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Shih, Ching-Hsien

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ON GRAPHIC SUBSPACES OF GRAPHIC SPACES

DISSERTATION

Presented in Partial Fulfillment of the Requirements for

the Degree of Doctor of Philosophy in the Graduate

School of The Ohio State University

By

Ching-Hsien Shih, B.S., M.S.

* * * * *

The Ohio State University

1982

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INTRODUCTION

Let $G$ and $H$ be two graphs with the same edge set, and $C(G)$ and $C(H)$ be the cycle spaces of $G$ and $H$ respectively. H. Whitney showed that if $C(G) = C(H)$ then $G$ and $H$ are 2-isomorphic. In this thesis, we try to find the relationships between $G$ and $H$ if $C(G)$ is a subspace of $C(H)$.

For connected graphs $G$ and $H$, if $C(G) \subseteq C(H)$ and $|V(H)| = |V(G)| - 1$ then $C(G)$ is a codimension-1 subspace of $C(H)$. One can easily check that the above conditions apply for the following examples.

(i) $H$ is obtained from $G$ by identifying two distinct vertices.

(ii) $G$ can be constructed from the wheel $W_n$ as follows. Start with $W_n$. Replace each of the $n$ triangles of $W_n$ by a graph whose vertices of attachment are the vertices of the triangle. Then $H$ is obtained from $G$ by "twisting" every pair of the neighboring subgraphs along the two common vertices.
(ii) $G$ can be constructed from $K_4$ as follows.

Start with $K_4$. Replace each of the four triangles of $K_4$ by a graph whose vertices of attachment are the vertices of the triangle. Then $H$ is obtained from $G$ by "twisting" every pair of the neighboring subgraphs along the two common vertices.

(See figure 2 and (iii) of the statements of theorem 1.)

Theorem 1 shows that up to 2-isomorphism, the above examples are all the possible cases for $C(G)$ to be a codimension-1 subspace of $C(H)$.

We prove theorem 1 by three cases. In the first case, we assume $G$ is 3-connected. By lemma 4, we can delete a vertex of $G$ to get a new graph $G'$, so that $C(G') = C(H|E(G'))$. Since $G'$ is 2-connected, we can use Whitney's 2-isomorphism theorem and Tutte's cleavage unit structure of 2-connected graphs to find the relationships between $G'$ and $H|E(G')$. We finish this case by properly replacing those edges we have deleted. In the second case, we assume $G$ is 2-connected. We
prove this case by using the results of the first case and induction. In the last case, we use the results of the first two cases and induction again to finish the proof.

If \( C(G) \) is a subspace of \( C(H) \) with codimension-\( k \) for \( k \geq 2 \), then we are not able to find all the possible relationships between \( G \) and \( H \). However, we found the relationships for some special cases.

Proposition 8 shows that if \( G \) is \((k + 3)\)-connected then up to 2-isomorphism, \( H \) can be obtained from \( G \) by identifying vertices.

Theorem 2 shows that if \( G \) is \((k + 2)\)-connected and \( G \) is not \( L(K_4) \), \( L(K_4) + t \), \( L(K_4) + \{t,t'\} \), then one of the following holds.

(i) There exists a graph \( G' \) obtained from \( G \) by identifying two distinct vertices, such that \( C(G') \leq C(H) \).

(ii) (a) \( G \) is constructed from two edge-disjoint subgraphs \( G_1 \), \( G_2 \) and \( k + 1 \) edges, such that \( G_1 \) and \( G_2 \) have one common vertex, and one end of each of the \( k + 1 \) edges
is in $G_1$ and the other end in $G_2$. Then $H$ is obtained from $G$ by deleting the $k + 1$ edges, splitting $G_1$ and $G_2$ at the common vertex, identifying the ends in $G$ of each of the $k + 1$ edges, then replacing the $k + 1$ edges so that the ends in $H$ of each are two distinct vertices obtained from splitting. (See figure 3 and (ii)(a) of the statement of theorem 2.)

(ii) (b) $G$ is constructed from two edge-disjoint subgraphs $G_1$, $G_2$ and $k$ edges, such that $G_1$ and $G_2$ have two common vertices, and one end of each of the $k$ edges is in $G_1$ and the other end in $G_2$. Then $H$ is obtained from $G$ by deleting the $k$ edges, twisting $G_1$ and $G_2$ along the two common vertices (See P. 8), identifying the ends in $G$ of each of the $k$ edges, then replacing the $k$ edges, so that the ends in $H$ of each are the two vertices we twist with. (See figure 4 and (ii) (b) of the statements of theorem 2.)

We prove theorem 2, by first using lemma 4 to find a vertex of $G$ to delete to obtain a new graph $G'$ so that $C(G')$ is a codimension-$(k - 1)$ subspace of $C(H|E(G'))$. Since $G'$ is
(k + 1) - connected, we can apply theorem 1 or induction to find the relationships between $G'$ and $H|E(G')$. Finally, we properly replace these edges we have deleted.

In theorem 3, we find the relationships between $G$ and $H$, if $G$ is the line graph of a complete graph. We prove theorem 3 by first applying theorem 2 and proposition 8 on certain subgraphs of $G$, then combining these subgraphs appropriately.
CHAPTER I

BACKGROUND

1. Graphs and Spaces

Let $E$ be a finite non-empty set. Consider $P(E)$, the set of all subsets of $E$, to be a vector space over $GF(2)$ (the addition is symmetric difference of sets). If $\Gamma$ is a subspace of $P(E)$ then the subspace of $P(E)$ orthogonal to $\Gamma$ is:

$$\Gamma^\perp = \{A \in P(E) : \forall F \in \Gamma, |A \cap F| \equiv 0 \pmod{2}\}.$$ 

A graph $G$ is a triple, which consists of a finite set $V(G)$ of vertices, a finite set $E(G)$ of edges, and a mapping associating to each edge an unordered pair of vertices called its ends. A loop is an edge with two identical ends. (See [L] P. 534.)

From now on, we assume that $G$ and $H$ are graphs such that $E(H) = E(G)$. For every $S \subseteq V(G)$, let $\delta_G(S)$, the coboundary of $S$, be the set of edges of $G$ which are not loops and have exactly one end in $S$. Let $K(G) = \{\delta_G(S) : S \subseteq V(G)\}$ and let $C(G) = \{E(H) : H$ is an even valency subgraph of $G\}$. Then $K(G)$
is the cocycle space of $G$ and $C(G)$ is the cycle space of $G$. An element of $C(G)$ ($K(G)$) is called a cycle (cocycle) of $G$.

We note that:

**Proposition 1.** $K(G)$ and $C(G)$ are subspaces of $P(E(G))$ and $K(G)^\perp = C(G)$. (See [WE].)

Then, $K(H)$ is a codimension-$k$ subspace of $K(G)$ if and only if $C(G)$ is a codimension-$k$ subspace of $C(H)$. We call a space graphic (cographic) if it is the cycle (cocycle) space of some graph. Any graph $H$ such that $C(H) = C(G)$ is called a realization of $C(G)$.

**Proposition 2.** If $G$ is connected then the set of coboundaries of all but any one singleton-vertices forms a base for $K(G)$.

Let $c(G)$ denote the number of connected components of $G$. Since $K(G)$ is the direct product of the cocycle spaces of its connected components, we have $\dim(K(G)) = |V(G)| - c(G)$.

2. **Whitney's 2-isomorphism theorem**

Two graphs $G_1$, $G_2$ are disjoint if $E(G_1) \cap E(G_2) = \emptyset$ and $V(G_1) \cap V(G_2) = \emptyset$, and $n$ graphs $G_1$, $G_2$, ..., $G_n$ are disjoint
if they are pairwise disjoint. For disjoint graphs $G_1, G_2, \ldots, G_n$, let $G_1 \cup G_2 \cup \ldots \cup G_n$ denote the new graph $G$ with $V(G) = V(G_1) \cup V(G_2) \cup \ldots \cup V(G_n)$, $E(G) = E(G_1) \cup E(G_2) \cup \ldots \cup E(G_n)$, such that each edge has the same ends as before.

**Identification of $n$ vertices** $v_1, v_2, \ldots, v_n$ of a graph $G$ results in a graph $G'$ with $V(G') = (V(G) - \{v_1, v_2, \ldots, v_n\}) \cup \{(v_1, v_2, \ldots, v_n)\}$ where $(v_1, v_2, \ldots, v_n)$ denotes a new vertex, $E(G') = E(G)$ and each edge $e \in E(G)$ has the same ends in $G'$ as in $G$ except if it has $v_i$ as end for some $i \in \{1, 2, \ldots, n\}$ in which case it has $(v_1, v_2, \ldots, v_n)$ instead. Thus each $(v_i, v_j)$-edge becomes a loop on $(v_1, v_2, \ldots, v_n)$. **Splitting a vertex $v$** of a graph $G$ into new vertices $v_1, v_2, \ldots, v_n$ results in a graph $G'$ with $V(G') = (V(G) - \{v\}) \cup \{v_1, v_2, \ldots, v_n\}$, $E(G') = E(G)$. We replace each $(v, u)$-edge $(u \in V(G) - \{v\})$ by an $(v_i, u)$-edge for exactly one $i$, $1 \leq i \leq n$ and replace each loop on $v$ by an $(v_j, v_k)$-edge for exactly one pair $(j, k)$, $1 \leq j \leq k \leq n$. (See \[I\] P. 535 and P. 539.)

For disjoint graphs $G_1, G_2, \ldots, G_n$ and distinct vertices $v_{1,1}, \ldots, v_{1,k_1}, v_{2,1}, \ldots, v_{2,k_2}, \ldots, v_{m,1}, \ldots, v_{m,k_m}$ of
Let $G_1, G_2, \ldots, G_n$ be the graph obtained from $G_1 \cup G_2 \cup \ldots \cup G_n$ by identifying $v_{i,1}, \ldots, v_{i,k_i}$ to a new vertex $(v_{i,1}, \ldots, v_{i,k_i})$, for $i = 1, 2, \ldots, m$. If there is no likelihood of confusion we shall use $v_{i,j}$ to denote the new vertex $(v_{i,1}, \ldots, v_{i,k_i})$, for some $j \in \{1, 2, \ldots, k_i\}$.

Let $G_1$ and $G_2$ be disjoint graphs, $v_1$ and $u_1$ be vertices of $G_1$, and $v_2$ and $u_2$ be vertices of $G_2$. The following operations are illustrated in figure 1.

**Cut-vertex identification.** $C(G_1, G_2) \rightarrow C(G_1, G_2; (v_1, v_2))$.

**Cut-vertex splitting.** $C(G_1, G_2; (v_1, v_2)) \rightarrow C(G_1, G_2)$. This is the inverse operation of cut-vertex identification, so that a graph can be split in this way only at a cut-vertex.

**Twisting.** $C(G_1, G_2; (v_1, v_2), (u_1, u_2)) \rightarrow C(G_1, G_2; (v_1, u_2), (v_2, u_1))$, with $|V(G_1)| \geq 3$ and $|V(G_2)| \geq 3$. We call $G_1$ and $G_2$, sometimes $E(G_1)$ and $E(G_2)$, the cells of the twist. We shall use $T_{E(G_1)}$ or $T_{E(G_2)}$ to denote the twist.
\[ V(G_1) - \{v_1, u_1\} \text{ and } V(G_2) - \{v_2, u_2\} \] are called the interiors of the cells. The interior of any cell is non-empty.

In this thesis, \( G = H \) means \( H \) can be obtained from \( G \) by a relabeling of vertices in \( G \).

Whitney's 2-isomorphism theorem. H. Whitney showed in [W] that \( C(G) = C(H) \) if and only if \( H \) can be obtained from \( G \) by a sequence of the above three operations. (See [W].)

It is natural to try to find the relationships between \( G \) and \( H \) when \( C(G) \) is a subspace of \( C(H) \) (or equivalently, \( K(H) \subseteq K(G) \)). The three theorems of this thesis state such relationships between \( G \) and \( H \) for the following cases.

1. \( C(G) \) is a codimension-1 subspace of \( C(H) \).

2. \( C(G) \) is a codimension-\( k \) subspace of \( C(H) \), \( k \geq 2 \) and \( G \) is \( (k + 2) \)-connected.

3. \( G \) is the line graph of a complete graph and \( C(G) \subseteq C(H) \).
3. **Surface Embeddings.**

If $H$ is a graph obtained from graph $G$ by identifying two distinct vertices in the same component, then $C(G)$ is a codimension-1 subspace of $C(H)$. Thus by proposition 1, $K(H)$ is a codimension-1 subspace of $K(G)$. If we apply this operation $k$ times, then we find a codimension-$k$ cographic subspace of a given cographic space. For a planar graph $G$, let $G^*$ be a planar dual of $G$. If $G^*$ can be embedded on a surface $S$ with Euler characteristic $2-k$, so that each face is simply connected, then the cocycle space of the $S$-dual of $G^*$ is a codimension-$k$ subspace of $K(G)$. But the above two operations cannot generate all the cographic subspaces of a cographic space $K(G)$. (See [R].)

**Example.** Let $G$ be the graph with vertex set \{z, 1, 2, 3, 4\} and edge set \{$g_1$, $g_2$, $g_3$, $g_4$, $h_1$, $h_2$, $h_3$, $h_4$\}, such that the ends of $g_1$ are 1, 4, the ends of $g_i$ are $i-1$, $i$ for $i = 2, 3, 4$, and the ends of $h_i$ are z, $i$ for $i = 1, 2, 3, 4$, and $H$ be the graph with vertex set \{a, b, c, d\}, such that the ends of $g_1$ and $g_3$ are a, d, the ends of $g_2$ and $g_4$ are b, c, the ends of $h_1$ are a, c, the ends of $h_2$ are a, b, the ends of $h_3$ are b, d, and the ends of $h_4$ are c, d.
It is easy to check that $K(H)$ is a codimension-1 subspace of $K(G)$, and $H$ cannot be obtained from $G$ by identifying two distinct vertices. If $H$ can be obtained from $G$ by taking a projective dual of $G^*$, then $\{h_1, h_2, e_1, e_3\}$ and $\{h_3, h_4, e_1, e_3\}$ must be face polygons of the projective embedding, since $\delta_H(a) = \{h_1, h_2, e_1, e_3\}$ and $\delta_H(d) = \{h_3, h_4, e_1, e_3\}$. Because $\{e_1, e_3\}$ is a path in $G^*$, the simply connected region bounded by $\{h_1, h_2, h_3, h_4\}$ in the projective embedding of $G^*$ contains the fifth vertex of $G^*$.

Therefore, $\{h_1, h_2, h_3, h_4\}$ is a face polygon of the embedding. Since $\{h_1, h_2, h_3, h_4\}$ is not a coboundary of any vertex of $H$, we have a contradiction. Thus $H$ cannot be obtained from $G$ by the two operations mentioned above.

P. D. Seymour made the following conjecture. (See [S].)

**Conjecture 1.** Let $G$ be a 2-connected graph, then there is a list of cycles of $G$, with each edge in precisely two of them.

This is equivalent to saying that a 2-connected graphic space contains a cographic subspace with full support. W.T. Tutte made the following stronger conjecture.

**Conjecture 2.** Any 2-connected graph can be embedded on a surface so that the dual graph contains no edge with two identical ends.
N. Robertson made an even stronger conjecture.

**Conjecture 3.** Any 2-connected graph can be embedded on a surface so that the dual graph is 2-connected.

But even a special case of conjecture 1 is still open.

**Conjecture 0.** Let G be a 2-connected graph without a Petersen minor, then there is a list of cycles of G, with each edge in precisely two of them.

Clearly, conjecture 3 implies conjecture 2, conjecture 2 implies conjecture 1, and conjecture 1 implies conjecture 0.

W.T. Tutte raised the following question. "When does there exists a codimension-1 cographic subspace of a space F?"

Theorem 1 solves the special case when F is also cographic.
CHAPTER II

STATEMENTS OF THE THEOREMS

1. The theorems.

Theorem 1. \( C(G) \) is a codimension-1 subspace of \( C(H) \) if and only if there exist connected graphs \( G^* \) and \( H^* \) such that \( C(G) = C(G^*) \), \( C(H) = C(H^*) \) and one of the following holds.

(i) \( H^* \) is obtained from \( G^* \) by identifying two distinct vertices.

(ii) There exist disjoint graphs \( G^*_1, G^*_2, \ldots, G^*_n \), for \( n \geq 3 \) and distinct vertices \( x_i, y_i, z_i \) of \( G^*_i \) for \( i = 1, 2, \ldots, n \), such that \( G^* = C(G^*_1, G^*_2, \ldots, G^*_n ; (y_1, x_2), (y_2, x_3), \ldots, (y_{n-1}, x_n), (y_n, x_1), (z_1, z_2, \ldots, z_n)) \), and \( H^* = C(G^*_1, G^*_2, \ldots, G^*_n ; (y_1, z_2, x_3), (y_2, z_3, x_4), \ldots, (y_{n-1}, z_n, x_1), (y_n, z_1, x_2)) \). (See figure 2.)

(iii) There exist four graphs \( G^*_1, G^*_2, G^*_3, G^*_4 \) and distinct vertices \( x_i, y_i, z_i \) of \( G^*_i \) for \( i = 1, 2, 3, 4 \), such that

\[
G^* = C(G^*_1, G^*_2, G^*_3, G^*_4 ; (x_1, y_2, z_4), (x_2, y_1, z_3), (x_3, y_4, z_2), \]

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(x_{1}', y_{2}', z_{3}, z_{4})), \text{ and } H^* = C(G_{1}', G_{2}', G_{3}', G_{4}'; (x_{1}', x_{2}', x_{3}, x_{4}'),
(y_{1}', y_{2}', y_{3}, y_{4}'), (z_{1}', z_{2}', z_{3}, z_{4}')). \text{ (See figure 2.)}

Remark. If \( G \) is 3-connected then we shall drop the * for \( G \) and \( G_i \)'s, since \( C(G) \) is uniquely graphic. We shall drop the * for \( H \) in (ii) and (iii) if \( G \) is 2-connected. (See the proof of theorem 1.)

If \( X \) is a set disjoint from \( V(G) \) then \( G + X \) denotes the graph obtained from \( G \) by adding the elements of \( X \) as vertices and joining all vertices of \( G \) to these elements. As usual if \( X = \{x\} \) is a singleton, we write \( G + X = G + x \).

Let \( K_n \) be the complete graph on \( n \) vertices, \( L(K_n) \) be the line graph of \( K_n \), and \( mK_n \) be the graph on \( n \) vertices such that there are exactly \( m \) edges joining each pair of distinct vertices. (See [L] p. 530 and p. 536.)

Theorem 2. If \( C(G) \) is a codimension-\( k \) subspace of \( C(H) \), \( k \geq 2 \) and \( G \) is \( (k+2) \)-connected, then one of the following holds.

(i) There exists a graph \( G' \) obtained from \( G \) by identifying two distinct vertices, such that \( C(G') \subseteq C(H) \).

(ii) There exists two disjoint graphs \( G_1, G_2 \), and distinct
vertices \( x_1, x_2, \ldots, x_{k+2} \in V(G_1), y_1, y_2, \ldots, y_{k+2} \in V(G_2) \)

for which either:

(a) \( k+1 \) edges \( e_1, e_2, \ldots, e_{k+1} \) exist with ends \( u_1 \) and \( v_1 \), \( u_2 \) and \( v_2 \), \ldots, \( u_{k+1} \) and \( v_{k+1} \) respectively, such that \( G = (G_1, G_2, e_1, e_2, \ldots, e_{k+1}; (x_1, u_1), (x_2, u_2), \ldots, (x_{k+1}, u_{k+1}), (y_1, v_1), (y_2, v_2), \ldots, (y_{k+1}, v_{k+1}), (x_{k+2}, y_{k+2}) \) and \( H = (G_1, G_2, e_1, e_2, \ldots, e_{k+1}; (x_1, y_1), (x_2, y_2), \ldots, (x_{k+1}, y_{k+1}), (x_{k+2}, u_1, u_2, \ldots, u_{k+1}), (y_{k+2}, v_1, v_2, \ldots, v_{k+1}) \), (See figure 3.) or

(b) \( k \) edges \( e_1, e_2, \ldots, e_k \) exist with ends \( u_1 \) and \( v_1, u_2 \) and \( v_2, \ldots, u_k \) and \( v_k \) respectively, such that \( G = (G_1, G_2, e_1, e_2, \ldots, e_k; (x_1, u_1), (x_2, u_2), \ldots, (x_k, u_k), (y_1, v_1), (y_2, v_2), \ldots, (y_k, v_k), (x_{k+1}, y_{k+1}), (x_{k+2}, y_{k+2}) \), and

\( H = (G_1, G_2, e_1, e_2, \ldots, e_k; (x_1, y_1), (x_2, y_2), \ldots, (x_k, y_k), (x_{k+1}, u_1, u_2, \ldots, u_k, y_{k+2}), (x_{k+2}, v_1, v_2, \ldots, v_k, y_{k+1}) \). (See figure 4.)

(iii) \( k = 2 \), \( G = L(K_4) \), \( H = 2K_4 \) and either:

(a) the edge with ends \( \{i, j\}, \{j, k\} \) in \( G \) has ends \( i, k \) in \( H \) for \( i, j, k \in \{1, 2, 3, 4\} \), or
(b) the edge with ends \( \{i, j\}, \{j, k\} \) in \( G \) has ends \( i, k \) in \( H \) for \( i, j, k \in \{1, 2, 3, 4\} \) with \( \{i, j\} \neq \{1, 2\} \neq \{j, k\} \), the edge with ends \( \{1, 2\}, \{1, h\} \) in \( G \) has ends \( 1, h \) in \( H \) and the edge with ends \( \{1, 2\}, \{2, h\} \) in \( G \) has ends \( 2, h \) in \( H \) for \( h \in \{3, 4\} \). (We may add an edge with ends \( \{1, 2\}, \{3, 4\} \) in \( G \) and \( 3, 4 \) in \( H \).

(iv) \( k = 3 \), \( G = L(K_4) + t \), \( H = 3K_4 \), the edges of \( L(K_4) \) act the same as (iii)(a) and the edge with ends \( t, \{i, j\} \) in \( G \) has ends \( i, j \) in \( H \) for all \( i, j \in \{1, 2, 3, 4\} \).

(v) \( k = 4 \), \( G = L(K_4) + \{t, t'\} \), \( H = 4K_4 \), the edges of \( L(K_4) + t \) act the same as (iv) and the edge with ends \( t', \{i, j\} \) in \( G \) has ends \( m, n \) in \( H \) for \( i, j \in \{1, 2, 3, 4\} \) and \( \{m, n\} = \{1, 2, 3, 4\} - \{i, j\} \).

Remark. Note that (iii)(b) is a special case of (ii). ((\( x_3, y_3 \) = \( \{1, 2\} \), \( x_4, y_4 \) = \( \{3, 4\} \).) We list it in (iii) because theorem 3 has a generalization of (iii).

In theorem 2, alternatives (i) and (ii) are the general cases, and the exceptional cases are derived from \( L(K_4) \). So it is interesting to discuss the case \( G = L(K_n) \).
Theorem 3. If $K(H) \subseteq K(L(K_n))$ for $n \geq 4$ then either:

(i) there exists $G'$, obtained from $L(K_n)$ by identifying two distinct vertices, such that $K(H) \subseteq K(G')$, or

(ii) $H = (n - 2)K_n$ and is such that either:

(a) the edge with ends $(i, j), (j, k)$ in $G$ has ends $i, k$ in $H$ for $i, j, k \in \{1, 2, \ldots, n\}$, or

(b) the edge with ends $(i, j), (j, k)$ in $G$ has ends $i, k$ in $H$ for $i, j, k \in \{1, 2, \ldots, n\}$ and $(i, j) \neq (1, 2) \neq (j, k)$, the edge with ends $(1, 2), (1, h)$ in $G$ has ends $1, h$ in $H$, and the edge with ends $(1, 2), (2, h)$ in $G$ has ends $2, h$ in $H$ for $h = 3, 4, \ldots, n$.

Remark. By proposition 1, we know that $C(G)$ is a codimension-$k$ subspace of $C(H)$ if and only if $K(H)$ is a codimension-$k$ subspace of $K(G)$. Therefore, theorem 3 is a generalization of (iii) of theorem 2.
1. Connectivity.

A loop is an edge with two identical ends. A loopless graph $G$ is said to be $k$-connected if $|V(G)| > k$ and for any deletion of $k-1$ vertices, the resulting graph is connected. The following theorems are immediate consequences of the definition and Whitney's 2-isomorphism theorem.

**Proposition 3.** If $G$ is $k$-connected then the valency of any vertex is at least $k$ and for any deletion of $m$ vertices, $0 < m < k$, the resulting graph is $(k-m)$-connected.

**Proposition 4.** If $G$ is 3-connected then $G$ is the only realization of $C(G)$.

**Proposition 5.** If $G$ is 2-connected and $H$ is a realization of $C(G)$ then $H$ can be obtained from $G$ by a sequence of twists.
The following proposition states basic facts about \( L(K_n) \).

**Proposition 6.** For \( n \geq 3 \), \( L(K_n) \) is \( 2(n-2) \)-connected and 
\[ |V(L(K_n))| = \frac{1}{2} n(n-1). \]

2. **Graphic subspaces.**

We want to find the relationship between \( G \) and \( H \) when \( C(G) \subseteq C(H) \) and there is no graphic space between \( C(G) \) and \( C(H) \). By applying vertex identification, we may assume that \( G \) is connected and will do this for the rest of this thesis. Also in what follows, paths are subgraphs with exactly two vertices of odd valency, called the ends of the paths.

**Lemma 1.** Suppose \( C(G) \subseteq C(H) \) and \( G' \) is the graph obtained from \( G \) by identifying two distinct vertices \( x, y \). Then there exists a path with ends \( x, y \) in \( G \) whose edge set is a cycle in \( H \) if and only if \( C(G') \subseteq C(H) \).

**Proof.** If \( C(G') \subseteq C(H) \) then the edge set of a path with ends \( x, y \) in \( G \) is certainly a cycle in \( G' \) hence a cycle in \( H \). Conversely, suppose such a path exists in \( G \), whose edge set is a cycle of \( H \). It is trivial to see that \( C(G) \subseteq C(G') \). By propositions 1 and 2, we have the codimension is one. So \( C(G') \) is generated by \( C(G) \) and the edge set of the path in \( G \) which is a cycle of \( H \), and so \( C(G') \subseteq C(H) \).
Remark. The existence of a path with ends $x$ and $y$ in $G$ whose edge set is a cycle in $H$ implies that the edge set of any path with ends $x$ and $y$ in $G$ is a cycle in $H$, because for any two paths with the same ends, the $(\text{mod } 2)$-union of their edge sets is a cycle. If there is no likelihood of confusion, we shall use $P$ to denote the edge set of a path $P$.

Lemma 2. If $C(G) \subseteq C(H)$ then either:

(i) there exists a graph $G'$, obtained from $G$ by identifying two vertices, such that $C(G') \subseteq C(H)$ or

(ii) the edge set of every triangle in $G$ is the edge set of a triangle in $H$.

Proof. Since the edge set of a triangle in $G$ is a cycle in $G$ it is also a cycle in $H$. If the edge set is not the edge set of a triangle in $H$, then at least one of the three edges is a loop in $H$. Thus (i) holds, by lemma 1. □

Let $G|S$ denote the induced graph of $G$ on a set $S$. Here $S$ is a subset of $V(G)$ or $E(G)$ depending on the context.

Lemma 3. If $C(G) \subseteq C(H)$, and $A \subseteq V(G)$ is such that $G|A$ is connected and $G|A = H|E(G|A)$ then either:
(i) there exists a graph $G'$, obtained from $G$ by identifying two vertices, such that $C(G') \subseteq C(H)$ or

(ii) for any $v \in V(G) - A$ such that $v$ is joined in $G$ to at least four vertices of $A$ we have $G|(A \cup \{v\}) = H|E(G|(A \cup \{v\}))$.

Proof. Assume (i) does not hold, and let $a_1, a_2, \ldots, a_n$ be all the vertices of $A$ joined to $v \in V(G) - A$ by edges, say $e_1, e_2, \ldots, e_n$ respectively. Consider a cycle in $G$ formed by $e_1, e_2$ and a path in $G|A$ joining $a_1$ and $a_2$. Since the path is still a path in $H$ joining $a_1$ and $a_2$ this implies that exactly one end of $e_1$ is either $a_1$ or $a_2$ in $H$. Similarly, exactly one end of $e_i$ is either $a_1$ or $a_i$ for $i = 2, 3, \ldots, n$. Since $n \geq 4$ this implies that $a_1$ is an end of $e_1$ in $H$. By lemma 1, the other end of $e_1$ in $H$ can not be in $A$. We still denote it by $v$. Similarly the ends of $e_i$ are $a_1$ and $v$ for $i = 2, \ldots, n$. This implies that $G|(A \cup \{v\}) = H|E(G|(A \cup \{v\}))$. □

A loop in $G$ is a cycle, hence a loop in $H$. Therefore, we may assume $G$ is loopless. If $e_1$ and $e_2$ are two edges with the same ends in $G$, then they form a cycle in $G$ hence a cycle in $H$. Therefore either they are loops in $H$ or they have the same ends in $H$. In the former case, if $G'$ is the graph obtained from $G$ by identifying the ends of $e_1$ then $C(G')$ is a graphic space.
between \( C(G) \) and \( C(H) \). In the latter case, we may consider these two edges as a single edge. Therefore, we may assume \( G \) is simple for the rest of this thesis.

**Lemma 4.** If \( C(G) \) is a codimension-k subspace of \( C(H) \) then there exist \( k \) vertices \( v_1, v_2, \ldots, v_k \in V(G) \) such that
\[
C(G - \{v_1, v_2, \ldots, v_k\}) = C(H|E(G - \{v_1, v_2, \ldots, v_k\})).
\]

**Proof.** By proposition 1, \( K(H) \) is a codimension-k subspace of \( K(G) \). By proposition 2, we choose vertices \( v_1, v_2, \ldots, v_n \in V(G), u_1, u_2, \ldots, u_{n-k} \in V(H) \) such that \( \delta(v_1), \delta(v_2), \ldots, \delta(v_n) \) form a base of \( K(G) \), and \( \delta(u_1), \delta(u_2), \ldots, \delta(u_{n-k}) \) form a base of \( K(H) \). By a basic linear algebra theorem, there exist \( k \) vertices of \( \{v_1, v_2, \ldots, v_n\} \), say \( v_1, v_2, \ldots, v_k \), such that \( \delta(v_1), \delta(v_2), \ldots, \delta(v_k), \delta(u_1), \delta(u_2), \ldots, \delta(u_{n-k}) \) form a base for \( K(G) \). Since \( \bigcup_{i=1}^{k} \delta(v_i) \) is the support of the space \( \langle \delta(v_1), \delta(v_2), \ldots, \delta(v_k) \rangle \), the restriction of
\[
\langle \delta(v_{k+1}), \delta(v_{k+2}), \ldots, \delta(v_n) \rangle \text{ on } E(G) - \bigcup_{i=1}^{k} \delta(v_i)
\]

same as the restriction of \( \langle \delta(u_1), \delta(u_2), \ldots, \delta(u_{n-k}) \rangle \) on
\[
E(G) - \bigcup_{i=1}^{k} \delta(v_i).
\]
Hence \( K(G - \{v_1, v_2, \ldots, v_k\}) = K(H|E(G - \{v_1, v_2, \ldots, v_k\})) \), and therefore \( C(G - \{v_1, v_2, \ldots, v_k\}) = \)
\[ C(H|E(G - \{v_1, v_2, \ldots, v_k\})) \]. □

**Remark.** If \( H \) is connected as well as \( G \), then
\[ V(H) = V(H|E(G - \{v_1, v_2, \ldots, v_k\})) \]. Therefore, we can construct
\( H \) from \( H|E(G - \{v_1, v_2, \ldots, v_k\}) \) by adding other edges back
properly.

**Proposition 7.** If \( C(K_n) \subseteq C(H) \) and \( E(H) = E(K_n) \) for
\( n \geq 5 \) then either:

(i) there exists a graph \( G' \), obtained from \( K_n \) by
identifying two distinct vertices, such that \( C(G') \subseteq C(H) \), or

(ii) \( H = K_n \).

**Proof.** We assume (i) does not hold. By lemma 1, any path in
\( K_n \) is not a cycle in \( H \). By lemma 2, any triangle in \( K_n \) is
a triangle in \( H \). Therefore, for any \( K_4 \)-subgraph \( G' \) of \( G \),
either \( C(G') = C(H|E(G')) \) or \( C(G') \) is a codimension-1 subspace
of \( C(H|E(G')) \).

**Case 1.** For some \( K_4 \) subgraph \( G' \), \( C(G') = C(H|E(G')) \).
Since \( G' \) is 3-connected, by proposition 4, \( G' = H|E(G') \).
Because any vertex in $V(K_n) - V(G')$ is joined to all four vertices of $G'$, by applying lemma 3 repeatedly, $H = K_n$.

Case 2. For any $K_4$ subgraph $G'$, $C(G')$ is a codimension-1 subspace of $C(H|E(G'))$. Evidently $|V(H|E(G'))| = 3$.

Let $e_1, e_2, e_3, e_4, e_5$ denote the edges in $K_n$ with ends 1 and 2, 1 and 3, 2 and 3, 4 and 1, 5 and 1, respectively. Since $e_4, e_1$ form a path in $K_n$, by lemma 1, they do not have the same ends in $H$. Similarly, $e_4, e_2$ do not have the same ends in $H$. Because $e_1, e_2, e_3$ form a triangle in $K_n$, by lemma 2, they form a triangle in $H$. By considering the $K_4$ subgraph of $K_n$ induced by $\{1, 2, 3, 4\}$, $e_4$ and $e_3$ have the same ends in $H$. Similarly $e_5$ and $e_3$ have the same ends in $H$. Hence $e_4$ and $e_5$ have the same ends in $H$, which contradicts lemma 1. □

Remark. In (i) $K_{n-1}$ is a spanning subgraph of $G'$, and $G'$ has some multiple edges and a loop incident with the new vertex. But a loop in $G'$ is a loop in $H$, and if $e_1$ and $e_2$ are two edges with the same ends in $G'$ then either they are loops in $H$ or they have the same ends in $H$. Therefore, we may again apply proposition 7 to $G'$ if $n-1 \geq 5$. 
Proposition 8. If \( G \) is \((k+3)\)-connected, \( H \) is a loopless connected graph and \( C(G) \) is a codimension-\( k \) subspace of \( C(H) \) then \( H \) is obtained from \( G \) by identifying vertices.

Proof. We will prove this by induction on \( k \). By lemma 4, there exist \( k \) vertices \( v_1, v_2, \ldots, v_k \) in \( G \) such that

\[
C(G - \{v_1, v_2, \ldots, v_k\}) = C(H|E(G - \{v_1, v_2, \ldots, v_k\})).
\]

By lemma 3, \( G - \{v_1, v_2, \ldots, v_k\} \) is 3-connected. By proposition 4,

\[
G - \{v_1, v_2, \ldots, v_k\} = H|E(G - \{v_1, v_2, \ldots, v_k\}).
\]

Therefore,

\[
V(H) = V(G) - \{v_1, v_2, \ldots, v_k\}.
\]

Since each \( v_1 \) is adjacent to at least four vertices of \( V(G) - \{v_1, v_2, \ldots, v_k\} \) and \( G \neq H \), by lemma 3, there exists a graph \( G' \), obtained from \( G \) by identifying two vertices, such that \( C(G') \) is a codimension-\(((k-2)+1)\) subspace of \( C(H) \). Since \( H \) is loopless, \( G' \) is loopless. Hence \( G' \) is \((k+2)\)-connected. By induction, \( H \) is obtained from \( G' \) by identifying vertices. Therefore, \( H \) is obtained from \( G \) by identifying vertices. \( \square \)


Let \( G \) be a 2-connected graph, and \( x, y \) be distinct vertices of \( G \). We say that \([x, y]\) is a hinge if and only if there exist two disjoint graphs \( G_1, G_2 \), and distinct vertices \( x_1, y_1 \in V(G_1) \),...
x_2, y_2 \in V(G_2) \text{ such that } G = C(G_1, G_2; (x_1, x_2), (y_1, y_2));

where (x_1, x_2) = x, (y_1, y_2) = y, G_1 \text{ is } 2\text{-connected, } |E(G_1)| \geq 2, \text{ and } |E(G_2)| \geq 2. \text{ Let } J \text{ be a fixed subgraph of } G, \text{ } P(G, J) \text{ be the class of all subgraphs of } G \text{ that are not subgraphs of } J \text{ and whose vertices of attachment are in } V(J). \text{ The minimal members of } P(G, J) \text{ are called } J\text{-components of } G. \text{ (See } T). \text{ For each hinge } [x, y], \text{ we assign one virtual edge with ends } x, y \text{ to each } [x, y] \text{-component containing more than one edge. We assign the same virtual edge to two } [x, y] \text{-components of } G \text{ if and only if they are the only } [x, y] \text{-components. W. T. Tutte showed in } T \text{ that } G \text{ can be decomposed into cleavage units, by "cutting" along the virtual edges. Each unit is a bond (a connected loopless graph on two vertices), a polygon, or a simple 3-connected graph. We can reconstruct } G \text{ by "gluing" the units together along the virtual edges. We can also reconstruct } G \text{ from any unit by properly replacing every virtual edge in the unit by the corresponding } [x, y] \text{-component or its complement. A twist on } G \text{ changes either the "orientation" of some virtual edges with the same ends or of the real and virtual edges of a path in a polygon unit. For each twist on a polygon unit, there is a corresponding twist on } G, \text{ and we shall use the same notation to denote them.}

Let } T_{E_1}, T_{E_2} \text{, be two twists on a 2-connected graph } G. \text{ We call them non-crossing if } T_{E_1}(G)|E_2 \text{ is connected. This is}
well-defined, because \( T_{E_1}(G)|_{E_2} \) is disconnected if and only if the vertices of attachment of \( T_{E_2} \) are interior vertices in different cells of \( T_{E_1} \). This implies the vertices of attachment of \( T_{E_1} \) are interior vertices in different cells of \( T_{E_2} \) and \( T_{E_1}(G)|_{E_2} \) is disconnected. It is easy to see that if \( T_{E_1} \) and \( T_{E_2} \) are non-crossing twists then \( T_{E_1} \) is a twist on \( T_{E_2}(G) \), \( T_{E_2} \) is a twist on \( T_{E_1}(G) \), and \( T_{E_1}(T_{E_2}(G)) = T_{E_2}(T_{E_1}(G)) \).

If \( G \) is 2-connected and \( C(G) = C(H) \) then by Whitney's 2-isomorphism theorem, \( H \) is obtained from \( G \) by a sequence of twists. Therefore, except for the cyclic order of the edges in some polygon units, the cleavage units of \( G \) are the same as the cleavage units of \( H \). Since two twists can cross only if they change the "orientation" of the same polygon unit, either \( H \) can be obtained from \( G \) by a sequence of non-crossing twists, or there exists a polygon unit of \( H \) which cannot be obtained from the corresponding unit of \( G \) by a sequence of non-crossing twists.

In the former case, we may assume that any two twists have at most one common vertex of attachment, because if \( T_{E_1} \) and \( T_{E_2} \) are non-crossing twists having two common vertices of attachment then 
\[ T_{E_1}(T_{E_2}(G)) = T_{E_1} + E_2(G) \].
Let \( T_1, T_2, \ldots, T_m \) be a sequence of non-crossing twists on \( G \). A cell \( G_i \) of a twist \( T_i \) is called an end-cell if the interior of \( G_i \) contains no vertex of attachment of other twists. Such a sequence has at least two end-cells.

Let \( x \) and \( y \) be two distinct vertices in \( G \). We say \( H \) is obtained from \( G \) by a sequence of twists \( T_1, T_2, \ldots, T_k \) not separating \( x \) and \( y \), followed by a sequence of non-crossing twists \( T_{k+1}, T_{k+2}, \ldots, T_m \) separating \( x \) and \( y \), if

\[
H = T_m(T_{m-1}(\ldots(T_1(G))\ldots)),
\]

any path \( P_{x,y} \) in \( G \) joining \( x \) and \( y \) is still a path in \( T_i(T_{i-1}(\ldots(T_1(G))\ldots)) \) for \( i = 1, 2, \ldots, k \), and any twist in \( T_{k+1}, T_{k+2}, \ldots, T_m \) separates the ends of \( P_{x,y} \) in \( T_k(T_{k-1}(\ldots(T_1(G))\ldots)) \).

**Lemma 5.** Let \( G \) be a 2-connected graph, and \( x \) and \( y \) be distinct vertices of \( G \). Then \( C(G) = C(H) \) if and only if \( H \) can be obtained from \( G \) by a sequence of twists not separating \( x \) and \( y \), followed by a sequence of non-crossing twists separating \( x \) and \( y \).

**Proof.** Since if \( H \) can be obtained from \( G \) by a sequence of twists then \( C(G) = C(H) \), we need only to prove the other direction.

Two twists are non-crossing unless they change the same polygon unit, and two non-crossing twists are commutative. Therefore, either
H can be obtained from G by a sequence of non-crossing twists or there exists a polygon unit of H which cannot be obtained from the corresponding unit of G by a sequence of non-crossing twists. Thus it is sufficient to prove the case when H is obtained from G by a sequence of twists which change the same polygon unit

\[ p = \{e_1, e_2, \ldots, e_n\} \] of G, where \( n \geq 4 \).

Let \( G_i \) be the subgraph of G corresponding to \( e_i \) for \( i = 1, 2, \ldots, n \). We may assume \( P|\{e_i, e_{i+1}\} \) is connected and \( x \in V(G_1), \ y \in V(G_k), \) for \( n \geq k \geq 2 \), since if \( x, y \in V(G_1) \) then H is obtained from G by a sequence of twists not separating x and y. Let \( P' \) be the polygon unit of H corresponding to \( P \), and \( P'_1, P'_2 \) be the two paths in \( P'|\{e_2, e_3, \ldots, e_n\} - e_k \) from the ends of \( e_1 \) to the ends of \( e_k \). Let \( C_1', C_2', \ldots, C_h' \) be the connected components of \( e_1 \cup P'_1 \cup e_k \cup (e_1, e_2, \ldots, e_k) \) ordered by the order of \( P_1' \), \( C_{h+1}', C_{h+2}', \ldots, C_m' \) be the connected components of

\[ e_1 \cup P'_2 \cup e_k \cup (e_1, e_2, \ldots, e_k) \] ordered by the order of \( P_2' \),

\( C_1', C_2', \ldots, C_{h-1}' \) be the connected components of \( P'_1|\{e_{k+1}, e_{k+2}, \ldots, e_n\} \) ordered by the order of \( P_1' \), and \( C_h', C_{h+1}', \ldots, C_{m-2}' \) be the connected components of \( P'_2|\{e_{k+1}, e_{k+2}, \ldots, e_n\} \) ordered by the order of \( P_2' \). Since any ordering of \( \{e_2, e_3, \ldots, e_k-1\} \) can
be obtained by a sequence of twists not separating x and y, we may assume that:

\[ C_1 - \{e_1', e_k\} = \{e_2', \ldots, e_{i_1}\} \]

\[ C_j = \{e_{i_{j-1}+1}', \ldots, e_{i_j}\} \text{ for } j = 2, 3, \ldots, h - 1, \]

\[ C_h - \{e_1', e_k\} = \{e_{i_{h-1}+1}', \ldots, e_{i_h}\}, \]

\[ C_{h+1} - \{e_1', e_k\} = \{e_{i_{h+1}'+1}', \ldots, e_{i_{h+1}}\}, \]

\[ C_j = \{e_{i_{j-1}+1}', \ldots, e_{i_j}\} \text{ for } j = h + 2, \ldots, m - 1, \]

\[ C_m - \{e_1', e_k\} = \{e_{i_{m-1}+1}', \ldots, e_{k-1}\}. \]

Similarly, we may assume \( C'_j \) contains consecutive edges of \( \{e_{k+1}', e_{k+2}', \ldots, e_n\} \) but in the reverse order for \( j = 1, 2, \ldots, m - 2 \).

(See figure 5.)

Let:

\[ A_j = ( \bigcup_{i=1}^{j-1} (C_i U C'_i) ) U C_j - e_k \text{ for } j = 1, 2, \ldots, h, \]

\[ B_j = ( \bigcup_{i=1}^j (C_i U C'_i) ) - e_k \text{ for } j = 1, 2, \ldots, h - 1, \]
\[ A_j = \bigcup_{i=1}^{j-2} \left( C_i \cup C_i' \right) \cup C_{j-1} \cup C_j - e_k \] for \( j = h+1, h+2, \ldots, m-1 \), and

\[ B_j = \bigcup_{i=1}^{j-1} \left( C_i \cup C_i' \right) \cup C_j - e_k \] for \( j = h+1, h+2, \ldots, m-1 \).

We have

\[ T_{A_1} \left( T_{B_1} \left( \ldots T_{A_h} \left( T_{B_{h+1}} \left( T_{A_{h+1}} \left( \ldots T_{B_{m-1}} \left( P \right) \ldots \right) \right) \right) \right) \right) = \mathcal{P}, \]

Hence \( T_{A_1}, T_{A_2}, \ldots, T_{A_{m-1}}, T_{B_1}, T_{B_2}, \ldots, T_{B_{h-1}}, T_{B_{h+1}}, \ldots, T_{B_{m-1}} \)

is a sequence of non-crossing twists and each twist, with the possible exceptions \( T_{A_1}, T_{B_{m-1}} \), corresponds to a twist in \( G \) separating \( x \) and \( y \). \( \square \)
CHAPTER IV

PROOFS OF THE THEOREMS

1. Proof of theorem 1.

Assume (i) does not hold. By lemma 1, any path in a realization of $C(G)$ is not a cycle in $H$. By lemma 4, there exists a vertex $v \in V(G)$ such that $C(G-v) = C(H\mid E(G-v))$. Let $G' = G - v$ and $H' = H\mid E(G-v)$.

Case 1. $G$ is 3-connected. By proposition 3, $G'$ is 2-connected.

Case 1.a. Suppose $H'$ can be obtained from $G'$ by a sequence of non-crossing twists $T_1, T_2, \ldots, T_m$. We may assume that $T_i$ and $T_j$ do not have the same vertices of attachment for distinct $i, j \in \{1, 2, \ldots, m\}$.

Claim 1. $T_i$ and $T_j$ have a common vertex of attachment for distinct $i, j \in \{1, 2, \ldots, m\}$.
Proof. If $T_i, T_j$ have no common vertex of attachment, then let $G'_i, G'_j$ be the cells of $T_i, T_j$ containing no vertex of $G'_i, G'_j$ respectively. Since $G$ is 3-connected, $v$ is joined to a vertex $v_i, v_j$ respectively, of the interior of $G'_i, G'_j$ by an edge $e_i, e_j$ in $G$. Let $P$ be a path in $G$ joining $v_i$ and $v_j$. Then $P, e_i, e_j$ form a cycle in $G$, hence a cycle in $H$. But $P$ breaks into at least three pieces in $H'$ as $T_1, T_2, \ldots, T_m$ are non-crossing. There is no way we can put $e_i$ and $e_j$ back so that $e_i, e_j, P$ form a cycle in $H$. □

Case 1.a.1. All twists have a common vertex of attachment $z$.

Case 1.a.1.1. There are three end-cells. We may assume there are four disjoint graphs $G'_1, G'_2, G'_3, G'_4$ and distinct vertices $y_1, y_2, z_1, z_2, z_3, z_4$ such that

$$G' = C(G'_1, G'_2, G'_3, G'_4); (x_1, y_1), (x_2, y_2), (x_3, y_3), (z_1, z_2, z_3, z_4),$$

and $G'_1, G'_2, G'_3$ are three end-cells for the sequence $T_1, T_2, \ldots, T_m$. (See figure 6.)

Since $G$ is 3-connected, there exists an edge $e_i$ in $G$ joining $v$ and a vertex $v_i \in V(G'_1) - (y_1, z_1)$ for
i = 1, 2, 3. Let \( P_i \) be a path in \( G_i' \) joining \( v_i \) and \( z_i \). Then \( P_i, P_2, e_1, e_2 \) form a cycle in \( G \) hence a cycle in \( H \). We note that \( v_1, v_2, v_3, z_1, z_2, z_3 \) represent different vertices in \( H \), therefore the ends of \( e_1 \) in \( H \) must be in \( \{z_1, z_2, v_1, v_2\} \). Similarly, by considering the cycle \( P_i, P_3, e_1, e_3 \) in \( G \), the ends of \( e_1 \) in \( H \) must be in \( \{z_1, z_3, v_1, v_3\} \), therefore the ends of \( e_1 \) in \( H \) are \( z_1 \) and \( v_1 \). However, \( P_1 \) and \( e_1 \) form a path in \( G \) and they form a cycle in \( H \), which contradicts lemma 1.

Case 1.a.1.2. There are exactly two end-cells.

We may assume there exist \( m + 1 \) disjoint graphs \( G_1', G_2', \ldots, G_m' \), and distinct vertices \( y_1, z_1 \in V(G_1'), x_2, y_2, z_2 \in V(G_2'), x_3, y_3, z_3 \in V(G_3'), \ldots, x_m, y_m, z_m \in V(G_m') \), such that \( G' = C(G_1', G_2', \ldots, G_m'; (y_1, x_2), (y_2, x_3), \ldots, (y_m, x_{m+1}), (z_1, z_2, \ldots, z_{m+1}), z = (z_1, z_2, \ldots, z_{m+1}) \)

and \( \bigcup_{i=1}^{k} G_i' \) is a cell of \( T_k \), for \( k = 1, 2, \ldots, m \). (See figure 7.) Clearly, \( G_1', G_m' \) are the two end-cells for the sequence of twists \( T_1, T_2, \ldots, T_m \). Let \( e_1 \) be an edge in \( G \) joining \( v \) and \( v_1 \in V(G_1') - \{y_1, z_1\} \), \( e_2 \) be an edge in \( G \)
joining $v$ and $v_2 \in V(G'_{m+1}) - \{z_{m+1}, x_{m+1}\}$, $P_1$ be a path in $G'$ joining $v_1$ and $z_1$, and $P_2$ be a path in $G_{m+1}$ joining $v_2$ and $z_{m+1}$. By considering the cycle $P_1, P_2, e_1, e_2$ in $G$, the ends of $e_1$ and $e_2$ in $H$ are $v_1, v_2, z_1, z_{m+1}$. If the ends of $e_1$ in $H$ are $v_1$ and $z_1$ then by lemma 1, we have a contradiction. If there is an edge in $G$ joining $v$ and a vertex of $V(G) - (V(G') \cup V(G'_{m+1}))$, then we have a contradiction by using an argument similar to the one in case 1.a.1.1. Let $e_3$ be an edge in $G$ joining $v$ and $v_3 \in V(G'_{m+1}) \cup V(G'_{m+1})$. Because $G$ is 3-connected, such an edge exists.

Case 1.a.1.2.1. The ends of $e_1$ in $H$ are $v_1$, $v_2$. Then the ends of $e_2$ in $H$ are $z_1$, $z_{m+1}$.

If $v_3 \in V(G'_{m+1}) - V(G'_{1})$ in $G$, then we have a contradiction, by considering the cycle in $G$ formed by $e_2, e_3$ and a path in $G_{m+1}$ joining $v_2$ and $v_3$. If $v_3 \in V(G'_{1})$ then the ends of $e_3$ in $H$ are $v_3$ and $v_2$, by considering the cycle in $G$ formed by $e_1, e_3$ and a path in $G_{1}$ joining $v_1, v_3$. Let $n = m+2$, $G_1 = G|V(G') \cup v$, $G_i = G'_1$ for $i = 2, 3, \ldots, n-1$, and $G_n = G|e_2$. Then (ii) holds.
Case 1.a.1.2.2. The ends of \( e_1 \) in \( \bar{H} \) are \( z_1, z_{m+1} \). This is similar to case 1.a.1.2.1, since the ends of \( e_2 \) in \( H \) are \( v_1, v_2 \).

Case 1.a.1.2.3. The ends of \( e_1 \) in \( H \) are \( v_2, z_1 \), and \( m > 1 \). If \( v_3 \in V(G'_1) - z_1 \) then we have a contradiction by considering the cycle in \( G \) formed by \( e_1, e_3 \) and a path in \( G'_1 \) joining \( v_1, v_3 \). Similarly, \( v_3 \notin V(G'_{m+1}) - z_{m+1} \). Therefore \( v_3 = z \) and the ends of \( e_3 \) in \( H \) are \( v_1, v_2 \). Let \( n = m + 3 \), \( G_i = G'_i \) for \( i = 1, 2, \ldots, n - 2 \), \( G_{n-1} = G|\{e_2, e_3\} \), and \( G_n = G|e_1 \). Then (ii) holds.

Case 1.a.1.2.4. The ends of \( e_1 \) in \( H \) are \( v_2, z_1 \), or \( v_2, z_{m+1} \), and \( m = 1 \). This is similar to case 1.a.1.2.3, since \( (z_1, z_2), (y_1, x_2) \) play similar roles in \( G \) and \( y_1, z_2 \) denote the same vertex in \( H \).

Case 1.a.1.2.5. The ends of \( e_1 \) in \( H \) are \( v_2, z_{m+1} \), and \( m > 1 \). If \( v_3 \in V(G'_1) \) then we have a contradiction by considering the cycle in \( G \) formed by \( e_1, e_3 \) and a path in \( G'_1 \) joining \( v_1, v_3 \). Similarly, \( v_3 \notin V(G'_{m+1}) \), which contradicts the existence of \( e_3 \).
Case 1.a.1.2.6. The ends of $e_1$ in $H$ are $v_1$, $z_{m+1}$, and $m=1$. This contradicts lemma 1, as we assume (i) does not hold.

Case 1.a.1.2.7. The ends of $e_1$ in $H$ are $v_1$, $z_{m+1}$, and $m>1$. Then the ends of $e_2$ are $z_1$, $v_2$ in $H$. If $v_3 \in V(G'_1)$ then the ends of $e_3$ in $H$ are $v_3$, $z_{m+1}$, by considering the cycle in $G$ formed by $e_1$, $e_2$ and a path in $G'_1$ joining $v_1$, $v_3$. Similarly, if $v_3 \in V(G'_{m+1})$ then the ends of $e_3$ in $H$ are $v_3$, $z_1$. Let $n = m+1$, $G_1 = G[V(G'_1) \cup \nu]$, $G_i = G_i'$ for $i = 2, 3, \ldots, n-1$, and $G_n = G[V(G'_n) \cup \nu]$ with any edge in $G$ joining $v$, $z$ deleted. Then (ii) holds.

This completes case 1.a.1.

Since any two twists have a common vertex of attachment, $m = 3$ when they do not all have the same common vertex of attachment. The remaining two cases are illustrated in figure 8.

Case 1.a.2. There exist disjoint graphs $G'_1$, $G'_2$, $G'_3$ and distinct vertices $x_1, y_1 \in V(G'_1)$, $x_2, y_2 \in V(G'_2)$, $x_3, y_3 \in V(G'_3)$ such that $G' = C(G'_1, G'_2, G'_3; (y_1, x_2), (y_2, x_3)$,
(y_3, x_1)), and G_i' is a cell of T_i for i = 1, 2, 3. Then

H' = C(G_1', G_2', G_3'; (x_1, y_2), (x_2, y_3), (x_3, y_1)).

Since G is 3-connected, there exists an edge e_i in G joining v and v_i ∈ V(G_i') - {x_i, y_i} for i = 1, 2, 3.

Let p_1 be a path in G_1' joining v_1 and y_1, and p_2 be a path in

G_2' joining v_2 and x_2. By considering the cycle e_1, e_2, p_1, p_2
in G, the ends of e_1 and e_2 in H are v_1, v_2, (x_3, y_1),
(x_2, y_3). Similarly, the ends of e_1 and e_3 are v_1, v_3,
(x_1, y_2), (x_2, y_3). Therefore the ends of e_1 in H are v_1,
(x_2, y_3), the ends of e_2 in H are v_2, (x_3, y_1), and the
ends of e_3 in H are v_3, (x_1, y_2). If e_1' is an edge in G
joining v and v_1' ∈ V(G_1'), then by considering the cycle in G
formed by e_1, e_1' and a path in G_1' joining v_1, v_1', the ends
of e_1' in H are v_1', (x_2, y_3). Similarly, if e_2' is an
edge in G joining v and v_2' ∈ V(G_2') then the ends of e_2' in
H are v_2', (x_3, y_1). If e_3' is an edge in G joining v and
v_3' ∈ V(G_3') then the ends of e_3' in H are v_3', (x_1, y_2). Let
n = 3, and G_i = G\mid V(G_i') U v with any edge in G joining v and
x_i deleted for i = 1, 2, 3. Then (ii) holds.
Case 1.a.3. There exist disjoint graphs $G'_1$, $G'_2$, $G'_3$, and distinct vertices $x_1, z_1 \in V(G'_1)$, $y_2, z_2 \in V(G'_2)$, $x_3, y_3 \in V(G'_3)$, $x_4, y_4, z_4 \in V(G'_4)$ such that $G' = C(G'_1, G'_2, G'_3, G'_4)$; $(x_1, y_2, z_4)$, $(x_3, y_3, z_1)$, $(x_4, y_4, z_2)$, $(z_4, z_2, z_4)$, $(y_3, y_4, y_2)$, $(x_4, x_3, x_1)$. The argument here is the same as case 1.a.2 except for the labels of vertices and the existence of $G'_4$. Thus (iii) holds.

Case 1.b. There exists a polygon unit $P$ of $G'$ such that the corresponding unit $P'$ in $H'$ cannot be obtained from $P$ by a sequence of non-crossing twists. Let $e_1, e_2, ..., e_m$ be the edges of the polygons, and $+$ be addition modulo $m$. We may assume that $e_i$ and $e_{i+1}$ are consecutive edges of $P$, for $i = 1, 2, ..., m$. We define recursively an equivalence relation $\sim$ on the edges of $P$ by the following rules:

(i) $e_i \sim e_i$ for $i = 1, 2, ..., m$,

(ii) if $e_i \sim e_{i+1} \sim ... \sim e_j$, $e_{j+1} \sim e_{j+2} \sim ... \sim e_k$ and $P'[e_i, e_{i+1}, ..., e_k]$ is connected, then $e_i \sim e_k$. 

(iii) if $e_i \sim e_j$ and $e_j \sim e_k$ then $e_i \sim e_k$.

(iv) if $e_i \sim e_j$ then $e_j \sim e_i$.

Let $C_1, C_2, ..., C_n$ denote the equivalence classes of edges of $P$.

Claim 2. Both $P|C_i$ and $P'|C_i$ are connected and $P'|C_i$ can be obtained from $P|C_i$ by a sequence of non-crossing twists on $P$ so that a cell of any twist in this sequence is in $C_i$.

Proof. We prove this claim by induction over successive stages of recursion in the definition of $\sim$. In stage 1, we define equivalence classes $C_{i1}, C_{i2}, ..., C_{in_1}$.

Stage 1. Let $n_1 = m$ and $C_{ij} = \{e_j\}$ for $j = 1, 2, ..., m$.

Clearly, $P|C_{ij}$ and $P'|C_{ij}$ satisfy the claim.

Stage i. We define the $i$-stage equivalence classes $C_{i1}, C_{i2}, ..., C_{in_i}$ by the following rule:

If two consecutive $(i-1)$-stage equivalence classes of $P$ are consecutive $(i-1)$-stage equivalence classes of $P'$ then
the edges of these two (i - 1)-stage classes belong to the same i-stage class.

Clearly, each i-stage class is a union of consecutive (i - 1)-stage classes of P and P' (in the same or reverse order in P and P'). Therefore, P_i^j and P'_i^j are connected for j = 1, 2, ..., n_i. Then P'_i^j can be obtained from P_i^j by a sequence of non-crossing twists on P so that a cell of any twist in this sequence is in C_i^j, since by induction, we can do the same thing for each (i - 1)-stage class, and then we only need to twist some of the (i - 1)-stage classes. All these twists are clearly non-crossing. It is also clear that the edges of an i-stage class are equivalent. □

We may assume C_i and C_{i+1} denote consecutive equivalence classes of P. Then by the definition of ~ and the argument in claim 2, P'_i | C_i U C_{i+1} is disconnected for i = 1, 2, ..., n - 1. Since P' cannot be obtained from P by a sequence of non-crossing twists, n ≥ 2. But then n ≥ 5, because the P'_i | C_i U C_{i+1} are disconnected. Let G'_i denote the subgraph of G' corresponding to C_i and let x_i, y_i be distinct vertices of G'_i for i = 1, 2, ..., n. We may assume that G' = C(G'_1, G'_2, ..., G'_n; (y'_1, x'_2), (y'_2, x'_3), ..., (y'_n-1, x'_n), (y'_n, x'_1)).
Claim 3. The integer \( n \) is odd, \( H' \mid E(G'_i) = G'_i \) for \( i = 1, 2, \ldots, n \), and \( H' = C(G'_1, G'_2, \ldots, G'_n; (y_1, x_3), (y_2, x_4), \ldots, (y_{n-1}, x_1), (y_n, x_2)). \) (See figure 9.)

Proof. Let \( + \) denote addition modulo \( n \). It is sufficient to show \( H' \mid E(G'_i) = G'_1 \) and that \( y_i, x_{i+2} \) represent the same vertex in \( H' \), for \( i = 1, 2, \ldots, n \).

If \( G'_i \) contains more than one edge then let \( f_i \) denote an edge in \( G \) joining \( v \) and \( v_1 \in V(G'_1) - \{x_i, y_i\} \). If \( G'_i \) contains only one edge then let \( \{g_i\} = E(G'_i) \). If both \( G'_i \) and \( G'_i+1 \) contain only one edge then let \( f_i \) denote the edge in \( G \) joining \( v \) and \( (y_i, x_{i+1}) \). Such edges always exist, since \( G \) is 3-connected. Let \( P_i \) denote a path in \( G'_i \) joining \( x_i \) and \( y_i \). If \( G'_i \) has more than one edge, then let \( P'_i \) denote a path in \( G'_i \) joining \( v_i \) and \( x_i \), and \( P''_i \) denote a path in \( G'_i \) joining \( v_i \) and \( y_i \). We may assume \( P_i = P'_i + P''_i \). Since \( P_1, P_2, \ldots, P_n \) form a cycle in \( G'_i \), they form a cycle in \( H' \). Therefore \( P_i \) is a path in \( H' \) for \( i = 1, 2, \ldots, n \).

Case 1.b.1. \( H' \mid E(G'_i) = G'_1 \) for \( i = 1, 2, \ldots, n \).
Case 1.b.1.1. \(|E(G_i^1)| > 1 \text{ and } |E(G_{i+2}^1)| > 1\).

Here \(f_i, P_i^1, P_{i+1}^1, P_{i+2}^1, f_{i+2}\) form a cycle in \(G\), and hence a cycle in \(H\), therefore \(H'|P_i^1 \cup P_{i+1}^1 \cup P_{i+2}^1\) has at most four odd valency vertices. Both \(H'|P_i^1 \cup P_{i+1}^1\), \(H'|P_{i+1}^1 \cup P_{i+2}^1\) are disconnected, hence \(H'|P_i^1 \cup P_{i+2}^1\) and \(H'|P_{i+2}^1\) are paths. This implies that \(y_i\) and \(x_{i+2}\) represent the same vertex in \(H'\).

Case 1.b.1.2. \(|E(G_i^1)| > 1 \text{ and } |E(G_{i+2}^1)| = 1\).

Case 1.b.1.2.1. \(|E(G_{i+3}^1)| = 1\). By considering the cycle formed by \(f_i, P_i^1, P_{i+1}^1, g_{i+2}^1, f_{i+2}\), \(H'|P_i^1 \cup g_{i+2}^1\) is a path. This implies that \(y_i\) and \(x_{i+2}\) represent the same vertex in \(H'\).

Case 1.b.1.2.2. \(|E(G_{i+3}^1)| > 1\). By considering the cycle formed by \(f_i, P_i^1, P_{i+1}^1, g_{i+2}^1, P_{i+3}^1, f_{i+3}\), \(H'|P_i^1 \cup P_{i+1}^1 \cup g_{i+2}^1 \cup P_{i+3}^1\) has at most four odd valency vertices. Since \(H'|P_i^1 \cup P_{i+1}^1\), \(H'|P_{i+1}^1 \cup g_{i+2}^1\) and \(H'|g_{i+2}^1 \cup P_{i+3}^1\) are disconnected, \(H'|P_i^1 \cup g_{i+2}^1\) and \(H'|P_{i+1}^1 \cup P_{i+3}^1\) are paths. This implies that \(y_i\) and \(x_{i+2}\) represent the same vertex in \(H'\).
Case 1.b.1.3. \( |E(G'_i)| = 1 \) and \( |E(G'_{i+2})| > 1 \).

This is similar to case 1.b.1.2.

Case 1.b.1.4. \( |E(G'_i)| = 1 \) and \( E(G'_{i+2})| = 1 \).

Case 1.b.1.4.1. \( |E(G'_{i-1})| > 1 \) and \( |E(G'_{i+3})| > 1 \). By the above arguments, \( y_{i-1} \) and \( x_{i+1} \) represent the same vertex in \( H' \), and \( y_{i+1} \) and \( x_{i+3} \) represent the same vertex in \( H \). By considering the cycle formed by \( f_{i-1}', P'_{i-1}', g_i', P'_{i+1}', g_{i+2}', P'_{i+3}' \), and \( f'_{i+3}' \), \( (g_i', g_{i+2}') \) is a path in \( H' \), and this implies that \( y_i \) and \( x_{i+2} \) represent the same vertex in \( H' \).

Case 1.b.1.4.2. \( |E(G'_{i-1})| > 1 \) and \( |E(G'_{i+3})| = 1 \). By the above arguments, \( y_{i-1} \) and \( x_{i+1} \) represent the same vertex in \( H' \). By considering the cycle formed by \( f_{i-1}' \), \( P''_{i-1}', g_i', P'_{i+1}', g_{i+2}' \), and \( f'_{i+2}' \), \( (g_i', g_{i+2}') \) is a path in \( H' \), and this implies that \( y_i \) and \( x_{i+2} \) represent the same vertex in \( H' \).

Case 1.b.1.4.3. \( |E(G'_{i-1})| = 1 \) and \( |E(G'_{i+3})| > 1 \). This is similar to case 1.b.1.4.2.
Case 1.b.1.4.4. \(|E(G'_{i-1})|=1, |E(G'_{i+3})|=1.\)

By considering the cycle \((f_{i-1}, g_i, f_{i+2}, g_{i+2}) \cup p_{i+1}\) and the fact that \(H'|g_i \cup p_{i+1}\), and \(H'|p_{i+1} \cup g_{i+2}\) are disconnected, \((g_i, g_{i+2})\) is a path in \(H'\), and this implies that \(y_i\) and \(x_{i+2}\) represent the same vertex in \(H\).

Case 1.b.2. \(H'|E(G'_i) \neq G'_i\) for some \(i \in \{1, 2, \ldots, n\}\). Then \(|E(G'_i)| > 1.\) By the arguments in case 1.b.1.1 and case 1.b.1.2, either \(H'|p'_i \cup p'_{i+2}\) or \(H'|p'_i \cup p'_{i+2}\) is a path. Since \(p_i\) is a path in \(H'|E(G'_i)\), by lemma 5, \(H'|E(G'_i)\) can be obtained from \(G'_i\) by a sequence of twists not separating \(x_i, y_i\). Either there exists such a sequence such that the twists of the sequence are non-crossing, or there exists a polygon unit \(Q\) of \(G'\) such that the corresponding unit \(Q'\) in \(H'\) cannot be obtained from \(Q\) by a sequence of non-crossing twists and the subgraph of \(G'\) corresponding to a certain edge of \(Q\) contains \(\bigcup_{j \neq i} E(G'_j)\).

Case 1.b.2.1. \(H'|E(G'_i)\) can be obtained from \(G'_i\) by a sequence of non-crossing twists not separating \(x_i, y_i\).

Case 1.b.2.1.1. There is an end-cell of this sequence of non-crossing twists such that \(y_i\) is not a vertex of this end-cell. We may assume \(v_i\) is an interior vertex of this
cell. This implies \( H'|P''_i \) has at least four odd valency vertices, therefore neither \( H'|P''_i \cup P_{i+2} \) nor \( H'|P''_i \cup P'_i \) is a path. We have a contradiction.

**Case 1.b.2.1.2.** There is an end-cell of this sequence of non-crossing twists such that \( y_1 \) is a vertex of this end-cell and \( x_1 \) is not a vertex of this end-cell. Since all twists do not separate \( x_1 \) and \( y_1 \), \( y_1 \) is a vertex of attachment of this end-cell. We may assume \( v_1 \) is an interior vertex of this cell. This implies \( P''_i \) is still a path in \( H' \), but none of the ends of \( P''_i \) in \( H' \) is a vertex of attachment of \( H'|E(G'_i) \) in \( H' \), hence neither \( H'|P''_i \cup P_{i+2} \) nor \( H'|P''_i \cup P'_i \) is a path. We have a contradiction.

**Case 1.b.2.1.3.** We may thus assume \( x_1 \) and \( y_1 \) are vertices of all end-cells of this sequence. Since there are at least two end-cells, this implies \( x_1 \) and \( y_1 \) are the vertices of attachments of all the cells of this sequence. Since \( G \) is 3-connected, there is an edge \( f'_i \) in \( G \) joining \( v \) and an interior vertex \( v'_i \) of a cell which does not contain \( v_1 \). Let \( P'''_i \) be a path in \( G'_i \) joining \( y_1 \) and \( v'_1 \). Then by the arguments in case 1.b.1 either \( H'|P'''_i \cup P_{i+2} \) or \( H'|P'''_i \cup P'_i \) is a path. In either alternative \( H'|P''_i \cup P_{i+2} \) and \( H'|P''_i \cup P'_i \)
are not paths, since for the two vertices of attachment of $H'|E(G'_1)$ in $H'$, one is in $H'|P''_i$ and the other is in $H'|P''_i$. We have a contradiction.

**Case 1.b.2.2.** There exists a polygon unit $Q$ of $G'$ such that the corresponding unit $Q'$ in $H'$ cannot be obtained from $Q$ by a sequence of non-crossing twists and the subgraph of $G'$ corresponding to a certain edge of $Q$ contains $\bigcup_{j \neq i} E(G'_j)$. Let $D_1, D_2, \ldots, D_n$, be the equivalence classes of edges of $Q$ which are obtained by claim 2, and $J_j$ be the subgraph of $G'$ corresponding to $D_j$ for $j = 1, 2, \ldots, n'$. Then $n' \geq 5$.

We may assume $G'|E(J_1) \cup E(J_{j+1})$ is connected for $j = 1, 2, \ldots, n'$. We have $G'|E(J_1) \subseteq E(J_1)$, and $v_i \in V(J_2) \cup V(J_3) - V(J_1)$. Then $H'|P''_1$ has at least two odd valency vertices in $H'|E(J_2)$, since both $H'|E(J_2) \cup E(J_1)$ and $H'|E(J_2) \cup E(J_3)$ are disconnected. This implies neither $H'|P''_1 \cup P_{i+2}$ nor $H'|P''_1 \cup P'_{i+2}$ is a path. We have a contradiction. This concludes case 1.b.2. □

**Claim 4.** If $f'_i$ is an edge in $G$ joining $v$ and $v'_i \in V(G'_i) - x_i$, then the ends of $f'_i$ in $H$ are $v'_i, (y_{i-1}, x_{i+1})$.

**Proof.**
Case l.b.3. \(|E(G_j)| = 1\), for \(j = 1, 2, \ldots, n\).

Let \(\{g_j\} = E(G_j)\). By considering the cycle \(\{f_i, e_i, e_{i-1}\}\) in \(G\), one end of \(f_i\) in \(H\) is either \((x_i, y_{i-2})\) or \((y_i, x_{i+2})\). By considering the cycle \(\{f_i, g_{i+1}, g_{i+2}, f_{i+2}\}\) in \(G\), the ends of \(f_i\) in \(H\) are in \([(x_{i+1}, y_{i-1}), (y_{i+1}, x_{i+3}), (x_{i+2}, y_i), (y_{i+2}, x_{i+4})]\).

Thus \((y_i, x_{i+2})\) is an end of \(f_i\) in \(H\). Similarly, by considering cycles \(\{f_i, e_{i+1}, f_{i+1}\}\) and \(\{f_i, e_i, e_{i-1}, f_{i-1}\}\), \((x_{i+1}, y_{i-1})\) is an end of \(f_i\) in \(H\). In this case, \(f_i' = f_i\), \(v_i' = y_i\), and the claim holds.

Case l.b.4. \(|E(G_j')| > 1\) for some \(j \in \{1, 2, \ldots, n\}\).

Since \(G\) is 3-connected, there exists an edge \(f_{j-1}'\) in \(G\) joining \(v\) and \(v_{j-1}' \in V(G_{j-1}') \cup V(G_{j-2}') - \{x_{j-2}', y_{j-1}'\}\), and there exists an edge \(f_{j+1}'\) in \(G\) joining \(v\) and \(v_{j+1}' \in V(G_{j+1}') \cup V(G_{j+2}') - \{x_{j+1}', y_{j+2}'\}\).

Case l.b.4.1. \(v_{j-1}' \in V(G_{j-1}') - y_{j-1}\) and \(v_{j+1}' \in V(G_{j+1}') - x_{j+1}\). Let \(R_{j-1}\) be a path in \(G_{j-1}'\) joining \(v_{j-1}'\) and \(y_{j-1}\). Then \(f_{j-1}', R_{j-1}', p_j', f_j\) form a cycle in \(G\), hence a cycle in \(H\). Therefore the ends of \(f_j\) in \(H\) are in...
\[ \{v_j, v'_{j-1}, (x_j, y_{j-2}), (y_{j-1}, x_{j+1})\} \]. Similarly, by considering a cycle in \( G \) containing \( f_j \) and \( f'_{j+1} \), the ends of \( f_j \) in \( H \) are in \( \{v_j, v'_{j+1}, (y_j, x_{j+2}), (x_{j+1}, y_{j-1})\} \). Thus the ends of \( f_j \) in \( H \) are \( v_j \) and \( (x_{j+1}, y_{j-1}) \).

**Case 1.b.4.2.** \( v'_{j-1} \in V(G'_{j-2}) - \{x_{j-2}, y_{j-2}\} \) and \( v'_{j+1} \in V(G_{j+1}) - x_{j+1} \). Let \( R_{j-2} \) be the path in \( G'_{j-2} \) joining \( v'_{j-1} \) and \( y_{j-2} \). By considering the cycle formed by \( f'_{j-1}, R_{j-2}, P_{j-1}, P'_j, \) and \( f_j \), the ends of \( f_j \) in \( H \) are in \( \{v_j, v'_{j-1}, (x_{j-1}, y_{j-3}), (y_{j-1}, x_{j+1})\} \). By the argument in case 1.b.4.1, the ends of \( f_j \) in \( H \) are also in \( \{v_j, v'_{j+1}, (y_j, x_{j+2}), (x_{j+1}, y_{j-1})\} \). Thus the ends of \( f_j \) in \( H \) are \( v_j \) and \( (x_{j+1}, y_{j-1}) \).

**Case 1.b.4.3.** \( v'_{j-1} \in V(G'_{j-1}) - y_{j-1} \) and \( v'_{j+1} \in V(G'_{j+2}) - \{x_{j+2}, y_{j+2}\} \). This is similar to case 1.b.4.2.

**Case 1.b.4.4.** \( v'_{j-1} \in V(G'_{j-2}) - \{x_{j-2}, y_{j-2}\} \) and \( v'_{j+1} \in V(G'_{j+2}) - \{x_{j+2}, y_{j+2}\} \). By the argument in case 1.b.4.2, the ends of \( f_j \) in \( H \) are in \( \{v_j, v'_{j-1}, (x_{j-1}, y_{j-3})\} \).
Similarly, the ends of \( f_j \) in \( H \) are in \( [v_j, v_{j+1}, x_{j+1}, y_{j-1}, (y_{j+1}, x_{j+3})] \), and so the ends of \( f_j \) in \( H \) are \( v_j \) and \( (y_{j-1}, x_{j+1}) \). This completes case 1.b.4.

In the above four cases, the ends of \( f_j \) in \( H \) are \( v_j \) and \((y_{j-1}, x_{j+1})\). If \( i = j \) then either \( f_i = f_i' \), or by considering a cycle in \( G \) containing \( f_i \) and \( f_i' \), the claim holds. We can assume \( i \neq j \). Let \( P_i \) be a path in \( G_i \) joining \( v_i \) and \( x_i \). By considering the cycle formed by \( f_i, P_i, P_{j+1}, \ldots, P_{i-1}, P_i, \) and \( f_i' \), the ends of \( f_i' \) in \( H \) are \( v_i' \) and \((y_{i-1}, x_{i+1})\). \( \square \)

Let \( G_i = G \setminus V(G_i) \cup v \) with any edge joining \( v \) and \( x_i \) deleted, for \( i = 1, 2, \ldots, n \). Then (ii) holds.

Case 2. \( G \) is 2-connected. We shall prove this case, with the additional properties that we may drop the * from \( H^* \) and \( H \) is connected in (ii) and (iii), by induction on \( |E(G)| \).

By case 1, we may assume the connectivity is 2. If \( G \) is a polygon, then \( H \) consists of two polygons with at most one common vertex. Let \( G^* \) be a polygon formed by \( E(G) \) such that the edges of the polygons are complementary paths in \( G^* \). Clearly, (i) holds. We may thus assume \( G \) is not a polygon. Then there exist
disjoint graphs in $G_1$, $G_2$, distinct vertices $x_1, y_1 \in V(G_1)$, $x_2, y_2 \in V(G_2)$, such that $G_1$ is 2-connected, $G = \text{C}(G_1, G_2; (x_1, x_2), (y_1, y_2))$ and $|E(G_1)| > 1 < |E(G_2)|$.

Let $H_1 = H|E(G_1)$, $H_2 = H|E(G_2)$. Clearly, $C(G_1) \subseteq C(H_1)$, $C(G_2) \subseteq C(H_2)$, and the codimensions are at most 1. Let $P_1$ be a path in $G_1$ joining $x_1$ and $y_1$, $P_2$ be a path in $G_2$ joining $x_2$ and $y_2$, and $D = P_1 + P_2$. Then $C(G) = \langle C(G_1), C(G_2), D \rangle$.

**Case 2.a.** Both codimensions are 1. Let $D_i$ be a cycle in $H_i$ such that $C(H_i) = \langle C(G_i), D_i \rangle$ for $i = 1, 2$. Then $C(H) = \langle C(G_1), C(G_2), D, D_1 \rangle$. Since $D_2 \not\in \langle C(G_1), C(G_2), D \rangle$ and $D_2 \not\in \langle C(G_1), C(G_2), D_1 \rangle$, there exist $C_1 \in C(G_1)$, $C_2 \in C(G_2)$ such that $D_2 = C_1 + C_2 + D_1 + D$. Thus $P_2 + C_2 + D_2 = P_1 + C_1 + D_1$. Since $G_1$ and $G_2$ are edge-disjoint, this implies $P_1 = C_1 + D_1$ and $P_2 = C_2 + D_2$. Therefore $P_1$ is a cycle in $H$.

By Lemma 1, we have a contradiction. This completes case 2.a.

**Claim 5.** If $C(H) = \langle C(H_1), C(H_2), D \rangle$ and $H_1$ is connected, then $P_1$ is a path in $H_1$.
Proof. By lemma 1, $P_1$ is not a cycle in $H_1$. If $H_1 | P_1$ has at least four odd valency vertices, then in order to make $D$ a cycle, a proper subset $D_2$ of $P_2$ must be a path joining two of the odd valency vertices of $H_1 | P_1$. Let $D_1$ be a path in $H_1$ joining the ends of $D_2$ in $H$. Since $D_1 + D_2$ is a cycle in $H$ and $C(H) = \langle C(H_1), C(H_2), D \rangle$, there exist $C_1 \in C(H_1)$, $C_2 \in C(H_2)$ such that $D_1 + D_2 = C_1 + C_2 + D$. Then $D_1 + C_1 + P_1 = D_2 + C_2 + P_2$. Because $E(H_1) \cap E(H_2) = \emptyset$, $D_1 + P_1 = C_1$ and $D_2 + P_2 = C_2$. This implies $P_1$ is a path in $H$, since $C_1$ is a cycle and $D_1$ is a path in $H$. We have a contradiction. \[\]

Case 2.b. $C(G_1) = C(H_1)$ and $C(G_2)$ is a codimension-1 subspace of $C(H_2)$. Since $G_1$ is 2-connected, $H_1$ is 2-connected. By claim 5, $P_1$ is a path in $H_1$, and hence $P_2$ is a path in $H_2$. Let $G'$ be the graph obtained from $G$ by replacing $G_1$ by an edge $e$ joining $x_2$ and $y_2$, and $H'$ be the graph obtained from $H$ by replacing $H_1$ by $e$ joining the ends of $P_2$. Clearly, $G'$ is 2-connected, $|E(G')| < |E(G)|$, and $C(G')$ is a codimension-1 subspace of $C(H')$. By induction, either there exist graphs $G'^*$ and $H'^*$ such that (i) holds for $G'^*$ and $H'^*$, with $C(G'^*) = C(G')$, and $C(H'^*) = C(H')$, or $H'$ is connected and there exists a graph $G'^*$ such that either (ii) or (iii) holds for $G'^*$ and $H'$, with $C(G'^*) = C(G')$, and
\[ C(H^*) = C(H'). \]

**Case 2.b.1.** There exist graphs $G^*$ and $H^*$ such that (i) holds, with\[ C(G^*) = C(G'), \]and \[ C(H^*) = C(H'). \] Let $H^*$ be the graph obtained from $H'*$ by replacing $e$ by $H_1$, and $G^*$ be the graph obtained from $G''*$ by replacing $e$ "properly" by $H_1$. Clearly, $C(G^*) = C(G)$, $C(H^*) = C(H)$, and (i) holds for $G^*$ and $H^*$. We have a contradiction.

**Case 2.b.2.** $H'$ is connected, and there exists a graph $G'^*$ such that either (ii) or (iii) holds for $G'^*$ and $H'$, with \[ C(G'^*) = C(G'), \]and \[ C(H'^*) = C(H'). \] Let $G^*$ be the graph obtained from $G'^*$ by replacing $e$ "properly" by $H_1$. Clearly, either (ii) or (iii) holds for $G^*$ and $H$, and $H$ is connected.

**Case 2.c.** $C(G_1)$ is a codimension-1 subspace of $C(H_1)$ and $C(G_2) = C(H_2)$. If $P_1$ is still a path in $H_1$, then we can follow the proof in case 2.b. We may thus assume $P_1$ is not a path in $H_1$. Then by claim 5, $H_1$ is disconnected. Therefore, by induction, there exist graphs $G_1'$ and $H_1'$ such that $C(G_1') = C(G_1)$, $C(H_1') = C(H_1)$, and $H_1'$ is obtained from $G_1'$ by identifying two distinct vertices $v_1, v_2$. 
Since $G_1$ is 2-connected, $G_1'$ is 2-connected. Because $H_1$ is disconnected, $H_1'$ is obtained from $H_1'$ by splitting at some cut vertices. The only possible cut vertex in $H_1'$ is the new vertex, therefore, $G_1' - \{v_1, v_2\}$ is disconnected. By lemma 1, a path in $G_1'$ joining $v_1$ and $v_2$ is not a path in any realization of $C(G)$, since such a path is a cycle in $H$. By lemma 5, $G_1$ is obtained from $G_1'$ by a sequence of twists and at least one of the twists separates $v_1$ and $v_2$. Therefore, $v_1$ and $v_2$ are in a unique polygon unit $R$ of $G_1'$. Let $Q$ be the polygon unit in $G_1$ corresponding to $R$. Let $e_1, e_2, \ldots, e_m$ be the edges of $Q$. We may assume $e_i$ and $e_{i+1}$ are consecutive edges in $Q$ for $i = 1, 2, \ldots, m-1$. Let $G_{1,i}$ be the subgraph of $G_1$ corresponding to $e_i$, and $u_i$ and $u_i'$ be the vertices of attachment of $G_{1,i}$, such that $u_i'$ and $u_{i+1}'$ represent the same vertex in $G_1$, for $i = 1, 2, \ldots, m-1$. Let $Q_i$ be a path in $G_{1,i}$ joining $u_i$ and $u_i'$ for $i = 1, 2, \ldots, m$.

If $x_1$ and $y_1$ are vertices of $G_{1,i}$, then any path in $G_1' \bigcup_{j \neq i} E(G_{1,j})$ joining $v_1$ and $v_2$ is a path in a realization of $C(G)$, since any ordering of $\{G_{1,1}, G_{1,2}, \ldots, G_{1,m}\}$ can be achieved by a sequence of twists in $G_1$ not separating $x_1$ and $y_1$. 
This is a contradiction, therefore, we may assume $x_1 \in V(G_{1,1}) - u'_1$, $y_1 \in V(G_{1,k}) - u_k$ for some $k$, and $1 < k < m$. Let $Q_x$ be a path in $G_{1,1}$ joining $x_1$ and $u'_1$, and $Q_y$ be a path in $G_{1,k}$ joining $y_1$ and $u_k$. We may choose all these paths so that $Q_x$ is a subpath of $Q_1$, $Q_y$ is a subpath of $Q_k$, and 

$$P_1 = \left( \bigcup_{i=2}^{k-1} Q_i \right) \cup Q_x \cup Q_y.$$  

If $x_1 = u_1$ then any ordering of $\{e_1, e_2, \ldots, e_{k-1}\}$ and $\{e_{k+1}, e_{k+2}, \ldots, e_m\}$ can be achieved by a sequence of twists in $G$ not separating $x_1$ and $y_1$. This means a path in $G'_{1,1}$ joining $v_1$ and $v_2$ in $G' \bigcup_{i \neq k} E(Q_i)$ is a path in a realization of $C(G)$. But such a path is a cycle in $H$, which contradicts lemma 1. Hence $x_1 \neq u_1$, and similarly, $y_1 \neq u_k$.

Since $H_1'$ is obtained from $G_1'$ by identifying $v_1$ and $v_2$ in the polygon unit, and $H_1$ is obtained from $H_1'$ by splitting at the new cut vertex, $H_1$ contains exactly two connected components $C_1, C_2$. Let $I = \{i : Q_i \subseteq E(C_1)\}$, $I_1 = \{1, 2, \ldots, k\} - I$, $I_2 = \{k+1, k+2, \ldots, m\} - I$. We may assume $l \in I$.

Claim 6. We can find a realization $G^*$ of $C(G)$ so that:
I - k, \ I_1, I_2, \ (k) form a partition of \ \{1, 2, ..., m\}, and all
the above graphs are connected.

\textbf{Proof.} Since \ \sum_{i=1}^{m} Q_i \ is a cycle in \ \textit{H}, \ Q_1 \ is a path in
\textit{H} \ for \ i = 1, 2, ..., m. Clearly, \ \sum_{i \in I} Q_i \ is a cycle in \ \textit{H}.

Because any ordering of \ \{e_2, e_3, ..., e_{k-1}\} \ and \ \{e_{k+1}, e_{k+2}, ..., e_m\} \ can be achieved by a sequence of twists in \ \textit{G}_1 \ not separating
\ x_1 \ and \ y_1, \ it follows that \ \sum_{i \in I_1} Q_i, \ \sum_{i \in I_2} Q_i \ and \ \sum_{i \in I-k} Q_i

are paths in a realization of \ \textit{C}(\textit{G}). \ If \ k \not\in I \ then \ \sum_{i \in I} Q_i \ is

a path in a realization of \ \textit{C}(\textit{G}). \ But \ \sum_{i \in I} Q_i \ is a cycle in \ \textit{H},

which contradicts lemma 1. Therefore, \ k \in I, \ and hence \ \textit{I}-k,
\ I_1, I_2, \ (k) \ form a partition of \ \{1, 2, ..., m\}.
If $P_1 \cap E(C_1)$ is a cycle in $C_1$ then $P_1 \cap E(C_1) = \bigcup_{i \in I} Q_i$.

But $1 \in I$, $Q_1 \subseteq E(C_1)$ and $Q_1 \cdot Q_2 \notin P_1$, so it follows that $P_1 \cap E(C_1)$ is not a cycle in $C_1$. If $P_1 \cap E(C_1)$ has at least four odd valency vertices, then by an argument similar to the proof of claim 5, we have a contradiction. Therefore, $P_1 \cap E(C_1)$ is a path in $C_1$. This implies $Q_x$ and $Q_y$ are paths in $C_1$. We still use $x_1$ to denote the end of $P_1 \cap E(C_1)$ which is in $H|E(G_{1,1})$, and $y_1$ to denote the end of $P_1 \cap E(C_1)$ which is in $H|E(G_{1,k})$. Since $P_1 \cap E(C_1)$ is a path in $H$ and $P_1$ is not a path in $H$, $P_1 \cap E(C_2)$ is not a cycle in $C_2$. Then by the proof of claim 5, $P_1 \cap E(C_2)$ is a path in $C_2$. This implies  

$$U_{i \in I_2} Q_i$$  

is a path in $C_2$, since $P_1 \cap E(C_2) = \bigcup_{i \in I_1} Q_i$ and  

$$(U_{i \in I_1} Q_i) \cup (U_{i \in I_2} Q_i)$$  

is a cycle in $C_2$. Therefore, 

$$H|U_{i \in I_1} E(G_{1,i}) \text{ and } H|U_{i \in I_2} E(G_{2,i})$$  

are connected. Clearly, 

$$H|U_{i \in I-k} E(G_{1,i}) \text{ and } H|E(G_{1,k})$$  

are connected.

Now $Q_i$ is a path in $H_1$, hence by lemma 5, $H|E(G_{1,i})$ can be obtained from $G_{1,i}$ by a sequence of twists not separating $u_i$ and $u_i'$ for $i = 1, 2, \ldots, m$. Since $Q_x$ is a path in $H|E(G_{1,1})$,
can be obtained from $G_{1,1}$ by a sequence of twists not separating $x_1$ and $u_1'$, and not separating $u_1$ and $u_1'$, hence not separating $x_1$ and $u_1$. Similarly, $H|E(G_{1,k})$ can be obtained from $G_{1,k}$ by a sequence of twists not separating $y_1$ and $u_k'$, not separating $y_1'$ and $u_k'$, and not separating $u_k$ and $u_k'$. Thus the claim holds.

Since $P_1 \cap E(C_1)$ is a path in $C_1$ and $P_1 \cap E(C_2) = \bigcup_{i \in I_1} Q_i$ is a path in $C_2$, $H|P_1$ has four odd valency vertices. Thus $H|P_2$ has four odd valency vertices. We may choose $P_2$ to be a simple path in $G_2$. Because $C(G)$ is a codimension-1 subspace of $C(H)$, $C(G|E(G_1) \cup P_2)$ is a subspace of $C(H|E(G_1) \cup P_2)$ with codimension at most one. This implies $|V(G|E(G_1) \cup P_2)| \leq |V(H|E(G_1) \cup P_2)| + 1$, by proposition 2 and the fact $G|E(G_1) \cup P_2$ is connected. Therefore $H|P_2$ consists of two disjoint simple paths $P_{2,1}$, $P_{2,2}$.

Let $G_{2,1}$, $G_{2,2}$, ..., $G_{2,m'}$ be the 2-connected blocks of $G_2$. Then $P_2 \cap E(G_{2,i})$ is still a path in $H$ and so $H|E(G_{2,i})$ can be obtained from $G_{2,i}$ by a sequence of twists in $G_{2,i}$ not separating the ends of $P_2 \cap E(G_{2,i})$ in $G_{2,i}$ for $i = 1, 2, ..., m'$. We may thus assume $G^*|E(G_{2,i}) = H|E(G_{2,i})$. Let
$I_1' = \{i : P_{2,1} \cap E(G_{2,1}) \neq \emptyset \}$, and $I_2' = \{i : P_{2,2} \cap E(G_{2,1}) \neq \emptyset \}$.

Since $G$ is 2-connected, $I_1'$ and $I_2'$ form a partition of

$\{1, 2, \ldots, m'\}$. Because any ordering of \{G_{2,1}, G_{2,2}, \ldots, G_{2,m'}\}

can be obtained by a sequence of twists in $G_2$ not separating $x_2$

and $y_2$, we may assume $G^*|U_{i \in I_1'} E(G_{2,i}) = H|U_{i \in I_1'} E(G_{2,i})$, and

$G^*|U_{i \in I_2'} E(G_{2,i}) = H|U_{i \in I_2'} E(G_{2,i})$. Because $P_{2,1}$ and $P_{2,2}$

are paths, the above two graphs are connected. Since the four odd

valency vertices of $H|P_1$ are $x_1$, $y_1$ and the two ends of

$C_2 |P_1 \cap E(C_2)$, we may assume $x_1$ is an end of $P_{2,1}$.

\textbf{Case 2.c.1. The other end of $P_{2,1}$ in $H$ is $y_1$.}

Then $C(G|E(G_1) U P_2)$ is a codimension-2 subspace of

$C(H|E(G_1) U P_2)$. We have a contradiction.

\textbf{Case 2.c.2. The other end of $P_{2,1}$ in $H$ is an end of $C_2 |P_1 \cap E(C_2)$}. Clearly, $H_1$ is connected. Let

$G^*_1 = G^* |( U_{i \in I-k} E(G_{1,i})) U ( U_{i \in I_1'} E(G_{2,i}))$

$G^*_2 = G^* |U_{i \in I_1} E(G_{1,i})$
\[ G^*_3 = G^*|E(G_1, k) \cup \bigcup_{i \in I_2} E(G_{2, i}) \], and

\[ G^*_4 = G^*| \cup_{i \in I_2} E(G_{1, i}) \].

We may choose \( G^* \) so that \( G^*_i = H|E(G^*_i) \) for \( i = 1, 2, 3, 4 \). Thus (ii) holds. (See figure 10.)

Case 2.d. \( C(G_1) = C(H_1) \) and \( C(G_2) = C(H_2) \).

Case 2.d.1. \( P_1 \) is a cycle in \( H_1 \). This contradicts lemma 1, since \( P_1 \) is a path in \( G \).

Case 2.d.2. \( P_1 \) is a path in \( H_1 \). Then \( P_2 \) is a path in \( H_2 \), and both \( H_1 \) and \( H_2 \) are connected. Since \( C(G) \) is a codimension-1 subspace of \( C(H) \), \|V(H)\| = |V(G)| - 1 and so \|V(H_1 \cap H_2)\| = 3 \). Let \( x, y \) be the ends of \( P_1 \) in \( H \), and \( z \) be the third common vertex of \( H_1 \) and \( H_2 \). Let \( G^*_1 \) be the graph obtained from \( H_1 \) by relabeling \( x, y, z \) by \( x_1, y_1, z_1 \) respectively and \( G^* = C(G^*_1, G^*_2; (x_1, x_2), (y_1, y_2)) \). Clearly, \( C(G^*) = C(G) \) and \( H \) is obtained from \( G^* \) by identifying \( z_1 \) and \( z_2 \). This contradicts our assumption.
Case 2.d.3. Assume $H_1|P_1$ has at least four odd valency vertices. By lemma 5, there exists a graph $G'_1$ obtained from $G_1$ by a sequence of twists not separating $x_1$ and $y_1$, and such that $H_1$ can be obtained from $G'_1$ by a sequence of twists non-crossing in $G'_1$ and separating $x_1$ and $y_1$. Since $C(G)$ is a codimension-1 subspace of $C(H)$ and $H_1$ is connected, $H_1|P_1$ has exactly four odd valency vertices. This implies all the non-crossing twists separate $x_1$ and $y_1$, and have a common vertex of attachment. Therefore, there exist disjoint graphs $G_{1,1}$, $G_{1,2}$, ..., $G_{1,m}$, and distinct vertices $x_1$, $v_1$, $z_1 \in V(G_{1,1})$, $u_2$, $v_2$, $z_2 \in V(G_{1,2})$, ..., $u_{m-1}$, $v_{m-1}$, $z_{m-1} \in V(G_{1,m-1})$, $y_1$, $u_m$, $z_m \in V(G_{1,m})$, such that:

$G'_1 = C(G_{1,1}, G_{1,2}, ..., G_{1,m}; (v_1, u_2), (v_2, u_3), ..., (v_{m-1}, u_m), (z_1, z_2, ..., z_m))$, and

$H_1 = C(G_{1,1}, G_{1,2}, ..., G_{1,m}; (z_1, u_2), (v_1, z_2, u_3), (v_2, z_3, u_4), ..., (v_{m-2}, z_{m-1}, u_m), (v_{m-1}, z_m))$.

Let $G_{2,1}$, $G_{2,2}$, ..., $G_{2,m'}$, $I'_1$, $I'_2$, $P_{2,1}$, $P_{2,2}$ be defined as in case 2.c.3. By the argument in case 2.c.3, there exists a graph $G^*$ such that:
\[ C(G) = C(G^*), \quad G^*|E(G_1') = G'_1, \]

\[ G^*| \bigcup_{i \in I_1'} E(G_{2,i}) = H| \bigcup_{i \in I_1'} E(G_{2,i}), \quad \text{and} \]

\[ G^*| \bigcup_{i \in I_2'} E(G_{2,i}) = H| \bigcup_{i \in I_2'} E(G_{2,i}). \]

Since the four odd valency vertices of \( H \mid P_1 \) are \( x_1, y_1, z_1, z_m \), the ends of \( P_{2,1} \) and \( P_{2,2} \) in \( H \) are \( x_1, y_1, z_1, z_m \). We may assume \( x_1 \) is an end of \( P_{2,1} \) in \( H \). By lemma 1, the other end is not \( z_1 \).

**Case 2.d.3.1.** The other end of \( P_{2,1} \) in \( H \) is \( z_m \). Then \( m > 2 \), by lemma 1. Let \( n = m \),

\[ G_1^* = G^*| \bigcup_{i \in I_1'} E(G_{2,i}) \cup E(G_{1,1}), \]

\[ G_n^* = G^*| \bigcup_{i \in I_2'} E(G_{2,i}) \cup E(G_{1,n}), \quad \text{and} \]

\[ G_j^* = G_{1,j}, \quad \text{for} \quad j = 2, 3, \ldots, n-1. \]

We may choose \( G^* \) so that \( G_i^* = H|E(G_i^*) \) for \( i = 1, 2, \ldots, n \).

Thus (ii) holds for \( G^* \) and \( H \), and \( H \) is connected.
Case 2.d.3.2. The other end of $P_{2,1}$ in $H$ is $y_1$. Let $n = m + 1$,

$$G^*_1 = G^*|\left( \bigcup_{i \in I_1} E(G_{2,i}) \right) \cup E(G_{1,1}) \ ,$$

$$G^*_n = G^*|\left( \bigcup_{i \in I_2} E(G_{2,i}) \right) \ , \text{ and}$$

$$G^*_j = G^*_{1,j} \ , \text{ for } j = 2, 3, \ldots, m \ .$$

We may choose $G^*$ so that $G^*_i = H|E(G^*_i)$ for $i = 1, 2, \ldots, n$.

Again (ii) holds for $G^*$ and $H$, and $H$ is connected.

Case 3. $G$ is separable. Let $G_1$ be a subgraph of $G$ with at most one vertex of attachment and $|E(G_1)| \geq 1 \leq |E(G) - E(G_1)|$.

We shall prove theorem 1 by induction on $|E(G)|$. Let $G_2 = G|(E(G) - E(G_1)) \ , \ H_1 = H|E(G_1) \ , \ H_2 = H|E(G_2) \ . \text{ Then}$

$C(G_1)$ and $C(G_2)$ are subspaces of $C(H_1)$ and $C(H_2)$ respectively.

Since $C(G)$ is the direct product of $C(G_1)$ and $C(G_2)$, at most one of the codimensions is one.

Case 3.a. $C(G_1)$ is a codimension-1 subspace of $C(H_1)$. By induction, there exist graphs $G^*_1, H^*_1$ such that one of (i), (ii), (iii) of theorem 1 holds. Let $G^* = G^*_1 \cup H^*_2 \ , \ H^* = H_1 \cup H_2 \ . \text{ Here } \cup \text{ denotes vertex-disjoint union.}$
Then one of (i), (ii), (iii) of theorem 1 holds.

Case 3.b. $C(G_2)$ is a codimension-1 subspace of $C(H_2)$.

This is similar to case 3.a.

Case 3.c. $C(G_1) = C(H_1)$ and $C(G_2) = C(H_2)$. We may choose $G_1$ to be a 2-connected block of $G$. Let $C \in C(H) - C(G)$.

Since $C(G)$ is a codimension-1 subspace of $C(H)$ and $H_1$ is 2-connected, $H|E(H_1) \cap C$ has at most two odd valency vertices, and hence is a path in $H$. Let $G^*$ be a graph obtained from $H$ by splitting $H_1$ and $H_2$ at one end of $H|E(H_1) \cap C$. Then clearly, $C(G) = C(G^*)$. But $C$ is a path in $G^*$, which contradicts lemma 1. $\square$
2. **Proof of theorem 2.**

We assume (i) does not hold. Then by lemma 1, any path in $G$ is not a cycle in $H$. We shall prove the theorem by induction on $k$.

By lemma 4, there exists a vertex $v$ in $G$, such that $C(G-v)$ is a codimension-$(k-1)$ subspace of $C(H|E(G-v))$. Let $G' = G-v$, and $H' = H|E(G')$. By proposition 3, $G'$ is $(k+1)$-connected.

**Case 1.** $k = 2$. Since $G'$ is 3-connected, (i) does not hold for $G'$. By theorem 1, either (ii) or (iii) of theorem 1 holds for $G'$ and $H'$. Then:

**Case 1.a.** Alternative (iii) of theorem 1 holds for $G'$ and $H'$. Then there exist disjoint graphs $G'_1, G'_2, G'_3, G'_4$, and distinct vertices $x_i, y_i, z_i \in V(G'_i)$ for $i = 1, 2, 3, 4$, such that:

$G' = C(G'_1, G'_2, G'_3, G'_4; (x_1, y_2, z_4), (x_2, y_1, z_3), (x_3, y_4, z_2), (x_4, y_3, z_1))$, and $H' = C(G'_1, G'_2, G'_3, G'_4; (x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4), (z_1, z_2, z_3, z_4))$. We may assume $|V(G'_i)| \geq 4$ for $i = 1, 2, 3, 4$. Otherwise, (ii) of theorem 1 holds for $G'$ and $H'$.

Since $G$ is 4-connected, there exists an edge $e_i$ in $G$.
joining \( v \) and \( v_i \in V(G'_i) - \{x_i, y_i, z_i\} \), for \( i = 1, 2, 3, 4 \).

Let \( P_{x_1}, P_{y_1}, P_{z_1} \) be paths in \( G'_i \) joining \( v_1 \) and \( x_1, v_1 \) and \( y_1, v_1 \) and \( z_1 \) respectively, for \( i = 1, 2, 3, 4 \). Then \( e_1, P_{x_1}, P_{y_2}, \) and \( e_2 \) form a cycle in \( G \), hence a cycle in \( H \).

Therefore, the ends of \( e_1 \) in \( H \) are in \( \{v_1, v_2, x_1, y_2\} \). Similarly by considering cycles \( \{e_1, e_3\} \cup P_{y_1} \cup P_{z_3} \) and \( \{e_1, e_4\} \cup P_{x_1} \cup P_{z_4} \) in \( G \), the ends of \( e_1 \) in \( H \) are in \( \{v_1, v_3, y_1, z_3\} \) and \( \{v_1, v_4, x_1, z_4\} \). Thus the ends of \( e_1 \) in \( H \) are in \( \{v_1, v_2, x_1, y_2\} \cap \{v_1, v_3, y_1, z_3\} \cap \{v_1, v_4, x_1, z_4\} \). Since \( e_1 \) is not a loop, we have a contradiction.

**Case 1.b.** Alternative (ii) of theorem 1 holds for \( G' \) and \( H' \). Then there exist disjoint graphs \( G'_1, G'_2, \ldots, G'_n \) for \( n \geq 3 \) and distinct vertices \( x_i, y_i, z_i \in V(G'_i) \), for \( i = 1, 2, \ldots, n \), such that:

\[
G' = C(G'_1, G'_2, \ldots, G'_n ; (y_1, x_2), (y_2, x_3), \ldots, (y_n, x_1), (z_1, z_2, \ldots, z_n)), \quad \text{and}
\]

\[
H' = C(G'_1, G'_2, \ldots, G'_n ; (y_1, z_2, x_3), (y_2, z_3, x_4), \ldots, (y_{n-1}, z_n, x_1), \]
\[
(y_{n-1}, z_1, x_2) \).
\]

A vertex in \( V(G'_i) - \{x_i, y_i, z_i\} \) is called an interior vertex of
Since \( G \) is \( h \)-connected, if \( G'_i \) has interior vertices, then there exists an edge \( f_i \) in \( G \) joining \( v \) and an interior vertex \( v_i \) of \( G'_i \). Let \( P_i \) be a path in \( G'_i \) joining \( v_i \) and \( z_i \). If \( G'_i \) and \( G'_{i+1} \) have no interior vertex, then let \( f_{i+1} \) be the edge in \( G \) joining \( v \) and \( (y_i, x_{i+1}) \), \( h_{i+1} \) be the edge in \( G \) joining \((z_1, z_2, \ldots, z_m)\) and \((y_i, x_{i+1})\). If \( G'_i \) has no interior vertex, then let \( g_i \) be the edge in \( G'_i \) joining \( x_i \) and \( y_i \). Thus the ends of \( h_{i+1} \) in \( H \) are \((y_i, z_{i+1}), (x_{i+1}, z_i)\), and the ends of \( g_i \) in \( H \) are \((x_i, z_{i-1}), (y_i, z_{i+1})\). Such edges always exist, since \( G \) is \( h \)-connected.

**Case 1.b.1.** Three of \((G'_1, G'_2, \ldots, G'_n)\), say \( G'_i, G'_j, G'_k \) have interior vertices. Since \( f_i, f_j, P_i, P_j \) form a cycle in \( G \), the ends of \( f_i \) in \( H \) are in \((v_i, v_j, z_i, z_j)\).

Similarly, the ends of \( f_i \) in \( H \) are in \((v_i, v_k, z_i, z_k)\). Therefore, \( v_i \) and \( z_i \) are the ends of \( f_i \) in \( H \), which contradicts lemma 1.

**Case 1.b.2.** Exactly two of \((G'_1, G'_2, \ldots, G'_n)\), say \( G'_1, G'_i \) have interior vertices. By considering the cycle formed by \( f_1, P_1, P_i \) and \( f_i \) in \( G \), the ends of \( f_i \) in \( H \) are in
\( \{v_1, v_1, z_1, z_1\} \). By lemma 1, at most one of \( v_1 \) and \( z_1 \) is a vertex of \( e_1 \) in \( H \).

**Case l.b.2.1.** \( i = 2 \) and \( n > 3 \). By considering the cycle formed by \( f_1, P_1, h_{n-1}, \) and \( f_{n-1} \) in \( G \), the ends of \( f_1 \) in \( H \) are in \( \{v_1, z_1, z_{n-1}, z_n\} \). Since \( n > 3 \), the ends of \( f_1 \) in \( H \) are \( v_1, z_1 \). We have a contradiction.

**Case l.b.2.2.** \( i = 2 \) and \( n = 3 \).

**Case l.b.2.2.1.** There exists an edge \( e \) in \( G \) joining \( v \) and \( v' \in V(G) - \{(y_1, x_2), (z_1, z_2), (v_1, v_2)\} \). We may assume \( v' \in V(G_1') \). By considering the cycle formed by \( e \), \( f_1 \) and a path in \( G_1' \) joining \( v_1 \) and \( v' \), exactly one end of \( f_1 \) in \( H \) is either \( v_1 \) or \( v' \). Thus \( v_1 \) is an end of \( f_1 \) in \( H \). By lemma 1, the other end of \( f_1 \) in \( H \) is neither \( z_1 \) nor \( (y_1, z_2) \). Thus \( v_1 \) and \( v_2 \) are the ends of \( f_1 \) in \( H \), and \( z_1 \) and \( z_2 \) are the ends of \( f_2 \) in \( H \). This implies that there is no edge in \( G \) joining \( v \) and a vertex in \( V(G_2') - \{v_2, x_2, z_2\} \).

If \( f \) is an edge in \( G \) joining \( v \) and \( u \in V(G_1') - v_1 \), then the ends of \( f \) in \( H \) are \( u, v_2 \). Let \( G_1 = G | V(G_1) \cup v \), \( G_2 = G_2' \), \( e_1 = f_2 \), and \( e_2 = g_3 \). Then (ii)(b) holds.
Case 1.b.2.2.2. There is no edge in $G$ joining $v$ and a vertex of $V(G) - \{(y_1, x_2), (z_1, z_2), v_1, v_2\}$.

Since $G$ is 4-connected, there exists edges $a_1, a_2$ in $G$ joining $v$ and $(y_1, x_2)$, $v$ and $(z_1, z_2)$ respectively. By considering the cycle in $G$ formed by $f_1, a_1$ and a path in $G_1'$ joining $v_1$ and $y_1$, exactly one end of $f_1$ in $H$ is either $v_1$ or $y_1$.

Similarly, exactly one end of $f_1$ in $H$ is either $v_1$ or $z_1$.

Thus either $v_1$ is an end of $f_1$ in $H$ or $y_1$ and $z_1$ are the ends of $f_1$ in $H$. If $v_1$ is an end of $f_1$ in $H$, then by lemma 1, $v_2$ is the other end. If $y_1$ and $z_1$ are the ends of $f_1$ in $H$, then $v_1$ and $v_2$ are the ends of $f_2$ in $H$. In both alternatives, (ii)(b) holds, by an argument similar to case 1.b.2.2.

Case 1.b.2.3. $i = 3$ and $n = 3$. This is similar to case 1.b.2.2.

Case 1.b.2.4. $i = 3$ and $n = 4$. Since $G$ is 4-connected, there exists an edge $f$ in $G$ joining $v$ and $u \in V(G) - \{v_1, v_2, (z_1, z_2)\}$. We may assume $u \in V(G_1')$. By considering the cycle in $G$ formed by $f_1, f$ and a path in $G_1'$ joining $u$ and $v_1$, exactly one end of $f_1$ in $H$ is either $v_1$ or $u$.

Thus $v_1$ is an end of $f_1$ in $H$. 

Case 1.b.2.4.1. The ends of $f_1$ in $H$ are $v_1, z_3$. This implies that the ends of $f_3$ in $H$ are $v_3, z_1$. If $f'$ is an edge in $G$ joining $v$ and a vertex $u'$ in $G_1'$, then by considering the cycle in $G$ formed by $f_1', f'$ and a path in $G_1'$ joining $v_1$ and $u'$, the ends of $f'$ in $H$ are $u'$ and $z_3$. Similarly, if $f'$ is an edge in $G$ joining $v$ and a vertex $u'$ in $G_3'$, then the ends of $f'$ in $H$ are $u'$ and $z_1$.

Let $G_1 = G|V(G'_1) \cup v$, $G_2 = G|V(G'_3) \cup v$ with any edge in $G$ joining $v$ and $(z_1, z_3)$ deleted, $e_1 = g_2$ and $e_2 = g_4$. Then (ii)(b) holds.

Case 1.b.2.4.2. The ends of $f_1$ in $H$ are $v_1, v_2$. Then the ends of $f_2$ in $H$ are $z_1, z_2$. This implies that there is no edge in $G$ joining $v$ and a vertex in $V(G'_2) - \{z_2, v_2\}$. If $f'$ is an edge in $G$ joining $v$ and $u' \in V(G'_1)$, then the ends of $f'$ in $H$ are $u'$ and $v_2$. Let $G_1 = G|V(G'_1) \cup v$, $G_2 = G'_3$, $e_1 = g_2$, $e_2 = g_4$, $e_3 = f_2$. Then (ii)(a) holds.

Case 1.b.2.5. $i = 3$ and $n > 4$. By considering the cycle in $G$ formed by $f'_1, p_1, h_4$ and $f'_4$, the ends
of $f_1$ in $H$ are in \{v_1, z_1, z_2, z_3\}. This implies that the ends of $f_1$ in $H$ are $v_1, z_1$. We have a contradiction.

**Case 1.b.2.6.** $i > 4$. This is similar to case 1.b.2.5. By considering the cycle in $G$ formed by $f_1', P_1', h_2$ and $f_2'$, we have a contradiction.

**Case 1.b.3.** *Exactly one of* \{$G_1', G_2', \ldots, G_n'$* *say* \$G_1'$, *has interior vertices.*

**Case 1.b.3.1.** $n = 3$. We may thus assume

$|E(G)| = |E(G_1')| + 3$. In this case $x_1, y_1, z_1$ play the same role.

Since $V(G_1') = V(G)$, by lemma 3, there are at most three edges joining $v$ and vertices of $G_1'$. Because $G$ is 4-connected, there are exactly three edges $f_1', f, f'$ in $G$ joining $v$ and vertices $v_1, u, u'$ in $G_1'$ respectively. By considering the cycle in $G$ formed by $f_1', P_1', h_2$ and $f_2'$, the ends of $f_1$ in $H$ are in \{v_1, z_1, (x_1, y_2), (y_1, x_3)\}. By lemma 1, $v_1$ cannot be an end of $f_1$ in $H$. By considering the cycle in $G$ formed by $f_1', f$ and a path in $G_1'$ joining $v_1$ and $u$, exactly one end of $f_1$ in $H$ is either $v_1$ or $u$. Thus $u$ is an end of $f_1$ in $H$. Similarly, $u'$ is an end of $f_1$ in $H$. This implies $u, u' \in \{x_1, y_1, z_1\}$. 
Since $x_1, y_1, z_1$ play the same role in this case, we may assume $u = x_1$ and $u' = y_1$. Let $G_1 = G_1', G_2 = G', e_1 = f_2', f, f', f_2}$, $e_1 = f_1'$, and $e_2 = h_2'$. Then (ii)(b) holds.

Case 1.b.3.2. \( n = 4 \). We may thus assume $|E(G)| = |E(G_1')| + 5$. By considering the cycles $\{f_1', h_2', f_2\} \cup P_1$ and $P_1 \cup \{f_1, h_3, f_3\}$ in $G$, the ends of $f_1$ in $H$ are in \( \{v_1, z_1, z_2, z_3\} \cap \{v_1, z_1, z_3, z_4\} = \{v_1, z_1, z_3\} \). By lemma 1, at most one of $v_1, z_1$ is an end of $f_1$ in $H$.

Case 1.b.3.2.1. The ends of $f_1$ in $H$ are $v_1, z_3$. If $f$ is an edge in $G$ joining $v$ and $u \in V(G_1') - v_1$, then by considering the cycle in $G$ formed by $f_1', f$ and a path in $G_1'$ joining $v_1$ and $u$, the ends of $f$ in $H$ are $u, z_3$. Let $G_1 = G_1' \cup v, G_2 = G_1' \{f_2, f_3, h_2, h_3, e_3\}, e_1 = g_2, \text{ and } e_2 = g_4$. Then (ii)(b) holds.

Case 1.b.3.2.2. The ends of $f_1$ in $H$ are $z_1, z_3$. Since $G$ is 4-connected, there is an edge $f$ in $G$ joining $v$ and $u \in V(G_1') - v_1$. Then one end of $f_1$ is either
v_1 or u. Thus u = z_1 and the ends of f in H are v_1, z_3.

Let G_1 = G', G_2 = G\{f_2, f_3, f, h_2, h_3, g_3\}, e_1 = f_1, e_2 = g_2,
and e_3 = g_4. Then (ii)(a) holds.

Case 1.b.3.3. n > 5. By considering the cycles
\[ P_1 \cup \{f_1, h_2, f_2\} \text{ and } P_1 \cup \{f_1, h_4, f_4\}, \]
the ends of f_1 in H are in \{v_1, z_1, z_2, z_3\} \cap \{v_1, z_1, z_4, z_5\} = \{v_1, z_1\}.
We have a contradiction.

Case 1.b.4. G'_i has no interior vertex for
i = 1, 2, ..., n. Clearly, G' = W_n, a wheel with hub
(z_1, z_2, ..., z_n). (See [T] P.102.)

Case 1.b.4.1. n = 3. Since G is 4-connected,
G = K_5. By proposition 7, we have a contradiction.

Case 1.b.4.2. n = 4. By considering the triangles \{f_1, g_2, f_2\} and \{f_1, g_1, f_4\},
one end of f_1 in H is either z_1 or z_3, the other end is either z_2 or z_4.

Case 1.b.4.2.1. The ends of f_1 in H are z_1, z_2. Then f_1 and h_1 form a cycle in H. However, they
form a path in $G$, which contradicts lemma 1.

**Case 1.b.4.2.2.** The ends of $f_1$ in $H$ are $z_1, z_4$. Then the ends of $f_2, f_3, f_4$ in $H$ are $z_3$ and $z_4$, $z_2$ and $z_3$, $z_1$ and $z_2$, respectively. If there exists an edge $f$ in $G$ joining $v$ and $(z_1, z_2, z_3, z_4)$, then the ends of $f$ in $H$ are $z_2, z_4$. Let $(z_1, z_2, z_3, z_4) = (3, 4), \ v = (1, 2), (y_1, x_2) = (1, 3), (y_2, x_3) = (2, 3), (y_3, x_4) = (2, 4), (y_4, x_1) = (1, 4), (y_4, z_1, x_2) = 1, (y_2, z_3, x_4) = 2, (y_3, z_4, x_1) = 3, \text{ and } (y_1, z_2, x_3) = 4$. Then (iii)(b) holds.

**Case 1.b.4.2.3.** The ends of $f_1$ in $H$ are $z_2, z_3$. This is similar to case 1.b.4.2.2.

**Case 1.b.4.2.4.** The ends of $f_1$ in $H$ are $z_3, z_4$. Then the ends of $f_2, f_3, f_4$ in $H$ are $z_1$ and $z_4$, $z_1$ and $z_2$, $z_2$ and $z_3$ respectively. Since $(f_1, h)$ is not a path in $H$, by lemma 2, there is no edge in $G$ joining $v$ and $(z_1, z_2, z_3, z_4)$. Now we label the vertices of $G$ and $H$ the same as case 1.b.4.2.2. Then $G = L(K_4)$, $H = 2K_4$ and thus (iii)(a) holds.
Case 1.b.  \( n > 4 \). By considering the cycles \( \{ f_1, h_1, h_3, f_3 \} \) and \( \{ f_1, h_1, h_4, f_4 \} \) in \( G \), the ends of \( f_1 \) in \( H \) are in \( \{ z_1, z_2, z_3, z_4 \} \cap \{ z_1, z_2, z_4, z_5 \} = \{ z_1, z_2, z_4 \} \).

By considering the triangles \( \{ f_1, g_2, f_2 \} \) and \( \{ f_1, g_1, f_1 \} \), one end of \( f_1 \) in \( H \) is either \( z_1 \) or \( z_3 \), the other end is either \( z_2 \) or \( z_4 \). Since \( n > 4 \), the ends of \( f_1 \) in \( H \) are \( z_1, z_2 \).

This implies \( \{ f_1, h_1 \} \) is a cycle in \( H \). However, \( \{ f_1, h_1 \} \) is a path in \( G \). We have a contradiction.

Case 2.  \( k = 3 \). By induction and the remark of theorem 2, either (ii) or (iii)(a) of theorem 2 holds for \( G' \) and \( H' \), since we assume (i) does not hold for \( G \) and \( H \).

Case 2.a. Alternative (ii)(a) of theorem 2 holds for \( G' \) and \( H' \). Then there exist two disjoint graphs \( G_1', G_2' \), distinct vertices \( x_1, x_2, x_3, x_4 \in V(G_1') \), \( y_1, y_2, y_3, y_4 \in V(G_2') \), and three edges \( e_1, e_2, e_3 \) with ends \( u_1 \) and \( v_1 \), \( u_2 \) and \( v_2 \), \( u_3 \) and \( v_3 \) respectively, such that:

\[
G' = (G_1', G_2', e_1, e_2, e_3 ; (x_1, u_1), (x_2, u_2), (x_3, u_3), (y_1, v_1)
(y_2, v_2), (y_3, v_3), (x_4, y_4)), \text{ and}
\]
\[ H' = (G_1', G_2', e_1, e_2, e_3; (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, u_1, u_2, u_3), (y_1, v_1, v_2, v_3)). \]

Since \( G \) is 5-connected, there exist at least five edges joining \( v \) and vertices of \( G' \). We may thus assume that there are at least three edges in \( G \) joining \( v \) and vertices of \( G'_1 \). Since \( G - (x_1, y_2, y_3, (x_4, y_4)) \) is connected, there exist two edges \( f_1, f_2 \) in \( G \) joining \( v \) and \( w_1 \in V(G'_1) - (x_1, x_4) \), \( v \) and \( w_2 \in V(G'_2) - (y_2, y_3, y_4) \) respectively. Clearly, \( w_1 \) and \( w_2 \) represent different vertices of \( H' \).

Let \( P_1 \) be a path in \( G'_1 \) joining \( w_1 \) and \( x_4 \), and \( P_2 \) be a path in \( G'_2 \) joining \( w_2 \) and \( y_4 \). By considering the cycle in \( G \) formed by \( f_1, P_1, P_2 \) and \( f_2 \), the ends of \( f_1 \) and \( f_2 \) in \( H \) are \( w_1, w_2, x_4, y_4 \). By lemma 1, at most one of \( w_1, x_4 \) is an end of \( f_1 \) in \( H \).

**Case 2.a.1.** The ends of \( f_1 \) in \( H \) are \( w_1, w_2 \).

Then the ends of \( f_2 \) in \( H \) are \( x_4, y_4 \). There is no edge in \( G \) joining \( v \) and \( w_3 \in V(G'_1) - (w_2, y_4) \), since \( f_2 \) and the path in \( G'_2 \) joining \( w_2, w_3 \) do not form a path in \( H \). If \( f_3 \) is an edge in \( G \) joining \( v \) and \( w_3 \in V(G'_1) - w_1 \), then by considering a cycle in \( G \) containing \( f_1 \) and \( f_3 \), the ends of \( f_3 \) in \( H \) are
\( w_3, w_2 \). Let \( G_1 = G \cup v \), \( G_2 = G' \), and \( e_4 = e_2 \). Then (ii)(a) holds.

\[ \text{Case 2.a.2. The ends of } f_1 \text{ in } H \text{ are } w_1, y_4. \]

Then the ends of \( f_2 \) in \( H \) are \( w_2, x_4 \). If \( f_3 \) is an edge in \( G \) joining \( v \) and \( w_3 \in V(G'_1) \), then the ends of \( f_3 \) in \( H \) are \( w_3, y_4 \). If \( f_3 \) is an edge in \( G \) joining \( v \) and \( w_3 \in V(G'_2) \), then the ends of \( f_3 \) in \( H \) are \( w_3, x_4 \). Let \( G_1 = G \cup v \), and \( G_2 = G \cup v \) with any edge joining \( v \) and \( y_4 \) deleted. Then (ii)(b) holds.

\[ \text{Case 2.a.3. The ends of } f_1 \text{ in } H \text{ are } x_4, y_4. \]

By an argument similar to case 2.a.1, there is no edge in \( G \) joining \( v \) and a vertex in \( V(G'_1) - \{w_1, x_4\} \). We have a contradiction.

This completes case 2.a.3.

In the following two cases, \( w_1 \) is not an end of \( f_1 \) in \( H \). By lemma 3, there are at most three edges in \( G \) joining \( v \) and vertices of \( G'_1 \). Since \( G \) is 5-connected, there exists an edge \( f_3 \) in \( G \) joining \( v \) and \( w_3 \in V(G'_2) - \{w_2, y_4\} \).

\[ \text{Case 2.a.4. The ends of } f_1 \text{ in } H \text{ are } w_2, x_4. \]

Then the ends of \( f_2 \) in \( H \) are \( w_1, y_4 \). By considering the cycle
in G formed by \( f_2, f_3 \) and a path in \( G'_2 \) joining \( w_2 \) and \( w_3 \),
one end of \( f_3 \) in \( H \) is either \( w_2 \) or \( w_3 \). This implies that
\( w_1 \) and \( w_3 \) represent the same vertex in \( H \). Similarly, let \( f_4 \)
be an edge in \( G \) joining \( v \) and \( w_4 \in V(G'_1) - \{w_1, x_4\} \), then \( w_2 \)
and \( w_4 \) represent the same vertex in \( H \). Since \( w_1 \in V(G'_1) - \{x_1, x_4\} \)
and \( w_2 \in V(G'_2) - \{y_2, y_3, y_4\} \), it follows that \( w_1 = x_i, w_3 = y_i \), for some
\( i \in \{2, 3\} \), \( w_2 = y_1 \), and \( w_4 = x_1 \). We may assume \( i = 2 \). We
note that the ends of the fifth edge in \( G \) must be \( v \) and \( (x_4, y_4) \).
Then \( G - \{y_1, y_2, y_3, (x_4, y_4)\} \) is disconnected, which contradicts
the fact \( G \) is 5-connected.

**Case 2.a.5.** The ends of \( f_1 \) in \( H \) are \( w_2, y_4 \).
Then the ends of \( f_2 \) in \( H \) are \( w_1, x_4 \). Since one end of \( f_2 \) is
either \( w_2 \) or \( w_3 \), \( w_1 \) and \( w_3 \) represent the same vertex in \( H \).
By an argument similar to case 2.a.4, we have a contradiction.

**Case 2.b.** Alternative (ii)(b) of theorem 2 holds for \( G' \)
and \( H' \). Then there exist two disjoint graphs \( G'_1, G'_2 \), distinct
vertices \( x_1, x_2, x_3, x_4 \in V(G'_1) \), \( y_1, y_2, y_3, y_4 \in V(G'_2) \), and
two edges \( e_1, e_2 \) with ends \( u_1 \) and \( v_1, u_2 \) and \( v_2 \) respectively,
such that:
\[ G' = C(G'_1, G'_2, e_1, e_2; (x_1, u_1), (x_2, u_2), (y_1, v_1), (y_2, v_2), (x_3, y_3), (x_4, y_4)), \]
\[ H' = C(G'_1, G'_2, e_1, e_2; (x_1, y_1), (x_2, y_2), (x_3, u_1, u_2, y_4), (x_4, v_1, v_2, y_3)). \]

Since \( G \) is 5-connected, we may assume there are at least three edges in \( G \) joining \( v \) and vertices of \( G'_1 \). Because \( G - \{x_1, y_2', (x_3, y_3), (x_4, y_4)\} \) is connected, there exist edges \( f_1, f_2 \) in \( G \) joining \( v \) and \( w_1 \in V(G'_1) - \{x_1, x_3, x_4\}, w_2 \in V(G'_2) - \{y_2, y_3, y_4\} \).

Let \( P_1 \) be a path in \( G'_1 \) joining \( w_1 \) and \( x_4 \), and \( P_2 \) be a path in \( G'_2 \) joining \( w_2 \) and \( y_4 \). By considering the cycle in \( G \) formed by \( f_1, P_1, P_2 \) and \( f_2 \), the ends of \( f_1 \) and \( f_2 \) in \( H \) are \( w_1, w_2, (x_3, y_4), (x_4, y_3) \). By lemma 1, at most one of \( w_1, x_3 \) is an end of \( f_1 \) in \( H \), and at most one of \( w_1, x_4 \) is an end of \( f_1 \) in \( H \).

**Case 2.b.1.** The ends of \( f_1 \) in \( H \) are \( w_1, w_2 \).

Then the ends of \( f_2 \) in \( H \) are \( y_3, y_4 \). Therefore, there is no edge in \( G \) joining \( v \) and a vertex of \( V(G'_2) - \{w_2, y_3, y_4\} \). Let \( f_3 \) be an edge in \( G \) joining \( v \) and \( w_3 \in V(G'_1) - w_1 \). By considering a cycle in \( G \) containing \( f_1 \) and \( f_3 \), the ends of \( f_3 \) in \( H \) are \( w_3 \) and \( w_2 \). Let \( \overline{G}_1 = G(V(G'_1) \cup v), \overline{G}_2 = G'_2 \), and
$e_3 = f_2$. Then (ii)(b) holds.

**Case 2.b.2.** The ends of $f_1$ in $H$ are $x_3, x_4$. Then the ends of $f_2$ in $H$ are $w_1, w_2$. This is similar to case 2.b.1.

**Case 2.b.3.** The ends of $f_1$ in $H$ are either $w_2, y_3$ or $w_2, y_4$. Then the ends of $f_2$ in $H$ are either $w_1, y_4$ or $w_1, y_3$. In this case, $w_1$ is not an end of $f_1$ in $H$. By lemma 3, there are at most three edges in $G$ joining $v$ and vertices of $G'_i$. Thus there exists an edge $f_3$ in $G$ joining $v$ and $w_3 \in V(G'_2) - \{w_2, y_3, y_4\}$. Then one end of $f_2$ in $H$ is either $w_2$ or $w_3$. Since $w_2 \in V(G'_2) - \{y_2, y_3, y_4\}$, this implies $w_1$ and $w_3$ represent the same vertex in $H$. Because $w_1 \in V(G'_1) - \{x_1, x_3, x_4\}$, it follows that $w_1 = x_2$ and $w_3 = y_2$. Similarly, there exists an edge $f_4$ in $G$ joining $v$ and $w_4 \in V(G'_1) - \{w_1, x_3, x_4\}$, and so $w_2 = y_1, w_4 = x_1$. Since the valency of $v$ in $G$ is at least 5, by the above argument, there exists a fifth edge $f_5$ in $G$ joining $v$ and a common vertex of $G'_1$ and $G'_2$ in $G$, say $(x_3, y_3)$. Since there are at most three edges in $G$ joining $v$ and vertices of $G'_i$ for $i = 1, 2$, there is no other edge in $G$ joining $v$ and a vertex in $V(G) - \{x_1, y_1, x_2, y_2, (x_3, y_3)\}$. Since $G$ is
5-connected, $G - \{x_1, x_2, x_3, x_4\}$ and $G - \{y_1, y_2, y_3, y_4\}$ are connected. Therefore $V(G'_1) = \{x_1, x_2, x_3, x_4\}$, $V(G'_2) = \{y_1, y_2, y_3, y_4\}$, and thus $G|V(G'_1) \cong K_4 \cong G|V(G'_2)$. When $f_1$ has ends $w_2, y_3$ in $G$, let $v = (1, 2, (x_4, y_4) = (3, 4), (x_1, y_1) = (1, 3), (y_1, v_1) = (1, 4), (x_2, v_2) = (2, 3), (y_2, v_2) = (2, 4), t = (x_3, y_3), (x_2, y_2) = 2, (x_3, y_3, u_2, y_4) = 3, and $(x_1, v_1, v_2, y_3) = 4$. Then $G = L(K_4) + t$, $H = 3K_4$, and (iv) holds. The other possibility is similar.

Case 2.c. Alternative (iii)(a) of theorem 2 holds. Then $G' = L(K_4)$ and $H' = 2K_4$. Since $G$ is 5-connected, each vertex $(i, j)$ of $G'$ is joined to $v$ by an edge $f_{i,j}$. Let $(i, j, h, h') = (1, 2, 3, 4)$. By considering the triangle formed by $f_{i,j}$, $f_{j,h}$, and the edge in $G$ joining $(i, j)$ and $(j, h)$, exactly one end of $f_{i,j}$ in $H$ is either $i$ or $h$. Similarly, exactly one end of $f_{i,j}$ in $H'$ is either $i$ or $h'$. Thus the ends of $f_{i,j}$ in $H$ are either $i, j$ or $h, h'$. Because of the symmetry of $G'$ and $H'$, we may assume the ends of $f_{i,j}$ in $H$ are $i, j$. Then the ends of $f_{j,h}$ in $H$ are $j, h$. Thus the ends of $f_{m,m'}$ in $H$ are $m, m'$ for all distinct $m, m' \in \{1, 2, 3, 4\}$. Let $v = t$. Then (iv) holds.
Case 3. $k \geq 4$. By induction, one of the following cases 3.a, 3.b, 3.c, or 3.d holds.

**Case 3.a.** Alternative (ii)(a) of theorem 2 holds for $G'$ and $H'$. Then there exist two disjoint graphs $G'_1$, $G'_2$, distinct vertices $x_1, x_2, \ldots, x_{k+1} \in V(G'_1)$, $y_1, y_2, \ldots, y_{k+1} \in V(G'_2)$, and $k$ edges $e_1, e_2, \ldots, e_k$ with ends $u_1$ and $v_1, u_2$ and $v_2, \ldots, u_n$ and $v_n$, respectively, such that:

$G' = (G'_1, G'_2, e_1, e_2, \ldots, e_k ; (x_1, u_1), (x_2, u_2), \ldots, (x_k, u_k), (y_1, v_1), (y_2, v_2), \ldots, (y_k, v_k), (x_{k+1}, y_{k+1}))$, and

$H' = (G'_1, G'_2, e_1, e_2, \ldots, e_k ; (x_1, y_1), (x_2, y_2), \ldots, (x_k, y_k), (x_{k+1}, v_1), (x_{k+1}, v_2), \ldots, (x_{k+1}, v_k), (y_{k+1}, u_1), (y_{k+1}, u_2), \ldots, (y_{k+1}, u_k))$.

The proof of this case is similar to case 2.a.

Since $G$ is $(k+2)$-connected and $k \geq 4$, there are at least six edges in $G$ joining $v$ and vertices of $G'$. We may thus assume there are at least three edges in $G$ joining $v$ and vertices of $G'_1$. Since $G - \{x_1, y_2, y_3, \ldots, y_k, (x_{k+1}, y_{k+1})\}$ is connected, there exist two edges $f_1, f_2$ in $G$ joining $v$ to $w_2 \in V(G'_1) - \{x_1, x_{k+1}\}$, and to $w_2 \in V(G'_2) - \{y_2, y_3, \ldots, y_{k+1}\}$, respectively. Clearly, $w_1$ and $w_2$ represent different vertices
in $H'$. By an argument similar to case 2.a, the ends of $f_1$ and $f_2$ in $H$ are $w_1', w_2', \ell_{k+1}', y_{k+1}'$, and at most one of $w_1$, $x_{k+1}$ is an end of $f_1$ in $H$.

Case 3.a.1. The ends of $f_1$ in $H$ are $w_1', w_2'$. Then the ends of $f_2$ are $x_{k+1}', y_{k+1}'$. Let $G_1 = G \mid V(G_1') \cup v$, $G_2 = G_2', e_{k+1} = f_2$. By an argument similar to case 2.a.1, (ii)(a) holds.

Case 3.a.2. The ends of $f_1$ in $H$ are $w_1', y_{k+1}'$. Then the ends of $f_2$ in $H$ are $w_2'$ and $x_{k+1}'$. Let $G_1 = G \mid V(G_1') \cup v$, $G_2 = G_2' \mid V(G_2') \cup v$ with any edge joining $v$ and $y_{k+1}'$ deleted. By an argument similar to case 2.a.2, (ii)(b) holds.

Case 3.a.3. The ends of $f_1$ in $H$ are $x_{k+1}', y_{k+1}'$. This is similar to case 2.a.3.

Case 3.a.4. The ends of $f_1$ in $H$ are $w_2', x_k'$. By an argument similar to case 2.a.4, the valency of $v$ in $G$ is at most 5, which contradicts the fact $G$ is $(k+2)$-connected and $k \geq 4$.

Case 3.a.5. The ends of $f_1$ in $H$ are $w_2', y_{k+1}'$. Again, the valency of $v$ in $G$ is at most 5. We have a contradiction.
Case 3.b. Alternative (ii)(b) of theorem 2 holds for $G'$ and $H'$. Then there exist two disjoint graphs $G_1', G_2'$, distinct vertices $x_1, x_2, \ldots, x_{k+1} \in V(G_1'), y_1, y_2, \ldots, y_{k+1} \in V(G_2')$, and edges $e_1, e_2, \ldots, e_{k-1}$ with ends $u_1$ and $v_1, u_2$ and $v_2, \ldots, u_{k-1}$ and $v_{k-1}$ respectively, such that:

$$G' = (G_1', G_2', e_1, e_2, \ldots, e_{k-1}; (x_1, u_1), (x_2, u_2), \ldots, (x_{k-1}, u_{k-1}); (y_1, v_1), (y_2, v_2), \ldots, (y_{k-1}, v_{k-1}); (x_k, y_k), (x_{k+1}, y_{k+1}))$$

$$H' = (G_1', G_2', e_1, e_2, \ldots, e_{k-1}; (x_1, y_1), (x_2, y_2), \ldots, (x_{k-1}, y_{k-1}); (x_k, y_{k+1}), (x_{k+1}, y_{k+1}); (y_k, y_{k+1}, v_1, v_2, \ldots, v_{k-1})).$$

Since the valency of $v$ in $G$ is at least $k+2$, we may thus assume there are at least $\left\lceil \frac{k+3}{2} \right\rceil$ edges in $G$ joining $v$ and vertices of $G_1'$. Since $G$ is $(k+2)$-connected, $G - \{x_1, y_2, \ldots, y_{k-1}, (x_k, y_k), (x_{k+1}, y_{k+1})\}$ is connected. Therefore, there exist two edges $f_1, f_2$ in $G$ joining $v$ to $w_1 \in V(G_1') - \{x_1, x_k, x_{k+1}\}$, and to $w_2 \in V(G_2') - \{y_2, y_3, \ldots, y_k\}$, respectively. Then the ends of $f_1$ and $f_2$ in $H$ are $w_1, w_2, (x_k, y_{k+1})$ and $(x_{k+1}, y_k)$. 
Case 3.b.1. The vertex \( w_1 \) is not an end of \( f_1 \) in \( H \). By lemma 3, there are at most three edges in \( G \) joining \( v \) and vertices of \( G'_i \). Thus \( k = 4 \) and there are exactly three edges in \( G \) joining \( v \) and vertices of \( G'_i \) for \( i = 1, 2 \). Therefore, neither \( (x_k, y_k) \) nor \( (x_{k+1}, y_{k+1}) \) is joined to \( v \) by an edge in \( G \). Let \( f_3, f_4 \) be the two edges in \( G \) joining \( v \) and \( w_3, w_4 \in V(G'_1) - \{w_1, x_k, x_{k+1}\} \). By considering a cycle in \( G \) formed by \( f_1, f_3 \) and a path in \( G'_i \) joining \( w_1 \) and \( w_3 \), one end of \( f_1 \) in \( H \) is either \( w_1 \) or \( w_3 \). Thus \( w_3 \) is an end of \( f_1 \) in \( H \). Since \( w_3 \notin \{w_1, x_k, x_{k+1}\} \), \( w_3 = w_2 \). Similarly, \( w_4 = w_2 \). We have a contradiction.

Case 3.b.2. The vertex \( w_1 \) is an end of \( f_1 \) in \( H \). By lemma 1, neither \( x_k \) nor \( x_{k+1} \) is the other end of \( f_1 \) in \( H \). Therefore \( w_1 \) and \( w_2 \) are the ends of \( f_1 \) in \( H \). Let \( G = G[V(G'_1) \cup v], G_2 = G'_2, e_k = f_2 \). By an argument similar to case 2.b.1, (ii)(b) holds.

Case 3.c. Alternative (iv) of theorem 2 holds for \( G' \) and \( H' \). Then \( G' = L(K_4) + t \) and \( H' = 3K_4 \). Since \( G \) is 6-connected, \( v \) is joined to every vertex of \( L(K_4) \). Let \( f_{i,j} \) be the edge in \( G' \) joining \( t \) and \( \{i, j\} \), and \( g_{i,j} \) be the edge in \( G \) joining
v and \( \{i, j\} \). By the argument in case 2.c, the ends of \( g_{i,j} \) in 
H are either \( i \) and \( j \) or \( k \) and \( k' \). By lemma 1, \( g_{i,j} \) and 
\( f_{i,j} \) cannot have the same ends. Therefore the ends of \( g_{i,j} \) in H 
are \( k, k' \). Since \( \{g_{i,j}, h_{i,j}\} \) is not a path in H, by lemma 
2, there is no edge in G joining \( v \) and \( t \). Let \( v = t' \). Then 
(\( v \)) holds.

Case 3.d. Alternative (\( v \)) of theorem 2 holds for \( G' \) 
and \( H' \). Then \( G = L(K_4) + (t, t') \) and \( H' = K_4 \). By the argument 
in case 3.c, there is no edge in G joining \( v \) and \( t \). Similarly, 
there is no edge in G joining \( v \) and \( t' \). Thus the valency of 
\( v \) in G is at most 6, which contradicts the fact G is 
7-connected. □

Assume (i) does not hold. Then by lemma 1, any path in \( L(K_n) \) cannot be a cycle in \( H \).

**Case 1.** \( n = 4 \). By proposition 6, \( L(K_4) \) is 4-connected with six vertices. If the codimension is less than 3, then by applying either proposition 8 or theorem 2, we have the conclusion. If the codimension is greater than 3, then proposition 2 implies that \( |V(H)| \leq 2 \). Applying lemma 2, by considering any triangle in \( L(K_4) \), we have a contradiction. Thus the codimension is 3. This implies that \( |V(H)| = 3 \). Let \( e_1, e_2, e_3, e_4, e_5 \) be edges of \( G \) having ends \( \{1, 2\} \) and \( \{1, 3\} \), \( \{1, 2\} \) and \( \{1, 4\} \), \( \{1, 3\} \) and \( \{1, 4\} \), \( \{2, 3\} \) and \( \{1, 4\} \) respectively. Then \( e_1, e_2, e_3 \) form a triangle in \( G \). By lemma 2, they form a triangle in \( H \). By lemma 1, \( e_4 \) cannot have the same ends as \( e_2 \) or \( e_3 \) in \( H \). This implies that \( e_4 \) has the same ends as \( e_1 \) in \( H \). Similarly, \( e_5 \) has the same ends as \( e_1 \) in \( H \). This implies \( e_4 \) and \( e_5 \) have the same ends in \( H \), which contradicts lemma 1.

**Case 2.** \( n = 5 \). By proposition 6, \( L(K_5) \) is 6-connected with ten vertices. By theorem 2, either the conclusion is true or the codimension is greater than 4. Thus \( |V(H)| \leq 5 \). By the discussion in case 1 and considering the \( L(K_4) \)-subgraph
\(|(|i, j| : i, j \in \{1, 2, 3, 4\}), |V(H)| \geq 4\) and (ii) of theorem 3 is true for the \(L(K_4)\)-subgraph. Let \(e_1, e_2, e_3\) be edges of \(G\) with ends \([5, 4]\) and \([3, 4]\), \([5, 3]\) and \([3, 4]\), \([2, 4]\) and \([3, 4]\) respectively. Lemma 2 implies that exactly one of 2 and 3 is an end of \(e_1\) in \(H\), since \(e_1\) and \(e_3\) are edges of a triangle of \(L(K_5)\) and the ends of \(e_3\) in \(H\) are 2 and 3. Similarly, by considering the triangle in \(L(K_5)\) formed by \([5, 4]\), \([3, 4]\) and \([1, 4]\), exactly one of 1 and 3 is an end of \(e_1\) in \(H\). Similarly, exactly one of 2 and 4 is an end of \(e_2\) in \(H\), and exactly one of 1 and 4 is an end of \(e_2\) in \(H\).

Case 2.a. \(V(H) = \{1, 2, 3, 4\}\). Then the ends of \(e_1\) in \(H\) are either 1 and 2 or 3 and 4, and the ends of \(e_2\) in \(H\) are either 1 and 2 or 3 and 4. But \(e_1\) and \(e_2\) are two edges of a triangle in \(L(K_5)\), and so by lemma 2, we have a contradiction.

Case 2.b. \(|V(H)| = 5\). We may thus assume \(V(H) = \{1, 2, 3, 4, 5\}\). Then the ends of \(e_1\) in \(H\) are 1 and 2, 3 and 4, or 3 and 5, and the ends of \(e_2\) in \(H\) are 1 and 2, 4 and 3, or 4 and 5.

Let \(e_4, e_5, e_6\) be the edges in \(L(K_5)\) with ends \([5, 3]\) and \([5, 4]\), \([5, 3]\) and \([5, 2]\), \([5, 4]\) and \([5, 2]\) respectively.
Since $e_1$, $e_2$, and $e_3$ form a triangle in $H$, by lemma 2, either 3 and 5 or 4 and 5 are the ends of $e_4$ in $H$. Thus 5 is an end of $e_4$ in $H$. Similarly, 5 is an end of $e_5$ and $e_6$. But $e_4$, $e_5$, and $e_6$ form a triangle in $L(K_5)$, and so by lemma 2, we have a contradiction.

Case 3. $n \geq 6$. Let $A_i = \{(i, j) : j \in \{1, 2, \ldots, n\} - i\}$ for $i \in \{1, 2, \ldots, n\}$. Let $G_i = G|A_i$, $G_{i,j} = G|A_i \cup A_j$, $H_{i,j} = H|E(G_i)$ and $H_{i,j} = H|E(G_{i,j})$, for distinct $i, j \in \{1, 2, \ldots, n\}$. Clearly $K(H_{i,j})$ is a subspace of $K(G_{i,j})$. Let $d_{i,j}$ denote the codimension. We also know that $G_i \cong K_{n-1}$.

Claim. $G_{i,j}$ is $(n-1)$-connected for distinct $i, j$.

Proof. For any two non-adjacent vertices say $(i, h), (j, k)$ in $L(K_n)$, the path starting from $(i, h)$ through $(i, j)$ to $(j, k)$ and paths starting from $(i, h)$ through $(i, m), (m, j)$ to $(j, k)$, for $m \in \{1, 2, \ldots, n\} - \{i, j\}$, are $n-1$ vertex-disjoint paths in $G_{i,j}$ joining $(i, h), (j, k)$. Therefore $G_{i,j}$ is $(n-1)$-connected. □

By proposition 7 and lemma 1, $H_{i,j} \cong G_i$ for $i = 1, 2, \ldots, n$, since $n-1 \geq 5$ and $K(H_{i,j})$ is a subspace of $K(G_i)$. Therefore,
\[ d_{i,j} = \dim(K(G_{i,j})) - \dim(K(H_{i,j})) \leq \dim(K(G_{i,j})) - \dim(K(H_i)) = n-2. \]

By theorem 2, either \( d_{i,j} \geq n-3 \) or \( d_{i,j} = 0 \), since \( G_{i,j} \) is \((n-1)\)-connected and (ii) does not hold.

**Case 3.a.** For some \( i, j \in \{1, 2, \ldots, n\} \), \( d_{i,j} = 0 \).

By proposition 4 and above claim this implies that \( G_{i,j} = H_{i,j} \).

Since for any \( v \in V(L(K_n)) - V(G_{i,j}) \) there are at least four vertices of \( G_{i,j} \), which are joined to \( v \), we can apply lemma 3 repeatedly to imply that \( H = L(K_n) \).

**Case 3.b.** For some \( i, j \in \{1, 2, \ldots, n\} \), \( d_{i,j} = n-2 \).

Then \( |V(H_{i,j})| = n-1 \), since \( |V(G_{i,j})| = 2n-3 \). Clearly, \( V(H_{i,j}) = V(H_i) \). Without loss of generality let \( V(H_i) = V(G_i) \).

Let \( e_k \) be the edge in \( L(K_n) \) joining \( \{i, j\}, \{i, k\} \), and \( e'_k \) be the edge in \( L(K_n) \) joining \( \{i, j\}, \{j, k\} \), for \( k \in \{1, 2, \ldots, n\} - \{i, j\} \).

**Case 3.b.1.** Suppose \( \{i, j\} \) is an end of \( e'_k \) in \( H_{i,j} \). Let \( \{i, h\} \) be the other end. Then \( e_h, e'_k \) form a cycle in \( H_{i,j} \). However \( e_h, e'_k \) form a path in \( G_{i,j} \) from \( \{i, h\} \) to \( \{j, k\} \), which contradicts lemma 1.
Case 3.b.2. Suppose \( (i, j) \) is not an end of \( e'_k \) in \( H_{i,j} \) for each \( k \in \{1, 2, \ldots, n\} - \{i, j\} \). Since \( G_j|\{e'_k : k \in \{1, 2, \ldots, n\} - \{i, j\}\} = H_j|\{e'_k : k \in \{1, 2, \ldots, n\} - \{i, j\}\} \) and \( |V(G_j|\{e'_k : k \in \{1, 2, \ldots, n\} - \{i, j\}\})| = n-1 = |V(H_{i,j})| \), any vertex of \( V(H_{i,j}) \) is an end of some \( e'_k \). We have a contradiction.

Case 3.c. Assume \( d_{i,j} = n-3 \) for distinct \( i, j \in \{1, 2, \ldots, n\} \). Since \( G_{i,j} \) is \((n-1)\)-connected, (ii) of theorem 2 holds for \( G_{i,j} \) and \( H_{i,j} \).

Case 3.c.1. Suppose (ii)(a) of theorem 2 holds for \( G_{i,j} \) and \( H_{i,j} \), for all distinct \( i, j \in \{1, 2, \ldots, n\} \). For each \( e_k \), in the notation of alternative (ii)(a) of theorem 2, there is at most one vertex in \( i, j \) which is joined to both ends of \( e_k \) by edges in \( G_{i,j} \). Since \( G_i \approx G_j \approx K_{n-1} \) and \( n-1 \geq 5 \), the edges with ends \( \{i, k\}, \{k, j\} \) in \( G_{i,j} \) are the only choices for the \( e_1, e_2, \ldots, e_{n-2} \) in (ii)(a) of theorem 2. Let \( a_k \) be the edge in \( G_{1,2} \) joining \( \{1, k\} \) and \( \{2, k\} \), \( b_k \) be the edge in \( G_1 \) joining \( \{1, 2\} \) and \( \{1, k\} \), and \( c_k \) be the edge in \( G_2 \) joining \( \{1, 2\} \) and \( \{2, k\} \), for \( k = 3, 4, \ldots, n \). Now we label the vertices of \( H_{1,2} \) so that 1 denotes the common vertex of \( a_3 \) and
c_3, 2 denotes the common vertex of a_3 and b_3, and k denotes the common vertex of b_k and c_k for k = 3, 4, ..., n. (See figure 11). We note that the ends of a_k in H are 1 and 2, for k = 3, 4, ..., n. Clearly, all edges of G_1,2 satisfy the rules of (ii)(a) of theorem 3. Since E(G_1,2) \cap E(G_1,3) \supseteq \{a_3, b_3, b_4, \ldots, b_n\}, which covers V(H_1,2), we have V(H_1,3) = V(H_1,2). By applying (ii)(a) of theorem 2 to G_1,3, all edges of G_1,3 satisfy the rules of (ii)(a) of theorem 3. Similarly, V(H_1,k) = V(H_1,2), and all edges of G_1,k satisfy the rules of (ii)(a) of theorem 3, for k = 3, 4, ..., n. Since E(L(K_n)) = \bigcup_{k=2}^{n} E(G_1,k), (ii)(a) holds.

Case 3.c.2. Suppose (ii)(b) of theorem 2 is true for some G_1,j and H_1,j. We may assume \( i, j = 1, 3 \). For each e_k in (ii)(b) of theorem 2, there are at most two vertices in G_1,3 which are joined to both ends of e_k. Since G_1 \cong G_3 \cong K_{n-1} and n-1 \geq 5, all but one of the edges with ends \( \{1, k\}, \{3, k\} \) are the only choices for the \( e_1, e_2, \ldots, e_{n-3} \) in (ii)(b) of theorem 2. We may assume \( (x_{n-2}, y_{n-2}) = (1, 2) \), and \( (x_{n-1}, y_{n-1}) = (1, 3) \). Let a_3 be the edge in G_1,3 joining \{1, 2\} and \{3, 2\}, b_3 be the edge in G_1 joining \{1, 3\} and \{1, 2\}, c_3 be the
edge in $G_3$ joining \{1, 3\} and \{3, 2\}, $a_k$ be the edge in $G_{1,3}$ joining \{1, k\} and \{3, k\}, $b_k$ be the edge in $G_1$ joining \{1, 3\} and \{1, k\}, and $c_k$ be the edge in $G_3$ joining \{1, 3\} and \{3, k\}, for $k = 4, 5, \ldots, n$. Now we label the vertices of $H_{1,3}$ so that 1 denotes the common vertex of $b_3$ and $c_3$, 2 denotes the common vertex of $a_3$ and $c_3$, 3 denotes the common vertex of $a_3$ and $b_3$, and $k$ denotes the common vertex of $b_k$ and $c_k$ for $k = 4, 5, \ldots, n$. (See figure 12.) We note that in this case (ii)(b) of theorem 2 is true for $G_{1,3}$ and $H_{1,3}$, where $(x_{n-2}, y_{n-2}) = (1, 2)$, and $(x_{n-1}, y_{n-1}) = (1, 3)$, exactly one end of $a_3$ in $H_{1,3}$ is in $H_1$, both ends of $a_k$ are in $H_1$ for $k = 4, 5, \ldots, n$, and $G_{1,3}$ contains all edges of $G_{1,3}$ that satisfy the rules of (ii)(b) of theorem 3. Since $E(G_{2,3}) \cap E(G_{1,3}) \supseteq \{a_3, c_3, c_4, \ldots, c_n\}$ which cover $V(H_{1,3})$, we have $V(H_{2,3}) = V(H_{1,3})$. Because $G_{2,3} | E(G_3) \cup b_3 = H_{2,3} | E(G_3) \cup b_3$, (ii)(b) of theorem 2 is true for $G_{2,3}$ and $H_{2,3}$, with $(x_{n-2}, y_{n-2}) = (1, 2)$, and $(x_{n-1}, y_{n-1}) = (2, 3)$. Then all edges of $G_{2,3}$ satisfy the rules of (ii)(b) of theorem 3. Let $e_k$ be the edge in $G_{1,k}$ joining \{1, 2\} and \{2, k\}. Since $E(G_{1,k}) \cap (E(G_{1,3}) \cup E(G_{2,3})) \supseteq \{e_k, b_3, b_4, \ldots, b_n\}$
which covers $V(H_{1,3})$, it follows that $V(H_{1,k}) = V(H_{1,3})$, for $k = 4, 5, \ldots, n$. Since all edges of $G_{2,3}$ satisfy the rules of (ii)(b) of theorem 3, the ends of $e_k$ in $H_{1,k}$ are 2 and $k$.

Then exactly one end of $e_k$ in $H_{1,k}$ is in $H_1$. However, both ends of $a_k$ in $H_{1,k}$ are in $H_1$. These remarks imply that (ii)(b) of theorem 2 is true for $G_{1,k}$ and $H_{1,k}$, with $(x_{n-2}, y_{n-2}) = (1, 2)$, and $(x_{n-1}, y_{n-1}) = (1, k)$. Then all edges of $G_{1,k}$ satisfy the rules of (ii)(b) of theorem 3. Since $E(L(K_n)) = \bigcup_{k=3}^{n} E(G_{1,k}) \cup E(G_{2,3})$, (ii)(b) holds. □
1. Another version of theorem 1.

**Proposition 9.** If alternative (iii) of theorem 1 holds for $G^*$ and $H^*$, then either (ii) of theorem 1 holds for some realization $G'^*$ of $C(G)$ and $H^*$, or there exists a Y-graph $Y\subseteq G_i^*$, (A Y-graph is a tree with three monovalent vertices.) such that $x_i$, $y_i$, $z_i$ are the monovalent vertices of $Y_i$, for $i = 1, 2, 3, 4$.

**Proof.** Otherwise, if there is no such Y-graph in $G_i^*$, say for $i = 4$, then all $[x_4, y_4, z_4]$-components of $G_4^*$ have at most two vertices of attachment.

**Case 1.** $G$ is 2-connected. Then all $[x_i, y_i, z_i]$-components of $G_i^*$ have exactly two vertices of attachment. Let $G_a$ be the union of $[\{x_i, y_i, z_i\} - a]$-components of $G_i^*$ which do not include $a$, for $a = x_i, y_i, z_i$. Then $G_i^* = G_{x_i} \cup G_{y_i} \cup G_{z_i}$. Let:
\[ G'_1 = C(g^*_1, g_{y^*_{14}}; (x_{1}, x_{4}), (z_{1}, z_{4})) , \]

\[ G'_2 = C(g^*_2, g_{x^*_{14}}; (y_{2}, y_{4}), (z_{2}, z_{4})) , \]

\[ G'_3 = C(g^*_3, g_{z^*_{14}}; (x_{3}, x_{4}), (y_{3}, y_{4})) , \]

and

\[ G'^{*} = C(g'_1, g'_2, g'_3; (x_{1}, y_{2}), (z_{2}, x_{3}), (y_{3}, z_{1}), (y_{1}, x_{2}, z_{3})) . \]

Clearly, \( C(g'^{*}) = C(g) \), and by a relabeling of the vertices of attachment of \( G'_1, G'_2, G'_3 \) in \( G'^{*} \), (ii) holds for \( G'^{*} \) and \( H'^{*} \).

**Case 2.** \( G \) is separable. This is similar to case 1. All the extra work involves properly placing the [a]-components of \( G^*_{14} \) which do not include any element of \( \{x_{4}, y_{4}, z_{4}\} - a \), for \( a = x_{1}, y_{1}, z_{1} \).

(See figure 13.) □

**Proposition 10.** If alternative (ii) of theorem 1 holds for \( G^* \) and \( H'^{*} \), then either (i) of theorem 1 holds for some realization \( G'^{*} \) of \( C(G) \) and \( H'^{*} \) of \( C(H) \), or there exists a Y-graph \( Y_1 \) of \( G^* \) \( U \) \( E(G^*_{14}) \), such that \( x_{1}, y_{1}, z_{1} \) are the monovalent vertices of \( Y_1 \), for \( i = 1, 2, \ldots, n \).

**Proof.**
Case 1. Every path in $G_i^*$ joining $x_i, y_i$ passes through $z_i$, for some $i \in \{1, 2, \ldots, n\}$. Then all $[x_i, y_i, z_i]$-components have at most two vertices of attachment. We may assume $i = n$.

Case 1.a. $G$ is 2-connected. Let $G_x$ be the union of the $[x_n, z_n]$-components of $G_n^*$ which do not include $y_n$, and $G_y$ be the union of the $[y_n, z_n]$-components of $G_n^*$ which do not include $x_n$. Then $G_n = G_x \cup G_y$. Let:

\[ G'_1 = C(g_1^*, G_x; (y_n, z_1), (z_n, x_1)) , \]
\[ G'_{n-1} = C(g_{n-1}^*, G_y; (z_n, x_n), (y_{n-1}, z_n)) , \]
\[ G'^* = C(g'_1, g_2^*, g_3^*, \ldots, g_{n-2}^*, g_{n-1}^*; (z_1, x_2), (y_1, z_2, x_3), \ldots, (y_{n-3}, z_{n-2}, x_{n-1}), (y_{n-2}, z_{n-1})) , \]
\[ H'^* = H^* . \]

Then $H'^*$ is obtained from $G'^*$ by identifying $x_1$ and $y_{n-1}$.

Clearly, $C(G'^*) = C(G)$. Thus (i) holds for $G'^*$ and $H'^*$.

(See figure 14.)

Case 1.b. $G$ is separable. This is similar to case 1.a. All the extra work involves properly placing the $[a]$-components of $G_n^*$ which do not include any element of $\{x_n, y_n, z_n\}$-a, for $a = x_n, y_n, z_n$. 
Case 2. There is a path in \( G_1^i \) joining \( x_i, y_i \) and not passing through \( z_i \), for \( i = 1, 2, \ldots, n \). If there is no such Y-graph in \( G_1^i \cup E(G_1^i) \), say for \( i = n \), then the vertices of attachment of each \([x_j, y_j, z_j]\) - component of \( G_j^* \) are in \([x_j, y_j]\) or \([z_j]\), for \( j = 2, 3, \ldots, n-2 \), the vertices of attachment of each \([x_1, y_1, z_1]\) - component are in \([x_1, z_1]\) or \([x_1, y_1]\), and the vertices of attachment of each \([x_{n-1}, y_{n-1}, z_{n-1}]\) - component are in \([y_{n-1}, z_{n-1}]\) or \([x_{n-1}, y_{n-1}]\).

Case 2.a. \( G \) is 2-connected. Let \( G_y \) be the union of the \([x_1, z_1]\) - components of \( G_1^* \) which do not include \( y_1 \), \( G_x \) be the union of the \([y_{n-1}, z_{n-1}]\) - components of \( G_{n-1}^* \) which do not include \( x_{n-1} \):

\[
G_1' = G_1^*|E(G_1^*) - E(G_y),
\]

\[
G_{n-1}' = G_{n-1}^*|E(G_{n-1}^*) - E(G_x),
\]

\[
G_n' = C(G_n^*, G_y, G_y; (y_{n-1}, z_n, x_1), (z_{n-1}, x_n), (y_n, x_1)),
\]

\[
x = (z_{n-1}, x_n), \ y = (y_n, x_1), \ z = (y_{n-1}, z_n, x_1), \ \text{and}
\]

\[
G_j' = G_j^*|E(G_j^*) \text{ for } j = 2, 3, \ldots, n-2.
\]
Case 2.a.1. \( n \) is odd. Let
\[
G^* = C(G_1', G_2', \ldots, G_n'; (y_1, x_3), (y_2, x_4), \ldots, (y_{n-2}, x), (y_{n-1}, x_1), (y, x_2)), \text{ and } H'^* = H^*. \quad (\text{See figure 15.})
\]

Then \( H'^* \) is obtained from \( G'^* \) by identifying \( z, (y_{n-1}, x_1) \), and \( C(G'^*) = C(G) \). Thus (ii) holds for \( G'^* \) and \( H'^* \).

Case 2.a.2. \( n \) is even. Let:
\[
G^* = C(G_1', G_2', \ldots, G_n'; (y_1, x_3), (y_3, x_5), \ldots, (y_{n-3}, x_{n-1}), (y_{n-1}, x_2), (y_2, x_4), (y_4, x_6), \ldots, (y_{n-4}, x_{n-2}), (y_{n-2}, x), (y, x_1)), \text{ and }
\]

\( H'^* \) be the graph obtained from \( G'^* \) by identifying \( (y_{n-1}, x_2) \) and \( (y_n, x_1) \). (See figure 16.)

Then \( C(G'^*) = C(G) \) and \( C(H'^*) = C(H) \). Thus (i) holds for \( G'^* \) and \( H'^* \).

Case 2.b. \( G \) is separable. This is similar to case 2.a.

All the extra work involves properly placing the \([x_j]\)-components, \([y_j]\)-components, and \([z_j]\)-components of \( G_j^* \), which do not include more

one element of \( \{x_j, y_j, z_j\} \), for \( j = 1, 2, \ldots, n-1 \). \( \square \)

Theorem 1'. Theorem 1 still holds, when we add to alternative
(iii) the condition, there exists a Y-graph $Y_i \subseteq G_i^*$, such that $x_i, y_i, z_i$ are the monovalent vertices of $Y_i$, for $i = 1, 2, 3, 4$, and to alternative (ii) the condition, there exists a Y-graph $Y_i$ of $G^* \cup E(G^*)$, such that $x_i, y_i, z_i$ are the monovalent vertices of $Y_i$, for $i = 1, 2, ..., n$.

Proof. This is an immediate consequence of theorem 1, proposition 9, and proposition 10.


It is clear, because all the cycles in $G$ are still cycles in $H$, that if one of the alternatives (ii), (iii), (iv), (v) holds for $G$ and $H$, then $C(G) \subseteq C(H)$. Suppose $G$ is $(k+2)$-connected, and there is a graph $G'$, such that $C(G) \subseteq C(G') \subseteq C(H)$. Then by proposition 8, $G'$ is obtained from $G$ by identifying vertices. By lemma 1, there is a path in $G$ which is a cycle in $G'$, and hence a cycle in $H$, which is not true for alternatives (ii), (iii), (iv), (v). Therefore, there is no graphic space between $C(G)$ and $C(H)$, when one of the alternatives (ii), (iii), (iv), (v) of theorem 2 holds for $G$ and $H$, and $G$ is $(k+2)$-connected.
3. Conjectures.

Due to the complexity of the proof, we cannot prove the following conjecture at present stage. But we strongly believe it is true.

**Conjecture 4.** If $G$ is $(k+1)$-connected, $C(G)$ is a codimension-$k$ subspace of $C(H)$, and $k \geq 5$ then either one of alternatives (i), (ii) in theorem 2 is true (but in (ii) we may delete $e_1$ from $G$ and $H$.) or one of the following holds:

(iii) $k = 5$, $G = L(K_5)$, $H = 3K_5$ and either:

(a) the edge with ends $(i, j), (j, k)$ in $G$ has ends $i, k$ in $H$, for $i, j, k \in \{1, 2, 3, 4, 5\}$, or

(b) the edge with ends $(i, j), (j, k)$ in $G$ has ends $i, k$ in $H$, where $(i, j) \neq [1, 2] \neq [j, k]$, the edge with ends $(1, 2), (1, h)$ in $G$ has ends $1, h$ in $H$, and the edge with ends $(1, 2), (2, h)$ in $G$ has ends $2, h$ in $H$, for $h \in [3, 4, 5]$.

(We may add an edge with ends $(1, 2), (i, j)$ in $G$ for $i, j \in [3, 4, 5]$. Then the ends of this edge are $i, j$ in $H$. We may delete the edge with ends $(3, 5), (4, 5)$ in $G$, if we have added two edges with ends $(1, 2), (3, 5)$, and $(1, 2), (4, 5)$ in $G$.)

(iv) $k = 6$, $G = L(K_5) + t$, $H = 4K_5$, the edges in $L(K_5)$
act the same as (iii)(a), and the edge with ends \( t, [i, j] \) in \( G \) has ends \( i, j \) in \( H \) for \( i, j \in \{1, 2, 3, 4, 5\} \).

**Remark.** If we delete \( e_1 \) from \( G \) and \( H \) in (ii), and let \( G' \) be the graph obtained from \( H \) by splitting the vertex \( (x_1, y_1) \), then we have \( C(G) \subseteq C(G') \subseteq C(H) \).

We make the following two conjectures, motivated by proposition 8.

**Conjecture 5.** Suppose \( C(G) \) is a codimension-\( k \) subspace of \( C(H) \), with no graphic space between them. Let \( G \) be 2-connected and \( G' \) be a maximal connected subgraph of \( G \) such that \( H|E(G') \) can be obtained from \( G' \) by a sequence of twists not separating the vertices of attachment of \( G' \) in \( G \). Then the number of vertices of attachment of \( G' \) in \( G \) is at most \( k + 2 \).

**Conjecture 6.** In conjecture 5, if the number of vertices of attachment of \( G' \) in \( G \) is \( k + 2 \), then there exists a graph \( G^* \) with \( C(G^*) = C(G) \) such that either alternative (i) or alternative (ii) of theorem 1 holds for \( G^* \) and \( H \), where we may replace the \( e_i \)'s by subgraphs of \( G \) with two vertices of attachment.

It is easy to prove that the intersection of any two maximal subgraphs, as described in conjecture 5, is a subgraph of \( G \) with at most two vertices of attachment.
The question remains, "Do there exist some theorems generalizing theorem 2 and conjecture 4 so that all the exceptional cases are derived from $L(K_n)$?"
Figure 1. Operations for 2-isomorphism
Figure 2. Alternatives (ii) and (iii) of theorem 1
Figure 3. Alternative (ii) (a) of theorem.
Figure 4. Alternative (ii) (b) of theorem 2
Figure 5. Illustration for the proof of lemma 5
Figure 6. Case 1.a.1.1 of the proof of theorem 1
Figure 7. Case 1.a.1.2 of the proof of theorem 1
Figure 8. Case 1.a.2 and case 1.a.3 of the proof of theorem 1.
Figure 9, Illustration for claim 3
Figure 10. Case 2.c.2 of the proof of theorem 1
Figure 11. Case 3,c,1 of the proof of theorem 3
Figure 12 Case 3.c.2 of the proof of theorem 3
Figure 11. Case 2 of proposition 9
Figure 1a, Case 1,a of proposition 10
Figure 15. Case 2.a.1 of proposition 10
Figure 16. Case 2.a.2 of proposition 19


