha' \alpha | a_{\alpha'} | a_{\alpha} \rangle \chi_{\alpha \beta \gamma \delta}^\dagger \chi_{\alpha' \beta' \gamma' \delta'} \]  \hspace{1cm} (2.291)

These statistical tensors transform on rotation of axis like the spherical harmonics \( Y_{\ell m} \) (cogrediently), i.e.,

\[ t'_{\ell m} = \sum_{\ell' m'} t_{\ell m} \delta_{\ell \ell'} \delta_{m m'} \]  \hspace{1cm} (2.292)

The principal advantage of introducing the statistical tensors is their behavior under rotation. The rotation of a density matrix requires two rotation matrix elements - the tensor requires one. The irreducible tensors of different ranks are not mixed by rotations, i.e., if all the components of a given rank are zero in one coordinate system, then they will remain zero in all coordinate systems. If the components of a given rank are not all zero (say, only \( t_{20} \) is non zero), then one may generate any component by rotations \( (t_{20} \rightarrow t_{22}, t_{24}, t_{22}, t_{20}, t_{2-2}, t_{2-2}) \), e.g., rotation with a Wien filter.

It is of some interest to examine the extent to which Eq. (2.291) is arbitrary. In looking for an irreducible representation of the density matrix operator, it is convenient to consider the way in which the density operator transforms under a rotation of axes, viz.,
\[ \rho_{s^m, s'^m} = \sum_{s_v, s'^v} \Delta_{s^m}^{s_v} \rho_{s_v, s'^v} \Delta_{s'^m}^{s'^v}. \quad (2.293) \]

The pair of rotation matrix elements may be reexpressed via the Clebsch Gordan series,
\[ \Delta_{s^m}^{s_v}(-)^{s'-m} \Delta_{s'^m}^{s_v} = (-)^{s'-m} \sum_{KQ} \langle s^v, s'-v | KQ \rangle \langle s^m, s'-m | KN \rangle \Delta_{KQ}^{KN}. \quad (2.294) \]

There is nothing discretionary about this expansion. However, eqn (2.294) may be written in another form, viz.,
\[ \Delta_{s^m}^{s_v} \Delta_{s'^m}^{s_v} = (-)^{s'-m} \sum_{KQ} \langle s^v, s'-v | KQ \rangle \langle s^m, s'-m | KN \rangle \Delta_{KQ}^{KN}. \quad (2.295) \]

where \( -q \rightarrow +q \) and \( -N \rightarrow +N \) and
\[ \Delta_{s^m}^{s_v} \Delta_{s'^m}^{s_v} = (-)^{m'-m} \Delta_{s^m}^{s_v} \Delta_{s'^m}^{s_v}. \quad (2.296) \]

have been used in eqn (2.295). On substituting both forms into Eq. (2.293) and using the orthogonality of the Clebsch Gordan coefficients, one obtains
\[ \sum_{s_v} \rho_{s^m, s'^m} (-)^m \langle s^v, s'-v | KN \rangle = \sum_{s_v} (-)^v \langle s^v, s'-v | KQ \rangle \rho_{s_v, s'^v} \Delta_{KQ}^{KN}. \quad (2.297) \]

and
\[ \sum_{s_v} \rho_{s^m, s'^m} (-)^m \langle s^v, s'-m | KN \rangle = \sum_{s_v} (-)^v \langle s^v, s'-v | KQ \rangle \rho_{s_v, s'^v} \Delta_{KQ}^{KN}. \quad (2.298) \]
Thus, one may define (temporarily) a contragredient statistical tensor,

$$\rho^{' \prime}_{p,q} = \sum_{s,s'} (-)^{s'} < s \mid s' \mid h_{p,q} > \rho_{s,v,s',v} \tag{2.299}$$

and a cogredient statistical tensor,

$$\rho^{' \prime \prime}_{n,b} = \sum_{s,s'} (-)^{s'} < s \mid s' \mid h_{n,b} > \rho_{s,v,s',v} \tag{2.300}$$

The tensor defined by Eq. (2.300) is Hermitian as may be seen by considering the definition of a Hermitian tensor operator,

$$T_{n,b} = (-)^{p-b} T_{n-b}^{+} \tag{2.301}$$

The choice of phase \( p \) is arbitrary; only the \((-)^{b} \) is essential, i.e.,

$$\rho^{' \prime \prime \prime}_{n,b} = \sum_{s,s'} (-)^{s'} < s \mid s' \mid h_{n,b} > \rho_{s,v,s',v} \tag{2.302}$$

$$= (-)^{b} \rho_{n-b} \tag{2.302}$$

since the density matrix is Hermitian. On the other hand from the conjugation property of the reduced matrix elements of a Hermitian tensor operator, one obtains

$$\sum_{s} < s \mid T_{n,b} \mid s' > = (-)^{s-s'} \sum_{s} < s \mid T_{n,b} \mid s' > \tag{2.303}$$
and
\[ \langle \alpha \omega, \beta \rho | 00 \rangle = \left( \frac{a^{-\alpha}}{a} \right) \delta_{\alpha, \beta} \delta_{\omega, -\rho} . \quad (2.304) \]

Thus as a matter of convenience a factor \( s(-)^s \) is included, viz.,
\[ t_{\eta \eta} = \frac{a}{s} \sum_{\nu \nu'} (-)^{s-\nu} \langle s', \nu' | s - \nu | \eta \rangle \rho_{s', \nu', \nu}. \quad (2.305) \]

One has selected,
\[ p = s - s', \quad (2.306) \]

and if the states are normalized, this phase choice and statistical factor lead to
\[ t_{oo} = 1 . \quad (2.307) \]

and
\[ t_{\eta \delta} = (-)^{s'-s} \delta \quad (2.308) \]

It is convenient to introduce tensor operators in order to discuss quantitatively the form that observables take in different coordinate systems. Eq. (2.291) may be written,
where the matrix elements are,
\[ (\Sigma^{+} \Sigma_{\theta})_{\mu'\mu} = (\Sigma_{\theta} \Sigma^{+})_{\mu'\mu} = (-)^{\theta} (\Sigma_{\theta} \Sigma^{+})_{\mu'\mu} \]
\[ = \frac{1}{8} (-)^{S-M} <\psi_{\mu'}|\psi_{\mu}| <\psi_{\mu'}|\psi_{\mu}| > . \]

Since,
\[ \text{Tr} \left( \Sigma_{\theta}^{+} \Sigma_{\theta} \right) = \lambda^2 \]

the statistical tensor, \( t_{\theta} \), may be identified as an expectation value of the operator
\[ t_{\theta} = \text{Tr} \left( \rho \Sigma_{\theta} \right) . \]

The hermiticity of the density matrix results in a condition on the statistical tensors:
\[ t_{\theta}^{*} = (-)^{\theta} t_{\theta-\theta} . \]

The parity operator, corresponding to reflection through the origin \((x,y,z \rightarrow -x,-y,-z)\) may be expressed as a reflection in a plane followed by a rotation of 180 degrees about an axis normal to the plane. If the Madison convention is adopted, the x-z plane is the scattering plane (diagram a). Reflection in this plane will reverse the
spin projection numbers (diagram b). Rotation about the y axis of 180 degrees completes the parity operation (diagram c).

The coordinate system used to describe the reaction in (c) is indicated - the y axis is out of the paper. Note that this is not the result of the parity operator acting on the original coordinate system. In all cases, the x axis is formed by requiring a right handed coordinate system. In comparing the coordinate systems in (a) and (c), one is related to the other by a rotation of 180 degrees about the y axis. Thus, if parity is conserved, the transformed system must be identical to the original system: this means that all coefficients must remain the same when each coordinate system is rotated 180 degrees about the y axis. Since,

\[ d_{6g},(m) = (-)^{l+\frac{8}{2}} S_{6g,-8} \quad , \quad (2.314) \]

the statement of parity conservation, if the y axis axis is chosen normal to the scattering plane, is
The hermiticity condition, Eq. (2.313), may be combined with the Madison convention parity relation, Eq. (2.315), to give

\[ t_{n_0} = (-)^{n_0} t_{n_0}^\dagger . \quad (2.316) \]

Thus for even (odd) \( k \), the coefficients are real (imaginary).

For definiteness, it will be assumed that the target is unpolarized, in forming the specific final density matrix,

\[ \rho_f^\dagger = A \rho_i^\dagger A^+ . \quad (2.317) \]

From Eq. (2.309), the initial density matrix may be expanded in terms of the set of operators \( (\mathcal{T}_{n_0}) \):

\[ \rho_f^\dagger = \frac{1}{A^2 A^2} \sum_{n_0} t_{n_0} A \mathcal{T}_{n_0} A^+ , \quad (2.318) \]

and the trace taken of \( \rho_f^\dagger \).

\[ \text{Tr}(\rho_f^\dagger) = \frac{1}{A^2 A^2} \left( \text{Tr}(A A^+) + \sum_{l=1}^{2S} t_{n_0} \text{Tr}(A \mathcal{T}_{n_0} A^+) \right) . \quad (2.319) \]

From Eqs. (2.285) and (2.287), this may be written.
The product of the initial intensity and $\text{Tr}(\rho_f^+)$ is the quantity that is measured in a particle counting device for an incident polarized beam. Note that $\text{Tr}(\rho_f^+)$ is not an observable in the sense of Eq. (2.285). An observable (e.g., differential cross section, analysing power, etc.) cannot depend on the characteristics of an incident beam of a given spin and energy (e.g., intensity, polarization, etc.).

The analysing tensors, $T_{n,q,b}$, defined by Eq. (2.320), have the same rotational properties as the $T_{n,q}$ — the general cross section is a scalar under rotations, since

$$
\sum_{m'} D_{m' n}^h(\epsilon) \, D_{m m}^h(\epsilon) = f_{m,n} .
$$

(2.321)

The analysing tensor obeys the hermiticity and parity relations given by Eqs. (2.313) and (2.315). From Eq. (2.313), the general cross section is real.

For some considerations, it is convenient to generalize the scattering amplitude, Eq. (2.282), to include the possibility that the initial beam is not incident along the z axis. It is only in this way that the azimuthal angle appears explicitly in the expressions for observables. The generalization consists of replacing Eq. (2.239) with,
The reaction amplitude, Eq. (2.282), is then given by

$$A_{cm} = \langle u | \sum_{Jm} \frac{\hat{H}_{cJm}}{\sqrt{\pi}} \langle c_{Jm}^{\dagger}|v \rangle \rangle. \quad (2.323)$$

From Eq. (2.287), the differential cross section is

$$\frac{d\sigma}{d\Omega} = \frac{L_{\gamma} \gamma_{\gamma}}{\Delta \Delta} \sum_{Jm} \left[ \frac{\hat{H}_{cJm}}{\sqrt{\pi}} \langle c_{Jm}^{\dagger}|v \rangle \right] \quad (2.324)$$

where

$$f = \frac{1}{2} \left[ 1 + (-)^{L+L'+P+P'+K} \right]. \quad (2.325)$$

To illustrate the gamma directional dependence, Eq. (2.324) may be written

$$\frac{d\sigma}{d\Omega}(\theta, \phi) = \sum_{\text{KQ}} \left[ A_{\text{KQ}} \Delta_{\text{KQ}}(\theta, \phi) \right]. \quad (2.326)$$

or,

$$\frac{d\sigma}{d\Omega}(\theta, \phi) = \sum_{\text{KQ}} \left[ A_{\text{KQ}} \gamma_{\text{KQ}}(\theta, \phi) \right]. \quad (2.327)$$

Averaging over angles due to a finite size gamma detector
with azimuthal symmetry may be included easily in this formalism. In an arbitrary coordinate system, Eq. (2.327) may be written

$$\frac{d\sigma}{d\omega} (\alpha, \beta) = \sum_{kq} A_{kq} \chi_k^\omega (\alpha, \beta) .$$

If the z axis of this coordinate system is chosen to coincide with the symmetry axis of the gamma detector (\(\Theta, \varphi\) in the original coordinate system), the cross section expression for gamma emission into an angle \(\Theta', \varphi'\) may be expressed in terms of Eq. (2.328) by rotating that coordinate system backwards into the original coordinate system, i.e.,

$$\frac{d\sigma}{d\omega} (\Theta', \varphi') = \sum_{kq} A_{kq} \sum_{Q'} \chi_k^{Q'} (\alpha, \beta) \frac{\Delta \varphi_{Q'} - \Theta_{Q'} - \varphi_{Q'}}{Q'}. \tag{2.329}$$

Now, \((\alpha, \beta)\) are the polar angles of the direction \((\Theta', \varphi')\) with respect to the direction \((\Theta, \varphi)\). The observed differential cross section is

$$\langle \frac{d\sigma}{d\omega} (\Theta, \varphi) \rangle = \int_{0}^{\varphi} d\varphi \int_{0}^{\Theta} d\Theta \frac{d\sigma (\Theta, \varphi)}{d\omega} \sin \Theta d\Theta . \tag{2.330}$$

where \(\varphi\) is the half angle subtended by the gamma counter, and \(r(\alpha)\) is the response function across the face of the detector - \(r(\alpha)\) is the detector efficiency for gamma rays received at an angle \(\alpha\) to the counter axis. Since the
gamma counter is assumed axially symmetric, only the \( \Omega = 0 \) terms in Eq. (2.330) survive integration over \( \beta \). Thus, Eq. (2.330) may be written

\[
\left< \frac{d\sigma}{d\Omega} (\theta, \omega) \right> = \int_0^\Phi \sin \alpha \, d\alpha \, r(\omega) \sum_{kq} A_{kq} \cdot P_k(\cos \alpha) \delta_{q,0}(\omega, \theta, 0). \tag{2.331}
\]

Defining

\[
f_k(\phi) = \int_0^\Phi \sin \alpha \, d\alpha \, r(\omega) P_k(\cos \alpha), \tag{2.332}
\]

The response function \( r(\alpha) \) may be formed so that

\[
f_{k=0}(\phi) = 1 \quad \tag{2.333}
\]

Thus, the average cross section may be written for cylindrically symmetric counters

\[
\left< \frac{d\sigma}{d\Omega} (\theta, \omega) \right> = \sum_{kq} f_k(\phi) A_{kq} Y^q_k(\theta, \omega), \tag{2.334}
\]

so that averaging has introduced \( k \) dependent attenuation factors. These factors depend only on detector characteristics and not on the particular cross section being observed.

From Eq. (2.320), the vector analysing power is

\[
\frac{\text{Tr}}{\frac{1}{A^2}} \left< \frac{d\sigma}{d\Omega} \right> = \frac{1}{A^2} \text{Tr} \left( A \, \mathcal{L}^+ \, A^+ \right), \tag{2.335}
\]
or,
\[
\frac{1}{T_{1q}} \frac{d\sigma}{d\alpha} = \frac{1}{A} \sum_{\omega_{\text{mA}}} A_{\text{em}a} \delta_{\text{em}a} \cdot (2.336)
\]

This expression may be developed into,
\[
\frac{1}{T_{1q}} \frac{d\sigma}{d\alpha} = \sum_{KN} B_{\text{kn}} B_{\text{no}} (k_Y), (2.337)
\]

where
\[
B_{\text{kn}} = \sqrt{3(\lambda/2)^2} \sum \sum \frac{\epsilon_1 \epsilon_2}{\lambda_1 \lambda_2} \Omega_{\text{kn}}^{*} \Omega_{\text{kn}}^{<}.
\]

It is clear from the form of Eq. (2.337) that the angle averaged form of this expression will involve the introduction of the same factor \( f_k (\phi) \) as in Eq. (2.336), viz.,

\[
\langle \frac{1}{T_{1q}} \frac{d\sigma}{d\alpha} \rangle = \sum_{KN} f_k (\phi) B_{\text{kn}} B_{\text{no}} (k_Y). (2.340)
\]

If the \( z \) axis is chosen to coincide with the incident wave vector \( \hat{k}_0 \), the cross section and analysing power
expressions are simplified. If, in addition, the y axis is chosen normal to the scattering plane \( \mathbf{k}_x \mathbf{k}_y \), then the analysing power expression simplifies further through application of Eq. (2.316). The results for an incident spin 1/2 particle may be written

\[
\frac{d\sigma}{d\Omega} = \frac{(2\Lambda)^2}{4\pi^2} \sum \left( a_n P_n^0 + b_n b_n P_n^1 \right), \quad (2.341)
\]

where the \( P_n^k \) are the associated Legendre functions\(^{12} \), and

\[
a_n = \frac{3 + \frac{\alpha a h}{(a+1)(n+1)}}{5} \sum \sum \frac{\ell'_l \ell'_l' \ell \ell'}{\ell l' \ell' \ell' \ell'} W(\ell \ell' \ell'; s k) \cdot W(\ell \ell' \ell'; s k) U_{\ell \ell'}^{s} U_{\ell' \ell}^{s'}, \quad (2.342)
\]

and

\[
b_n = \frac{3 + \frac{\alpha a h}{(a+1)(n+1)}}{5} \sum \sum \frac{\ell'_l \ell'_l' \ell \ell'}{\ell l' \ell' \ell' \ell'} W(\ell \ell' \ell'; s k) \cdot W(\ell \ell' \ell'; s k) U_{\ell \ell'}^{s} U_{\ell' \ell}^{s'}, \quad (2.343)
\]

These expressions are in agreement with the corresponding forms found via the Devons and Goldfarb approach\(^4 \), with the important exception that in place of a general reduced matrix element, one now has a well defined collision matrix element corresponding to transitions in the internal and external regions.
An interesting consequence of the division of the total volume of integration into an internal and external part is the appearance of resonance phenomena in the external region (channel resonances). This is easily seen by considering the elastic scattering collision matrix element in the form (one level for simplicity):

\[
U_{ls,ls} = e^{i(\omega_l - \omega_s)} \left\{ 1 + i \frac{\sqrt{s}}{E_r + \Delta_r - E - i \frac{\Gamma}{2}} \right\}^{(2.345)}
\]

The external radial integral, Eq. (2.271), may be split into a hard sphere part and a channel resonance part, explicitly,

\[
R_{el}(h.s.) = \sum_{\alpha} \int_{0}^{\infty} E_{\alpha} \frac{G_{\alpha}}{h \nu} \left[ T_{el} - e^{i(\omega_l - \omega_s)R_{el}} \right]^{(2.346)}
\]

and

\[
R_{el}(c.r.) = \frac{i e^{i(\omega_l - \omega_s)\sqrt{s}}}{E_r + \Delta_r - E - i \frac{\Gamma}{2}} \sum_{\alpha} \int_{0}^{\infty} E_{\alpha} \frac{G_{\alpha}}{h \nu} \theta_{\alpha} \quad (2.347)
\]

The total collision matrix element, Eq. (2.219), may be written in the form

\[
U_{ls,ls} = \frac{i N_b v_l e^{i(\omega_l - \omega_s)\sqrt{s}}}{E^l - E - i \frac{\Gamma}{2}} \left( \Gamma_{b} v_{l} + \Theta_{l}^{b} \Theta_{l s} T \right) + U_{el}(h.s.) \quad (2.348)
\]

where for a unique bound state orbital angular momentum \( \ell' \)
The hard sphere is identical to Eq. (2.270), except for the replacement of the full elastic scattering phase shift with the hard sphere phase shift in Eq. (2.271). The first term in Eq. (2.348) gives rise to the observed resonance; the effective gamma width being written as the sum of the internal and channel contributions, respectively. In the case of nucleon capture, it is clear that the channel contribution to the effective gamma width is strongest for single particle resonances \( \Theta_{2s} \sim 1 \) to single particle bound states \( \Theta_{2s} \sim 1 \). It is to be emphasized, however, that a relegation of resonance type to one term or the other is not appropriate, since these terms refer to two distinct regions in space, i.e., the single particle nature of a state would be contained in the first (internal) term as well. The implication of the second (external) term is that the presence of final states that can be reached by the emission of dipole radiation and which have \( \Theta_{2s} \sim 1 \) will cause anomalously large resonances—corresponding to the superposition of external transitions on top of the usual internal transitions.

There remains in the internal term a sum over the proper levels of the compound state. These levels may be classed as energy local and energy distant levels; where local is defined essentially by the incident particle
energy. The distant terms in the dispersion sum can be taken into account if a model of the interaction between the incident particle and the target nucleus is assumed. In R matrix theory, models such as the strong interaction model and intermediate interaction model have been introduced.

The strong interaction model implies that there is no appreciable free motion of the incident particle inside the internal region. Instead the particle undergoes collisions as soon as it enters the internal region. This model is usually represented quantitatively by the assumption that the signs of the partial width amplitudes, $\sqrt{\lambda}$, are random such that the cross terms in an expression such as Eq. (2.206) sum to zero. If, in addition, boundary conditions are imposed on the proper states of the nucleus such that the logarithmic derivative matrix, $L^0$, becomes essentially the penetration matrix, $P$, then, since the background R matrix occurs in combination with $L^0$, any background R matrix may be safely ignored if the penetration matrix is sufficiently small (as at low energies). Thus the strong interaction model implies that the distant levels can be ignored entirely.

In the intermediate interaction model, the incident projectile retains its single particle identity to the extent that the proper states $\lambda$ may be usefully expanded in terms of product states of the target nucleus and inci-
dent particle moving in the average potential of the target nucleus. This model admits the possibility of incident particle motion in the internal region without collisions. The average potential, from R matrix theory, contains an imaginary part and is nonlocal. It has been demonstrated that these considerations lead to a form of the distant level R matrix such that the hard sphere radial integral, Eq. (2.346), may be replaced with a comparable integral extending into the internal region. The final bound state and initial state single particle wave functions in the internal region are solutions of the radial Schrödinger equation with the average potential, V+iW. In principle, the parameters of this average potential may be determined by requiring that it describe the energy positions and widths (W) of the giant resonances. If these parameters are not too strongly energy dependant, they could be extended downward into the energy region of interest. It is to be emphasized, however, that this average potential (optical potential) should not be determined by, say, matching low energy elastic phase shifts, since these phase shifts are not exclusively single particle in nature. In other words, that part of the low energy phase shifts that deviates from hard sphere scattering is determined by many particle hole configurations, i.e., compound nucleus formation elastic scattering in addition to shape elastic scattering. The next section is concerned with an implementa-
tion of these ideas, as well as closer examination of the electromagnetic operator.

G. INTERMEDIATE INTERACTION MODEL

The fundamental starting point is the formula for the scattering amplitude given by first order perturbation theory for a transition from a continuum state to a discrete state with the emission of a circularly polarized photon ($E = \pm 1$) along the direction of $k - \gamma$ (projection $E$ referred to the $k - \gamma$ axis):

$$\beta_{E m_A; m_a m_A}(k - \gamma) = \langle \chi_f | \beta_E(k - \gamma) | \chi_i \rangle.$$  \hfill (2.350)

The initial wavefunction (neglecting exchange effects) is

$$\chi_i = \frac{1}{\sqrt{L}} \sum_{m_a m_A} \lambda_{m_A}^{m_a} \lambda_{m_A}^{m_a} \Psi_{m_a m_A}^{(\pm)}(kr),$$  \hfill (2.351)

where $\lambda_{m_A}^{m_a}$ is the internal wave function for the target particle (core); $\lambda_{m_A}^{m_a}$ is the spin function for the incident particle; and $\Psi_{m_a m_A}^{(\pm)}(kr)$ is the wavefunction of relative motion of particle and core, generalized to include the possibility of spin flip during elastic scattering. The spin orbit interaction, restricted to couplings like $1\cdot s$, which causes the non conservation of the individual projections of $l$ and $s$, involves only the spin of the incident particle and not the core spin ($A$). The core is only
the source of a mean nuclear field in this approach.

Defining the $z$ axis as being along the incident projectile's direction of motion, the wave function of relative motion is

$$
\chi_{m_1 m_2}^{(z)}(k r) = \frac{\sqrt{4\pi}}{k r} \sum_{lm} Y_{l}^{m_1 m_2} c_{l 0} \chi_{l m}^{(r)}(k r).
$$

(2.352)

The resultant angular momentum, $j$, and its $z$ component, $m_1$, are conserved during the scattering by the spin orbit potential, whereas the individual $z$ components of $l$ and $m$ are not. When $m_1 = -l$ (called spin-flip*), the orbit is tilted - the projection of $l$ is $m_l$. The partial waves $\chi_{l}^{(r)}(k r)$ are solutions to the radial Schrödinger equation,

$$
\left\{ \frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} - \frac{2m}{\hbar^2} (U_{c} + U_{e} + U_{l}) \right\} \chi_{l}^{(r)} = 0,
$$

(2.353)

where at the origin,

$$
\chi_{l}^{(r)}(0) = 0,
$$

(2.354)

and for asymptotic $r$, i.e., beyond the range of the nuclear potentials, the radial wave becomes

$$
\chi_{l}^{(r)} \rightarrow \frac{i}{2} \left[ I_{l} - n_{l} 0_{l} \right],
$$

(2.355)

where $I_{l}$ and $0_{l}$ are defined in Eq.s (2.144) and (2.145).
\( \eta^* \) is the reflection coefficient or scattering matrix element for the \((l, j)\) wave. Note that lower case \( j \) is the total angular momentum only for spin zero targets. In terms of quantities defined previously, \( \eta^* \) is, for charged particles

\[
\eta^* = e^{i \left( \omega - \omega_l \right)} \left\{ \frac{1 - L^l x^l / x_l^l}{1 - L x / x} \right\}_{r = a_c}^\infty
\]

(2.356)

where \( a_c \) is understood to be the asymptotic radius, i.e., the radius beyond which the scattering matrix element \( \eta^* \) does not change within some prescribed precision. The \( \eta^* \) are determined by numerically integrating Eq. (2.353) and matching the function (and its derivative) to the form of Eq. (2.355) at \( r = a_c \). In Eq. (2.353), the potentials are taken to be

\[
u = -V f(x_o) + i \left[ 4 W \frac{d}{dx} \frac{d}{dx} f(x_o) \right], \quad (2.357)
\]

and

\[
U_c = \begin{cases} \frac{2 \sqrt{2} e^2}{3 R_c} (3 - \frac{r^2}{R_c^2}) & ; r < R_c \\ \frac{1}{2 \sqrt{2} e} \frac{e^2}{r} & ; r \geq R_c \\
\end{cases} \quad (2.358)
\]

and

\[
U_c^* = \left( \frac{\hbar}{m_c} \right)^2 V_{s_o} (\vec{r} \cdot \vec{E}) \frac{1}{r} \frac{d}{dr} f(x_o), \quad (2.359)
\]
where $\mathbf{\sigma}$ is the Pauli spinor, and $\mathcal{Q}$ is in units of $\pi$.

$$f(x_{ij}) = (1 + e^{x_{ij}})^{-1}; \quad x_{ij} = \frac{r_{ij} - r_{i} A^3_{ij}}{a_{ij}}$$

and the $\pi$-meson Compton wavelength is

$$\left( \frac{k}{m_{\pi} c} \right)^2 = 0.00 \; f_{\pi}^2$$

and $V$, $a_{1}$, and $V_{50}$ are taken to be constants (in units of energy). The initial wave function is normalized to unit probability flux area. Details of the solution method of the free particle Schrödinger equation are given in Appendix A.

The final state bound wave function is that for a single particle with spin orbit coupling in a central field vector coupled to the core particle.

$$\Psi_{1}^{\mu} = \sum_{j} \langle \mathbf{\bar{a}}^{\lambda} \mathbf{a}_{j} | \mathbf{\lambda} \mathbf{M}_{0} \rangle \mathbf{I}_{\lambda}^{\mu} \mathbf{w}_{\lambda}^{\mu} \; \mathbf{w}_{\mu}^{\mu} / r$$

Standard conventions are maintained with respect to time reversal, and the radial function $u_{E}^{\mu}$ is one of the eigen solutions to the bound state Schrödinger equation; corresponding to the binding energy $E_{i}$ one has

$$\left\{ \frac{d^2}{dr^2} - \frac{\varphi(E_{i})}{r^2} - \frac{Q(\mu + 1)}{r^2} - \frac{\varphi(\mu + \mu + \mu)}{r^2} \right\} u_{E}^{\mu} = 0$$
with boundary conditions: at the origin,

\[ u^0_2(0) = 0, \]  \hspace{1cm} (2.364)

and at an asymptotic radius,

\[ \frac{1}{r} \ln u^0_2 \]  \hspace{1cm} (2.365)

where \( W_n(\eta, \rho) \) is the Whittaker function, for charged particles, evaluated at the binding energy. For a given binding energy, the real part of the potential, Eq. (2.357), is adjusted to produce the eigensolutions with a specified number of nodes. The bound state radial wavefunction is normalized,

\[ \int_0^\infty |u^0_2|^2 \, dr = 1. \]  \hspace{1cm} (2.366)

Details of this calculation are in Appendix B.

The operator in Eq. (2.350) is

\[ \beta_\varepsilon(\hbar^2) = \sqrt{\frac{\pi e^4}{\hbar^2}} \Omega^M_{\varepsilon}(\Omega), \]  \hspace{1cm} (2.367)

where \( \rho^M \) is the density of final states for each \( \varepsilon \), and

\[ \Omega^M_{\varepsilon}(\Omega) = \frac{1}{2\pi c} \frac{m \Omega}{\Lambda \omega} \right \{ \frac{L^*}{L M^*} \left[ \frac{L^*}{L M^*} \right]^2 + \frac{L^*}{L M^*} \right \} \phi^M_{\varepsilon}(\Omega) \Phi^0_v \]  \hspace{1cm} (2.368)
Here \( \Omega = (\varphi, \theta, 0) \), i.e., the azimuthal and polar angles of the gamma radiation measured from the incident beam's direction. \( \Omega = (\theta', \varphi') \) are the center of mass angles giving the direction in which the scattered particles separate, and \( G_L \) is defined by

\[
G_L = \frac{2}{\sqrt{2}} \left( k_1 r^L \mathcal{J}_{L-1}(k_2 r) - L \mathcal{J}_L(k_2 r) \right) + \frac{1}{\sqrt{2}} \left( k_2 r^L \mathcal{J}_{L-1}(k_1 r) - L \mathcal{J}_L(k_1 r) \right),
\]

where

\[
h_i = \frac{h \varphi}{m_i}(m_i + m_2).
\]

When the binding energy of the nucleon (cluster) is small, the radial extent of the wavefunction can be large, i.e., the radial integral in Eq. (2.350) converges very slowly. The use of the long wavelength approximation for the electromagnetic interaction operator is not justified in these cases. At the same time, it is not necessary to consider the operator in the generality implicit in Eqs (2.263) through (2.266), where a sum over all nucleons is suggested. Rather, the operator should be of a two body type, since the matrix elements in Eq. (2.270) are evaluated only in the external region where the nuclei are separate, and in Eq. (2.350), the target particle is a spectator. There are a number of ways in which to establish the form of this operator. It is convenient to begin with the
formalism of Ref.s (20) and (85) for this purpose.

In the absence of an electromagnetic field, the hamiltonian for a system of particles may be written as:

\[ H = T_0(P) + H_0(\mathcal{A}_\alpha), \]  \hspace{1cm} (2.370)

where \( P \) (and \( B \)) are center of mass operators; and \( \mathcal{A}_\alpha \) is a set of operators representing the internal variables. \( T_0 \) is the kinetic energy operator for the center of mass. In an external electromagnetic field described by \( \mathcal{A}(\vec{r}) \), the hamiltonian, Eq. (2.370), becomes:

\[ H = T\{\vec{P}, \mathcal{A}(\vec{r})\} + H\{\mathcal{A}_\alpha, \mathcal{A}(\vec{r})\}, \]  \hspace{1cm} (2.371)

where braces are used to indicate that \( H \) and \( T \) are functionals, e.g.,

\[ T\{\vec{P}, \mathcal{A}(\vec{r})\} = T\{\vec{P} - \sum_\alpha \frac{e\alpha}{c} \mathcal{A}(\vec{r})\}. \]  \hspace{1cm} (2.372)

If the field is weakly coupled to the system, an expansion of \( H\{\mathcal{A}_\alpha, \mathcal{A}(\vec{r})\} \) may be used, i.e.,

\[ H\{\mathcal{A}_\alpha, \mathcal{A}(\vec{r})\} = H_0(\mathcal{A}_\alpha) + H_1\{\mathcal{A}_\alpha, \mathcal{A}(\vec{r})\} + \]  \hspace{1cm} (2.373)

\[ + H_2\{\mathcal{A}_\alpha, \mathcal{A}(\vec{r})\} \ldots \]

Only the linear functional \( H_1\{\mathcal{A}_\alpha, \mathcal{A}(\vec{r})\} \) is responsible for one photon processes. Defining
\[ q = \frac{i}{\hbar c} \sum_{\omega} E_{\omega} G(\vec{r}_\omega), \quad (2.374) \]

where \( G(\vec{r}_\omega) \) is an arbitrary gauge function. The hamiltonian is gauge invariant if

\[ H\left\{ \omega, \vec{A}, \vec{G} \right\} = e^{\frac{1}{2}} H\left\{ \omega, \vec{A}, \vec{G} \right\} e^{-\frac{1}{2}}, \quad (2.375) \]

i.e., if a gauge transformation on the potentials is equivalent to a canonical (unitary) transformation of \( H^5 \). As a unitary transformation does not change physical results, transition probabilities are independent of the gauge. The kinetic energy operator \( T(\vec{p}, \vec{A}) \) is independently gauge invariant. \( \vec{A} \) may be set to zero and both sides of Eq. (2.375) expanded,

\[ H_0(\omega) + H_1\left\{ \omega, \vec{A}, \vec{G} \right\} + H_2\left\{ \omega, \vec{A}, \vec{G} \right\} + \ldots = H_0(\omega) + \left\{ G, H_0 \right\} + \frac{1}{2!} \left\{ G, \left\{ G, H_0 \right\} \right\} + \ldots \quad (2.376) \]

where the bracket is a commutator. Terms of equal order may be equated, and in particular,

\[ H_1\left\{ \omega, \vec{A}, \vec{G} \right\} = \left\{ G, H_0 \right\}. \quad (2.377) \]

This is a condition on the hamiltonian, since \( G \) (hermitian) is not necessarily a gauge function.
The vector potential for the radiation field is taken to be,
\[ \vec{A}(\vec{r}) = \sum_{h, b} \frac{\gamma \theta}{\hbar \nu} \left( e^{i \vec{k} \cdot \vec{r}} a_{h, b} \right) + e^{i \vec{k} \cdot \vec{r}} a_{h, b}^\dagger, \]
(2.378)
where \( a_{h, b}^\dagger \) and \( a_{h, b} \) are creation and annihilation operators for a photon with wave number \( k \) and polarization \( q \).

For single photon processes, \( \vec{A} \) is to be treated only in first order. So, the interaction term between the radiation field and internal motion of the system is from Eq. (2.373)
\[ I = \sum_{h, b} \frac{\gamma \theta}{\hbar \nu} H_{1} \left[ \frac{\vec{r}}{\hbar \nu}, (e^{i \vec{k} \cdot \vec{r}} a_{h, b} \right] \]
(2.379)
where \( \vec{r} = \vec{R} + \vec{r}', \vec{r}' \) being the distance from the center of mass of the system to the point where the field is to be evaluated. Since \( \vec{R} \) commutes with \( \vec{E} \), the interaction term for the emission of a photon of wavenumber \( k \) and polarization \( e \) is,
\[ I' = \frac{\gamma \theta}{\hbar \nu} e^{i \vec{k} \cdot \vec{r}} H_{1} \left[ \frac{\vec{r}}{\hbar \nu}, e^{i \vec{k} \cdot \vec{r}} \right] \]
(2.380)

Since the total wavefunction of the system may be written as
\[ \Psi = e^{i \vec{E} \cdot \vec{R}} \Psi_{\text{int}}, \]
(2.381)
the factor \( e^{-ik\cdot r} \) establishes conservation of momentum, viz.,

\[
(\mathbf{T}, \mathbf{T}) = \delta (\mathbf{k} - \mathbf{k}' - \mathbf{k}) (\mathbf{T}, \mathbf{T})
\] (2.382)

where \( \mathbf{T}' \) is that part of Eq. (2.380) that is composed of internal variables only. One may decompose the vector plane wave into irrotational and solenoidal parts, i.e.,

\[
e^b e^{-ik\cdot r} = \mathbf{s} + i \mathbf{\tilde{s}' x} (\mathbf{k} \times \mathbf{s}'),
\] (2.383)

where

\[
\mathbf{s} = \int \sum_c e^b e^{-ik\cdot \mathbf{r}} \, d\mathbf{r},
\] (2.384)

and

\[
\mathbf{\tilde{s}'} = \frac{1}{m} \int \sum_c e^b e^{-ik\cdot \mathbf{r}} \, d\mathbf{r},
\] (2.385)

Thus, the internal interaction is

\[
\mathbf{T}' = -\frac{e^{\mathbf{\gamma} \cdot \mathbf{k}}}{\omega V} \sum \left(-i \mathbf{h}\right)^{L-1} \left\{ \frac{1}{i} \left[ \mathbf{H}_0, \Delta_{\mathbf{H}_0} \right] - i \mathbf{H}_0 \left[ \mathbf{S}_0, (e^b \mathbf{r} \times \mathbf{\tilde{\omega}}) \times \mathbf{\tilde{\omega}} \right] \right\}
\] (2.386)

where
\[ D_L = \sum \epsilon_{\alpha}(e_\beta \cdot \vec{r}') (k \cdot \vec{r})^{L-1} / (L! \cdot h^{L-1}) \] \hspace{1cm} (2.387) 

and 

\[ \bar{\omega}_L = L \cdot \vec{r}' (k \cdot \vec{r})^{L-1} / [(L+1)! \cdot h^{L-1}] \] \hspace{1cm} (2.388)

\( D_L \) is the electric \( 2^L \) pole moment in a non irreducible representation. At low energies and small \( r \), this expansion converges rapidly, so only the lowest of the terms which contribute to a transition need be considered. However, for \( kr > 1 \), not only the lowest terms contribute but there is an infinite series of higher terms contributing for a given transition, i.e., given \( \Delta L \) and parity change.

For large values of \( kr \), the multipole moments are best described on the basis of the angular momentum and parity of the radiation field produced in the transition. The irreducible representations of these fields are the multipole fields. For this reason, it is necessary to write the interaction term as 

\[ I'' = \sqrt{\frac{\mu c}{\hbar \nu}} \text{H} \left\{ \epsilon_{\alpha} e_\beta e^{-i k \vec{r}}' \right\} + \cdots \] \hspace{1cm} (2.389)

\[ I'' = \sqrt{\frac{\mu c}{\hbar \nu}} \text{H} \left\{ \epsilon_{\alpha} e_\beta e^{-i k \vec{r}}' \right\} \sum_{lm} \frac{d_{lm}}{\sin \theta} \bar{A}_{\alpha l m} e^{i m \theta} + \cdots \] \hspace{1cm} (2.390)

where the transverse vector plane wave has been expanded in
terms of electric and magnetic multipole fields in a solenoidal gauge. In this gauge, these fields constitute a complete set for transverse (solenoidal) vectors. These fields are defined in Eqs. (2.73) and (2.74).

By the linearity of $H$, Eq. (2.390) may be written,

$$I^{''} = I^{''}(e) + I^{''}(m), \quad (2.391)$$

where

$$I^{''}(e) = i \sqrt{\frac{e}{2n}} \sum_{L,m} ^{L,m} H \{ \omega_{j}, \ A_{lm}^{e*} \}, \quad (2.392)$$

and similarly for $I^{''}(m)$. Using Eq. (2.44), it may be shown that

$$A_{lm}^{e*} = \frac{(-i)^{L+2}}{4(L+1)} \left( \frac{1}{r} \nabla (G_{l} Y_{m}^{*}) + k r d_{L} Y_{m}^{*} \right), \quad (2.393)$$

where

$$G_{l} = \frac{d}{dr} (r d_{L}). \quad (2.394)$$

So, the linear functional part of Eq. (2.392) may be written,

$$H \{ \omega_{j}, A_{lm}^{e*} \} = \frac{(-i)^{L+2}}{4(L+1)} \left[ \frac{1}{r} H \{ \omega_{j}, \nabla (G_{l} Y_{m}^{*}) \} + H \{ \omega_{j}, k r d_{L} Y_{m}^{*} \} \right]. \quad (2.395)$$
From Eq. (2.377)

\[ H_1 \{ \psi_n, \mathcal{G}_L, \psi_{m*}^r \} = \left[ S, H_0 \right] = i \frac{d}{dt}, \quad (2.396) \]

where

\[ f = \frac{i}{\hbar c} \sum_{\alpha} e_{\alpha} G_{L}(\alpha) \psi_{m*}^r (\theta_{\alpha}, \varphi_{\alpha}). \quad (2.397) \]

So, Eq. (2.392) becomes

\[ I''(e) = \sqrt{\frac{2 \pi \hbar c}{\kappa \nu}} \sum_{LM} \left[ \frac{H_0 \cdot f}{\hbar \{L(L+1)\}} \right] \left[ \frac{H_1 \{ \psi_n, L \mathcal{G}_L, \psi_{m*}^r \} \{L(L+1)\}}{\hbar \{L(L+1)\}} \right]. \quad (2.398) \]

Eq. (2.398) may be used to define

\[ I''(e) = \sum_{LM} \left( \dot{P}_L^m + \dot{S}_L^m \right), \quad (2.399) \]

where

\[ \dot{P}_L^m = \sqrt{\frac{\kappa}{\mathbb{w} \nu}} \sum_{\alpha} e_{\alpha} G_{L}(\alpha) \psi_{m*}^r \frac{L(L+1)}{\hbar \{L(L+1)\}} \sum_{\alpha} e_{\alpha} G_{L}(\alpha) \psi_{m*}^r \quad (2.400) \]

and

\[ \dot{S}_L^m = -\sqrt{\frac{\kappa}{\mathbb{w} \nu}} \sum_{\alpha} e_{\alpha} G_{L}(\alpha) \psi_{m*}^r \frac{L(L+1)}{\hbar \{L(L+1)\}} H_1 \{ \psi_n, L \mathcal{G}_L, \psi_{m*}^r \} \quad (2.401) \]
The quantity $P^\omega$ has the properties associated with the Siegert theorem; its form is independent of any characteristics of the nucleon other than its charge and position. The secondary term $S^\omega$ depends explicitly on the form of the hamiltonian. For small $kr$, it is one order (in $kr$) smaller than $P^\omega$; hence it is usually neglected. This is the condition under which the Siegert theorem applies. But for large $r$, the secondary term cannot be neglected on this basis. Generally, the form of the electric moment depends explicitly on the form of the hamiltonian. Under all conditions, the magnetic moment depends explicitly on the form of the hamiltonian.

The first point to be examined in this connection is the influence of the interaction terms on the motion of the center of mass term: $T[\Pi,\Lambda]$. For this purpose it is necessary to have an explicit hamiltonian. An ordinary two particle hamiltonian is assumed - the potential is a function of relative separation and is velocity independent. Although an $\hbar$s term in the potential may be incorporated with the prescription of replacing the mechanical momentum with the canonical momentum for a charged particle in an external electromagnetic field, in the Wood-Saxon potential, the $\hbar$s term is small and so its inclusion would be an unwarranted complication. For an ordinary hamiltonian, the internal interaction term is
\[
\psi_{\text{int}} = e^{i \vec{\omega} \cdot \vec{r} / \mu}\left\{ \frac{e^{it\mu \vec{e} \cdot \vec{r}}}{\mu^2 c} \right\} e^{-i \vec{p} \cdot \vec{r}} + \frac{\mu}{\hbar \omega}\left[ g_{\alpha \alpha} \left( \frac{\vec{e} \times \vec{e}^\dagger}{\hbar \omega} \right) \cdot \frac{e^{i \vec{e} \cdot \vec{r}}}{\mu} \right]
\]

where the transversity of the field has been used \((\vec{k} \cdot \vec{e}^\dagger = 0)\); and \(\mu\) is the reduced mass; and \(g_{\alpha \alpha}\) is the spin factor for the \(\alpha\) th particle. The external interaction term is
\[
\psi_{\text{int}} = \frac{1}{\mathcal{M}} \left( \frac{\omega_{\text{c}}}{\mu c} \right) \left\{ e^{i \vec{e} \cdot \vec{r}} \cdot \vec{\nabla}_r \right\} \frac{\mu}{\hbar \omega} g_{\alpha \alpha} \left( \frac{\vec{e} \times \vec{e}^\dagger}{\hbar \omega} \right) \cdot \left\{ \frac{e^{i \vec{e} \cdot \vec{r}}}{\mu} \right\}
\]

where \(\mathcal{M} = m_1 m_2\); \(\hbar\) is in units of \(\mu\) and can be represented in terms of the Pauli matrices. An external transition amplitude is of the form, from Eq. (2.381),
\[
\beta_{\text{ext}} \sim \left( \frac{\mu}{\hbar c} e^{i \vec{e} \cdot \vec{r}} \right) \psi_{\text{int}} \left( \frac{\mu}{\hbar c} e^{i \vec{e} \cdot \vec{r}} \right).
\]

On substituting Eq. (2.403) into Eq. (2.404) one obtains
\[
\beta_{\text{ext}} \sim \left( \frac{\mu}{\hbar c} e^{i \vec{e} \cdot \vec{r}} \right) \left( \frac{\mu}{\hbar c} e^{i \vec{e} \cdot \vec{r}} \right) \left( \frac{\mu}{\hbar c} e^{i \vec{e} \cdot \vec{r}} \right) \left( \frac{\mu}{\hbar c} e^{i \vec{e} \cdot \vec{r}} \right)
\]

The first part of this expression may be compared directly with Eq. (30) of Ref. (20). The relevant factor of Eq.
(2.405) may be expanded in powers of \( kr \), i.e.,
\[
\phi_{F_{1\mu}} \frac{i e \gamma_c}{\epsilon} e^{i \mathbf{k} \cdot \mathbf{r}} \phi_{F_{1\mu}} = \frac{L}{2} \frac{i e \gamma_c}{\epsilon} \left( \phi_{F_{1\mu}} \phi_{F_{1\mu}} + \phi_{F_{1\mu}} \phi_{F_{1\mu}} + \cdots \right) \tag{2.406}
\]

The first term in this expansion corresponds to electric dipole transitions in the internal transition amplitude as may be seen from Eq. (2.402). But the internal coordinate wavefunctions are orthogonal, and thus the first term in Eq. (2.406) gives no contribution to the center of mass (external) transition amplitude. The leading term for external contributions corresponds to an internal quadrupole transition. But if quadrupole internal transitions in \( B(\text{ext}) \), then dipole internal transitions in \( B(\text{int}) \) and so the external contribution is negligible in comparison. The conclusion of Ref. (20) is: in general, \( B(\text{ext}) \) contributes in one higher order than \( B(\text{int}) \) and so may always be neglected for the single photon processes. For \( kr \) not much less than 1, this argument may be altered slightly by expanding the plane wave as

\[
e^{i \mathbf{k} \cdot \mathbf{r}} = \sum_{\ell m} \frac{L}{\sqrt{4\pi}} j_{\ell} (kr) y_{\ell m}^* \phi_{Lm} \phi_{Lm}^* \tag{2.407}
\]

So that Eq. (2.405) may be written (retaining only the first term).
And a $B(\text{int})$ may be formed similarly (retaining only that term corresponding to $E_0$ defined in Eq. (2.400)),

$$B(\text{int}) \sim \delta(E - E' - \hbar \omega) \frac{1}{\hbar} \sum_{\ell m} \left[ \sum_{\alpha} (-i) \right] L_{\alpha} \left[ \sum_{\ell m} \left( \chi_{\alpha\ell m} \chi_{\alpha\ell m}^* \right) \right]$$

(2.409)

The appearance of $\omega$ in this last equation (from Eq. (2.396)) is significant; it differs from $c/k$ by only the recoil of the system and will henceforth be taken equal to $\omega$. The integral of Eq.s (2.408) and (2.409). However, the coefficients of these two expressions are in the approximate ratio of

$$\frac{\hbar}{c}$$

Thus, $B(\text{ext})$ is negligible for low energies.

It is of interest to examine the relative magnitudes of the primary and secondary terms of Eq. (2.398), i.e.,

$$[H_0, f], \text{ and } H_f \left\{ \xi_\alpha, \xi_{\alpha}^* \right\}$$

For an ordinary Hamiltonian, the secondary term has the
\[ h\{\alpha, \varphi, \theta, \varphi'\} = - \frac{i}{\alpha} \sum \left[ \frac{e^{ik\alpha}}{2im^2c} \left( \bar{\alpha} \cdot \bar{\beta} + \beta \cdot \bar{\alpha} \right) + \frac{m}{m_\alpha} \sum e^{ik\alpha} g_{\alpha} \bar{\sigma}_\alpha \cdot (\bar{\sigma} \times \beta) \right] \]

where

\[ \bar{\beta} = k \bar{r} + j_L \gamma^*_L, \]

and all coordinates are relative to the center of mass.

The first term on the right hand side of Eq. (2.410) may be written,

\[ \left[ \bar{\alpha} \cdot \bar{\beta} + \beta \cdot \bar{\alpha} \right] = 3 k \alpha d_L \gamma^*_L + h_r \gamma^*_L \frac{dj_L}{d\alpha} \]

so that Eq. (2.410) becomes

\[ h\{\alpha, \varphi, \theta, \varphi'\} = - \frac{i}{\alpha} \sum \left[ \frac{e^{ik\alpha}}{2im^2c} \left( h \gamma^*_L (3d_L + \]

\[ + \frac{dJ_L}{d\alpha} + \sigma d_L \frac{d\beta}{d\alpha} + \frac{m}{m_\alpha} \hat{\lambda}_L (\bar{\sigma}_\alpha \cdot \gamma^*_L) \right) \right] \]

From Eq. (2.396), the primary term is

\[ \left[ \frac{H_0}{\hbar} \right] = \frac{\hbar \omega}{\hbar} \frac{i}{k_c} \sum e^{ik\alpha} (d_L + r \frac{dJ_L}{d\alpha}) \gamma^*_L \]

Once again, the integrands are comparable but the coefficients of these terms are in the ratio of

\[ \frac{E_\gamma}{\alpha mc^2} \]
Thus, the secondary term is negligible for an ordinary Hamiltonian at low energies.

From Eq. (2.391), the magnetic part of the interaction is,

\[ \Omega_{\mu\nu} = \frac{2i\pi c}{\omega} \mathcal{L} \left[ \frac{\partial}{\partial \omega} \right]_{\mu\nu} \tilde{A}^{\*} \mathcal{L} \left( j_{\omega} \right) \]  \hspace{1cm} (2.415)

where

\[ \tilde{A}^{\*} = \frac{L(j_{\omega})^{L+1}}{L(L+1)} \tilde{L}(j_{\omega}) \]  \hspace{1cm} (2.416)

For an ordinary Hamiltonian, the interaction term is

\[ H_{I_{\omega}, I_{\nu}} (\tilde{L}(j_{\omega} \tilde{L}^*)^*) = \sum \frac{m}{\omega} \left\{ \frac{e\tilde{L}^*}{\omega m_\alpha^2 c} \left( \tilde{L}(j_{\omega} \tilde{L}^*)^* \right) \cdot \hat{\nabla} + \right. \]

\[ + \frac{m_\alpha}{\omega} \tilde{\mu}_\alpha \tilde{\tau}_\alpha \cdot \left[ \tilde{L}(j_{\omega} \tilde{L}^*)^* \right] \} \] \hspace{1cm} (2.417)

where the identity \( \hat{\nabla} \cdot \tilde{L}(j_{\omega} \tilde{L}^*) = 0 \) has been used. The first term on the right hand side of Eq. (2.417) may be written

\[ \left[ \tilde{L}(j_{\omega} \tilde{L}^*)^* \right]^* \hat{\nabla} - \frac{m_\alpha}{\tilde{\mu}_\alpha} \hat{\nabla} \left( j_{\omega} \tilde{L}^* \right) \cdot \tilde{\mu}_\alpha \] \hspace{1cm} (2.418)

where \( \tilde{\mu}_\alpha = \tilde{\tau}_\alpha \tilde{L}^\* \) and the coordinates are relative to the center of mass of the two body system. The second term on the right hand side of Eq. (2.417) is recognized as being
proportional to the electric vector potential. So, from Eq. (2.393)

\[
\nabla \times \left[ \psi \left( \phi^* \right) \right] = i \left\{ \nabla \left( \phi \phi^* \right) + \kappa^2 \nabla \phi \right\} \tag{2.419}
\]

It has been shown that the second term on the right hand side is negligible compared to the first. So Eq. (2.419) may be written

\[
H_\alpha \left\{ \phi_\alpha, \left( \phi^* \right) \right\} = \sum_{\alpha} \left\{ \frac{e_\alpha}{i m_\alpha c} \frac{m_\alpha}{\kappa} \nabla \phi \phi^* \phi \right\} + i \frac{m_\alpha}{\kappa} \nabla \phi \phi^* \phi \left( \phi \phi^* \right) \tag{2.420}
\]

Thus, Eq. (2.415) may be written for an ordinary hamiltonian,

\[
H_{\text{lm}} = 2 \pi \frac{e_\alpha}{m_\alpha} \sum_{l=0}^{L+1} \frac{\delta^l}{\sqrt{l!(L+l)!}} \left\{ \frac{e_\alpha}{i m_\alpha} \nabla \phi \phi^* \phi \right\} \tag{2.421}
\]

and from Eq. (2.399)

\[
H_{\text{le}} = 2 \pi c \left\{ \frac{e_\alpha}{m_\alpha} \sum_{L=0}^{L+1} \frac{\delta^L}{\sqrt{L!(L+l)!}} \right\} \tag{2.422}
\]

These expressions are in a coordinate frame with the center of mass as origin. The change to relative coordinates,
\[ \tau_i \rightarrow -\frac{e}{m} \tau, \quad \theta, \varphi \rightarrow \theta, \varphi \]

\[ \tau_z \rightarrow \frac{m_i}{m} \tau, \quad \theta, \varphi \rightarrow \pi - \theta, \pi + \varphi \]

transforms Eq. (2.422) into

\[ \mathcal{L}^\mu(\vec{r}) = 2\pi c \left( \frac{\hbar}{m} \right)^2 \sum_{l=0}^{\infty} \frac{\Delta_{2l+1}^l}{\Delta_{2l+1}^l} e^{i\mu k \cdot \vec{r}} \left\{ \begin{array}{l}
2 \left[ k \times \frac{d}{dx} J_{l+1}(k \cdot x) - \frac{d}{dx} J_l(k \cdot x) \right] + \\
+ (-1)^l \left[ k \times \frac{d}{dx} J_{l+1}(k \cdot x) - \frac{d}{dx} J_l(k \cdot x) \right] \end{array} \right\} , \quad (2.423) \]

where

\[ h_\mu = \frac{\hbar}{m} \frac{m_i}{m} \frac{m_i}{(m_i + m)} \], \quad r = |\vec{r} - \vec{r}_i| \quad (2.424) \]

and \( z_i \) is the charge of the particle in units of e. Eq. (2.423) reduces to standard form in the long wavelength approximation (\( kr << 1 \)) apart from an irrelevant factor of \( -q \). The factor of \( -q \) arises from an alternate phase choice in the original definition of the vector potentials (cf. Eq. (2.23)). This factor will reappear, in a compensating manner, in the magnetic term, such that the relative sign of the electric and magnetic terms is the same as that of the standard expression (Ref. (16)). The long wavelength form of Eq. (2.423) is
The magnetic interaction term in relative coordinates is more complicated. The result is

\[ \mathcal{I}^{(m)} = \frac{e}{\hbar c} \sum_{\ell m} e^{\frac{z}{\hbar c}} \Psi_{\ell m} \left( \mathbf{L} \cdot \mathbf{L}_{\ell m} \right) \frac{1}{\sqrt{\ell + 1}} \sum_{\ell' m'} e^{\frac{z}{\hbar c}} \Psi_{\ell' m'} \left( \mathbf{L} \cdot \mathbf{L}_{\ell' m'} \right) \frac{1}{\sqrt{\ell' + 1}} \]

(2.426)

where

\[ A_{\ell m} = \frac{1}{\ell + 1} \int \frac{1}{x^2 - m^2} \, d\mathbf{x} \left[ \mathbf{L} \cdot \mathbf{L}_{\ell m} \right] \Psi_{\ell m} \left( \mathbf{L} \cdot \mathbf{L}_{\ell m} \right) \frac{1}{\sqrt{\ell + 1}} \sum_{\ell' m'} e^{\frac{z}{\hbar c}} \Psi_{\ell' m'} \left( \mathbf{L} \cdot \mathbf{L}_{\ell' m'} \right) \frac{1}{\sqrt{\ell' + 1}} \]

(2.427)

\[ B_{\ell m} = \frac{1}{\ell + 1} \int \frac{1}{x^2 - m^2} \, d\mathbf{x} \left[ \mathbf{L} \cdot \mathbf{L}_{\ell m} \right] \Psi_{\ell m} \left( \mathbf{L} \cdot \mathbf{L}_{\ell m} \right) \frac{1}{\sqrt{\ell + 1}} \sum_{\ell' m'} e^{\frac{z}{\hbar c}} \Psi_{\ell' m'} \left( \mathbf{L} \cdot \mathbf{L}_{\ell' m'} \right) \frac{1}{\sqrt{\ell' + 1}} \]

(2.428)

\[ C_{\ell m} = -\frac{1}{\ell + 1} \int \frac{1}{x^2 - m^2} \, d\mathbf{x} \left[ \mathbf{L} \cdot \mathbf{L}_{\ell m} \right] \Psi_{\ell m} \left( \mathbf{L} \cdot \mathbf{L}_{\ell m} \right) \frac{1}{\sqrt{\ell + 1}} \sum_{\ell' m'} e^{\frac{z}{\hbar c}} \Psi_{\ell' m'} \left( \mathbf{L} \cdot \mathbf{L}_{\ell' m'} \right) \frac{1}{\sqrt{\ell' + 1}} \]

(2.429)

\[ D_{\ell m} = \frac{1}{\ell + 1} \int \frac{1}{x^2 - m^2} \, d\mathbf{x} \left[ \mathbf{L} \cdot \mathbf{L}_{\ell m} \right] \Psi_{\ell m} \left( \mathbf{L} \cdot \mathbf{L}_{\ell m} \right) \frac{1}{\sqrt{\ell + 1}} \sum_{\ell' m'} e^{\frac{z}{\hbar c}} \Psi_{\ell' m'} \left( \mathbf{L} \cdot \mathbf{L}_{\ell' m'} \right) \frac{1}{\sqrt{\ell' + 1}} \]

(2.430)
\[ E_V = -\left\{ \begin{array}{c} (l+1) \\ 2(2l+1) \end{array} \right\} Y_{m,m+1} \bar{Y}_{l+1,m} \] 

(2.431)

and

\[ E_V = \left\{ \begin{array}{c} l \\ 2L-1 \end{array} \right\} Y_{m,m-1} \bar{Y}_{l-1,m} \] 

(2.432)

For \( L=1 \) and \( kr<<1 \), this expression reduces to standard form. The long wavelength form of Eq. (2.426) is

\[ I''(m) = \sqrt{\frac{\hbar c}{\omega}} (\gamma e + \delta m \omega) \left[ \frac{2I}{m^2} + \frac{2\epsilon}{m^2} \right] \] 

(2.433)

Comparison of the relative magnitudes of the electric and magnetic interaction terms shows that

\[ \frac{I''(m)}{I''(e)} \sim \frac{\epsilon_\gamma}{\omega c^2} \] 

Thus, for an ordinary Hamiltonian, the magnetic interaction term is negligible at low energies.

The reaction amplitude, Eq. (2.350), may now be evaluated for the two-body electric dipole operator. The result is

\[ \beta_{m_a m_a} = \frac{1}{\hbar^2} \left( \begin{array}{c} -j \cdot a \cdot m + \omega \cdot 1 \\ j \cdot (a \cdot m) \end{array} \right) \] 

\[ \left( \begin{array}{c} \delta \cdot m \cdot m \cdot 1 \cdot m \end{array} \right) \] 

(2.434)
The Clebsch-Gordan coefficient embodies the usual selection rule for $E1$ transitions, and the dimensionless radial integral is,

$$ R_{j_{1}j_{2}j_{3}} = \frac{1}{\pi} \left( \frac{e^{2} k_{\gamma}}{E_{\gamma}} \right)^{\frac{1}{2}} \int_{0}^{\infty} r^{j_{3}} \, \mathcal{G}(r) \, \mathcal{X}(r) \, \, .$$

(2.436)

$S_{j}$ is a spectroscopic factor.

The reaction amplitude, Eq. (2.283), may be limited to resonant transitions involving only local levels by including just that part of the collision matrix, i.e., the effective gamma width (the term in parentheses in Eq. (2.348)) in Breit Wigner expressions. This local resonance amplitude may be added to the direct capture reaction amplitude to form the total reaction amplitude,

$$ T_{e_{m}^{B}:m_{A}^{m}} = A_{e_{m}^{B}:m_{A}^{m}} + B_{e_{m}^{B}:m_{A}^{m}} \, \, \, e^{i \epsilon}.$$  

(2.437)

There is no a priori reason to assert that the coherence term, $c$, is independent of the projection quantum numbers or is energy independent; this presumption is made only for simplicity. In this form, the amplitude $B$, in the internal region, represents a background contribution to the local
resonant transitions. If the potential used to form $B$ were correct, one would expect this term to dominate the sum, Eq. (2.437), in certain energy regions - corresponding to the formation of broad single particle resonances. If the interior contribution, were removed from the scattering amplitude $B$, this term would become the hard sphere scattering amplitude.

For a polarized beam and unpolarized target, the reaction cross section is

$$\frac{d\sigma}{d\Omega} = \frac{1}{A^2} \sum_{\text{channels}} T_{\text{c.a.}} \cdot T_{\text{c.a.}}^*$$

or

$$\frac{d\sigma}{d\Omega} = \left( \frac{d\sigma}{d\Omega} \right)_{\text{c.a.}} + \left( \frac{d\sigma}{d\Omega} \right)_{\text{b.c.}} + \left( \frac{d\sigma}{d\Omega} \right)_{\text{int.}} \tag{2.439}$$

Each of the terms in Eq. (2.439) may, for spin 1/2 projectiles, be written in the form

$$\frac{d\sigma}{d\Omega} = \frac{2A^2}{Z^2 A^2} \left\{ \sum_{\text{channels}} a_h P_h + \sum_{\text{channels}} b_h P^*_h \right\} \tag{2.440}$$

The compound nucleus $a$'s and $b$'s are given in Eq.s (2.342) and (2.343); the collision matrix contains only local resonances in this case. The remaining terms are:
\[ a_{h}^{k} = \frac{\gamma}{2} \sum \left( -\frac{1}{k^{k}} \right)^{1/2} \langle u, v \mid \pi \rangle \langle \pi, \rho \mid a \rangle \]  \hspace{2cm} (2.441)

\[ W(i, j, d, l; j', h) W(d, l, j, 2l; a, h), \]

and
\[ b_{h}^{k} = \frac{\gamma}{2} \sum \left( \frac{1}{k^{k}} \right)^{1/2} \langle u, v \mid \pi \rangle \langle \pi, \rho \mid a \rangle \]

where the radial integral \( \langle u, v \mid \pi \rangle \langle \pi, \rho \mid a \rangle \) is defined in Eq. (2.436). The combination \( (u, v, \pi) \) is written to indicate that in the evaluation of this product, \( j' = l' = j \) and \( \ell_{1} + \ell_{2} = \text{even} \). The restriction on \( j' \) follows from geometry, and the second restriction follows from the assumption that there is no parity mixing in the final
The interference terms are

\[ a^I_k = \frac{1}{\Omega} \sum (-)^{L_1+L_2+\Delta} \frac{2^L_2}{L_2} \frac{L_2^2}{L_2} \delta_{L_1} \delta_{L_2} \delta_{\Delta} \langle k_{0} | \mathcal{U}_{1} \mathcal{U}_{2} \rangle \]

\[ = \pi \frac{\delta_{L_2}}{\sqrt{L_2}} \frac{\delta_{L_2}}{\sqrt{L_2}} \delta_{L_1} \delta_{L_2} \delta_{\Delta} \langle k_{0} | \mathcal{U}_{1} \mathcal{U}_{2} \rangle \]

\[ \mathcal{U}_{1} \mathcal{U}_{2} = \langle L_1 | L_{2-1} | k_{0} \rangle \sum b^2 \langle \frac{L_1}{L_2} \frac{L_2}{L_2} \delta_{\Delta} \rangle \]

\[ \langle f_{1}, f_{2}, h_{2}, f_{1} | f_{1}, f_{2}, h_{2}, f_{1} \rangle \]

where \( \Delta \) is as defined earlier in Eq. (2.444) and \( \mathcal{U}_{1}^I \) is the local collision matrix element defined in terms of the effective gamma width.
The analysis of the previous chapter amounts to the incorporation of certain background terms into the ordinary analysis of radiative capture reactions. It was shown that in the simplest case (strong interaction model), a hard sphere background contribution, arising from integration in the entrance channel, is always required. If, in addition, there are elastic scattering resonances, channel radiative capture resonances will also occur. The implication of these background terms (occasionally dominating the foreground terms) on the observation of interference effects in radiative capture reactions is easily seen by considering the form of the background contribution, i.e.,

$$u^J = u^{(\text{int})} + u^{(\text{ext})} \quad (3-1)$$

The background (external) collision matrix element for certain values of spin and parity is occasionally the only collision matrix element for that value of spin and parity. Thus, in general, there will be interference terms consisting of the product of internal matrix elements of a given $J^\pi$ with external collision matrix elements with different $J^\pi$. If this background contribution, corresponding to the
formation of virtual states of given spin and parity, were not included, the predicted interference observables would necessarily be incomplete.

Within R matrix theory, this is not an unusual idea - background contributions of every spin and parity could be included in an analysis of reaction data. In the strong interaction model, it is commonly assumed, however, that only local resonance values of $J^\pi$ will contribute, and neglecting the background for that $J^\pi$ is felt to be tantamount to neglecting the background for $J^\pi$ values not within the set of local resonances. It will be seen that inclusion of the strongest background terms having $J^\pi$ values differing from the local $J^\pi$ values has a dramatic effect on those observables arising from interference between collision matrix elements of differing spin and parity (e.g., $p^*1^2C$). Thus, the common approach throughout the applications of the strong interaction model will be the formation of collision matrix elements, whether from the internal or external region.

If it is desired to account for the resonance peaks in terms of internal and external contributions to the reaction, then it is necessary to have elastic scattering phase shifts to account for the distorted part of the outgoing wave. If these phase shifts are to be extrapolated into an adjacent energy region, it is also highly desirable that they are of a rigorous form, i.e., no ad hoc backgrounds in
place of hard sphere scattering (of any radius). It is occasionally found (e.g., $p^+{^{16}}O$) that the external resonance contribution is entirely adequate to account for a gamma resonance structure. On the other hand, resonances may also be parametrized with the effective gamma partial width (Eq. (2.348)), if elastic scattering phase shifts are not available. In all cases considered, the external contribution to radiative capture reactions is calculated explicitly. As was shown in Chapter II, the existence of the external contribution is model independent and must always be taken into account.

The data that are analyzed have been extracted from published graphs. With several trivial exceptions, no further reduction of the data has been performed. Depending on the format of the graph, the error in extracting data in this manner can range from insignificant ($p^+{^{12}}C$) to appreciable (e.g., in cases where the original data have been plotted on a logarithmic scale). Under no circumstances, however, could the extracted data deviate from the original data by more than an error bar. It is felt that the essence of the conclusions drawn will not be affected by these occasional inaccuracies. The criterion of best fit to the data is that the usual chi-squared function shall be minimized.

The examples were chosen according to the following criteria: data should be available over an energy region containing both resonance and background structure (e.g.,
excitation curves over several MeV) or over an energy region fully characterized via measurement of the interference observables (e.g., differential cross section and analysing power angular distributions); and the resonance structure should not be such as to obfuscate the direct reaction component, i.e., the resonance structure should be sufficiently simple that ambiguities do not arise from inadequate information about the resonances, both elastic scattering resonances and radiative capture resonances. In other words, it is desired to keep the conclusions drawn as unqualified as possible. While it has been noted that elastic phase shift data is not absolutely necessary for this mode of analysis, i.e., a hard sphere backdrop in conjunction with effective gamma partial widths may be used, it is preferred to account for as much of the observed gamma resonance structure as possible in terms of the distorted initial wave and this requires elastic phase shift data.

A. $^3$He($^4$He,$^7$Be)

In the strong interaction model, the absence of resonance phenomena implies that radiative capture proceeds entirely in the external region. This set of circumstances constitutes a limiting case of the strong interaction formalism of Chapter II, i.e., the total collision matrix element is the external collision matrix element only. As the
external collision matrix elements are known, with the exception of a quantity proportional to the energy independent reduced partial width of the bound state, all observables are determined to within a scaling factor.

If external transitions are limited to those of an electric dipole nature, then from Eq. (2.270) (in particular the Clebsch Gordan coefficient: \(<Q0,10|Q^*0>\), only certain partial waves of the continuum wavefunction are involved (and these partial waves need not reflect the presence of an internal wavefunction). An example of these conditions is afforded by the elastic scattering of \(^3\)He from \(^4\)He. The final bound states of \(^7\)Be are characterized with an \(L^*=1\) (cf. Fig. III-1).

Fig. III-1 Level diagram for \(^7\)Be indicating the two possible gamma transitions in the region of the experiment of Ref. (29) as well as various Q values involved and the locations and quantum numbers of the various excited states.

From the Clebsch Gordan coefficient \(<L0,10|L^*0>\), one
need be concerned with only the S and D wave elastic scattering phase shifts. The S wave phase shift over a bombarding energy range of 2.5 to 12 MeV is accurately described by scattering from a charged hard sphere of radius 2.8 fm. The D wave phase shifts (J=2±1/2) are approximately equal over a bombarding energy range of 2.5 to 12 MeV and are well represented by scattering from a charged hard sphere of radius 2.8 fm. The P and F partial waves cannot contribute to electric dipole transitions to \( l^1 = 1 \) bound states. Note that the channel radius used is extremely small.

The work of Ref. (16) is concerned with radiative capture to the ground and first excited state of \(^7\text{Be}\). Capture to the ground state of \(^7\text{Be}\) involves the following external collision matrix elements

\[
U_{l=0}^\gamma, U_{l=2}^{3\gamma}, U_{l=2}^{5\gamma},
\]

in an obvious notation. Capture to the first excited state of \(^7\text{Be}\), at a binding energy of 1.587 - 1.432 MeV, involves the following external collision matrix elements

\[
U_{l=0}^\gamma, U_{l=2}^{7\gamma}.
\]

Basically, these collision matrix elements are energy dependent integrals - the integrand being a product of a two body normalized bound wave function, the two body elec-
tric dipole operator and the distorted plane wave in the entrance channel. An example of one radial integrand is given in Fig. III-2.

![Radial integrand graph]

Fig. III-2. Radial integrand representing in part the \( \frac{5}{2}^+ \) collision matrix element for the reaction \(^3\text{He}(^4\text{He}, Q)^7\text{Be} \) at 1.556 MeV.

The energy in Fig. III-2 is the center of mass energy. The individual elements of the radial integrand are plotted separately in the top figure. The horizontal dashed line is the bound state wave function. The vertical dashed line is the electric dipole operator. The solid lines are the real and imaginary parts of the distorted plane wave. The zero line on this graph may be found by the converging hard sphere distorted plane wave lines, i.e., the wave function is zero at 2.8 fm. Note that, in the case of a real phase
shift, the real and imaginary parts of the continuum wave function are not linearly independent - one part is obtained from the other with the multiplication of some energy dependent phases. The product of these three functions is plotted in the bottom figure. The dashed line is the imaginary part of the integrand. It is seen that, at this particular incident energy, the radial integrand is substantial up to around 15 fm. For aesthetic reasons, the plot is terminated at a radius determined by the radial integrand dropping to 1 percent of its maximum absolute value - the actual calculation of the integral extends 3 or 4 times beyond this. Convergence of all integrals is assured via the Simpson-Romberg procedure.

In Fig. III-3, the real and imaginary parts of the radial integral corresponding to the 1/2+ external collision matrix element for E1 direct transitions from a 1/2+ initial wave to the 3/2- ground state in 7Be are plotted. The smoothness of the integral with energy is characteristic of the hard sphere nature of the initial wave function and accounts for the smoothly varying character of the cross section. Note also the vanishing of this function as the energy approaches zero - the behavior is a property of the hard sphere phase shifts employed. The remaining radial integrals are plotted in Figs. III-4 through III-7. The 3/2+ and 5/2+ radial integrals are identical because the phase shifts are J independent. There is only a minor
Fig. III-3 $^3$He($^4$He, $^7$Be) $^7$Be radial integral corresponding to the $1/2^+$ external collision matrix element.

The difference in Fig.s III-3 and III-6; and in Fig.s III-4 and III-7. This difference results from a greater binding energy in the final state for $E1$ transitions to the ground state.

In Fig. III-8, eighteen representative points are selected from Ref. (29) and plotted. These points correspond to the sum of the two radiative capture transitions possible in the $^3$He($^4$He, $^7$Be) $^7$Be reaction at the bombarding energies of Ref. (29), i.e.,

$$
\sigma\left(\bar{\tau}_0 + \bar{\tau}_1\right) = 4\pi \left[ A_0(\bar{\tau}_0) + A_0(\bar{\tau}_1) \right].
$$

(3-2)
The theoretical predictions in Fig. III-8 are made without the renormalizing factor $N_E$ (cf. Eq. (2.270)) in order to parallel the calculations of Ref. (16).

In Fig. III-9, the ratio of the cross section for transitions to the first excited state to the cross section for transitions to the ground state for the $^3$He($^4$He, $\gamma$) $^7$Be reaction is plotted. The experimental points, taken from Ref. (29), correspond to

$$\frac{\sigma(\gamma_1)}{\sigma(\gamma_0)} = \frac{A_0(\gamma_1)}{A_0(\gamma_0)}.$$  \hspace{1cm} (3.3)

The predictions (crosses) in both Figs. III-8 and III-9 are
Fig. III-5 $^3\text{He}(^3\text{He}, \alpha)$ $^7\text{Be}$ radial integral corresponding to the $5/2^+$ external collision matrix element.

the result of a least squares fitting with two free parameters. These parameters affect the normalization of the theoretical curves, not their shapes. It is clear that agreement is excellent over the entire energy range. This is the work of T.A. Tombrello and is repeated as a check of the fundamental program. Minor variations of Tombrello's approach may be found in Refs. (33) and (34).

Given the manner in which the data were extracted from the semi-logarithmic graph in Ref. (16), i.e., eighteen out of forty or so points were selected as representative of the whole, and the neglect of error bars, it is clear that the criterion of best fit to the two sets of data will
Fig. III-6 $^3\text{He} (^4\text{He}, \gamma)^7\text{Be} (*)$ radial integral corresponding to the $1/2^+$ external collision matrix element.

probably differ from the criterion used in Ref. (16). Therefore, only approximate agreement is expected in the absolute normalizing values. However, perfect agreement is expected in the branching ratios. Tombrello found normalizing factors of 1.25 and 1.05. The theoretical curves in Figs III-8 and III-9 correspond to normalizing factors of 1.31 and 1.08. There is perfect agreement in the branching ratios, since these numbers are independent of the normalization and the data; reflecting only the wave functions, operators, and manner of integration. The neglect of M1 and E2 transitions, requiring P and F wave initial wave functions, is therefore justified on the basis that Tom-
Tombrello did include these contributions. Tombrello arrived at the same conclusion: M1 and E2 transitions are negligible. It is a matter of some interest that the D wave contributions are essential to a good fit at the lowest energies.

The normalizing factors in Tombrello's work are identified as being

$$\Theta_J^2 = \left( \frac{2}{3} \right) \frac{ma^2}{E^2} \sigma_J^2$$

(3.4)

where $\sigma_J^2$ is the reduced width of the bound state. These are different from the normalizing factors occurring in a
Fig. III-8 The total cross section in microbarns for the $^3\text{He}(^4\text{He}, y)^7\text{Be}$ reaction as a function of center of mass energy. The experimental points (ellipses) are from Ref. (29) and correspond to the sum of direct transitions to the ground and first excited states. The theoretical predictions are shown as crosses.

treatment based on explicit bound state normalization, i.e., the requirement that the bound state shall have unit probability of being somewhere in space (Eq. (2.221)). These normalizing factors are, from Eq. (2.229),

$$\frac{\Theta_{J^P}^2}{\left\{ 1 + \sum \frac{\Theta_e^2}{\epsilon_c} \int_0^\infty \frac{\epsilon^2}{\epsilon_c} \, d\epsilon \right\}}$$

The factor of $(2/3)$ in Eq. (3.4) is a matter of taste; the bound state wavefunction of Ref. (16) is identical to that
Fig. III-9 The branching ratio as a function of center of mass energy for the $^3\text{He}(^4\text{He}, \gamma )^7\text{Be}$ reaction.

of Chapter II with the exception of the renormalizing term $N_b$. In fact, it is impossible to fit the data of Figs III-8 and III-9 with the renormalizing term included, unless the channel radius is changed from the abnormally small 2.8 fm to a more reasonable 4.2 fm.

The normalizing factor given by Eq. (3.4) may take on any value given by the ratio of the unscaled theoretical prediction to the experimental data. It is unsatisfactory that this normalizing factor is unbounded in this manner, as this implies an arbitrary normalization for the bound state wavefunction. In contrast, an upper limit for the normalizing factor is provided by the renormalizing term.
and therefore this term must be included in the external collision matrix elements. In the limit of very large channel radius, the renormalizing term, Nb, approaches 1, and the two normalizing terms become identical. In this limit, the value quantity and external bound wavefunctions (Eqs (2.228) and (2.229)) are zero. It is trivial to observe that direct extraction of the fundamental quantity \( \Theta_4^2 \) from fits to a single set of data is not possible if the renormalizing factor is included. The earlier normalization term used by Tombrello does not correspond to the normalization term of Chapter II.

B. \( _{12}^3 \text{C}(p, \gamma)_{13}^4 \text{N} \)

C. Rolfs has noted an unusual feature in the history of the analyses of the \( _{12}^3 \text{C}(p, \gamma)_{13}^4 \text{N} \) reaction\(^{22}\). In early analyses of this reaction (see, for example, Ref. (17)), it was found that a non-resonant background contribution to the reaction was required to supplement transitions of a Breit Wigner nature. On the other hand, Young et al.\(^{35}\) in a more recent analysis found that the reaction features in the beam energy range that they investigated could be well explained by resonance formation alone; without involving an additional background component. Rolfs' own investigation was in part motivated by these conflicting results.

Rolfs found\(^{22}\) that the conclusions drawn by Young et al.\(^{35}\) are not unique; that their data can also be fitted
equally well with the inclusion of the direct capture process. From the perspective of an analysis of gamma excitation curves at 0° and 90°, Holfs is entirely correct; but it is regrettable that he did not make a stronger statement regarding the validity of the analysis conducted by Young et al. They analysis is, in fact, seriously flawed.

It remains a matter of some curiosity that the fits to the A0, A1 and A2 differential cross section coefficients in Ref. (35) are achieved with precisely the (incorrect) formulas quoted in Ref. (35). Unusual features of these expressions only become noticeable when forming the differential cross section at 0°. In other words, Young et al. did incorporate background contributions to the $^{12}\text{C}(p, ^{13}\text{N})$ reaction (albeit unwittingly).

The initial analysis of the $^{12}\text{C}(p, ^{13}\text{N})$ reaction was carried out in the present work within the limits of conventional resonance theory, i.e., the same as Chapter II, but with the collision matrix elements limited to the internal region. Internal background contributions were included in the form of a diagonal energy dependant $R^0$ matrix, to determine whether in fact a background is essential to account for this reaction. From Eq. (2.287), the expression for the differential cross section is
The excitation and resonance energies for the first and second excited states are from Ref. (22). Also shown are expected direct capture transitions in the $^{12}\text{C}(p,\gamma)^{13}\text{N}$ reaction to the ground state and the three low lying unbound states. The latter states in turn disintegrate predominantly into the $p+^{12}\text{C}$ channel due to their large proton widths22.

\[ \omega_{0}(\theta,\phi) = \frac{\alpha^2}{8} \left[ \mu_{0}^2(\theta,\phi) \omega_{01} + \alpha_1^2 \mu_{02}^2(\theta,\phi) \right. \]
\[ \times \left[ \omega_{m1} + \beta_1^2 \omega_{m2} + \beta_2 \omega_{me} \right] + \alpha_1 \omega_{01} \mu_{02} \right) + \]
\[ \times \left[ \omega_{m2} + \beta_2 \omega_{m2} + \beta_2 \omega_{me} \right] \cos(\delta(\theta) - \delta(\phi)) + \alpha_2^2 \mu_{02} \right) \frac{\alpha_1}{8} \right] \]
\[ \times \left[ \omega_{m1} + \beta_1 \omega_{m2} + \beta_1 \omega_{me} + \beta_2 \omega_{me} \right] \]
\[ \times \cos(\delta(\theta) - \delta(\phi)) + \alpha_2 \omega_{01} \mu_{02} \right] \omega_{01} \mu_{02} \left[ \omega_{m2} + \beta_2 \omega_{me} \right] \cos(\delta(\theta) - \delta(\phi)) \right] \]

where,

\[ \alpha_1^2 = \frac{\Gamma(\omega_{01})}{\Gamma(\omega_{01})} = \left( \frac{\Gamma_{\omega_{01}}}{\Gamma_{\omega_{01}}} \right) \left( \frac{E_0}{E} \right)^3 \left( \frac{\alpha_1}{E} \right)^3, \]

Fig. III-10  Low lying levels in $^{13}\text{N}$ from Ref. (22).
\[ \beta_i^2 = \frac{\Gamma_i(E_{\gamma})}{\Gamma_{(m_{1})}} = \frac{\Gamma_i(E_{\gamma})}{\Gamma_{(m_{1})}} \left( \frac{E_{\gamma}}{3.55} \right)^5 \left( \frac{3.51}{E_{\gamma}} \right)^3, \quad (3.7) \]

\[ \alpha_i^2 = \frac{\Gamma_i(E_{\gamma})}{\Gamma_i(E_{\gamma})} = \frac{\Gamma_i(E_{\gamma})}{\Gamma_{(m_{2})}} \left( \frac{E_{\gamma}}{3.55} \right)^5 \left( \frac{3.51}{E_{\gamma}} \right)^3, \quad (3.8) \]

\[ \beta_2^2 = \frac{\Gamma_i(E_{\gamma})}{\Gamma_{(m_{2})}} = \frac{\Gamma_i(E_{\gamma})}{\Gamma_{(m_{2})}} \left( \frac{E_{\gamma}}{3.55} \right)^7 \left( \frac{3.55}{E_{\gamma}} \right)^5, \quad (3.9) \]

and

\[ \Gamma_i(E_{\gamma}) = \Gamma_{i(E_{\gamma})} \left( \frac{E_{\gamma}}{3.37} \right)^3. \quad (3.10) \]

The energy dependence of the gamma partial widths is approximately accounted for by keeping explicit the factors \( E_{\gamma}^{2L+1} \), from Eq. (2.262). Note that from Eq. (2.216), the excitation energy dependence of the gamma partial width is contained entirely in the operator part of this expression. Thus, the extraction of the factor \( E_{\gamma}^{2L+1} \) is an excellent approximation of the partial width energy dependence in the internal region, i.e., \( \Gamma_{\gamma} \) is approximately energy independent.

The 3.55 MeV 5/2+ level in \(^{13}\)N is formed by the capture of a D wave incident proton by the \(^{12}\)C target and decays by either M2 or E3 radiation to the 1/2- ground state (cf. Fig. III-10). The 3/2- level is formed by the capture of P
wave incident protons and decays by either M1 or E2 
radiation to the ground state. The 1/2+ level is formed by 
the capture of S wave incident protons and decays by E1 
radiation to the ground state. The notation of Ref. (35) 
is maintained in Eq. (3.5) et seq. to facilitate compari­ 
son. In Eq. (3.5)

$$\mu(J) = \frac{\pm \sqrt{g}}{(E-E_0)^2 + r^2/4}^{\frac{1}{2}}$$  (3.11)

and

$$\beta(J) = (\beta(J) + \omega e - \omega e)$$  (3.12)

where

$$\beta(J) = \tan^{-1} \left[ \frac{2(E-E_0)}{r} \right]$$  (3.13)

and

$$\omega e = \tan^{-1} \left( \frac{E_0}{G(e)} \right)$$  (3.14)

Eo is that value of E for which E - E0 = 0. In this and 
subsequent work, the boundary condition is chosen so that 
the shift function is zero at resonance. Amendments to 
resonance level positions taken from work with different 
boundary conditions are made appropriately. Following Ref.
(35), the \( \mu (j) \) are considered positive quantities and the indeterminacy of the sign of the interference terms is taken to be contained in the coefficients \( \alpha \). The signs of the \( \alpha \) coefficients are deduced by comparing the differential cross section expression with experimental measurements. The sign of \( \alpha_1 \), for example, is unambiguously determined by considering that part of Eq. (3.5) corresponding to the coefficient of the first order Legendre polynomial \( (\cos \Theta) \), viz.,

\[
A_{1}(\Theta) = \frac{x^2}{4} \Gamma^{2}(\Theta) \left\{ \alpha_1 \mu \epsilon \mu_3 - \alpha_2 \cos \left[ \delta(3\epsilon) - \delta(3\mu) \right] \right. \\
v \left[ (2 + 2\sqrt{3} \beta_1) + \alpha_1 \alpha_2 \mu_1 \mu_3 \cos \left[ \delta(3\epsilon) - \delta(3\mu) \right] \right] \\
v \left[ 6.24 + 1.2 \beta_1 + 6.79 \alpha_1 \alpha_2 \beta_1 \beta_2 \mu_1 \mu_3 \mu_5 \cos \left[ \delta(3\epsilon) - \delta(3\mu) \right] \right].
\]

It is seen from Eq. (3.15) that a reversal of the sign of \( \alpha_1 \) inverts the entire pattern of the predicted \( A1 \) coefficient.

The principle error in the angular correlation expression of Ref. (35) is now apparent. It amounts to a + sign in front of the hard sphere phase shift in Eq. (3.12). There is no discretion in this sign; reversing it is tantamount to replacing outgoing waves with incoming waves. In the work of Ref. (35), the background contribution consists of twice the negative hard sphere phase shift for both the \( 1/2^+ \) and \( 3/2^- \) levels. The source of this error is easily
traced. Young et al.\textsuperscript{35} have lifted the two resonance expression from Ref. (36), and this expression contains the indicated misprint. Young et al. quote Blatt and Biedenharn\textsuperscript{17} as the theoretical source for their phase shift expressions. However, there is no fundamental difference between Blatt and Biedenharn and Lane and Thomas\textsuperscript{3} in regard to their phase shift expressions.

Some of the angle dependant quantities in Eq. (3.5) are

\begin{equation}
\omega_{m_1} = 4\rho_0 - \omega \rho_2, \quad (3.15)
\end{equation}

\begin{equation}
\omega_2 = 4\rho_0 + \omega \rho_2, \quad (3.16)
\end{equation}

\begin{equation}
\omega_{m_1} \omega_2 = \omega \sqrt{3} \rho_2, \quad (3.17)
\end{equation}

\begin{equation}
\omega_{e_1} = \omega \rho_0, \quad (3.18)
\end{equation}

\begin{equation}
\omega_{e_1 m_1} = \omega \rho_1, \quad (3.19)
\end{equation}
\[ \omega_{E_2} = 2 \sqrt{3} P_1 \]
\[ \omega_{m_2} = 6 P_0 + \frac{24}{7} P_2 - \frac{24}{7} P_4, \]
\[ \omega_{m_{m_2}} = \frac{3}{13} (9 P_1 - 4 P_3)/5, \]
\[ \omega_{E_2 m_2} = \frac{6}{5} \left( P_1 + 4 P_3 \right) \frac{12}{7} \left( P_0 + \frac{4}{5} P_3 \right) \]

and
\[ \omega_{E_1 m_2} = \sigma \sqrt{3} P_2, \]

where the \( P_n \) angular functions are the ordinary Legendre polynomials. These expressions are consistent with Ref. (4). The coefficients of Ref. (35) are shown in traces where they are in error, Eqs (3.20) and (3.23). It has been shown in the present work that the errors in the angular coefficients are source of major effects, and that the incorrect angular coefficients in conjunction with the incorrect phase shifts do produce the predicted curves of Ref. (35), i.e., there are no unintentional misprints in this article that would mitigate the conclusions drawn.
There are additional inconsistencies and inaccuracies in Ref. (35) that warrant discussion.

One of the inaccuracies is associated with the acceptance of the approximate ωσ values of Ref. (38). Seagrave found the thick target yield ratio, from annihilation radiation in the reaction $^{12}$C(p,γ)$^{13}$N(β⁺)$^{13}$C, at Ep=2.0 MeV and at Ep=1.0 MeV to be 2.39±.05. The thick target yield at Ep=1.0 MeV is 7.6x10⁻¹⁰ disintegrations per incident proton. From the very approximate relations of Ref. (39), i.e.,

$$\omega_{\sigma} = \frac{\sigma_{e} \mu^{2} E_{p}}{2\pi \hbar^{2} \omega p}$$

$$\omega_{\sigma} = \frac{\lambda^{2}}{8\pi \omega^{2}} \frac{\Gamma_{p} \Gamma_{x}}{\Gamma_{0}}$$

(3.25)

where Y(∞) is the maximum thick target yield (or full step in the thick target yield curve), σe is the target stopping cross section (units of, say, ev·cm²), and λ is the center of mass proton wavelength, one may find an ωσ value for the first 1/2+ resonance of .67 ev, which is the value used in Ref. (35). However, it is well known that this level (Ep=457 keV) is not amenable to the virtually energy independent analysis epitomized by Eq. (3.25). The thick target yield at Ep=2.0 MeV is 1.1x10⁻⁹ disintegrations per incident proton. This number follows from the ratio Y(2.0 MeV)/Y(1.0 MeV)=2.39 and the observation that, for a total width of 70 keV for the 3/2- level,
only 95.5 percent of the full yield of this level is apparent at Ep=2.0 MeV, i.e.,
\[ \frac{\gamma(2 \text{ meV})}{\gamma(1 \text{ meV})} = \frac{\gamma(\frac{\gamma}{2}) + 0.955 \gamma(\frac{\gamma}{2})}{\gamma(\gamma)} \]  
(3.26)

Thus a ratio of yields due to the first and second levels may be found. From Eq. (3.25) and the yield due to the second level, the $\omega \gamma$ value is 1.39 eV. To relate the $\omega \gamma$ values to the ratios of partial widths, it is necessary to determine the branching ratios for decay of any level.

In the immediate vicinity of the second resonance (Ep=1.70 MeV), the gamma ray angular distribution is dominated by the $3/2^-$ resonance alone. Therefore, neglecting other levels, the ratio of the zeroth and second Legendre polynomial coefficients is
\[ \frac{A_0}{A_2} = \frac{4 + 4\beta_2^2}{-2 + 2\beta^2 + 4\beta^2 \beta_1} \]  
(3.27)

Using the experimentally determined ratio $A_2/A_0=-0.05$, it follows that $\beta_1=-0.092$ (i.e., 8.4 percent E2 mixture). The other possible solution to Eq. (3.27) (i.e., $\beta_1=1.4$) can be excluded on transition strength arguments. It is seen that the $3/2^-$ level decays predominantly through $M1$ radiation. The $E2$ branch may be neglected in first approx-
imation in order to form the partial width ratio $\alpha_i$.

From Eq. (3.6),

$$\alpha_i^2 = \frac{\Gamma_{\text{em}}(2.37)}{\Gamma_{\text{em}}(3.51)} \left( \frac{3.51}{3.51} \right)^3 = \frac{(1.39/2)}{.67} \left( \frac{2.37}{3.51} \right)^3 = 0.319 \quad (3.28)$$

Thus, the analysis of Young et al. is based on a total width for the $^{13}$N second excited state of 70 keV. It is recognized in Ref. (35) that a total width for the $J/2-$ level of 51.5 keV (lab) and a total cross section at resonance of 37 ub produces an $\omega\gamma$ of 1.06 eV, from Eq. (3.25). However, this amendment is not incorporated in the analysis. The actual total width for the $J/2-$ level from the source quoted by Young et al. is 60 keV (lab). It is also noted that a 90° differential cross section of 3.71 ub/sr and an $A2/A0$ ratio of -0.65 is not consistent with a total cross section of 37.5 ub at $E_p=1.70$ MeV. To sum up, there is room for improvement in the $^{12}$C($p, \gamma$)$^{13}$N reaction analysis. However, the $\alpha_i$ and $\beta_i$ partial width ratios, determined in this manner, are excellent starting parameters in a general search program.

A program was written to search on the energy independent part of the parameters $\alpha_i$, $\beta_i$, $\alpha_\omega$, and $\beta_\omega$ with the objective of fitting only the $A0$, $A1$ and $A2$ coefficients as given by Young et al. (35), covering the incident proton energy range of 1.5 to 2.0 MeV. The data were shifted back by half the target thickness (24 keV at
Fly. 11.1-11 \(^{12}\text{C}(p, \alpha)^{13}\text{N}\) AO coefficient over the energy region of the \(E_p=1.70\) MeV resonance.

\(E_p=1.70\) MeV to approximately account for energy loss in the target. Normalization of the Legendre coefficients of Young et al. was taken to be \(0.052\) ub/sr/\(mm\) (x-1 for AO; x-2 for A1; x-1 for A2). This number is based on a total cross section at \(E_p=1.70\) MeV of 35 ub, and a 90\(^\circ\) differential cross section of 3.7 ub/sr. The resonance parameters were taken from Ref. (40), as amended by Ref. (41). The resonance partial width \(\Gamma_{2}\) (E1,1/2\(^+\)) was determined by evaluating the theoretical AO coefficient at \(E_p=457\) keV and demanding that the resulting total cross section be 130 ub. The \(\omega_{2}\) value for the 1/2\(^+\) level was found to be 1.5 eV, in agreement with Ref. (42). Excellent fits were
Fig. III-12 $^{12}\text{C}(p, \gamma)^{13}\text{N}$ A1 coefficient over the energy region of the $E_p=1.70$ MeV resonance.

achieved to the experimental A0, A1, and A2 coefficients with the inclusion of the 1/2+ and 3/2- levels and an energy linear 1/2+ and 3/2- background contribution to the reaction. The theoretical expressions used were rigorous, as given in Ref. (3) for a diagonal background $R$ matrix. From this analysis, the 5/2+ level (manifested clearly in $p^{+12}\text{C}$ elastic scattering) could not be strictly excluded as playing a small role in the formation of the A1 coefficient (in particular). The quality of the data, however, is such that the addition of the 5/2+ level for this reaction cannot be justified. Were the 5/2+ level contributing to this reaction in a significant manner ($M2,E3$), the Legendre
coefficients necessary to fit the cross section data would exceed $k=2$ (to $k=4$).

It was not possible to fit the analysing powers of Ref. (42) with the aforementioned levels and background. The general characteristic of the predicted analysing powers was their smallness in comparison to the experimental analysing powers.

Application of the formalism of Chapter II is particularly straightforward in this case. The isolated resonance internal collision matrix elements are of the form

$$U_{l=0}^{\gamma=+}(E_1) \quad U_{l=1}^{3/2-}(E_1) \quad U_{l=1}^{5/2-}(E_2).$$
The external collision matrix elements are

\[ \mathcal{A}_{0}^{\gamma_{1}^{+}}(E_{1}) \quad \mathcal{A}_{1}^{\gamma_{1}^{+}}(E_{1}) \]

The representation of the D wave direct capture to the \( l^s=1 \) final bound state corresponds to a new collision matrix element \( (3/2^{+} E1) \), vis a vis the case where only resonance levels are the source of collision matrix elements \( (1/2^{+} E1, 3/2^{-} M1, 3/2^{-} E2) \).

The part of the analysing power-differential cross section product corresponding to the coefficient of the first
The coefficients of this expression agree with Ref. (43), as amended by Ref. (44). The first external collision matrix element, i.e., \( U(1/2^+) \), adds to the already present element arising from the 1/2+ resonance at \( E_p=457 \text{ keV} \) and thus has little effect on the overall interference pattern.
in the vicinity of the $3/2^-$ resonance. However, as is seen from Eq. (3.30), the $U(3/2^+)$ external collision matrix element interferes with the $M1$ and $E2$ transitions from the $3/2$ resonance and is the source of a sizable analysing power.

The $^{12}\text{C}(p, \gamma)^{13}\text{N}$ A0 coefficient is plotted as a function of the incident proton energy in Fig. III-11. The ellipses are the data of Ref. (35); the horizontal dashed line is the A0 coefficient as calculated with only the external collision matrix elements; the vertical dashed line is the A0 coefficient as calculated with only the internal collision matrix elements; and the solid line is the complete A0 coefficient. It is not suggested that...
there is any significance in this split — it is done merely for convenience, i.e., to serve as an internal check of the programming results. In other words, the solid line represents the best fit to the data. In the absence of the internal or external contributions to this reaction, the best fit would necessarily place the effective horizontal or vertical dashed lines elsewhere. This is to say that the parameters necessary for a best fit with the combined internal and external contributions are not necessarily equivalent to, say, parameters extracted from a best fitting in terms of a resonance with background analysis. The background here is not variable, aside from an overall...
energy independant scaling factor. Nevertheless, one may draw limited conclusions from the placement of the vertical and dashed lines.

From Fig. III-11, the direct capture background to the total cross section ($4 \pi \alpha_0$) is virtually negligible over the resonance. The $A_2$ coefficient, in Fig. III-13, is also affected negligibly by the direct capture contribution. The $A_1$ coefficient in Fig. III-12, however, exhibits the direct capture contribution to this interference coefficient to the extent that the entire pattern is shifted downward in energy. The $A_1$ coefficient in terms of the participating collision matrix elements is

$$A_1 = \frac{\beta^2}{4} \left[ 2 \text{Re} \left[ u^{3/2} \epsilon \nu \right] + \frac{2}{\sqrt{3}} \text{Re} \left[ u^{3/2} \epsilon \nu \right] + \frac{2}{5} \text{Re} \left[ u^{3/2} \epsilon \nu \right] + u^{3/2} \epsilon \nu \right] - 2 \text{Re} \left[ u^{3/2} \epsilon \nu \right] \right].$$

Thus the drop through zero in Fig. III-12 is clearly seen to result from the direct capture component interfering with the dominant M1 transition.

The differential cross section at 0° and 90° is not sensitive to the magnitude or sign of the external contributions to this reaction. The $A_1$ coefficient does not appear in the 90° differential cross section expression and is masked by the larger $A_0$ and $A_2$ coefficients at 0°. Specif-
ically, the differential cross section at $0^\circ$ is

$$\frac{d\sigma}{d\omega}(0^\circ) = \frac{A^2}{4} \left\{ |U_{L^+,E^*}|^2 + |U_{L^-,E^*}|^2 + 3 |U_{L^+,E^2}|^2 + 3 |U_{L^-,E^2}|^2 + 2 \text{Re} \left[ U_{L^+,E^1} U_{L^-,E^1} \right] + \frac{2}{3} \text{Re} \left[ U_{L^+,E^2} U_{L^-,E^2} \right] - 3 \text{Re} \left[ U_{L^+,E^1} U_{L^+,E^2} \right] \right\},$$

while the differential cross section at $90^\circ$ is

$$\frac{d\sigma}{d\omega}(90^\circ) = \frac{A^2}{4} \left\{ |U_{L^+,E^1}|^2 + \frac{3}{2} |U_{L^-,E^1}|^2 + \frac{3}{2} |U_{L^+,E^2}|^2 + \frac{3}{2} |U_{L^-,E^2}|^2 + \frac{3}{3} \text{Re} \left[ U_{L^+,E^1} U_{L^+,E^2} \right] - 2 \text{Re} \left[ U_{L^+,E^1} U_{L^-,E^2} \right] \right\}.$$

On simply considering the large number of sources in each of Eqs. (3.32) and (3.33), it is clear that an analysis based on a consideration of the differential cross section at $0^\circ$ and $90^\circ$ alone can not reasonably arrive at unambiguous conclusions regarding the contribution of the direct capture mechanism. Thus it is only on the basis of interference terms, in any energy region, that the external contribution to the reaction is determined.

External contributions are herein limited to electric dipole contributions; thus an additional (A3) coefficient arises from interference between the $3/2^-$ ($L=2$, $p=2$, $Q=1$)
resonance and the virtual $3/2^+$ resonance (L=1, p=2, $\lambda=2$).
The A3 coefficient is plotted in Fig. III-14. The small size of this coefficient is completely consistent with its neglect in the analysis of Young et al. \textsuperscript{35}.

The analysis program consisted of the variation of six parameters with the intent of minimizing the chi-squared function composed of the experimental A0, A1 and A2 coefficients and the corresponding theoretical quantities. Where possible, the starting parameters in the search were taken from Ref. (35). The variable parameters were the dimensionless reduced partial width of the bound state, the $1/2^+$ E1 partial width at resonance, the $3/2^-$ E2 partial width at resonance, the $3/2^-$ M1 partial width at resonance, a quantity representing the deviation from the input proton partial width as given in Ref. (40) for the $3/2^-$ resonance, and a quantity representing the deviation from the resonance energy as given in Ref. (40) for the $3/2^-$ resonance position. The sum over the remaining bound channels at $E=1.944$ MeV in the renormalizing factor Mb and the sum over negative energy channels at the incident excitation energy in the shift function were neglected for simplicity. An additional free parameter, corresponding to the sum over the remaining bound channels in the factor Mb, could have been included. However, for the $^{13}$N ground state, this additional parameter is expected to be small due to the relatively high separation (binding) energies of other neg-
native energy channels (integral over the external region of these tightly bound configurations is small) and due to the large \( (p+^{12}\text{C}) \) parentage in the \( ^{13}\text{N} \) ground state deduced from stripping and other reactions, and therefore of only negligible influence on the overall normalization factor. The neglect of negative energy channels in the shift function is essentially on the same basis.

![Graph showing external collision matrices](image)

**Fig. III-18** \( ^{12}\text{C}(p, \gamma)^{13}\text{N} \) External collision matrix elements

The best fit parameters are a 0.2 keV shift in the resonance energy; a 6 percent change from Ref. (40) in the proton partial width; an \( E2/M1 \) ratio, the square root of gamma partial widths at resonance, of \(-1.03\) (originally this quantity was \(-0.92\)); an \( \omega\gamma \) value for the first resonance
Fig. III-19 \[ ^{12}\text{C}(p, \gamma)^{13}\text{N} \] Analysing power at 30° over the energy region of the $E_p=1.70$ MeV resonance.

$(1/\omega)$ of 0.35 ev; and a dimensionless reduced width for the p+\[ ^{12}\text{C} \] channel in the $^{13}\text{N}$ ground state of 0.41. These values are not unusual or suspicious with the apparent exception of the small $\omega\gamma$ value for the $^{13}\text{N}$ first excited state.

In fact, within the theory of Chapter II, the size of this quantity, representing internal transitions, is not unusual. The gamma resonance at $E_p=457$ keV corresponds to a strong S wave proton resonance, and thus, in contrast to the situation at $E_p=1.70$ MeV, an electric dipole channel resonance is expected. So the large gamma resonance at $E_p=457$ keV is expected to result from external electric transitions as well as internal transitions. The high
energy tail of the 1/2+ channel resonance at $E_p=457$ keV is incorporated in this analysis, dealing with transitions in the vicinity of 1.70 MeV, through the 1/2+ elastic scattering phase shifts. As the resonance part of the $S$ wave phase shift is negligible in the vicinity of 1.70 MeV, the background in this energy region is not sensitive to the parameters characterizing the 1/2+ $E_p=457$ keV resonance (or any other resonance). Since the initial wave function is distorted primarily by hard sphere phases in this energy region, the external collision matrix elements are straight lines in this limited energy region (cf. Fig. III-18).
Fig.III-21 $^{12}\text{C}(p,\alpha)^{13}\text{N}$ Analyzing power at $112.5^\circ$ over the energy region of the $E_p=1.70$ MeV resonance.

The analyzing powers for the $^{12}\text{C}(p,\alpha)^{13}\text{N}$ reaction in the vicinity of the second resonance are taken from Ref. (42), Fig.s III-19 through III-21. The analyzing powers of Ref. (42) have been shifted forward by 12 keV and multiplied by the differential cross sections as determined by the preceding analysis. The resulting angular distributions of the analyzing power-differential cross section product have been least squares fitted with the $m=1$ associated Legendre polynomials of orders $k=1$ and $k=3$. Terms of order $k=1$ through $k=3$ are expected in this expansion (Fig.s III-15 through III-17); however, as there are only three data points in the analyzing power angular distribu-
tions of Ref. (42), the \( k=2 \) coefficient was neglected on the basis that it was the least significant in this energy region, corresponding to the interference of background terms. The \( k=3 \) coefficient, on the other hand, involves interference between the \( 3/2^- \) resonance and background, and is larger.

In Figs. III-15 through III-21, the solid line is the prediction for these corresponding quantities resulting from the fit to only the \( A_0, A_1 \) and \( A_2 \) coefficients of Ref. (35). The theoretical curves are merely overlaid on the analysing power data; the analysing power data have not been included in the chi-squared minimization program. It is encouraging to note that if the large size of these analysing powers is accurate (indicated in Fig. III-15), the extranuclear direct capture mechanism, as formulated in Chapter II, is entirely adequate to describe the \(^{12}\text{C}(p, \gamma)^{13}\text{N} \) reaction over the energy region of the second resonance.

C. \(^{16}\text{O}(p, \gamma)^{17}\text{F} \)

Due to the tightly bound \(^{16}\text{O} \) core, the two observed bound states at \( \text{Ex}(J^\pi)=0(5/2^+) \) and \( \text{Ex}(J^\pi)=.495(1/2^+) \) are expected to have large reduced widths (Fig. III-22). So the direct capture mechanism is expected to play a dominant role in radiative capture to these states.
Fig. III-22  $^{17}$F energy level diagram. Only the first four energy levels are shown.

The $^{16}$O(p,γ)$^{17}$F reaction has been investigated in some detail (see, for example, Refs. (21) and (46)) and interpreted in terms of the direct capture mechanism. While there is qualitative agreement in the interpretation of radiative capture transitions below $E_p=3.0$ MeV, terminating in the ground state, Rolfs' conclusion\(^\text{21}\) in regard to the pronounced feature near $E_p=2.66$ MeV in the yield curve for gamma transitions terminating in the first excited state is contrary to the conclusion reached by Domingo\(^\text{46}\).

Rolfs asserted that this anomaly is the result of interference between a Breit Wigner amplitude for the 2663 keV resonance and the direct capture amplitude. Rolfs appar-
ently used hard sphere phases in forming his initial wave function over the entire energy region. Domingo asserted that this anomaly is explicable entirely in terms of transitions occurring in the external region just as a consequence of the resonant part of the $P(1/2)$ partial wave*. From Chapter II, the discrepancy is primarily semantics. Rolfs apparently chose to work with an effective gamma width, whereas Domingo elected to keep the external contributions to the reaction separate from the internal contributions. The effective gamma width, Eq. (2.348), is composed of an internal and external part—the external part being entirely calculable in terms of the elastic scattering resonance parameters.

It was decided to reanalyze the data with the inclusion of an internal resonance collision matrix element to see if, in fact, this source was superfluous as Domingo maintained. There is no evidence to suggest that the internal contribution was incorporated correctly in the analysis of Ref. (46). First, however, a preliminary calculation consisting of radiative capture to the ground state is described.

The proton binding energy of the ground state of $^{17}\text{F}$ is only 601 keV. Thus the proton radial wavefunction extends outside the nuclear radius with significant amplitude (Fig. III-23).
The radial integrals in Figs. III-24 through III-26 are smooth functions of energy since the $P(3/2)$, $F(5/2)$ and $F(7/2)$ partial waves have no resonating components in this energy range. The $5/2^-$ level in $^{17}$F at $E_p=3.47$ MeV is very narrow (<1.5 keV), has very small radiation and particle widths and makes negligible contribution to the $F(5/2)$ phase shift several hundred keV lower in energy. The $3/2^-$ level at $E_p=4.35$ MeV ($\Gamma(CM)=226$ keV) is broad and may decay to the ground state via E1 transitions, and so was included in the initial parameter variation analysis. It provides the only internal collision matrix element for $^{17}$F.
Fig. III-24 $^{160}(p, \alpha)^{176}$ radial integral representing the $3/2^-$ external collision matrix element as a function of energy.

capture to the ground state. The external collision matrix elements are of the form

$$\mathcal{W}_{2-3}^{3/2^-}, \quad \mathcal{W}_{2-3}^{5/2^-}, \quad \mathcal{W}_{2-3}^{7/2^-},$$

and were constructed with the level parameters of Ref. (47), including the channel radius and unusual shift functions.

The results of an essentially one parameter fit are shown in Fig. III-27. The data are from Ref. (21); a gamma detector finite solid angle is implicit in the data ($Q^2 = .94$). The normalizing factor for the direct capture
transition (DC→0) was found to be .41. This may be compared with the normalizing factors of Holfs (.44) and Domingo (.38). The theoretical A2 coefficients quoted by Holfs in Ref. (21), viz., A2=+.13, +.28, and +.34 at Ep=1.10, 1.30, and 2.40 MeV, are far too large for the DC→0 transition, if these numbers are in ub/sr. However, these numbers have the correct magnitude for the DC→495 E1 transition, which is a P→S transition describable by only an A2 coefficient (i.e., a sin^2 θ angular distribution).

The first state in ^17F (1/2+, J'=0) is bound by 106 keV against proton escape. Consequently, the single particle
bound state wave function has significant amplitude to very large radii. At low energies, where penetrabilities inhibit the incident proton from approaching the nucleus, it is seen that the smallness of the incident wave near the nuclear surface in conjunction with a large bound state tail causes the radial integrand maximum to occur at a radius of order 30 fm. (Fig. III-28).

As the incident energy increases, the incident proton moves closer to the nuclear surface and consequently the radial integrand peaks much closer to the surface (Fig. III-29). In Figs. III-28 and III-29, the deviation of the
Fig. III-27 $^{16}\text{O}(p, \gamma)^{17}\text{F}$ differential cross section at $90^\circ$ (top) and $0^\circ$ for the direct capture gamma transitions to the ground state (data are from Ref. (21)). The solid line through the data points represents the theoretical predictions of the present work.

The integrand at $r=5.10$ fm from zero reflects the contribution of the internal wave function at these energies. Thus, at 2.282 MeV, the internal contribution (1/2- resonating at the cm energy 2.505 MeV) is negligible, since the zero intercept reflects the hard sphere phase shift. However, at 2.917 MeV (beyond the 1/2- resonance), the small width of this resonance (19 keV (cm)) is still felt in the $P(1/2)$ phase, i.e., the radial integrand does not intercept the $r=5.10$ fm point at zero. The internal wavefunction near the resonance is the source of the internal collision
matrix element. The significance of the internal wavefunction with respect to all gamma transitions is measured by the degree to which an internal collision matrix element is necessary in describing observables.

In Fig. III-30 and III-31, the radial integrals are plotted as a function of energy. The structure in the 1/2-radial integral results from the resonating phase shift in the incident wave at $E_p = 2.663$ MeV. This structure is a channel resonance. The internal collision matrix elements for gamma transitions to the first excited state in the parameter variation program are represented by

![Graph showing radial integrand with energy and radius axes.](image-url)
Fig. III-29 \(^{16}O(p, \gamma)^{17}F(*)\) radial integrand at \(E(\text{cm})=2.917\) MeV corresponding to the formation of a \(1/2^-\) external collision matrix element.

\[
U_{l=1}^{\gamma^-} (0.66 \text{ meV}) \quad U_{l=1}^{3\gamma^-} (4.35 \text{ meV})
\]

The resonance parameters are from Ref. (47). The external collision matrix elements are represented by

\[
U_{l=1}^{\gamma^-} \quad U_{l=1}^{3\gamma^-}
\]

The results of the analysis are shown in Fig. III-32. The data are from Ref. (21) and reflect a finite counter
Fig. III-30 \(^{16}\text{O}(p, \gamma)^{17}\text{F}(\gamma)\) 1/2- radial integral attenuation coefficient of \(Q^2=0.94\). Otherwise, the yield at 0° at low energy (away from the resonance) would be zero, since the \(P\rightarrow S\) transition is described by a \(\sin^2 \Theta\) angular distribution \((\Delta O+Q^2\Delta \Theta=0, \text{if } Q^2=1)\).

It has proven impossible to extract, from the figures in Ref. (21), the DC->495 differential cross section data over the 1/2- resonance region near \(E_p=2.66\) MeV — the scale is such that 42 keV are compressed to a millimeter. So the easier to extract data of Ref. (46) are plotted in Fig. III-33 only to suggest the resonance structure in this region at 90°. The data of Ref. (46) are uncorrected for solid angle and target thickness (9 keV for 2.66 MeV pro-
Fig. III-33 $^{16}O(\mu, \gamma)^{17}P(*)$ 3/2- radial integral

tons); the analysis of Ref. (46) consisted of modifying the theoretical expressions to conform with the experimental conditions, which are not given in any detail. Fig. III-33 is a blow-up of Fig. III-32 in the vicinity of 2.66 MeV and the data of Ref. (46) have been shifted back by 4.5 keV. The target of Ref. (46) was not of uniform density (oxide layer on tungsten); the energy spread of the incident beam at target was approximately 3 keV at 2.66 MeV. The data in Fig. III-33 have been normalized at the low energy end with the data of Ref. (21).

In Fig. III-33, the vertical dashed line represents the predicted differential cross section at 90° and 0° arising
Fig. III-32 $^{16}O(p, \gamma)^{17}F(\gamma)$ differential cross section at $90^\circ$ (top) and $0^\circ$. The data are from Ref. (21). The solid line through the data points represents the theoretical predictions.

from internal transitions alone; the horizontal dashed line arising from external transitions alone; and the solid line reflects the differential cross section formed from the sum of the (presumed) internal and external collision matrix elements.

Given the nature of the cross section data in this energy region, it is difficult to draw conclusions regarding the necessity of an internal gamma transition. From Fig. III-33, the qualitative features of the data are reproduced by gamma transitions in the external region alone. The data in Fig. III-33 were not included in the
chi-squared minimization program. For purposes of illustration, the gamma width for the 3.11→.50 MeV resonant transition was fixed at 12 meV²¹. The effect of this arbitrary choice is quickly damped on either side of the resonance.

The A₀, A₂, and B₂ predicted coefficients are shown in Figs. III-34 through III-36 in the vicinity of the 2.66 MeV resonance with the parameters and conventions of Fig. III-33. An analysing power is possible (Fig. III-36) even when gamma transitions are limited only to the external region. This is a trivial observation, since an alternative
Fig. III-34 \(^{16}\)O(p, \gamma \gamma)\(^{17}\)F(*) A0 coefficient in the vicinity of the 2.66 MeV resonance.

View of the capture process is that the gamma transitions are arising from interactions within an extended nucleus (to approximately 100 fm); it happens that in a shell of this volume \((r>a(c))\), the initial and final wavefunctions and interaction operator are known.

The normalization factor for the DC->495 E1 transition was found to be .45. The corresponding factor found by Rolfs\(^{21}\) is .44; and by Domingo\(^{46}\) is .57. The angular distribution for the DC->495 E1 transition is, well below the \(E_p=2.66\) MeV resonance,

\[
\frac{d\sigma}{d\Omega} = A_2 [Q_2 - 1] - Q_2 A_2 \frac{3}{2} \sin^2 \theta .
\]  
(3.34)
Angular distributions for the DC→495 transitions at £p=1.10, 1.80, and 2.40 MeV are plotted in Ref. (21) as a function of sin² Θ (from the text - Fig. 8 in Ref. (21) appears to be mislabelled). The graphs for the DC→495 transition have a small positive intercept and a positive slope. For Q²=.94, it is seen from Eq. (3.34) that these features can occur only for a negative A2 coefficient. The A2 coefficients in the present work were found to be -.12, -.28, and -.39 ub/sr at £p=1.10, 1.80, and 2.40 MeV (approximately). These compare very favorably with the experimental 'A2' coefficients from Ref. (21) at these energies: .15±.02, .29±.03, and .37±.04 (a positive A2
Additional radiative capture data in the form of high resolution cross section and analysing power angular distributions over the 2.66 MeV resonance would be useful in resolving the question of whether internal gamma transitions are significant or whether, as now seems possible, the gamma transition at 2.66 MeV is describable in terms of extranuclear transitions alone, which was Domingo's assertion. Certainly, in order to assess the magnitude and sign of an internal wavefunction by partial width measurements, it is necessary to account for that part of the gamma tran-
position that occurs in the external region.

Before drawing conclusions in regard to the approaches followed by Rolfs and Domingo, it is worthwhile to recall that the extranuclear capture process is only weakly model dependant (relying on the validity of an expansion of the bound state wavefunction in terms of two body configurations\(^3\)). The electromagnetic operators in a nuclear potential free region are completely known and elastic scattering phase shifts are measurable (emphasis is on gamma transitions). On the other hand, internal gamma transitions rely on wavefunction and operator models for their description. Thus, accounting for a gamma resonance in terms of the nearly model independant extranuclear process, which was the thrust of Domingo's work, is important. Although the equivalence between Rolfs' and Domingo's approach has been shown in Chapter II, resorting to an effective gamma partial width as Rolfs has implicitly done is felt to be an unnecessary parametrization of the 2.66 MeV gamma resonance. It is the contention of the present work that the nearly model independant extranuclear capture process is identifiable as a proportion of all gamma transitions and thus is reaction mechanism revealing.

D. \(^{20}\text{Ne}(p, \gamma)^{21}\text{Na}\)

The excitation energies as well as the spins and parities of the low lying states in the nucleus \(^{21}\text{Na}\) are fairly
well known (Fig. III-37). Recently, high resolution studies (Ref.s (50), (51) and (52)) of the p+²⁰Ne radiative capture reaction were prompted by the apparent lack of correspondence between mirror levels for the mass 21 system.

Rolfs\textsuperscript{52} supplements experimental work that included the discovery of new resonances occurring in the radiative capture reaction with a reaction mechanism analysis. An analysis pointing toward identification of interference sources is not as forbidding as it first appears, since many of the states below Ep=2.0 MeV are very narrow and may be ignored in this type of analysis. The most prominent feature in the differential cross section data at 90° for gamma tran-
sitions to the first excited state in $^{21}_{}$Na ($5/2^+$ at 332 keV) is not an artifact of a thick target. Rather, it appears to be an interference pattern involving a broad resonance. Bolfs has assigned an energy and width to this resonance based on his direct capture analysis in conjunction with resonance formation. The energy and width do not agree with the identification of Ref. (51), based on a high resolution gamma experiment in conjunction with time-of-flight spectra for the $^{20}_{}$Ne($d,n$)$^{21}_{}$Na reaction. Bolfs acknowledged that a rigorous explanation of these discrepancies is not possible as no detailed excitation curves around this broad resonance are shown in Ref. (51), and then reevaluated the time of flight spectrum shown in Fig. 4 of Ref. (51), using the known excitation energies for the calibration of the neutron peaks on both sides of the peak of interest (no. 8), to arrive at an excitation energy and width much closer to those of his own work. Fig. 4 of Ref. (51) is uncorrected for the energy dependance of the detector's efficiency. If the neutron peaks are not too distorted over a region of several hundred keV, the 52 keV displacement that Bolfs required corresponds to a displacement in the time of flight spectrum of Ref. (51) of 2 millimeters. This is a large displacement. Since the width is large and the peak is not sharp, the discrepancy between the assignments of Ref.s (51) and (52) remains. Therefore, it is of some interest to reexamine this reaction in light
of the results of Chapter II. The $^{20}\text{Ne}(p, \alpha)^{21}\text{Na}$ data are from Ref. (52).

In the energy region between $E(\text{cm})=486$ keV and $E(\text{cm})=1976$ keV, there are six known levels. There are two sets of two levels with the same spin and parity assignments, i.e., 3/2$^-$ at $E_p=1830$ keV and $E_p=1311$ keV; and 5/2$^+$ at $E_p=1955$ keV and $E_p=1169$ keV. The remaining two states are: 9/2$^+$ at $E_p=417$ keV and 5/2$^-$ at $E_p=1504$ keV. No variation of parameters is possible with the 1504 keV resonance as no measurements were made in this vicinity due to nearby $^{22}\text{Ne}(p, \alpha)^{23}\text{Na}$ resonances$^{52}$. The total width of this state is 2.6 eV$^{50}$ and consequently its influence in adjacent energy regions is minimal.

The 9/2$^+$ state at $E_p=417$ keV is assumed to have a total width dominated by the gamma partial width. The gamma partial width for this state is taken to be 24 meV (from its analogue state at 2866 keV in $^{21}\text{Ne}^{52}$). From $\omega_p = 0.062$ meV and a spin parity assignment of 9/2$^+$, the proton partial width is 0.012 meV$^{52}$. This state has a substantial branch (40 percent) to the bound state at 332 keV and is assumed to decay by E2 transitions. Given the narrowness of this state, i.e., it is included in the excitation curve of Ref. (52) and labelled 'thick-target' resonance yield, the parameters for this resonance are not varied in the chi-squared minimization program.
The $5/2^+$ states at $E_p=1169$ keV and $E_p=1955$ keV have branches to the 332 keV bound state (via M1 transitions) of < 3 percent and 63 percent respectively\(^5^0\). The radiative and particle partial widths are taken to be: 12 meV and 15.5 eV for the lowest $5/2^+$ state, and 510 meV and 3930 eV for the $5/2^+$ state at $E_p=1955$ keV\(^5^0\). While a multiple level gamma resonance expression has been implicitly defined\(^3\), the small total width of the lower $5/2^+$ level does not warrant this complication. The total collision matrix for the two $5/2^+$ states is taken to be the sum of two Breit-Wigner forms (an interference term composed of these two levels would influence only the immediately adjacent area of the lowest $5/2^+$ resonance due to its small width and therefore is neglected). The resonance gamma parameters taken from Ref. (50) are not corrected for angular distribution effects - the error limits on the gamma widths of Ref. (50) are 30 percent. However, the major parameters in the chi-squared minimization program are left as variables in fitting Rolfs' data, i.e., the lower $5/2^+$ level parameters are held fixed; the upper $5/2^+$ level's D wave proton partial width is taken equal to the total width\(^5^2\) and the M1 partial width and total width for this level are allowed to vary. Although this state has been observed as a resonance in inelastic proton scattering, the inelastic partial width is negligible compared to the elastic partial width\(^5^2\).
The $3/2^-$ states at $E_p=1311$ keV and $E_p=1830$ keV decay to the $332$ keV state in $^{21}$Na via E1 transitions with branching percentages of 60 and 63 percent$^5$, respectively. The radiative widths for the lower and upper states are taken to be $14$ meV$^50$ and .5 eV$^52$. The total width of the $3/2^-$ state at $E_p=1311$ keV is 193 eV$^50$. The total width of the $3/2^-$ state at $E_p=1830$ keV is initially taken to be 171 keV$^52$. Following Ref.s (50) and (52), the $Q=1$ proton partial width is taken equal to the total width; other positive energy channels (radiative partial widths) in the total width and shift factor are neglected and all negative energy channels are neglected. The neglect of negative energy channels in the shift function is equivalent to neglecting the volume integral of the probability density in the part of the external region with negative energy channels, i.e., one is neglecting the function

$$ N = \sum_{c^-} \gamma_{c^-}^2 \frac{dS_{c^-}}{dE} . \tag{3.35} $$

Except near a threshold for a positive energy channel (where $S(c^-)$ has anomalous behavior), $N$ is expected to be small compared to one. Note that the sum over negative energy channels can not include the configuration $\{p^+^{20}\text{Ne}\}$, which is the (implicit) sum over all permutations of 11 indistinguishable protons and 10 indistinguishable neutrons resulting in this partition. $^{21}\text{Na}$ in combination with a
gamma ray forms a positive energy channel and is being neglected on the basis of small partial widths (included in the numerator but only rarely in the denominator of the collision matrix element). Once the gamma energy is removed from the system, the bound configuration \( \{p + ^{20}\text{Ne}\} \) is included explicitly in the external collision matrix element, consistent with momentum conservation laws, just as a consequence of the initial wavefunction resembling this partition of particles. The bound channels in Eq. (3.34) are just the ordinary ones, e.g., \( \{^4\text{He} + ^{17}\text{F}\} \) sharing an energy equal to the initial energy and bound to the extent that the wavefunction does not extend into the external region with significant amplitude. This bound channel occurs again at a reduced energy in the renormalizing factor for the final bound state. The shift factor for the retained channel is not neglected since the upper \( 3/2^-\) level is broad.

The \( 5/2^-\) state at \( E_p=1504 \text{ keV} \) decays to the \( 332 \text{ keV} \) first excited state via \( E1 \) transitions (64 percent\(^{50}\)). The excitation curve of Ref. (52) omits the energy region of this resonance. The total width of this state is 2.6 \( \text{eV} \) and the gamma partial width is 11 \( \text{meV} \). No variation of these parameters is permitted; the level is included as given.

The internal collision matrix elements for the reaction \( ^{20}\text{Ne}(p, \gamma)^{21}\text{Na} \) in the energy region \( E(\text{cm})=486 \text{ keV} \) to
The external collision matrix elements, corresponding to direct capture to an \( \Lambda^t = 2 \) final bound state, may be expressed in terms of the \( P(3/2) \), \( F(5/2) \), and \( F(7/2) \) elastic phase shifts. A common channel radius (4.83 fm) and the level parameters used in forming these phase shifts are taken from Ref. (50). The \( P(3/2) \) phase shift over this energy region includes both resonances, the Coulomb and hard sphere phase shifts. The \( F(5/2) \) phase shift is formed
in terms of the single $E_p=1504$ keV resonance, the Coulomb and hard sphere phase shifts. And the $F(7/2)$ phase shift is taken to be the Coulomb and hard sphere phases. The resonance analysis of Ref. (50) is conventional, i.e., no special difficulties are mentioned, but limited to data only in the immediate energy vicinity of the resonances they measured and which included only three angles. The upper 3/2- level is not included in the analysis of Ref. (50) except as a background term near the lower 3/2- level.

In Fig. III-38, the real and imaginary parts of the 3/2- radial integral are plotted as a function of energy. The channel resonance at the position of the upper 3/2- level
occurs as a resonance structure in this integral. The lower 3/2- resonance at around 1250 keV in Fig. III-38 has been stepped over.

The energies at which the integrals are evaluated are the incident proton energies taken from the excitation curve of Ref. (52). It is seen in Fig. III-39 that the 5/2- external collision matrix element is fairly smooth in this energy region. The 5/2- resonance has been partially stepped over. The 7/2- external collision matrix element (Fig. III-40) is in fact smooth over this energy region. The 5/2- and 7/2- external collision matrix elements are not functions of unknown resonance parameters (phase
shifts). On the other hand, the form of the $3/2^-$ external collision matrix element near $E(\text{cm}) = 1600 \text{ keV}$ is completely determined by these parameters. In an ideal situation, only the radiative partial width would be unknown in this type of calculation. However, the resonance position and total width are also unknown in this case. So an estimate of the position and width of the upper $3/2^-$ level is input and the remaining parameters optimized relative to this estimate. While it is entirely possible to recalculate the initial $3/2^-$ wave function and redo the integrals through each iteration of the chi-squared minimization program, it is felt that the very limited data base, i.e., insufficient
constraints, could produce meaningless results.

In Fig. 111-41, the $^{20}\text{Ne}(p, \gamma)^{21}\text{Na}^*(\gamma)$ differential cross section at 90° is plotted as function of energy. The data are from Ref. (52). The dashed line corresponds to the best fit to the data with four free parameters in the vicinity of the interference structure. The scaling factor was established at the low energy end of the data. The total lab width of the upper 3/2- resonance is 180 keV and the resonance energy is that of Ref. (51), some 50 keV below the position assigned in Ref. (52). More data just above 1.6 MeV would be helpful in confirming this analysis.

E. INTERMEDIATE INTERACTION APPLICATION

In principle, the incorporation of distant levels into the collision matrix elements through introduction of the optical potential solutions in the internal region should constitute a refinement of the strong interaction approach. While it is clear that the necessity of a refinement is not indicated in the examples heretofore considered, one could anticipate situations in which an internal background single step contribution to radiative capture reactions is required to either boost the background on which resonances sit or to account for broad structure in the spectra, e.g., radiative capture in heavy nuclei. In general, computations of this sort are extremely complex.
Fig. III-42 Continuum wavefunction variation with decreasing asymptotic radii for $(\ell, j) = (0, -5)$

Fig. III-43 Continuum wavefunction variation with decreasing asymptotic radii for $(\ell, j) = (2, 1.5)$
The intermediate interaction model to be applied is that given in Chapter II. This approach is a generalization of the Rolfs' model\(^1\) to include a complex optical potential with a spin-orbit term. The data used in this application are the \(^{12}\text{C}(p, \gamma)^{13}\text{N} A_0, A_1, \text{and } A_2\) coefficients of Ref. (35), given in section B of this Chapter and analyzed there using the strong interaction model. An alternative, equivalent description of this data in terms of the more elaborate intermediate interaction model is expected to be possible for suitable choices of the optical model parameters. For purposes of illustrating this model, the global optical potential of Ref. (19) is chosen. The energy dependence of the dynamic parameters of this optical potential is ignored over the small energy range (<500 keV) of this application in order to decrease computer running time.

The most time consuming part of the preparatory phase of the intermediate interaction program is the numerical integration of the differential equations governing the behavior of the continuum and bound wavefunctions (Appendixes A and B) in order to form the direct collision matrix elements (Eq. (2.444)).

In Fig.s III-42 and III-43, the real and imaginary parts of a number of continuum radial wavefunctions for the pertinent \((Q,j)\) values for \(E1\) radiative capture to a bound \(1p(1/2)\) state are plotted on top of one another. Each of these radial wavefunctions corresponds to a particular
Fig. III-44  Radial integrand at $E(\text{cm})=1.405$ MeV for $(Q, j) = (0, -5)$

Fig. III-45  Radial integrand at $E(\text{cm})=1.405$ MeV for $(Q, j) = (2, 1.5)$
phase shift. The first radial wavefunction plotted corresponds to a phase shift (for a particular value of \((L, j)\)) determined at an asymptotic radius which was determined by an examination of the optical potential. Subsequent (and overplotted) radial wavefunctions are the result of decreasing the matching (so called asymptotic) radius from which the phase shift and radial wavefunction are calculated. This calculation produces the phase shift and radial wavefunction as a function of the matching radius (Appendix A). The degree to which the radial wavefunction depends on the matching radius is reflected in the extent to which the plot of a radial wavefunction derived from one matching radius is indistinguishable from a radial wavefunction derived from another matching radius, i.e., the lines (radial wavefunction) become thicker as the matching radius is decreased, reflecting an altered wavefunction.

In the example plotted in Fig. III-42, the initial matching radius is approximately 10 fm. (the point at which the real part of the optical potential had fallen to a small fraction of the Coulomb and centrifugal potentials). The matching radius is decreased until the phase shift (for \((L, j)=(0,-5)\)) deviates by more than 1 part in a 1000 from the phase shift (either the real or imaginary part) calculated at approximately 10 fm. In the example plotted in Fig. III-42 \(((L, j)=(0,-5))\), the smallest matching radius occurs at approximately 7 fm. In the example plotted in
Fig. III-46 The real and imaginary parts of the reflection coefficient (top) and radial integrals (bottom) for $(\ell, j, \ell', j') = (0, .5, 1, .5)$ as a function of energy.

Fig. III-47 The real and imaginary parts of the reflection coefficient (top) and radial integrals (bottom) for $(\ell, j, \ell', j') = (2, 1.5, 1, 1.5)$ as a function of energy.
Fig. III-43 \((\ell, j) = (2, 1.5)\), the smallest matching radius occurs at approximately 5 fm. The purpose of this (visual) examination was merely to establish (in an approximate manner) what could be called, in \(R\) matrix theory, the channel dependant internal region. Parenthetically, a phase shift change of 1 part in 100 results in a significant broadening of the radial wavefunction at small radii, and thus the transition integrals would be altered significantly were the criterion of 1 part in 100 adopted. The channel radii were determined at the lowest energy in the range of interest, which is less than 500 keV wide.

In Fig.s III-44 and III-45, the factors of the radial integrands are plotted as a function of radius. The vertical dashed line (top) is the wavefunction for a proton bound by 1.944 MeV in the \(1P(1/2)\) shell in \(^{13}\text{N}\). The horizontal dashed line (top) is the electric dipole operator. The solid lines (top) are the real and imaginary parts of the continuum wavefunction for specified \((\ell, j)\). An external region (larger interval between points in Fig.s III-44 and III-45) is defined in this approach just for computational ease - there is no special significance to the boundary. The real (solid line) and imaginary parts of the radial integrand are plotted in the bottom half of Fig.s III-44 and III-45. These curves may be used to establish which is the real part of the continuum radial wavefunction in the top half of these figures. Thus, for
Fig. III-48 $^{12}\text{C}(p, \gamma)^{13}\text{N} A0$ coefficient as a function of energy.

Fig. III-49 $^{12}\text{C}(p, \gamma)^{13}\text{N} A1$ coefficient as a function of energy.
example, the real part of the continuum radial wavefunction in the top half of Fig. III-44 is the solid line that is positive at large radii. Near the origin, the real part of the integrand is small; the surface absorption term in the potential gives rise to a large imaginary part of the integrand. The loss of particles out of the entrance channel into the compound state is absent in Rolfs' model, which uses a real square well potential.

In the bottom half of Fig.s III-46 and III-47, the real (ellipses) and imaginary (rectangles) parts of the internal radial integral and the real (diamonds) and imaginary (triangles) parts of the external radial integrals are plotted as a function of energy. The total direct reaction contribution to the process is the sum of these integrals. The smoothness of these functions with respect to energy variation is retained, i.e., there is no difference in this regard from the strictly external radial integral of the strong interaction model. However, the magnitudes of these integrals considerably exceed those of the strong interaction radial integrals.

In Fig.s III-48 through III-50, the $A_0$, $A_1$, and $A_2$ coefficients of Ref. (35) are plotted along with the best coefficients attainable with this intermediate interaction model for a given set of optical potential parameters. In Fig. III-51, the $B_1$ coefficient is plotted. In these figures, the compound nucleus contribution to the various
Fig. III-50 \( ^{12}\text{C}(p, \gamma)^{13}\text{N} \) A2 coefficient as a function of energy.

Fig. III-51 \( ^{12}\text{C}(p, \gamma)^{13}\text{N} \) B1 coefficient as a function of energy.
coefficients is signified with a plus sign; the direct contribution with a cross; the interference contribution with an asterisk; and the total coefficient with a rectangle. The effective partial gamma widths have been allowed to vary in the chi-squared minimization program. It is seen that the A0, and A2 fits are adequate. The calculated A1 coefficient does not achieve the magnitude of the experimental A1 coefficient below Ep=1.70 MeV. Since the 1/2* and 3/2* background contributions are weighted with a single parameter (spectroscopic factor), the poor fits of Figs. III-49 and III-51 are the results of introducing the two background terms in the relative proportions shown in Figs. III-46 and III-47.

It is not to be concluded that the intermediate interaction model presented here cannot account for the $^{12}$C(p, $\gamma$)$^{13}$N reaction. Given the numbers of variables in this model (among these are the geometric and dynamic parameters of the optical potential), it is likely that a fit could be achieved. The questionable significance of the parameters associated with a good fit did not seem to justify the expenditure of computer time necessary to obtain such parameters.
IV. CONCLUSIONS

An application of B-matrix theory to radiative capture reactions has been presented. The development began with the formulation of photon wavefunctions and photon collision matrix elements. The photon collision matrix elements, representing the dynamics of a radiative capture transition from an initial to a final state, were divided into two contributions according to whether the spatial integration involved the internal or external region. This treatment parallels R-matrix theory, in which it is assumed that there is some radius beyond which the nuclear potential is negligible and the wavefunctions in this external region are described by known Hamiltonians. Consequently, the collision matrix elements corresponding to gamma transitions in the external region are very nearly model independent. In contrast, the internal collision matrix elements are completely dependent on a model of nuclear interactions for their evaluation. In the internal region, the initial wavefunction was obtained from the R-matrix formalism. Through the initial internal wavefunction, nuclear interaction models were introduced and discussed.
It was seen in Chapter III that very satisfactory fits to radiative capture data at low excitation energy in light nuclei were obtained in terms of the strong interaction formalism of Chapter II. Specifically, the internal transitions were represented entirely in terms of compound nucleus formation and the external transitions in terms of matrix elements, whose calculation requires knowledge of the elastic scattering phase shifts but not of the optical potential. Although not indicated explicitly, it is clear that nothing in the present work prevents an internal background contribution from being incorporated, in addition to the extranuclear transitions, in the usual R-matrix fashion (the internal background contributions implicitly represent other reaction mechanisms). No internal background was necessary to account for the reactions treated in Chapter III. From this fact, one may conclude that a direct internal reaction mechanism is not discernable at low energies in these nuclei.

The interesting and difficult problem of CN/DI merger in the internal region when both mechanisms are comparable in magnitude has not been addressed rigorously in the present work. Further work continues on this point. In light nuclei at low energies, the concept of compound nucleus formation is probably not the best way to account for nuclear excitations via coupling to the incident nucleon and thus that the R-matrix theory may well not be the most appropri-
ate formalism within which to work. The advantages of $\hbar$-matrix theory (explicit energy dependance in many cases that permits a systematic approach in which large quantities of data are parameterized in terms of energy independent but nonetheless 'black-box' type quantities) are therefore lost in a more nearly microscopic approach, in which the coupling of the entrance channel to the highly complex compound nucleus states is analyzed in more detail by considering intermediate stages (doorway states) in the coupling process (Ref.s 86, 87, 88, 89). On the other hand, the clarity of the physical processes involved in nuclear scattering in a microscopic theory seems well worth the additional effort.
APPENDIX A

SOLUTION OF THE FREE PARTICLE SCHRODINGER EQUATION

The Schrödinger equation for the system is given by

\[
\left[ -\frac{\hbar^2}{2m} \nabla^2 + u \right] \psi = \varepsilon \psi, \tag{A.1}
\]

where,

\[
m = \frac{m_0 m_A}{m_0 + m_A}, \tag{A.2}
\]

is the reduced mass, \( m_0 \) and \( m_A \) being the masses of the incident and target particles, respectively, while

\[
E = E_{\text{lab.}} \cdot \frac{m_A}{m_0 + m_A}, \tag{A.3}
\]

is the energy of the center of mass system, \( E_{\text{lab.}} \) being the lab energy of the incident particle.

The optical potential is here given by\(^\text{19}\),

\[
U(r) = V_{\text{cm}} - V_f(x_0) + \left( \frac{k}{m_0 c} \right)^2 V_{\text{so}} (\vec{\phi} \cdot \vec{L}) \cdot \left( \frac{1}{r} \frac{\partial}{\partial r} f(x_0) - i \left[ W_f(x_0) - 4 W_d \frac{1}{c x_0} f(x_0) \right] \right), \tag{A.4}
\]

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where

$$V_{\text{coul}} = \begin{cases} \frac{2\alpha Z A e^2}{r} ; & r \geq R_c \\ \frac{2\alpha Z A e^2}{\partial R_c} \left(3 - \frac{r^2}{R_c^2}\right) ; & r < R_c \end{cases} \quad (A.5)$$

$$R_c = r_c A^{1/3} \quad (A.6)$$

$$f(x_i) = (1 + e^{x_i})^{-1}, \quad x_i = \left(\frac{r_i - r_c A^{1/3}}{a_i}\right) \quad (A.7)$$

The length factor introduced in the spin orbit term is the \(\pi\)-meson Compton wavelength \(=1.4 \text{ fm}\), \(\vec{\sigma}\) is the Pauli spin operator, and \(\vec{L}\) is in units of \(\text{fm}\). The operators \(\vec{\sigma}\) and \(\vec{L}\) represent, respectively, the spin and orbital angular momentum operators of the incident particle. The total angular momentum of the incident particle is given by \(j = \vec{L} + \vec{S}\), where \(S = \vec{\sigma}/2\) for protons and neutrons. The various terms in Eq. (A.4) represent the Coulomb, central real, spin orbit (real) and central imaginary potentials. \(V(\text{coul})\) is the Coulomb potential of a spherical, uniform charge distribution of radius \(R_c\). The functions \(f(x_0)\), \(f(x_0)\), \(f(x_d)\) and \(f(x_{so})\) are Woods-Saxon form factors with appropriate radius and diffusivity parameters. The central imaginary potential corresponds to either volume, surface or volume plus surface absorption of particles out of the
entrance channel. The form of the optical potential is purely phenomenological, although the existence of a nonlocal complex scattering potential is well justified. In general, the dynamic parameters \( V, V_{so}, W, \) and \( W_d \) are functions of the bombarding energy.

Separation of the variables appearing in the complete Schrödinger equation leads to the usual radial equation for each value of the orbital and total angular momentum, \( \ell \) and \( j \):

\[
\left\{ \frac{1}{r^2} \frac{d}{dr} \left[ r^2 \frac{d}{dr} \right] - \frac{\ell (\ell + 1)}{r^2} - \frac{\ell}{\ell + 1} \left( \frac{\ell}{\ell + 1} \left( \frac{\ell}{\ell + 1} \right) \right) \right\} \chi_{\ell}^j = 0 ,
\]

where

\[
g(r) = -E + V_{coul} - V_{f}(\ell) - i \left[ W_f(\ell) - \right. \\
- \left. + W_{\ell} \frac{d}{dr} f(\ell) \right] ,
\]

and

\[
h(r) = \left( \frac{\hbar}{m_n c} \right)^2 \frac{d}{dr} f(\ell) \left[ \ell (\ell + 1) - (\ell + 1) \right] \]

The functions \( g(r) \) and \( h(r) \) are well behaved at the origin. \( \chi_{\ell}^j (r) \) is the radial wave function which must satisfy the following boundary conditions: it must vanish at the origin, and the asymptotic form of the solution must match a distorted plane wave evaluated at the point \( r=a(c) \) beyond which the nuclear potential is negligible is comparison to
the centrifugal and Coulomb potentials, i.e.,

\[ \chi_2^l(r) = -\frac{\omega}{2}\left(\frac{G_l}{r} - iF_l\right) - \frac{i}{2}\left(\frac{G_l}{r} + iF_l\right) \]  \(, \quad (n.11) \)

where \( G_l \) and \( F_l \) are, respectively, the irregular and regular normalized Coulomb wave functions; \( \omega \) is the Coulomb phase shift and \( s^*_l \) is the nuclear phase (complex). Other considerations influence the determination of the asymptotic radius for S wave neutrons. All of the physically measured quantities may be expressed in terms of the nuclear phase shift. The asymptotic radius is determined by requiring a change (to, say, 1 part in a 1000) in the nuclear phase shift as the matching radius is decreased. Beyond this radius, the phase shift is constant with respect to the matching radius.

The numerical solution of the radial Schrödinger equation is complicated by the centrifugal and spin orbit singularities at the origin. These singularities effectively rule out boundary value techniques (matrix equations) for the solution; thus requiring an initial value approach coupled with at least one iteration to determine the size of the solution. The size of the solution is important, since the wave function is required, not just the phase shift.

A Taylor series is used to determine the solution in the vicinity of the origin. Using these starting values, the
solution is extended over the radial range via a Taylor series method, i.e., with specified $X(0)$ and $X'(0)$, the second derivative is obtained from Eq. (A.8), higher derivatives by differentiation, and both the function and its first derivative are calculated at the new point $d$ away by

$$
\chi_{n+1} = \chi_n + d \chi'_n + \frac{d^2}{d^2} \chi''_n + \ldots \quad (A.12)
$$

which are the quantities needed for the continuation of the process from the point $n+1$ to $n+2$.

There are certain advantages to this method. First, if one is prepared to compute enough derivatives, any truncation error can be avoided. It happens that the computation of higher derivatives of the Woods-Saxon form factor is particularly easy to program. Second, one is not restricted to equal radial intervals in the extension of the solution over the range. One drawback to this method is, in general, that the Taylor series may converge slowly and many terms might be needed in the series for the function and its derivative. It happens that for the potential function given in Eq. (A.4), the series converges rapidly beyond the first several points. Another drawback to this method is that it is very slow, generally, and thus this method is not to be found in programs optimizing the various optical model parameters (cf. Ref. (55)). In the present situation, however, only 30 or 40 accurate integrations
of the radial Schrödinger equation are required and slowness is not apparent. Any integration method may be used after the starting values near the origin are obtained.

Determination of the starting values is based on the fact that the nuclear and Coulomb potentials are negligible in the immediate vicinity of the origin in comparison to the centrifugal potential \(1/r^2\) so that the regular wave function behaves as follows:

\[
\lim_{r \to 0} \chi^d(r) = r^l+1 .
\]  
(A.13)

Hence, a power series solution of Eq. (A.8) must have the form

\[
\chi^d(r) = r^l+1 \left( a_o + a_1 r + \ldots \right). \]  
(A.14)

The \(v\)th derivative of the radial Schrödinger equation (Eq. (A.8)) is:

\[
\chi^{(v)}(r) = \sum_{h=0}^{v} \frac{v!}{h! (v-h)!} \left\{ (-1)^h (h+1)! \frac{\ell(\ell+1)}{r^{2+\ell}} + g^h \right\} \chi^{(v-h)}(r) + \sum_{m=0}^{h} \frac{h!}{m!} \frac{h^m (-1)^{h-m}}{r^{h-m+1}} \chi^{(v-h)}(r) ,
\]  
(A.15)

where the functions \(g(r)\) and \(h(r)\) as well as \(\chi(r)\) have superscripts to signify the derivative order, i.e., for a
superscript equal to 1, the term corresponds to the first derivative of the function. Eq. (A.15) may be expressed in terms of the series solution, Eq. (A.14), and \( r \) set to zero to provide a recurrence relation for the coefficients \( a(n) \), viz.,

\[
a_n \left\{ (n+1)(n+2) - \ell (\ell+1) \sum_{k=0}^{n+\ell+1} \frac{(-1)^k (n+\ell+1)!}{h! (h+2)(n+\ell-1-k)!} \right\} = \\
\sum_{n=0}^{N-1} \frac{g h^{(0)}}{u!} a_{n-2} - n + \sum_{n=0}^{\infty} \frac{1}{h! (n+\ell-1-k)!} \cdot
\]

The first couple terms are for \( l = 1 \):

\[
a_1 = \left[ h^0 a_0 + \frac{1}{4} \right] \quad \text{(A.17)}
\]

\[
a_2 = \left[ g^0 a_0 + h^1 a_0 + h^0 a_1 \right] / 10 \quad \text{(A.18)}
\]

It is seen that the spin-orbit term is responsible for the 'first derivative' coefficient \( a(1) \). Also, in general, only the normalization coefficient \( a(0) \) is unknown. This coefficient may be selected arbitrarily at first, affecting only the size of the solution. In fact, it is initially set to \((1, 0, 0)\) for reasons which will become apparent.

To within a normalization constant, the solution is known in the vicinity of the origin. The solution may be
extended over the range via the Taylor series,

\[ \psi(r+d) = \sum_{n=0}^{\infty} \frac{\psi(n)}{n!} d^n, \quad (A.19) \]

in combination with the radial Schrödinger equation.

Defining

\[ F(n) = \frac{\psi_n}{n!}. \quad (A.20) \]

The \( n \)th derivative of the radial equation, Eq. (A.8), may be written

\[ F'(n+1) = \sum_{h=0}^{n} \binom{n}{h} \frac{n^2}{h^2} \frac{F(n-h)}{(n-h+1)(n-h+2)} + \]

\[ + \sum_{h=0}^{n} \binom{n}{h} \frac{n^2}{h^2} \frac{G(h) F(n-h)}{(n-h+1)(n-h+2)} + \sum_{h=0}^{n} \sum_{m=0}^{n} \frac{H(m)}{(n-h+1)(n-h+2)} \]

\[ \times \frac{h-m}{n^2-m+1} F(n-h), \quad (A.21) \]

where

\[ G(h) = g^h(n) / h! \quad (A.22) \]

and

\[ H(m) = h^m(n) / m! \quad (A.23) \]

\( F(0) \) and \( F(1) \) are determined by the function \( \psi \) and its
derivative at the previous point; and \( r = nd, \) \( d \) being the interval size. The recursion relation for the function \( F, \) Eq. (A.21), is started at \( r = 3d. \) No general relations have been found relating the step size, \( d, \) to the number of derivatives included in the evaluation of the function at any point for a desired accuracy.

The matching condition at the radius \( r = a(c) \) is

\[
\chi_n + \alpha_n \chi'_n = \beta_n, \tag{A.24}
\]

where

\[
\alpha_n = -\frac{O(a_c)}{O'(a_c)}, \quad \beta_n = -\frac{1}{O'(a_c)}, \tag{A.25}
\]

and

\[
O(r) = e^{-i\omega \epsilon} (G_\epsilon(r) + i f_\epsilon(r)). \tag{A.26}
\]

All derivatives are with respect to \( r. \) The quantity on the left hand side of Eq. (A.24) is recognized as the slope of the function

\[
G(a_\epsilon) = \chi_n + \alpha_n \chi'_n - \beta_n \tag{A.27}
\]

with respect to \( a(0), \) the normalization constant, provided the constant \( a(0) \) is \((1, 0).\) Thus, Newton's method for
converging on a root is particularly trouble free here. The desired normalization is

\[ a_0 = \frac{\beta_n}{\chi_n + \alpha_n \chi'_n} \]  \hspace{1cm} (A.26)

This normalization to the size of the asymptotic functions, \( I \) and \( O \), is achieved by re-integrating Eq. (A.8) with \( a(0) \) given by Eq. (A.28).

A general program which computes efficiently the Coulomb functions for any of the parameters \( \eta, \rho, \) and \( l \) represents a considerable undertaking. A program has been written for computing the Coulomb functions over an extensive range of these parameters using the techniques and formulas given by Probery (Ref. (54)), and this program has been used throughout these calculations with the necessity of only one modification in one of the asymptotic region algorithms. (One of the semi-convergent series used in the program cuts off when the terms begin to increase - there was no provision for underflows before this point was reached.)
APPENDIX B

SOLUTION OF THE BOUND PARTICLE SCHRODINGER EQUATION

The radial Schrödinger equation for negative energy is

\[
\left\{ \frac{d^2}{dr^2} - \frac{Q_2(r)}{r^2} - \frac{2 \mu}{\hbar^2} \left[ \Re \left( \gamma \right) + \frac{\hbar \omega}{r} \right] \right\} \psi \rho = 0, \quad (B.1)
\]

where

\[
\Re \left( \gamma \right) = \nu_{\text{coul}} - q \nu_f \left( \omega_0 \right) + |E_b|, \quad (B.2)
\]

where \( E_b \) is the binding energy of the nucleon in consideration relative to the ground state of the nucleon plus target; \( q \) is a scaling factor for the real potential to produce the observed binding energy; and other quantities are as defined in Appendix A.

The boundary conditions on the function \( u(r, \rho) \) are: the function must be zero at the origin, and the function and its first derivative shall match the Whittaker function and its first derivative at a point where the nuclear potential is negligible. The asymptotic radius is generally selected to be the largest of the channel radii, \( a(c) \), found in Appendix A. (As long as the matching point exceeds a mini-
num radius, how far beyond that minimum is the matching point is of no importance.) The function $u(Q,j)$ is normalized by the requirement that its square integral be one, where the infinite radius is defined as one of the points in a range where an asymptotic expression for the Whittaker function holds. No detailed attention has been given to the lack of orthogonality between the initial (continuum) and final (bound) wave functions, as a result of using different single particle hamiltonians. This question has been addressed in Refs. (57) and (58).

The equation for $u(Q,j)$ represents an eigenvalue problem of boundary value type. As with the free particle equation, the singularities at the origin suggest that an initial value method of solution be pursued. The integration of Eq. (3.7) starts at $r=a(c)$ with the values of the Whittaker function and its derivative, i.e.,

\[ u^d_c(a_c) = W_0(n_b, \rho_b) \bigg| \phi_b = a_c \sqrt{\frac{\omega \mu E_b}{\hbar}} , \]

\[ \frac{d\phi_b}{dr} \bigg|_{r=0} = - \frac{dW}{dr} \bigg|_{r=0} \rho_b = a_c \sqrt{\frac{\omega \mu E_b}{\hbar}} , \]

where

\[ R = a_c - r \]
The fourth order Runge-Kutta method is used with these two points to generate starting values for the Milne predictor-corrector process. The integration is (essentially) restarted at \( r=a(c) \) in order to retain necessary flexibility in the interval sizes on either side of \( r=a(c) \). The external interval size is generally much larger than the internal interval size - appropriate to the complexity of the functions in these regions (Fig. B-1).

![Fig. B-1](image)

*Fig. B-1* Single particle radial wave functions as a function of radius. The vertical dashed line corresponds to a neutron bound in a \( 3P(1/2) \) level \((n+207\text{Pb})\). The external-internal boundary is discerned by the spacing of points.

The Milne method is adequate for extending the solution in the internal region (non-variable closely spaced points) to one point in front of \( r=0 \). At this time, an extrapolation...
tion to $r=0$ is performed via a Lagrangian form for $l=0$ points; thus forming a function of $q$,

$$S(q) = \sum_{n=1}^{2k+3} C_n \chi_n(q),$$

where

$$C_i = \frac{\Pi r(i)}{\Pi r(i)},$$

and

$$\Pi r(i) = (r - r_0) \ldots (r - r_{i-1})(r - r_{i+1}) \ldots (r - r_m).$$

The parameter $q$ is initially set to zero - no bound states are expected in the absence of an attractive potential - and then stepped through an increasing, finely spaced set of $q$'s; evaluating the function $S(q)$.

In Fig. B-2, the zeros of the function $S(q)$ are indicated (eigenfunctions of Eq. (B.1)). The evaluation of the function $S(q)$ stops when the $q$-axis crossings equals the number of (input) nodes in the bound state wave function. At this time, a false position algorithm (interpolative, guaranteed convergence) is used to refine the location of the zero crossing.

Proceeding these calculations, a determination of the Whittaker function in the external region and establishing
Fig. B-2 Schematic representation of the function \( S(j) \)

An 'infinity' are made. The Whittaker function is the solution to the differential equation

\[
\frac{d^2W}{d\rho^2} - \left\{ 1 + \frac{2\eta}{\rho} + \frac{\ell(\ell+1)}{\rho^2} \right\} W = 0 , \quad (B.9)
\]

where

\[
\eta = \frac{2\epsilon_2^2 A e^2}{\hbar \nu} , \quad \rho = \hbar \nu , \quad (B.10)
\]

which vanishes at infinity; dominating at small \( \rho \). So, provided a reasonably accurate representation of the function is found at large \( \rho \), lack recurrence from large \( \rho \).
gives the Whittaker function accurately over the range. The asymptotic expansion of the Whittaker function for large \( \rho \) is given by the semi-convergent series

\[
W_\ell(n, \rho) = \frac{e^{-\rho}}{(2\rho)^n} \left\{ \frac{1}{\rho} + \sum_{m=1}^{\infty} \frac{\left\{ \frac{(l+n)^2}{(l+n+1)^2} \cdots \frac{(l+n)^2}{(l+n-m+1)^2} \right\}}{m! (2\rho)^m} \right\} \tag{B.11}
\]

which is evaluated at a tentative 'infinite' radius. If Eq. (B.11) does not converge within the required accuracy, then the 'infinite' radius is increased. The numbers derived from the asymptotic expression, Eq. (B.11), are checked further with the integral representation of the Whittaker function, viz.,

\[
W_\ell(n, \rho) = \frac{(-\rho)^{-n} e^{-\rho}}{\Gamma(l+n+1)} \int_0^\infty t^{l+n} e^{-t} \left( 1 + \frac{t}{\rho} \right)^{-n-l} dt \tag{B.12}
\]

The size of the error involved in not taking the integral, Eq. (B.12), to infinity is easily found for various \( \ell \); the error is made negligible. The integration proceeds via computation with different interval sizes together with linear combinations of successive estimates to improve the approximations obtained (Simpson-Romberg procedure). The approximations to the asymptotic Whittaker function found
by both methods, Eqs (B.11) and (B.12), typically agree to seven figures.

Since the intervals in this external region are variable and usually large, the Fox and Mayer deferred correction process is employed in the back recurrence procedure. With this method, one can use a relatively large interval, take full subsequent account of the truncation errors and know that one has achieved this. The solution is extended three points beyond the point of principle interest (r=a(c)) and convergence is required for the principle point (r=a(c)) - which is the furthest point inward from the asymptotic region. Convergence is usually achieved in less than four iterations - depending on the overall extent of the region of integration.
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