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CONTRIBUTIONS TO RATIONAL HOMOTOPIE THEORY

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CONTRIBUTIONS TO RATIONAL HOMOTOOPY THEORY

Dissertation

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

By

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INTRODUCTION

The technique of doing homotopy theory at the rational level first found expression in the fundamental papers ([21] and [22]) of Serre. It was not until the late 1960's, however, that Serre's ideas were codified by the works of Quillen [18] and Sullivan. Quillen's methods, though neither explicit nor easily accessible, showed that the rational homotopy theory of 1-connected spaces could be understood entirely within an algebraic framework. Sullivan, on the other hand, approached the subject in a more classical fashion, explicitly constructing a space to reflect all the rational characteristics of a given 1-connected homotopy type. In fact, Sullivan's construction, which proceeds inductively through the Postnikov tower, generalizes, providing a "localization" of a given space for any chosen set of rational primes.

The connection between Quillen's work and that of Sullivan remained rather obscure until Sullivan re-examined the problem from a viewpoint whose origins lie in the investigations of DeRham [7], Thom [25] and Whitney [30]. DeRham, of course, showed that the real cohomology algebra of a smooth manifold could be computed using the associated
(differential graded) algebra of smooth differential forms. It was in the work of Thom and Whitney on the real commutative cochains problem, however, that the first indications arose of a deeper relationship between forms and homotopy type. At that time, though, there was no precise way to use the full algebraic content of the real algebra of forms to obtain homotopy information. This difficulty, as we now know, is related to the impossibility of constructing a localized Postnikov tower over the real numbers. Sullivan adapted the Thom-Whitney approach to the rational situation by associating to a given space a differential graded algebra of rational polynomial forms which computes the rational cohomology algebra of the space. Moreover, via his theory of minimal models and the previous construction of the rationalized Postnikov tower, Sullivan showed that rational homotopy type is mirrored completely in the algebra of rational forms.

Chapter 0 is devoted to a brief description of the aspects of Sullivan's theory that are used in the sequel. In particular, we discuss the homotopy theory of differential graded algebras, the notion of minimal DGA's and the spatial realization functor from DGA's to simplicial spaces. These are the essential ingredients in Sullivan's solution of the rational commutative cochains problem and the consequent algebraic characterization of rational homotopy type.
Chapter 1 employs Sullivan's techniques to study the rational homotopy theory of group actions. Although actions of finite groups provide the main focus of our results, several generalizations are also obtained pertaining to actions of infinite groups of arithmetic type. In particular, we find that the rational homotopy theory of certain group actions may be completely understood by examining the induced action on rational cohomology. Also considered in this chapter is the problem of replacing homotopy actions (homotopy equivariant maps) by topological actions (equivariant maps).

Chapter 2 is concerned with the study of abstract Riemannian structures on differential graded algebras. Sullivan had indicated earlier that a Riemannian metric on a smooth manifold provides a canonical construction of the minimal model. In our abstract situation we provide the complete details of the construction and a few generalizations. The canonical model is used to obtain some of the results of Chapter 1 from the viewpoint of the Riemannian structure.

Chapter 3 is devoted to studying certain aspects of the notions of formality and coformality in differential graded algebras and spaces. In particular, we prove that the mapping cone (homotopy fibre) of a formal (coformal) map is a formal (coformal) space. Here, also, the homology decomposition of a minimal DGA is introduced as a dual concept to the Postnikov decomposition and it is shown that,
just as the coformality of a space is related to the coformality of its Postnikov pieces, the formality of a DGA is related (by a rather precise obstruction theory) to the formality of its homology sections. We interpret the obstructions to formality from this point of view.
§ 1. Differential Graded Algebras

Throughout we shall work over \( \mathbb{Q} \), the field of rational numbers, although many of the results have extensions to any field of characteristic 0.

**Def. 1:** A **differential graded algebra (DGA)** is a pair \((A, d_A)\) consisting of a graded vector space \( A = \bigoplus A^i i \) and a linear map \( d_A : A \to A \) of degree 1 satisfying:

1. \( A \) possesses an associative multiplication (i.e., for each \( i, j ; A^i \otimes A^j \to A^{i+j} \)) such that \( x \cdot y = (-1)^{ij} y \cdot x \) where \( x \in A^i, y \in A^j \).
2. \( d_A \) is a graded derivation (i.e., \( d_A(x \cdot y) = d_A x \cdot y + (-1)^i x \cdot d_A y \) where \( x \in A^i \)).
3. \( d_A \) is a differential (i.e., \( d_A^2 = 0 \)).

The cohomology algebra of \( A \) is defined in the usual way,
\[ H^i(A) = Z^i(A)/B^i(A) , \]

where \( Z^i(A) = \{ a \in A^i \mid da = 0 \} \) and \( B^i(A) = \{ a \in A^i \mid db = a \text{ for some } b \in A^{i-1} \} \). The former is referred to as the \( i^{th} \) cocycles and the latter as the \( i^{th} \) coboundaries. The product induced from \( A \) makes \( H(A) = \bigoplus_{i} H^i(A) \) into a DGA with zero differential.

\( A \) is connected if \( A^0 = \mathbb{Q} \), \( H \)-connected if \( H^0(A) = \mathbb{Q} \) and \( 1 \)-connected if it is connected and \( H^1(A) = 0 \). \( A \) is said to be of finite type if \( H^i(A) \) is finite dimensional for each \( i \). Unless otherwise stated this condition on our DGA's is presumed throughout.

\( L_n(V) \) shall denote the DGA freely generated on the vector space \( V \) in dimension \( n \). Hence, \( L_n(V) \) is a polynomial algebra on \( V \) if \( n \) is even and an exterior algebra on \( V \) if \( n \) is odd. The differential in both cases is taken to be zero.

Let a DGA \( A \) be connected. Let \( A^+ = \bigoplus_{i>0} A^i \).

The graded vector space of indecomposables is defined to be, \( Q(A) = A^+/A^+ \cdot A^+ \). The differential of \( A \), \( d \), is said to be decomposable if \( d(A^+) \subseteq A^+ \cdot A^+ \).

An elementary extension of \( (A,d_A) \) is a DGA \( (B,d_B) \) where \( B = A \otimes L_n(V) \) with differential \( d_B|_A = d_A \) and
\[ d_B(V) \subseteq A. \] Note that if \( d_A \) is decomposable, then \( d_B \) is decomposable if and only if \( d_B(V) \subseteq A_+^* A_+^* \). If \( \tau : L_{n+1}(V) \to A \) is a DGA map, then we may form the elementary extension \( A \otimes_{\tau} L_n(V) \), where \( d \big|_A = d_A \) and \( d(v) = \tau(v) \) for \( v \in V \). The differential here is decomposable exactly when \( \tau(V) \subseteq A_+^* A_+^* \).

Def. 2: A DGA \( M \) is said to be minimal if it can be written as an increasing union of sub-DGA's,

\[
Q \subseteq M_1 \subseteq M_2 \subseteq \ldots \subseteq M_n \subseteq \ldots \subseteq \bigcup_{i=0}^{n+1} M_i = M
\]

where \( M_{n+1} \) is obtained from \( M_n \) by an elementary extension with decomposable differential.

Note that if \( M \) is minimal, then \( M \) is free as an algebra and \( d_M \) is decomposable. Indeed, if \( M \) is 1-connected, then these conditions are necessary and sufficient for \( M \) to be minimal. Moreover, if \( M \) is 1-connected, then the terms in the decomposition of Def. 2 have the form,

\[
M_1 = 0, \quad M_2 = L_2(V_2)
\]

and

\[
M_{n+1} = M_n \otimes_{\tau} L_{n+1}(V_{n+1})
\]
In particular, note that in this case each elementary extension in the decomposition is formed by adjoining a vector space in one higher degree than the previous stage. If \( M \) is a minimal DGA, then the graded vector space of indecomposables \( Q(M) \) is called the cohomotopy of \( M \).

**Def. 3:** If \( A \) is a DGA, then the pair \((M, \rho)\) consisting of a minimal DGA \( M \) and a DGA-map \( \rho : M \to A \) is called a minimal model for \( A \) if \( \rho \) induces isomorphisms of cohomology.

**Remark 1:** A mapping of DGA's \( f : A \to B \) is called a quasi-isomorphism if it induces isomorphisms of cohomology. Hence, we may refer to \( \rho \) above by this terminology.

**Theorem 2** (See [10], [2] or [24]): Every \( H \)-connected DGA has a minimal model unique up to isomorphism.

An important fact about minimal DGA's that is used frequently in the sequel is the

**Prop. 3:** If \( f : M \to N \) is a quasi-isomorphism of minimal DGA's, then \( f \) is an isomorphism.
For a proof see [2] or [13].

We now turn to the homotopy theory of DGA's. The notion of homotopy that we use is that of Sullivan [24]. Two other definitions have been given (see [10] or [13]), including the original definition of Deligne et al [5]. Halperin [10] has shown that on the overlap of their domains of definition the three notions coincide.

Let $M$ be a minimal DGA. Since $M$ is freely generated as a graded algebra, we may write $M = L(x_i)$ for a set of generators $\{x_i\}$. Let $M^I$ denote the algebra $L(x'_i, x''_i, \hat{x}_i)$, where $x'_i$ and $x''_i$ are two copies of $x_i$ and $\hat{x}_i$ has degree, $|\hat{x}_i| = |x_i| - 1$ (recall $|x| = \text{degree of } x$). In order to define a differential on $M^I$ we adapt a technique of Baues and Lemaire [1] from chain algebras to DGA's. Define a map $s: M \to M^I$ by $s(x_i) = \hat{x}_i$ and extend to a derivation of degree $-1$ by

$$s(x_1 x_2) = \hat{x}_1 x_2'' + (-1)^{|x_1|} x_1' \hat{x}_2.$$ 

A differential may then be defined on $M^I$ by

$$d(x'_i) = (dx_1)' , \quad d(x''_i) = (dx_1)'' \quad \text{and}$$

$$d(\hat{x}_i) = x'_i - x''_i - sdx_i.$$
and then extending to a derivation. It is routine to check that $d^2 = 0$. From the last formula we obtain,

$$sd(x^1) - ds(x^1) = x'_1 - x''_1,$$

showing that the two embeddings

$$x_1 \mapsto x'_1 \quad x_1 \mapsto x''_1$$

of $M$ in $M^I$ are cochain homotopic. These embeddings provide a natural map $M \otimes M \to M^I$.

**Def. 4:** $f, g: M \to B$ are homotopic if there exists $H: M^I \to B$ making the following diagram commutative:

$$
\begin{array}{c}
M \otimes M \\
\downarrow \\
M^I \\
\end{array} 
\xrightarrow{e} 
\begin{array}{c}
B \\
\end{array}

\begin{array}{c}
H \\
\end{array}
$$

where $e$ is defined by,

$$e(x_1 \otimes 1) = f(x_1), \quad e(1 \otimes x_j) = g(x_j).$$
Sullivan shows that this definition of homotopy does not depend on the choice of generators. He also proves,

**Prop. 4:** Homotopy is an equivalence relation.

**Remark 5:** This notion of homotopy may be extended to any DGA built up from another DGA by elementary extensions. In a sense, this is a version of relative homotopy. Also Prop. 4 indicates that for the class of DGA's we consider this definition of homotopy has advantages over the original. The original notion could only be shown to be an equivalence relation when the DGA's were minimal. As we have said, though, our notion of homotopy is an equivalence relation on a much wider class of DGA's.

We now consider several fundamental extension and lifting results in the homotopy theory of DGA's. The first is immediate.

**Prop. 5:** Let \( f: A \rightarrow B \) be a mapping of DGA's. The obstructions to an extension \( \tilde{f}: A \otimes L_n(V) \rightarrow B \) are cohomology classes \( [f(dx_i)] \in H^{n+1}(B) \), where \( \{x_i\} \) is a basis for \( V \).
Corollary 6: (Extension Theorem) If $H^{n+1}(B) = 0$, then any $f: A \to B$ has an extension $\tilde{f}: A \otimes L_n(V) \to B$, for any elementary extension $A \otimes L_n(V)$.

The homotopical version of Prop. 5 requires the re-collection of the notion of the algebraic mapping cone of cochain complexes. Let $\theta: A \to B$ be a DGA map (or even a mapping of complexes). Form the complex,

$$C^n = A^n \oplus B^{n-1}$$

with differential $d(a,b) = (-da, db + \theta(a))$. The relative cohomology of $\theta: A \to B$ is then defined to be the cohomology of the complex $(C^n, d)$ and is denoted $H(B,A)$. The obvious maps fit $H(B,A)$ into a long exact sequence,

$$\ldots \to H^{n-1}(B) \to H^n(B,A) \to H^n(A) \to H^n(B) \to \ldots$$

Now consider the following homotopy commutative diagram,

$$\begin{array}{ccc}
M & \xrightarrow{f} & A \\
\downarrow & & \downarrow \\
M \otimes L_n(V) & \xrightarrow{\tilde{f}} & B \\
\end{array}$$
where a basis for $V$ is $\{x_i\}$ and a homotopy $H$ is specified. We have,

**Prop. 7:** The obstructions to the existence of
\[ \tilde{f}: M \cong L_n(V) \to A \text{ extending } f \text{ with } \theta \circ \tilde{f} = \overline{f} \]
are cohomology classes,
\[ [f(dx_1), -\overline{f}(x_1) + Hsdx_1] \in H^{n+1}(B,A). \]

From the long exact cohomology sequence associated to the algebraic mapping cone construction, we see that if $\theta : A \to B$ is a quasi-isomorphism, then $H^n(B,A) = 0$ for each $n$. Recalling that a minimal algebra is constructed via a sequence of elementary extensions, Prop. 7 takes the form,

**Corollary 8.** (Homotopy Lifting Theorem for Minimal DGA's):
In the following diagram, if $M$ is minimal and $\theta$ is a quasi-isomorphism, then the indicated lifting exists;
Remark 9:

(1) We may strengthen Corollary 8 (see [5]) to say that $\theta$ induces a bijection of homotopy sets $[M,A] \leftrightarrow [M,B]$.

(2) If $\theta$ is surjective, then it is not hard to show that we may choose $\tilde{f}$ so that $\theta \cdot \tilde{f} = f$.

(3) Given $f: A \to B$, Corollary 8 and Remark 9 (1) show that we have a homotopy commutative diagram,

\[
\begin{array}{ccc}
M_A & \xrightarrow{\tilde{f}} & M_B \\
\downarrow{\rho_A} & & \downarrow{\rho_B} \\
A & \xrightarrow{f} & B
\end{array}
\]

where $\tilde{f}$ is unique up to homotopy. $\tilde{f}$ is sometimes referred to as the minimal model of $f$.

(4) In (3) if $f$ is a quasi-isomorphism, then so is $\tilde{f}$ since $\rho_A$ and $\rho_B$ are. By Prop. 3, though, this implies that $\tilde{f}$ is an isomorphism. Hence, $A$ and $B$ have isomorphic minimal models.

Def. 5: $A$ and $B$ are said to be of the same homotopy type if $M_A \sim M_B$. 
Def. 6: The DGA $\mathcal{A}$ is called formal if it has the same homotopy type as $H(\mathcal{A})$.

Remark 10: It is immediate from Def. 5 that $\mathcal{A}$ is formal if and only if there exists a quasi-isomorphism $
abla: M_\mathcal{A} \rightarrow H(\mathcal{A})$. We note that other characterizations of formality have been given (e.g., [5] and [10]).

§2. Q-Polynomial Forms and Spatial Realization

In attempting to generalize DeRham's work from manifolds to arbitrary spaces, Sullivan took the first step by defining appropriate forms for simplicial complexes. Specifically (see [9]), let $K$ be a simplicial complex. Then $K$ is the union of $n$-simplices $\Delta^n$, where we think of $\Delta^n$ as,

$$\Delta^n = \{(t_0, \ldots, t_n) | 0 \leq t_i \leq 1, \sum_{i=0}^{n} t_i = 1\}.$$ 

The $t_i$ are called barycentric co-ordinates. Take all forms in $\mathbb{R}^{n+1}$, restricted to $\Delta^n$, of the form,
where \( \phi_{i_1 \ldots i_j} \) is a polynomial in the \( t_i \) with coefficients in \( \mathbb{Q} \). There are two relations,

\[
\sum_{i=0}^{n} t_i = 1
\]

\[
\sum_{i=0}^{n} dt_i = 0
\]

The DGA so constructed is denoted \( A(\Delta^n) \).

**Def. 7:** Let \( K \) be a simplicial complex. Then

\[
A(K) = \{ (\omega_\sigma)_{\sigma \in K} | \omega_\sigma \in A(\sigma) \text{ and } \omega_\sigma \big|_\tau = \omega_\tau \text{ if } \tau \text{ is a face of } \sigma \}
\]

So \( A(K) \) is a collection of forms, one for each simplex \( \sigma \) of \( K \), which are compatible under restriction to faces.
The differential and wedge product may be defined simplexwise, so $A(K)$ is a DGA over $Q$. Now, just as for DeRham's Theorem, integration provides a map,

$$I: A(K) \longrightarrow C^*(K),$$

where we understand $C^*(K)$ to have rational coefficients. We have the fundamental

**Theorem 11:** $I$ induces an isomorphism of cohomology algebras.

The functor $A(\ )$ can, in fact, be defined on simplicial sets (compare (2)). If $X$ is an arbitrary space we have

**Def. 8:** $A(X) = A(S(X))$, where $S(X)$ is the singular complex of $X$ over $Q$.

The minimal model of $A(X)$ is denoted by $M(X)$. Sullivan proves,
Theorem 12: The homotopy category of rational nilpotent spaces of finite type is equivalent to the category consisting of isomorphism classes of minimal DGA's and homotopy classes of maps between them.

Corollary 12: If $X$ and $Y$ are two nilpotent spaces, then $[X^o, Y^o]$ is in bijective correspondence with $[M(Y), M(X)]$, where $X^o, Y^o$ are the respective rationalizations.

Furthermore, Sullivan shows that there is a precise correspondence between the construction of a rational space via the Postnikov Tower and the construction of a minimal DGA via elementary extensions. Sullivan [24] also defines a functor $\langle \rangle$ adjoint to $A(\ )$. $\langle \rangle$ is a functor from DGA's to simplicial sets and is called the spatial realization. Again, Bousfield and Gugenheim [2] made this notion precise and proved,

Theorem 13:

1. If $X$ is a rational nilpotent simplicial space of finite type, then there is an adjunction map

$$\phi_X : X \to \langle M(X) \rangle$$

which is a rational homotopy equivalence.
(2) If $A$ is a DGA, then there is an adjunction map

$$\psi_Y : A \to M(\langle A \rangle)$$

which is a quasi-isomorphism.

The geometric realization of Milnor, $|\cdot|$, is described in [14] and is a functor from simplicial sets to topological spaces adjoint to the singular complex functor $S(\cdot)$. In particular, the adjunction maps,

$$|S(X)| \longrightarrow X$$
$$Y \longrightarrow S(|Y|)$$

$(X \in \text{Top}, \ Y \in S)$ are weak equivalences. Apply the geometric realization to Theorem 13 to obtain

$$X \leftarrow |S(X)| \longrightarrow |\langle M(X) \rangle|$$

where both maps induce rational cohomology isomorphisms for $X$ a nilpotent space of finite type. In this situation, then, $X$ and $|\langle M(X) \rangle|$ have the same rational homotopy type. Finally, we note the extension of Def. 6 to spaces.
Def. 6': A space $X$ is said to be formal if its algebra of Q-polynomial forms $A(X)$ is.
CHAPTER 1

LIFTING HOMOTOPY ACTIONS IN RATIONAL
HOMOTOPY THEORY

In [4] George Cooke considered the problem of determining when a homotopy action of a group \( G \) on a space \( X \) may be replaced by an equivalent topological action of \( G \) on a space \( Y \) of the same homotopy type as \( X \). His solution was to provide a sequence of obstructions to such a replacement. Moreover, he considered various special cases where the obstructions would vanish due to restrictive group-theoretical or geometrical hypotheses.

In this paper we deal with similar questions, but from the point of view of rational homotopy theory. We exploit the algebraic group structure of the group of automorphisms of Sullivan's minimal model as described in [24]. There is some overlap with Cooke's work. For example, restricted to the class of spaces considered here, his theorem 2.3 (for the case \( R = \mathbb{Q} \)) is the first part of our theorem 3. For the most part, however, our objectives are somewhat different than Cooke's. We go beyond the lifting of actions to consider the problem of replacing a homotopy
(or even a cohomology) equivariant map by an equivalent equivariant map. The bulk of the paper is devoted to a study of actions on minimal differential graded algebras and equivariant maps between them. Geometric consequences are derived via Sullivan's spatial realization functor (see [24] or [2]).

The paper is organized as follows. In section 1 we give basic definitions and describe the main geometric results obtained. Section 2 is devoted to the construction of the most basic object in our study, the equivariant minimal model. A more elaborate concept of G-minimal model has been described by Georgia Triantafillou in [28]. This notion is truer to the spirit of G-equivariant homotopy theory, but does not easily lend itself to the applications we have in mind. Although our G-minimal model is not the same as that of Triantafillou, the construction when G is finite seems implicit in her work. Following [24], section 3 describes the structure of \( \text{Aut}(M_x) \) for a nice space \( X \). Several algebraic and geometric consequences are then derived. In section 4 we construct obstructions to replacing a homotopy equivariant automorphism of a minimal DGA by an equivariant automorphism in the same homotopy class. For this purpose we require the notion of non-abelian cohomology as described in [20].
§ 1. The Main Geometric Results

Adopting Cooke's notation, we denote by $G(X)$ the space of self homotopy equivalences of $X$. $E(X) = \pi_0(G(X))$ shall denote the discrete group of homotopy classes of self homotopy equivalences of $X$.

**Def. 1:** An action of a group $G$ on a space $X$ up to homotopy is a map $\nu : G \times X \to X$ such that, 1) $\nu_g = \nu(g, -)$ is a homotopy equivalence for each $g \in G$ and 2) $\hat{\nu} : G \to E(X)$ defined by $\hat{\nu}(g) = \nu_g$ is a group homomorphism. If $\hat{\nu}$ is injective say that $\nu$ is effective up to homotopy.
Our spaces shall always be of the homotopy type of CW complexes. If $X$ is nilpotent, then we may construct its rationalization $X_\mathbb{Q}$. The universal property of localization implies that a homotopy equivalence $X \rightarrow X$ induces a homotopy equivalence $X_\mathbb{Q} \rightarrow X_\mathbb{Q}$ uniquely determined up to homotopy. This provides a homomorphism

$\theta : E(X) \rightarrow E(X_\mathbb{Q})$. Now, say that a space $X$ has finite total homotopy dimension if

$$\sum_{i=0}^{\infty} \dim(\pi_i(x) \otimes \mathbb{Q}) < \infty.$$ 

A key result that will allow us to study homotopy actions rationally was proved by Sullivan [24] and (independently) Wilkerson [31]. We list it as,

**Theorem 1:** Let $X$ be nilpotent and either a finite complex or of finite total homotopy dimension. Then $E(X_\mathbb{Q})$ is an algebraic group, $\text{Im}(\theta)$ is an arithmetic subgroup of $E(X_\mathbb{Q})$ and $\ker(\theta)$ is finite.

Now, for $X$ satisfying the hypotheses of Theorem 1, the structure of $E(X_\mathbb{Q})$ allows us to make the following definition.

**Def. 2:** A homotopy action $\mu : G \times X \rightarrow X$ is called **rationally reductive** if $\text{Im}(\theta \circ \hat{\mu})$ is contained in a maximal reductive subgroup of $E(X_\mathbb{Q})$. If $\theta \circ \hat{\mu}$ is injective, then say that $\mu$ is **rationally effective**.
If $G$ is a finite group, then $\text{Im}(\theta \ast \mu)$ must also be finite. Hence, since any finite subgroup of an algebraic group is reductive and any reductive subgroup is contained in a maximal one, we see that any homotopy action of a finite group is rationally reductive.

Let $(X,\mu)$ and $(Y,\nu)$ be (homotopy) $G$-spaces. That is, $X$ and $Y$ are provided with respective (homotopy) $G$-actions $\mu$ and $\nu$. A map $\phi: X \to Y$ is homotopy equivariant if

\[
\begin{array}{ccc}
X & \xrightarrow{\mu} & X \\
\downarrow \phi & & \downarrow \phi \\
Y & \xrightarrow{\nu} & Y
\end{array}
\]

is homotopy commutative for each $g \in G$. Similarly, $\phi: X \to Y$ is homology equivariant if the diagram induces a commutative diagram of cohomology groups for each $g \in G$.

Now we can give the rational homotopy theoretic version of equivalence of actions analogous to the usual notion in transformation group theory.

**Def 3:** Let $\mu_i: G \times X_i \to X_i$, $i = 1, 2$ be (homotopy) actions.

1) $\mu_1$ is said to be (homotopy) elementary equivalent to $\mu_2$ if there exists an (homotopy) equivariant map
\( \phi : X_1 \to X_2 \) which is a homotopy equivalence. 2) \( \mu_1 \) is said to be rationally (homotopy) elementary equivalent to \( \mu_2 \) if there exists an (homotopy) equivariant map \( \phi : X_1 \to X_2 \) which is a rational homotopy equivalence.

With the exception of homotopy elementary equivalence, the definitions above are not equivalence relations. Hence, the various notions of "equivalence" considered will be those generated by the appropriate notions of elementary equivalence.

**Def. 4:** The triple \( (A, \nu, \phi) \) is said to be a (rational) geometric realization of a homotopy action \( \mu : G \times X \to X \) if \( \nu : G \times A \to A \) is a topological action of \( G \) on \( A \) and \( \phi : X \to A \) is a (rational) homotopy elementary equivalence.

In the same spirit consider the following situation. Let \( X \) be a \( G \)-space and let \( f : X \to X \) be a (rational) homotopy equivalence. Assume that \( f \) is homotopy equivariant.

**Def. 5:** If there exist \( G \)-spaces \( A_i \), equivariant maps which are (rational) homotopy equivalences \( \phi^{2i+1} : A_{2i+1} \to A_{2i} \), \( \phi^{2i} : A_{2i-1} \to A_{2i} \) and an equivariant map \( f_n : A_n \to A_n \) so that the diagram
is homotopy commutative, then we say that the system

\((A_i, \phi_i, f_n)\) is a (rational) equivariant realization of

\(f\) of length \(n\).

Our main geometric result concerns the two types of
realizations described above. However, before we can state
the theorem we must explain the notion of \(H\)-conjugacy. If

\(\mu\) is a homotopy action of \(G\) on \(X\), then the induced
action on the cohomology of \(X\) shall be denoted by \(H\mu\).
If \(H\mu\) and \(H\nu\) are two induced actions, they are said
to be \(H\)-conjugate if there exists an automorphism

\(T \in \text{Aut}(M_X)\) satisfying,

\[ T^* \cdot H\mu(g) \cdot T^{*-1} = H\nu(g) \]

for each \(g \in G\). Obstructions to the existence of an auto-
morphism of a minimal DGA realizing a given automorphism
of its cohomology algebra have been constructed by Halperin
and Stasheff in [11]. Thus we consider that part of the
problem effectively computable.

\textbf{Def. 6:} A space \(X\) is said to be \textbf{nice} if its minimal
model \(M_X\) is finitely generated.
Spheres, compact Lie groups, and, more generally, nilpotent spaces of finite total homotopy dimension are examples of nice spaces.

As a consequence of the main Theorem 2 of section 4 we have the following geometric application.

**Theorem 3:** Let $X$ be a nice space and assume $G$ is a finite group. Then,
1) Any homotopy action of $G$ on $X$ has a rational geometric realization.
2) Any two rational geometric realizations of two $H$-conjugate homotopy actions of $G$ on $X$ are rationally equivalent.
3) Any rational homotopy equivalence $f: X \to X$ which is homotopy equivariant has a rational equivariant realization of length 2.

We note that in Theorem 3 we may replace "homotopy action" by "homology action" in 1) and "homotopy equivariant" by "homology equivariant" in 3). These observations will be apparent from the proofs.

An extension of Theorem 3 for non-finite $G$ is given in section 5. There, also, we consider the case of non-discrete $G$ as follows.
Let $N(g_1, g_2)$ denote the subgroup of the topological group $G$ generated by $g_1$ and $g_2$. Via restriction, any action of $G$ on a space $X$ induces an action of $N = N(g_1, g_2)$ on $X$.

**Theorem 4**: Let $G$ be a topological group acting via $\mu$ on a nice space $X$. If $\mu|_N$ is rationally reductive and $\mu_{g_1}, \mu_{g_2}$ induce the same map on rational cohomology, then they are rationally homotopic.

**Corollary 5**: If $G$ is compact and $\mu_{g_1}, \mu_{g_2}$ induce the same map on rational cohomology, then they are rationally homotopic.

We note here that all the results stated above for nice spaces also hold true for finite nilpotent complexes (which may, of course, have non-finitely generated minimal models). The technicalities involved in adapting proofs for this class of spaces would unduly lengthen the paper; hence, we have decided to remain within the framework of nice spaces.
§ 2. The C-Semisimple G-Minimal Model

**Def. 7**: A pair \((A, \mu)\) is called a G-DGA if \(A\) is a differential graded algebra and \(\mu\) is a homomorphism from \(G\) to the group of DGA automorphisms of \(A\), \(\mu : G \to \text{Aut}(A)\).

Hence \(\mu\) provides an action of \(G\) on \(A\). Restricting the action to each \(A_i\) defines a representation of \(G\) which we denote by \(\mu_i\).

Let \(L_n(V)\) denote the free (graded) algebra on the vector space \(V\) concentrated in dimension \(n\). To consider \(L_n(V)\) as a DGA we give it the differential \(d = 0\). The DGA \(L_{n+1}(V) \otimes L_n(V)\) is the tensor product of the graded algebras \(L_{n+1}(V)\) and \(L_n(V)\) where \(V\) and \(\overline{V}\) are two copies of the same vector space \(V\). The differential is defined by,

\[
dv = 0 \text{ for } v \in V, \quad d\overline{v} = v \text{ for } v \in V
\]

where \(\overline{V}\) corresponds to \(v\) under the shift of dimension.

Now, given a DGA \(A\) and a DGA map \(\tau : L_{n+1}(V) \to A\) we may construct the pushout
The differential in $A \otimes _T L_n(\overline{V})$ is defined by $d|_A = d_A$ and $d|_V = \tau |_V$. $A \otimes _T L_n(\overline{V})$ is called an elementary extension of $A$. If $A$ is free, then $A \otimes _T L_n(\overline{V})$ is also free. Similarly, if $d_A$ and $\tau$ are decomposable $(d_A(A) \subset A^+ \cdot A^+)$, then so is $d$ on $A \otimes _T L_n(\overline{V})$.

Def. 8: A DGA $M$ is called minimal if $M$ may be written as an increasing union of sub-DGA's constructed from the ground field by elementary extensions with decomposable differential. That is, $Q \subset M_1 \subset M_2 \subset \ldots \subset \bigcup _i M_i = M$ with $M_{i+1} = M_i \otimes _T L_{i+1}(V)$ and $\tau_i$ decomposable.

We say that a DGA $A$ has a minimal model $M$ if $M$ is a minimal DGA and there is a DGA map $\rho: M \rightarrow A$ inducing an isomorphism of cohomology. A DGA $A$ is said to be connected if $A^0 = Q$ and homologically connected if $H^0(A) = Q$. It is known that any homologically connected DGA has a minimal model which is unique up to isomorphism. Now, given a $G$-DGA $(A, \mu)$, we would like to construct a minimal DGA $M$ with a $G$-action, denoted $\mu$, and an equivariant DGA map $\rho: M \rightarrow A$ inducing isomorphisms of cohomology. Such a construction will be called a $G$-minimal model for $(A, \mu)$. Although it seems that the procedure cannot be carried out in general, we
present a situation sufficient for the construction of the
G-minimal model.

Let \( \mathcal{C} \) be a class of representations of a group \( G \)
satisfying the following properties:

I. Each irreducible representation of \( G \) is in \( \mathcal{C} \).

II. Each representation in \( \mathcal{C} \) is a direct sum of
irreducible representations.

III. The tensor product of two representations in \( \mathcal{C} \)
is also in \( \mathcal{C} \).

IV. Any sum of representations in \( \mathcal{C} \) is also in \( \mathcal{C} \).

An easy consequence is the following,

**Lemma A:** Let \((V, \xi)\) and \((W, \eta)\) be representations of
\( G \) with \( f: V \to W \) equivariant.

1) If \( f \) is surjective and \( \xi \in \mathcal{C} \), then
\( \eta \in \mathcal{C} \) and there exists an equivariant splitting \( g: W \to V \).

2) If \( f \) is injective and \( \eta \in \mathcal{C} \), then \( \xi \in \mathcal{C} \)
and there exists an equivariant splitting \( h: W \to V \).

**Def. 9:** A \( G \)-DGA \((A, \mu)\) is called \( \mathcal{C} \)-semisimple if
\( \mu_i \in \mathcal{C} \) for each \( i \).

**Def. 10:** Let \((A, \mu)\) be a \( G \)-DGA. A \( G \)-minimal model for
\((A, \mu)\) is a \( G \)-DGA \((M, \overline{\mu})\) so that
1) \( M \) is a minimal
DGA and 2) there exists an equivariant DGA map
\( \rho : M \to A \) inducing an isomorphism of cohomology.

**Theorem B:** Let \((A,\mu)\) be a \(G\)-semisimple \(G\)-DGA. Then there exists a \(G\)-semisimple \(G\)-minimal model for \((A,\mu)\).

(Sketch of proof)

Define \( M(0) = Q \) and \( \rho(1) = 1 \). Assume that \( M(n) \) has been constructed along with \( \rho_n; M(n) \to A \) satisfying:

1) \( M(n) \) is a minimal \(G\)-semisimple \(G\)-DGA and \( \rho_n \) is equivariant.
2) \( \rho_n \) induces an isomorphism in cohomology in dimension \( \leq n \).
3) \( \rho_n \) induces an injection in cohomology in dimension \( n+1 \).

The construction of \( M(n+1) \) and \( \rho_{n+1} \) satisfying 1-3 will complete the proof by induction. The three equivariant
surjections $\mathbb{Z}^{n+2}(M(n)) \rightarrow H^{n+2}(M(n))$, $\mathbb{Z}^{n+1}(A) \rightarrow H^{n+1}(A)$, $A^{n+1} \rightarrow B^{n+2}(A)$ each have equivariant splittings provided by Lemma A. We denote these splittings by,

$\alpha : H^{n+2}(M(n)) \rightarrow \mathbb{Z}^{n+2}(M(n))$, $\beta : H^{n+1}(A) \rightarrow \mathbb{Z}^{n+1}(A)$

and $\Lambda : B^{n+2}(A) \rightarrow A^{n+1}$.

$G$ acts on $A$ and $M(n)$ by DGA automorphisms, so $\text{im} \ \rho^* \subseteq H^{n+1}(A)$ and $\ker \rho^* \subseteq H^{n+2}(M(n))$ are $G$-stable subspaces. By $C$-semisimplicity $\text{im} \ \rho^*$ has a $G$-complement $V_1$. That is, $\text{im} \ \rho^* \oplus V_1 = H^{n+1}(A)$. Denote $\ker \rho^*$ by $V_2$.

Define $M(n+1) = M(n) \otimes L_{n+1}(V_1 \oplus V_2)$ where $\tau = L_{n+2}(V_1 \oplus V_2) \rightarrow M(n)$ is defined by $\tau \big|_{V_1} = 0$ and $\tau \big|_{V_2} = \alpha \big|_{V_2}$. (Note that here we have suppressed the bar over and $\bar{\sigma}$.) Define $\rho_{n+1} : M(n+1) \rightarrow A$ by:

$\rho_{n+1} \big|_{M(n)} = \rho_n$, $\rho_{n+1} \big|_{V_1} = \beta \big|_{V_1}$ and $\rho_{n+1} \big|_{V_2} = \Lambda \cdot \rho_n \cdot \alpha \big|_{V_2}$. Properties 1-3 follow in a straightforward manner. $C$-semisimplicity of $M(n+1)$ is a consequence of Lemma A and axioms III and IV of a $C$-class.

Q.E.D.

Now consider the following situation. The following diagram consists of $G$-DGA's and equivariant maps commuting up to a homotopy which is also equivariant.
Prop. C: If $\phi$ is an isomorphism of cohomology, then there exists an equivariant lifting

$$\tilde{f}: M \otimes \Gamma_n L(V) \rightarrow A$$

such that $\tilde{f}|_M = f$ and $\phi \cdot \tilde{f}$ is $G$-homotopic to $\bar{f}$.

Note that since $L_n(V_1 \oplus V_2) = L_n(V_1) \otimes L_n(V_2)$ we need only consider the case where $V$ is irreducible. If $M$ is minimal, then as a graded algebra we may write $M = L(X_1)$. Then we may define $M^\Gamma = L(x'_1, x''_1, \hat{x}_1)$ with differential,

$$d(x'_1) = (dx_1)', \quad d(x''_1) = (dx_1)''$$

$$d\hat{x}_1 = x'_1 - x''_1 - sdx_1.$$
Here $s$ is a map $s: M \to M^I$ with $s(x_i) = \hat{x}_i$ and extended via $s(xy) = s(x)y'' + (-1)^{|x|} x's(y)$. Two DGA maps $f, g: M \to A$ are said to be homotopic if there exists $H: M^I \to A$ so that $H(x_1) = f(x_1)$ and $H(x''_1) = g(x_1)$.

If $M$ has $G$-action $\mu$, then we may define a $G$-action on $M^I$ by, $\mu_g(x') = (\mu(x'))'$, $\mu_g(x'') = (\mu(x''))''$ and $\mu_g(x_i) = s(\mu_g(x_i))$. This allows us to consider the notion of $G$-homotopy.

Denote by $C^* = A^* \otimes B^{*-1}$ the mapping cone construction of chain complexes given by a map $\varnothing: A \to B$. $C^*$ has differential defined by $d(a,b) = (-da, db+\varnothing(a))$.

Let $(x_1, \ldots, x_n)$ be a basis for $V$. Define $F: V \to C^{n+1}$ by

$$F(x_i) = -(f \cdot \tau(x_i), \bar{f}(x_i) + Hsdx_i)$$

where $H$ is the $G$-homotopy of the diagram. $F$ is clearly equivariant since all of the defining maps are. By Schur's Lemma, $\text{Im } F = 0$ or $F$ is an isomorphism onto its image. If $\text{Im } F = 0$, then an explicit (and simple) construction of $f$ is possible. We consider the case $\text{Im } F \neq 0$.

It is easily checked that $F(x_1)$ is a cocycle in $C^{n+1}$. Therefore, since $\varnothing$ induces an isomorphism of cohomology (implying $H^*(C) = 0$), there must exist a non-
trivial coboundary which "kills" $F(x_1)$. Hence $Z^n(c) \nsubseteq d^{-1}(\text{Im } F)$. Again invoke Schur's Lemma to see that $d: d^{-1}(\text{Im } F) \rightarrow \text{Im } F$ is a surjection. Lemma A implies that there is an equivariant splitting $\alpha$. Hence $d \mid\alpha(\text{Im } F) \rightarrow \text{Im } F$ is an isomorphism.

Define $\bar{T} : M \otimes L_n(v) \rightarrow A$ by, $\bar{T} | M = f$ and $\bar{T} | v = p_A \circ d|^{-1} \circ F$ where $p_A : C^n \rightarrow A^n$ is the projection.

Define $\bar{H} : (M \otimes L_n(v))^I \rightarrow B$ by, $\bar{H} | M^I = H$

$\bar{H}(x'_1) = \emptyset \circ \bar{f}(x_1)$, $\bar{H}(x''_1) = \bar{f}(x_1)$ and $\bar{H}(x'_1) = -p_B \circ d|^{-1} \circ F(x_1)$ where $p_B : C^n \rightarrow B^{n-1}$ is the projection. The required properties are easily checked.

Q.E.D.

Corollary D: The $G$-minimal model is unique up to isomorphism.

Proof:

Let $M_1$ and $M_2$ be two $G$-minimal models for $(A, \mu)$. At each stage of $M_1$ we have a lifting problem:

\[
\begin{array}{ccc}
M_1(n) & \xrightarrow{\sim} & A^n \\
\downarrow & & \downarrow \rho_1 \\
M_1(n) & \xrightarrow{\sim} & L_{n+1}(v) \\
\downarrow & & \downarrow \rho_2 \\
M_2 & \xrightarrow{\sim} & A
\end{array}
\]
Since \( \rho_2 \) induces cohomology isomorphisms, Prop. C guarantees that the lifting \( \tilde{\rho} \) exists. A lifting at each stage provides a global lifting,

\[
\begin{array}{ccc}
M_1 & \rightarrow & M_2 \\
\downarrow & & \downarrow \\
\rho_1 & \rightarrow & \rho_2 \\
\end{array}
\]

The diagram commutes up to homotopy and since \( \rho_1, \rho_2 \) induce cohomology isomorphisms, then so must \( \rho \) also. However, a map of minimal DGA's inducing cohomology isomorphisms is an isomorphism. Therefore \( \rho \) is the required isomorphism of minimal models.

Q.E.D.

We have constructed the \( G \)-minimal model under the rather general assumption of \( \mathcal{C} \)-semisimplicity. Our major applications, however, center on the case where \( G \) is finite. In this situation we may take \( \mathcal{C} \) to be the class of all representations of \( G \). I-IV are clearly satisfied. In particular, if \( G \) is finite and \( \mu : G \times X \rightarrow X \) is a topological action of \( G \) on \( X \), then
there is an induced action of $G$ on the singular complex $S(X)$ of $X$ via semi-simplicial automorphisms. The naturality of Sullivan's $Q$-polynomial functor (see [24]) then provides an action of $G$ on the DGA $A(X)$. Since $G$ is finite, the action is semisimple and the $G$-minimal model may be constructed. Hence we obtain a chain of equivariant constructions reflecting both the $Q$-homotopy type of $X$ and the $Q$-homotopy theory of the action $\mu$.

§ 3. The Structure of $\text{Aut}(M_x)$ and Some Results on DGA $G$-Actions

Let $M$ be a minimal DGA. We consider three automorphism groups associated to $M$. $\text{Aut}(M)$ = the group of DGA automorphisms of $M$, $h\text{-Aut}(M)$ = the group of homotopy classes of automorphisms of $M$ (see the proof of Prop. C for the notion of homotopy of DGA maps), $H\text{-Aut}(M)$ = the group of cohomology automorphisms of $M$ induced by automorphisms of $M$ itself. Sullivan [24] proves that the following relations hold between these groups. There are surjections, $\pi_1 : \text{Aut}(M) \to h\text{-Aut}(M)$, $\pi_2 : h\text{-Aut}(M) \to H\text{-Aut}(M)$ and $\pi = \pi_2 \circ \pi_1 : \text{Aut}(M) \to H\text{-Aut}(M)$ such that $U_1 = \text{Ker } \pi_1$, $U_2 = \text{Ker } \pi_2$ and $U_0 = \text{Ker } \pi$ are unipotent subgroups.
If \( M \) is finitely generated, then Sullivan observes that each of the three automorphism groups has the structure of an algebraic matrix group. Thus we may apply Mostow's Levi Decomposition \([15]\) to see that \( \text{Aut}(M) = R \rtimes U \), \( \text{h-Aut}(M) = R \rtimes U/U_1 \) and \( \text{H-Aut}(M) = R \rtimes U/U_o \). Hence \( R \) is the reductive part of \( \text{Aut}(M) \) and the semi-direct product factors \( U, U/U_1 \) and \( U/U_o \) are the unipotent radicals of their respective groups. The significant feature here is that all three groups have the same reductive part \( R \). We note that \( R \) is not unique. However, if \( \text{Aut}(M) = R' \ltimes U \), then there exists \( s \in \text{Aut}(M) \) with \( sR's^{-1} = R \).

**Def. 11:** 1) An action of \( G \) on \( M \) is a homomorphism 
\[ \mu : G \rightarrow \text{Aut}(M) \].

2) A homotopy action of \( G \) on \( M \) is a homomorphism 
\[ \nu : G \rightarrow \text{h-Aut}(M) \].

3) A homology action of \( G \) on \( M \) is a homomorphism 
\[ \nu : G \rightarrow \text{H-Aut}(M) \].

**Def. 12:** An action (h-action, H-action) of \( G \) on \( M \) is said to be reductive if \( \mu(G) \subset R \) for some maximal reductive subgroup \( R \subset \text{Aut}(M) \) (h-Aut(M), H-Aut(M)).

Note that here and in the rest of the paper we use the abbreviations "h-action" and "H-action" in place of
"homotopy action" and "homology action" respectively.

**Def. 13:** Let \( \mu_1 \) and \( \mu_2 \) be actions (h-actions, H-actions) of \( G \) on \( M \). Say that \( \mu_1 \) is conjugate (h-conjugate, H-conjugate) to \( \mu_2 \) if there exists \( \tau \in \text{Aut}(M) \) (h-Aut(M), H-Aut(M)) with \( \tau \cdot \mu_1(g) \cdot \tau^{-1} = \mu_2(g) \) for each \( g \in G \).

These notions of conjugacy, of course, correspond to the usual definition of equivalence of representations.

Now suppose that \( \mu: G \to \text{Aut}(M) \) is reductive. Assume that \( \mu(G) \subseteq R' \), where \( R' \) is a maximal reductive subgroup. As we have seen, there exists \( s \in \text{Aut}(M) \) with \( s \mu(G)s^{-1} = sR's^{-1} = R \). Thus, up to conjugacy we may assume that all the reductive actions we consider actually have their images in \( R \). Of course, similar remarks apply to both h-conjugacy and H-conjugacy.

Before we examine the consequences of the definitions above, we recall the multiplicative structure of the semi-direct products under consideration. Given \((r,u)\) and \((r',u')\) in \( R \times U \) we have, \((r,u)(r',u') = (rr',u+r\cdot u')\).

The operation in \( U \) is not commutative, but is denoted by + to distinguish it from the action of \( R \) on \( U \), denoted by *. Also, recall that \((r,u)^{-1} = (r^{-1},r^{-1}\cdot(-u))\).

If \( \mu: G \to \text{Aut}(M) \) is an action of \( G \) on \( M \), then \( h\mu \) and \( H\mu \) shall denote the respective induced actions
on homotopy and cohomology. Now we may formulate a basic algebraic result.

Prop. E: Let $(M,\mu_1)$ and $(M,\mu_2)$ be G-DGA's with $M$ finitely generated minimal. Suppose $\mu_1$ and $\mu_2$ are reductive actions. If $H\mu_1$ is $H$-conjugate to $H\mu_2$, then $\mu_1$ is conjugate to $\mu_2$.

Proof:

We may assume $\mu_i(G) \subseteq R$ for $i = 1, 2$. Therefore we write $\mu_i(G) = (\mu_i(g), 0) \in \text{Aut}(M) = R \circ U$. The reductive part $R$ remains invariant under the projection to $H\text{-Aut}(M)$; hence, we write $H\mu_i(g) = (\mu_i(g), \bar{0})$, where "-" denotes the projection to $U/U_0$.

$H\mu_1 \sim H\mu_2$ implies that there exists $\tau = (\rho, \sigma) \in H\text{-Aut}(M)$ with $\tau \cdot H\mu_1 \cdot \tau^{-1} = H\mu_2$. Computing, we have

$$(\rho, \sigma)(\mu_1, \bar{0})(\rho^{-1}, \rho^{-1}, (\bar{\sigma}))$$

$$= (\rho \mu_1 \rho^{-1}, \sigma + \rho \mu_1 \rho^{-1}, (\bar{\sigma}))$$

$$= (\mu_2, \bar{0}) .$$

Now $\rho \in R \subseteq \text{Aut}(M)$ and we have,

$$(\rho, 0)(\mu_1, 0)(\rho^{-1}, 0) = (\rho^{-1} \rho^{-1}, 0)$$

in $\text{Aut}(M)$. Hence $\mu_1 \sim \mu_2$.

Q.E.D.
As an immediate consequence of Prop. E we obtain its geometric analogue.

**Prop. E':** Suppose $G$ is finite and $(X, \mu_1), (X, \mu_2)$ are $G$-spaces with $X$ nice. If $H\mu_1$ is $H$-conjugate to $H\mu_2$, then $\mu_1$ is rationally equivalent to $\mu_2$.

**Proof:**

Since $G$ is finite we may construct the $G$-minimal models $(M, \mu_1)$ and $(M, \mu_2)$. Because the rational cohomology of $X$ is reflected by that of $M$, the induced actions $H\mu_1$ are the same as the induced actions $H\mu_1$. Hence, $H\mu_1$ is $H$-conjugate to $H\mu_2$. Now, $X$ nice and $G$ finite imply that $M$ is finitely generated and the actions are reductive. We then see that $\mu_1$ is conjugate to $\mu_2$ by Prop. E. Hence there exists $s \in \text{Aut}(M)$ with $s^{-1}\mu_1 \cdot s = \mu_2$. Applying Sullivan's spatial realization functor followed by Milnor's geometric realization functor we obtain an equivariant map,

$$\langle s \rangle : (|<M>|, |<\mu_2>|) \to (|<M>|, |<\mu_1>|).$$

Now, as we have mentioned previously, $\mu_1$ induces an action $S\mu_1$ on the singular complex $S(X)$ of $X$. By naturality of the geometric realization, the standard homotopy equivalence $|S(X)| \to X$ is equivariant. The
action of \( \mathbb{S} \mu_1 \) on \( S(X) \) induces an action \( \mathbb{A} \mu_1 \) on \( A(X) \), the Q-polynomial forms on \( S(X) \). Spatial realization \( \langle \rangle \) is adjoint to \( \mathbb{A} \), providing a map \( S(X) \to \langle A(X) \rangle \) which is equivariant by the naturality of the functors involved. Similarly, the equivariance of \( M \to A(X) \) is preserved under both concepts of realization. Hence we have a chain of equivariant maps which are rational homotopy equivalences,

\[
(X, \mu_1) \to (|S(X)|, |S\mu_1|) \to (|M|, |\mu_1|) \to (S, X) f (|S\mu_2|) \to (X, \mu_2).
\]

Therefore \( \mu_1 \) is rationally equivalent to \( \mu_2 \).

Q.E.D.

The singular functor and Milnor's geometric realization are classical. The functorial nature of the Q-polynomial form functor \( \mathbb{A} \) and the spatial realization \( \langle \rangle \) is discussed in detail in [27].

With a view toward later application we now present a slight generalization of Prop. E'.

**Corollary F**: Let \( G \) be finite and \( (X, \mu), (Y, \nu) \) \( G \)-spaces with \( X \) and \( Y \) nice. If there is an isomorphism of minimal models which is homology equivariant, then \( (X, \mu) \) is rationally equivalent to \( (Y, \nu) \).
Proof:

Let \( T : M \rightarrow M \) be an isomorphism satisfying
\[
T^* \cdot H\mu \cdot T^{-1} = H\nu .
\]
Define an action \( \omega : G \rightarrow \text{Aut}(M_X) \) by
\[
\omega = T^{-1} \cdot \nu \cdot T .
\]
Computing, we obtain
\[
H\omega = T^{-1} \cdot H\nu \cdot T = T^{-1} \cdot H\nu \cdot T^{-1} \cdot T^* = H\mu .
\]
Thus, by Prop. E there exists \( S \in \text{Aut}(M_X) \) with
\[ S^{-1} \cdot \omega \cdot S = \mu . \]
Then,
\[
\mu = S^{-1} \cdot \omega \cdot S = S^{-1} \cdot T^{-1} \cdot \nu \cdot T \cdot S =
\]
\[
= (TS)^{-1} \cdot \nu \cdot (TS) ,
\]
Therefore, \( (M_X, \mu) \sim (M_Y, \nu) \). As in the proof of Prop. E',
spatial and geometric realizations provide a rational equivalence \( (X, \mu) \sim (Y, \nu) \).

Q.E.D.

§ 4. Non-Abelian Cohomology and Obstructions to Equivariance

Let \( M \) be a minimal DGA and let \( f \in \text{Aut}(M) \).

Suppose that \( \mu \) is an action of \( G \) on \( M \) and that \( f \)
is homotopy equivariant with respect to $\mu$. We consider under what conditions we may replace $f$ by an equivariant automorphism in the same homotopy class as $f$. Our objective is to inject some notion of "naturality" into the $G$-minimal model construction.

Recall that $U_1 = \ker(\pi_1: \text{Aut}(M) \to \text{h-Aut}(M))$, the subgroup of automorphisms homotopic to the identity. Hence, the coset $fU_1$ consists of automorphisms homotopic to $f$. $G$ acts on $fU_1$ because $f$ is homotopy equivariant. A fixed point under this action is simply an equivariant map in the homotopy class of $f$. Thus we have reduced our original question to one of the existence of fixed points under a $G$-action on the coset $fU_1$. The rest of this section is devoted to reformulating and solving this problem in a purely algebraic context.

To simplify notation we shall write $gfg^{-1}$ for the action of $g \in G$ on $f$ in place of the proper $\mu_g f g^{-1}$. Assume that $fu \in fU_1$ is a fixed point under the action of $G$. Then

$$fu = gfg^{-1} = gfg^{-1} gug^{-1}.$$ 

Now, $f$ is homotopy equivariant so there is a unique $\sigma(g) \in U_1$ with $gfg^{-1} = f \sigma(g)$. Also, note that
\[ f\sigma(gh) = ghfh^{-1}g^{-1} \]
\[ = g\sigma(h)g^{-1} \]
\[ = gfg^{-1}\sigma(h)g^{-1} \]
\[ = f\sigma(g)\sigma(h)g^{-1} . \]

Thus, \( \sigma(gh) = \sigma(g)\sigma(h)g^{-1} . \)

Returning to our first equation,
\[ fu = gfg^{-1}gug^{-1} = f\sigma(g)gug^{-1} . \]

Thus,
\[ 1 = u^{-1}\sigma(g)gug^{-1} . \]

The two relations we have derived are well known in the study of non-abelian cohomology of groups. We shall briefly discuss this theory in order to formulate our obstruction results.

Let \( G \) be a group acting on another group \( U \). In order to make apparent the connection with the discussion above, we assume that \( G \) and \( U \) are contained in some larger group and that \( G \) acts on \( U \) via conjugation. Define,
\[ Z^1(G;U) = \{ \sigma: G \to U \mid \sigma(gh) = \sigma(g)\sigma(h)g^{-1} \} \]
We define an equivalence relation \( \sim \) on \( Z^1(G;U) \) by saying that \( \sigma \) is cohomologous to \( \lambda \) (\( \sigma \sim \lambda \)) if there is \( u \in U \) with \( \lambda(g) = u^{-1}\sigma(g)ug^{-1} \). The quotient of \( Z^1(G;U) \) under the equivalence relation \( \sim \) is denoted \( H^1(G;U) \) and is called the first non-abelian cohomology set of \( G \) with coefficients in \( U \). \( H^1(G;U) \) does not necessarily have a law of composition and, therefore, is not in general a group. The cocycle that is constant at 1 (the "zero cocycle") does provide a distinguished element, however. That such an element exists allows us to make use of the usual notion of exactness for sequences of pointed sets. If \( A \) and \( B \) are \( G \)-groups with \( A \leq B \), then \( C = B/A \) is also a \( G \)-group. In [20] Serre derives the exact sequence of pointed sets,

\[
1 \to A^G \to B^G \to C^G \to H^1(G;A) \to H^1(G;B) \to H^1(G;C) .
\]

The only property that we shall use is that if \( H^1(G;A) = 0 = H^1(G;C) \), then \( H^1(G;B) = 0 \).

Now let us return to our original situation. For a homotopy equivariant automorphism \( f \) we obtained the relation, \( \sigma(gh) = \sigma(g)g \sigma(h)g^{-1} \). Thus \( f \) defines a cocycle \( \sigma \in Z^1(G;U_1) \). Furthermore, we saw that \( fu \in fU_1 \) is equivariant if and only if the relation
$1 = u^{-1} \sigma(g) g u g^{-1}$ is satisfied. This equation simply expresses that fact that $\sigma$ is cohomologous to the zero cocycle. Hence the cocycle $\sigma$ associated to a homotopy equivariant automorphism $f$ is exactly the obstruction to equivariance we desire. The next question, of course, is to consider under what conditions $\sigma \sim 1$ (i.e., $[\sigma] = 0 \in H^1(G;U)$). We work with the least general (but most practical) case; namely, when $H^1(G;U) = 0$.

**Lemma G:** If $U$ is abelian, then $H^1(G;U)$ corresponds to the usual definition of group cohomology.

**Proof:**

Since $U$ is abelian we use additive notation. The cohomology of $G$ with coefficients in $U$ is usually defined as the group of crossed homomorphisms,

$$Z_x^1(G;U) = \{ \sigma : G \to U \mid \sigma(gh) = \sigma(g) + g \cdot \sigma(h) \}$$

modulo the principal crossed homomorphisms,

$$B_x^1(G;U) = \{ \lambda_u \in Z_x^1(G;U) \mid \lambda_u(g) = g \cdot u - u \}.$$

Of course in our situation the action $g \cdot u = g u g^{-1}$. It is then clear that (written in additive notation)

$$Z^1(G;U) = Z_x^1(G;U).$$

Now $\alpha \sim \beta$ in $Z^1(G;U)$ if and only if $\beta(g) = -u + \alpha(g) + g u g^{-1}$ (written additively).
Because \( U \) is abelian we may write \( \beta(g) = \alpha(g) + gug^{-1}u \).

But this simply says that \( \beta = \alpha + \lambda_u \). Hence \( \beta \sim \alpha \)
in \( Z_x(G;U) \). Thus the two notions of equivalence are actually the same; consequently \( H^1(G;U) = H^1_x(G;U) \).

Q.E.D.

Before we can state the next result we need to recall several facts about unipotent algebraic groups over \( \mathbb{Q} \).

A good reference is Chapter IV of [6].

**Facts:** Let \( U \) be a unipotent \( \mathbb{Q} \)-algebraic group. Then,

1) Every closed subgroup of \( U \) is unipotent.

2) Every quotient of \( U \) is unipotent.

3) If \( U \) is abelian, then \( U \cong \mathbb{Q}^k \), where \( k = \dim U \).

Also, since \( U \) is unipotent it is nilpotent and therefore has a finite lower central series,

\[ U = C^1 \supset C^2 \supset \ldots \supset C^s \supset 1, \]

where \( C^{i+1} = [C^i, U] \). Each \( C^i \) is closed and hence, unipotent by Fact 1). By Fact 2) each quotient \( C^i/C^{i+1} \) is unipotent. Also, \( [C^i, C^i] \subset [C^i, U] = C^{i+1} \) so \( C^i/C^{i+1} \) is abelian. By Fact 3) \( C^i/C^{i+1} \cong \mathbb{Q}^k \) for some \( k \). Now assume \( G \) acts on \( U \) leaving each \( C^i \) invariant.
Lemma H: If $H^1(G;W) = 0$ for every $QG$-module $W$, then $H^1(G;U) = 0$.

Proof:

$1 = C^{s+1} = [C^s, U]$ implies that $C^s \subseteq Z(U)$, the center of $U$. Because $C^s$ is unipotent, $C^s = Q^k$. By assumption $H^1(G;C^s) = 0$.

Now consider $C^{i+1} \rightarrow C^i \rightarrow C^i/C^{i+1} = D^i$. Inductively we assume that $H^1(G;C^{i+1}) = 0$. $H^1(G;D^i) = 0$ since $D^i = Q^k$. Applying Serre's exact sequence, we obtain $H^1(G;C^i) = 0$. By induction on the lower central series of $U$, $H^1(G;U) = 0$.

Q.E.D.

Examples of groups satisfying the condition of Lemma H include, finite groups, $Sp_n(Z)$ $n \geq 2$ and $SL_n(Z)$ $n \geq 3$. Therefore, if one of these groups acts on a minimal DGA, then the obstructions to equivariance vanish and any homotopy equivariant automorphism may be replaced by an equivariant automorphism in the same homotopy class.

We would now like to formalize this theory. Consider the sequence,

$\begin{align*}
(A) \quad Aut_C(M) \xrightarrow{\psi} h-Aut_C(M) \xrightarrow{\Sigma} H^1(G;U_1) \end{align*}$
where $\text{Aut}_G(M) =$ the group of equivariant automorphisms of $M$ and $\text{h-Aut}_G(M) =$ the group of homotopy classes of homotopy equivariant automorphisms of $M$. We define $w$ in the obvious way, $w(f) = [f]$. The definition of $\Sigma$ relies on our earlier discussion. $\Sigma([f]) = [\sigma] \in H^1(G; U_1)$, where $\sigma$ is the cocycle associated to the homotopy equivariant automorphism $f$.

**Prop. I:** $\Sigma$ is well defined and the sequence (A) is exact at the middle joint.

**Proof:**

Let $[f_1] = [f_2]$ and suppose $\Sigma([f_1]) = [\sigma]$ and $\Sigma([f_2]) = [\lambda]$. Then we have, $gf_1g^{-1} = f\sigma(g)$ and $gf_2g^{-1} = f\lambda(g)$. Also, $f_1 = f_2u$ since $[f_1] = [f_2]$. Computing, we obtain

$$gf_1g^{-1} = f_1\sigma(g)$$

$$= gf_2ug^{-1}$$

$$= gf_2g^{-1}gug^{-1}$$

$$= f_2\lambda(g)gug^{-1}$$

$$= f_1u^{-1}\lambda(g)gug^{-1}.$$
Hence, \( \sigma(g) = u^{-1}\lambda(g)gu^{-1} \); \( \sigma \sim \lambda \) and hence, \([\sigma] = [\lambda] \). Therefore \( \Sigma \) is well defined.

If \( f \) is equivariant, then \( gfg^{-1} = f \), so the associated cocycle is the zero cocycle. Hence, \( \Sigma(w(f)) = 0 \).

Now let \( \Sigma([f]) = 0 \). Hence, the cocycle \( \sigma \) associated to \( f \) is cohomologous to the zero cocycle. Thus, there exists \( u \in U_1 \) with \( \sigma(g) = u^{-1}guy^{-1} \) for each \( g \in G \). Hence,

\[
\sigma(g)gu^{-1}g^{-1} = u^{-1} \\
\sigma g(g)gu^{-1}g^{-1} = fu^{-1} \\
gfg^{-1}gu^{-1}g^{-1} = fu^{-1} \\
gfu^{-1}g^{-1} = fu^{-1}.
\]

Therefore \( fu^{-1} \) is equivariant and \( [fu^{-1}] = [f] \) since \( u \in U_1 \). Thus \( w(fu^{-1}) = [f] \).

Q.E.D.

The exactness of the sequence (A) simply expresses the solution to the equivariance problem that we have discussed. In this spirit we may consider the two exact sequences,

\[
(B) \quad \text{Aut}_G(M) \rightarrow H^1(G; \text{Aut}_G(M)) \rightarrow H^1(G; U_1) \]
The exactness of (B) and (C) is proved in an analogous fashion to that of (A) and expresses the solution to an associated equivariance problem. We leave the exact formulations to the reader. We collect the algebraic results of sections 3 and 4 in a theorem.

**Theorem 2:** Let $G$ be a group and $M$ a finitely generated minimal DGA. Then,

1) Any reductive homotopy action of $G$ on $M$ may be lifted to an action of $G$ on $M$.

2) Any two reductive actions of $G$ on $M$ which are $H$-conjugate are conjugate.

3) The sequences,

\[
\begin{align*}
(A) & \quad \text{Aut}_G(M) \to h\text{-Aut}_G(M) \to H^1(G;U_1) \\
(B) & \quad \text{Aut}_G(M) \to \text{H-Aut}_G(M) \to H^1(G;U_0) \\
(C) & \quad h\text{-Aut}_G(M) \to \text{H-Aut}_G(M) \to H^1(G;U_2)
\end{align*}
\]

are exact at the middle joint.

4) If $H^1(G;W) = 0$ for every $QG$-module $W$, then any homotopy equivariant automorphism of $M$ may be lifted to an equivariant automorphism of $M$ within its homotopy class.
Proof:

1) Let \( \mu : G \to h\text{-}Aut(M) \) be a reductive homotopy action. Hence \( \mu(G) \subset R' \subset h\text{-}Aut(M) \) for some maximal reductive subgroup \( R' \). As we have observed, there exists \( s \in h\text{-}Aut(M) \) so that \( s(\mu(G))s^{-1} \subset sR's^{-1} = R \). Hence we obtain homotopy action, denoted \( s\mu s^{-1} \), by defining \( s\mu s^{-1}(g) = s\mu g s^{-1} \). By Sullivan's theorem we know that, when restricted to \( R \subset Aut(M) \), the projection \( p : Aut(M) \to h\text{-}Aut(M) \) is an isomorphism. Thus \( p \mid_{R}^{-1} \) exists. Also, we note that by the definition of \( h\text{-}Aut(M) \), there exists \( S \in Aut(M) \) such that \( p(S) = s \). Choosing such an \( S \) we define an action \( \overline{\mu} \) of \( G \) on \( M \) by,

\[
\overline{\mu}_{g} = s^{-1}(p \mid_{R}^{-1}(s\mu g s^{-1}))S.
\]

Then,

\[
\overline{\mu}_{g} = pS^{-1}p \mid_{R}^{-1}(s\mu g s^{-1})pS
\]

\[
= s^{-1}s_{g}s^{-1}s
\]

\[
= \mu_{g}.
\]

Hence, \( \overline{\mu} \) is the required lifting of \( \mu \).

2) This is Prop. E.

3) This is Prop. I and the discussion following.

4) Apply Lemma H to 3).

Q.E.D.
§5. Geometric Implications of Theorem 2.

Let \( X \) be a nice space. We choose, once and for all, a minimal model for \( X \) and denote it by \( M \). As we have seen, \( \text{Aut}(M) \approx R \times U \) and \( E(X_0) \approx h-\text{Aut}(M) \approx R \times U/U_1 \).

Proof of Theorem 3:

1) A homotopy action \( \tilde{\mu} : G \to E(X) \) \( (\mu : G \times X \to X) \)
induces a homotopy action \( \theta \circ \tilde{\mu} : G \to E(X_0) \approx h-\text{Aut}(M) \)
\( (\theta \mu : M \to M) \). Via the spatial and geometric realization functors \( (\bigtriangleup) \) and \( | \ | \) respectively) we obtain the homotopy commutative diagram for each \( g \in G \),

\[
\begin{array}{ccc}
X & \longrightarrow & |S(X)| \\
\downarrow \mu_g & & \downarrow |s\mu_g| \\
X & \longrightarrow & |S(X)|
\end{array}
\begin{array}{ccc}
& \longrightarrow & |\langle M \rangle| \\
\downarrow & & \downarrow |\langle \theta \mu_g \rangle| \\
& \longrightarrow & |\langle M \rangle|
\end{array}
\]

Now \( X \) has the homotopy type of a CW complex, so \( X \to |S(X)| \) may be inverted up to homotopy. This does not change the homotopy commutativity of the diagram.

Thus, \( X \longrightarrow |\langle M \rangle| \) homotopy commutes.

\[
\begin{array}{ccc}
X & \longrightarrow & |\langle M \rangle| \\
\downarrow \mu_g & & \downarrow |\langle \theta \mu_g \rangle| \\
X & \longrightarrow & |\langle M \rangle|
\end{array}
\]
Since $G$ is finite the homotopy action $\mu$ is rationally reductive. Therefore, $\theta \mu$ is reductive as an $h$-action on $M$. By Theorem 2 we may lift $\theta \mu$ to an action, denoted $\mu^\sim$, on $M$ so that for each $g \in G$

\[
\begin{array}{ccc}
M & \xrightarrow{\mu^\sim} & M \\
id & \downarrow & \downarrow \text{id} \\
M & \xrightarrow{\theta \mu} & M
\end{array}
\]

commutes up to homotopy. Hence, the following diagram also commutes up to homotopy for each $g \in G$.

\[
\begin{array}{ccc}
|\langle \mu^\sim \rangle| & \xrightarrow{\langle M \rangle} & |\langle M \rangle| \\
\text{id} & \uparrow & \uparrow \text{id} \\
|\langle M \rangle| & \xrightarrow{|\theta \mu^\sim|} & |\langle M \rangle|
\end{array}
\]

Combining this diagram with the one above we obtain,

\[
\begin{array}{ccc}
X & \xrightarrow{\mu^\sim} & |\langle M \rangle| \\
\downarrow \mu^\sim & & \downarrow |\langle \mu^\sim \rangle| \\
X & \xrightarrow{|\langle M \rangle|}
\end{array}
\]

commuting up to homotopy for each $g \in G$. Thus $X \xrightarrow{\langle M \rangle}$ is the rational geometric realization desired.
2) Let $\mu_1$ be two $H$-conjugate homotopy actions on $X$ and let $\bar{\theta}_i: (X, \mu_1) \to (A_i, \nu_i) \ i = 1, 2$ be rational geometric realizations for the $\mu_1$. Now $G$ is finite, so we may construct the $G$-minimal models $M_1$, $M_2$ for $A_1$, and $A_2$ respectively. The $\bar{\theta}_i$ determine, up to homotopy, mappings of minimal models (also denoted $\bar{\theta}_i$), $\theta_i: M_1 \to M_2$, where $M$ is the chosen minimal model for $X$. $\bar{\theta}_i$ induces isomorphisms of cohomology and therefore, by minimality of $M_1$ and $M_2$, is an isomorphism.

The idea of the proof is straightforward. Recalling Corollary F, we see that in order to show that $(A_1, \nu_1)$ is rationally equivalent to $(A_2, \nu_2)$ it is enough to provide an isomorphism of minimal models $M_1 \to M_2$ which is homology equivariant. As we saw in the proof of 1), a homotopy action $\mu_1$ on $X$ induces a homotopy action $\theta \mu_1$ on $M$. Because $\bar{\theta}_1$ is a rational geometric realization, the following diagrams are homotopy commutative.

\[
\begin{align*}
\begin{array}{ccc}
M_1 & \xrightarrow{\theta_1} & M \\
\downarrow \nu_1 & & \downarrow \theta \mu_1 \\
M_1 & \xrightarrow{\nu_1} & M \\
\end{array}
\quad \quad \quad
\begin{array}{ccc}
M & \xleftarrow{\theta_2} & M_2 \\
\downarrow \theta \mu_2 & & \downarrow \nu_2 \\
M & \xleftarrow{\theta_2} & M_2 \\
\end{array}
\end{align*}
\]

Now $\mu_1$ and $\mu_2$ are assumed to be $H$-conjugate.

Therefore, there exists $T \in \text{Aut}(M)$ so that $T^* \cdot H_{\mu_1} \cdot T^{-1} =$
= Hμ₂. That is, the following diagram commutes at the level of cohomology,

\[
\begin{array}{ccc}
M & \xrightarrow{T} & M \\
\downarrow{\theta\mu_1} & & \downarrow{\theta\mu_2} \\
M & \xrightarrow{T} & M
\end{array}
\]

(Note that \(H\theta\mu_1 = H\mu_1\) and \(H\nu_1 = H\nu_1\).) Putting the three diagrams together we obtain a diagram

\[
\begin{array}{ccc}
M_1 & \xrightarrow{\theta_1} & M & \xrightarrow{T} & M & \xleftarrow{\theta_2} & M_2 \\
\downarrow{\theta\mu_1} & & \downarrow{\theta\mu_2} & & \downarrow{\nu_2} \\
M_1 & \xrightarrow{\theta_1} & M & \xrightarrow{T} & M & \xleftarrow{\theta_1} & M_2
\end{array}
\]

commuting on the induced cohomology diagram. Hence,

\[
H\nu_2 (\theta_2^{-1} \circ T \circ \theta_1)^{*} = (\theta_2^{-1} \circ T \circ \theta_1)^{*} \circ H\nu_1
\]

\[
H\nu_2 = (\theta_2^{-1} \circ T \circ \theta_1)^{*} \circ H\nu_1 \circ (\theta_2^{-1} \circ T \circ \theta_1)^{-1}\]

Thus, \(\theta_2^{-1} \circ T \circ \theta_1 : M_1 \to M_2\) is the desired homology equivariant isomorphism of minimal models.
3) Let \( f: X \to X \) be a rational homotopy equivalence. 
\( f \) induces a map of \( \mathbb{Q} \)-polynomial forms which we also denote by \( f: A(X) \to A(X) \). (Here we use the simpler notation \( A(X) \) as shorthand for the more proper \( A(S(X)) \).) The lifting property for minimal DGA's implies that there is a homotopy commutative diagram,

\[
\begin{array}{ccc}
M & \xrightarrow{F} & M \\
\Downarrow{\rho} & & \Downarrow{\rho} \\
A(X) & \xrightarrow{\rho \circ \rho} & A(X)
\end{array}
\]

where \( M \xrightarrow{\rho} A(X) \) is the minimal model of \( A(X) \) and \( F \) is unique up to homotopy. The homotopy commutativity of the diagram implies that since \( f \) and \( \rho \) induce isomorphisms of cohomology, then so also must \( F \). However, \( M \) is minimal, so \( F \) is an automorphism.

We have assumed that \( f \) is homotopy equivariant, so we obtain \( gfg^{-1} = f \) for each \( g \in G \). The map \( \rho \) is equivariant by the construction of the \( G \)-minimal model; hence, \( \varphi \rho g^{-1} = \rho \). Using these relations we compute,

\[
\begin{align*}
\rho \circ gFg^{-1} &= g\rho g^{-1}gFg^{-1} \\
&= g\rho Fg^{-1} \\
&= gf\rho g^{-1} \\
&= gf^{-1}\rho g^{-1} \\
&= f\rho .
\end{align*}
\]
Since \( F \) is, up to homotopy, the unique map satisfying \( \rho F = f \rho \), we must have \( gFg^{-1} = F \) for each \( g \in G \).

Hence, \( F \) is homotopy equivariant. Because \( G \) is finite, Theorem 2 4) implies that there exists \( \overline{F} \in \text{Aut}(M) \) with \([\overline{F}] = [F]\) and which is equivariant. Taking spatial and geometric realizations we obtain an equivariant map \(| \overline{F} \rangle : |\langle M \rangle| \rightarrow |\langle M \rangle|\). The diagram

\[
\begin{array}{c}
X \leftarrow |S(X)| \rightarrow |\langle M \rangle| \\
\downarrow f \downarrow \phantom{f} \phantom{f} \\
X \leftarrow |S(X)| \rightarrow |\langle M \rangle|
\end{array}
\]

displays the rational equivariant realization of length 2.

Q.E.D.

The rational geometric realization constructed in the proof of 1) is called the **canonical realization** of the homotopy action \( \mu \). The construction of the canonical realization makes clear that we do not require \( G \) to be finite as long as we know that \( \mu \) is rationally reductive. Hence, removing the restriction on the group, but restricting the type of action possible, allows the formulation of an analog of Theorem 3 1).
Under the same hypotheses we might consider the analog of Theorem 3 2). Note that in this case we are not assured of the existence of the $G$-minimal model. Hence, we have little hope of obtaining a result for arbitrary rational geometric realizations. However, we may formulate a result analogous to Theorem 2 2) by restricting our attention to canonical realizations. The proof is, of course, exactly the same with the realizations $(A_1, \nu_1)$ replaced by canonical realizations $(|\langle M \rangle|, |\langle \nu_1 \rangle|)$. We collect our observations in a theorem.

**Theorem 3':** Let $G$ be a group and $X$ a nice space. Then,

1) Any rationally reductive homotopy action of $G$ on $X$ has a rational geometric realization.

2) Any two $H$-conjugate rationally reductive homotopy actions of $G$ on $X$ have rationally equivalent canonical realizations.

Theorem 4 and Corollary 5 provide further generalizations of our basic situation. We proceed to prove these results now.
**Proof of Theorem 4:**

\[ \mu |_N \] is assumed to be rationally reductive, so we suppose \( \theta^* \hat{\mu}(N) \subset R \). In particular,

\[ [\mu_{g_1}] [\mu_{g_2}]^{-1} \in R. \]

Now, the projection \( p : h-Aut(M) \to H-Aut(M) \), when restricted to \( R \), is an isomorphism. But,

\[ p( [\mu_{g_1}] [\mu_{g_2}]^{-1} ) = \mu_{g_1}^* \mu_{g_2}^* = 1 \]

since \( \mu_{g_1} \), \( \mu_{g_2} \) induce the same map on cohomology. Hence,

\[ [\mu_{g_1}] [\mu_{g_2}]^{-1} = 1 \text{ since } p|_R \text{ is injective. That is,} \]

\[ [\mu_{g_1}] = [\mu_{g_2}] ; \mu_{g_1} \text{ is rationally homotopic to } \mu_{g_2}. \]

Q.E.D.

**Proof of Corollary 5:**

If \( G \) is compact, then \( \pi_0(G) \) is finite. If \( g \) and \( h \) are in the same component of \( G \), then \( \mu_g \) and \( \mu_h \) are homotopic. In particular, we obtain a factorization,
Since $\hat{\theta}_\mu$ factors through a finite group we see that $\mu$ is rationally reductive. Applying Theorem 4 completes the proof.

Q.E.D.
In [24] Sullivan observes that the construction of the minimal model for a Riemannian manifold $X$ may be made canonical. In other words, for a Riemannian manifold there are unique choices to be made in the various steps of the construction of the minimal model. It is the existence of a specific Riemannian metric on $X$ which restricts the choices uniquely. Now, the minimal model of an arbitrary space is determined only up to isomorphism. This lack of determinacy is bothersome because we only achieve some vestige of functoriality when we descend to the level of homotopy. The canonical model remedies this problem at least in the restricted situation of lifting isometries from a DGA with Riemannian structure to its model.

In this chapter we give an abstract version of a Riemannian metric in the context of differential graded algebras. Then, supplying the details that Sullivan omits in the observation referred to above, we show that a DGA with Riemannian structure admits a canonical minimal model.
We may also define a Riemannian structure within the context of differential graded algebras with group action (i.e., C-DGA's). In this situation we may provide an alternate construction of the equivariant minimal model. The restricted naturality provided by the canonical model plays the central role in this construction. Using the canonical model and in the spirit of the previous chapter, we may prove basic results characterizing certain DGA-maps by their effect on rational cohomology.

§ 1. Riemannian Structures and Canonical Models

A Riemannian metric on a smooth manifold provides an inner product, not only for tangent and cotangent vectors, but for the spaces of differential forms of any given degree. In the general DGA situation we shall explicitly construct these induced inner products.

Let $M$ be a smooth compact manifold without boundary. Also, let $d$ denote the exterior differential on forms, $d^*$ the adjoint of $d$ with respect to the inner product inherited from a chosen Riemannian metric for $M$ and $H^i$ the space of harmonic $i$-forms. Then we have,

Theorem (Kodaira-Hodge): There is an orthogonal direct sum decomposition,
\[ A^i(M) = dA^{i-1}(M) \oplus d^*A^{i+1}(M) \oplus H^i, \]

where \( A^i(M) \) = smooth \( i \)-forms on \( M \).

From this theorem we abstract the following definition.

**Def. 1:** Let \( A = (A^i,d_i) \) be a differential graded algebra. A has a **Riemannian structure** \( (A,\langle \rangle) \) if,

1) Each \( A^i \) is provided with a given inner product \( \langle \rangle \).

2) With respect to \( \langle \rangle \), for each \( i \) there are orthogonal decompositions,

\[ A^i = C^i \oplus Z^i, \quad Z^i = B^i \oplus H^i \]

where \( Z^i = \ker d_i \) and \( B^i = \text{im } d_{i-1} \).

**Prop. 1:** Any DGA may be given a Riemannian structure.

**Proof:**

Take vector space decompositions,

\[ A^i = C^i \oplus Z^i, \quad Z^i = B^i \oplus H^i \]

and choose inner products for \( C^i, B^i \) and \( H^i \). Define the inner product \( \langle \rangle \) on \( A^i \) by restricting to the chosen
inner products on \( C^i \), \( B^i \), and \( H^i \) and require that it be orthogonal with respect to the decomposition.

Q.E.D.

Now let \( V \) be a vector space with a given inner product \( \langle \cdot, \cdot \rangle \). In the classical situation of exterior algebra if \( \lambda = \alpha_1 \wedge \ldots \wedge \alpha_p \), \( \sigma = \beta_1 \wedge \ldots \wedge \beta_p \) are elements of \( \Lambda^p(V) \) with \( \alpha_i, \beta_i \in V \), then define
\[
\langle \lambda, \sigma \rangle = \det(\langle \alpha_i, \beta_j \rangle).
\]
Extending bilinearly, this definition defines an inner product on \( \Lambda^p(V) \). The properties of an inner product are verified using standard properties of the determinant. For example, switching \( \alpha_i \) and \( \alpha_j \) gives \( -\langle \lambda, \sigma \rangle \) on the left (by bilinearity) and \( -\det(\langle \alpha_i, \beta_j \rangle) \) on the right since \( \det \) is alternating. Also, the symmetry condition of \( \langle \cdot, \cdot \rangle \) is satisfied because \( \det(A) = \det(A^t) \) for any matrix \( A \) and its transpose \( A^t \).

Recall that \( L_n(V) \) denotes the graded algebra freely generated by the vector space \( V \) in dimension \( n \). As a DGA \( L_n(V) \) is provided with the differential \( d = 0 \). If \( n \) is odd, then we are essentially in the classical situation described above. Hence, the inner product on \( V \) may be extended to \( L_n(V) \) by defining
\langle \lambda, \sigma \rangle = \det(\langle \alpha_i, \beta_j \rangle)

with \( \alpha_i, \beta_j \in V \) and extending bilinearly.

The case where \( n \) is even is similar, but we may not use the determinant. \( L_n(V) \) is a symmetric algebra here and, thus switching \( \alpha_i \) and \( \alpha_j \) in \( \lambda \) has no effect on \( \lambda \). Of course, such a switch would still place a negative in front of \( \det(\langle \alpha_i, \beta_j \rangle) \). Hence, we desire a function similar to the determinant, but which is not alternating. Such a function is the so-called permanent of a matrix. If \( A \) is a \( p \times p \) matrix, then

\[
\text{perm}(A) = \sum_{\sigma \in S_p} a_{1\sigma(1)}a_{2\sigma(2)} \cdots a_{p\sigma(p)} .
\]

This definition is similar to that of the determinant, but the factor \( \text{sgn}(\sigma) \) in each term is excluded. The permanent is invariant under permutations of rows and columns. Hence, switching \( \alpha_i \) and \( \alpha_j \) in \( \lambda \) has no effect on \( \lambda \). Also, the permanent is multilinear and \( \text{perm}(A) = \text{perm}(A^t) \). Properties of the permanent different from those of the determinant are given in [19]. Using these properties listed above, however, we obtain an inner product on \( L_n(V) \) (\( n \) even) by defining,

\[
\langle \lambda, \sigma \rangle = \text{perm}(\langle \alpha_i, \beta_j \rangle)
\]
where $\alpha_i, \beta_j \in V$ and $\lambda = \alpha_1 \ldots \alpha_p$ and $\sigma = \beta_1 \ldots \beta_p$.

Now, given inner products on vector spaces $V$ and $W$ we must explain how an inner product is obtained on $V \otimes W$. Let $v_1 \otimes w_1$, $v_2 \otimes w_2$ be simple elements of $V \otimes W$ and suppose the inner products on $V$ and $W$ are denoted $(\cdot, \cdot)$ and $(\cdot, \cdot)$ respectively. Define

$$[v_1 \otimes w_1, v_2 \otimes w_2] = (v_1, v_2) \cdot (w_1, w_2).$$

Extending bilinearly, $[\cdot, \cdot]$ defines an inner product on $V \otimes W$. If $V_1, \ldots, V_n$ are vector spaces with respective inner products $(\cdot, \cdot)_i$, $i = 1, \ldots, n$, then for simple elements $v_1 \otimes \ldots \otimes v_n$, $w_1 \otimes \ldots \otimes w_n$ the multilinear extension of $$(v_1 \otimes \ldots \otimes v_n, w_1 \otimes \ldots \otimes w_n) = \prod_i (v_i, w_i)_i$$ defines an inner product on $V_1 \otimes \ldots \otimes V_n$.

For the sake of exposition we now describe the case $A = L(V \otimes W) = L_n(V) \otimes L_m(W)$. The inner products on $V$ and $W$ provide inner products for $L_n(V)$ and $L_m(W)$. Because $L_n(V)$ and $L_m(W)$ are freely generated (and hence, so is $A$) any monomial in $A$ is a product of elements of $V$ and $W$. Say that a monomial is in standard form if all the factors from $V$ precede all the factors from $W$. Of course, any monomial is equal to one in standard form.
If \( x = v_1 \cdots v_p \cdot w_1 \cdots w_q \), \( y = \overline{v}_1 \cdots \overline{v}_r \cdot \overline{w}_1 \cdots \overline{w}_s \), then define an inner product by,

\[
\langle x, y \rangle = \begin{cases} 
0 & \text{if } r \neq p, s \neq q \\
= \langle v_1 \cdots v_p, v_1 \cdots v_p \rangle \cdot \langle w_1 \cdots w_q, w_1 \cdots w_q \rangle & \text{if } r = p, s = q.
\end{cases}
\]

Extending bilinearly to sums of monomials provides the inner product on \( A \). It is sufficient to define \( \langle \cdot \rangle \) on monomials in standard form since there is an essentially unique standard form. The only permutations of factors possible is within \( V \) or \( W \). Then, depending on whether the dimension is odd or even respectively, the determinant or permanent handles the switch of factors.

The same arguments apply in the more general case \( A = L(V_1 \oplus V_2 \oplus V_3 \oplus \ldots) \), but the notation becomes cumbersome. If \( V_1, V_2, V_3 \cdots \) is a chosen ordering of vector spaces, then again any monomial may be put in standard form. That is, factors belonging to the same \( V_1 \) may be grouped together in position corresponding to the place of \( V_1 \) in the ordering. An inner product may then be defined as in the simpler case of \( L(V \oplus W) \).

The previous discussion is crucial to the construction of a canonical model for a DGA with Riemannian structure.
It is to this construction that we now turn; in order to simplify exposition we assume that $A$ is a 1-connected DGA with Riemannian structure. As usual, suppose that $A$ is of finite type.

**Theorem A:** If $A$ is a 1-connected DGA with Riemannian structure, then there exists a canonical minimal model $M$ with canonical map $\rho: M \to A$ inducing isomorphisms on cohomology.

**Proof:**

Define $\rho: Q \to A^0$ by $\rho(1) = 1$ and induce the proper inner product on $Q$ via the isomorphism $\rho^*: Q \cong H^0(A) \cong H^0 \subseteq A^0$.

Now, as an inductive hypothesis assume that $M(n-1)$ has been constructed canonically with $\rho_{n-1}: M(n-1) \to A$ (also canonical) inducing isomorphisms in cohomology up to dimension $n-1$ and an injection is cohomology in dimension $n$. Also assume that $M(n-1)$ has a canonically constructed inner product.

In order to construct $M(n)$ we must find canonical splittings for the three surjections:

(i) $\mathbb{Z}^{n+1}(M(n-1)) \to H^{n+1}(M(n-1))$

(ii) $\mathbb{Z}^{n}(A) \to H^{n}(A)$

(iii) $A \overset{d}{\to} B^{n+1}(A)$.
(i) Since $M(n-1)$ has an inner product we have a canonical orthogonal sum $W \oplus B^{n+1}(M(n-1)) = Z^{n+1}(M(n-1))$. The projection $\pi: Z^{n+1} \rightarrow H^{n+1}(M(n-1))$ induces an isomorphism $W \cong H^{n+1}(M(n-1))$, giving $H^{n+1}(M(n-1))$ an inner product structure. Hence we obtain an orthogonal sum $(\ker \rho^*_n)^\perp \oplus B^{n+1}(M(n-1))$. Let $V_2 = (\ker \rho^*_n)$. Now $\pi_W^{-1}$ gives a canonical splitting $H^{n+1}(M(n-1)) \rightarrow W \oplus Z^{n+1}$. Call this composition $\alpha$ and note that $\alpha(V_2) \oplus \rho^*_n \oplus B^{n+1} = Z^{n+1}$ may be obtained as an orthogonal sum using the inner product.

(ii) Using the Riemannian structure of $A$ we have an isomorphism $h: H^n(A) \rightarrow H^n$. Now $i: H^n \rightarrow Z^n(A)$ and we may then define $\beta = i \circ h: H^n(A) \rightarrow Z^n(A)$ which splits the projection. The map $h$ is canonical since there is a unique "harmonic" element in each cohomology class. Hence the splitting is canonical.

(iii) $A^n = C^n \oplus B^n \oplus H^n$ using the Riemannian structure. Now $\ker d_n = B^n \oplus H^n$, so $d_n|_{C^n}$ is an isomorphism onto $B^{n+1}(A)$. Hence, if $a \in B^{n+1}$, then there is a unique $c_a \in C^n$ such that $dc_a = a$. Therefore define a splitting $\Delta: B^{n+1} \rightarrow A^n$ by $\Delta(a) = c_a$. 
Now, the inner product on $H^n(A)$ induced from that on $H^n$ gives a (canonical) orthogonal sum, $H^n(A) = \text{im } \rho_n^* \oplus V_1$. Now use the inner products on $V_1$ and $V_2$ to obtain an inner product on $V_1 \oplus V_2$ so that the sum is orthogonal.

Now construct the elementary extension,

$$M(n) = M(n-1) \otimes \tau L_n(V_1 \oplus V_2)$$

where $\tau : L_{n+1}(V_1 \oplus V_2) \to M(n-1)$ is defined by

$$\tau\Big|_{V_1} = 0 \quad \text{and} \quad \tau\Big|_{V_2} = \alpha \Big|_{V_2}.$$

Recall that $d(m \otimes 1) = dm \otimes 1$ and $d(1 \otimes v) = \tau(v) \otimes 1$. Hence the differential on $M(n)$ is canonical since $\alpha$ is. Note that $d\Big|_{V_2}$ is an isometry because the inner product structure on $H^{n+1}(M(n-1))$ was induced via the projection $\pi \Big|_W$. Note that to satisfy the induction hypothesis we must have an (canonical) inner product on $M(n)$. This follows, however, from the discussion on extending inner products from $V$ to $L(V)$.

Now we proceed to define $\rho_n$ canonically. The definitions of $\rho_n$ repeat the ones in the unstructured case, so induce cohomology isomorphisms.
Since \( \rho_n \) is constructed entirely from canonical maps, it too is canonical. Hence the inductive hypotheses are fulfilled for \( M(n) \) and \( \rho_n : M(n) \to A \). Continuing inductively, we are done.

Q.E.D.

Remark 2: \( H^i(A) \) is finite dimensional since \( A \) is of finite type. Therefore, the construction implies that \( M^i \) is finite dimensional for each \( i \). Hence, the induced inner product on \( M \) actually provides a Riemannian structure for \( M \).

Recall that the usual passage from a DGA to its minimal model is not functorial. There are two main problems:

1) the minimal model is determined only up to isomorphism, and
2) maps between DGA's may be lifted to their minimal models only up to homotopy. Now consider the category \( \langle \cdot \rangle \)-DGA, whose objects are DGA's with Riemannian structure and whose morphisms are isometries. The construction of the canonical model for a Riemannian DGA overcomes 1). The
following proposition handles 2) and makes clear that the association of a canonical model to a Riemannian DGA is, in fact, a functor from \( \langle \gamma \rangle \)-DGA to \( \langle \gamma \rangle \)-DGA.

**Prop. 3:** Let \( A \) have a Riemannian structure and suppose \( f : A \rightarrow A \) is an isometry. Then there exists a canonical isometry \( \tilde{f} : M \rightarrow M \) so that the following diagram strictly commutes:

\[
\begin{array}{ccc}
M & \xrightarrow{\tilde{f}} & M \\
\downarrow{\rho} & & \downarrow{\rho} \\
A & \xrightarrow{f} & A
\end{array}
\]

**Proof:**

We employ the splittings derived earlier. Clearly \( \tilde{f} \) exists on \( Q \). Since \( M^k = H^k \), \( \tilde{f} \) exists on the first non-zero cohomological dimension \( k \) as well.

As an inductive hypothesis assume that \( \tilde{f}_{n-1} : M(n-1) \rightarrow M \) has been constructed with \( \rho \circ \tilde{f}_{n-1} = f \circ \rho_{n-1} \).

Define \( \tilde{f}_n \) as follows:

\[
\begin{align*}
\tilde{f}_n \bigg|_{M(n-1)} &= \tilde{f}_{n-1} \\
\tilde{f}_n \bigg|_{V_1} &= h^{-1} \cdot f \cdot h \\
\tilde{f}_n \bigg|_{V_2} &= \pi \circ \tilde{f}_{n-1} \circ \alpha
\end{align*}
\]
where \( h : H^n(A) \to H^n \), \( \alpha : H^{n+1}(M(n-1)) \to Z^{n+1}(M(n-1)) \), 
\( \pi : Z^{n+1}(M(n-1)) \to H^{n+1}(M(n-1)) \). Hence, \( f_n : M(n) = M(n-1) \otimes_{\mathbb{L}} L_n(V_1 \otimes V_2) \to M \).

We must show that \( \rho \circ f_n = f \cdot \rho_n \). Thus,

\[
\rho \circ f_n \bigg|_{V_1} = \rho \circ h^{-1} \circ f \circ h \bigg|_{V_1}
\]

\[
= h \circ h^{-1} \circ f \circ h \bigg|_{V_1} \quad \text{since} \quad \rho \bigg|_{V_1} = h \bigg|_{V_1}
\]

\[
= f \circ h \bigg|_{V_1}
\]

\[
= f \circ \rho_n \bigg|_{V_1} \quad \text{since} \quad \rho_n \bigg|_{V_1} = h \bigg|_{V_1}
\]

\[
\rho \circ f_n = \rho \circ \pi \circ f_{n-1} \circ \alpha \bigg|_{V_2}
\]

\[
= \Delta \circ \rho_{n-1} \circ \alpha \circ \pi \circ f_{n-1} \circ \alpha \bigg|_{V_2}
\]

\[
= \Delta \circ \rho_{n-1} \circ f_{n-1} \circ \alpha \bigg|_{V_2}
\]

\[
= \Delta \circ f \circ \rho_{n-1} \circ \alpha \bigg|_{V_2}
\]

\[
= f \circ \Delta \circ \rho_{n-1} \circ \alpha \bigg|_{V_2}
\]

\[
= f \circ \rho_n \bigg|_{V_2}
\]
In this computation we have used the facts that, restricted to \( V_2 \), \( \alpha \cdot \pi = \text{id} \) and \( f \cdot \Delta = \Delta \cdot f \). Hence \( \rho_n \cdot f_n = f \cdot \rho_n \).

Q.E.D.

In several obvious places in the proof of the proposition we have used the following,

**Observation 4:** If \( f \) is an isometry of a DGA \( A \) with Riemannian structure, then \( f \) preserves the orthogonal sum decomposition of \( A^i \) for each \( i \).

**Proof:**

Since \( f \) is a DGA map, \( Z^i \) and \( B^i \) are clearly preserved. We show that \( C^i \) and \( H^i \) are preserved. First, let \( f(x) = z \in Z^i \). Then \( f(dx) = df(x) = dz = 0 \). \( f \) is an isomorphism so \( dx = 0 \). Hence, for each \( z \in Z^i \), there exists \( x \in Z^i \) with \( f(x) = z \). Now let \( c \in C^i \). Then, for \( z \in Z^i \)

\[
\langle fc, z \rangle = \langle fc, fx \rangle = \langle c, x \rangle = 0.
\]

Hence \( fc \) is orthogonal to \( Z^i \) which implies that \( fc \in C^i \). A similar proof holds for \( H^i \).

Q.E.D.
From the definition of \( \tilde{f}_n \) we see that the only new information is contained in \( \tilde{f}_n \mid V_1 \). Now \( V_1 \) is essentially a subspace of \( H^n \), so we see that \( \tilde{f} \) is determined by the effect of \( f \) on \( H^n \), the "harmonic" elements in the decomposition of \( A^n \). Equivalently, \( \tilde{f} \) is determined by \( f^* \), the induced map on cohomology. In a later section we shall study this phenomenon in a more detailed fashion.

**Remark 5:** If \( \tilde{f} \) and \( \tilde{g} \) are the canonical lifts of \( f \) and \( g \) respectively, then it is easy to verify that the canonical lift of \( (\tilde{f} \circ \tilde{g}) \) is simply \( \tilde{f} \circ \tilde{g} \). Therefore, if a group \( G \) acts on \( (A, \langle \cdot, \cdot \rangle) \) so that each \( g \in G \) acts as an isometry, then the group action may be lifted to the canonical minimal model.

**Remark 5** allows us to connect the notion of Riemannian structure to the previous chapter. We consider this in the next section.
§2. G-Riemannian Structures

In this section we describe an extension of the notion of Riemannian structure within the framework of differential graded algebras with group action. This extension will generalize to the case of DGA’s the particular situation of a group acting as isometries of a Riemannian manifold.

Def. 2: Let G be a group acting as automorphisms of a DGA A. A is said to have a G-Riemannian structure (A, ⟨⟩) if 1) (A, ⟨⟩) is a Riemannian structure on A and 2) ⟨⟩ is G-invariant (i.e., for each g ∈ G, ⟨gx, gy⟩ = ⟨x, y⟩).

Prop. 6: If A has a G-Riemannian structure, then C^i, B^i, and H^i are G-stable for each i.

Proof:

By 2) of the definition of a G-Riemannian structure we see that each g ∈ G acts as an isometry of A. By Observation 4 each g then preserves the orthogonal sum decomposition.

Q.E.D.
It is not possible, in general, to provide a G-DGA with a G-Riemannian structure. In certain restricted cases this is possible, however.

Def. 3: An action of a group G on a DGA with Riemannian structure \((A, \langle \rangle)\) is said to be R-semisimple if,

1) For each \(i\), the restricted homomorphism \(G \to \text{Aut}(A^1)\) is a direct sum of irreducible representations,

2) Each irreducible representation is unitary with respect to \(\langle \rangle\).

Prop. 7: Let \((A, \langle \rangle)\) be a DGA with Riemannian structure. If \(G\) acts R-semisimply on \((A, \langle \rangle)\), then there exists a G-Riemannian structure on \(A\).

Proof:

By 1) of Def. 3 we may find a decomposition into \(G\)-stable subspaces, \(A^1 = C^1 \oplus B^1 \oplus H^1\). Here, of course, the direct sums may not be orthogonal. Also by 1), \(C^1 = \oplus \xi_{\alpha}\), \(B^1 = \oplus \eta_{\beta}\) and \(H^1 = \oplus \rho_{\gamma}\) as direct sums of irreducible representations.

Define a new inner product on \(A\) by:

\[
(x,y) = \langle x,y \rangle \quad \text{if} \quad x,y \in \xi_{\alpha}, \quad x,y \in \eta_{\beta}, \quad x,y \in \rho_{\gamma},
\]

\[
= 0 \quad \text{otherwise}.
\]
The effect of this definition is to keep the inner product unchanged on the irreducible representations \( \xi_\alpha, \eta_\beta \) and \( \rho_\gamma \) and to make all such direct sums orthogonal. Note that by 2) of Def. 3, \((\cdot,\cdot)\) is G-invariant. Hence \( A^i = C^i \oplus B^i \oplus H^i \) is a G-Riemannian structure for \( A \).

Q.E.D.

Corollary 8: If \( G \) is a finite group acting on a DGA \( A \), then \( A \) has a G-Riemannian structure.

**Proof:**

Choose a Riemannian structure on \( A \) (by virtue of Prop. 1), \((A,\langle \cdot,\cdot \rangle)\). Define a new inner product by,

\[
(x,y) = \sum_{g \in G} \langle gx,gy \rangle.
\]

Clearly, \((\cdot,\cdot)\) is G-invariant; hence, each \( g \in G \) is unitary. Since \( G \) is finite each representation \( G \to \text{Aut}(A^i) \) is a direct sum of irreducible representations and, since each \( g \in G \) is unitary, each irreducible representation is unitary with respect to \((\cdot,\cdot)\). Thus \( G \) acts R-semisimply on \((A,\langle \cdot,\cdot \rangle)\). By Prop. 7 we are done.

Q.E.D.
As a corollary to Theorem A, Remark 2, Prop. 3 and Remark 5, we have an alternate construction of the equivariant minimal model.

Prop. 8: If \((A, \langle \cdot, \cdot \rangle)\) is a \(G\)-Riemannian structure, then \(M\) is a (canonical) \(G\)-minimal model for \(A\) with an induced \(G\)-Riemannian structure.

§ 3. Some Results on Liftings

This section is devoted to proving two basic propositions about the liftings constructed in Prop. 2.

Prop. 9: Let \((A, \langle \cdot, \cdot \rangle)\) be a DGA with Riemannian structure and suppose \(f, g: A \to A\) are isometries. Then, \(\tilde{f} = \tilde{g}\) if and only if \(f \big|_{H^i} = g \big|_{H^i}\) for each \(i\).

Prop. 10: \(\tilde{f} = \tilde{g}\) if and only if \(f^* = g^*\).

We may prove these results by simply inspecting the definition of \(\tilde{f}\) in Prop. 2. However, our techniques will further exemplify the usefulness of the algebraic group structure of \(|\text{Aut}(M)|\).

Let \(M\) be a minimal DGA with Riemannian structure \((M, \langle \cdot, \cdot \rangle)\). From the previous chapter recall the definitions
of $\text{Aut}(M)$ and $\text{H-Aut}(M)$. Also, recall that both of these groups have algebraic group structures with the same reductive parts. Since $M$ has a Riemannian structure we may consider the algebraic subgroup of isometries $\text{Isom}(M) \subset \text{Aut}(M)$. Define $\text{H-Isom}(M) = \text{H-Aut}(M) \cap \text{Isom}(\text{H}(M))$, where the inner product on $\text{H}^1(M)$ is induced from that on $\text{H}^1$ via the isomorphism $h: \text{H}^1(M) \to \text{H}^1$.

Prop. 11: $\text{Isom}(M)$ is a reductive subgroup of $\text{Aut}(M)$.

Proof:

Recall that an algebraic group is reductive if its unipotent radical reduces to the identity. Also recall that the unipotent radical is the (unique) largest connected normal unipotent subgroup of the algebraic group in question. We show that an isometry cannot be unipotent; hence, $\text{Isom}(M)$ contains no unipotent subgroup except for the identity subgroup.

Let $U = I + N$, with $N$ nilpotent. Assume that $U$ is an isometry; hence $U^{-1} = U^t$. Now, $U^t = I + N^t = U^{-1}$ and,

\[ I = UU^t \]

\[ = (I+N)(I+N^t) \]

\[ = I+N+N^t+NN^t. \]
Also,

\[ I = U^t U \]

\[ = (I+N^t)(I+N) . \]

\[ = I+N^t+N+N^tN . \]

Comparing, we obtain \( NN^t = N^tN \). Now \( N \) is nilpotent, so suppose \( N^k = 0 \), \( N^{k-1} \neq 0 \). Then, since \( N \) and \( N^t \) commute,

\[ (NN^t)^k = N^k(N^t)^k = 0 . \]  

Hence \( NN^t \) is nilpotent. This implies that every eigenvalue of \( NN^t \) is 0. Thus \( \text{tr}(NN^t) = 0 \).

Now let \( N = (a_{ij}) \), \( N^t = (b_{ij}) \) with \( b_{ij} = a_{ji} \) be matrix representations of \( N \) and \( N^t \). Also let

\( NN^t = (c_{ij}) \). Then,

\[ c_{ij} = \sum_k a_{ik}b_{kj} = \sum_k a_{ik}a_{jk} . \]

In particular,

\[ c_{ii} = \sum_k a_{ik}a_{ik} = \sum_k a_{ik}^2 . \]

Therefore, \( \text{tr}(NN^t) = \sum a_{ik}^2 = 0 \). Thus, \( a_{ik} = 0 \) for all \( i,k \). Hence \( N = 0 \) and therefore \( U = I \). Hence any unipotent isometry is the identity.

Q.E.D.
Since $\text{Isom}(M)$ is reductive it is contained in a maximal reductive subgroup $R$. We may choose this subgroup as the reductive part of the Levi Decomposition of $\text{Aut}(M)$ and $H-\text{Aut}(M)$.

**Corollary 12:** $\text{Isom}(M) \to H-\text{Isom}(M)$ is an injection.

**Proof:**

Consider the following commutative diagram:

\[
\begin{array}{c}
\text{Aut}(M) \\
\uparrow \sim \\
R \\
\uparrow \\
\text{Isom}(M)
\end{array} \quad \begin{array}{c}
\longrightarrow \\
\sim \\
\longrightarrow \\
\longrightarrow \\
\text{H-Isom}(M)
\end{array}
\]

Since the middle horizontal map is an isomorphism the restriction to $\text{Isom}(M)$ is an injection.

Q.E.D.

We now give a simple proof of Prop. 10. The proof of Prop. 9 then consists merely of identifying $H^i(M)$ with $H^i$ for each $i$. 
Proof of Prop. 10:

Suppose \( f, g : A \to A \) are isometries with \( f^* = g^* \).

Let \( \tilde{f} \) and \( \tilde{g} \) be the respective lifts to the canonical minimal model \( M \) of \( A \). \( \tilde{f} \) and \( \tilde{g} \) are isometries of \( M \) and \( \tilde{f}^* = \tilde{f}^* = g^* = \tilde{g}^* \). Corollary 12 then implies that \( \tilde{f} = \tilde{g} \).

Q.E.D.
CHAPTER III

SOME RESULTS ON FORMALITY AND COFORMALITY IN DIFFERENTIAL GRADED ALGEBRAS

In [5] Deligne, Griffiths, Morgan and Sullivan introduced the concepts of formal differential graded algebra and formal space. There it was shown that any compact Kähler manifold is formal. Intuitively, this means that the R-homotopy type of a compact Kähler manifold is determined by its DeRham cohomology algebra. Sullivan [24] and (independently) Neisendorfer and Miller [16] then showed that formality over R implies formality over Q. Of course, at least for nilpotent Kähler manifolds, the Q-homotopy type has a well defined representation as a CW homotopy type. Hence, it was shown that the rationalization of a nilpotent compact Kähler manifold is a formal consequence of its rational cohomology algebra.

Formality was studied in more depth by Neisendorfer and Miller [16] from the point of view of both differential graded algebras and (following Quillen [18]) differential graded Lie algebras. It should be noted
that in this paper also appeared the notion of co-formality accompanied by several basic results about it.

Of course, the next step in the study of formality was the extension of the concept to maps. This was accomplished in several papers (see [29] or [8]). It was then shown by Felix and Tanré [8] that the mapping cone of a formal map is a formal space. Their method of proof employed Quillen’s Lie algebra model and the Lie algebra condition for formality expressed by Neisendorfer and Miller; that is, that the differential of the minimal Lie algebra model is quadratic.

In this chapter we present another proof of the Felix-Tanré result using differential graded algebras. Also, a definition of coformality is given which may then be easily generalized to a version for maps. Then, using our proof of the mapping cone result as a model, we prove the analogous result that the homotopy fibre of a coformal map is a coformal space.

The DGA proof of the Felix-Tanré Theorem relies upon a suitable definition of weak cofibration in the context of differential graded algebras. The first section is devoted to a study of such weak cofibre sequences. A dual notion is defined in §3 and used to prove the dual
of the mapping cone result. Weak cofibrations find later application in the study of homology decompositions of minimal DGA's. The homology decomposition is defined in § 5 where it is shown to prove a new technique for the study of formality. In this chapter we assume that all spaces and DGA's are 1-connected.

§ 1. Weak Cofibrations in Differential Graded Algebras

Consider the following situation. Suppose that the following diagram of cochain complexes is cochain homotopy commutative.

\[
\begin{array}{ccc}
M & \xrightarrow{\phi} & A \\
\downarrow{g} & & \downarrow{f} \\
N & \xrightarrow{\theta} & B
\end{array}
\]

Let \( P \) and \( C \) denote the respective algebraic mapping cones of \( g \) and \( f \). That is,

\[
p^n = M^n \oplus N^{n-1}, \quad c^n = A^n \oplus B^{n-1}
\]

with respective differentials,
\[ d_p(m,n) = (-d_m, d_n + g(m)) \]

\[ d_c(a,b) = (-d_a, d_b + f(a)) \]

Let \( H \) be a (fixed) cochain homotopy making the diagram homotopy commutative. \( H \) satisfies,

\[ dH + Hd = \theta^*g - f \cdot \phi \]

Define \( \lambda: Z^1(P) \to Z^1(C) \) by,

\[ \lambda(m,n) = (\phi(m), \theta(n) + H(m)) \]

Then \( \lambda \) induces \( \Lambda = \lambda^*: H^*(g) = H^*(P) \to H^*(C) = H^*(f) \). In particular, note that this construction may be applied to a DGA-homotopy commutative diagram of DGA's (since a DGA homotopy induces a cochain homotopy). Restricting the situation further, let \( N \) be the DGA with \( Q \) in degree 0 and all other degrees trivial. Denote this DGA by \( Q \). It is easy to see that, in this case, \( H^*(g) = H^*(M) \). Said differently, if \( M \to A \to B \) is a sequence of minimal DGA's with \( f \cdot \phi = 0 \), then we obtain \( \Lambda: H^*(M) \to H^*(f) \).

**Def. 1:** A sequence of minimal DGA's \( M \to A \to B \) is called a weak cofibration (or weak cofibre sequence) if,
1) $f \ast \phi = 0$ and
2) $\Lambda$ is an isomorphism.

The following two algebraic results enable us to give the most important example of a weak cofibration of DGA's.

**Lemma 1:** Let $A \to B \to C$ be a sequence of cochain complexes. Then,

1) If $\phi$ induces isomorphisms of cohomology, then $H^*(C_\phi) \simeq H^*(C_{\varnothing})$, where $C_\phi$, $C_{\varnothing}$ are the respective algebraic mapping cones.

2) If $\Theta$ induces isomorphisms of cohomology, then $H^*(C_\phi) = H^*(C_{\varnothing})$, where $C_\phi$, $C_{\varnothing}$ are the respective algebraic mapping cones.

**Proof:**

We prove 1) only. 2) follows immediately. Consider the commutative diagram,

\[
\begin{array}{ccc}
A & \xrightarrow{\phi} & C \\
\downarrow{\phi} & & \downarrow{\lambda} \\
B & \xrightarrow{\Theta} & C \\
\end{array}
\]
Define $\lambda$ by, $\lambda(a,c) = (\phi(a), c)$. Now,

\[ d\lambda(a,c) = d(\phi(a), c) \]
\[ = (-d\phi(a), dc + \theta\phi(a)) \]
\[ = \lambda(-da, dc + \theta\phi(a)) \]
\[ = \lambda d(a,c) . \]

Thus $\lambda$ is a cochain map. The 5-lemma applied to the long exact cohomology sequences associated to the rows then shows that $\lambda^*$ is an isomorphism.

Q.E.D.

Lemma 2: If $\phi$ and $\theta$ induce cohomology isomorphisms in the diagram (*), then $\Lambda$ is an isomorphism.

Proof:

It is well known that two cochain homotopic maps induce the same map on the cohomology of their respective mapping cones. Hence, for (*) we have, $H^*(g) \cong H^*(\theta g) \cong H^*(f\phi) \cong H^*(f)$, where the first and third isomorphisms follow from Lemma 1 and the second from the opening remarks. It is not hard to see that the isomorphism displayed is in fact $\lambda$ .

Q.E.D.
Prop. 3: If $X \rightarrow Y \rightarrow C_f$ is a mapping cone sequence of spaces, then $M(C_f) \rightarrow M(Y) \rightarrow M(X)$ is a weak cofibre sequence of DGA's.

Proof: By standard results of minimal model theory, $j^* f = 0$ implies that $M(f) \cdot M(j) = 0$. Recall that the integration map $I: A(\cdot) \rightarrow C(\cdot)$ from $\mathbb{Q}$-polynomial forms to rational cochains is a mapping of cochain complexes inducing isomorphisms of cohomology. We have a homotopy commutative diagram of cochain complexes,

```
\begin{align*}
  & M(Y) \rightarrow A(Y) \rightarrow C(Y) \\
  & M(f) \downarrow A(f) \downarrow C(f) \\
  & M(X) \rightarrow A(X) \rightarrow C(X).
\end{align*}
```

We then obtain a cochain homotopy commutative diagram,
Then,

\[ H^*(M(C_f)) = H^*(C(C_f)) \]

\[ = H^*(C(f)) \]

\[ = H^*(M(f)) \text{ by Lemma 2.} \]

Q.E.D.

Now consider the following diagram of spaces,

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{j} & & \downarrow{g} \\
Z & \xleftarrow{h} & C_f
\end{array}
\]

Mapping Cone Extension Theorem (MCET): The map \( h \) exists if and only if \( g \circ f \neq 0 \). A basic result that will prove of use in later sections is the

Prop. 4: Let \( M \rightarrow A \rightarrow B \) be a weak cofibre sequence. Then, for \( U \) minimal, any \( g: U \rightarrow A \) with \( f \circ g = 0 \) has a lifting \( \beta: U \rightarrow M \) such that \( \phi \circ \beta = g \).

Proof:

Take spatial and geometric realizations to obtain the following diagram.
Now \( f \phi = 0 \) since \( M + A + B \) is a weak cofibration. Hence \( |\langle \phi \rangle| \cdot |\langle f \rangle| = 0 \). By the MCET, \( h \) exists. Since \( h \) induces an isomorphism of cohomology by the 5-lemma, 
\[
|\langle M \rangle| \cong C|\langle f \rangle|.
\]
The MCET also implies the existance of \( h' \). Define \( b = h' \cdot h^{-1} : |\langle M \rangle| \to |\langle U \rangle| \). Taking minimal models we obtain a homotopy commutative diagram,

\[
\begin{array}{ccc}
M & \xrightarrow{\phi} & A \\
\downarrow{B} & & \downarrow{f} \\
U & \xleftarrow{g} & B \\
\end{array}
\]

Note: It is also possible to prove Prop. 4 directly via obstruction theory.
§ 2. The Mapping Cone of a Formal Map

Recall that a differential graded algebra $A$ is said to be *formal* if there is a quasi-isomorphism,

$$\theta_A : M_A \to H^*(M_A),$$

from the minimal model of $A$ to its cohomology. $\theta_A$ is said to be a formalization for $M_A$.

Various characterizations of formality have been given (see [5], [1] and [16]). One that will be important for later applications is the following result of Sullivan [24] (see also, Shiga [23]).

**Theorem 5:** The following are equivalent:

1) $M$ is formal

2) $\text{Aut}(M) \xrightarrow{\pi} \text{Aut}(H^*(M))$ is surjective

3) There exists a grading automorphism in the image of $\pi$.

Recall that $\text{Aut}(\ )$ refers to the DGA automorphisms of $M$ and $H^*(M)$. Also recall that a grading automorphism $\tau : H^*(M) \to H^*(M)$ is an automorphism of the form,

$$\tau(x) = t |x|_x$$

where $|x|$ is the degree of $x$ and $t$ is a rational number not equal to $\pm 1$. 
We now relativize the notion of formality to maps in the following definition.

**Def. 2:** Let \( f : M \to N \) be a mapping of minimal DGA's. The map \( f \) is said to be formal if there exist formalizations \( \theta_M, \theta_N \) and a homotopy commutative diagram,

\[
\begin{array}{ccc}
M & \xrightarrow{f} & N \\
\downarrow^{\theta_M} & & \downarrow^{\theta_N} \\
H^*(M) & \xrightarrow{f^*} & H^*(N)
\end{array}
\]

**Remark 6:**

1) Note that this definition requires \( M \) and \( N \) to be formal.

2) Since \( M \) and \( N \) are formal, the lifting property for minimal algebras implies that the diagram

\[
\begin{array}{ccc}
M & \xrightarrow{\tilde{f}^*} & N \\
\downarrow^{\theta_M} & & \downarrow^{\theta_N} \\
H^*(M) & \xrightarrow{f^*} & H^*(N)
\end{array}
\]

may always be filled in by an \( \tilde{f}^* \) with \( \theta_N \tilde{f}^* = f^* \theta_M \). The map \( \tilde{f}^* \), however, may not be homotopic to \( f \). This is
the essence of the definition; \( f \) is formal if it is contained in the homotopy class of the unique (up to homotopy) lifting \( \tilde{f}^* \). In this sense, \( f \) is determined by its effect on cohomology.

3) A map \( f: A \to B \) between not necessarily minimal DGA's is called formal if the homotopy-unique lifting \( \tilde{T}: \mathcal{M}_A \to \mathcal{M}_B \) is formal. This situation is referred to by some authors by saying that \( f \) and \( f^* \) have the same minimal model.

4) Although Vigue-Poirrier initially defined formality of maps in [29], the first in-depth investigation was undertaken by Felix-Tanre in [8].

As stated in the introduction, the purpose of this section is to give a proof of the Felix-Tanre mapping cone result totally within the framework of DGA's. We list the result as *.

**Theorem 7 (Felix-Tanre):** If \( f: X \to Y \) is a formal map between 1-connected spaces, then the mapping cone \( C_f \) is a formal space.

Before we can prove this theorem we must discuss several technical aspects of the mapping cone sequence as well as describe a certain construction in DGA's.

* Recall Def. 6' of Chapter 0.*
Lemma 8: If \( X \rightarrow Y \rightarrow C_f \) is a cofibre sequence of 1-connected spaces, then \( (C_f)_o = C_{f_o} \), where \( (\ )_o \) denotes the rationalization of a space and \( f_o \) is the rationalization of the map \( f \).

Proof:

We fill in the following diagram with maps

\[ \lambda: C_f \rightarrow C_{f_o}, \quad \sigma: (C_f)_o \rightarrow C_{f_o} \]

so that it is homotopy commutative.

\[
\begin{array}{ccc}
X & \rightarrow & Y \\
\downarrow X_o & & \downarrow Y_o \\
X_o & \rightarrow & Y_o \\
\end{array}
\quad \begin{array}{ccc}
& \rightarrow & C_f \\
& \downarrow & \downarrow \sigma \\
& (C_f)_o & \rightarrow C_{f_o}
\end{array}
\]

Existence of \( \lambda \): The composition \( X \rightarrow X_o \rightarrow Y_o \rightarrow C_{f_o} \) is nullhomotopic. Thus, by the MCET, there is an extension \( \lambda: C_f \rightarrow C_{f_o} \).

Existence of \( \sigma \): We first show that \( C_{f_o} \) is 0-local. It is sufficient to show this for homology. Consider the long exact homology sequences:
where τ is defined by, τ(x) = tx for t a non-zero rational number. \( H_i(X_0) \) and \( H_i(Y_0) \) are 0-local for all i, so \( \tau \) is an isomorphism on these groups. The 5-Lemma then shows that \( \tau \) is an isomorphism on \( H_1(C_{f_o}) \).

Hence, \( C_{f_o} \) is 0-local.

The universal property of localization then, homotopy-uniquely, fills in the diagram,

```
\[ C_f \rightarrow C_{f_o} \]
```

Now, localization preserves exactness so \( X_0 \rightarrow Y_0 \rightarrow (C_{f})_0 \) has a long exact cohomology sequence associated with it.

Using \( \sigma \), compare this sequence with that of \( X_0 \rightarrow Y_0 \rightarrow C_{f_o} \). The 5-Lemma then shows that \( \sigma^* \) is an isomorphism.

Since both spaces are 1-connected, \( C_{f_o} \cong (C_{f})_0 \).

Q.E.D.
Now let $X \to Y \to C_f$ be a cofibre sequence. Then there is a co-operation map,

$$c : C_f \to C_f \Sigma X$$

given by collapsing the $1$-level in $CX$. It is well known that the composition

$$c : C_f \to C_f \Sigma X \to C_f$$

is homotopic to the identity and that the composition

$$c : C_f \to C_f \Sigma X \to \Sigma X$$

is homotopic to $q : C_f \to \Sigma X$, the map collapsing $Y$ to a point. The map $c$ provides an action,

$$c^* : H^{n-1}(X) \times H^n(C_f) \to H^n(C_f).$$

Lemma 9 (Corollary 15.10 of [12]):

1) $q^* = \partial$, the connecting homomorphism in the long exact cohomology sequence of $X \to Y \to C_f$.

2) If $(\alpha, \mu) \in H^{n-1}(X) \times H^n(C_f)$, then $c^*(\alpha, \mu) = \alpha + \partial \mu = \alpha + q^*(\mu)$. 
We now recall the definition of the "wedge product" of DGA's. If \( \{ A_i \} \) is a collection of DGA's, each equipped with an augmentation \( \varepsilon_i : A_i \to Q \), define

\[
\vee A_i = Q \oplus \bigoplus \ker \varepsilon_i.
\]

If \( X \) is a basepointed space, then the inclusion of the basepoint \( x_0 \to X \) provides an augmentation on \( Q \)-polynomial forms \( A(X) + A(x_0) = Q \). As noticed in [2], \( A(\cdot) \) takes colimits into limits. In particular,

**Lemma 10:** If \( \{ X_i \} \) is a collection of basepointed spaces, then \( A(\vee X_i) \cong \vee A(X_i) \).

**Remark 11:**

1) If the \( A_i \) are all connected \( (A^0 = Q) \) and the augmentation is the identity on \( A^0 \) and zero on \( A^1 \), \( i \neq 0 \), then \( \vee A_i \) is the DGA with \( Q \) in degree 0 and \( \vee A^1 \) in higher degrees.

2) We may apply 1) to the minimal models \( M_i \) of \( A_i \). Now \( \vee M_i + \vee A_i \) is a quasi-isomorphism, but \( \vee M_i \) may not be minimal since non-trivial relations are introduced by the direct sum structure in higher degrees.
Prop. 12: If \( \bigvee X_i \) is a wedge of spaces, then there is a quasi-isomorphism \( \tilde{\theta}: M(\bigvee X_i) \rightarrow VM(X_i) \).

Proof: Using Lemma 10 and Remark 11 2) we obtain:

\[
\begin{array}{ccc}
M(\bigvee X_i) & \rightarrow & A(\bigvee X_i) \\
\downarrow & & \downarrow \\
\tilde{\theta} & & VA(X_i) \\
& & \downarrow \\
& & M(X_i)
\end{array}
\]

The existence of \( \tilde{\theta} \) is assured by the lifting property for minimal DGA's since the vertical map is a quasi-isomorphism. The vertical and horizontal maps are quasi-isomorphisms; hence, so is \( \tilde{\theta} \).

Q.E.D.

In particular, note that we have a quasi-isomorphism

\[ \tilde{\theta}: M(C_f \vee \Sigma X) \rightarrow M(C_f)VM(\Sigma X) . \]

Prop. 12 is the first step in the proof of Theorem 7. There remain two rather technical steps which construct formal DGA's whose wedge product has minimal model isomorphic to \( M(C_f) \). Now, if \( \{ X_i \} \) is a collection of formal spaces, then Prop. 12 provides a composition,
which is a quasi-isomorphism, hence the wedge of formal spaces is formal; we will have then shown that $M(C^r)$ is formal. We begin with the following:

**Lemma 13:** There exists a formal minimal DGA $U$ and maps $\alpha: U \to M(Y), \beta: U \to M(C^r)$ such that $M(j) \circ \beta = \alpha$, $\text{Im}(\alpha^*) = \text{Im}(j^*)$ and $\alpha^*$ is injective.

**Proof:** $\text{ker } f^*$ has the structure of a graded vector space with multiplication. Adjoin $Q$ in degree 0 to obtain a DGA denoted by $K$. Let $U = M(K)$. $U$ is obviously formal since $U = M(K) \to K \cong H^*(K)$ is a quasi-isomorphism. Note that $K \cong H^*(K)$ since $K \subset H^*(Y)$ and the differential on $H^*(Y)$ is zero.

Now $Y$ is formal, so we have a diagram

\[
\begin{array}{c}
M(Y) \xrightarrow{\alpha} U \\
\downarrow{\theta^*_Y} \downarrow{1} \\
H^*(Y) \xleftarrow{1} K
\end{array}
\]

where $\alpha$ exists by the lifting property of minimal DGA's. Without loss of generality we may assume that $\theta^*_X = \text{id}_{H^*(X)}$
and \( \theta_Y^* = \text{id}_{H^*(Y)} \). Thus we obtain,
\[
in(\alpha^*) = K = \ker f^* = \text{im}(j^*)
\]

where the last equality follows from exactness in the co-

homology sequence for \( \xrightarrow{f} Y \rightarrow C_f \).

Using the formality of \( f \) we obtain a homotopy

commutative diagram,

\[
\begin{array}{ccc}
M(f) & \alpha & U \\
M(X) & \downarrow \theta_X & M(Y) \\
H^*(X) & f^* & \downarrow \theta_Y \\
& H^*(Y) & \downarrow i \\
& K & \rho
\end{array}
\]

Now, \( f^* \cdot i \cdot \rho = 0 \) since \( i(K) = \ker f^* \). Thus
\( \theta_X \cdot M(f) \cdot \alpha = 0 \). Now, \( U \) is minimal and \( \theta_X \) is a quasi-

isomorphism, so there is a bijection of homotopy sets
\( [U, H^*(X)] \leftrightarrow [U, M(X)] \) by the lifting property of minimal

DGA's. Under this bijection the zero classes correspond.

Hence \( M(f) \cdot \alpha = 0 \). Propositions 3 and 4 then provide

a lifting \( \beta: U \rightarrow M(C_f) \) with \( M(j) \cdot \beta = \alpha \).

Q.E.D.

**Remark 14:** By construction, \( \alpha^* \) and \( \beta^* \) are injective

since \( i: K \rightarrow H^*(Y) \) is. Now \( j^* \cdot \beta^* = \alpha^* \), so \( j^* \mid_{\beta^*(U)} \)
is injective also. Therefore, $\text{im}(j^*) = g^*(U)$. We see that, cohomologically, $U$ approximates a piece of $M(C_f)$.

We now consider how to approximate the remaining portion of $M(C_f)$. If $V$ is a graded vector space, then we may consider it as a DGA (adjoining $Q$ in degree 0 if necessary) with trivial multiplication. When we consider a DGA so constructed we shall denote it by $T(V)$. As convenient shorthand, we write $M(V)$ in place of the more proper $M(T(V))$.

Consider $q^*: H^*(E_X) \to H^*(C_f)$. As vector spaces, we obtain a decomposition,

$$H^*(E_X) = \ker q^* \oplus W$$

for some complement $W$. The inclusion $W \subset H^*(E_X)$ induces $T(W) \subset T(H^*(C_X))$. Now, $T(H^*(C_X)) \simeq H^*(C_X)$ since all cup products vanish due to the co-$H$ structure of $E_X$. We have a homotopy commutative diagram,

$$
\begin{array}{ccc}
M(i) & \longrightarrow & M(C_X) \\
\downarrow \rho & & \downarrow \theta_{C_X} \\
M(W) & \longrightarrow & H^*(C_X)
\end{array}
$$

\[ \begin{array}{c}
\rho \\
\downarrow \iota \\
\theta_{C_X}
\end{array} \]
using the fact that $\Sigma X$ is formal with $M(H^*(\Sigma X)) = M(\Sigma X)$. Again assuming without loss of generality that $\theta^* = \text{id}_{H^*(\Sigma X)}$ we have,

$$\text{im}(M(i^*)) = \text{im}(i^*) = W.$$

**Remark 15:** From the exact cohomology sequence, $W = \text{im}(q^*) = \ker j^*$. Hence, $W$ is the required cohomological complement to $\beta(U)$ in $M(C_f)$. Again note that $M(W)$ is clearly formal since $T(W) = H^*(T(W))$.

We now give the proof of Theorem 7.

**Proof of Theorem 7:**

By the preceding discussion, Prop. 12 and Lemma 13, we obtain a homotopy commutative diagram,

$$
\begin{align*}
M(C_f) \cong X & \xrightarrow{\phi} M(C_f)^{(\Sigma X)} \\
\beta VM(i) & \quad \eta \\
M(C_f) VM(\Sigma X) & \xrightarrow{\beta VM(i)} UVM(W) \xleftarrow{M(UVM(W))}
\end{align*}
$$
by the lifting property for minimal DGA's. Note that
\[ \text{im}(\eta^*) = \beta^*(U) \oplus W. \] Now, by Remark 14, \( \beta^*(U) \oplus \ker j^* = H^*(C_f) \) and by Remark 15, \( \beta^*(U) \oplus q^*(W) = H^*(C_f) \).

Consider the composition,
\[ M(UVM(W)) \xrightarrow{\eta} M(C_f \vee X) \xrightarrow{M(c)} M(C_f). \]

Lemma 9 implies that \( M(c)^*(\text{im} \eta^*) = c^*(\beta^*(U) \oplus W) = \beta^*(U) \oplus q^*(W) = H^*(C_f) \). Hence \( M(c)^* \eta \) is a quasi-isomorphism. Since \( M(C_f) \) and \( M(UVM(W)) \) are minimal this implies that \( M(C_f) \cong M(UVM(W)) \). Recognizing the formality of \( M(UVM(W)) \) completes the proof.

Q.E.D.

Remark 16:
1) Considering the form of the solution we see that \( C_f \) is decomposed as a wedge product \( AVB \), where \( B \) is a wedge of spheres.

2) A more careful analysis of the proof would show that \( j: Y \to C_f \) is a formal map.

3) If \( f: X \to Y \) is formal, then \( H^*(C_f) = \ker f^* \oplus s \text{coker} f^* \), where \( s \) suspends (i.e., raises the degree of) \( \text{coker} f^* \) by one. This is then a necessary condition for the formality of \( f \).
Examples:

1) Using Remark 16.3, we see that the Whitehead product map $[i,i] : S^5 \to S^3 \vee S^3$ is not formal; $C_f = S^3 \times S^3$ and $H^*(S^3 \times S^3) \neq \ker f^* \oplus s \operatorname{coker} f^*$.

2) Perhaps the simplest example of a non-formal map if the Hopf map $H : S^3 \to S^2$. The cofibre $C_H$ is $\text{CP}(2)$ and $\text{CP}(2)$ is formal because its cohomology algebra is of the form symmetric algebra/Borel Ideal ($H^*(\text{CP}(2)) = \mathbb{Q}[U]/(U^3)$). For details see [2] or [16]. Note also that $H^*(\text{CP}(2)) = H^*(S^2) \oplus S H^*(S^3) = \ker H^* \oplus s \operatorname{coker} H^*$. However, if $H$ were formal, then we would have a diagram,

\[
\begin{array}{ccc}
M(S^2) & \xrightarrow{H^*} & M(S^3) \\
\downarrow & & \downarrow \\
H^*(S^2) & \xrightarrow{H^*} & H^*(S^3)
\end{array}
\]

commuting up to homotopy. $H^* = 0$ implies that $M(H) = 0$ since the vertical maps are quasi-isomorphisms. Hence the Hopf map would be rationally homotopic to zero. We know, however, that $H$ generates $\pi_3(S^2) = \mathbb{Z}$ and is, therefore, rationally essential. Thus, $H$ cannot be formal.
Examples of Formal Maps:

1) Deligne et al [5] show that holomorphic maps between compact Kähler manifolds are formal.

2) Suppose that a DGA $A$ has a Riemannian structure as described in Chapter 2. In particular, for each $i$, $A^i = C^i \oplus Z^i$ and $Z^i = B^i \oplus H^i$. Let $f$ be an isometry with respect to the given Riemannian structure. In particular, $f$ preserves the given decompositions.

Without using the Riemannian structure, we have a homotopy commutative diagram,

```
\begin{array}{ccc}
M & \xrightarrow{f} & M \\
\downarrow{\rho} & & \downarrow{\rho} \\
A & \xrightarrow{f} & A
\end{array}
```

by the lifting property of minimal DGA's.

Suppose now that the "harmonic elements" $H = \oplus H^i$, form a sub-DGA of $A$. That is, we assume that the product of harmonic forms is harmonic. Since $H \cong H^*(A)$, the inclusion $H \rightarrow A$ is a quasi-isomorphism inducing an isomorphism of minimal models $M_H \rightarrow M$, where $\tilde{i}$ is the homotopy-unique lift of $i$. 
Since \( f \) is an isometry, \( f(H) \subseteq H \) and we obtain the homotopy commutative diagram,

\[
\begin{array}{ccc}
M_H & \longrightarrow & M_H \\
\downarrow & & \downarrow \\
H & \longrightarrow & H
\end{array}
\]

Consider the following diagram where the sides and bottom are homotopy commutative.

\[
\begin{array}{ccc}
M_H & \xrightarrow{f_H} & M_H \\
\downarrow & \sim \downarrow & \downarrow \\
M_H & \xrightarrow{\lambda} & M \\
\downarrow & \downarrow & \downarrow \\
H & \xrightarrow{f} & A
\end{array}
\]

We wish to show that the top is homotopy commutative.

Computing, we obtain
\[
\rho \circ \widetilde{\iota} \circ f_H = \iota \circ \lambda \circ f_H \\
= \iota \circ f \circ \lambda \\
= f \circ \iota \circ \lambda \\
= f \circ \rho \circ \widetilde{\iota} \\
= \rho \circ f \circ \iota 
\]

Now \( \rho \) is a quasi-isomorphism and \( M_H \) is minimal; hence, \([ M_H, A ] \rightarrow [ M_H, M ] \). Thus \( \widetilde{\iota} \circ f_H = \widetilde{\iota} \circ \iota \).

Then we obtain the following homotopy commutative diagram,

\[
\begin{array}{ccc}
M & \xrightarrow{\iota^{-1}} & M_H \\
\downarrow f & & \downarrow f_H \\
M & \xrightarrow{\sim^{-1}} & M_H \\
\end{array}
\quad
\begin{array}{ccc}
& \xrightarrow{\lambda} & \\
\downarrow f & & \downarrow f \\
& \xrightarrow{\sim^{-1}} & \\
\end{array}
\quad
\begin{array}{ccc}
M_H & \rightarrow & H \\
\downarrow \lambda & & \downarrow \iota \\
M_H & \rightarrow & H \\
\end{array}
\quad
\begin{array}{cc}
\sim & \xrightarrow{H^*(A)} \\
\downarrow f^* & & \downarrow f^* \\
H & = & H^*(A) \\
\end{array}
\]

Hence \( f \) is a formal map.

We have presented the abstract version of the situation of an isometry acting on a riemannian symmetric space. For in this case the product of harmonic forms is harmonic; hence, the space is formal \((M(X) \sim M_H \rightarrow H = H^*(X))\). Thus,
the above argument shows that an isometry of a riemannian symmetric space is formal.

§ 3. Weak Fibrations in Differential Graded Algebras and Coformalization.

In this section we dualize the notions of § 1. The idea of coformality is also presented as a dual to the concept of formality within the context of differential graded algebras. This preliminary material leads, in § 4, to the statement and proof of the dual of Theorem 7.

Def. 1':

(i) A sequence of DGA's \( C \rightarrow D \rightarrow N \) is called a fibration (of DGA's) if

1) \( N = L(V) \) is freely generated on the graded vector space \( V \) as an algebra.

2) \( D \cong C \otimes L(V) \) as algebras.

3) \( q \) and \( i \) are the obvious inclusion and projection respectively and \( d(V_J) \subset C \otimes L(\otimes V) \).

(ii) A sequence of DGA's \( A \rightarrow^p B \rightarrow^q M \) is called a weak fibration (or weak fibre sequence) of DGA's if there exists a fibration \( A \rightarrow D \rightarrow N \) and quasi-isomorphisms \( \Psi : B \rightarrow D \), \( \Phi : M \rightarrow N \) so that
commutes up to homotopy.

Extending the minimal model theory, Halperin \[10\] shows that any DGA map may be approximated by a fibration of DGA's. We note that in his terminology a fibration is a special type of Koszul-Sullivan extension.

Prop. 3': If \( F \rightarrow E \rightarrow B \) is a fibration (up to homotopy) of 1-connected spaces, then \( M(B) \rightarrow M(E) \) \( M(\iota) \rightarrow M(F) \) is a weak fibre sequence of DGA's.

Proof:

Halperin \[10\] (see also \[26\]) shows that for any (Serre) fibration we have a commutative diagram,

\[
\begin{array}{ccc}
A(B) & \rightarrow & A(E) \\
\uparrow & & \uparrow \\
M(B) & \rightarrow & M(B) \otimes L(V) \\
\uparrow & & \uparrow \\
& & L(V)
\end{array}
\]
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where the vertical maps are quasi-isomorphisms and \( A(\cdot) \) is Sullivan's \( Q \)-polynomial form functor. Consider the homotopy commutative diagram,

\[
\begin{array}{c}
\xymatrix{
M(B) & M(E) & M(F) \\
\downarrow \text{id} & \downarrow A(E) & \downarrow A(F) \\
M(B) & M(B) \otimes L(V) & L(V)
}
\end{array}
\]

where the dotted arrows exist by the lifting property for minimal algebras. Hence we obtain,

\[
\begin{array}{c}
\xymatrix{
M(B) & M(E) & M(F) \\
\downarrow & \downarrow & \downarrow \\
M(B) & M(B) \otimes L(V) & L(V)
}
\end{array}
\]

where the vertical maps are quasi-isomorphisms. Thus, \( M(B) \to M(E) \to M(F) \) is a weak fibration.

Q.E.D.

Lemma: If \( A \overset{p}{\to} B \overset{j}{\to} M \) is a weak fibration, then

\[ |\langle M \rangle| \to |\langle B \rangle| \to |\langle A \rangle| \]

is a fibration up to homotopy.
Proof:

We, of course, assume 1-connectedness throughout.

Since $A \to B \to M$ is a weak fibration, we have

\[
\begin{array}{ccc}
A & \to & B \to M \\
\downarrow \phi & & \downarrow \psi \\
A & \to & D \to N
\end{array}
\]

where $N = L(V)$ and $D = A \otimes L(V)$ as algebras. As shown in [2], $A \to A \otimes L(V) = D$ is a cofibration of DGA's and by Lemma 8.2 of [2], $|\langle D \rangle| \to |\langle A \rangle|$ is a Serre fibration.

Its fibre is clearly $|\langle N \rangle|$. We then have,

\[
\begin{array}{ccc}
|\langle N \rangle| & \to & |\langle D \rangle| \to |\langle A \rangle| \\
\downarrow & & \downarrow \\
|\langle M \rangle| & \to & |\langle B \rangle| \to |\langle A \rangle|
\end{array}
\]

where the vertical maps are quasi-isomorphisms. Since this is a diagram of rational spaces this implies that the vertical maps are homotopy equivalences. Thus, $|\langle M \rangle| \to |\langle B \rangle| \to |\langle A \rangle|$ is homotopy equivalent to a fibration.

Q.E.D.
Note that we have extended our use of spatial realization to non-minimal DGA's. That this is meaningful is shown in §12 of [2].

We now formulate and prove the dual version of Prop. 4. Consider the following diagram of spaces, where the top row is a (homotopy) fibration:

\[
\begin{array}{ccc}
F & \xrightarrow{j} & E \xrightarrow{p} B \\
\downarrow{h} & & \downarrow{g} \\
Z & & \\
\end{array}
\]

**Homotopy Fibre Lifting Theorem (HFLT):** The map \( h \) exists if and only if \( p \circ g = 0 \).

**Prop. 4':** Let \( A \rightarrow B \rightarrow M \) be a weak fibre sequence. Then, for \( U \) minimal, any \( g: B \rightarrow U \) with \( g \circ p = 0 \) has an extension \( \beta: M \rightarrow U \) such that \( \beta \circ j = g \).

**Proof:**

Taking spatial and geometric realizations, we obtain a homotopy fibration,

\[
|\langle M \rangle| \rightarrow |\langle B \rangle| \rightarrow |\langle A \rangle|.
\]
Hence we have,

\[
\begin{array}{c}
\langle M \rangle \\
\downarrow h \\
\langle U \rangle \\
\end{array} \quad \begin{array}{c}
\langle B \rangle \\
\uparrow \quad \langle A \rangle \\
\langle g \rangle \\
\end{array} \quad \begin{array}{c}
|p| \\
\end{array}
\]

with \(|p| \circ |g| = 0\) (since \(gp = 0\)). By the HFLT, \(h\) exists with \(|j| \circ h = |g|\). Letting \(\beta = M(h)\) we obtain,

\[\begin{array}{c}
A \quad p \quad B \\
\downarrow \quad j \quad \downarrow M \\
\quad \quad \quad g \\
\quad \quad \quad U \quad \quad \quad \beta
\end{array}\]

with \(\beta \circ j = g\).

Q.E.D.

Now let \(M\) be a minimal DGA. If \(M\) is formal, then there is a quasi-isomorphism \(M \rightarrow H^\bullet(M)\). Intuitively, the cohomology of a DGA is what might be referred to as the formalization of the DGA. To dualize the concept of formality, we must first construct a DGA reflecting the cohomotopy of a given minimal algebra \(M\) instead of its cohomology. This DGA is then the coformalization of \(M\).
Suppose that \( V = \bigoplus_{i=2}^{\infty} V^i \) is a graded vector space provided with a map of degree 1, \( D: V \rightarrow V \otimes V \), where
\[
(V \otimes V)^n = \sum_{i+j=n} V^i \otimes V^j.
\]
Upon taking duals with respect to \( \mathbb{Q} \), we obtain \( V^* = \bigoplus (V^i)^* \) and
\[
\alpha_{ij}: V^i \otimes V^j \to V^{i+j-1}^*
\]
where \( \alpha_{ij} = (p_{ij} \circ D)^* \); \( p_{ij}: \sum_{i+j=n+1} V^i \otimes V^j \to V^i \otimes V^j \) is the projection for each \( i \) and \( j \). The structure so defined is similar to that of a graded Lie algebra. However, there is a shift in degrees under multiplication and we have not yet referred to a Jacobi Identity.

Def. 3: A pair \((V, D)\) is called a **Co-Whitehead Algebra** if,

1) \( V = \bigoplus V^i \) is a graded vector space
2) \( D: V \rightarrow V \otimes V \) is a graded vector space map of degree 1
3) For each \( i \) and \( j \), \( \alpha_{ij} \) satisfies the Jacobi Identity (see [17] for a precise version).
Main Example: The graded vector space of homotopy groups of a rationalized space is a Whitehead algebra (see [17]). It is not hard to see, then, that the cohomotopy of the space, together with the dual of the Whitehead product map, is a co-Whitehead algebra (see [17]).

Now let \((V, D)\) be a co-Whitehead algebra and let \(\pi(V) = \bigotimes L_1(V^1)\) denote the graded commutative algebra freely generated by \(V\). Extend \(D\) to a derivation of \(\pi(V)\). Condition 3) of Def. 3 is equivalent to the fact that \(D^2 = 0\). Hence \((\pi(V), D)\) is a DGA. \(D\) is decomposable since the first non-trivial degree of \(V\) is 2. Thus \(\pi(V)\) is minimal. \(D\) has the precise form, \(D(X_i) = \sum X_r X_s\) where \(\{X_i\}\) is a homogeneous basis for \(V\).

Suppose that \(f: V \rightarrow W\) is a map of graded vector spaces. \(f\) is a map of co-Whitehead algebras if \(V\) and \(W\) are co-Whitehead algebras and \((f \otimes f)^* D_V = D_W^* f\). This condition implies that the induced map \(\pi(f): \pi(V) \rightarrow \pi(W)\) is a DGA map. That is, \(\pi(f)^* D_V = D_W^* \pi(f)\).

The association, \((V, D) \mapsto (\pi(V), D)\) defines a functor from co-Whitehead algebras to DGA's.

Let \(M\) be a minimal DGA freely generated by \(\{X_i\}\). Let \(Q(M) = M_+/M_+^* M_+\) denote the indecomposables of \(M\).
In [16] it is shown that the map \( D : Q(M) \rightarrow Q(M) \otimes Q(M) \),

\[
x_1 \mapsto \frac{1}{2} \sum x_s \otimes x_t + (-1)^{|s||t|} x_t \otimes x_s
\]

is independent of the basis chosen for \( Q(M) \). (Here, \( d_M(x_1) = \sum x_s x_t + P \), where \( P \) consists of sums of monomials of length 3 or greater in the \( x_i \).) Now to any minimal DGA there corresponds a \( Q \)-homotopy type. Under this correspondence, \( D \) is dual to the Whitehead product. Hence \( \alpha_{ij} \) satisfies the Jacobi identity and \((Q(M),D)\) is a co-Whitehead algebra. Denote the DGA \( \pi(Q(M)) \) by \( \pi(M) \). The differential has the form \( D(x_1) = \sum x_s x_t \).

**Def. 4:** A minimal DGA \( M \) is **coformal** if \( (M,d_M) \cong (\pi(M),D) \).

A specific isomorphism \( (M,d_M) \cong (\pi(M),D) \) is called a coformalization of \( (M,d_M) \).

The most important feature of the construction of \( \pi(M) \) is its dependence only on the cohomotopy of \( M \) and the Whitehead product of the associated \( Q \)-homotopy type. This is the proper dualization to the idea of formality.
Def. 5: 1) A DGA \( A \) is said to be **coformal** if there is a DGA-isomorphism \( \varphi : M_A \rightarrow \pi(M_A) \), where \( M_A \) is the minimal model of \( A \).

2) A space is **coformal** if its DGA of \( Q \)-polynomial forms is coformal.

Remark 17: Informally, a space is coformal if its \( Q \)-homotopy type is determined by its graded vector space of rationalized homotopy groups together with its Whitehead product structure. All higher order Whitehead products vanish in a coformal space, dualizing the fact that Massey products vanish in a formal space.

Let \( f : M \rightarrow N \) be a DGA map of minimal algebras and let \( Q(f) : Q(M) \rightarrow Q(N) \) be the induced map on cohomotopy. Again representing \( M \) and \( N \) by \( Q \)-homotopy types, the naturality of the Whitehead product (and its dual) implies that \( (Q(f) \circ Q(f)) \cdot D_M = D_N \cdot Q(f) \). Thus, \( Q(f) \) is a mapping of co-Whitehead algebras. Hence we obtain

\[ \pi(f) : \pi(M) \rightarrow \pi(N). \]

Remark 18: If \( f \simeq g : M \rightarrow N \), then \( Q(f) = Q(g) \); that is, homotopic maps induce the same map on cohomotopy. Thus \( \pi(f) = \pi(g) \).
This remark allows us to make the following definition.

**Def. 6:** A DGA map \( f : A \to B \) is said to be **coformal** if there exist isomorphisms \( \theta_A, \theta_B \) such that the following diagram commutes up to homotopy:

\[
\begin{array}{ccc}
M_A & \xrightarrow{M(f)} & M_B \\
\downarrow{\theta_A} & & \downarrow{\theta_B} \\
\pi(M_A) & \xrightarrow{\pi(f)} & \pi(M_B)
\end{array}
\]

where \( M_A, M_B \) are the respective minimal models of \( A \) and \( B \) and \( \pi(f) = \pi(M(f)) \).

**Def. 7:** A map of spaces \( f : X \to Y \) is said to be **coformal** if the induced map \( A(f) : A(Y) \to A(X) \) is.

**Examples:**

1) **Spheres are coformal:** \( M(S^{2n}) = L(x_{2n}, y_{4n-1}) \) with \( dy = x^2, dx = 0 \) and \( M(S^{2n+1}) = L(X_{2n+1}) \) with \( dx = 0 \).

2) **Eilenberg-MacLane spaces are coformal.** This follows because \( H^*(K(\pi,n)) \) is free, implying \( M(K(\pi,n)) \cong H^*(K(\pi,n)) \) with zero differential.
3) $S^3 \cup S^3 \cup g \mathbb{C}$ is coformal (but not formal), where $g = [[i, i_2], i_1]$. For details, see [8].

4) Fat Wedges of spheres are not coformal because they are models for the construction of higher order Whitehead products.

5) $CP(n)$ is not coformal. $M(CP(n)) = (L(x_2, y_{2n+1}), \text{dy} = x^{n+1})$.

6) If $f: M \rightarrow N$ is a DGA map between minimal algebras, the $\pi(f): \pi(M) \rightarrow \pi(N)$ is, essentially, the linear part of $f$. Say that $f$ preserves generators if $M = L(x_i), N = L(y_j)$ and $f(x_i) = \sum \alpha_j y_j$, where $\alpha_j \in \mathbb{Q}$.

Prop.: If $M$ and $N$ are coformal and $f$ preserves generators, the $f$ is coformal. Simply note that $f = \pi(f)$.

7) The Hopf maps are coformal. Consider $H: S^3 \rightarrow S^2$. $M(S^3) = (L(z_3), d = 0)$ and $M(S^2) = (L(x_2, y_3), dx = 0, dy = x^2)$. $H$ induces $M(H): M(S^2) \rightarrow M(S^3)$: $M(H)(x) = 0$, $M(H)(y) = z$. $H$ preserves generators, so it is coformal. (Of course this is obvious by a direct application of the definition!)
8) Consider the following result:

**Prop.**: If \( f : X \to Y \) is zero on rational homotopy groups, but non-zero in rational homology, then it is not coformal.

**Proof**: 
Suppose \( f \) is coformal. We have the diagram,

\[
\begin{array}{ccc}
M(Y) & \xrightarrow{M(f)} & M(X) \\
\downarrow & & \downarrow \\
\pi(Y) & \xrightarrow{\pi(f)} & \pi(X)
\end{array}
\]

where the vertical maps are isomorphisms. Now, by hypothesis, \( \pi(f) = 0 \). Hence \( M(f) = 0 \). But then \( f^* = 0 \), contradicting the hypothesis.

Q.E.D.

Consider: \( M_1 = (L(x_2), d=0) \), \( M_2 = (L(y_4), d=0) \). Let \( f : M_2 \to M_1 \) be defined by, \( f(y) = x^2 \). \( M_1 \) and \( M_2 \) are obviously coformal and \( f \) satisfies the hypotheses of the proposition. Hence \( f \) is not coformal. This situation is realized on the space level by squaring the generator of \( H^2(K(Q,2)) = \langle x \rangle \). Then \( x^2 \in H^4(K(Q,2)) = [K(Q,2), K(Q,4)] \).
§ 4. The Fibre of a Coformal Map

The purpose of this section is to prove the following dual version of Theorem 7.

\textbf{Theorem 7':} If $F \rightarrow E \rightarrow B$ is a fibration up to homotopy of 1-connected spaces and $p$ is a coformal map, then $F$ is a coformal space.

Let $X$ be a space. The following notation will be convenient: $Q(X) = Q(M(X))$ and $\pi(X) = \pi(M(X))$. If $f: X \rightarrow Y$, then $Q(f) = Q(M(f))$, $\pi(f) = \pi(M(f))$ and $Q(X/Y) = Q(X)/\text{im } Q(f)$.

Now suppose that $F \rightarrow E \rightarrow B$ is a homotopy fibration. By Prop. 3', $M(B) \rightarrow M(E) \rightarrow M(F)$ is a weak fibration of DGA's. Consider the sequence $Q(B) \xrightarrow{Q(p)} Q(E) \rightarrow Q(E/B)$. Recall that $(Q(B), D_B)$ and $(Q(E), D_E)$ are co-Whitehead algebras. The following result allows us to give such a structure to $Q(E/B)$.

\textbf{Lemma 19:} There exists $D_{E/B}: Q(E/B) \rightarrow Q(E/B) \otimes Q(E/B)$ giving $Q(E/B)$ the structure of a co-Whitehead algebra so that $c$ is a co-Whitehead map.
**Proof:**

The structure maps $D_E, D_B$ are dual to the respective Whitehead products. Using the naturality of the Whitehead product we obtain a commutative diagram,

$$
\begin{array}{c}
Q(B) \\ Q(p) \\
\downarrow \\
Q(E) \\
\end{array}
\begin{array}{c}
\xrightarrow{D_B} \\
\xrightarrow{D_E} \\
\downarrow \\
\xrightarrow{D_{E/B}} \\
\end{array}
\begin{array}{c}
Q(B) \otimes Q(B) \\
Q(p) \otimes Q(p) \\
Q(E) \otimes Q(E) \\
Q(E/B) \otimes Q(E/B) \\
\end{array}
$$

Then $\text{im}(Q(p)) \subseteq \ker((\varepsilon \otimes \varepsilon) \cdot D_E)$. Hence, there is a factorization (denoted $D_{E/B}$),

$$
\begin{array}{c}
Q(E) \\
\downarrow \varepsilon \\
Q(E/B) \\
\end{array}
\begin{array}{c}
\xrightarrow{D_E} \\
\xrightarrow{D_{E/B}} \\
\downarrow \\
\end{array}
\begin{array}{c}
Q(E) \otimes Q(E) \\
Q(E/B) \otimes Q(E/B) \\
\end{array}
\begin{array}{c}
\xrightarrow{\varepsilon \otimes \varepsilon} \\
\end{array}
$$

Now, $D^*_E$ satisfies the Jacobi identity and $\varepsilon^*, (\varepsilon \otimes \varepsilon)^*$ are injective (since $\varepsilon$ and $\varepsilon \otimes \varepsilon$ are surjective). We have,
We then obtain a DGA map \( \pi(\epsilon) : \pi(E) \to \pi(E/B) \).

Note that \( \pi(\epsilon) \circ \pi(p) = 0 \).

Now let \( F \xrightarrow{1} E \xrightarrow{p} B \) be a fibration. There is an operation of \( \Omega B \) on \( F \), \( c : \Omega B \times F \to F \) inducing an action on homotopy groups, \( \pi_n(\Omega B) \times \pi_n(F) \to \pi_n(F) \).

Let \( i : \Omega B \to \Omega B \times F \) be the natural inclusion and let \( s : \Omega B \to F \) be the composition \( c \circ i \).

Lemma 9' (Prop. 15.8' of [12]):

1) \( S_* = \partial \), the connecting homomorphism in the long exact homotopy sequence of the fibration.

2) If \( (\mu, \alpha) \in \pi_{n+1}(B) \times \pi_n(F) \), then \( c_*(\mu, \alpha) = \alpha + \partial \mu = \alpha + S_*(\mu) \).
Now \( c : \Omega B \times F \to F \) induces a homotopy-unique map, \( M(c) : M(F) \to M(\Omega B \times F) \cong M(F) \otimes M(\Omega B) \). On cohomotopy we obtain, \( Q(c) : Q^n(F) \to Q^n(F) \oplus Q^n(\Omega B) \). Lemma 9' implies that,

\[ Q(c)(h) = (h, \beta^*(h)) \]

where \( \beta^* \) is the dual of the connecting homomorphism \( \beta \).

**Proof of Theorem 7':**

Consider the diagram (which exists since \( p \) is formal),

\[
\begin{array}{ccc}
M(B) & \xrightarrow{M(p)} & M(E) \\
\downarrow{\theta_B} & & \downarrow{\theta_E} \\
\pi(B) & \xrightarrow{\pi(p)} & \pi(E) \\
\end{array}
\quad \xrightarrow{\beta} \quad
\begin{array}{ccc}
M(F) & \xrightarrow{M(j)} & M(F) \\
\downarrow{\pi(E)} & & \downarrow{\pi(E)} \\
\pi(E/B) & \xrightarrow{\pi(E/B)} & \pi(E/B) \\
\end{array}
\]

\( \beta \) exists by Prop. 4' since the top row is a weak fibration and \( \pi(E)^* \theta_E^* M(p) = \pi(E)^* \pi(p)^* \theta_B = 0 \). Note that \( \text{im}(Q(\beta)^* Q(j)) = \text{im}(Q(\epsilon)) = Q(E)/\text{im} Q(p) = Q(E/B) \). Hence \( Q(\beta) \) is surjective.

Now consider the exact cohomotopy sequence,

\[
\begin{array}{cccc}
\beta^* & Q(j) & Q(p) \\
\ldots & Q(\Omega B) & Q(F) & Q(E) & Q(B)
\end{array}
\]
$M(\Omega B) = H^\ast(\Omega B)$ since the cohomology of $\Omega B$ is free. Hence the differential on $M(\Omega B)$ is zero and $\Omega B$ is coformal.

Let $W = \text{im } \partial^\ast \subseteq Q(\Omega B)$. As graded vector spaces, we have a splitting $\alpha: Q(\Omega B) \to W$. Since $d = 0$ in $M(\Omega B)$, then $\partial_{\Omega B} = 0$ and we obtain $\pi(\alpha): \pi(\Omega B) \to \pi(W)$.

Now $Q(F) = \text{im } \partial^\ast \oplus \ker \partial^\ast \cong W \oplus Q(E/B)$ as graded vector spaces. Consider the composition,

$$M(c) \to M(F) \otimes M(\Omega B) \xrightarrow{\partial \otimes \pi(\alpha)} \pi(E/B) \otimes \pi(W).$$

The preceding descriptions of $Q(c)$, $Q(\beta)$ and $Q(F)$ show that the induced map on cohomotopy is an isomorphism.

$(Q(\beta) \otimes Q(\alpha)) \circ Q(c)$ is clearly surjective. It is injective since $Q(\beta)$ is injective on $\text{im } Q(j) = Q(E/B)$ and $Q(\alpha)\partial^\ast(h) \neq 0$ for $h \in \text{im } \partial^\ast \cong Q(F)/\text{im } Q(j)$.

Since $M(F)$ and $\pi(E/B) \otimes \pi(W)$ are minimal, then $M(F) = \pi(E/B) \otimes \pi(W)$. The latter is coformal by definition. Hence $M(F)$ is coformal.

Q.E.D.

\textbf{Remark 20:}

1) We have actually decomposed $F_0$ as a product $A \times B$, where $B$ is a product of Eilenberg-MacLane spaces.
2) A careful analysis of the proof shows that $j: F \to E$ is a coformal map.

3) A necessary condition for $p: E \to B$ to be coformal is, $Q(F) = \ker Q(p) \otimes \text{coker } Q(p)$.

§ 5. The Homology Decomposition of a 1-Connected DGA

If $M$ is a 1-connected minimal DGA, then we may decompose it by degree into a sequence of DGA's,

$$Q \subset M(2) \subset M(3) \subset \ldots \subset M(n) \subset \ldots \subset M$$

where $M(k+1) = M(k) \otimes \Lambda L_{k+1}(V)$ and $\Lambda$ is decomposable. In terms of cohomotopy, $M(n)$ approximates $M$ up to degree $n$. In fact, $\pi(M)(n) = \pi(M(n))$, so coformality of $M$ implies coformality of $M(n)$. It is a consequence of Sullivan's general theory that the above sequence corresponds to the Postnikov decomposition of a space. The coformality of a space, then, implies the coformality of each of its Postnikov terms.

In this section we dualize the above decomposition. The construction imitates the homology decomposition of a space. Indeed, the homology decomposition of a space provides
a homology decomposition of the associated minimal model. Using our methods we reprove a result of Toomer [27] (though in a much weaker form) that states that the homology decomposition of a rational space is essentially unique. We also apply the technique to give a criterion for formality. Let $M$ be a 1-connected minimal DGA.

**Def. 7:** The pair $(N,i)$ consisting of a DGA $N$ and a map $i: M \to N$ is called a homology $n$-section of $M$ if,

1) $H^i(N) = 0$ for $i > n$ and

2) $i$ induces cohomology isomorphisms in dimensions $\leq n$.

**Prop. 21:** For each $n$ there exists a homology $n$-section of $M$.

**Proof:**

Let $V_1 = H^{n+1}(M)$ and choose a splitting $\tau_1: V_1 \to Z^{n+1}(M)$. Extend to $\tau_1: L_{n+1}(V_1) \to M$ and form $M \otimes L_n(V_1)$. Now, $d(l \otimes V) = \tau_1(V)$, so the addition of $V_1$ kills $H^{n+1}(M)$. No new cohomology is created in dimension $n+1$ since $M$ is 1-connected. Hence, $H^{n+1}(M^1) = 0$ where $M^1 = M \otimes L_n(V_1)$. Also, $M \to M^1$ induces cohomology isomorphisms up to dimension $n$. 
In the same fashion, let \( V_2 = \mathbb{H}^{n+2}(M^1) \) and choose a splitting \( \tau_2: V_2 \rightarrow \mathbb{Z}^{n+2}(M^1) \). Again, \( \mathbb{H}^{n+2}(M^2) = 0 \) where \( M^2 = M^1 \otimes_{\tau_2} L_{n+1}(V_2) \), and \( M + M^2 \) induces cohomology isomorphisms up to dimension \( n \).

Continuing this process we obtain

\[
M[n] = M \otimes \bigotimes_{i} L_{n+1-i}(V_i),
\]

where \( V_k = \mathbb{H}^{n+k}(M^{k-1}) \) and \( \tau_k \) is a suitable splitting. The inclusion, \( M \rightarrow M[n] \) is the desired \( i \). (We denote it by \( i_n \)).

Q.E.D.

Of course, the existence of homology \( n \)-sections does not necessarily allow us to decompose \( M \). We require some sort of compatibility condition. In order to achieve this we recall the following result from Chapter 0.

**Extension Theorem (ET):** Let \( f: M \rightarrow A \) be a DGA map and suppose \( M \) is extended to \( M \otimes_{\tau} L_n(V) \). Let \( V = \langle x_1, \ldots, x_k \rangle \). Then the obstructions to extending \( f \) to \( \bar{f}: M \otimes_{\tau} L_n(V) \rightarrow A \) are

\[
[f(\tau x_i)] \in \mathbb{H}^{n+1}(A), \quad i = 1, \ldots, k
\]

As an immediate application we have,
Prop. 22: Let $M[n]$ and $M[n-1]$ be homology sections constructed in Prop. 21. Then there is a commutative diagram,

\[ \begin{array}{ccc}
M & \xrightarrow{i} & M[n] \\
\downarrow{\epsilon_n} & & \downarrow{\epsilon_n} \\
M[n-1] & \xleftarrow{i_{n-1}} & M[n-1]
\end{array} \]

Proof:
Consider the diagram,

\[ \begin{array}{ccc}
M & \xrightarrow{i_{n-1}} & M[n-1] \\
\downarrow{\epsilon_n} & & \\
M & \xrightarrow{\otimes} & L_\infty(V_1)
\end{array} \]

The obstructions to the existence of the indicated lifting lie in $H^{n+1}(M[n-1]) = 0$. Proceeding similarly for each $M^i$ provides a map $\epsilon_n$.

Q.E.D.

Remark 23:
1) Note that since $i_n$ and $i_{n-1}$ induce cohomology isomorphisms in dimensions $\leq n-1$, then so does $\epsilon_n$.
2) Choices were made in the construction of $M[n]$. 
If $M[n]$ and $M[n]'$ are the results of different choices, then again appealing to the ET we obtain,

![Diagram](image)

$\psi$ is then a quasi-isomorphism. Hence, $M[n]$ and $M[n]'$ are of the same homotopy type. That is, their minimal models are isomorphic. It is in this sense that $M[n]$ is well defined.

For technical reasons we prefer to define homology decompositions in terms of minimal DGA's. We do this, in particular, to make use of our notion of weak cofibration. $M[n]$ is not in general minimal, so we consider its minimal model, which we denote by $M_n$. By the lifting property of minimal models we have,

![Diagram](image)

and
Lemma 24: The following diagram commutes up to homotopy,

\[
\begin{array}{ccc}
M[n] & \xrightarrow{\rho_n} & M^n \\
\varepsilon_n \downarrow & & \downarrow \varepsilon_n \\
M[n-1] & \xleftarrow{\rho_{n-1}} & M_{n-1}
\end{array}
\]

\(\varepsilon_n\) then induces cohomology isomorphisms in dimensions \(\leq n-1\).

**Proof:**

Consider the diagram,

\[
\begin{array}{ccc}
M[n] & \xrightarrow{i_n} & M^n \\
\varepsilon_n \downarrow & & \downarrow \varepsilon_n \\
M[n-1] & \xleftarrow{i_{n-1}} & M_{n-1}
\end{array}
\]

\[
\begin{array}{ccc}
M[n] & \xrightarrow{\rho_n} & M^n \\
\varepsilon_n \downarrow & & \downarrow \varepsilon_n \\
M[n-1] & \xleftarrow{\rho_{n-1}} & M_{n-1}
\end{array}
\]

\[
\begin{array}{ccc}
M[n] & \xrightarrow{i_n} & M^n \\
\varepsilon_n \downarrow & & \downarrow \varepsilon_n \\
M[n-1] & \xleftarrow{i_{n-1}} & M_{n-1}
\end{array}
\]

\[
\begin{array}{ccc}
M[n] & \xrightarrow{\rho_n} & M^n \\
\varepsilon_n \downarrow & & \downarrow \varepsilon_n \\
M[n-1] & \xleftarrow{\rho_{n-1}} & M_{n-1}
\end{array}
\]
Now \( \rho_{n-1} \) is a quasi-isomorphism, so there is a bijection \( [M, M[n-1]] \cong [M, M_{n-1}] \). Thus, to show \( \tilde{\epsilon}_n \circ i_n \cong i_{n-1} \), it is sufficient to show \( \rho_{n-1} \circ \tilde{\epsilon}_n \circ i_n \cong \rho_{n-1} \circ i_{n-1} \).

Using the homotopy commutativity of the other triangles, we obtain the computation,

\[
\rho_{n-1} \circ \tilde{\epsilon}_n \circ i_n = \epsilon_n \circ \rho_{n} \circ i_n
\]

\[
= \epsilon_n \circ i_n
\]

\[
= i_{n-1}
\]

\[
= \rho_{n-1} \circ i_{n-1}.
\]

Q.E.D.

**Def. 8:** A homology decomposition of a 1-connected minimal DGA \( M \) consists of a homotopy commutative diagram of minimal DGA's,

\[
\begin{array}{c}
M_{k+1} \\ \downarrow \epsilon_{k+1} \\
M_k \\
\downarrow \epsilon_k \\
M_{k-1} \\
\downarrow i_{k} \\
\vdots \\
M_2 \\
\downarrow i_2 \\
M \\
\end{array}
\]
such that for each $k$, $(M_k, i_k)$ is a homology $k$-section of $M$. Such a system is denoted by $(M_k, i_k, ε_k)$.

Remark 25: Although $(M[n], i_n, ε_n)$ is not a homology decomposition (since $M[n]$ is not minimal), we shall refer to it as the canonical decomposition. The homology decomposition $(M_n, i_n, ε_n)$ shall be called the canonical homology decomposition.

Example: Let $X$ be a 1-connected space. Take a homology decomposition, $X_2 ⊆ X_3 ⊆ \ldots \subseteq X$, where $i_k: X_k \rightarrow X$ induces homology isomorphisms up to dimension $k$ and $H_i(X_k) = 0$ for $i > k$. Now, $H^i(X_k; Q) = \text{Hom}(H_1(X_k), Q)$, so $M(i_k): M(X) \to M(X_k)$ induces cohomology isomorphisms up to dimension $k$ and $H^i(M(X_k)) = H^i(X_k; Q) = 0$ for $i > k$. Piecing together the homotopy commutative triangles,

\[
\begin{array}{c}
M(X_k) \to M(X_{k-1}) \\
\downarrow M(i_k) \downarrow \quad \quad \quad \downarrow M(i_{k-1}) \\
M(X) & \end{array}
\]

provides a homology decomposition of $M(X)$. 
Of course we have not used the full structure of the homology decomposition of a space. We have not considered the \( k' \)-invariants and the associated pushouts,

\[
\begin{array}{ccc}
K'(H_n X, n-1) & \xrightarrow{k'} & X_{n-1} \\
\downarrow & & \downarrow \\
CK'(H_n X, n-1) & \longrightarrow & X_n
\end{array}
\]

\( K'(H_n X, n-1) \rightarrow X_{n-1} \rightarrow X_n \) is a cofibre sequence with \( k' \) homologically trivial. Our next task is to incorporate these ideas into the framework of DGA homology decompositions.

It is technically convenient, for the moment, to work with a homology \((n-1)\)-section \( M[n-1]' \). \( M[n-1]' \) is constructed from \( M[n] \) by first killing \( H^n(M[n]) = H^n(M) \) and then killing any new cohomology created at a previous stage. That is, \( M[n-1]' = M[n] \otimes \otimes L_{n-i}(Q_i) \), where

\[
Q_k = H^{n-k+1}(M[n] \otimes \otimes L_{n-i}(Q_i)) \quad \text{and} \quad \lambda_k \quad \text{is a suitable splitting.}
\]

Of course, \( M[n-1] \simeq M[n-1]' \) (i.e., \( M[n-1] \simeq M[n-1]' \)).

We now construct the DGA equivalent of a Moore space, \( L_{n-1}(V) \).
\[ L_{n-1}(V) = L_{n-1}(V) \otimes \bigotimes_{i=1}^{k-1} L_{n+1-1}(P_i) \]

where \( P_k = \sigma_k \) and \( \sigma_k \) is a suitable splitting. Then, \( H^i(L_{n-1}(V)) = 0 \) for \( i \neq n-1 \) and \( H^{n-1}(L_{n-1}(V)) = V \).

Now consider the sequence of DGA's with \( V = H^n(M) \),

\[
\begin{align*}
M[n] & \xrightarrow{\alpha} M[n-1] \xrightarrow{\ell} L_{n-1}(V) \\
\end{align*}
\]

where \( \alpha \) is the inclusion and \( \ell \) is defined by

\[
\ell \big|_{M[n]} = 0, \quad \ell \big|_{Q_1} = \text{id}_{H^n(M)},
\]

and extended by the ET.

Clearly, \( \ell \circ \alpha = 0 \) by definition.

Prop. 26: \( \Lambda : H^*(M[n]) \longrightarrow H^*(\ell) \) is an isomorphism.

**Proof:**

In this case \( \lambda : Z(M[n]) \rightarrow Z(\ell) \) takes the form, \( \lambda(x) = (\alpha(x), 0) \), since \( \ell \circ \alpha = 0 \).
$H^i(\mathcal{L}) = 0$ for $i > n$: Let $(a,b) \in Z^i(\mathcal{L})$. Then
d(a,b) = (-da, db + \ell(a)) = (0, 0). Hence, $da = 0$ and
db + \ell(a) = 0. Now, $H^i(M[n-1]'') = 0$ for $i \geq n$, so
$a = dc$ for some $c$. Then $db + \ell(a) = db + \ell(dc) = d(b + \ell(c)) = 0$. Thus,
$b + \ell(c) \in Z^{i-1}(L_{n-1}(V))$. But $H^i(L_{n-1}(V)) = 0$
for $i > n$, so there exists $e$ with $de = b + \ell(c)$. Therefore,
\[d(-c, e) = (dc, de - \ell(e)) = (b, 0) = (a, b).\]
Hence, every cocycle is a coboundary for $i > n$; $H^i(\mathcal{L}) = 0$.

$\Lambda$ is an isomorphism for $i \leq n$: For $i < n-1$, $(L_{n-1}(V))^i = 0$. Hence $H^i(\mathcal{Q}) \cong H^i(M[n-1]'') \cong H^i(M[n])$ for $i \leq n-1$
(since $C^i_\mathcal{L} = (M[n-1]'')^i \oplus (L_{n-1}(V))^{i-1}$).

$\Lambda$ is an isomorphism for $i = n$:
(Injective) Let $\lambda(x) = (\alpha(x), 0) = d(c, 0)$ with
$(c, 0) \in C^{n-1}_\mathcal{L}$. Then $d(-c) = \alpha(x)$ and $\ell(c) = 0$. Hence
$c \in \text{im}(\alpha)$ since $\mathcal{L} \bigg|_{Q_1} \neq 0$. If $\alpha(c') = -c$, then
(since $\alpha$ is injective) $dc' = x$. Thus $\Lambda$ is injective.
Let \((a, b) \in \mathbb{Z}^n(L)\). Now, \(b \in V\), so there exists \(q \in Q_L\) with \(L(q) = b\). Let \(c = a + dq \in \mathbb{Z}^n(M[n])\). Then, \((a, b) = (c - dq, L(q)) = (c, 0) + d(q, 0)\). Thus \(A\) is surjective.

Q.E.D.

We have a diagram

\[
\begin{array}{cccc}
M_n & \xrightarrow{\varepsilon} & M_{n-1} & \xleftarrow{\psi} \\
\downarrow{\alpha} & & \downarrow{\rho} & \\
M[n] & \xrightarrow{\varepsilon} & M[n-1] & \xleftarrow{\psi} \\
\downarrow{\alpha} & & \downarrow{\rho} & \\
L_{n-1}(V) & & & \\
\end{array}
\]

where \(\psi\) is constructed as in Remark 23 and \(\tilde{\psi}\) is the lifting to minimal models. Now \(\psi\) is a quasi-isomorphism, so \(\tilde{\psi}\) is an isomorphism. We have,

\[
\tilde{\rho} \circ \psi \circ \tilde{\alpha} \simeq \psi \circ \rho \circ \tilde{\alpha}
\]

\[
\simeq \psi \circ \alpha \circ \rho
\]

\[
\simeq \varepsilon \circ \rho
\]

\[
\simeq \tilde{\rho} \circ \varepsilon
\]
By homotopy uniqueness of lifting for minimal models, we obtain $\tilde{\psi} \cdot \tilde{\alpha} = \tilde{\varepsilon}$.

Then in the sequence $M_n \rightarrow M_{n-1} \rightarrow \tilde{L}_{n-1}(V)$ we obtain

$$
\ell \cdot \rho^* \cdot \tilde{\psi} \cdot \tilde{\varepsilon} = \ell \cdot \rho^* \cdot \tilde{\psi} \cdot \tilde{\psi} \cdot \tilde{\alpha} = \ell \cdot \rho^* \cdot \tilde{\alpha} = \ell \cdot \alpha \cdot \rho = 0 \text{ since } \ell \cdot \alpha = 0 .
$$

$A$ has the form, $H^*(M_n) \cong H^*(M[n]) \cong H^*(\ell) = H^*(\ell \cdot \rho^* \cdot \tilde{\psi} \cdot \tilde{\psi}^{-1})$, where the final isomorphism follows by Lemma 1. Hence we have,

**Prop. 27:** $M_n \xrightarrow{\sigma} M_{n-1} \xrightarrow{k} \tilde{L}_{n-1}(V)$ is a weak cofibre sequence, where $\sigma = \tilde{\varepsilon}$ and $k = \ell \cdot \rho^* \cdot \tilde{\psi} \cdot \tilde{\psi}^{-1}$.

Note that since $\sigma = \tilde{\varepsilon}$, this sequence fits into the canonical homology decomposition of $M$. 
This is the desired imitation of the spatial homology decomposition.

We now consider some consequences of the construction.

Prop. 28: If $M$ is formal, then $M_n$ is formal for each $n$.

Proof:

Consider the diagram,

\[
\begin{array}{ccc}
M & \xrightarrow{0} & H^*(M) \\
\downarrow & & \downarrow \\
M \otimes_T L_n(V_1) & & \\
\end{array}
\]

where $p$ is the projection sending all cohomology in dimensions $\geq n+1$ to zero. The lifting exists since all obstructions lie in $H^{n+1}(M[n]) = 0$. Continuing to apply the ET, we obtain a lifting,

\[
\begin{array}{ccc}
M & \xrightarrow{p^*0} & H^*(M[n]) \\
\downarrow & & \downarrow \\
M[n] & & \\
\end{array}
\]
Now \( \tilde{\theta} \) is a quasi-isomorphism since \( i_n \) and \( p \cdot \theta \)
induce cohomology isomorphisms in dimensions \( \leq n \). Composing with \( \rho : M \rightarrow M[n] \) provides a quasi-isomorphism
\( \tilde{\theta} \circ \rho : M_n \rightarrow H^*(M[n]) = H^*(M_n) \).

Q.E.D.

In [3] Brown and Copeland given an example of a 1-connected space having two homology decompositions not all of whose \( n \)-sections are homotopy equivalent. The key point in this non-uniqueness result is the existence of torsion in the homotopy of the space. Toomer [27] shows that such a situation cannot arise if the space is rational (i.e., 1-connected with each homology group a finite dimensional rational vector space).

Prop. 29: If \( X \) is a rational space, then any two homology decompositions have \( n \)-sections of the same homotopy type.

Proof:

Let \( X_2 \subset X_3 \subset \ldots \subset X \) and \( \overline{X}_2 \subset \overline{X}_3 \subset \ldots \subset X \)
be two homology decompositions of \( X \). Passing to minimal models, the ET applied to the canonical decomposition provides a homotopy commutative diagram,
where $M_k$ is the minimal model of $M(X)[k]$. The vertical maps and the right horizontal map are quasi-isomorphisms. Hence, so are $M_k \rightarrow M(X_k)$ and $M_k \rightarrow M(\overline{X}_k)$. $M_k$, $M(X_k)$ and $M(\overline{X}_k)$ are minimal, so we obtain isomorphisms $M(X_k) \cong M_k \cong M(\overline{X}_k)$. Since $X_k$, $\overline{X}_k$ are rational, $X_k = \overline{X}_k$.

Q.E.D.

Remark 30: The proof of Prop. 29 actually shows that any two n-sections of a 1-connected space have the same rational homotopy type. Rationally, then, homology decompositions are unique.

Our final application makes use of the refined structure (Prop. 27) of the canonical homology decomposition to indicate how a formal space may be built by formal n-sections.
We begin by defining a map \( \tau : \frac{L_{n-1}(V)}{L_{n-2}(V)} \) in the following way. Let \( t \in \mathbb{Q} / \{0\} \). Define \( \tau|_V(v) = t^n \cdot v \) and extend freely. Consider

\[
\begin{array}{ccc}
L_{n-1}(V) & \xrightarrow{\tau} & \frac{L_{n-1}(V)}{L_{n-2}(V)} \\
\downarrow & & \downarrow \\
L_{n-1}(V) \otimes L_n(P) & \xrightarrow{\sigma} & L_n(P)
\end{array}
\]

\( \tau \) exists by the ET since \( H^{n+1}(\frac{L_{n-1}(V)}{L_{n-2}(V)}) = 0 \). Also, \( \tau \) is unique up to homotopy since the obstructions to the existence of a homotopy \( \tau_1 \approx \tau_1' \) lie in \( H^n(\frac{L_{n-1}(V)}{L_{n-2}(V)}) = 0 \). Continuing in this fashion we obtain a homotopically unique map \( \tau : \frac{L_{n-1}(V)}{L_{n-2}(V)} \). Note that \( \tau^* \) has the form, \( \tau^*(v) = t^n \cdot v \).

**Theorem 31**: Let \( M_n \xrightarrow{q} M_{n-1} \rightarrow \frac{L_{n-1}(V)}{L_{n-2}(V)} \) be a weak cofibre sequence with \( V = H^n(M) \) and suppose \( M_{n-1} \) is formal. There exists a homotopy commutative diagram.

\[
\begin{array}{ccc}
M_{n-1} & \xrightarrow{k} & \frac{L_{n-1}(V)}{L_{n-2}(V)} \\
\downarrow{\tau} & & \downarrow{\tau} \\
M_{n-1} & \xrightarrow{k} & \frac{L_{n-1}(V)}{L_{n-2}(V)}
\end{array}
\]
where $T$ is a lift of the grading automorphism associated to some $t$ if and only if $M_n$ is formal.

**Proof:**

($\Rightarrow$) Consider the diagram

\[ \begin{array}{ccc}
M_n & \overset{\sigma}{\rightarrow} & M_{n-1} \\
\downarrow{\bar{T}} & & \downarrow{T} \\
M_n & \overset{\sigma}{\rightarrow} & M_{n-1} \\
\downarrow{\bar{T}} & & \downarrow{T} \\
M_n & \overset{\sigma}{\rightarrow} & M_{n-1} \\
\end{array} \]

$k \circ \sigma = \tau \circ k \circ \sigma = 0$. Since $\sigma$ induces isomorphisms in cohomology in dimensions $\leq n-1$, then $\bar{T}^*([x]) = \tau |x| [x]$. We need only check dimension $n$.

Since $H^n(M_n) \cong H^*(k)$ we check the situation on $H^n(k)$.

Let $(a, b) \in Z^n(k)$. Then $a \in Z^n(M_{n-1})$, $b \in V$.

Computing,

\[ T(a, b) = (Ta, Tb) \]

\[ = (t^n a + d\alpha', t^n b) \]

\[ = t^n (a, b) + d(-\alpha', 0) \]
Hence, on cohomology $(\overline{T}^*)^n([x]) = t^n[x]$. Therefore, $\overline{T}$ is a lifting of the grading automorphism associated to $t$. Theorem 5 then insures that $M^n$ is formal.

$(\Leftarrow)$ If $M^n$ is formal, then Prop. 28 insures that $M^n[k]$ is formal for each $k$. Hence, the minimal model $(M^n)_k$ is formal. Consider the homotopy commutative diagram,

\[
\begin{array}{ccc}
M^n & \xrightarrow{\sigma} & M^{n-1} \\
\downarrow{\omega} && \downarrow{\iota} \\
(M^n)_{n-1} & \xrightarrow{\rho} & M^n[n-1]
\end{array}
\]

where $\omega$ and $\eta$ are constructed by the appropriate lifting and extension theorems. Let $\emptyset = \eta \circ \rho$. Since $\emptyset$ is a quasi-isomorphism of minimal DGA's, then $\emptyset$ is an isomorphism. Hence, $(M^n)_{n-1}$ formal implies that $M^{n-1}$ is formal. This is not sufficient to allow us to construct the required diagram, however.

Now, given a lifting of a grading automorphism $\overline{T}_\lambda : M^n \to M^n$, extension and lifting results provide a homotopy commutative diagram,
where \( T_{\lambda} = \emptyset \cdot \xi_{\lambda} \cdot \emptyset^{-1} \). Now \( \sigma = \eta \cdot i = \eta \cdot \rho \cdot \omega = \phi \cdot \omega \), so that if

\[
\begin{array}{ccc}
\text{M}_n & \xrightarrow{\omega} & (\text{M}_n)_{n-1} \\
\downarrow T_{\lambda} & & \downarrow T_{\lambda} \\
\text{M}_n & \xrightarrow{\omega} & (\text{M}_n)_{n-1}
\end{array}
\]

homotopy commutes then we obtain the desired diagram. To show \( \bar{t}_{\lambda} \cdot \omega = \omega \cdot \bar{T}_{\lambda} \) we use the fact that \( \rho \) induces a bijection of homotopy sets \( [\text{M}_n, \text{M}_n(\text{M}_n)] \leftrightarrow [\text{M}_n, (\text{M}_n)_{n-1}] \).

\[
\begin{array}{ccc}
\text{M}_n & \xrightarrow{\rho \cdot \omega} & \text{M}_n(\text{M}_n)_{n-1} \\
\downarrow \bar{T}_{\lambda} & & \downarrow \bar{T}_{\lambda} \\
\text{M}_n & \xrightarrow{\rho \cdot \omega} & \text{M}_n(\text{M}_n)_{n-1}
\end{array}
\]

\[
\begin{align*}
\rho \cdot \bar{t}_{\lambda} \cdot \omega &= \bar{t}_{\lambda} \cdot \rho \cdot \omega = \bar{t}_{\lambda} \cdot i = i \cdot \bar{T}_{\lambda} \\
\rho \cdot \omega \cdot \bar{T}_{\lambda} &= i \cdot \bar{T}_{\lambda}
\end{align*}
\]
Hence $\rho \circ \bar{\tau}_\lambda \circ \omega = \rho \circ \omega \circ \bar{T}_\lambda$ and consequently $\bar{\tau}_\lambda \circ \omega = \omega \circ \bar{T}_\lambda$.

Thus

\[
\begin{array}{ccc}
M_n & \longrightarrow & M_{n-1} \\
\downarrow \phi \circ \omega = \sigma & & \downarrow \phi \circ \omega = \sigma \\
\overline{T}_\lambda & & T_\lambda \\
\downarrow & & \downarrow \\
M_n & \longrightarrow & M_{n-1}
\end{array}
\]

homotopy commutes. Since $T_\lambda$ is clearly a lifting of a grading automorphism, we are done.

Q.E.D.

**Example:** Consider the homology decomposition of $CP(2): S^3 \xrightarrow{H} S^2 \xrightarrow{I} CP(2)$, where $H$ is the Hopf map. Recall; $M(S^3) = (L(z_3), dx=0, dy=x^2)$ and $M(S^2) = (L(x_2,y_3), dx=0, dy=x^2)$ and $M(CP(2)) = (L(u_2,v_5), du=0, dv=u^3)$. $M(H)$ is defined by; $M(H)(x) = 0$, $M(H)(y) = z$.

Now, $T: M(S^3) \to M(S^3)$ is defined by $T(z) = t^4 z$ and $T: M(S^2) \to M(S^2)$ is defined by; $T(X) = t^2 x$ and $T(Y) = t^4 y$. It is easy to check that the following diagram commutes.
Now $M(i): M(\mathbb{C}P(2)) \to M(S^2)$ is defined by

$M(i)(u) = x$ and $M(i)(v) = xy$.

$M(H) \cdot T \cdot M(i) = 0$, so a $\overline{T}$ exists by Prop. 4. But we may construct it directly. Define $\overline{T}$ by,

$\overline{T}(u) = t^2u$, $\overline{T}(v) = t^6v$.

$\overline{T}$ lifts the grading automorphism associated to $t$. Therefore (as we already knew, of course) $\mathbb{C}P(2)$ is formal.

Remark 32: Suppose that the cohomology of $M$ vanishes after some dimension (as, for example, is the situation if $M$ is the minimal model of a finite CW complex): Taking a homology decomposition of $M$, we see that there are a finite number of "obstructions" to the formality of $M$. 
These "obstructions" are diagrams of the sort displayed in Theorem 31.

We now give a more precise formulation of these results. Let $H$ denote the homotopy set $[M_{n-1}, \mathbb{L}_{n-1}(V)]$, where $M_n \to M_{n-1} \to \mathbb{L}_{n-1}(V)$ is the cofibre sequence considered above. We may provide $H$ with the structure of a $\mathbb{Q}$-vector space using the bijection between DGA and spatial homotopy sets and the fact that the realization of $\mathbb{L}_{n-1}(V)$ is a Moore space.

Adjoin the appropriate identity automorphisms to the set of grading automorphisms and to the set of liftings of grading automorphisms of $M_{n-1}$. Denote the latter group by $G$. The grading automorphism associated to $\lambda$ is denoted by $\tau_\lambda$; a lifting by $T_\lambda$. There is an action of $G$ on $H$,

$$G \times H \to H \quad (T_\lambda, f) \to f \cdot T_\lambda.$$ 

The orbit of $k: M_{n-1} \to \mathbb{L}_{n-1}(V)$ is denoted by $\Theta(k)$.

Let $\Gamma(k)$ denote the "curve" in $H$ described by,

$$\gamma: \mathbb{Q} \to H \quad \lambda \to \tau_\lambda \cdot k.$$ 

Note that $\gamma$ is well defined since $\tau_\lambda$ is unique up to homotopy.
The existence of a homotopy commutative diagram,

\[
\begin{array}{c}
M_{n-1} \xrightarrow{k} L_{n-1}(V) \\
\downarrow T_{\lambda} \downarrow \tau_{\lambda} \downarrow \tau_{\mu} \\
M_{n-1} \xrightarrow{k} L_{n-1}(V)
\end{array}
\]

clearly implies that \( \Theta(k) \cap \Gamma(k) \neq \{ k \} \). We wish to show that the reverse implication also holds. The only difficulty that arises is the possibility of a non-trivial intersection \( k \cdot T_{\lambda} = \tau_{\mu} \cdot k \). Considering the diagram,

\[
\begin{array}{c}
M_n \xrightarrow{\sigma} M_{n-1} \xrightarrow{k} L_{n-1}(V) \\
\downarrow T \downarrow \downarrow \tau_{\mu} \\
M_n \xrightarrow{\sigma} M_{n-1} \xrightarrow{K} L_{n-1}(V)
\end{array}
\]

we see that \( T \) cannot be a lifting of a grading automorphism. Hence, we cannot infer that \( M_n \) is formal in this situation. Thus, we must show that the intersection above is impossible. We begin with the,
Lemma 33: Suppose $T: M \to M$ is a DGA automorphism of a minimal DGA $M$ such that $T^*([x]) = \lambda [x]$ for all $[x] \in H^i(M)$, $i \leq k$. Let $M' = \{ x \in M | Tx = \lambda^p x \text{ for some } p \}$. Then $M'$ is a sub-DGA of $M$ and the inclusion $M' \to M$ induces cohomology isomorphisms in dimensions $\leq k$.

Proof:

Clearly $M'$ is a sub-DGA of $M$. We show that the inclusion $M'_C \to M_C$ of complexifications induces the appropriate cohomology isomorphisms. Because $H^*(M')$ and $H^*(M)$ are rational vector spaces the desired result then follows. The following result is easily proved.

Sublemma: If $0 \to V \to W$ is a short exact sequence of vector spaces and $T: V \to V$ is a linear isomorphism leaving $U$ invariant, then $p_V(T) = p_U(T) \cdot p_W(T)$, where $p(T)$ is the characteristic polynomial of $T$.

Now consider the short exact sequence $B^i \to Z^i \to H^i$, $i \leq k$ of coboundaries, cocycles and cohomology of the complexification of $M$. Decompose each of these complex vector spaces into eigenspaces,

$$B = \bigoplus_{\sigma} B_{\sigma}, \quad Z = \bigoplus_{\alpha} Z_{\alpha}, \quad H = H_{\lambda}.$$
The last decomposition follows by hypothesis on $T$. By the sublemma we must have $\alpha = \lambda^i$ for some $\alpha$. Hence $Z_\lambda \subset Z$. Now, $B_\lambda$ may be zero, but we surely have the following commutative diagram,

$$
\begin{array}{cccc}
B & \xrightarrow{j} & Z & \xrightarrow{p} & H = H_\lambda \\
& \cup & \cup & \searrow & \\
& B_\lambda & \xrightarrow{j} & Z_\lambda & \\
\end{array}
$$

Linear transformations preserve eigenspaces, so (since $H = H_\lambda$) $p(Z_\alpha) = 0$ for all $\alpha \neq \lambda^i$. Thus $\overline{p}$ is surjective since $p$ is. Also, if $j(B_\sigma) \cap Z_\lambda \neq 0$, then $\sigma = \lambda^i$. Hence, $Z_\lambda / B_\lambda \to H$ is an isomorphism. Since $Z_\lambda / B_\lambda = H^1(M'_C)$, we are done.

Q.E.D.

Prop. 34: If $(**)$ is homotopy commutative, then $k = 0$.

Proof:

Let $M'$ denote the sub-DGA of $M_n$ considered in the Lemma above; denote the inclusion $M' \to M_n$ by $i$. By
the lifting result for minimal DGA's (since \( \sigma \cdot i \) is a quasi-isomorphism), we obtain a homotopy commutative diagram,

\[
\begin{array}{ccc}
M_{n-1} & \xrightarrow{id} & M_{n-1} \\
\downarrow{\sigma \cdot i} & & \downarrow{\sigma \cdot i} \\
M' & \xrightarrow{\beta} & M'
\end{array}
\]

We then obtain,

\[
k = k \cdot \text{id}_{M_{n-1}}
\]

\[
= k \cdot \sigma \cdot i \cdot \beta
\]

\[
= 0 \quad \text{since } k \cdot \sigma = 0 .
\]

Q.E.D.

Remark 35: If \( k \neq 0 \), then \( M_n \) is obviously formal.

The analogous situation in the spatial version of the homology decomposition is that of attaching a wedge of Moore spaces (which are rationalized spheres).

Hence we may suppose that (**) is homotopy commutative only if \( \lambda = \mu \). Thus, a non-trivial intersection \( \Theta(k) \cap \Gamma(k) \neq \{k\} \) has the desired form, \( k \cdot T_\lambda \equiv \tau_\lambda \cdot k \).
We then immediately obtain a more precise version of Theorem 31.

**Theorem 31':** Let $\sigma_n \to M_{n-1} \to L_{n-1}(V)$ be a weak cofibre sequence and suppose that $M_{n-1}$ is formal. Then $\emptyset(k) \cap \Gamma(k) \neq \{k\}$ if and only if $M_n$ is formal.
BIBLIOGRAPHY


8. Y. Félix and D. Tanré, Sur la formalité des applications, Publications IRMA Lille, 3 no. 2 (1981), 1-44.


28. G.V. Triantafillou, Equivariant minimal models, (to appear in *Trans. AMS*).
